BIDUAL OCTAHEDRAL RENORMINGS AND STRONG REGULARITY IN BANACH SPACES

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Abstract. We prove that every separable Banach space containing ℓ_1 can be equivalently renormed so that its bidual space is octahedral, which answers, in the separable case, a question in Godefroy (1989) [5]. As a direct consequence, we obtain that every dual Banach space, with a separable predual, failing to be strongly regular (that is, without convex combinations of slices with diameter arbitrarily small for some closed, convex and bounded subset) can be equivalently renormed with a dual norm to satisfy the strong diameter two property (that is, such that every convex combination of slices in its unit ball has diameter two).

1. Introduction

The existence of isomorphic copies of ℓ_1 in a Banach space has been a central topic in Banach space theory, in fact there are well known isomorphic characterizations, independent of the considered norm in the space, as the one given by H. Rosenthal in [14]. Also, there are purely geometrical characterizations, dependent of the considered norm in the space. For example, in [13], B. Maurey shows a celebrated characterization of separable Banach spaces containing isomorphic copies of ℓ_1 as those separable Banach spaces X satisfying that there exists a $x^{**} \in X^{**} \setminus \{0\}$ such that

$$\|x + x^{**}\| = \|x - x^{**}\| \quad \text{for all } x \in X.$$

The latter characterization is closely related to the study of ℓ_1-types (also studied by V. Kadets, V. Shepelska and D. Werner in [10]) and fails in the non-separable case [13]. Later, octahedral norms were introduced in an unpublished paper by G. Godefroy and B. Maurey [7], and these kind of norms are used by G. Godefroy and N. Kalton in [6] in relation with the ball topology in Banach spaces, with applications to the study of unique preduals.

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We recall that the norm $\|\cdot\|$ on a Banach space X (or X) is called *octahedral* if, for every finite-dimensional subspace E of X and every $\varepsilon > 0$, there is a $y \in S_X$ such that

$$\|x + y\| \geq (1 - \varepsilon)\big(\|x\| + \|y\|\big) \quad \text{for all } x \in E.$$

In [6, Lemma 9.1] it is proved that a separable Banach space X is octahedral if, and only if, there exists a $x^{**} \in X^{**} \setminus \{0\}$ such that

$$\|x + x^{**}\| = \|x\| + \|x^{**}\| \quad \text{for all } x \in X.$$

It is clear that ℓ_1 (as L_1 or $C([0,1])$) is octahedral, and also it is easy to find non-octahedral equivalent norms in ℓ_1. However, if we allow the space to be renormed, then G. Godefroy proved the following general characterization:

Theorem (see [5, Theorem II.4]). *Let X be a Banach space. The following assertions are equivalent:*

(i) X contains a subspace isomorphic to ℓ_1.
(ii) there exists an equivalent norm $|\cdot|$ in Xsuch that $(X, |\cdot|)$ is octahedral.
(iii) there exists an equivalent norm $|||\cdot|||$ in X and $x^{**} \in X^{**} \setminus \{0\}$ such that

$$|||x + x^{**}||| = |||x||| + |||x^{**}||| \quad \textit{for all } x \in X..$$

Observe that the condition (iii) above is stronger that the condition given by the aforementioned characterization given by B. Maurey, however the above theorem is isomorphic in nature.

It is known that if the bidual X^{**} is octahedral, then so is X. However, the converse is not true in general. The space $C([0,1])$ is a natural example with an octahedral norm and non-octahedral bidual norm. Therefore a natural problem was posed by Godefroy in [5]:

Question 1 (see [5, p. 12]). *If X contains an isomorphic copy of ℓ_1, does there always exist an equivalent norm on X such that the bidual X^{**} is octahedral?*

The main aim of this note is to give a positive answer in the separable case to the abovementioned question.

At present it is known that there is a close relationship between octahedral norms and other geometrical properties in Banach spaces, such as for example, the Daugavet property [11] or the diameter two properties. A Banach space X has the strong diameter two property if every convex combination of slices in B_X, the unit ball of X, has diameter two. It is known that a dual Banach space X^* is octahedral if, and only if, X satisfies the strong diameter two property [1], which gives a complete duality relation between these two properties.

Recall that a Banach space is said to be strongly regular if every closed, bounded and convex subset of X contains convex combinations of slices with diameter arbitrarily small. Strong regularity is a weaker isomorphic property than the well known Radon-Nikodym property in Banach spaces (see [4] for background). It is known that X^* is strongly regular if and only if X does not contain isomorphic copies of ℓ_1 [4, Corollary VI.18]. Hence, Question 1 is a particular case (a dual case in fact) of the general open question:

Question 2 (see [2]). *Can every Banach space failing to be strongly regular be equivalently renormed such that it has the strong diameter two property?*

Let us now to describe the organization of the paper. After some notation and preliminaries, we start the Section 2 with the definition of an octahedral set in a Banach space, which tries to be a localization of octahedrality for subsets in a Banach space, giving local octahedrality properties in the bidual. A such set can be obtained in a dual Banach space, whenever the predual satisfies the strong diameter two property, exploiting the dual relation between octahedrality and the strong diameter two property. We also give at Proposition 2.4 a general result which produces an octahedral bidual norm in a Banach space from another Banach space with an octahedral set. The Section 3 studies the properties of a subset constructed by M. Talagrand [15] in the dual of $C(\Delta)$ the space of continuous functions on Cantor set Δ, with the strong diameter two property. The initial interest of this set was to answer affirmatively to the question posed by J. Diestel and J. Uhl, about the relation between w^*-dentability of w^*-compact subsets of a dual Banach space and the existence of ℓ_1-copies in the predual. Finally, we conclude at Lemmas 3.3 and 3.4 the existence of an isometric ℓ_1-sequence in $C(\Delta)$ which is an octahedral set in some subspace of $C(\Delta)$ equipped with a different norm. The Section 4 implements the general results of Section 1 for the space $C(\Delta)$ and the octahedral set obtained in Section 3 to get at Theorem 4.1 an equivalent norm in $C(\Delta)$ with octahedral bidual norm. Finally, using the good embedding of $C(\Delta)^*$ in the dual of every separable Banach space with ℓ_1-copies, we get at Theorem 4.2 that every separable Banach space with isomorphic copies of ℓ_1 has an equivalent norm with octahedral bidual norm, which answers in the separable case the Question 1. As a direct consequence we get at Corollary 4.3 that a dual Banach space, with separable predual, failing strong regularity can be equivalently renormed, with a dual norm, to satisty the strong diameter two property, which is a partial answer to Question 2 (in fact an answer to the dual case). We finish with some other consequences around the ball topology on Banach spaces and questions.

We pass now to introduce some notation. We consider only real Banach spaces. For a Banach space X, X^* denotes the topological dual of X, B_X and S_X stand for the closed unit ball and unit sphere of X, respectively, and w, respectively w^*, denotes the weak and weak-star topology in X, respectively X^*. For a subspace Y of X, $Y^\perp : \{f \in X^* : f(Y) = \{0\}\}$, which is a subspace of X^*. Then $Y^{\perp\perp}$ (the perp of $Y^\perp$) is a subspace of X^{**}. By $\operatorname{lin} A$ we denote the linear span of the subset A of X. A slice of a set C in X is a set of X given by

$$S(C, x^*, \alpha) = \{x \in C\colon x^*(x) > \sup x^*(C) - \alpha\},$$

where $x^* \in S_X^*$ and $\alpha > 0$. A w^*-slice of a set C of X^* is a slice of C determined by elements of X seen in X^{**}. By j we will denote the canonical embedding of X into X^{**}. If Y is a subspace of X^*, then $X_{|Y}$ will denote the set of functionals in $j(X)$ restricted to Y, that is, $X_{|Y} := \{x_{|Y} : x \in X \subset X^{**}\}$.

2. Previous results

From [9, Proposition 2.1] we know that a Banach space X is octahedral if, and only if, for every $n \in \mathbb{N}$, $\varepsilon > 0$, and for every $x_1, \ldots, x_n \in S_X$ there is $y \in S_X$ such that

$$\|x_i + y\| \geq 2 - \varepsilon \quad \text{for every } i \in \{1, \ldots, n\}.$$

It would be natural to call then X-octahedral to a subset A of vectors in S_X, satisfying that for every $n \in \mathbb{N}$, $\varepsilon > 0$, and for every $x_1, \ldots, x_n \in S_X$ there is $a \in A$ such that

$$\|x_i + a\| \geq 2 - \varepsilon \quad \text{for every } i \in \{1, \ldots, n\}.$$

In the case X is separable and octahedral, it is essentially proved in [10], using ℓ_1-types techniques, that it is possible to choose a X-octahedral subset as a ℓ_1-sequence.

In the case X^{**} is octahedral, then the w^*-lower semicontinuity of the norm in X^{**} gives that it is possible to choose a X^{**}-octahedral set A in such way that $A \subset X$. The next easy lemma shows different ways to get the octahedrality of X^{**}.

Lemma 2.1. *Let X be a Banach space. The following are equivalent:*

(i) *X^{**} is octahedral.*

(ii) *for every $n \in \mathbb{N}$, $x_1^{**}, \ldots, x_n^{**} \in S_{X^{**}}$, and $\varepsilon > 0$, there is $y^{**} \in S_{X^{**}}$ such that*

$$\|x_i^{**} + y^{**}\| \geq 2 - \varepsilon \quad \textit{for every } i \in \{1, \ldots, n\}.$$

(ii') *for every $n \in \mathbb{N}$, $x_1^{**}, \ldots, x_n^{**} \in S_{X^{**}}$, and $\varepsilon > 0$ there is $y \in S_X$ such that*

$$\|x_i^{**} + y\| \geq 2 - \varepsilon \quad \textit{for every } i \in \{1, \ldots, n\}.$$

(iii) *for every $r \geq 1$, $n \in \mathbb{N}$, $x_1^{**}, \ldots, x_n^{**} \in rB_{X^{**}}$, and $\varepsilon > 0$ there is $y^{**} \in B_{X^{**}} \setminus \{0\}$ such that*

$$\|x_i^{**} + y^{**}\| \geq (1 - \varepsilon)(\|x_i^{**}\| + 1) \quad \textit{for every } i \in \{1, \ldots, n\}.$$

(iii') for every $r \geq 1$, $n \in \mathbb{N}$, $x_1^{**}, \dots, x_n^{**} \in rB_{X^{**}}$, and $\varepsilon > 0$ there is $y \in B_X \setminus \{0\}$ such that

$$\|x_i^{**} + y\| \geq (1-\varepsilon)(\|x_i^{**}\| + 1) \quad \textit{for every } i \in \{1, \dots, n\}.$$

Proof. The equivalence of (i) and (ii) is done in [9]. Obviously (ii') $\Rightarrow$ (ii), (iii') $\Rightarrow$ (iii), and (i) $\Rightarrow$ (iii).

(iii')$\Rightarrow$(ii'). Let $n \in \mathbb{N}$, $x_1^{**}, \dots, x_n^{**} \in S_{X^{**}}$, and $\varepsilon > 0$. By (iii') there is a $y \in B_X \setminus \{0\}$ such that

$$\|x_i^{**} + y\| > (1 - \frac{\varepsilon}{4})(1+1) = 2 - \frac{\varepsilon}{2} \quad \text{for every } i \in \{1, \dots, n\}.$$

Therefore $\|y\| > 1 - \frac{\varepsilon}{2}$ and

$$\|x_i^{**} + \frac{y}{\|y\|}\| \geq \|x_i^{**} + y\| - \|y - \frac{y}{\|y\|}\| \geq 2 - \frac{\varepsilon}{2} - (1 - \|y\|) > 2 - \varepsilon$$

for every $i \in \{1, \dots, n\}$.

(iii)$\Rightarrow$(iii'). Let $r \geq 1$, $n \in \mathbb{N}$, $x_1^{**}, \dots, x_n^{**} \in rB_{X^{**}}$, and $\varepsilon > 0$. By (iii) there is a $y^{**} \in B_{X^{**}} \setminus \{0\}$ such that

$$\|x_i^{**} + y^{**}\| > (1-\varepsilon)(\|x_i^{**}\| + 1) \quad \text{for every } i \in \{1, \dots, n\}.$$

By Goldstine's theorem there is a net $\{y_\lambda\} \subset B_X \setminus \{0\}$ such that y_λ weak* converges to y^{**}. Finally, from the weak* lower semicontinuity of the norm in X^{**}, we deduce that there is a λ_0 such that

$$\|x_i^{**} + y_{\lambda_0}\| > (1-\varepsilon)(\|x_i^{**}\| + 1) \quad \text{for every } i \in \{1, \dots, n\}.$$

□

The equivalence of (i) and (iii') in Lemma 2.1 motivates the following definition.

Definition 2.2. Let X be a Banach space and fix B a closed, convex, and bounded subset of X^{**}. A subset $A \subset B_X \setminus \{0\}$ is called an *octahedral set for X^{**} in B* if for every $n \in \mathbb{N}$, $x_1^{**}, \dots, x_n^{**} \in B$, and $\varepsilon > 0$ there is an element $a \in A$ such that

$$\|x_i^{**} + a\| \geq (1-\varepsilon)(\|x_i^{**}\| + 1) \quad \text{for every } i \in \{1, \dots, n\}.$$

In the case $B = B_{X^{**}}$ we will say that A is an octahedral set for X^{**}, without mentioning $B_{X^{**}}$.

Observe that an octahedral set for X^{**} is a subset of X giving the octahedrality in X^{**} and so in X.

As we say in the introduction there is a complete duality relation between, octahedrality and strong diameter two property. In fact, a Banach space X satisfies the strong diameter two property (SD2P in short) if, and only if, X^* is octahedral. The next lemma exploits this dual relation to get octahedral subsets from SD2P subsets.

Lemma 2.3. *Let X be a Banach space. Assume that there is $A \subset B_X \setminus \{0\}$ such that for every $n \in \mathbb{N}$, $0 < \varepsilon < 2$, and every average of slices of B_{X^*}, $\frac{1}{n}\sum_{i=1}^n S(B_{X^*}, x_i^{**}, \varepsilon)$, there exist $x_i^*, y_i^* \in S(B_{X^*}, x_i^{**}, \varepsilon)$ and $x \in A$ such that*

$$\left(\frac{1}{n}\sum_{i=1}^n x_i^* - \frac{1}{n}\sum_{i=1}^n y_i^*\right)(x) > 2 - \varepsilon.$$

Then for every $n \in \mathbb{N}$, $x_1^{**}, \dots, x_n^{**} \in S_{X^{**}}$, and $\varepsilon > 0$ there exists $y \in A$ such that

$$\|x_i^{**} + y\| > 2 - \varepsilon \quad \text{for every } i \in \{1, \dots, n\},$$

and so A is an octahedral set for X^{**}.

Proof. Let $n \in \mathbb{N}$, $x_1^{**}, \dots, x_n^{**} \in S_{X^{**}}$, and $\varepsilon > 0$. Consider now the convex combination of slices $\frac{1}{n}\sum_{i=1}^n S(B_{X^*}, x_i^{**}, \frac{\varepsilon}{2n})$, then by our assumption there are $x_i^*, y_i^* \in S(B_{X^*}, x_i^{**}, \frac{\varepsilon}{2n})$ and a $x \in A$ such that

$$\left(\frac{1}{n}\sum_{i=1}^n x_i^* - \frac{1}{n}\sum_{i=1}^n y_i^*\right)(x) > 2 - \frac{\varepsilon}{2n}.$$

Then $x_i^*(x) - y_i^*(x) > 2 - \frac{\varepsilon}{2}$ for every for every $i \in \{1, \dots, n\}$ and so $x_i^*(x) > 1 - \frac{\varepsilon}{2}$ for every for every $i \in \{1, \dots, n\}$.

Finally, we obtain that

$$\|x_i^{**} + x\| \geq x_i^{**}(x_i^*) + x_i^*(x) > 1 - \frac{\varepsilon}{2n} + 1 - \frac{\varepsilon}{2} \geq 2 - \varepsilon.$$

Therefore, by the equivalence of (ii') and (iii') in Lemma 2.1, A is an octahedral set for X^{**}. ☐

Our strategy will be to renorm a Banach space with an bidual octahedral norm, starting from another Banach space with a good octahedral subset. The next proposition is a stability property under renorming, in this direction.

Proposition 2.4. *Let $(X, \|\cdot\|_X)$ be a Banach space. Assume that there exists a Banach space $(Y, \|\cdot\|_Y)$ and a bounded linear operator $S\colon X \to Y$ with $\|S\| > 0$ such that the following conditions hold:*

(A1) there exists an octahedral set $B \subset B_Y$ for Y^{**};
(A2) there exists Z a subspace of X such that $S|_Z : (Z, \|\cdot\|_X) \to (\widehat{B}, \|\cdot\|_Y)$ is an onto isomorphism, where $\widehat{B}$ denote the closed linear span B.

Then there is an equivalent norm in X such that the bidual X^{**} is octahedral.

Proof. As S is an onto isomorphism, we can renorm Z so that S is a onto isometry. Now, we can extend the above norm to an equivalent norm $\|\cdot\|$ on X. Define a new norm on X by

$$p(x) = \|Sx\|_Y + \|x + Z\| \quad \text{for all } x \in X.$$

It is clear that p is a seminorm. As $S|_Z$ is an one to one, then p is a norm. Denote by $r := \|S\|$. In order to see that p is an equivalent norm $\|\cdot\|$ we show that $\frac{1}{r+2}\|x\| \leq p(x) \leq (r+1)\|x\|$ for every $x \in X$. The upper estimate is clear. For the lower estimate, assume that $x \in X$ with $\|x\| = 1$. If $\|x+Z\| \geq \frac{1}{r+2}$ then it is clear that $p(x) \geq \frac{\|x\|}{r+2}$. Assume now that $\|x+Z\| < \frac{1}{r+2}$, then there is a $z \in Z$ with $\|x-z\| < \frac{1}{r+2}$. Hence, $\|z\| \geq \|x\| - \|x-z\| > \frac{r+1}{r+2}$ and

$$\begin{aligned} p(x) &\geq \|Sx\|_Y \geq \|Sz\|_Y - \|S(x-z)\|_Y \\ &\geq \|Sz\|_Y - \|S(x-z)\|_Y \\ &\geq \|z\| - r\|x-z\| \geq \frac{r+1}{r+2} - \frac{r}{r+2} = \frac{\|x\|}{r+2}. \end{aligned}$$

Note that (X,p) is isometric to a subspace of $Y \oplus_1 X/Z$, thus its bidual $(X,p)^{**}$ is isometric to a subspace of $Y^{**} \oplus_1 X^{**}/Z^{\perp\perp}$. Denote by $\tilde{p}$ the bidual norm of $(X,p)^{**}$, that is,

$$\tilde{p}(x^{**}) = \|S^{**}x^{**}\|_{Y^{**}} + \|x^{**} + Z^{\perp\perp}\| \quad \text{for all } x^{**} \in X^{**}.$$

Finally, we prove that $(X^{**}, \tilde{p})$ has an octahedral norm. Let $n \in \mathbb{N}$, $x_1^{**}, \ldots, x_n^{**}$ in X^{**} with $\tilde{p}(x_i^{**}) = 1$, and $\varepsilon > 0$. By assumption (A1), B is an octahedral set for Y^{**}, thus we can find $b \in B$ such that $\|S^{**}x_i^{**} + b\|_{Y^{**}} \geq \|S^{**}x_i^{**}\|_{Y^{**}} + 1 - \varepsilon$ for every $i \in \{1, \ldots, n\}$. Find now an element $z \in Z$ such that $b = Sz$. Since $S^{**}z = Sz = b$, we have now

$$\begin{aligned} \tilde{p}(x_i^{**} + z) &= \|S^{**}x_i^{**} + S^{**}z\|_{Y^{**}} + \|x_i^{**} + z + Z^{\perp\perp}\| \\ &\geq \|S^{**}x_i^{**}\|_{Y^{**}} + 1 - \varepsilon + \|x_i^{**} + Z^{\perp\perp}\| \\ &= \tilde{p}(x_i^{**}) + 1 - \varepsilon = 2 - \varepsilon \end{aligned}$$

for every $i \in \{1, \ldots, n\}$. $\square$

The next proposition gives a dual Banach space Y with the SD2P from a "good" subset with the SD2P inside another Banach space X^*.

Proposition 2.5. *Let X be a Banach space, $C \subset B_{X^*}$ a weak* compact and convex set, and Y be the linear span of C. Assume that every convex combination of slices of C has diameter two, then there is a (non necessarily equivalent) dual norm $|\cdot|$ in Y satisfying $|\cdot| \geq \|\cdot\|_Y$ such that $(Y, |\cdot|)$ has the strong diameter two property and $B_{(Y,|\cdot|)} = \operatorname{co}(C \cup -C)$. Furthermore, if i is the inclusion map from $(Y, |\cdot|)$ into X^*, and i^* is the adjoint operator from X^{**} into $(Y, |\cdot|)^*$, then $i^*(X)$ is a dense subspace of $(Y, |\cdot|)_*$, the predual of $(Y, |\cdot|)$, with $\|i^*\| \leq 1$. Finally, if Y is closed in X^*, then Y is w^*-closed and $|\cdot|$ and $\|\cdot\|_Y$ are equivalent norms.*

Proof. Define $K := \operatorname{conv}(C \cup -C)$. Then K is a weak* compact and convex set of B_{X^*} such that every convex combination of slices of K has diameter two [1, Lemma 2.12]. Now, for every $\varepsilon > 0$, we define an equivalent norm $\|\cdot\|_\varepsilon$ in X^* whose new unit ball is the set $B_{(X^*,\|\cdot\|_\varepsilon)} = K + \varepsilon B_{X^*}$.

Denote by

$$Z := \{x^* \in X^* \colon \sup_{\varepsilon>0} \|x^*\|_\varepsilon < \infty\}$$

and define a norm on Z by $|x^*| := \sup_{\varepsilon>0} \|x^*\|_\varepsilon$ for every $x^* \in Z$. Observe that $B_{(Z,|\cdot|)} \subset (1+\varepsilon)B_Z$ for every ε and thus $|\cdot| \geq \|\cdot\|_Z$. Note that $(Z, |\cdot|)$ is isometric to the diagonal subspace of the ℓ_∞-sum of the family of Banach spaces $\{(X^*, \|\cdot\|_\varepsilon) : \varepsilon > 0\}$. Then Z is a Banach space.

Now $B_{(Z,|\cdot|)} = K$, that is, $K = \bigcap_{\varepsilon>0}(K+\varepsilon B_{X^*})$. Indeed, clearly $K \subset K + \varepsilon B_{X^*}$ for every $\varepsilon > 0$. For the converse, if $x^* \in \bigcap_{\varepsilon>0}(K + \varepsilon B_{X^*})$, then there are $k_\varepsilon \in K$ and $x^*_\varepsilon \in B_{X^*}$ such that $x^* = k_\varepsilon + \varepsilon x^*_\varepsilon$ for every $\varepsilon > 0$. Hence, $\|x^* - k_\varepsilon\| = \varepsilon\|x^*_\varepsilon\| \leq \varepsilon$ for every $\varepsilon > 0$. Thus, $d(x^*, K) = 0$ and $x^* \in K$, because K is closed.

From the equality $B_{(Z,|\cdot|)} = K$, we get that $Z = \operatorname{lin} B_{(Z,|\cdot|)} = \operatorname{lin} K = Y$. Then $(Y, |\cdot|)$ is a Banach space whose unit ball is compact in a locally convex and separated topology in Y, the weak-star topology of X^* on Y, and so $(Y, |\cdot|)$ is a dual space whose predual is the closure in $(Y, |\cdot|)^*$ of $X_{|Y} := \{x_{|Y} : x \in X \subset X^{**}\}$ (see [12]). Now it is clear that $i^*(X)$ is a dense subspace of $(Y, |\cdot|)_*$ and $\|i^*\| \leq 1$, since $\|i\| \leq 1$. In the case Y is closed in X^*, we have that $|\cdot|$ is an equivalent norm in $Y = Z$, since Y is closed in X, $(Z, |\cdot|)$ is complete and $|\cdot| \geq \|\cdot\|_Z$, and so Y is w^*-closed, applying Banach-Dieudonné Theorem.

Finally, let us show that Y has the strong diameter two property. Let $\delta > 0$, $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in (0, 1)$, and $f_1, \ldots, f_n \in S_{(Y,|\cdot|)^*}$. We call again f_i to the Hahn-Banach extension of f_i to X for every i, and note that the supremum on $K = B_{(Y,|\cdot|)}$ of f_i and the extension agree for every i. We will show that there are $x^*_1, x^*_2 \in \frac{1}{n}\sum_{i=1}^n S(B_{(Y,|\cdot|)}, f_i, \alpha_i)$ such that $|x^*_1 - x^*_2| > 2 - \delta$.

Since every convex combination of slices of K has $\|\cdot\|$-diameter 2, then there are $x^*_1, x^*_2 \in \frac{1}{n}\sum_{i=1}^n S(K, f_i, \alpha_i)$ such that $\|x^*_1 - x^*_2\| > 2 - \delta$. From $B_{(Y,|\cdot|)} = K \subset B_{(Y,\|\cdot\|)}$, we conclude that

$$x^*_1, x^*_2 \in \frac{1}{n}\sum_{i=1}^n S(B_{(Y,|\cdot|)}, f_i, \alpha_i)$$

and

$$|x^*_1 - x^*_2| \geq \|x^*_1 - x^*_2\| > 2 - \delta.$$

□

3. Talagrand set

We begin by introducing some notations and facts from [15, Theorem 4.6], where a "good" subset in $C(\Delta)^*$ is constructed with the SD2P.

Consider the natural numbers $s \geq 3$ and let $(N_s)_{s\geq 3}$ be a partition of $\mathbb{N}$ into disjoint infinite sets. Now fix $s \geq 3$. For $I \subset \mathbb{N}$, $i \in \mathbb{N}$, define on the ith copy of $\{0,1\}$, denoted by $\{0,1\}_i$, a measure

$$\nu_{s,I}^{(i)} = \begin{cases} \frac{1}{s}\delta_0^{(i)} + \frac{s-1}{s}\delta_1^{(i)} & \text{if } i \in I \\ \frac{s-1}{s}\delta_0^{(i)} + \frac{1}{s}\delta_1^{(i)} & \text{if } i \notin I. \end{cases}$$

Now for $J \subset \mathbb{N}$, $I \subset J$, and $p \in J$, define

$$\mu_{s,I}^{J} = \bigotimes_{i\in J} \nu_{s,I}^{(i)},$$

$$\bar{\mu}_{s,I}^{J,p} = \delta_0^{(p)} \otimes \left(\bigotimes_{i\in J\setminus\{p\}} \nu_{s,I}^{(i)} \right), \quad \bar{\bar{\mu}}_{s,I}^{J,p} = \delta_1^{(p)} \otimes \left(\bigotimes_{i\in J\setminus\{p\}} \nu_{s,I}^{(i)} \right),$$

and

$$\rho_I^J = \bigotimes_{s\geq 3} \mu_{s,I\cap N_s}^{J\cap N_s}.$$

If $J = \mathbb{N}$, then we will use the notation $\rho_I := \rho_I^{\mathbb{N}}$.

We also consider the operator $T_J \colon C(\Delta_J) \to C(\Delta_J)$ defined by $T_J(f)(I) = \rho_I^J(f)$, for every $I \subset J$, where we have identified Δ_J with the power set $\mathcal{P}(J)$. One can easily check that $T_J^* \colon \mathcal{M}(\Delta_J) \to \mathcal{M}(\Delta_J)$ satisfies $T_J^*(\delta_I^J) = \rho_I^J$, where δ_I^J is the Dirac measure at I on Δ_J, and then $T_J^*(\theta) = w^*\text{-}\int_{\Delta_J} \rho_I^J \, d\theta(I)$. If $J = \mathbb{N}$, then we will use the notation $T := T_{\mathbb{N}}$.

Note that the identification of Δ with $\mathcal{P}(\mathbb{N})$ is done in the following way: if $x \in \{0,1\}^{\mathbb{N}}$ then we see x as the element in Δ given by $I_x = \{n \in \mathbb{N} : x(n) = 1\}$; if $I \subset \mathbb{N}$ then we can see I as the element $x_I \in \Delta$ given by $x_I(n) = 1$ if $n \in I$ and $x_I(n) = 0$ in otherwise. In this way, we have

$$\nu_{s,I\cap N_s}^{(i)} = \begin{cases} \frac{1}{s}\delta_0^{(i)} + \frac{s-1}{s}\delta_1^{(i)} & \text{if } \ x_I(i) = x_{N_s}(i) = 1 \\ \frac{s-1}{s}\delta_0^{(i)} + \frac{1}{s}\delta_1^{(i)} & \text{if otherwise.} \end{cases}$$

Denote by P_Δ the set of probability measures on Δ. In [15, Theorem 4.6] it was proven that $\mathcal{C} := T^*(P_\Delta)$ is a convex w^*-compact set of P_Δ with the property that every convex combination of slices of $\mathcal{C}$ has diameter two.

The next lemma and proposition are part of the proof of [15, Theorem 4.6] and give one of the tools to find a key octahedral set. We include it here for sake of completeness.

Lemma 3.1 (see [15, Lemma 4.8])**.** *For every $s \geq 3$, every $\delta > 0$, there exists $k = k(s,\delta)$ such that:*

For every $n \in \mathbb{N}$, every $J \subset \mathbb{N}$, $|J| \geq kn$, every $(\varphi_i)_{i\in\{1,\dots,n\}} \in \mathcal{M}(\Delta_J)^*$, $\|\varphi_i\| \leq 1$, there exists $p \in J$ such that

$$\sup_{I\in\{\emptyset,J\}} \sup_{i\in\{1,\dots,n\}} |\varphi_i(\bar{\mu}_{s,I}^{J,p} - \bar{\bar{\mu}}_{s,I}^{J,p})| < \delta.$$

Proposition 3.2 (see the proof of [15, Theorem 4.6]). *Let $n \in \mathbb{N}$ and $S_1, \dots, S_n$ be slices of $\mathcal{C}$. Then for every $s \geq 3$ there exist $J \subset \mathbb{N}$, $|J| \geq n$, and $p \in J$ such that*

$$\sigma_1 := \frac{1}{n}\left(\sum_{i=1}^{l} \mu_{s,J}^{J} \otimes \bar{\gamma}_i + \sum_{i=l+1}^{n} \mu_{s,\{p\}}^{J} \otimes \bar{\gamma}_i\right) \in \frac{1}{n}\sum_{i=1}^{n} S_i,$$

$$\sigma_2 := \frac{1}{n}\left(\sum_{i=1}^{l} \mu_{s,J\setminus\{p\}}^{J} \otimes \bar{\gamma}_i + \sum_{i=l+1}^{n} \mu_{s,\{\emptyset\}}^{J} \otimes \bar{\gamma}_i\right) \in \frac{1}{n}\sum_{i=1}^{n} S_i,$$

where $l \in \{1, \dots, n\}$ and $\bar{\gamma}_i$ is a probability measure for every $i \in \{1, \dots, n\}$.

Proof. Observe first that for every $J \subset \mathbb{N}$, $|J| < \infty$, every $\sigma \in \mathcal{C}$ is a convex combination of elements of $\mathcal{C}$ of the form $\rho_I^J \otimes \gamma_I$, where I runs through the subsets of J, and $\gamma_I = T^*_{\mathbb{N}\setminus J}(\theta_I)$ for some probability measure θ_I on $\Delta_{\mathbb{N}\setminus J}$.

Let $n \in \mathbb{N}$, $\delta > 0$, $\varphi_1, \dots, \varphi_n \in \mathcal{M}(\Delta)^*$ with $\|\varphi_i\| = 1$, and consider

$$S(\mathcal{C}, \varphi_i, \delta) = \{\sigma \in \mathcal{C} \colon \varphi_i(\sigma) > M_i - \delta\},$$

where $M_i = \sup_{\sigma\in\mathcal{C}} \varphi_i(\sigma)$.

For every $s \geq 3$, let $J_0 \subset N_s$, $|J_0| = n2^n k$, where $k = k(s, \delta/2)$ is given by Lemma 3.1. For every $i \in \{1, \dots, n\}$, choose $\sigma_i \in \mathcal{C}$ such that $\varphi_i(\sigma_i) > M_i - \delta/2$. By the above observation (applied to σ_i) and a convexity argument, we may find $I_i \subset J_0$ such that $\varphi_i(\rho_{I_i}^{J_0} \otimes \gamma_{i,I_i}) > M_i - \delta/2$.

By a cardinality argument, there exists $J \subset J_0$, $|J| \geq nk(s, \delta/2)$, which satisfies either $J \subset I_i$ or $J \subset J_0 \setminus I_i$ for every $i \in \{1, \dots, n\}$. If we put $\bar{\gamma}_i = \mu_{s,I_i\setminus J}^{J_0\setminus J} \otimes \gamma_{i,I_i}$, we have, since $J_0 \subset N_s$, that $\rho_{I_i}^{J_0} \otimes \gamma_{i,I_i} = \mu_{s,J\cap I_i}^{J} \otimes \bar{\gamma}_i$.

Define now elements $\psi_i \in \mathcal{M}(\Delta_J)^*$ by $\psi_i(\sigma) = \varphi_i(\sigma \otimes \bar{\gamma}_i)$, and apply Lemma 3.1 to find a $p \in J$ such that

$$\sup_{I\in\{\emptyset,J\}} \sup_{i\in\{1,\dots,n\}} \psi_i(\bar{\mu}_{s,I}^{J,p} - \bar{\bar{\mu}}_{s,I}^{J,p}) < \delta/2.$$

Without loss of generality we can suppose that $J \subset I_i$ for $i \in \{1, \dots, l\}$ and $J \subset J_0 \setminus I_i$ for $i \in \{l+1, \dots, n\}$. By a convexity argument we deduce that

$$\{\mu_{s,J}^{J} \otimes \bar{\gamma}_i, \mu_{s,J\setminus\{p\}}^{J} \otimes \bar{\gamma}_i\} \subset S_i \quad \text{for } i \in \{1, \dots, l\}$$

and

$$\{\mu_{s,\{p\}}^{J} \otimes \bar{\gamma}_i, \mu_{s,\emptyset}^{J} \otimes \bar{\gamma}_i\} \subset S_i \quad \text{for } i \in \{l+1, \dots, n\}.$$

□

Let $p \in \mathbb{N}$ and, in the following, denote by $f_p\colon \Delta \to \mathbb{R}$ the function

$$f_p(x) = \begin{cases} -1 & \text{if } x(p) = 0 \\ 1 & \text{if } x(p) = 1, \end{cases}$$

where $x \in \Delta$. Note that $f_p \in C(\Delta)$ with $\|f_p\| = 1$ for every $p \in \mathbb{N}$.

In the following, we denote by $\mathcal{Y}$ be the linear span of $\mathcal{C}$ and $|\cdot|$ the norm on $\mathcal{Y}$ given by Lemma 2.5. Recall that, from this lemma, $\mathcal{Y}$ is a dual Banach space with the strong diameter two property satisfying $B_{(\mathcal{Y},|\cdot|)} = \mathcal{K} = \mathrm{co}(\mathcal{C} \cup -\mathcal{C})$. Denote by $(\mathcal{Y}_*, |\cdot|_*)$, the predual of $(\mathcal{Y}, |\cdot|)$.

Lemma 3.3. *The set $\{f_{p|\mathcal{Y}}\colon p \in \mathbb{N}\}$ is an octahedral set for $(\mathcal{Y}_*, |\cdot|_*)^{**}$.*

Proof. Note first that, since $B_{\mathcal{Y}} = \mathcal{K} = \{\lambda a - (1-\lambda)b\colon \lambda \in [0,1], a, b \in \mathcal{C}\}$, then it suffices to prove that $\{f_{p|\mathcal{Y}}\}_{p\in\mathbb{N}}$ is an octahedral set in $\mathcal{C}$ for $(\mathcal{Y}_*, |\cdot|_*)^{**}$

Indeed, assume that $\{f_{p|\mathcal{Y}}\}_{p\in\mathbb{N}}$ is an octahedral set in $\mathcal{C}$ for $(\mathcal{Y}_*, |\cdot|_*)^{**}$, and fix $n \in \mathbb{N}$, $\varepsilon > 0$, and $S_1, \ldots, S_n$ slices in $\mathcal{K}$. Observe that every slice S_i has nonempty intersection with $\mathcal{C}$ or $-\mathcal{C}$, since $\mathcal{K} = \mathrm{co}(\mathcal{C} \cup -\mathcal{C})$. Let $I := \{i \in \{1, \ldots n\} : S_i \cap \mathcal{C} \neq \emptyset\}$ and $J := \{1, \ldots n\} \setminus I$. Then the subset

$$A := \frac{1}{n} \sum_{i\in I} S_i \cap \mathcal{C} + \frac{1}{n} \sum_{i\in J} S_i \cap (-\mathcal{C}) \subset \frac{1}{n} \sum_{i=1}^{n} S_i$$

is an average of slices in $\mathcal{C}$. As we are assuming that $\{f_{p|\mathcal{Y}}\}_{p\in\mathbb{N}}$ is an octahedral set in $\mathcal{C}$ for $(\mathcal{Y}_*, |\cdot|_*)^{**}$, there are $x_i^*, y_i^* \in S_i \cap \mathcal{C} \subset S_i$ for every $i \in I$,$x_i^*, y_i^* \in (-S_i) \cap \mathcal{C} \subset S_i$ for every $i \in J$, and $p \in \mathbb{N}$ such that

$$\frac{1}{n} \sum_{i=1}^{n} (x_i^* - y_i^*)(f_p) > 2 - \varepsilon,$$

which gives that $\{f_{p|\mathcal{Y}}\colon p \in \mathbb{N}\}$ is an octahedral set in $B_{\mathcal{Y}} = \mathcal{K}$ for $(\mathcal{Y}_*, |\cdot|_*)^{**}$

Let's prove now that $\{f_{p|\mathcal{Y}}\}_{p\in\mathbb{N}}$ is an octahedral set in $\mathcal{C}$ for $(\mathcal{Y}_*, |\cdot|_*)^{**}$. For this we will apply Lemma 2.3.

Let $n \in \mathbb{N}$, $S_1, \ldots, S_n$ be slices of $\mathcal{C}$, and $\varepsilon > 0$. Find $s \geq 3$ such that $4/s < \varepsilon$. By Proposition 3.2 there exist $J \subset \mathbb{N}$, $|J| \geq n$, and $p \in J$ such that

$$\sigma_1 := \frac{1}{n} \left(\sum_{i=1}^{l} \mu_{s,J}^J \otimes \bar{\gamma}_i + \sum_{i=l+1}^{n} \mu_{s,\{p\}}^J \otimes \bar{\gamma}_i \right) \in \frac{1}{n} \sum_{i=1}^{n} S_i,$$

$$\sigma_2 := \frac{1}{n} \left(\sum_{i=1}^{l} \mu_{s,J\setminus\{p\}}^J \otimes \bar{\gamma}_i + \sum_{i=l+1}^{n} \mu_{s,\{\emptyset\}}^J \otimes \bar{\gamma}_i \right) \in \frac{1}{n} \sum_{i=1}^{n} S_i,$$

where $l \in \{1, \ldots, n\}$ and $\bar{\gamma}_i$ is a probability measure for every $i \in \{1, \ldots, n\}$.

Therefore

$$
\begin{aligned}
&\langle \sigma_1-\sigma_2, f_p\rangle \\
&= \left\langle \frac{1}{n}\left(\sum_{i=1}^{l}\mu^J_{s,J}\otimes\bar\gamma_i+\sum_{i=l+1}^{n}\mu^J_{s,\{p\}}\otimes\bar\gamma_i\right)\right. \\
&\quad \left. -\frac{1}{n}\left(\sum_{i=1}^{l}\mu^J_{s,J\setminus\{p\}}\otimes\bar\gamma_i+\sum_{i=l+1}^{n}\mu^J_{s,\{\emptyset\}}\otimes\bar\gamma_i\right), f_p\right\rangle \\
&= \frac{1}{n}\left\langle (\mu^J_{s,J}-\mu^J_{s,J\setminus\{p\}})\otimes\left(\sum_{i=1}^{l}\bar\gamma_i\right)+(\mu^J_{s,\{p\}}-\mu^J_{s,\emptyset})\otimes\left(\sum_{i=l+1}^{n}\bar\gamma_i\right), f_p\right\rangle \\
&= \frac{s-2}{s}\left\langle (\delta_1^{(p)}-\delta_0^{(p)})\otimes\left[\frac{l}{n}\left(\mu^{J\setminus\{p\}}_{s,J\setminus\{p\}}\otimes\left(\frac{1}{l}\sum_{i=1}^{l}\bar\gamma_i\right)\right)\right.\right. \\
&\quad \left.\left. +\frac{n-l}{n}\left(\mu^{J\setminus\{p\}}_{s,\emptyset}\otimes\left(\frac{1}{n-l}\sum_{i=l+1}^{n}\bar\gamma_i\right)\right)\right], f_p\right\rangle \\
&= 2\frac{s-2}{s} > 2-\varepsilon.
\end{aligned}
$$

In the last equality we use the Fubini theorem, and the fact that the measure which appears between brackets

$$
\left[\frac{l}{n}\left(\mu^{J\setminus\{p\}}_{s,J\setminus\{p\}}\otimes\left(\frac{1}{l}\sum_{i=1}^{l}\bar\gamma_i\right)\right)+\frac{n-l}{n}\left(\mu^{J\setminus\{p\}}_{s,\emptyset}\otimes\left(\frac{1}{n-l}\sum_{i=l+1}^{n}\bar\gamma_i\right)\right)\right]
$$

is a probability measure, since it is a convex combination of probability measures. $\square$

Denote by j the canoncial embedding of $C(\Delta)$ into $C(\Delta)^{**}$. Observe now that, from Lemma 2.5 we have for every $f\in C(\Delta)$ that

$$
f_{|\mathcal{Y}}\in\mathcal{Y}_* \text{ and } |f_{|\mathcal{Y}}|_*\le \|f_{|\mathcal{Y}}\|_{(\mathcal{Y},\|\cdot\|_{C(\Delta)^*})^*}.
$$

In the following lemma we will collect some of the properties of f_p. In particular, the linear spans of f_p in $C(\Delta)$, $j(C(\Delta))|_{\mathcal{Y}}$, and $\mathcal{Y}_*$ are all isomorphic to ℓ_1.

Lemma 3.4. *The functions f_p satisfy the following:*

(1) $1=\|f_p\|_{C(\Delta)}\ge\|\, f_p|_{\mathcal{Y}}\,\|_{(\mathcal{Y},\|\cdot\|_{C(\Delta)^*})^*}\ge|f_{p|\mathcal{Y}}|_*$ *for every $p\in\mathbb{N}$.*

(2) For every $n\in\mathbb{N},\alpha_1,\dots,\alpha_n\in\mathbb{R}$

$$
\Big\|\sum_{i=1}^{n}\alpha_i f_{p_i}\Big\|_{C(\Delta)}=\sum_{i=1}^{n}|\alpha_i|.
$$

(3) For every $n\in\mathbb{N},\alpha_1,\dots,\alpha_n\in\mathbb{R}$

$$
\sum_{i=1}^{n}|\alpha_i|\ge\Big\|\sum_{i=1}^{n}\alpha_i\, f_{p_i}|_{\mathcal{Y}}\Big\|_{(\mathcal{Y},\|\cdot\|_{C(\Delta)^*})^*}\ge\Big|\sum_{i=1}^{n}\alpha_i f_{p_i|\mathcal{Y}}\Big|_*\ge\frac{1}{3}\sum_{i=1}^{n}|\alpha_i|.
$$

Proof. (1) is clear from the preceeding comment.

(2). Let $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R} \setminus \{0\}$, and $p_1, \ldots, p_n \in \mathbb{N}$. Since $\|f_{p_i}\|_{C(\Delta)} = 1$, we clearly have that $\| \sum_{i=1}^n \alpha_i f_{p_i} \|_{C(\Delta)} \leq \sum_{i=1}^n |\alpha_i|$. For the other inequality, choose an $x_0 \in \Delta$ such that

$$x_0(p_i) = \begin{cases} 0 & \text{if } \ \alpha_i < 0 \\ 1 & \text{if } \alpha_i > 0. \end{cases}$$

Hence,

$$\Big\| \sum_{i=1}^n \alpha_i f_{p_i} \Big\|_{C(\Delta)} \geq \sum_{i=1}^n \alpha_i f_{p_i}(x_0) = \sum_{i=1}^n |\alpha_i|.$$

(3). Let $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R} \setminus \{0\}$, and $p_1, \ldots, p_n \in \mathbb{N}$. Find $s_1, \ldots, s_n \geq 3$ such that $p_i \in N_{s_i}$. Observe that the estimates

$$\sum_{i=1}^n |\alpha_i| \geq \Big\| \sum_{i=1}^n \alpha_i \ f_{p_i}|_{\mathcal{Y}} \Big\|_{(\mathcal{Y}, \|\cdot\|_{C(\Delta)^*})^*} \geq \Big| \sum_{i=1}^n \alpha_i f_{p_i|\mathcal{Y}} \Big|_*$$

are clear, because $\|f_{p_i}\|_{C(\Delta)} = 1$ and $\|\cdot\|_{(\mathcal{Y}, \|\cdot\|_{C(\Delta)^*})^*} \geq |\cdot|_*$ on $j(C(\Delta)_{|\mathcal{Y}}$. For the lower estimate choose an $I \subset \mathbb{N}$ such that $p_i \in I$ if $\alpha_i > 0$ and $p_i \notin I$ if $\alpha_i < 0$. Consider the measure

$$\rho_I = \bigotimes_{s \geq 3} \bigotimes_{i \in N_s} \nu^{(i)}_{s, I \cap N_s}$$

and denote by

$$\mu_k := \bigotimes_{s \geq 3} \bigotimes_{i \in N_s \setminus \{p_k\}} \nu^{(i)}_{s, I \cap N_s}.$$

Notice that μ_k is a probability measure for every $k \in \{1, \ldots, n\}$. By the Fubini theorem we have

$$\begin{aligned}
\Big| \sum_{i=1}^n \alpha_i f_{p_i} \Big|_* &\geq \Big| \sum_{i=1}^n \alpha_i \int_\Delta f_{p_i} d\rho_I \Big| \\
&= \Big| \sum_{i=1}^n \alpha_i \int_\Delta f_{p_i} d\Big(\nu^{(p_i)}_{s_i, I \cap N_{s_i}} \otimes \mu_i \Big) \Big| \\
&= \Big| \sum_{i=1}^n \alpha_i \int_{\{0,1\}^{\mathbb{N} \setminus \{p_i\}}} \int_{\{0,1\}_{p_i}} f_{p_i} d\nu^{(p_i)}_{s_i, I \cap N_{s_i}} d\mu_i \Big| \\
&= \Big| \sum_{\alpha_i > 0} \alpha_i \int_{\{0,1\}^{\mathbb{N} \setminus \{p_i\}}} \int_{\{0,1\}_{p_i}} f_{p_i} d\Big(\frac{1}{s_i} \delta^{(p_i)}_0 + \frac{s_i - 1}{s_i} \delta^{(p_i)}_1 \Big) d\mu_i \\
&+ \sum_{\alpha_j < 0} \alpha_j \int_{\{0,1\}^{\mathbb{N} \setminus \{p_j\}}} \int_{\{0,1\}_{p_j}} f_{p_j} d\Big(\frac{s_j - 1}{s_j} \delta^{(p_j)}_0 + \frac{1}{s_j} \delta^{(p_j)}_1 \Big) d\mu_j \Big| \\
&= \Big| \sum_{\alpha_i > 0} \alpha_i \Big(\frac{s_i - 1}{s_i} - \frac{1}{s_i} \Big) + \sum_{\alpha_j < 0} \alpha_j \Big(\frac{1}{s_j} - \frac{s_j - 1}{s_j} \Big) \Big|
\end{aligned}$$

$$= \sum_{\alpha_i>0} |\alpha_i| \Big(\frac{s_i - 2}{s_i} \Big) + \sum_{\alpha_j<0} |\alpha_i| \Big(\frac{s_j - 2}{s_j} \Big)$$
$$\geq \frac{1}{3} \sum_{i=1}^{n} |\alpha_i|.$$

□

4. Main results

Let us summarize the key properties we have proved in the above sections.

(1) (Proposition 2.5) There is a (non closed) subspace $\mathcal{Y}$ of $C(\Delta)^*$ and a dual norm $|\cdot|$ in $\mathcal{Y}$ with the SD2P, such that $\|\cdot\| \leq |\cdot|_{C(\Delta)^*}$ on $\mathcal{Y}$. Furthermore, the restriction to $\mathcal{Y}$ of every element of $C(\Delta)$ is an element in $\mathcal{Y}_*$, the predual of $(\mathcal{Y}, |\cdot|)$.

(2) (Lemmas 3.3 and 3.4) There is an isometric ℓ_1-sequence $\{f_p\}$ in $C(\Delta)$ such that $\{f_{p|\mathcal{Y}}\} \subset \mathcal{Y}_*$ is an octahedral set for $(\mathcal{Y}, |\cdot|)^*$ and also it is an isomorphic ℓ_1-sequence.

The results of the above sections allow us to get now the bidual octahedral renorming for $C(\Delta)$.

Theorem 4.1. *The Banach space $C(\Delta)$ admits an equivalent norm such that its bidual norm is octahedral.*

Proof. Denote by $j_{\mathcal{Y}}$ the natural bounded linear map from $C(\Delta)$ to $j(C(\Delta))|_{\mathcal{Y}} := \{x_{|\mathcal{Y}} : x \in C(\Delta) \subset C(\Delta)^{**}\}$ and by i the inclusion map from $j(C(\Delta))|_{\mathcal{Y}}$ to $\mathcal{Y}_*$. Hence, $S := i \circ j_{\mathcal{Y}}$ is a bounded linear map from $C(\Delta)$ to $\mathcal{Y}_*$. Let Z (respectively, Z_*) be the subspace given by the closed linear span of f_p in $C(\Delta)$ (respectively, by $\{f_{p|\mathcal{Y}}\}$ in $(\mathcal{Y}_*, |\cdot|_*)$).

By Lemma 3.4, $S|_Z$ is an onto isomorphism from $Z \subset C(\Delta)$ onto $(Z_*, |\cdot|_*)$, and $\{f_{p|\mathcal{Y}}\}$ is an octahedral set for $(\mathcal{Y}_*, |\cdot|_*)^{**}$ from Lemma 3.3. Now, applying Proposition 2.4 we are done. □

As a consequence, we get the main announced result, which answer the Question 1 in the separable case.

Theorem 4.2. *If X is a separable Banach space containing a subspace isomorphic to ℓ_1, then there is an equivalent norm in X such that the bidual X^{**} is octahedral.*

Proof. Assume that X contains a subspace isometric to ℓ_1. From [3, Theorem 2] (see also [8]) we know that there is a closed subspace Y of X such that $C(\Delta)$ is isometric to X/Y. Denote by π the quotient map from X to X/Y. By the proof of Theorem 4.1 there is an equivalent norm $|\cdot|$ on X/Y such that its bidual is octahedral, moreover there exists an octahedral set $\{w_p : p \in \mathbb{N}\} \subset B_{(X/Y,|\cdot|)}$ whose linear span W is isomorphic to ℓ_1.

Since $\pi(B_X^\circ) = B_{X/Y}^\circ$, then there is a bounded sequence $\{z_p\} \subset X$ such that $\pi(z_p) = w_p$ for every $p \in \mathbb{N}$. Denote by Z the closed linear span of $\{z_p\}$. From the boundedness of π and $\{z_p\}$ we conclude the existence of $K, L, M > 0$ such that for every $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ we have that

$$K\sum_{i=1}^{n} |\alpha_i| \leq \Big|\sum_{i=1}^{n} \alpha_i w_{p_i}\Big| \leq L\Big\|\sum_{i=1}^{n} \alpha_i z_{p_i}\Big\| \leq M\sum_{i=1}^{n} |\alpha_i|.$$

Thus $\pi|_Z \colon Z \to (W, |\cdot|)$ is an onto isomorphism. Now, applying Proposition 2.4 we are done. □

Notice that the separability assumption is only used in the above proof, to assure that for every separable Banach space X with ℓ_1-copies, there is a linear and bounded operator from X onto $C(\Delta)$. This last assertion is false in the non-separable case, for example is known to be false if $X = \ell_\infty$, since every separable quotient of ℓ_∞ is reflexive [8]. Then the above proof does not work in the general case.

Recall that a Banach space is said to be (w^*-)strongly regular if every closed, bounded and convex subset of X contains convex combinations of (w^*-)slices with diameter arbitrarily small. It is known that X^* is strongly regular if and only if X does not contain isomorphic copies of ℓ_1, and also it is known that strong regularity and w^*-strong regularity are equivalent properties in dual Banach spaces [4, Corollary VI.18].

The next consequence collects the different characterizations of octahedrality, strong diameter two property and strong regularity among the existence of isomorphic copies of ℓ_1 and it is a dual answer to Question 2, in the setting of separable predual.

Corollary 4.3. *Let X be a separable Banach space. The following are equivalent:*

- *(i) X contains a subspace isomorphic to ℓ_1.*
- *(ii) X^* fails to be strongly regular.*
- *(iii) X^* fails to be w^*-strongly regular.*
- *(iv) X has an equivalent octahedral norm.*
- *(v) X has an equivalent norm such that every convex combination of w^*-slices in B_{X^*} has diameter two.*
- *(vi) For every $\varepsilon > 0$ there is an equivalent norm in X such that every convex combination of slices in B_{X^*} has diameter, at least, $2-\varepsilon$.*
- *(vii) there exists an equivalent norm in X such that X^* has the strong diameter two property.*
- *(viii) there exists an equivalent norm in X such that X^{**} is octahedral.*

Proof. The equivalence between the assertions (i) to (vi), and between (vii) and (viii), were written in [1]. The equivalence between (i) and (viii) is the Theorem 4.2. □

A dual Banach space X^* is said to have the w^*-strong diameter two property (w^*-SD2P) if every convex combination of w^*-slices in B_{X^*} has diameter two. Observe that the above result gives, in particular, that SD2P and w^*-SD2P in X^* are equivalent under renorming, whenever X is separable. However, these two properties are not equivalent (see [1]).

Recall that the ball topology $b(X)$ on a Banach space X, is the coarsest topology on X so that the norm closed balls of X are $b(X)$-closed. Another consequence of Theorem 4.2 is the following (see [6]).

Corollary 4.4. *Let X be a separable Banach space and denote by $b(X^{**})_1$ the ball topology in X^{**} restricted to $B_{X^{**}}$. If X contains a subspace isomorphic to ℓ_1, then there is an equivalent norm in X so that, not only $b(X^{**})_1$ fails to be Hausdorff, but every pair of nonempty $b(X^{**})_1$-open subsets of $B_{X^{**}}$ has nonempty intersection.*

Proof. The fact that b is not Hausdorff, under the hypotheses of the above corollary, is a consequence of Theorem 9.3 in [6]. The last conclusion in the above corollary is a consequence of Theorem 4.2 joint to the equivalence between (1) and (3) in Lemma 9.1 of [6], which is valid in the non-separable case with the same proof. □

We will end this note with a couple of questions for the non-separable case. We do not know whether Question 1 is separably determined nor whether it is possible to renorm ℓ_∞ or $L_\infty[0,1]$ such that its bidual is octahedral.

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References

[1] J. Becerra Guerrero, G. López-Pérez, and A. Rueda Zoca, *Octahedral norms and convex combination of slices in Banach spaces*, J. Func. Anal. **266** (2014), 2424–2435.

[2] J. Becerra Guerrero, G. López-Pérez, and A. Rueda Zoca, *Extreme differences between weakly open subsets and convex combinations of slices in Banach spaces*, Adv. Math. **269** (2015), 56–70.

[3] S. J. Dilworth, M. Girardi, and J. Hagler, *Dual Banach spaces which contain isometric copy of L_1*, Bull. Pol. Acad. Sci. Math. **48** (2000), 1–12.

[4] N. Ghoussoub, G. Godefroy, B. Maurey, and W. Schachermayer, *Some topological and geometrical structures in Banach spaces*, Mem. Amer. Math. Soc. **378** (1987).

[5] G. Godefroy, *Metric characterization of first Baire class linear forms and octahedral norms*, Studia Math. **95** (1989), 1–15.

[6] G. Godefroy and N. J. Kalton, *The ball topology and its applications*, Contemporary Math. **85** (1989), 195–237.

[7] G. Godefroy and B. Maurey. *Normes lisses et anguleuses sur les espaces de Banach séparables*, unpublished preprint.

[8] J. Hagler and C. Stegall, *Banach spaces whose duals contain complemented subspaces isomorphic to* $C[0,1]^*$, J. Func. Anal. **13** (1973), 233–251.

[9] R. Haller, J. Langemets, and M. Põldvere. *On duality of diameter 2 properties*, J. Conv. Anal. **22** (2015), no. 2, 465483.

[10] V. Kadets, V. Shepelska, and D. Werner, *Thickness of the unit sphere, ℓ_1-types, and the almost Daugavet property*, Houston J. Math. **37** (2011), 867–878.

[11] V. Kadets, R. Shvidkoy, G. Sirotkin, and D. Werner, *Banach spaces with the Daugavet property*, Trans. Amer. Math. Soc. **352** (2000), 855–873.

[12] S. Kaijser, *A note on dual Banach spaces*, Math. Scand. **41** (1977), 325-330.

[13] B. Maurey, *Types and ℓ_1-subspaces*, Longhorn Notes, Texas Functional Analysis Seminar, Austin, Texas 1982/1983.

[14] H. Rosenthal, *A characterization of Banach spaces containing l^1*, Proc. Nat. Acad. Sci. U.S.A., **71** (1974), 2411-2413.

[15] W. Schachermayer, A. Sersouri, and E. Werner, *Moduli of nondentability and the Radon–Nikodým property in Banach spaces*, Israel J. Math. **65** (1989), 225–257.

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