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Geometry with octahedral norms and nonlinear geometry of Banach spaces

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Abstract

In the present thesis, several problems of diverse nature surrounding the theory of Banach spaces are addressed. Even though full solutions have not always been achieved, constituent advance has been delivered in most of the cases.

We are going to subdivide this varied project into three general topics: Isomorphic theory, nonlinear geometry and isometric theory.

In respect to the isomorphic theory of Banach spaces, we are mainly interested in properties such as the Krein-Milman property (KMP) and the Radon-Nikodym property (RNP), as well as linear and nonlinear approximation properties. Firstly, we generalize a well-known result by W. Schachermayer on the equivalence between the KMP and RNP under another property called strong regularity. Then, we turn on approximation properties and seize the opportunity to relate the classical version of those properties with their nonlinear counterparts. Specifically, we give a nonlinear description of the Finite Dimensional Decomposition for dual spaces.

Secondly, we approach a problem risen by G. Godefroy and N. Ozawa in 2014 where they wonder whether every separable Banach space has a generating compact and convex Lipschitz retract (GCCR). We find sufficient conditions for this retract to exist as well as a necessary condition under a 'mild' assumption on the assymptotic shape of the GCCR. We also give a complete solution to the Hölder version of that problem, as well as a construction of a Banach space without good GCCR's. Strongly connected to the latter problem is N. Kalton's problem on approximability of separable spaces or, equivalently, the study of the Bounded Approximation property in Lipschitz free spaces over nets of separable Banach spaces. We amend a previously known result from P. Hájek and M. Novotný studying a retractional structure in nets for finite dimensional spaces, spaces with a basis containing c_0 and separable $\mathcal{L}_\infty$-spaces.

Finally, we focus on isometric properties of Banach spaces. We start giving a metric characterization of the Daugavet property in vector-valued Lipschitz spaces and then turn on proving that every vector-valued Lipschitz space over a metric space which is not bounded and uniformly discrete is the dual of an octahedral space (the vector-valued Lipschitz free space). We finish this project studying topological stability of open sets under averages on a convex set. In addition to being a highly exclusive property of the unit ball, it is purely geometric and characterizes the family of ℓ_1-preduals among all L_1-preduals.

Keywords: Banach spaces, Nonlinear geometry, octahedral norms, Retractions, Lipschitz, Approximation property.

Resumen

En la presente tesis se abordan varios problemas de diversa naturaleza alrededor de la teoría de espacios de Banach. Aunque no siempre se han podido encontrar soluciones completas, en la mayoría de casos se ha conseguido un avance constitutivo.

Vamos a dividir este variado proyecto en tres temas generales: Teoría isomorfa, geometría no lineal y teoría isométrica.

Con respecto a la teoría isomorfa de espacios de Banach, nuestro principal interés reside en propiedades como la propiedad de Krein-Milman (KMP) o la propiedad de Radon-Nikodym (RNP), así como propiedades de aproximación lineales y no lineales. Primero generalizamos un conocido resultado de W. Schachermayer sobre la equivalencia entre la KMP y la RNP bajo la llamada regularidad fuerte. Después, pasamos a estudiar propiedades de aproximación, relacionando la versión clásica de estas propiedades con su contrapartida no lineal. Concretamente, damos una descripción no lineal de la existencia de descomposiciones finito-dimensionales en espacios duales.

En segundo lugar, enfocamos un problema planteado por G. Godefroy y N. Ozawa en 2014, donde se preguntan si en todo espacio de Banach separable existe un retracto Lipschitz convexo, compacto y generador (GCCR). Encontramos condiciones suficientes para que este retracto exista así como condiciones necesarias bajo una 'leve' condición sobre el tamaño asintótico del GCCR. También damos una solución completa a la versión Hölder de dicho problema y construimos un espacio de Banach sin buenos GCCRs. Fuertemente relacionado con este último problema está el problema de N. Kalton sobre la aproximabilidad de espacios separables, equivalentemente, la propiedad de aproximación acotada en espacios Lipschitz libres sobre redes de espacios de Banach separables. Se mejora un resultado de P. Hájek y M. Novotný, estudiando una estructura retraccional en redes de espacios finito-dimensionales, espacios con base conteniendo a c_0 y espacios $\mathcal{L}_\infty$ separables.

Finalmente, nos centramos en propiedades isométricas de espacios de Banach. Empezamos dando una caracterización métrica de la propiedad de Daugavet en espacios Lipschitz vector-valuados y luego pasamos a probar que el espacio Lipschitz vector-valuado sobre un espacio métrico que no sea a la vez acotado y uniformemente discreto es el dual de un espacio octaedral (su espacio Lipschitz libre vector-valuado). Finalizamos este proyecto estudiando la estabilidad topológica de conjuntos abiertos de un convexo bajo medias. Esta propiedad en la bola unidad, además de ser muy exclusiva, es puramente geométrica y caracteriza a la fimilia de preduales de ℓ_1 de entre los más generales preduales de L_1.

Palabras clave: Espacios de Banach, Geometría no lineal, Normas octaedrales, Retracciones, Lipschitz, Propiedad de aproximación.

Abstrakt

V této práci se zabýváme několika problémy z různých oblastí teorie Banachových prostorů. Některé z těchto problémů jsou zde zcela vyřešeny, u ostatních bylo dosaženo podstatného pokroku.

Naše práce je rozdělena do tří částí: isomorfní teorie, nelineární geometrie a isometrická teorie.

V isometrické části se především zabýváme Krein-Milmanovou (KMP) a Radon-Nikodýmovou (RNP) vlastností, spolu s lineárními a nelineárními vlastnostmi aproximace. Nejprve provedeme zobecnění známého výsledku W. Schachermayera o ekvivalenci KMP a RNP za předpokladu takzvané silné regularity. Poté obrátíme svoji pozornost na aproximační vlastnost a studujeme souvislost mezi klasickými aproximačními vlastnostmi a jejich nelineárními verzemi. Přesněji, předvedeme nelineární charakterizaci vlastnosti FDD pro duální prostory.

Ve druhé části studujeme problém Godefroy a Ozawy z roku 2014 o existenci generujícího konvexního kompaktu (GCCR) v každém separabilním Banachově prostoru. Nalezneme postačující podmínky pro existenci takového kompaktu a též podmínky nutné, za jistého dodatečného předpokladu na asymptotické chování zmíněného kompaktu. Předvedeme též úplné řešení pro Holderovskou verzi problému a dále konstrukci Banachova prostoru s GCCR s neočekávanými vlastnostmi. Tato problematika úzce souvisí se známým problémem N. Kaltona o aproximabilitě separabilních Banachových prostorů, nebo v ekvivalentní formulaci o omezené aproximační vlastnosi (BAP) Lipschitzovsky volných (Lipschitz free) prostorů nad sítěmi v separabilních Banachových prostorech. Pokračujeme tak ve studiu retrakční struktury, započatém Hájkem a Novotným, pro případ prostorů se Schauderovou bází obsahujících c_0 a separabilních $\mathcal{L}_\infty$-prostorů.

Ve třetí části nakonec studujeme isometrické vlastnosti Banachových prostorů. Nejprve dokážeme metrickou charakterizaci Daugavetovy vlastnosti pro vektorově hodnotové Lipschitzovské prostory a poté dokážeme, že každý takový prostor nad metrickým prostorem, jenž není omezený a unifomně diskrétní, je duálem k oktahedrálnímu prostoru (vektorově hodnotovému Lipschitzovsky volnému prostoru). Náš projekt je zakončen studiem topologické stability otevřených množin na konvexní průměrování na konvexních množinách. Tato vlastnost je velmi limující a charakterizuje třídu ℓ_1-preduálů mezi všemi L_1-preduály.

Klíčová slova: Banachových prostorů, Nelineární geometrie, oktaedrická norma, retrakce, Lipschitz, aproximační vlastnost.

Acronyms

c.c.b. Closed convex and bounded

KMP Krein-Milman property

RNP Radon-Nikodym property

CPCP Convex point of continuity property

SR Strong regularity

SRO Strong regularity for open sets

DdP Diameter d property

w Weak topology

FDD Finite dimensional decomposition

BAP Bounded approximation property

AP Approximation property

GCCR Generating compact convex Lipschitz retract

URED Uniformly rotund in every direction

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Chapter 1

Introduction

1.1 Main goals of the thesis

This thesis is focused on the study of the geometry of Banach spaces. Banach spaces are a common setting in a wide variety of situations and problems in mathematics such as PDE's or machine learning and there has been an intense research of its geometry since the early 20th century. As a consequence of so many years of development, this branch of mathematics has been studied from several points of view. Therefore, there are plenty of subdivisions of Banach spaces theory but, for the purpose of this thesis, two different criteria are given to distinguish between problems:

- The isomorphic or isometric nature of the problem. Even though interest in isometric properties has been shown getting on in years, the most classical approach to this matter is the so-called isomorphic theory of Banach spaces. Isomorphic properties of Banach spaces are those properties stable by isomophisms between Banach spaces whereas the isometric theory deals with properties stable by the much stronger surjective isometries.
- The level of linearity. The linear structure present in the very definition of a Banach space inevitably leads to its study being centered on properties given by linear operators or functionals. This approach has been challenged in the recent and not so recent past by people studying nonlinear mappings. In order to have a quantitative estimate on the level of continuity of such mappings, Lipschitz and uniform continuity are in the heart of nonlinear geometry of Banach spaces.

"At present I propose above all to gather results about linear operators defined in certain general spaces" *S. Banach, 1932*

In this thesis we carry out different projects concerning the latter contrasting possibilities of research in the geometry of Banach spaces. Here we present a list of ambitious goals. The aim of this dissertation is not solving all of them but approaching them and trying to give some new insights.

1.1.1 Isomorphic theory of Banach spaces

Let us develop here a more detailed explanation of the fragment of the isomorphic theory that we are interested in.

Firstly, we focus on the geometry of the closed convex and bounded subsets of a Banach space (c.c.b. sets for short). In this regard, the Krein-Milman and Radon-Nikodym properties

focus on the 'amount of vertices' and the 'quality' of these vertices in c.c.b. sets. More precisely, a c.c.b. set C is said to enjoy the Krein-Milman property (KMP for short) if every c.c.b. subset of C has an extreme point whereas C is said to enjoy the Radon-Nikodym property (RNP for short) whenever every c.c.b. subset of C has a denting point, which is a stronger version of an extreme point. Despite the seemingly stronger nature of the RNP over the KMP, it is a long-standing problem to show whether these two properties are actually different or not. This leads to the first goal of the dissertation.

(G_1) Delve into the known results on the relationship between the KMP and the RNP trying to give new insights to the problem.

On the other hand, one of the most challenging difficulties of the Banach spaces theory is the lack of compactness in the unit ball of infinite dimensional spaces. As humans, when tackling a problem we would like to have an intuition that resembles the nature and geometry around as. Therefore, having a finite approximation of the elements of study is extremely desirable. Since Banach's times several questions concerning the finite (or compact) approximation of Banach spaces have attracted the attention of experts all around the globe. In this respect, several approximation properties have been defined, some of them are the existence of a Schauder basis (motivated by the concept of basis in finite dimensional spaces), the existence of a finite dimensional decomposition (FDD for short) or the bounded approximation property (BAP for short). All these properties may be defined by means of uniformly bounded finite-rank linear operators pointwise converging to the identity operator on a Banach space. This leads to our second goal of the section.

(G_2) Study approximation properties in some classes of Banach spaces and its structural implications.

1.1.2 Nonlinear Geometry of Banach spaces

On the one hand, in 1932 Mazur and Ulam moved off the mainstream of assuming linearity in all mappings and went on proving that every surjective isometry (not necessarily linear) between two Banach spaces is indeed affine [MU32]. Hence, the only onto isometries preserving 0 are the linear ones. On the other hand, if no quantitative assumption on the level of continuity of the surjection is considered, nothing may be derived since Kadec proved in 1966 that all separable infinite dimensional Banach spaces are homeomorphic [Kad67]. Therefore, the central nonlinear morphisms that generally appear when avoiding linearity are Lipschitz or uniformly continuous.

One of the topical issues in the linear theory is the study of linear bounded projections and complemented subspaces of Banach spaces, extremely connected to the aforementioned approximation properties. It is therefore natural to study its nonlinear counterpart, the Lipschitz and uniformly continuous retractions. In this setting we find a question by Godefroy and Ozawa in [GO14, Question 3] which is one of the main goals of this thesis.

(G_3) Do all separable Banach spaces have a Lipschitz retraction onto a compact and convex subset whose closed linear span is the whole space?

We call such a retract a *generating compact convex Lipschitz retract* (GCCR for short). In fact, if the retraction is λ-Lipschitz for some $\lambda \geqslant 1$ we will call it a λ-GCCR. It is worth mentioning that the existence of a GCCR in a Banach space X would immediately imply the existence of a sequence of equi-Lipschitz mappings into X with compact range and pointwise converging to the identity, a property initially considered in [Kal12] that may be seen as a nonlinear approximation property. Kalton also considered in [Kal12] a relaxation of the mentioned nonlinear approximation property where the latter sequence is asked to be equi-uniformly continuous instead of equi-Lipschitz. From now on, we will say that X is *Lipschitz approximable* if the above is satisfied for an equi-Lipschitz sequence and *approximable* if the sequence is just equi-uniformly continuous. In [Kal12, Problem 1] the following question, which constitutes another important goal of this thesis, arises:

(G_4) Are all separable Banach spaces approximable?

Finally, since linear approximation properties have been far more studied than nonlinear ones it is desirable to find connections between linear and nonlinear approximation properties in the spirit of the Ribe program (see [Nao12]). We therefore address the following problem.

(G_5) Find relations between linear and nonlinear approximation properties.

1.1.3 Isometric properties of Banach spaces

In this subsection we are interested in properties that depend completely on the norm of a Banach space. Some of these properties are the octahedrality of the norm and the Daugavet property. A Banach space is said to be octahedral if for every finite collection of elements $x_1, \ldots, x_n$ of the sphere there is another element x in the sphere which is almost ℓ_1-perpendicular to all of them, that is, for every $\varepsilon > 0$, one may find such an x satisfying $\|x + x_i\| \geqslant 2 - \varepsilon$ for every $i = 1, \ldots, n$. On the other hand, a Banach space X is said to enjoy the Daugavet property if every rank-one operator $T : X \to X$ satisfies $\|I + T\| = 1 + \|T\|$. We will be specially intrigued by the behaviour of these properties in the space $\mathrm{Lip}_0(M, X)$ of X-valued Lipschitz mappings defined over a metric space M, where X is any Banach space.

(G_6) Give a characterisation of the metric spaces M whose Lipschitz space $\mathrm{Lip}_0(M, X)$ has the Daugavet property or is the dual of an octahedral space for every Banach space X.

Finally, we study a very restrictive property for closed convex subsets of Banach spaces called weak stability. A c.c.b. set C is said to be weakly stable if any average of weakly open sets of C is again weakly open in C. If C is the unit ball of a Banach space X then weak stability of C implies that X^* is octahedral. Far from trivial, there are very few known examples of Banach spaces whose unit ball is weakly stable. Hence, our last goal will be the following:

(G_7) Find classes of Banach spaces with a weakly stable unit ball.

1.2 Background and motivation of the thesis

In this section we are going to motivate all the goals stated in Section 1.1.

1.2.1 Isomorphic theory of Banach spaces

The question of whether the KMP implies the RNP may be consider by now a classical open problem in functional analysis. Many well-known mathematicians have tried to solve it unsuccessfully and very little progress has been achieved in the recent past. However, several important advancements have been made. Let us state here some of the most important ones.

A real breakthrough was obtained in 1975 by Huff and Morris when they showed that RNP and KMP are equivalent for dual Banach spaces in [HM75] (here we say that a Banach space enjoys the RNP or KMP if its unit ball enjoys it). In 1980, Bourgain proved that the RNP is determined by spaces with an FDD, showing that if every subspace with FDD of a given Banach space enjoys the RNP then the space itself also enjoys it. He used a highly nontrivial construction in [Bou80] which was later clarified by Schachermayer in [Sch87a], who also proved in [Sch87b] the equivalence between KMP and RNP for spaces with unconditional basis.

The main result of [Sch87b] is in the core of the theory around the KMP and RNP and states that under strong regularity the latter properties are equivalent, where a c.c.b. set C is said to be strongly regular if every c.c.b. subset of C has arbitrarily small convex combinations of weakly open sets. This resembles a local result by Bor-Luh Lin, Pei-Kee Lin and Troyanski in [LLT88] that every extreme point which is a point of continuity is indeed a denting point. We will try to build on this result by Schachermayer and give some new insights to the problem.

Now, in order to discuss the background of our second goal of the thesis (G_2), let us change the setting and focus on a very different kind of isomorphic properties, the approximation properties.

Let X be a Banach space and consider a sequence of linear operators $T_n : X \to X$. We state here some properties for such a sequence (T_n):

(P_1) - Exists $\lambda > 0$ such that $\|T_n\| \leqslant \lambda$ for every $n \in \mathbb{N}$.
- $\dim(T_n(X)) < \infty$ for every $n \in \mathbb{N}$.
- $\|T_n(x) - x\| \to 0$ for every $x \in X$.

(P_2) $T_n \circ T_m = T_{\min(n,m)}$ for every $n, m \in \mathbb{N}$.

(P_3) $\dim(T_n(X)) = n$ for every $n \in \mathbb{N}$.

Let X be separable. If it is possible to find a sequence (T_n) satisfying (P_1), (P_2) and (P_3) we say that X has a *Schauder basis*. We will say that X has an FDD whenever (T_n) can be found satisfying only (P_1) and (P_2). Finally, we will say that X has the λ-BAP if just (P_1) holds. The 1-BAP is often referred as MAP whereas BAP will mean λ-BAP for some $\lambda \geqslant 1$. Also, if there is a net $T_\alpha : X \to X$ of finite-rank linear operators converging to the identity operator uniformly on compacta then we say that X has the *approximation property* (AP for short).

These concepts have been around the world of Banach spaces for a long time but it was not until Per Enflo gave his counterexample in [Enf73b] to the approximation property that we got to know a separable Banach space without any of the aforementioned approximation properties. He first constructed in [Enf73a] a Banach space with basis constant strictly grater

than one, a result that he finally improved in [Enf73b] by means of a similar approach. Since then, many results on approximation properties have appeared, showing that none of the implications below can be reversed

$$\text{Schauder basis} \;\Rightarrow\; \text{FDD} \;\Rightarrow\; \text{BAP} \;\Rightarrow\; \text{AP}.$$

See [Sza87], [Rea] and [FJ73]. However, one of the still open questions from Grothendieck's times is whether there is a dual Banach space with the AP but without the MAP. We will refer to this question as the AP VS MAP problem which is motivated by Grothendieck's result that every separable dual space with the AP enjoys the MAP (see [Gro55]).

1.2.2 Nonlinear geometry of Banach spaces

Regarding our third goal (G_3) (completely central in this dissertation), it is worth mentioning that very little was known at the beginning of this project. Godefroy and Ozawa gave in [GO14, Question 3] a simple construction of a GCCR for spaces with unconditional basis. However, their construction does not seem to work when unconditionality is removed.

As we have already mentioned, GCCR's are connected to the Lipschitz approximability of a Banach space which is in turn seemingly stronger than being approximable. After Kalton showed in [Kal12, Theorem 4.6], by means of a very ingenious argument, that if X^* is separable then both X and X^* are approximable, a possitive solution to question (G_3) and specially (G_4) sounds conceivable. However, we will show in Section 1.3 some evidence suggesting that at least question (G_3) might have a negative answer. During the process of tackling these goals, a geometric study of the 'asymptotic smallness' of compact sets is going to be crucial for us, as we will see in Section 1.3.

An important tool in nonlinear geometry of Banach spaces and general metric geometry is the Lipschitz-free space over a metric space (M, d_M), often denoted as $\mathcal{F}(M)$ (see [GK03] for some background on Lipschitz-free spaces). Indeed, Kalton proved in [Kal12, Theorem 4.4] that X is approximable if, and only if, the Lispchitz-free space over a net of X has the BAP. This constitutes a connection between the nonlinear structure of a Banach space and the linear structure of a Lipschitz-free space, building hence a bridge between goals (G_2) and (G_4). In this setting, Hájek and Novotný addressed in [HN17] the problem of finding Schauder basis in Lipschitz-free spaces over nets in Banach spaces. Indeed, they showed the existence of this basis for nets in spaces with unconditional basis, leaving open the more general case of spaces with Schauder basis. They obtain that result by findidng a retractional structure in the underlying net.

It is worth mentioning that both a positive and a negative answer to Kalton's problem would be highly appreciated. Indeed, a counterexample to (G_4) would solve the aforementioned AP VS MAP problem by means of a renorming of ℓ_1 without the MAP (its dual space would have the AP but not the MAP). Following a similar direction to that of [Kal12, Theorem 4.4] we find a previous result by Godefroy and Kalton showing that a Banach space X enjoys the BAP if, and only if, the Lipschitz-free space $\mathcal{F}(X)$ enjoys it [GK03, Theorem 5.3]. Therefore,

the stability of the BAP by Lipschitz isomorphisms is derived. This result suggests that goal (G_5) may be quite interesting.

1.2.3 Isometric properties of Banach spaces

In this part of the thesis we have to focus on 3 different isometric properties. The equation satisfied in the Daugavet property is known as *Daugavet equation* because I. Daugavet proved in [Dau63] that every compact operator on $\mathcal{C}([0,1])$ satisfies it. Since then, many examples of Banach spaces enjoying the Daugavet property have appeared. E.g. $\mathcal{C}(K)$ for a perfect compact Hausdorff space K; $L_1(\mu)$ and $L_\infty(\mu)$ for a non-atomic measure μ; or preduals of Banach spaces with the Daugavet property (see [KKW03, KSSW00, Wer01] for a detailed treatment of the Daugavet property).

Here, we will focus on the space $\mathrm{Lip}_0(M,X)$ since it is known that $\mathrm{Lip}_0(M)$ has the Daugavet property if, and only if, the metric space M is length. The natural question that arises in this setting is therefore whether the same metric characterisation works in the vector valued case, that is, whether the space $\mathrm{Lip}_0(M,X)$ being Daugavet is equivalent to M being length regardless which X is used as codomain. This problem (goal (G_6)) has been around since the latter characterisation of Lipschitz spaces was found in [GLPRZ18a, Theorem 3.5] and it has been explicitly asked in [GLPRZ18a, Question 1] and [LRZ20, Remark 3.8]. Also, in [GLPRZ18b, Theorem 2.4] the predual of $\mathrm{Lip}_0(M,X)$ is shown to be octahedral for every complete metric space M which is not uniformly discrete and bounded, provided X is a Hilbert space. They then ask in [GLPRZ18b, Question 3.2 (1)] whether the predual of $\mathrm{Lip}_0(M,X)$ is octahedral for every Banach space X whenever M is a complete metric space which is not both uniformly discrete and bounded.

Now, we move on to the third isometric property that we are studying: weak stability and its analogue in the norm topology referred here as *stability*. In general, stability is a restrictive property with several implications. First of all, stability gives solution to the abstract Dirichlet problem [CM75], and secondly, it is a key to the study of extremal operators, since for a compact and Hausdorff space K and a Banach space X with stable unit ball, a norm one element $f \in C(K,X)$ is an extreme point of the unit ball if, and only if, every value of f is an extreme point in the unit ball of X [Gra92].

Weak stability has attracted the attention of some researchers in geometry of Banach spaces and new examples of weak stable unit balls of Banach spaces have emerged. For example, in [AL18] and [HKPo19] it is proved that the unit ball of $C_0(K)$, for a Hausdorff and locally compact space K, is weak stable if, and only if, K is scattered (see also [AGH$^+$20] and [GFP20]). As it can be seen in the above references, the current interest is to know how big the class of Banach spaces with a weak stable unit ball is, and the natural questions in the setting of the above papers is to know what preduals of L_1 or what $C_0(K,X)$ spaces have a weak stable unit ball. This is what will be aimed for (G_7). Moreover, Lindenstrauss seemed to be interested in giving geometric characterizations for important families of L_1-preduals among all L_1-preduals and the weak stability of the unit ball is, as we will see in Section 1.3, a very good candidate for the family of ℓ_1-preduals.

1.3 Progress achieved

In this section we proceed to show the advances obtained during the development of this thesis. We will present a summary of the contents of each of the papers that were written as a consequence of all the time spent carrying out the different projects commented in Section 1.1. The order in which these are presented is the same order in which the goals above appear and it is not necessarily chronological.

Each of the subsections 1.3.1, 1.3.2, 1.3.3, 1.3.4, 1.3.5, 1.3.6, 1.3.7, 1.3.8 and 1.3.9 below respectively correspond to the papers [LPM24], [HM23a], [HM23b], [Med24], [Med23], [HM22], [HM23c], [MRZ23] and [LPM22].

1.3.1 The equivalence between CPCP and strong regularity under the Krein-Milman property

Let X be a Banach space, D is a closed convex and bounded subset of X and τ is a locally convex topology (τ has a countable basis of open subsets) in D containing the relative weak topology on D. For every subset C of D we will denote $\tau|_C$ the induced topology of τ in D. With this notation, given $\delta > 0$, we say that D has the

i) *τ-convex point of continuity property* (τ-CPCP) if, for every nonempty convex subset C of D and every $\varepsilon > 0$ there is a nonempty $\tau|_C$-open set with diameter less than ε.

ii) *τ-strongly regular* (τ-SR) if, for every nonempty convex subset C of D and every $\varepsilon > 0$ there is a convex combination of nonempty $\tau|_C$-open sets with diameter less than ε.

iii) *τ-strongly regular for open subsets* (τ-SRO) if, for every nonempty and convex τ-open subset O of D and for every $\varepsilon > 0$ there is a convex combination of nonempty $\tau|_O$-open subsets with diameter less than ε.

iv) *τ-Diameter δ property* (τ-DδP) if every nonempty τ-open subset of D has diameter at least $\delta > 0$.

We managed to show that, with the above properties, the following analogue of Schachermayer's Theorem 2.1 in [Sch87b] holds.

Theorem 1.3.1. *Let X be a separable Banach space and D a τ-strongly regular closed, bounded and convex subset of X without the τ-CPCP, where τ is a locally convex topology on D containing the weak topology on D. Then, there is a closed convex subset of D without extreme points, that is, D fails KMP.*

Also, digging into the proof of Theorem 1.3.1 the following isometric version of Theorem 1.3.1 is deduced.

Corollary 1.3.2. *Let X be a separable Banach space and D a closed, bounded and convex subset of X with the τ-DδP for some positive δ, where τ is a locally convex topology on D containing the weak topology on D. If D is τ-SRO then D fails KMP.*

In the spirit of goal (G_1), since RNP and KMP are equivalent when w-CPCP and SR are not [Sch87b], the above theorem suggests exploring a new way to proceed: starting from a Banach space X failing CPCP, is it possible to construct a locally convex topology τ on the unit ball of X, B_X, containing the weak topology so that B_X still fails τ-CPCP but satisfies τ-SR? A positive answer to this question would show that RNP and KMP are indeed equivalent. Complementarily, if X fails CPCP there is a c.b.c subset C of B_X with the w-DδP for some $\delta > 0$, where w stands for the weak topology on C. Is it possible to construct τ in C such that C still enjoys τ-D$\widetilde{\delta}$P for some $\widetilde{\delta} > 0$ but it is τ-SRO? This would also solve the KMP VS RNP problem.

In this regard we have the following result.

Theorem 1.3.3. *There exists a locally convex topology τ in B_{c_0} containing the weak topology such that B_{c_0} has the τ-DδP for $\delta = 2$ and is τ-SRO.*

1.3.2 Compact retractions and Schauder decompositions in Banach spaces

We focus now on goals (G_2) and (G_3). Firstly, we need to give some definitions.

Let X be a separable Banach space. X is said to have the π-property if there is a uniformly bounded sequence of finite rank projections (S_n) on X pointwise converging to the identity on X.

We will say that $\beta = (e_n) \subset X$ is a fundamental sequence if $[e_n] = X$ and we will denote $E_n^\beta = [e_i]_{i=1}^n$.

Let $\beta = (e_n)$ be a fundamental sequence of X and $K \subset X$ a bounded subset. We define the following concepts:

- The sequence of inner radii $(r_n(\beta, K))$ given by

$$r_n(\beta, K) = \sup\left\{r \geqslant 0 \ : \ B_{E_n^\beta}(x, r) \subset K \cap E_n^\beta \ , \ x \in X\right\} \quad \forall n \in \mathbb{N}.$$

- The sequence of heights $(h_n(\beta, K))$ given by

$$h_n(\beta, K) = \sup\left\{d(x, K \cap E_n^\beta) \ : \ x \in K\right\} \quad \forall n \in \mathbb{N}.$$

Definition 1.3.4. We say that a bounded subset K of a separable Banach space X is *Small* if there exist an $\varepsilon \in (0, 1)$, a fundamental sequence $\beta = (e_n)$ in X and a strictly increasing $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$0 < \frac{h_{\sigma(n)}(\beta, K)}{r_{\sigma(n)}(\beta, K)} \leqslant \frac{1}{2\sigma(n)^2\Big(\big(1 + \frac{2}{\varepsilon}\big)^{\sigma(n)} + 2\Big)} \quad \forall n \in \mathbb{N}. \tag{1.3.1}$$

If a GCCR K of a Banach space X is Small, we say it is a Small GCCR. With this in mind, we prove the following results.

Theorem 1.3.5. *If a Banach space X has an FDD then X admits a Small GCCR.*

Theorem 1.3.6. *If a Banach space X admits a Small GCCR then X has the π-property.*

Since the π-property is known to be equivalent to the existence of an FDD for dual spaces, we have the following characterization.

Corollary 1.3.7. *Let X be a dual Banach space. Then X admits a Small GCCR if, and only if, X has an FDD.*

Now, regarding general Lipschitz or uniformly continuous retracts, the following generalization of the Lindenstrauss-Tzafriri characterization of Hilbertian spaces is obtained.

Theorem 1.3.8. *Let X be a Banach space. Then X is isomorphic to a Hilbert space if and only if every convex and compact subset of X is a Lipschitz retract of X.*

Hence, there is no hope of fulfilling (G_3) without any assumption on the shape of the compact set. On the other hand, we will see that every separable Banach space satisfy the condition of Theorem 1.3.8 for the weaker uniform continuity instead of Lipschitz continuity.

A normed space X is said to be uniformly rotund in the direction $z \in X$ if whenever (x_n) and (y_n) are two sequences in X such that

1. $||x_n|| = ||y_n|| = 1$ for every $n \in \mathbb{N}$,
2. $\lim_{n\to\infty} ||\frac{x_n+y_n}{2}|| = 1$,
3. There is a sequence of real numbers (r_n) such that $x_n - y_n = r_n z$ for every $n \in \mathbb{N}$,

then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.
If X is uniformly rotund in the direction z for every $z \in S_X$ then we say that X is uniformly rotund in every direction (URED for short).

Theorem 1.3.9. *Let X be a separable Banach space and $K \subset B_X$ be a compact convex subset. If X is URED then the nearest point map from B_X onto K is unique and uniformly continuous.*

Corollary 1.3.10. *If X is separable then every compact and convex subset K of X is a uniformly continuous retract of X.*

1.3.3 Retractions and the bounded approximation property in Banach spaces

We keep here the trend of finding connections between goals (G_2) and (G_3) in the spirit of (G_5). In particular, we will find necessary conditions for a separable space to admit a GCCR and a slightly weaker condition to that of Theorem 1.3.6 is obtained for a more general family of GCCR's.

Let K be a bounded subset of a separable Banach space X. We say that K is *Flat* if there is $\varepsilon > 0$ and a fundamental sequence β in X such that the sequence $(h_n(\beta, K))$ of heights satisfy

$$\liminf_n (3+\varepsilon)^n h_n(\beta, K) = 0. \tag{1.3.2}$$

If such set K is also a GCCR we will call it a Flat GCCR. It is worth mentioning that condition (1.3.2) is far less restrictive than the condition given in (1.3.1) for the definition of smallness (Definition 1.3.4).

In this setting we have the following Theorem 1.3.11

Theorem 1.3.11. *Let X be a separable Banach space. If X admits a Flat GCCR then X enjoys the BAP.*

1.3.4 On the constant of Lipschitz approximability

Judging from the results obtained in subsections 1.3.2 and 1.3.3, the existence of a counterexample to (G_3) in the family of spaces without approximation properties sounds more than reasonable. Motivated by this fact, we now explore the path traced by Enflo when constructing his counterexmple for the linear approximation property in the seek of a counterexample to (G_3). He first produced a Banach space with basis constant strictly greater than one in [Enf73a]. Analogously, it would be interesting to know if there is a separable Banach space with no λ-GCCR's for some $\lambda > 1$. This is the problem that we address here.

Theorem 1.3.12. *There is $\lambda > 1$ and a separable Banach space X isomorphic to ℓ_1 such that X does not have any λ-GCCR.*

It is worth mentioning that the fact that X in Theorem 1.3.12 is isomorphic to ℓ_1 is not just purely anecdotal. Indeed, X can be shown to have basis constant strictly greater than 1 which is somehow close to failing the MAP. This, as mentioned before, would solve the AP VS MAP problem.

1.3.5 Compact Hölder retractions and nearest point maps

In this subsection, a positive result to a Hölder version of question (G_3) is given. This could potentially be connected to (G_4).

Given K a bounded subset of a separable Banach space and a sequence of positive numbers (h_n). We will say that K is *h_n-flat* if there is a fundamental sequence β in X such that $h_n(\beta, K) \leqslant h_n$ for every $n \in \mathbb{N}$.

Proposition 1.3.13. *For every decreasing sequence of positive real numbers (h_n) and every separable Banach space X there is an h_n-flat compact convex subset of X whose closed linear span is X.*

Theorem 1.3.14. *Let $\alpha \in (0,1)$. Then, every $20^{\frac{n}{\alpha-1}}$-flat subset of a Banach space X is an α-Hölder retract of X.*

Corollary 1.3.15. *For every separable Banach space X and every $\alpha \in (0,1)$ there is a generating compact convex set K of X which is an α-Hölder retract of X. Moreover, for each $\varepsilon > 0$, K can be chosen to be Flat.*

Corollary 1.3.16. *For every separable Banach space X without the BAP and for every $\alpha \in (0,1)$ there is a generating compact convex subset of X which is an α-Hölder retract of X but not a Lipschitz retract of X.*

1.3.6 Schauder basis in Lipschitz free spaces over nets in Banach spaces

We now move on to the goal (G_4) which is in turn connected to (G_2) as shown by Kalton in [Kal12, Theorem 4.4]. That is, we will focus on approximation properties of the Lipschitz-free space over a net in a Banach space. More specifically, as stated in the title of this subsection, we will seek for Schauder basis in this setting, where we find two main questions.

Question 1.3.17. *If X is a Banach space with Schauder basis, does the Lipschitz-free space over a net of X have Schauder basis?*

Question 1.3.18. *If X is a Banach space with Schauder basis, does the Lipschitz-free space over the entire grid of X have Schauder basis?*

For us, a net of a separble Banach space X is a subset N of X which is uniformly discrete and discretely dense. More precisely, N is *uniformly discrete* whenever there is $a > 0$ such that $d(x, y) \geqslant a$ for every $x, y \in N$, $x \neq y$, whereas we say that N is *discretely dense* if there is $b > 0$ such that for every $x \in N$ there is $y \in N$ with $d(x, y) \leqslant b$. Also, if X has a Schauder basis (e_n) then the entire grid given by that basis is defined as the set

$$G = \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} \mathbb{Z} e_k.$$

Our approach to this problem is completely nonlinear since we will work with the retractional structure of these nets.

Definition 1.3.19. Let M be a countable metric space and $K \geqslant 1$. A K-retractional basis of M is a sequence of retractions $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n : M \to M$ satisfying,

1. $\varphi_n(M) = M_n := \bigcup_{j=1}^{n} \{x_j\}$,
2. $\bigcup_{j=1}^{\infty} \{x_j\} = M$,
3. φ_n is K-Lipschitz for every $n \in \mathbb{N}$,
4. $\varphi_m \circ \varphi_n = \varphi_{\min(m,n)}$ for every $m, n \in \mathbb{N}$.

We will say that M has a retractional basis whenever it has a K-retractional basis for some $K \geqslant 1$.

This is a reasonable approach due to the fact that the existence of a K-retractional basis in a countable metric space M forces the existence of a Schauder basis with constant $\lambda = K$ in $\mathcal{F}(M)$ (as shown in [HN17]). We present here a solution to Question 1.3.18 and several positive results concerning Question 1.3.17.

Theorem 1.3.20. *The entire grid given by a Schauder basis of a Banach space has a retractional basis.*

Theorem 1.3.21. *Let $n \in \mathbb{N}$ and let N be a net of an n-dimensional Banach space. Then N has a K-retractional basis where K does not depend on n.*

Theorem 1.3.22. *Let N be a net of a Banach space with a Schauder basis. If $c_0 \subset X$ then N has a retractional basis.*

Theorem 1.3.23. *The net N of a Banach space with a c_0-like FDD has a retractional basis.*

Here c_0-like FDD stands for spaces which are c_0-sums of finite dimensional spaces or isomorphic to them.

1.3.7 Schauder basis in Lipschitz free spaces over nets of $\mathcal{L}_\infty$-spaces

While all the positive results for Question 1.3.17 above need the containment of c_0 in the space, we present here some examples of nets in Banach spaces that do not contain c_0. We follow the same nonlinear approach of finding retractional basis but the construction developed here is completely different.

A Banach space X is said to be an $\mathcal{L}_\infty$-space if there is $\lambda > 0$ such that every finite dimensional subspace F of X is contained in another subspace E of X which is λ-isomorphic to ℓ_∞^n for some $n \in \mathbb{N}$.

Theorem 1.3.24. *A net N in a separable infinite dimensional $\mathcal{L}_\infty$-space has a retractional basis.*

1.3.8 A characterisation of the Daugavet property in spaces of vector-valued Lipschitz functions

Finally, we turn our attention to the isometric part of the dissertation. Here, goal (G_6) will be completely achieved.

Theorem 1.3.25. *If M is a complete length metric space and X is a Banach space then $\mathrm{Lip}_0(M, X)$ has the Daugavet property.*

For the next result we need to introduce the vector-valued Lipschiz-free space. Given M a metric space and X a Banach space, the *vector-valued Lipschitz-free* space is going to be the projective tensor product

$$\mathcal{F}(M)\widehat{\otimes}_\pi X.$$

In the same way that $\mathcal{F}(M)$ is the canonical predual of $\mathrm{Lip}_0(M)$, the vector-valued Lipschitz-free space also satisfies $(\mathcal{F}(M)\widehat{\otimes}_\pi X)^* \equiv \mathrm{Lip}_0(M, X)$.

Theorem 1.3.26. *Let M be a metric space. If M is not a bounded and uniformly discrete space then $\mathcal{F}(M)\widehat{\otimes}_\pi X$ is octahedral for every Banach space X.*

1.3.9 A characterization of the weak topology in the unit ball of purely atomic L_1-preduals

Let us introduce in detail the last isometric properties we will be working with. Given C a c.c.b. subset of a Banach space X, we say that C is

1) *stable* if for every pair of relatively norm open subsets U_1, U_2 of C, the set $\frac{U_1+U_2}{2}$ is again relatively norm open in C.
2) *weakly stable* if for every pair of relatively weakly open subsets U_1, U_2 of C, the set $\frac{U_1+U_2}{2}$ is again relatively weakly open in C.

It is worth mentioning that weakly stable unit balls enjoy extremely strong diameter 2 properties such as the attaining symmetric strong diameter 2 property, that is, given any finite family of slices, there exists a direction such that all these slices contain a line segment of length 2 in this direction. Therefore, to certain extent, weak stability may be seen as an extreme negation of the RNP.

Theorem 1.3.27. *Let X be a real or complex isometric predual of a $L_1(\mu)$ space for some measure μ. Then B_X is weakly stable if, and only if, μ is purely atomic.*

As we said in the introduction, we are also interested in getting examples of (weak) stability in the class of $C_0(K, X)$.

Theorem 1.3.28. *Let X be a stable and weakly stable Banach space and K a scattered Hausdorff topological space. Then $C_0(K, X)$ is stable and weakly stable.*

Theorem 1.3.28 turns Theorem 2.5 (b) of [AGH$^+$20] into a complete characterization since weak and norm stability agree on finite dimensional spaces. With the same techniques used here we solve Question (a) from [AGH$^+$20].

Chapter 2

The equivalence between CPCP and strong regularity under the Krein-Milman property

Ginés López-Pérez and Rubén Medina. The equivalence between CPCP and strong regularity under Krein-Milman property. J. Funct. Anal., 286(4): Paper No. 110273, 2024. [LPM24].

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THE EQUIVALENCE BETWEEN CPCP AND STRONG REGULARITY UNDER KREIN-MILMAN PROPERTY

GINÉS LÓPEZ-PÉREZ AND RUBÉN MEDINA

Abstract. We obtain a result in the spirit of the well-known W. Schachermeyer and H. P. Rosenthal research about the equivalence between Radon-Nikodym and Krein-Milman properties, by showing that, for closed, bounded and convex subsets C of a separable Banach space, under Krein-Milman property for C, one has the equivalence between convex point of continuity property and strong regularity both defined for every locally convex topology on C, containing the weak topology on C. Then, under Krein-Milman property, not only the classical convex point of continuity property and strong regularity are equivalent, but also when they are defined for an arbitrary locally convex topology containing the weak topology. We also show that while the unit ball B of c_0 fails convex point of continuity property and strong regularity (both defined for the weak topology), threre is a locally convex topology τ on B, containing the weak topology on B, such that B still fails convex point of continuity property for τ, but B surprisingly enjoys strong regularity for τ-open sets. Moreover, B satisfies the diameter two property for the topology τ, that is, every nonempty τ-open subset of B has diameter two even though every τ-open subset of B contains convex combinations of relative τ-open subsets with arbitrarily small diameter, that is, B fails the strong diameter two property for the topology τ. This stresses the known extreme differences up to now between those diameter two properties from a topological point of view.

1. Introduction

There are three families of subsets in each bounded, convex, and non-empty subset C of a Banach space X that are highly relevant to understand the weak topology w of the space X and its structural properties: the family of slices of C (a subbasis of the weak topology), the family of relatively weakly open subsets of C (or a basis of the weak topology) and the family of convex combinations of slices or relatively weakly open subsets of C (a π-basis of the weak topology). From the isomorphic point of view, these three families of subsets give rise to three widely studied isomorphic

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properties in Banach spaces, the Radon-Nikodym property (RNP) for C (existence of slices of arbitrarily small diameter in each bounded and convex subset of C), the convex point of continuity property (CPCP) for C (existence of relatively weakly open subsets of arbitrarily small diameter in each bounded and convex subset of C) and strong regularity (SR) for C (existence of convex combinations of slices or relatively weakly open subsets with arbitrarily small diameter in each bounded and convex subset of C). When C is the unit ball B_X of a Banach space X, it is said that X verifies RNP, CPCP, or is strongly regular, respectively (see [7], [11] for background). Another widely studied property related to the previous ones is the Krein-Milman property (KMP). It is said that the subset C verifies the KMP if each nonempty, closed, bounded and convex subset of C has some extreme point, and when C is the unit ball of a space it is said that the space itself verifies the KMP. It is known that RNP implies CPCP and CPCP implies SR, and that these three properties are different [2]. It is also known that RNP implies KMP, and it is a famous problem still open today whether KMP implies RNP. Research on this problem has obtained some important partial answers (see for example [7], [8], [9], [12], [13], [14], [16]), although in general the most relevant result in this regard has been to be able to relate the properties defined above, obtaining that for a closed, bounded, convex and strongly regular subset C of a Banach space X, C verifies the RNP as long as C verifies the KMP ([17], [15]). Consequently, the problem of the equivalence between the RNP and the KMP is solved for subsets where the RNP and the SR are not equivalent, or if desired, this problem is solved for subsets in which the CPCP and the SR are not equivalent, since RNP implies CPCP. In short, the previous result can be stated by saying that under the KMP, one has that CPCP and SR are equivalent properties. Since the CPCP and the SR are defined in terms of families of subsets relevant to the weak topology, it is then natural to wonder if translating the definitions of the CPCP and SR for a topology other than the weak one may draw similar results. In this note we study this possibility, showing that for a locally convex topology that contains the weak topology, it is also obtained that the new CPCP and SR properties for the above topology are equivalent under the KMP, which generalizes the results known up to now. As a consequence, a new strategy to obtain the equivalence between the RNP and KMP is given: starting from a Banach space X failing CPCP for the weak topology (otherwise we already know that RNP and KMP are equivalent for X), it would be enough to build a locally convex topology τ on the unit ball B_X of X, satisfying that for this new topology B_X still fails CPCP while verifying SR, which implies, applying the previously announced result, that X does not enjoy KMP either. In fact, if one starts with a Banach space whose unit ball has nonempty relatively weakly open subsets with uniformly positive diameter, we prove that it is enough to assume SR for open subsets in order to get the failure of KMP. The above strategy is based on the possibility of building a suitable topology for which the CPCP and SR properties are not equivalent. The most ambitious question would be whether this can be done for any space without the classical CPCP, but we don't have an answer to this question for now. However, another natural question is whether this can be done in some space where CPCP and SR properties are equivalent for the

weak topology, or at least for some space where the unit ball does not satisfy either of the above two properties. The answer in this case is affirmative, since it is shown in this note that there is a locally convex topology on the unit ball B of the space c_0, containing the weak topology, such that B is SR for the family of open subsets of this topology, but it does not verify CPCP for the same topology. Of course, such a topology is not the weak topology because B is neither CPCP nor SR for the weak topology.

From the geometric point of view, there are also widely studied properties as a non-isomorphic counterpart to the properties we have talked about so far, they are the so-called diameter 2 properties, which are also defined through the same families of subsets of the unit ball of a Banach space relevant to the weak topology discussed above. A Banach space verifies the diameter two property for slices (slice-D2P), diameter two property (D2P) or strong diameter two property (SD2P) if each slice, nonempty weak open, or convex combination of slices or nonempty weak open subsets, respectively, of the unit ball of the space has diameter 2. These properties are related to other well-known properties, such as the Daugavet property and octahedrality, and dualize as non-differentiability properties of the dual norm. Furthermore, it is known that the three previous properties are different and it is clear that SD2P implies D2P and D2P implies slice-D2P (see [3], [4], [6]). Precisely, to obtain the difference between the D2P and SD2P properties, an equivalent norm was constructed in the space c_0 in such a way that each nonempty weak open subset of the unit ball of c_0 has a diameter 2 for the new norm, while for such a norm it is possible to find convex combinations of slices or nonempty weak open subsets with arbitrarily small diameter [4]. It is natural then, as previously stated for the CPCP and SR properties, to ask if it is possible to obtain a similar result to the above for the unit ball of c_0 using the usual norm of c_0, but changing the definition of D2P and SD2P to another proper topology. The answer is yes, in fact, for the previously mentioned topology built on the unit ball of c_0, it is obtained that the unit ball of c_0 verifies the D2P for this topology while each nonempty open subset for this topology contains convex combinations of relatively open subsets for the same topology with arbitrarily small diameter, so in particular it does not satisfy SD2P for the same topology. Of course, the diameters are measured here with the usual norm of c_0. Therefore, the differences between the D2P and SD2P properties are even greater than what has been known so far if they are defined for more general topologies, even when keeping the natural norm of the space.

2. Main results

Let X be a Banach space, D is a closed convex and bounded subset of X and τ is a locally convex topology (τ has a countable basis of open subsets) in D containing the relative weak topology on D. For every subset C of D we will denote $\tau|_C$ the induced topology of τ in D. With this notation, we recall the definitions for CPCP and SR. We say that D has the

i) τ-convex point of continuity property (τ-CPCP) if, for every bounded and convex subset C of D and every $\varepsilon > 0$ there is a nonempty $\tau|_C$-open set with diameter less than ε.
ii) τ-strongly regular (τ-SR) if, for every bounded and convex subset C of D and every $\varepsilon > 0$ there is a convex combination of nonempty $\tau|_C$-open sets with diameter less than ε.
iii) τ-strongly regular (τ-SR) for open subsets if, for every nonempty and convex τ-open subset O of D and for every $\varepsilon > 0$ there is a convex combination of nonempty $\tau|_O$-open subsets with diameter less than ε.
iv) Krein-Milman property (KMP) if every nonempty closed and convex subset of D has some extreme point.

When τ is the weak topology restricted to D, we use CPCP, SR and SR for open subsets, and omit the reference to the topology. Also, when D is the unit ball of a space X and τ is defined on X, we say that X satisfies τ-CPCP, τ-SR or τ-SR for open subsets, respectively.

As we said in the introduction, it is well known that D satisfies CPCP whenever D is SR and satisfies KMP, that is, under KMP, CPCP and SR are equivalent. Our mail goal is to get the same result for another topology other than the weak one. We start with a couple of easy lemmas, omitting the elementary proof of the first one.

Lemma 2.1. Let E be a closed subset of a Banach space X and let (S_n) be a decreasing sequence of subsets of E. If $\operatorname{diam}(S_n) \to 0$ then there is $x \in E$ such that every sequence $(x_n) \subset E$ with $x_n \in \overline{S_n}$ for every $n \in \mathbb{N}$ converges to x.

Lemma 2.2. If D is τ-strongly regular and C is a nonempty convex subset of D, then, for every $\varepsilon > 0$ and every nonempty set $O \in \tau|_C$ there is a a finite family of nonempty sets $O_1, \dots O_n \in \tau|_C$, with $O_i \subset O$, such that

$$\operatorname{diam}\Big(\sum_{i=1}^{n} \frac{1}{n} O_i\Big) < \varepsilon.$$

Proof. Since τ is locally convex and C is convex we get that $\tau|_C$ is locally convex. Hence, there is a nonempty convex set $\widetilde{O} \in \tau|_C$ with $\widetilde{O} \subset O$. Now, as D is τ-strongly regular, we deduce that there are nonempty sets $O_1, \dots, O_n \in \tau|_{\widetilde{O}}$ such that

$$\operatorname{diam}\Big(\sum_{i} \frac{1}{n} O_i\Big) < \varepsilon.$$

We are now done because $O_i \in \tau|_{\widetilde{O}} \subset \tau|_C$ for every $i = 1, \dots, n$. □

Inspired by Schachermayer's work in [17], we introduce in the following definition the subdiameter (SD) in order to estimate from bellow the diameter of convex combinations of weakly open subsets, taking into account the coefficients of the convex combinations.

Notation. Given some countable set S, we denote $\ell_1(S)$ the summable functions from S to $\mathbb{R}$. We also denote $S^+_{\ell_1(S)}$ the positive face of the sphere of $\ell_1(S)$, that is,

$$S^+_{\ell_1(S)} = \Big\{(x_k)_{k\in S} \ : \ x_k \geq 0 \text{ for } k \in S \text{ and } \sum_{k\in S} x_k = 1\Big\}.$$

If $\#S = n \in \mathbb{N}$ then $\ell_1(S)$ is lattice isometric to $\ell_1(\{1,\dots,n\})$, which we denote simply by ℓ_1^n.

Definition 2.3. Let C be a convex subset of D and $n \in \mathbb{N}$. Given a n-tuple of nonempty sets $(O_i)_{i=1}^n \subset \tau|_C$ and $\lambda \in S^+_{\ell_1^n}$, we define the sub-diameter of $\sum_i \lambda(i)O_i$ as

$$SD(\lambda,(O_i)) = \inf\Big\{\operatorname{diam}\Big(\sum_i \lambda(i)U_i\Big) \ : \ U_i \in \tau|_{O_i}\setminus\{\emptyset\} \text{ for } 1\leq i\leq n\Big\}.$$

The next two lemmas use the definition of subdiameter to get a relation between the diameter of a convex combination of weak open subsets and the distance from a finite set to these convex combination.

Lemma 2.4. Let C be a convex subset of D, $\varepsilon > 0$, $n,k \in \mathbb{N}$, $\lambda \in S^+_{\ell_1^n}$, $\{x_1,\dots,x_k\} \subset X$ and a n-tuple of nonempty sets $(\widetilde{O}_i)_{i=1}^n \subset \tau|_C$. Then, for $1 \leq i \leq n$ there is a nonempty $\tau|_{\widetilde{O}_i}$-open set O_i such that:

$$\operatorname{diam}\Big(\sum_i \lambda(i)O_i\Big) \leq SD(\lambda,(O_i)) + \varepsilon, \tag{2.1}$$

$$d\Big(\{x_1,\dots,x_k\}, \sum_i \lambda(i)O_i\Big) \geq \frac{SD(\lambda,(O_i))}{2} - \varepsilon. \tag{2.2}$$

Proof. By definition, for each $i = 1,\dots,n$ there must be $O_i^0 \in \tau|_{\widetilde{O}_i}\setminus\{\emptyset\}$ such that

$$\operatorname{diam}\Big(\sum_i \lambda(i)O_i^0\Big) \leq SD(\lambda,(\widetilde{O}_i)) + \varepsilon/2 \leq SD(\lambda,(O_i^0)) + \varepsilon/2. \tag{2.3}$$

We will be done if we find $O_i \in \tau|_{O_i^0}\setminus\{\emptyset\}$ for $1\leq i\leq n$ satisfying (2.2). Indeed, we would have from (2.3) that

$$\operatorname{diam}\Big(\sum_i \lambda(i)O_i\Big) \leq \operatorname{diam}\Big(\sum_i \lambda(i)O_i^0\Big) \leq SD(\lambda,(O_i^0)) + \varepsilon \leq SD(\lambda,(O_i)) + \varepsilon,$$

and thus (2.1) would also hold.

Let us then prove by induction on $k \in \mathbb{N}$ that there is for every $i \leq n$ a nonempty set $O_i \in \tau|_{O_i^0}$ satisfying (2.2).

First step of the induction ($k = 1$): Clearly, there are $a,b \in \sum_i \lambda(i)O_i^0$ and $x^* \in S_{X^*}$ such that

$$x^*(a-b) \geq SD(\lambda,(O_i^0)) - \varepsilon/2.$$

We may assume without loss of generality that the last inequality holds together with $x^*(x_1) \leq x^*\big(\frac{a+b}{2}\big)$ (changing sign of x^* and swapping the roles of a and b if necessary). We define now

$$O_i^1 = O_i^0 \cap \{x \in D \,:\, x^*(x) > \sup x^*(O_i^0) - \varepsilon/2\}.$$

Clearly, $O_i^1 \in \tau|_{O_i^0} \setminus \{\emptyset\}$ for every i. Let us prove that $O_1^1, \dots O_n^1$ satisfy (2.2) for $k = 1$:

If $y \in \sum_i \lambda(i) O_i^1$ then $y = \sum_i \lambda(i) y_i$ where $y_i \in O_i^1$. Therefore,

$$\begin{aligned} x^*(y) = \sum_i \lambda(i) x^*(y_i) > \sum_i \lambda(i) \sup x^*(O_i^0) - \varepsilon/2 \geq \sup x^*\Big(\sum_i \lambda(i) O_i^0\Big) - \varepsilon/2 \\ \geq x^*(a) - \varepsilon/2. \end{aligned}$$

Hence, for every $y \in \sum_i \lambda(i) O_i^1$,

$$\begin{aligned} \|y - x_1\| \geq x^*(y - x_1) \geq x^*(a) - x^*\Big(\frac{a+b}{2}\Big) - \varepsilon/2 = x^*\Big(\frac{a-b}{2}\Big) - \varepsilon/2 \\ \geq \frac{SD(\lambda, (O_i^0))}{2} - \frac{3\varepsilon}{4}. \end{aligned} \tag{2.4}$$

Clearly, from (2.3) we have that

$$SD(\lambda, (O_i^1)) \leq \operatorname{diam}\Big(\sum_i \lambda(i) O_i^1\Big) \leq \operatorname{diam}\Big(\sum_i \lambda(i) O_i^0\Big) \leq SD(\lambda, (O_i^0)) + \varepsilon/2.$$

Therefore, putting together this last inequality and (2.4) we finish this step.

Inductive step: Let us assume that for some $m \in \mathbb{N}$ there are nonempty sets $O_1^m, \dots, O_n^m \in \tau|_{O_i^0}$ satisfying equation (2.2) for $k = m$. We may then construct $O_1^{m+1}, \dots O_n^{m+1} \in \tau|_C \setminus \{\emptyset\}$ with $O_i^{m+1} \subset O_i^0$ and satisfying equation (2.2) for $k = m+1$. The construction is analogous to the one given for the case $k = 1$. We just take $x^* \in S_{X^*}$ and $a, b \in \sum_i \lambda(i) O_i^m$ such that $x^*(a-b) \geq SD(\lambda, (O_i)) - \varepsilon$ and $x^*(x_{m+1}) \leq x^*\big(\frac{a+b}{2}\big)$. Then, the sets given by

$$O_i^{m+1} = O_i^m \cap \{x \in D \,:\, x^*(x) > \sup x^*(O_i^m) - \varepsilon/2\}$$

make the trick. Hence, (2.2) holds for $O_i = O_i^k$ for every $i \leq n$. □

Lemma 2.5. Let C be a convex subset of D, $\varepsilon > 0$, $n, k \in \mathbb{N}$ and $\{x_1, \dots, x_k\} \subset X$. If $(\widetilde{O}_i)_{i=1}^n \subset \tau|_C$, then there is another n-tuple of nonempty $\tau|_C$-open sets $(O_i)_{i=1}^n$ with $O_i \subset \widetilde{O}_i$ such that

$$\operatorname{diam}\Big(\sum_{i=1}^n \lambda(i) O_i\Big) \leq 2d\Big(\{x_1, \dots, x_k\}, \sum_{i=1}^n \lambda(i) O_i\Big) + \varepsilon \quad \forall \lambda \in S^+_{\ell_1^n}.$$

Proof. Let us consider $(\lambda_i)_{i=1}^N$ an $\frac{\varepsilon}{4n}$-net of $S^+_{\ell_1^n}$. It is now straightforward to construct, by recursively using Lemma 2.4, n-tuples $(O_i^j)_{i=1}^n$ for $j \leq N$ satisfying

(1) $\emptyset \neq O_i^1 \subset \widetilde{O}_i$ and $\emptyset \neq O_i^{j+1} \subset O_i^j$ for every $j < N$ and $i \leq n$.

(2) $\operatorname{diam}\Big(\sum_{i=1}^n \lambda_j(i)O_i^j\Big) \leq 2d\Big(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda_j(i)O_i^j\Big) + \varepsilon/4$ for $j \leq N$.

We take $O_i = O_i^N$ for $i = 1,\dots,n$ so that for every $j \leq N$,

$$\begin{aligned}\operatorname{diam}\Big(\sum_{i=1}^n \lambda_j(i)O_i\Big) \leq& \operatorname{diam}\Big(\sum_{i=1}^n \lambda_j(i)O_i^j\Big)\\ \leq& 2d\Big(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda_j(i)O_i^j\Big) + \varepsilon/4\\ \leq& 2d\Big(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda_j(i)O_i\Big) + \varepsilon/4.\end{aligned} \tag{2.5}$$

If we take now $\lambda \in S^+_{\ell_1}$ arbitrary, then there must be $j \leq N$ such that $\|\lambda_j - \lambda\|_{\ell_1^n} \leq \frac{\varepsilon}{4n}$. It is straightforward to check that

$$\operatorname{diam}\Big(\sum_{i=1}^n \lambda(i)O_i\Big) \leq \operatorname{diam}\Big(\sum_{i=1}^n \lambda_j(i)O_i\Big) + \varepsilon/2,$$

$$d\Big(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda(i)O_i\Big) \geq d\Big(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda_j(i)O_i\Big) - \varepsilon/4.$$

We are therefore done since

$$\begin{aligned}\operatorname{diam}\Big(\sum_{i=1}^n \lambda(i)O_i\Big) \leq& \operatorname{diam}\Big(\sum_{i=1}^n \lambda_j(i)O_i\Big) + \varepsilon/2\\ \overset{(2.5)}{\leq}& 2d\Big(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda_j(i)O_i\Big) + 3\varepsilon/4\\ \leq& 2d\Big(\{x_1,\dots,x_k\}, \sum_{i=1}^n \lambda(i)O_i\Big) + \varepsilon.\end{aligned}$$

□

Now, we are ready to get our main result

Theorem 2.6. Let X be a separable Banach space and D a τ-strongly regular closed, bounded and convex subset of X without the τ-CPCP, where τ is a locally convex topology on D containing the weak topology on D. Then, there is a closed convex subset of D without extreme points, that is D fails KMP.

Proof. Let C be a bounded and convex subset of D and $\delta > 0$ such that every $\tau|_C$-open set has diameter greater than $\delta > 0$. We assume without loss of generality that $C \subset B_X$. Our aim is to find out a closed bounded and convex subset of D without extreme points. We will split the proof into three different steps. The first two steps will deal with the construction of a particular set $E \subset D$ whereas in the final step we will prove that E is the sought set without extreme points.

Since X is separable we may consider a sequence (x_n) dense in X. We are going to inductively define a sequence of indexes sets (Ω_n) satisfying that $\Omega_1 = \{1, \ldots, m_1\}$ and $\Omega_{n+1} = \Omega_n \times \{1, \ldots, m_{n+1}\}$ for every $n \in \mathbb{N}$ where $(m_n) \subset \mathbb{N}$.

Step 1. We are going to define inductively the sequence (Ω_n) altogether with a family of nonempty $\tau|_C$-open sets indexed by $\Omega = \bigcup_n \Omega_n$, $(O_\omega)_{\omega\in\Omega}$ satisfying the following properties for every $n \in \mathbb{N}$,

(1) $O_{(\omega,i)} \subset O_\omega$ for every $\omega \in \Omega_n$ and $i \in \{1, \ldots, m_{n+1}\}$.
(2) For every $\lambda \in S^+_{\ell_1(\Omega_n)}$,

$$\operatorname{diam}\Big(\sum_{\omega\in\Omega_n} \lambda(\omega) O_\omega\Big) \leq 2d\Big(\{x_1, \ldots, x_n\}, \sum_{\omega\in\Omega_n} \lambda(\omega) O_\omega\Big) + 2^{-n}.$$

(3) For every $\omega \in \Omega_n$,

$$\operatorname{diam}\Big(\sum_{i=1}^{m_{n+1}} \frac{1}{m_{n+1}} O_{(\omega,i)}\Big) \leq 2^{-n}.$$

Let us proceed with the inductive construction. For n=1 we consider $m_1 = 1$. We use Lemma 2.5 with $n = k = 1$, $\widetilde{O}_1 = C$, $\varepsilon = 1/2$ and $\lambda \in S^+_{\ell_1(\{1\})} = \{1\}$. Thus, there is a nonempty $\tau|_C$-open set O_1 satisfying (2). Now, provided that for some $n \in \mathbb{N}$ we have $\Omega_n = \{1, \ldots, m_1\} \times \cdots \times \{1, \ldots, m_n\}$ and O_ω for every $\omega \in \Omega_n$ such that $\{O_\omega\}_{\omega\in\Omega_n}$ satisfy (1)-(3), let us define m_{n+1} and $O_{(\omega,i)}$ for every $(\omega, i) \in \Omega_n \times \{1, \ldots, m_{n+1}\} = \Omega_{n+1}$ satisfying properties (1), (2) and (3). First, by Lemma 2.2 there is $m_{n+1} \in \mathbb{N}$ such that for every $\omega \in \Omega_n$ there are m_{n+1} nonempty $\tau|_C$-open sets $(\widetilde{O}_{(\omega,i)})_{i=1}^{m_{n+1}}$ such that

$$\operatorname{diam}\Big(\sum_{i=1}^{m_{n+1}} \frac{1}{m_{n+1}} \widetilde{O}_{(\omega,i)}\Big) \leq 2^{-n} \quad \text{and} \quad \widetilde{O}_{(\omega,i)} \subset O_\omega \quad \forall i = 1, \ldots, m_{n+1}. \tag{2.6}$$

Now, by Lemma 2.5, for every $\omega \in \Omega_{n+1}$ there is a nonempty $\tau|_C$-open set $O_\omega \subset \widetilde{O}_\omega$ satisfying property (2). Clearly, by (2.6), $(O_\omega)_{\omega\in\Omega_{n+1}}$ also satisfies property (3). We may assume that O_ω is convex for every $\omega \in \Omega$ without loss of generality because τ is locally convex.

Step 2. We are finally ready to construct the closed convex subset of D without extreme points. Let us first pick for every $\omega \in \Omega$ an element $y_\omega \in O_\omega$. Now, if $\omega \in \Omega_n$ for some $n \in \mathbb{N}$ we define for every $k \geq n + 3$ the element

$$z^k_\omega = \sum_{\widetilde{\omega}\in\Omega^\omega_k} \frac{1}{m_{n+1} \ldots m_k} y_{\widetilde{\omega}},$$

Where $\Omega^\omega_k = \{\widetilde{\omega} \in \Omega_k \ : \ \widetilde{\omega}(i) = \omega(i) \ \ \forall i \leq n\}$. Notice that $\#\Omega^\omega_k = m_{n+1} \cdots m_k$. We claim that $(z^k_\omega)_k$ is a Cauchy sequence for every $\omega \in \Omega$. From property (1) we deduce that $O_{\widetilde{\omega}} \subset O_\omega$ for every $\widetilde{\omega} \in \Omega^\omega_k$. Now, for every $q \in \{n + 2, \ldots, k - 1\}$ we

have $\Omega_k^\omega = \bigcup_{\widetilde{\omega}\in\Omega_q^\omega} \Omega_k^{\widetilde{\omega}}$ and therefore

$$\begin{aligned}\sum_{\widetilde{\omega}\in\Omega_k^\omega} \frac{1}{m_{n+1}\dots m_k} O_{\widetilde{\omega}} &= \sum_{\widetilde{\widetilde{\omega}}\in\Omega_q^\omega} \sum_{\widetilde{\omega}\in\Omega_k^{\widetilde{\widetilde{\omega}}}} \frac{1}{m_{n+1}\dots m_q m_{q+1}\dots m_k} O_{\widetilde{\omega}} \\ &= \sum_{\widetilde{\widetilde{\omega}}\in\Omega_q^\omega} \frac{1}{m_{n+1}\dots m_q} \sum_{\widetilde{\omega}\in\Omega_k^{\widetilde{\widetilde{\omega}}}} \frac{1}{m_{q+1}\dots m_k} O_{\widetilde{\omega}}.\end{aligned} \tag{2.7}$$

Hence, it follows that

$$\begin{aligned}\sum_{\widetilde{\omega}\in\Omega_k^\omega} \frac{1}{m_{n+1}\dots m_k} O_{\widetilde{\omega}} &\subset \sum_{\widetilde{\omega}\in\Omega_q^\omega} \frac{1}{m_{n+1}\dots m_q} O_{\widetilde{\omega}} \\ &= \sum_{\widetilde{\omega}\in\Omega_{q-1}^\omega} \frac{1}{m_{n+1}\dots m_{q-1}} \sum_{i=1}^{m_q} \frac{1}{m_q} O_{(\widetilde{\omega},i)}.\end{aligned} \tag{2.8}$$

Equation (2.7) yields that if $n \in \mathbb{N}$ and $\omega \in \Omega_n$,

$$z_\omega^k \in \sum_{\widetilde{\omega}\in\Omega_q^\omega} \frac{1}{m_{n+1}\dots m_q} O_{\widetilde{\omega}} \qquad \forall q,k \in \mathbb{N}\ ,\ n+2 \le q \le k-1. \tag{2.9}$$

On the other hand, if $q \ge n+2$,

$$\begin{aligned}&\operatorname{diam}\Big(\sum_{\widetilde{\omega}\in\Omega_q^\omega} \frac{1}{m_{n+1}\dots m_q} O_{\widetilde{\omega}} \Big) \\ &\overset{(2.8)}{\le} \sum_{\widetilde{\omega}\in\Omega_{q-1}^\omega} \frac{1}{m_{n+1}\dots m_{q-1}} \operatorname{diam}\Big(\sum_{i=1}^{m_q} \frac{1}{m_q} O_{(\widetilde{\omega},i)} \Big) \overset{(3)}{\le} 2^{-q+1}.\end{aligned} \tag{2.10}$$

Therefore, for every $k, k' \ge q+1$,

$$\|z_\omega^k - z_\omega^{k'}\| \le \operatorname{diam}\Big(\sum_{\widetilde{\omega}\in\Omega_q^\omega} \frac{1}{m_{n+1}\dots m_q} O_{\widetilde{\omega}} \Big) \overset{(2.10)}{\le} 2^{-q+1}.$$

This proves that $(z_\omega^k)_k$ is a Cauchy sequence for every $\omega \in \Omega$. We are then allowed to consider its limit $z_\omega \in D$. Our sough set finally is

$$E = \overline{\operatorname{co}}\big((z_\omega)_{\omega\in\Omega}\big).$$

Step 3. We just need to show that E does not have any extreme point.

Claim 1. We claim that for every $a \in E$ there is a sequence $(a_k)_k$ where $a_k \in \operatorname{co}\big((z_\omega)_{\omega\in\Omega_k}\big)$ for every $k \in \mathbb{N}$ such that $a_k \xrightarrow{\|\cdot\|} a$.

It is enough to show that if $\omega \in \Omega_n$ for some $n \in \mathbb{N}$ then for every $q > n$ it holds that

$$z_\omega = \sum_{\widetilde{\widetilde{\omega}} \in \Omega_k^\omega} \frac{1}{m_{n+1} \cdots m_q} z_{\widetilde{\widetilde{\omega}}} \in \operatorname{co}\big((z_{\widetilde{\widetilde{\omega}}})_{\widetilde{\widetilde{\omega}} \in \Omega_q}\big). \tag{2.11}$$

Indeed, if (2.11) holds, then for every $x \in \operatorname{co}\big((z_\omega)_{\omega\in\Omega}\big)$ there is $N \in \mathbb{N}$ such that for every $q \geq N$, $x \in \operatorname{co}\big((z_{\widetilde{\omega}})_{\widetilde{\omega}\in\Omega_q}\big)$ and the claim is immediately deduced. Let us show the proof of (2.11). Let $\omega \in \Omega_n$ and $q > n$ be fixed. From (2.9) we know that

$$z^k_{\widetilde{\widetilde{\omega}}} \in \sum_{\widetilde{\omega} \in \Omega^{\widetilde{\widetilde{\omega}}}_{k-1}} \frac{1}{m_{q+1} \dots m_{k-1}} O_{\widetilde{\omega}} \qquad \forall \widetilde{\widetilde{\omega}} \in \Omega_q, \ \ \forall k \geq q+3.$$

Therefore, for every $k \geq q+3$ it follows that

$$\begin{aligned} \sum_{\widetilde{\widetilde{\omega}} \in \Omega_q^\omega} \frac{1}{m_{n+1} \dots m_q} z^k_{\widetilde{\widetilde{\omega}}} &\in \sum_{\widetilde{\widetilde{\omega}} \in \Omega_q^\omega} \frac{1}{m_{n+1} \dots m_q} \sum_{\widetilde{\omega} \in \Omega^{\widetilde{\widetilde{\omega}}}_{k-1}} \frac{1}{m_{q+1} \dots m_{k-1}} O_{\widetilde{\omega}} \\ &\overset{(2.7)}{=} \sum_{\widetilde{\omega} \in \Omega^\omega_{k-1}} \frac{1}{m_{n+1} \dots m_{k-1}} O_{\widetilde{\omega}} =: S_k. \end{aligned} \tag{2.12}$$

Clearly, from (2.9) we know that $z^k_\omega \in S_k$ for every $k \geq q+3$. From (2.8) we get that $S_{k+1} \subset S_k$ for every $k \geq q+3$ and from (2.9) we know that $\operatorname{diam} S_k \to 0$. Hence, by Lemma 2.1

$$z_\omega = \lim_k z^k_\omega = \lim_k \sum_{\widetilde{\widetilde{\omega}} \in \Omega_q^\omega} \frac{1}{m_{n+1} \dots m_q} z^k_{\widetilde{\widetilde{\omega}}} = \sum_{\widetilde{\widetilde{\omega}} \in \Omega_q^\omega} \frac{1}{m_{n+1} \dots m_q} z_{\widetilde{\widetilde{\omega}}},$$

which finally proves the claim.

Let us take $a \in E$ and a sequence (a_k) converging to a as in the claim. Then, for every $k \in \mathbb{N}$ there is $\lambda_k \in S^+_{\ell_1(\Omega_k)}$ such that $a_k = \sum_{\omega \in \Omega_k} \lambda_k(\omega) z_\omega$ for every $k \in \mathbb{N}$. Let us prove that a is not an extreme point in E. First, we extend λ_k to $\bigcup_{n \leq k} \Omega_n$ as follows. If $\omega \in \Omega_n$ with $n \leq k$ we define

$$\lambda_k(\omega) = \sum_{\widetilde{\omega} \in \Omega_k^\omega} \lambda_k(\widetilde{\omega}).$$

We may assume (taking subsequence if necessary) that for every $n \in \mathbb{N}$, the sequence $(\lambda_k|_{\Omega_n})_k$ is pointwise convergent. Therefore, we define $\lambda : \Omega \to [0,1]$ as

$$\lambda(\omega) = \lim_k \lambda_k(\omega) \qquad (\omega \in \Omega).$$

Clearly, $\lambda|_{\Omega_n} \in S^+_{\ell_1(\Omega_n)}$ for every $n \in \mathbb{N}$. Moreover, if $\omega \in \Omega_n$ then

$$\sum_{\widetilde{\omega} \in \Omega_m^\omega} \lambda(\widetilde{\omega}) = \lim_k \sum_{\widetilde{\omega} \in \Omega_m^\omega} \lambda_k(\widetilde{\omega}) = \lim_k \lambda_k(\omega) = \lambda(\omega) \quad \forall m \geq n. \tag{2.13}$$

Let us now define for every $m, n \in \mathbb{N}$ with $m \geq n$ and every $\omega \in \Omega_n$ the element

$$a^m(\omega) = \begin{cases} \sum_{\widetilde{\omega} \in \Omega_m^\omega} \frac{\lambda(\widetilde{\omega})}{\lambda(\omega)} z_{\widetilde{\omega}} & \text{if } \lambda(\omega) > 0, \\ z_\omega & \text{otherwise.} \end{cases}$$

Claim 2. For every $\omega \in \Omega$ the sequence $(a^m(\omega))_m$ is convergent, moreover, if we take $a(\omega) = \lim_m a^m(\omega)$ then $a = \sum_{\omega \in \Omega_n} \lambda(\omega) a(\omega)$ for every $n \in \mathbb{N}$.

If we fix $n \in \mathbb{N}$, we have that $a_k \in \sum_{\omega \in \Omega_n} \lambda_k(\omega) O_\omega$ for every $k \in \mathbb{N}$ so that $a \in \overline{\sum_{\omega \in \Omega_n} \lambda(\omega) O_\omega}$. By property (2) we deduce that

$$\begin{aligned} \operatorname{diam}\Big(\sum_{\omega \in \Omega_n} \lambda(\omega) O_\omega \Big) &\leq 2d\Big(\{x_1, \dots, x_n\}, \sum_{\omega \in \Omega_n} \lambda(\omega) O_\omega \Big) + 2^{-n} \\ &\leq 2d(\{x_1, \dots, x_n\}, a) + 2^{-n} \xrightarrow{n \to \infty} 0. \end{aligned} \tag{2.14}$$

Also, if $m > n$ and $\omega \in \Omega_n$ with $\lambda(\omega) > 0$ then

$$\sum_{\widetilde{\omega} \in \Omega_m} \lambda(\widetilde{\omega}) O_{\widetilde{\omega}} = \lambda(\omega) \sum_{\widetilde{\omega} \in \Omega_m^\omega} \frac{\lambda(\widetilde{\omega})}{\lambda(\omega)} O_{\widetilde{\omega}} + \sum_{\widetilde{\widetilde{\omega}} \in \Omega_n \setminus \{\omega\}} \lambda(\widetilde{\widetilde{\omega}}) \sum_{\widetilde{\omega} \in \Omega_m^{\widetilde{\widetilde{\omega}}}} \frac{\lambda(\widetilde{\omega})}{\lambda(\widetilde{\widetilde{\omega}})} O_{\widetilde{\omega}},$$

so that,

$$\operatorname{diam}\Big(\sum_{\widetilde{\omega} \in \Omega_m} \lambda(\widetilde{\omega}) O_{\widetilde{\omega}} \Big) \geq \lambda(\omega) \operatorname{diam}\Big(\sum_{\widetilde{\omega} \in \Omega_m^\omega} \frac{\lambda(\widetilde{\omega})}{\lambda(\omega)} O_{\widetilde{\omega}} \Big).$$

Therefore, by (2.14) we have

$$\lim_m \operatorname{diam}\Big(\sum_{\widetilde{\omega} \in \Omega_m^\omega} \frac{\lambda(\widetilde{\omega})}{\lambda(\omega)} O_{\widetilde{\omega}} \Big) = 0 \quad \forall \omega \in \Omega_n \text{ with } \lambda(\omega) > 0. \tag{2.15}$$

Let us prove that the sequence of subsets $\Big(\sum_{\widetilde{\omega} \in \Omega_m^\omega} \frac{\lambda(\widetilde{\omega})}{\lambda(\omega)} O_{\widetilde{\omega}} \Big)_m$ is decreasing for every $\omega \in \Omega$ in order to use Lemma 2.1. Clearly, by (2.13), property (1) and the fact that $O_{\widetilde{\omega}}$ is convex for every $\widetilde{\omega} \in \Omega$, we have that

$$\sum_{\widetilde{\widetilde{\omega}} \in \Omega_{m+1}^{\widetilde{\omega}}} \frac{\lambda(\widetilde{\widetilde{\omega}})}{\lambda(\widetilde{\omega})} O_{\widetilde{\widetilde{\omega}}} \subset O_{\widetilde{\omega}} \qquad \forall \widetilde{\omega} \in \Omega_m, \quad \forall m \in \mathbb{N}.$$

Then, if $\omega \in \Omega_n$ for some $n \in \mathbb{N}$ and $m \geq n$,

$$\sum_{\widetilde{\omega} \in \Omega_{m+1}^\omega} \frac{\lambda(\widetilde{\omega})}{\lambda(\omega)} O_{\widetilde{\omega}} = \sum_{\widetilde{\omega} \in \Omega_m^\omega} \sum_{\widetilde{\widetilde{\omega}} \in \Omega_{m+1}^{\widetilde{\omega}}} \frac{\lambda(\widetilde{\widetilde{\omega}})}{\lambda(\omega)} O_{\widetilde{\widetilde{\omega}}} = \sum_{\widetilde{\omega} \in \Omega_m^\omega} \frac{\lambda(\widetilde{\omega})}{\lambda(\omega)} \sum_{\widetilde{\widetilde{\omega}} \in \Omega_{m+1}^{\widetilde{\omega}}} \frac{\lambda(\widetilde{\widetilde{\omega}})}{\lambda(\widetilde{\omega})} O_{\widetilde{\widetilde{\omega}}} \subset \sum_{\widetilde{\omega} \in \Omega_m^\omega} \frac{\lambda(\widetilde{\omega})}{\lambda(\omega)} O_{\widetilde{\omega}},$$

which means that $\Big(\sum_{\widetilde{\omega} \in \Omega_m^\omega} \frac{\lambda(\widetilde{\omega})}{\lambda(\omega)} O_{\widetilde{\omega}} \Big)_m$ is decreasing for every $\omega \in \Omega$, as we wanted. In particular, taking $\omega = 1 \in \Omega_1$, this shows that the sequence of sets $\Big(\sum_{\omega \in \Omega_m} \lambda(\omega) O_\omega \Big)_m$ is a decreasing sequence.

By Lemma 2.1 the sequence $(a^m(\omega))_m$ is convergent in E since $a^m(\omega) \in \sum_{\widetilde{\omega}\in\Omega_m^\omega} \frac{\lambda(\widetilde{\omega})}{\lambda(\omega)} O_{\widetilde{\omega}}$. Let us denote $a(\omega)$ its limit. We just need to prove that $a = \sum_{\omega\in\Omega_n} \lambda(\omega)a(\omega)$ for every $n \in \mathbb{N}$. If we fix $n \in \mathbb{N}$ and denote $a^m = \sum_{\omega\in\Omega_n} \lambda(\omega)a^m(\omega)$ for every $m \geq n$, it is enough to show that $a^m \to a$. By (2.15), we know that $\lim_m \operatorname{diam}\Big(\sum_{\omega\in\Omega_m} \lambda(\omega)O_\omega\Big) = 0$. For every $m \geq n$ we have that $a \in \overline{\sum_{\omega\in\Omega_m} \lambda(\omega)O_\omega}$ and

$$a^m = \sum_{\substack{\omega\in\Omega_n \\ \lambda(\omega)>0}} \lambda(\omega)a^m(\omega) = \sum_{\substack{\omega\in\Omega_n \\ \lambda(\omega)>0}} \lambda(\omega) \sum_{\widetilde{\omega}\in\Omega_m^\omega} \frac{\lambda(\widetilde{\omega})}{\lambda(\omega)} z_{\widetilde{\omega}} = \sum_{\widetilde{\omega}\in\Omega_m} \lambda(\widetilde{\omega}) z_{\widetilde{\omega}} \in \sum_{\omega\in\Omega_m} \lambda(\omega)O_\omega.$$

Therefore, by Lemma 2.1, $a^m \to a$ and Claim 2 is proven.

Finally, we wanted to show that E has no extreme point, that is, we want to show that our previously picked element $a \in E$ is not an extreme point in E. Notice that by Claim 2, it is enough to show that there is $\omega \in \Omega$ with $\lambda(\omega) > 0$ such that $a(\omega) \neq a$.

Let us argue by contradiction. If we deny the above statement then for every $\omega \in \Omega$ with $\lambda(\omega) > 0$ we have $a(\omega) = a$. Therefore, $a \in O_\omega$ for every $\omega \in \Omega$ with $\lambda(\omega) > 0$. From (2.13), we know that there must be a sequence $(\omega_n) \subset \Omega$ such that for every $n \in \mathbb{N}$, $\omega_{n+1} \in \Omega_{n+1}^{\omega_n}$ and $\lambda(\omega_n) > 0$. Hence, $a \in O_{\omega_n}$ for every $n \in \mathbb{N}$. By property (2) we obtain that $d(a, X) \geq \delta/2$ which is clearly false.

□

We want to remark that while part of the techniques used in the proof of the above theorem appear in [17], we don't use the essential approach in [17] about the positive face of the unit sphere of L_1 via the construction of an operator from L_1 to the space X, in order to get the subset without extreme points.

Observe that the hypothesis on D (non τ-CPCP) in the above theorem is used first to get a subset C so that every nonempty weakly open subset of C has diameter at least δ, for some positive δ. From this moment, the τ-SR is used to find convex combinations of relatively τ-open subsets with diameter arbitrarily small inside every relative τ-open subset of C. Then, starting with D satisfying that every nonempty τ-open subset of D has diameter at least $\delta > 0$ (say D has the τ-Diameter δ Property, τ-DδP for short), we have with the same proof that D fails KMP whenever D is τ-SR for open subsets.

Corollary 2.7. Let X be a separable Banach space and D a closed, bounded and convex subset of X with the τ-DδP for some positive δ, where τ is a locally convex topology on D containing the weak topology on D. If D is τ-SR for open subsets then D fails KMP.

It is worth mentioning that SR for open subsets is strictly weaker than SR. Indeed, every Banach space with a LUR norm has a unit ball which is SR for open subsets

while it may not be SR (for instance, a LUR renorming of c_0). This underlines the fact that SR for open subsets is in general not stable under isomorphisms whereas SR is. Hence, Corollary 2.7 is a more precise and isometric statement of Theorem 2.6. On the other hand, it is known that if a closed, bounded and convex subset C of a Banach space X fails CPCP, then there is an equivalent norm on X so that the new unit ball has the DδP for some $\delta > 0$ [5].

In order to get a nonseparable analogous result of the above theorem, recall that a topological space X is said to have countable tightness if for every $A \subset X$ and for every $x \in \overline{A}$ there is a countable subset $B \subset A$ such that $x \in \overline{B}$. The key to get the nonseparable result is to have the separable determination of τ-CPCP, which is proved in the next lemma.

Lemma 2.8. Let X be a Banach space and D a closed, bounded and convex subset of X without τ-CPCP, where τ is a locally convex topology on D with countable tightness containing the weak topology on D. Then there is a separable, closed, bounded and convex subset of D without τ-CPCP.

Proof. If D fails τ-CPCP there is a convex subset A of D and a $\delta > 0$ such that every relatively τ-open subset of A has diameter at least δ, so $a \in \overline{A \setminus B(a, \frac{\delta}{2})}^{\tau}$ for every $a \in A$. As τ has countable tightness, for every $a \in A$ there is D_a a countable subset of $A \setminus B(a, \frac{\delta}{2})$ such that $a \in \overline{D_a}^{\tau}$. For $a_0 \in A$ define $B_1 = \{a_0\} \cup co(D_{a_0})$ and $B_{n+1} = B_n \cup co(\cup_{a \in B_n} D_a)$ for every n. Now put $B = \cup_n B_n$. Then $\overline{B}^{\tau}$ is a separable, closed, bounded and convex subset of D. Observe that B is convex since $B_n \subset co(\cup_{a \in B_n} D_a)$ and $B_{n+1} = B_n \cup co(\cup_{a \in B_n} D_a)$ for every n. It is enough to prove that every relatively τ-open subset of B has diameter, at least, $\frac{\delta}{2}$. Indeed, if U is a relatively τ-open subset of B then there is $x \in U \cap B_n$ for some n and $x \in \overline{D_x}^{\tau}$. Then there is $y \in D_x \cap B \cap U$. Since $\|x - y\| > \frac{\delta}{2}$, one has $\text{diam}(U) \geq \frac{\delta}{2}$ and we are done. □

Finally, using the above lemma joint to theorem 2.6 we get the following

Corollary 2.9. Let X be a Banach space and D a τ-strongly regular closed, bounded and convex subset of X without τ-CPCP, where τ is a locally convex topology on D with countable tightness containing the weak topology on D. Then, there is a closed and convex subset of D without extreme points, that is D fails KMP.

It is worth mentioning that the weak topology on any Banach space has countable tightness regardless the space being separable or not [7].

3. Non-CPCP and SR locally convex topology on the unit ball of c_0

Since the problem about the equivalence between RNP and KMP, as we said in the introduction, is solved when CPCP and SR are not equivalent, the above theorem suggests exploring a new way to proceed: starting from a Banach space X failing CPCP, is it possible to construct a locally convex topology τ on the unit ball of X,

B_X, containing the weak topology so that B_X still fails τ-CPCP but satisfies τ-SR? Notice that a positive answer to this question would show that RNP and KMP are indeed equivalent. We don't know the answer to this question, but a first step is to answer the above question for the unit ball of c_0, B_{c_0}, since it fails CPCP. In fact, every nonempty weakly open subset of B_{c_0} has diameter 2, that is B_{c_0} enjoys the DδP for $\delta = 2$. The interesting fact is that B_{c_0} also fails to be SR even for open sets so that the weak topology does not make the trick.

In this section we give a positive answer to the latter question for B_{c_0} and for that purpose we state and prove the following lemma.

Lemma 3.1. Let $k \in \mathbb{N}$, $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$ countable family and $A_0 \in \mathcal{A}$. There is a partition $\{A_0^1, \dots A_0^k\}$ of A_0 satisfying that for every $F \in \mathcal{A}^{<\omega}$ and $1 \leq i \leq k$, if $A_0 \cap \bigcap_{A\in F} A$ is infinite, so is $A_0^i \cap \bigcap_{A\in F} A$.

Proof. Since $\mathcal{A}$ is countable, so is $\mathcal{F} = \{F \in \mathcal{A}^{<\omega} \ : \ A_0 \cap \bigcap_{A\in F} A \text{ is infinite}\}$. Hence, we may label its elements as $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ where $F_1 = \{A_0\}$ and we also label $A_0 = \{a_j\}_{j\in\mathbb{N}}$ (the case when A_0 is not infinite is trivial). We now construct inductively a family $\{A_n^i\}_{\substack{i\leq k\\ n\in\mathbb{N}}}$ of finite subsets of A_0 with the following properties for every $p, q \in \mathbb{N}$,

(1) For $i, j \leq k$, $A_p^i \cap A_q^j = \emptyset$ if either $i \neq j$ or $p \neq q$.
(2) $\#\big(A_p^i \cap \bigcap_{A\in F_m} A\big) \geq 1$ for every $m \leq p$ and $i \leq k$.
(3)
$$A_0 \setminus \Big(\bigcup_{\substack{m\leq p\\ i\leq k}} A_m^i \Big) \subset \{a_j\}_{j>kp}.$$

We take $A_1^i = \{a_i\}$ for $i = 1, \dots, k$ which satisfy the above properties for $p = q = 1$. Assume that A_m^i has been defined for every $i \leq k$ and $m \leq n-1$ satisfying the above properties for $p, q \leq n-1$. We aim to define A_n^i for $i \leq k$ such that $\{A_m^i\}_{\substack{i\leq k\\ m\leq n}}$ satisfy properties (1) to (3) for $p, q \leq n$. For that purpose, if $A_0 \setminus \Big(\bigcup_{\substack{i\leq k\\ m\leq n-1}} A_m^i \Big) = \{a_{\sigma(j)}\}_{j\in\mathbb{N}}$ we take $A_{n,1}^i = \{a_{\sigma(i)}\}$ and inductively $A_{n,m}^i = A_{n,m-1}^i \cup \{a_{\tau(i)}\}$ where

$$\Big(A_0 \cap \bigcap_{A\in F_m} A\Big) \setminus \Big(\bigcup_{\substack{i\leq k\\ r\leq n-1}} A_r^i \cup \bigcup_{i\leq k} A_{n,m-1}^i \Big) = \{a_{\tau(j)}\}_{j\in\mathbb{N}}. \tag{3.1}$$

We rename $A_{n,n}^i = A_n^i$ for $i = 1, \dots, k$. Let us prove that we reached our goal, that is, $\{A_m^i\}_{\substack{i\leq k\\ m\leq n}}$ satisfy properties (1) to (3) for $p, q \leq n$.

Property (1): If $p, q \leq n-1$ then by the induction hypothesis it is clear that (1) holds true. Otherwise, we may assume that $p = n$. If $q < p = n$ and $i, j \leq k$ then by construction, $A_{n,1}^i \cap A_q^j = \emptyset$ since $a_{\sigma(i)} \notin A_q^j$. If we assume that $A_{n,m-1}^i \cap A_q^j = \emptyset$, we need to show that $A_{n,m}^i \cap A_q^j = \emptyset$. Indeed, $A_{n,m}^i \cap A_q^j = \big(A_{n,m-1}^i \cup \{a_{\tau(i)}\}\big) \cap A_q^j = \{a_{\tau(i)}\} \cap A_q^j = \emptyset$ since again by construction $a_{\tau(i)} \notin A_q^j$. This shows that $A_n^i \cap A_q^j = A_{n,n}^i \cap A_q^j = \emptyset$. Finally, if $q = p = n$ and $i \neq j \leq k$ then $a_{\sigma(i)} \neq a_{\sigma(j)}$

and hence $A^i_{n,1} \cap A^j_{n,1} = \emptyset$. We assume that $A^i_{n,m-1} \cap A^j_{n,m-1} = \emptyset$ and show that $A^i_{n,m} \cap A^j_{n,m} = \emptyset$. Since $a_{\tau(i)} \neq a_{\tau(j)}$ and $a_{\tau(i)}, a_{\tau(j)} \notin A^i_{n,m-1} \cup A^j_{n,m-1}$ it is clear that $A^i_{n,m} \cap A^j_{n,m} = \big(A^i_{n,m-1} \cup \{a_{\tau(i)}\}\big) \cap \big(A^j_{n,m-1} \cup \{a_{\tau(j)}\}\big) = \emptyset$. Hence, $A^i_n \cap A^j_n = A^i_{n,n} \cap A^j_{n,n} = \emptyset$.

Property (2)**:** If $p < n$ then by the induction hypothesis $\#\big(A^i_p \cap \bigcap_{A\in F_m} A\big) \geq 1$ for every $m \leq p$ and $i \leq k$. Let then $p = n$ and $i \leq k$. In this case $a_{\tau(i)} \in A^i_{n,m} \subset A^i_n$. By (3.1), it is clear that $a_{\tau(i)} \in \bigcap_{A\in F_m} A$ and therefore $a_{\tau(i)} \in A^i_n \cap \bigcap_{A\in F_m} A$.

Property (3)**:** By the induction hypothesis we have that

$$\{a_{\sigma(j)}\}_{j\in\mathbb{N}} = A_0 \setminus \Big(\bigcup_{\substack{m\leq n-1\\ i\leq k}} A^i_m \Big) \subset \{a_j\}_{j>k(n-1)}.$$

This means that $\sigma(1) \geq k(n-1)+1$ and hence $\sigma(j) \geq \sigma(1)+j-1 \geq k(n-1)+j$. Taking into account that last inequality, we have that

$$\begin{aligned} A_0 \setminus \Big(\bigcup_{\substack{m\leq n\\ i\leq k}} A^i_m \Big) =& \Big(A_0 \setminus \Big(\bigcup_{\substack{m\leq n-1\\ i\leq k}} A^i_m \Big) \Big) \setminus \Big(\bigcup_{i\leq k} A^i_n \Big) \\ =& \{a_{\sigma(j)}\}_{j\geq 1} \setminus \Big(\bigcup_{i\leq k} A^i_{n,n} \Big) \subset \{a_{\sigma(j)}\}_{j\geq 1} \setminus \Big(\bigcup_{i\leq k} A^i_{n,1} \Big) \\ =& \{a_{\sigma(j)}\}_{j>k} \subset \{a_{k(n-1)+j}\}_{j>k} = \{a_j\}_{j>kn}. \end{aligned}$$

This finishes the induction.

Finally, we define $A^i_0 = \bigcup_{n\in\mathbb{N}} A^i_n$ for every $i \leq k$. It only remains to prove that $A^1_0, \ldots, A^k_0$ satisfy the statement of the lemma. From (1) it is clear that $A^i_0 \cap A^j_0 = \emptyset$ for $i \neq j$. It is also immediate from (3) that $\bigcup_{i=1}^k A^i_0 = A_0$ so that $\{A^1_0, \ldots, A^n_0\}$ forms a partition of A_0. Finally, if $F \in \mathcal{F}$ then combining properties (1) and (2) we have that

$$\#\big(A^i_0 \cap \bigcap_{A\in F} A\big) \overset{(1)}{=} \sum_{n=1}^{\infty} \#\big(A^i_n \cap \bigcap_{A\in F} A\big) \overset{(2)}{=} +\infty.$$

□

Now, we pass to show the existence of an appropriate topology on B_{c_0}. From now on we say that an interval $I \subset [-1,1]$ is a proper open subinterval whenever it is open as a subset of $[-1,1]$ and different from $[-1,1]$.

Theorem 3.2. There exists a locally convex topology τ in B_{c_0} containing the weak topology such that B_{c_0} is τ-strongly regular for open subsets, but B_{c_0} has the τ-Diameter δ Property for $\delta = 2$.

Proof. We will proceed by induction, constructing an increasing sequence $(\beta_n)_{n\in\mathbb{N}}$ of countable convex subbases of topologies in B_{c_0} where β_1 is a subbase of the weak topology and such that for every $n \in \mathbb{N}$ the following properties are satisfied:

(1) For every nonempty $U \in \beta_n$ there is $A_U \subset \mathbb{N}$ such that

$$U = \{(x_k) \in B_{c_0} \ : \ x_k \in I_k, \ \forall k \in \mathbb{N} \setminus A_U\},$$

where I_k is a nonempty proper open subinterval of $[-1,1]$.

(2) For every $F \in \mathcal{F}(\beta_n) := \{G \in \beta_n^{<\omega} \ : \ \bigcap_{U\in G} U \neq \emptyset\}$, the set $\bigcap_{U\in F} A_U$ is infinite.

(3) If $n > 1$ then for every $F \in \mathcal{F}(\beta_{n-1})$ there exist $U_1, \dots, U_n \in \beta_n \setminus \{\emptyset\}$ such that $U_i \subset \bigcap_{U\in F} U$ for $i = 1, \dots, n$ and

$$\operatorname{diam}\left(\sum_{i=1}^{n} \frac{1}{n} U_i\right) \leq \frac{4}{n}.$$

Clearly, the weak topology has a convex countable subbase β_1 satisfying properties (1) and (2) (notice that property (3) does not apply when $n = 1$). Now, for the inductive step, assume that β_{n-1} is a convex countable subbase containing β_1 and satisfying properties (1) and (2). Since $\mathcal{F}(\beta_{n-1})$ is countable, we label its elements as $\mathcal{F}(\beta_{n-1}) = \{F_m\}_m$. Now we rename $\bigcap_{U\in F_m} A_U = A_m$ so that by (1),

$$\bigcap_{U\in F_m} U = \{(x_k) \in B_{c_0} \ : \ x_k \in I_{k,m}, \ \forall k \in \mathbb{N} \setminus A_m\},$$

for some proper open subintervals $I_{k,m}$ of $[-1,1]$. We define β_n by means of another inductive argument. Specifically, we construct now an increasing sequence of subbases $\{\beta_n^p\}_{p\in\mathbb{N}}$ such that β_n^p enjoys the following properties for every $p \in \mathbb{N}$,

(4) $\beta_n^p = \{U_p^1, \dots, U_p^n\} \cup \beta_n^{p-1}$ with $U_p^1, \dots, U_p^n$ pairwise disjoint subsets of $\bigcap_{U\in F_p} U$ (taking $\beta_n^0 = \beta_{n-1}$).

(5) β_n^p satisfies properties (1) and (2).

(6) $\operatorname{diam}\left(\sum_{i=1}^{n} \frac{1}{n} U_p^i\right) \leq \frac{4}{n}$.

Let us construct β_n^1 and show the above properties for $p = 1$.

We use here Lemma 3.1 with $k = n$, $\mathcal{A} = A_1 \cup \{A_U\}_{U\in\beta_{n-1}}$ and $A_0 = A_1$. Therefore, there exists $\{A_1^1, \dots, A_1^n\}$ a partition of A_1 such that for $i = 1, \dots, n$,

$$A_1^i \cap \bigcap_{U\in F_1\cup F} A_U \text{ is infinite} \quad \forall F \in \beta_{n-1}^{<\omega} \text{ with } F_1 \cup F \in \mathcal{F}(\beta_{n-1}). \tag{3.2}$$

We define n pairwise disjoint subsets of $\bigcap_{U\in F_1} U$ as

$$U_1^i = \{(x_k) \in B_{c_0} \ : \ x_k \in I_{k,1}^i \text{ if } k \in \mathbb{N} \setminus A_1, \ x_k \in (-1/n, 1/n) \text{ if } k \in A_1 \setminus A_1^i\},$$

where $I_{k,1}^1, \dots, I_{k,1}^n \subset I_{k,1}$ are pairwise disjoint nonempty open intervals of the same length.

Claim. We claim that $\beta_n^1 := \{U_1^1, \dots, U_1^n\} \cup \beta_{n-1}$ is a subbase satisfying (4), (5) and (6).

Clearly, β_n^1 satisfies by definition property (4) so that we just have to show (5) and (6). Let us first tackle property (5). It is immediate from the definitions and

inductive assumptions that β_n^1 satisfies (1) with $A_{U_1^i} = A_1^i$ for $i = 1, \dots, n$. Let us focus on property (2). Consider then $F \in \mathcal{F}(\beta_n^1)$. If $F \in \beta_{n-1}^{<\omega}$ then from the induction hypothesis we know that (2) is satisfied. Otherwise, there is $1 \leq i \leq n$ such that $U_1^i \in F$. Since $\bigcap_{U\in F} U \neq \emptyset$ and $U_1^1, \dots, U_1^n$ are pairwise disjoint, necessarily $F \setminus \{U_1^i\} \in \beta_{n-1}^{<\omega}$. Now, since $U_1^i \subset \bigcap_{U\in F_1} U$ we obtain that

$$\bigcap_{U\in F_1\cup F\setminus\{U_1^i\}} U = \left(\bigcap_{U\in F_1} U\right) \cap \left(\bigcap_{U\in F\setminus\{U_1^i\}} U\right) \subset U_1^i \cap \left(\bigcap_{U\in F\setminus\{U_1^i\}} U\right) = \bigcap_{U\in F} U \neq \emptyset.$$

Hence, $F_1 \cup F \setminus \{U_1^i\} \in \mathcal{F}(\beta_{n-1})$ and by (3.2) we have that $\bigcap_{U\in F} A_U \supset A_1^i \cap \bigcap_{U\in F_1\cup F\setminus\{U_1^i\}} A_U$ is infinite . This finally proves that β_n^1 enjoys property (2) and hence (5). Let us finish the proof of the claim by showing that β_n^1 enjoys property (6). Indeed, if we pick $(x_k^{i,1})_k, (y_k^{i,1})_k \in U_1^i$ for $i = 1, \dots, n$, then $|x_k^{i,1} - y_k^{i,1}| \leq \frac{2}{n}$ whenever $k \notin A_1^i$. Therefore, taking into account that $A_1^1, \dots, A_1^n$ are pairwise disjoint we get that

$$\left\| \sum_{i=1}^n \frac{1}{n}(x_k^{i,1})_k - \sum_{i=1}^n \frac{1}{n}(y_k^{i,1})_k \right\| = \frac{1}{n} \sup_k \left| \sum_{i=1}^n (x_k^{i,1} - y_k^{i,1}) \right| \leq \frac{1}{n} \sup_k \left(2 + \sum_{\substack{i=1 \\ k\notin A_1^i}}^n |x_k^{i,1} - y_k^{i,1}| \right) \leq \frac{4}{n}.$$

This finishes the proof of the claim.

Let us now get into the inductive step. Assume β_n^{p-1} has been defined for some $p > 1$ enjoying properties (4), (5) and (6).We use Lemma 3.1 with $k = n$, the countable set $\mathcal{A} = \{A_p\} \cup \{A_U\}_{U\in\beta_n^{p-1}}$ and $A_0 = A_p$ and obtain a partition $\{A_p^1, \dots, A_p^n\}$ of A_p such that for every $F \in (\beta_n^{p-1})^{<\omega}$ with $F_p \cup F \in \mathcal{F}(\beta_n^{p-1})$ we have that $A_p^i \cap \bigcap_{U\in F_p\cup F} A_U$ is infinite (this is due to the fact that β_n^{p-1} satisfy (2)). Now, we define for $i = 1, \dots, n$ the set,

$$U_p^i = \{(x_k) \in B_{c_0} \ : \ x_k \in I_{k,p}^i \text{ if } k \in \mathbb{N} \setminus A_p, \ x_k \in (-1/n, 1/n) \text{ if } k \in A_p \setminus A_p^i\},$$

where $I_{k,p}^1, \dots, I_{k,p}^n \subset I_{k,p}$ are pairwise disjoint nonempty open intervals of the same length. Following the same reasoning as in the case $p = 1$, we know that $\beta_n^p := \{U_p^1, \dots, U_p^n\} \cup \beta_n^{p-1}$ satisfies properties (4), (5) and (6). Finally, β_n is defined as the union $\beta_n := \bigcup_p \beta_n^p$. It is straightforward to see that β_n satisfies properties (1), (2) and (3) and hence the inductive construction is complete.

We are then ready to define our sought topology τ as the topology generated by the subbase $\beta = \bigcup_n \beta_n$. τ is clearly locally convex and contains the weak topology. Let us check that B_{c_0} is τ-strongly regular but does not enjoy the τ-CPCP.

In order to show that B_{c_0} fails τ-CPCP we will prove that every nonempty τ-open subset of B_{c_0} has diameter 2. For that, observe that the family of finite intersections of elements in β is a basis for τ, so we will be done if we get that every set of the form $\bigcap_{U\in F} U$ is either empty or has diameter 2 for every $F \in \beta^{<\omega}$. There has to be some $n \in \mathbb{N}$ such that $F \in \beta_n^{<\omega}$. If we assume that $\bigcap_{U\in F} U$ is nonempty then $F \in \mathcal{F}(\beta_n)$ and the set $\bigcap_{U\in F} A_U$ is infinite from (2). Now, from (1), if $U \in F$ the elements in U are sequences of B_{c_0} without restrictions on coordinates from A_U, so there is a k^th coordinate (in fact, there are infinitely many coordinates) where the

elements of U have no restrictions on its values, other than belonging to B_{c_0}, for every $U \in F$. Then, $\operatorname{diam}(\bigcap_{U\in F} U) = 2$.

In order to show that B_{c_0} is τ-SR for open subsets, let O be a nonempty τ-open set and $\varepsilon > 0$. Then, there is $F \in \beta^{<\omega}$ such that $\emptyset \neq \bigcap_{U\in F} U \subset O$. Therefore, there is $n \in \mathbb{N}$ such that $F \in \mathcal{F}(\beta_n)$. Consider $m > n$ such that $4/m < \varepsilon$. Since $(\beta_n)_n$ is increasing, $F \in \mathcal{F}(\beta_{m-1})$ and thus by property (3) there are $U_1, \ldots, U_m \in \beta \setminus \{\emptyset\}$ such that $U_i \subset \bigcap_{U\in F} U$ for $i = 1 \ldots, n$ and

$$\operatorname{diam}\left(\sum_{i=1}^{m} \frac{1}{m} U_i\right) \leq \frac{4}{m} < \varepsilon.$$

□

We think that it would be interesting to know if the same result holds for the unit ball of L_1.

Observe that the topology defined in the above theorem can be defined on the whole space c_0. Moreover, it is even possible to define a topology τ on ℓ_∞, following the same above ideas, containing the weak-star topology on ℓ_∞ so that every relatively τ-open subset of the unit ball has diameter 2, while the unit ball contains convex combinations of relatively τ-open subsets with arbitrarily small diameter.

In [4] (see also [1]) the authors proved that every Banach space containing c_0 can be equivalently renormed satisfying D2P so that the new unit ball contains convex combinations of relatively weakly open subsets of arbitrarily small diameter, showing the extreme difference between D2P and SD2P. The latter result stresses this extreme difference from a topological point of view, enlarging the family of open subsets with diameter 2 and obtaining the existence of convex combinations of relatively open subsets with arbitrarily small diameter, without renorming.

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Chapter 3

Compact retractions and Schauder decompositions in Banach spaces

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COMPACT RETRACTIONS AND SCHAUDER DECOMPOSITIONS IN BANACH SPACES

PETR HÁJEK AND RUBÉN MEDINA

Abstract. Let X be a separable Banach space. We give an almost characterization of the existence of a Finite Dimensional Decomposition (FDD for short) for X in terms of Lipschitz retractions onto generating compact subsets K of X.

In one direction, if X admits an FDD then we construct a Lipschitz retraction onto a small generating convex and compact set K. On the other hand, we prove that if X admits a "small" generating compact Lipschitz retract then X has the π-property. It is still unknown if the π-property is isomorphically equivalent to the existence of an FDD.

For dual Banach spaces this is true, so our results give a characterization of the FDD property for dual Banach spaces X.

We give an example of a small generating convex compact set which is not a Lipschitz retract of $C[0,1]$, although it is contained in a small convex Lipschitz retract and contains another one.

We characterize isomorphically Hilbertian spaces as those Banach spaces X for which every convex and compact subset is a Lipschitz retract of X.

Finally, we prove that a convex and compact set K in any Banach space with a Uniformly Rotund in Every Direction norm is a uniform retract, of every bounded set containing it, via the nearest point map.

1. Introduction

The study of retractions is a large area of research in topology and nonlinear analysis with many applications. In the uniform or Lipschitz setting the authoritative monograph [BL00] contains many fundamental results and the general point of view.

The results in our paper were originally motivated by the natural question asked by Godefroy and Ozawa in [GO14] (and then subsequently in [God15a], [God15b], [God20], and [GLP19]) whether every separable Banach space admits a generating convex and compact Lipschitz retract (GCCR, for short). The set C is called generating if the closed linear span of C is the whole space X. We will say that C is a λ-GCCR if it is a GCCR and

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there exists a λ-Lipschitz retraction from X onto C. A positive answer to this problem would immediately imply that every separable Banach space is Lipschitz approximable, solving an important open problem of Kalton. Using a rather short, but ingenious argument, Kalton [Kal12] showed that indeed if X has a separable dual (or it is itself a separable dual) then X is approximable. But his compact identity-approximating mappings are far from being retractions. In [GO14] it is noted that the retractions can be constructed rather easily if X has an unconditional basis, but the same approach fails even for spaces with a Schauder basis.

Our first result (Theorem 3.3) is that a Banach space X with an FDD admits a GCCR. Moreover, if X has a monotone FDD, then it has a $(5+\epsilon)$-GCCR, for every $\epsilon > 0$. If X has a monotone Schauder basis, then it has a $(1+\epsilon)$-GCCR, for every $\epsilon > 0$. Our GCCR is roughly speaking a diamond shaped set, and the retraction mapping is somehow aligned with the canonical projections but it is certainly not the nearest point map.

The compact sets used in our arguments are in some sense small. Intuitively this means that they are contained in sufficiently small neighbourhoods of its finite dimensional sections. The smallness condition restricts the asymptotic behaviour of the compact set, in a certain sense, but otherwise it leaves a considerable freedom as regards its possible shape.

Our second result (Theorem 4.7) is that if a Banach space admits a small generating Lipschitz retract (not necessarily convex) then it has the π-property. Note that a Banach space with the metric π-property has an FDD [Joh70], see also [Cas01] Thm. 6.4, and it is still open if the π-property implies metric π-property (and hence FDD) under an equivalent renorming. So our previous results combined together are possibly a characterization of the existence of an FDD property for a separable Banach space X in terms of the existence of Lipschitz retractions onto small GCCR. In the case of dual spaces the characterization holds true, thanks to Theorem 1.3 in [JRZ71].

The proof is based on several deep ingredients. Of course, one would like to use the (Gateaux) differentiation theory for Lipschitz mappings in separable Banach spaces in order to pass from the compact Lipschitz retraction to its linearization with a small range. However, to do this directly using the existence of points of Gateaux differentiability does not seem possible. Instead, our proof takes a detour, and produces the finite rank linear projections (needed for the π-property) indirectly, only proving their existence. We first pass from the small retract K to a finite dimensional subspace E_n, which contains the bulk of the points of K, in a certain sense. Of course, we do not immediately have in our hands a good Lipschitz retraction from X onto E_n, but using the "Lipschitzization" of coarse Lipschitz mappings due to Bourgain [Bou87] (in the formulation of Begun [Beg99]) we obtain a good Lipschitz almost retraction to E_n from finite dimensional subspaces G of X, of controlled dimension, which contain E_n. For the next step we use the deep and yet unpublished result, due to Vitali Milman, communicated to us by Bill Johnson. Namely, the projection constant $\lambda(E_n, X)$ of a n-dimensional

subspace E_n of X is witnessed by $\lambda(E_n, G)$ for a certain finite dimensional subspace G of X, of dimension roughly 3^n. This unexpected fact makes our subsequent argument quantitatively independent of the Banach space X. At this point we may use the differentiation theory and averaging in the finite dimensional setting, applied to the finite dimensional approximate version of our retraction (crucially using the smallness assumption for K) to produce a good linear projection from G to E_n. But this implies that $\lambda(E_n, X)$ cannot be too large, which eventually yields the π-property for X.

Since the π-property is equivalent to the FDD property for dual Banach spaces [JRZ71], as a corollary to our above results we obtain a retractional characterization of an FDD in the class of dual Banach spaces.

We remark that our techniques above are applicable to small compact sets only, and we do not know if analogous results hold for general convex compact subsets of X.

Using the same approach, we also give a variant of the result of Godefroy and Kalton in [GK03], who characterized the BAP property by means of a sequence of finite rank Lipschitz mappings pointwise convergent to the identity. Namely, using the finite rank Lipschitz retractions we characterize the π-property.

We also note that if there is a Lipschitz retraction of X onto a convex compact set K, and Y is a closed linear subspace of X spanned by K, which is linearly complemented in its bidual Y^{**}, then Y is a complemented subspace of X. This means that a natural way of getting the Lipschitz retraction from X onto K is to simply compose the linear projection from X onto Y with a Lipschitz retraction from Y onto K.

In the next section, we give an example of a small compact convex set K in the space $C[0, 1]$, which is contained in a small GCCR, and contains another small GCCR, but such that K is not a GCCR for the space $C[0, 1]$. This example underlines the subtleness of the existence of Lipschitz retractions. Moreover, by the results of the last section, the set K is a uniform retract for the space $C[0, 1]$.

In our final section we first give a new characterization of isomorphically Hilbertian spaces as those Banach spaces for which every convex and compact subset is a Lipschitz retract.

Then we proceed to the problem of uniformly continuous retractions onto convex compact sets. Our main result is that if a Banach space is equipped with a URED norm, then every convex and compact subset is a uniformly continuous retract, from any bounded superset, with respect to the nearest point map. We recall that every separable Banach space admits an equivalent URED renorming. This result implies, in particular, that every convex and compact set is an absolute uniform retract, a fact recently established in [CCW21]. The case of a general (nonseparable) space X is also treated.

2. Preliminaries

Our notation and terminology is standard. We work in the setting of real Banach spaces throughout this note. For the general concepts and results of Banach space theory we refer to [Fab+11]. The background on the various forms of the approximation property can be found in [Cas01]. For the theory of Lipschitz and uniform retracts we refer to the first two chapters in the monograph [BL00].

Let us now pass to some definitions and results which are used heavily in our note. We start with the formulation of several classical concepts related to the approximation property, of increasing generality.

A Schauder basis for a real Banach space X is a sequence $(x_n) \subset X$ with the property that for every $x \in X$, there exists a unique sequence $(\alpha_n) \subset \mathbb{R}$ such that

$$\left|\left| x - \sum_{i=1}^{n} \alpha_i x_i \right|\right| \xrightarrow{n\to\infty} 0.$$

In this case we say that X has a Schauder basis and we call the projections $P_n(x) = \sum_{i=1}^{n} \alpha_i x_i$ the natural projections of the Schauder basis. If (x_n) is a Schauder basis in a Banach space X, then we denote by (x_n^*) the coordinate functionals, which form a basic sequence in the dual space X^*.

Definition 2.1. A sequence (E_n) of finite dimensional subspaces of a Banach space X is called the finite dimensional decomposition (FDD for short) if for every $x \in X$ there is a unique sequence $x_n \in E_n$ so that

$$\left|\left| x - \sum_{i=1}^{n} x_i \right|\right| \xrightarrow{n\to\infty} 0.$$

In this case we say that X has an FDD and we call the projections $P_n(x) = \sum_{i=1}^{n} x_i$ the natural projections of the FDD.

If X is a Banach space with a Schauder basis (resp. FDD) then the natural projections of the basis (resp. FDD) are uniformly bounded. Moreover, X can be equivalently renormed so that this uniform bound is 1. If the uniform bound is 1 (with respect to the given norm) we say that the Schauder basis (resp. FDD) is monotone.

Definition 2.2. A Banach space X is said to have the π-property if there is a uniformly bounded net of finite rank projections (S_α) on X converging strongly to the identity on X. If this uniform bound is $\lambda \geq 1$ then we say that X has the π_λ-property. In the case when $\lambda = 1$ we say that X has the metric π-property.

Proposition 2.3 ([Cas01])**.** *For a separable Banach space X and $\lambda \geq 1$, the following are equivalent:*

- *X has the π_λ-property*

- *There is a sequence of λ-bounded finite rank projections (S_n) on X pointwise converging to the identity such that*

$$S_m S_n = S_n \quad \forall m \geq n.$$

As we mentioned above, it seems to be an open problem whether the π-property is equivalent to the existence of an FDD for a separable Banach space.

Definition 2.4. Let X be a Banach space. If there is a uniformly bounded net of finite rank operators (T_α) on X tending strongly to the identity on X, then we say that X has the bounded approximation property (BAP for short). If $\lambda \geq 1$ is a uniform bound for the net then we say that X has the λ-bounded approximation property (λ-BAP for short). We refer to the 1-BAP as the metric approximation property (MAP for short).

Definition 2.5. Let X be a Banach space. If there is a net of finite rank operators (T_α) on X converging to the identity on X uniformly on compacta, then we say that X has the approximation property (AP for short).

Definition 2.6. A Banach space X is said to have the compact approximation property if for every $\varepsilon > 0$ and every compact set K in X there is a compact operator $T \in \mathcal{L}(X)$ so that $||Tx - x|| \leq \varepsilon$ for all $x \in K$.

For an arbitrary Banach space X, the previously defined concepts are ordered from the strongest to the weakest, that is,

$$\text{Schauder basis} \Rightarrow \text{FDD} \Rightarrow \pi\text{-property} \Rightarrow \text{BAP} \Rightarrow \text{AP} \Rightarrow \text{CAP}.$$

With the possible exception of FDD $\Rightarrow$ π-property , none of the above implications can be reversed ([Sza87],[Rea], [FJ73] and [Wil92]).

Let us pass to the non-linear approximation properties.

Definition 2.7. A map T from a metric space M into another metric space N is said to be Lipschitz if there exists some $\lambda > 0$ such that

$$d(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in M.$$

We say that λ is a Lipschitz constant for T and we call the infimum of all Lipschitz constants for T the Lipschitz norm of T, that is,

$$||T||_{Lip} = \inf\left\{\lambda > 0,\ \text{Lipschitz constant for } T\right\} = \sup_{x,y \in M, x \neq y} \frac{d\big(T(x), T(y)\big)}{d(x, y)}.$$

If $\lambda > 0$ is a Lipschitz constant for T then we say that T is λ-Lipschitz.

Definition 2.8. Let X be a Banach space. If there is a net of finite rank Lipschitz maps (T_α) on X, whose Lipschitz norms are uniformly bounded, converging uniformly on compacta to the identity on X, then we say that X has the Lipschitz bounded approximation property. If this net is bounded by $\lambda \geq 1$ then we say that X has the λ-Lipschitz bounded approximation property.

Theorem 2.9 ([GK03] Theorem 5.3)**.** *Let X be an arbitrary Banach space. Then the following conditions are equivalent:*

- *X has the λ-BAP.*
- *The Lipschitz free space $\mathcal{F}(X)$ has the λ-BAP.*
- *X has the λ-Lipschitz bounded approximation property.*

Definition 2.10. We say that a complete metric space M is approximable whenever there is a subadditive map $\omega : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0} \omega(t) = 0$ so that for every finite set $E \subset M$ and every $\varepsilon > 0$ we can find a uniformly continuous map $\psi : M \to M$ such that $d(x, \psi(x)) < \varepsilon$ for every $x \in E$, $\psi(M)$ is relatively compact and the modulus of continuity of ψ is bounded by ω. If $\omega(t) = Lt$ for some $L > 0$ then we say that M is Lipschitz approximable.

If M is separable then it is easy to see that M is approximable (resp. Lipschitz approximable) if and only if there is an equi-uniformly continuous (resp. equi-Lipschitz) sequence of maps $\psi_n : M \to M$ with relatively compact range such that $\lim_{n\to\infty} d(x, \psi_n(x)) = 0$ for every $x \in X$.

These concepts were introduced and studied in several papers, [Kal04], [Kal12] and [GLZ14].

Kalton proved in [Kal12] that every Banach space with a separable dual (or a separable dual space itself) is approximable. It is still an open problem if every separable Banach space is approximable. Godefroy, on the other hand, observed in [God20] that the compact approximation property implies that the space is Lipschitz approximable, showing that the Lipschitz approximability is a strictly weaker property than the AP.

In the next sections, we are going to make a repeated use of the following concepts.

Definition 2.11. Let M be a metric space and $N \subset M$. A retraction from M onto N is a map $R : M \to N$ such that $R\big|_N = Id_N$. In this case we say that N is a retract of X. If R is Lipschitz (resp. uniformly continuous) then we say that R is a Lipschitz (resp. uniformly continuous) retraction onto N and N is a Lipschitz (resp. uniformly continuous) retract of M.

If N is a Lipschitz (resp. uniformly continuous) retract of M for every metric space M containing it, we say that N is an absolute Lipschitz (resp. uniformly continuous) retract.

We will say that a subset K of a Banach space X generates X whenever the closed linear span of K is equal to X. A Lipschitz retract K of a Banach space X that is convex, compact, and generates X is going to be called a generating convex compact retract (GCCR for short) of X.

Note that the existence of a GCCR implies that the space X is Lipschitz approximable.

Definition 2.12. Let M be a metric space, $N \subset M$. We say that $R : M \to N$ is a proximity map (or a nearest point map) onto N if

$$d(R(x), x) = \inf_{y \in N} d(y, x) \quad \forall x \in M.$$

A proximity map may not exist in some situations and if it exists it may not be unique. If M is a uniformly convex Banach space then this map is known to be unique and uniformly continuous whenever N is a closed convex subset [Bjo79].

Definition 2.13. A normed space X is said to be uniformly rotund in the direction $z \in X$ if whenever (x_n) and (y_n) are two sequences in X such that

(1) $||x_n|| = ||y_n|| = 1$ for every $n \in \mathbb{N}$,
(2) $\lim_{n\to\infty} \left|\left|\frac{x_n+y_n}{2}\right|\right| = 1$,
(3) There is a sequence of real numbers (r_n) such that $x_n - y_n = r_n z$ for every $n \in \mathbb{N}$,

then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.
If X is uniformly rotund in the direction z for every $z \in S_X$ then we say that X is uniformly rotund in every direction (URED for short).

3. Compact Lipschitz retracts

The problem whether every separable Banach space has a GCCR was posed in [GO14]. In fact, having a GCCR for a Banach space X gives a lot of information. For instance, it implies that X is Lipschitz approximable and that X has the BAP if and only if $\mathcal{F}(K)$ has the BAP by Theorem 2.9. Our goal in this section is a construction of a certain diamond shaped convex and compact set K, which generates the Banach space X, together with a Lipschitz retraction from X onto K. Our construction is performed under the assumption that X has a FDD so in particular it gives a positive answer to the question asked in [GLP19], whether Pelczynski's space has a GCCR. In order to make the construction work, we are forced to make K in some sense small. The method breaks down if we try to replace the FDD by a weaker concept, e.g. a Markushevish basis. As we will find out in the subsequent parts of our note, there is a good reason for that. Namely, the kind of retractions we are using here imply that X has the π-property (which is possibly equivalent to having an FDD).

Our approach is to define some Lipschitz retractions onto increasing finite-dimensional sections of K, so that then we proceed by taking a limit of these mappings to define the final retraction onto K. All these initial retractions will be defined as compositions of little perturbations of the natural projections of the FDD. To this end, we first prove some Lemmas stating the Lipschitz behaviour of the perturbed projections.

Let us proceed with the construction. Let $(X, ||\cdot||)$ be a Banach space with a monotone FDD $(X_n)_{n\in\mathbb{N}}$ whose natural projections are $(P_n)_{n\in\mathbb{N}}$. For every $x \in X$ and $i \in \mathbb{N}$ we are going to denote $x_i = (P_i - P_{i-1})(x)$ where

$P_0 \equiv 0$. Let $(r_n)_{n\in\mathbb{N}}$ be an arbitrary decreasing sequence of positive real numbers. For every $m \in \mathbb{N}$ we define the function $f_m : X \to \mathbb{R}$, by setting $f_1 : X \to \{r_1\}$ to be the constant function, and

$$f_m(x) = r_m \left(1 - \sum_{i=1}^{m-1} \frac{||x_i||}{r_i}\right),\ m \geq 2.$$

We will define for $n \in \mathbb{N}$ the following subsets of X

$$K = \overline{\text{co}}\left(\bigcup_{k\in\mathbb{N}} r_k B_{X_k}\right) \quad , \quad K_n = \overline{\text{co}}\left(\bigcup_{k=1}^{n} r_k B_{X_k}\right).$$

Finally, given $n, m \in \mathbb{N}$ such that $m \leqslant n$ we consider $E_{n,m} = P_m(X) \cup K_n$, and $F_{n,m} : E_{n,m} \to E_{n,m-1}$ given by

$$F_{n,m}(x) = \begin{cases} x\ , & \text{if } \sum_{i=1}^{m} \frac{||x_i||}{r_i} \leqslant 1, \\ P_{m-1}(x)\ , & \text{if } \sum_{i=1}^{m-1} \frac{||x_i||}{r_i} \geqslant 1, \\ P_{m-1}(x) + \frac{x_m}{||x_m||} f_m(x)\ , & \text{if } \sum_{i=1}^{m-1} \frac{||x_i||}{r_i} < 1 < \sum_{i=1}^{m} \frac{||x_i||}{r_i}, \end{cases}$$

where, if $m = 1$, we set $\sum_{i=1}^{0} \frac{||x_i||}{r_i} = 0$.

One may see every $F_{n,m}$ as a perturbation of P_{m-1} restricted to $E_{n,m}$. The next Lemma 3.1 states that it is possible to approximate the Lipschitz behaviour of $F_{n,m}$ by $F_{n-1,m}$.

Lemma 3.1. *There exists a sequence $(A_m)_{m\in\mathbb{N}} \subset \mathbb{R}^+$ such that*

$$f_m \ \text{ is } \ \frac{r_m A_{m-1}}{r_{m-1}}\text{-Lipschitz} \quad \forall m > 1,$$

and if $m, n \in \mathbb{N}$ such that $m \leq n-1$ then,

$$||F_{n,m}(x) - F_{n,m}(y)|| \leq \frac{r_n A_n}{r_{n-1}}||x-y|| + ||F_{n-1,m}\circ P_{n-1}(x) - F_{n-1,m}\circ P_{n-1}(y)||,$$

for every $x, y \in E_{n,m}$.

Proof. Just consider $A_m = 2m$. Then,

$$||P_m(x)||_{\ell_1(\{X_i\}_{i=1}^m)} \leq A_m ||P_m(x)|| \quad \forall x \in X,$$

so we have that

$$\begin{aligned} |f_m(x) - f_m(y)| &= \left| r_m \left(\sum_{i=1}^{m-1} \frac{||x_i|| - ||y_i||}{r_i} \right) \right| \leq \frac{r_m}{r_{m-1}} \sum_{i=1}^{m-1} ||x_i - y_i|| \\ &\leq \frac{r_m A_{m-1}}{r_{m-1}} ||P_{m-1}(x-y)|| \leq \frac{r_m A_{m-1}}{r_{m-1}} ||x-y||. \end{aligned}$$

Now, to prove the second part of the lemma, let us define $G_{n-1,m} : E_{n,m} \to E_{n-1,m-1}$ by

$$G_{n-1,m}(x) = F_{n-1,m}(P_{n-1}(x)) \quad \forall x \in E_{n,m}.$$

Taking into account the previous definitions, it is immediate that

$$G_{n-1,m}(x) = F_{n,m}(P_{n-1}(x)) \quad \forall x \in E_{n,m}.$$

Hence, it holds that $g_n := F_{n,m} - G_{n-1,m} = P_n - P_{n-1}$.
Now, if $x \neq y \in E_{n,m}$ are such that $\sum_{i=1}^{m} \frac{||x_i||}{r_i} \leq 1$ and $\sum_{i=1}^{m} \frac{||y_i||}{r_i} \leq 1$ then

$$\frac{||F_{n,m}(x) - F_{n,m}(y)||}{||x-y||} = 1.$$

If that is not the case, then we may assume that $\sum_{i=1}^{m} \frac{||y_i||}{r_i} > 1$, so $||y_n|| = 0$ and $F_{n,m}(y) = F_{n,m}(P_{n-1}(y)) = G_{n-1,m}(y)$. It follows that

$$\frac{||F_{n,m}(x) - F_{n,m}(y)||}{||x-y||} \leq \frac{||g_n(x)|| + ||G_{n-1,m}(x) - G_{n-1,m}(y)||}{||x-y||} \quad .$$

It is enough to prove that $\frac{||g_n(x)||}{||x-y||} \leq \frac{r_n A_n}{r_{n-1}}$. If $\sum_{i=1}^{n} \frac{||x_i||}{r_i} > 1$ then $||x_n|| = 0$ and it is obviously true. Otherwise,

$$\frac{||x_n||}{r_n} \leq 1 - \sum_{i=1}^{n-1} \frac{||x_i||}{r_i} \leq \sum_{i=1}^{n-1} \frac{||y_i|| - ||x_i||}{r_i} \leq \sum_{i=1}^{n-1} \frac{||x_i - y_i||}{r_i} \leq \frac{\sum_{i=1}^{n-1} ||x_i - y_i||}{r_{n-1}}.$$

So it is true that

$$||x-y||_{\ell_1(\{X_i\}_{i=1}^n)} \geq ||x_n|| \frac{r_{n-1}}{r_n}.$$

Finally,

$$\frac{||g_n(x)||}{||x-y||} \leq \frac{||x_n|| A_n}{||x-y||_{\ell_1(\{X_i\}_{i=1}^n)}} \leq \frac{||x_n|| A_n}{||x_n|| \frac{r_{n-1}}{r_n}} \leq \frac{r_n A_n}{r_{n-1}}.$$

■

Proposition 3.2. *For every $j \in \mathbb{N}$ and $x, y \in P_j(X)$ the following implications hold true:*

(3.1)

$$\sum_{i=1}^{j-1} \frac{||x_i||}{r_i}, \sum_{i=1}^{j-1} \frac{||y_i||}{r_i} \leq 1 \Rightarrow ||F_{j,j}(x) - F_{j,j}(y)|| \leq \left(5 + \frac{r_j A_{j-1}}{r_{j-1}}\right) ||x-y||,$$

(3.2)

$$\sum_{i=1}^{j-1} \frac{||y_i||}{r_i} > 1 \Rightarrow ||F_{j,j}(x) - F_{j,j}(y)|| \leq \left(1 + \frac{r_j A_{j-1}}{r_{j-1}}\right) ||x-y||.$$

Proof. First we prove (3.1) case by case:

If $\sum_{i=1}^{j} \frac{||x_i||}{r_i} > 1$ and $\sum_{i=1}^{j} \frac{||y_i||}{r_i} > 1$, then from the definition of f_j we know that $f_j(y) < ||y_j||$ so

$$\begin{aligned}
&\frac{\left\| x_j||y_j||f_j(x) - y_j||x_j||f_j(y) \right\|}{||x_j||\,||y_j||} \\
&\leq \frac{\left\| x_j||y_j||f_j(x) - x_j f_j(y)||x_j|| \right\| + \left\| x_j f_j(y)||x_j|| - y_j||x_j||f_j(y) \right\|}{||x_j||\,||y_j||} \\
&= \frac{\left| f_j(x)||y_j|| - f_j(y)||x_j|| \right|}{||y_j||} + \frac{||x_j - y_j||f_j(y)}{||y_j||} \\
&\leq \frac{\left| f_j(x)||y_j|| - f_j(y)||y_j|| \right| + \left| f_j(y)||y_j|| - f_j(y)||x_j|| \right|}{||y_j||} + ||x_j - y_j|| \\
&= |f_j(x) - f_j(y)| + \frac{f_j(y)||x_j - y_j||}{||y_j||} + ||x_j - y_j|| \\
&\leq \frac{r_j A_{j-1}}{r_{j-1}}||x - y|| + 2||x_j - y_j|| \leq \left(4 + \frac{r_j A_{j-1}}{r_{j-1}}\right)||x - y||.
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
&||F_{j,j}(x) - F_{j,j}(y)|| \\
&\leq ||P_{j-1}(x - y)|| + ||(F_{j,j} - P_{j-1})(x) - (F_{j,j} - P_{j-1})(y)|| \\
&= ||P_{j-1}(x - y)|| + \left|\left| \frac{x_j}{||x_j||} f_j(x) - \frac{y_j}{||y_j||} f_j(y) \right|\right| \\
&= ||P_{j-1}(x - y)|| + \frac{\left\| x_j||y_j||f_j(x) - y_j||x_j||f_j(y) \right\|}{||x_j||\,||y_j||} \\
&\leq \left(5 + \frac{r_j A_{j-1}}{r_{j-1}}\right)||x - y||.
\end{aligned}$$

If $\sum_{i=1}^{j} \frac{||x_i||}{r_i} \leq 1$ and $\sum_{i=1}^{j} \frac{||y_i||}{r_i} > 1$, then from the definition of f_j it follows that $||x_j|| \leq f_j(x)$ and $||y_j|| - f_j(y) > 0$ so

$$\begin{aligned}
\frac{\left\| x_j||y_j|| - y_j f_j(y) \right\|}{||y_j||} &\leq \frac{\left\| x_j||y_j|| - y_j||y_j|| \right\| + \left\| y_j||y_j|| - y_j f_j(y) \right\|}{||y_j||} \\
&= ||x_j - y_j|| + \big(||y_j|| - f_j(y)\big) \\
&= ||x_j - y_j|| + \big(||y_j|| - ||x_j||\big) + \big(||x_j|| - f_j(y)\big) \\
&\leq 2||x_j - y_j|| + \big(f_j(x) - f_j(y)\big) \\
&\leq \left(4 + \frac{r_j A_{j-1}}{r_{j-1}}\right)||x - y||.
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
&||F_{j,j}(x) - F_{j,j}(y)|| \\
&\leq ||P_{j-1}(x-y)|| + ||(F_{j,j} - P_{j-1})(x) - (F_{j,j} - P_{j-1})(y)|| \\
&= ||P_{j-1}(x-y)|| + \left|\left| x_j - \frac{y_j}{||y_j||} f_j(y) \right|\right| \\
&= ||P_{j-1}(x-y)|| + \frac{\left\| \, x_j||y_j|| - y_j f_j(y) \, \right\|}{||y_j||} \\
&\leq \left(5 + \frac{r_j A_{j-1}}{r_{j-1}}\right) ||x-y||.
\end{aligned}$$

The case when $\sum_{i=1}^{j} \frac{||x_i||}{r_i} \leq 1$ and $\sum_{i=1}^{j} \frac{||y_i||}{r_i} \leq 1$ is trivially true because $F_{j,j}$ acts as the identity, so we have proven (3.1).

Finally, we prove (3.2) distinguishing between 3 different cases:

If $\sum_{i=1}^{j-1} \frac{||x_i||}{r_i} \geq 1$ then $F_{j,j}(x) - F_{j,j}(y) = P_{j-1}(x-y)$ so it is straightforward.

If $\sum_{i=1}^{j-1} \frac{||x_i||}{r_i} < 1 < \sum_{i=1}^{j} \frac{||x_i||}{r_i}$ then, as $1 \leq \sum_{i=1}^{j-1} \frac{||y_i||}{r_i}$, it holds that

$$\begin{aligned}
|f_j(x)| &= r_j \left(1 - \sum_{i=1}^{j-1} \frac{||x_i||}{r_i}\right) \\
&\leq r_j \left(\sum_{i=1}^{j-1} \frac{||y_i|| - ||x_i||}{r_i}\right) = |f_j(x) - f_j(y)| \leq \frac{r_j A_{j-1}}{r_j} ||x-y||.
\end{aligned}$$

Hence,

$$\begin{aligned}
||F_{j,j}(x) - F_{j,j}(y)|| &= \left|\left| P_{j-1}(x-y) + \frac{x_j}{||x_j||} f_j(x) \right|\right| \\
&\leq \left(1 + \frac{r_j A_{j-1}}{r_{j-1}}\right) ||x-y||.
\end{aligned}$$

Finally, if $\sum_{i=1}^{j} \frac{||x_i||}{r_i} \leq 1$, then

$$\begin{aligned}
||x_j|| &\leq r_j \left(1 - \sum_{i=1}^{j-1} \frac{||x_i||}{r_i}\right) \leq r_j \left(\sum_{i=1}^{j-1} \frac{||y_i|| - ||x_i||}{r_i}\right) = |f_j(x) - f_j(y)| \\
&\leq \frac{r_j A_{j-1}}{r_{j-1}} ||x-y||,
\end{aligned}$$

and so,

$$||F_{j,j}(x) - F_{j,j}(y)|| = ||P_{j-1}(x-y) + x_j|| \leq \left(1 + \frac{r_j A_{j-1}}{r_{j-1}}\right) ||x-y||.$$

■

Theorem 3.3. *There is a sequence $(q_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that for every Banach space X with a monotone FDD (X_n), and every decreasing sequence $(r_n) \subset \mathbb{R}^+$ satisfying $\frac{r_n}{r_{n-1}} \leq q_n$, the set*

$$K = \overline{\text{co}}\left(\bigcup_{n\in\mathbb{N}} r_n B_{X_n}\right)$$

is a compact Lipschitz retract of X.

Proof. Consider some $\delta > 0$ and $(\delta_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that

$$\prod_{n\in\mathbb{N}} (1+\delta_n) \leq 1 + \delta,$$

and consider a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that the sequence given by

$$(\alpha_n)_{n\in\mathbb{N}} := \left(\sum_{k=n+1}^{\infty} a_k A_k\right)_{n\in\mathbb{N}}$$

verifies that

$$\alpha_n \leq \frac{\delta_n}{2} \qquad \forall n \in \mathbb{N}.$$

Now we set

$$q_n = \min\left\{a_n, \frac{\delta_n}{2A_{n-1}}\right\}.$$

Suppose that (r_n) is a sequence as in the statement of the Theorem. Given any $n \in \mathbb{N}$ we define the following retraction

$$F_n = F_{n,1} \circ \cdots \circ F_{n,n} \circ P_n : X \to K_n.$$

For a given $x \in X$ and $n \in \mathbb{N}$ we set $\widetilde{m} = \max\left\{k \in \{1,\dots,n+1\} \ : \ \sum_{i=1}^{k-1} \frac{||x_i||}{r_i} \leq 1\right\}$ so that it is possible to compute $F_n(x)$ as

$$F_n(x) = \begin{cases} P_{\widetilde{m}-1}(x) + \frac{x_{\widetilde{m}}}{||x_{\widetilde{m}}||} f_{\widetilde{m}}(x), & \text{if } \widetilde{m} \leq n, \\ P_n(x) & \text{if } \widetilde{m} = n+1. \end{cases}$$

Now, if $x, y \in P_n(X)$, we claim that $||F_n(x) - F_n(y)|| \leq 5(1+\delta)$. Indeed, let us consider

$$m = \max\left\{k \in \{1,\dots,n+1\} : \sum_{i=1}^{k-1} \frac{||x_i||}{r_i} \leq 1, \sum_{i=1}^{k-1} \frac{||y_i||}{r_i} \leq 1\right\}.$$

If $m = n + 1$ then $F_n(x) - F_n(y) = x - y$.

If $m = n$ then $F_n(x) = F_{n,n}(x)$ and $F_n(y) = F_{n,n}(y)$ and we use Proposition 3.2 to finish this case.

If $m = n - 1$ then $F_n(x) = F_{n,n-1}(F_{n,n}(x))$ and $F_n(y) = F_{n,n-1}(F_{n,n}(y))$, and we know from the definition of m that x or y verifies (3.2) of Proposition 3.2 for $j = n$. Hence, using Lemma 3.1 together with Proposition 3.2 we get that

$$\begin{aligned}
||F_n(x) - F_n(y)|| &= ||F_{n,n-1}(F_{n,n}(x)) - F_{n,n-1}(F_{n,n}(y))|| \\
&\leq \frac{r_n A_n}{r_{n-1}} ||F_{n,n}(x) - F_{n,n}(y)|| \\
&+ \left|\left|(F_{n-1,n-1} \circ P_{n-1})(F_{n,n}(x)) - (F_{n-1,n-1} \circ P_{n-1})(F_{n,n}(y))\right|\right| \\
&\leq \left(5 + \frac{r_{n-1}A_{n-2}}{r_{n-2}} + \frac{r_n A_n}{r_{n-1}}\right) ||F_{n,n}(x) - F_{n,n}(y)|| \\
&\leq \left(5 + \frac{r_{n-1}A_{n-2}}{r_{n-2}} + \frac{r_n A_n}{r_{n-1}}\right)\left(1 + \frac{r_n A_{n-1}}{r_{n-1}}\right) ||x - y|| \\
&\leq \left(5 + \frac{\delta_{n-1}}{2} + \alpha_{n-1}\right)\left(1 + \frac{\delta_n}{2}\right) ||x - y|| \leq 5(1 + \delta)||x - y||.
\end{aligned}$$

Otherwise, if $m \leq n - 2$ then

$$F_n(x) = F_{n,m} \circ \cdots \circ F_{n,n}(x) \text{ and } F_n(y) = F_{n,m} \circ \cdots \circ F_{n,n}(y).$$

Also, from the definition of m we get that, if $n \geq p \geq m + 2$, then the point $F_{n,p} \circ \cdots \circ F_{n,n}(x)$ or the point $F_{n,p} \circ \cdots \circ F_{n,n}(y)$ satisfies (3.2) of Proposition 3.2 for $j = p - 1$. Also the point x or the point y satisfies (3.2) for $j = n$. Having this in mind and using the same argument as in the previous step, we check that

$$\begin{aligned}
&||F_n(x) - F_n(y)|| = ||F_{n,m} \circ \cdots \circ F_{n,n}(x) - F_{n,m} \circ \cdots \circ F_{n,n}(y)|| \\
&\leq \left(5 + \frac{r_m A_{m-1}}{r_{m-1}} + \sum_{k=m+1}^{n} \frac{r_k A_k}{r_{k-1}}\right) ||F_{n,m+1} \circ \cdots \circ F_{n,n}(x) - F_{n,m+1} \circ \cdots \circ F_{n,n}(y)|| \\
&\leq \left(5 + \frac{r_m A_{m-1}}{r_{m-1}} + \sum_{k=m+1}^{n} \frac{r_k A_k}{r_{k-1}}\right)\left(1 + \frac{r_{m+1}A_m}{r_m} + \sum_{k=m+2}^{n} \frac{r_k A_k}{r_{k-1}}\right) \\
&\cdot ||F_{n,m+2} \circ \cdots \circ F_{n,n}(x) - F_{n,m+2} \circ \cdots \circ F_{n,n}(y)|| \leq \cdots \\
&\cdots \leq \left(5 + \frac{r_m A_{m-1}}{r_{m-1}} + \sum_{k=m+1}^{n} \frac{r_k A_k}{r_{k-1}}\right)\left(\prod_{j=m+1}^{n-1}\left(1 + \frac{r_j A_{j-1}}{r_{j-1}} + \sum_{k=j+1}^{n} \frac{r_k A_k}{r_{k-1}}\right)\right) \\
&\cdot ||F_{n,n}(x) - F_{n,n}(y)||
\end{aligned}$$

$$\leq 5\left(\prod_{j=m}^{n-1}\left(1+\frac{r_jA_{j-1}}{r_{j-1}}+\sum_{k=j+1}^{n}\frac{r_kA_k}{r_{k-1}}\right)\right)\left(1+\frac{r_nA_{n-1}}{r_{n-1}}\right)||x-y||$$

$$\leq 5\left(\prod_{j=m}^{n-1}\left(1+\frac{\delta_j}{2}+\alpha_j\right)\right)\left(1+\frac{\delta_n}{2}\right)||x-y|| \leq 5\prod_{j=m}^{n}(1+\delta_j)||x-y||$$

$$\leq 5(1+\delta)||x-y||.$$

It is easy to see that $\forall x \in P_n(X)$, if $k > n$ then $F_k(x) = F_n(x)$ so we may define the following map

$$F : \bigcup_{n\in\mathbb{N}} P_n(X) \to \text{co}\left(\bigcup_{n\in\mathbb{N}} r_nB_{X_n}\right),$$

$$F(x) = \lim_{n\to\infty} F_n(x),$$

which is a $5(1+\delta)$-Lipschitz retraction. Considering now $R : X \to K$ as the extension of F to the whole X, we are done. ∎

Remark 3.4. We may actually choose $q_n = \frac{1}{n2^{n+1}}$. This arises from choosing $\delta_n = 2^{-n+1}$ and $a_k = \frac{1}{k2^{k+1}}$, so that $\alpha_n = 2^{-n}$.

Notice that the restriction of the previous retraction R to one of the blocks of the FDD is the radial projection. Hence, it is not possible to obtain an estimate for the Lipschitz norm of R better than 2 for general FDD spaces. Next, we are going to treat the special case when the blocks of the FDD are of dimension 1, that is when X has a Schauder basis, which leads to a much better estimate on the Lipschitz norm of the retraction.

From now on, $dim X_n = 1$ for every $n \in \mathbb{N}$ and $(e_n, e_n^*)_{n\in\mathbb{N}}$ is a monotone Schauder basis in X such that $||e_n|| = 1$ and $\langle e_n\rangle = X_n$. From now on $r_n \in \mathbb{R}^+$ is a decreasing sequence, (P_n) is the sequence of projections of the basis and we keep denoting $x_i = (P_i - P_{i-1})(x)$ for every $x \in X$. We are going to keep the previous definition for $F_{n,m}$ and f_m, as well as for $E_{n,m}$. The main difference between this case and the general one is stated in Proposition 3.5.

Proposition 3.5.

$$F_{m,m} \;\; \textit{is} \;\; \left(1+\frac{r_mA_{m-1}}{r_{m-1}}\right)\textit{-Lipschitz} \quad \forall m \in \mathbb{N}.$$

Proof. We are going to prove it case by case. We have already proved in Proposition 3.2 the case when $x, y \in E_{m,m} = P_m(X)$ are such that $\sum_{i=1}^{m-1}\frac{||y_i||}{r_i} > 1$ or $\sum_{i=1}^{m-1}\frac{||x_i||}{r_i} > 1$, so let us assume throughout the proof that $\sum_{i=1}^{m-1}\frac{||x_i||}{r_i} \leq 1$ and $\sum_{i=1}^{m-1}\frac{||y_i||}{r_i} \leq 1$.

If $\sum_{i=1}^{m} \frac{||x_i||}{r_i} \leq 1$ and $\sum_{i=1}^{m} \frac{||y_i||}{r_i} \leq 1$ then $F_{m,m}(x) - F_{m,m}(y) = x - y$ so it is straightforward that $||F_{m,m}(x) - F_{m,m}(y)|| = ||x - y||$.

If $\sum_{i=1}^{m} \frac{||x_i||}{r_i} \leq 1$ and $\sum_{i=1}^{m} \frac{||y_i||}{r_i} > 1$, then we split this case into 2 different subcases. For these subcases we are going to set

$$t = \frac{e_m^*(x) - \frac{e_m^*(y)}{|e_m^*(y)|} f_m(y)}{e_m^*(x) - e_m^*(y)}$$

whenever $e_m^*(x) \neq e_m^*(y)$ and $t = \infty$ otherwise.

Subcase $t \notin [0, 1]$.
In this subcase $|e_m^*(y)| = ||y_m|| > f_m(y) \geq 0$ so then $e_m^*(x) \neq 0$ because otherwise $t = \frac{f_m(y)}{|e_m^*(y)|} \in [0, 1]$. We claim that in this subcase $\frac{e_m^*(x)}{|e_m^*(x)|} = \frac{e_m^*(y)}{|e_m^*(y)|}$. Indeed, if not, multiplying and dividing t by $\frac{e_m^*(x)}{|e_m^*(x)|} = \frac{|e_m^*(x)|}{e_m^*(x)} = -\frac{|e_m^*(y)|}{e_m^*(y)}$, it is easy to see that

$$t = \frac{|e_m^*(x)| + f_m(y)}{|e_m^*(x)| + |e_m^*(y)|} \in [0, 1],$$

which is not possible. This means that $\frac{e_m^*(x)}{|e_m^*(x)|} = \frac{e_m^*(y)}{|e_m^*(y)|}$ so now it can be easily seen that

$$t = \frac{|e_m^*(x)| - f_m(y)}{|e_m^*(x)| - |e_m^*(y)|}.$$

As $|e_m^*(x)| - f_m(y) \geq |e_m^*(x)| - |e_m^*(y)|$ we claim that $|e_m^*(x)| - f_m(y) \geq 0$. In fact, we check case by case and obtain the following scheme

$$\begin{cases} t = \infty & \Rightarrow\ |e_m^*(x)| - |e_m^*(y)| = 0 \ \Rightarrow\ |e_m^*(x)| - f_m(y) \geq 0, \\ t < 0 & \Rightarrow\ |e_m^*(x)| - |e_m^*(y)| < 0 \ \Rightarrow\ |e_m^*(x)| - f_m(y) > 0, \\ t > 1 & \Rightarrow\ |e_m^*(x)| - |e_m^*(y)| > 0 \ \Rightarrow\ |e_m^*(x)| - f_m(y) > 0. \end{cases}$$

Then,

$$\left| e_m^*(x) - \frac{e_m^*(y)}{|e_m^*(y)|} f_m(y) \right| = |e_m^*(x)| - f_m(y) \leq |f_m(x) - f_m(y)| \leq \frac{r_m A_{m-1}}{r_{m-1}} ||x - y||,$$

and we have

$$\begin{aligned} ||F_{m,m}(x) - F_{m,m}(y)|| &= \left|\left| P_{m-1}(x - y) + \left(x_m - \frac{y_m}{||y_m||} f_m(y) \right) \right|\right| \\ &\leq ||x - y|| + \left| e_m^*(x) - \frac{e_m^*(y)}{|e_m^*(y)|} f_m(y) \right| \\ &\leq ||x - y|| + \frac{r_m A_{m-1}}{r_{m-1}} ||x - y||. \end{aligned}$$

The subcase $t \in [0,1]$ is simpler. Due to the convexity of the norm and the fact that $||P_{m-1}|| = 1$ we have

$$||F_{m,m}(x) - F_{m,m}(y)|| = ||P_{m-1}(x-y) + t(x_m - y_m)|| \leq ||x-y||,$$

so we are finally done with both subcases. Now, if $\sum_{i=1}^{m} \frac{||x_i||}{r_i} > 1$ and $\sum_{i=1}^{m} \frac{||y_i||}{r_i} > 1$ then $||x_m|| > 0$ and $||y_m|| > 0$ and we are also spliting this case into 2 subcases for technical reasons:

Subcase $\frac{x_m}{||x_m||} = \frac{y_m}{||y_m||}$. Here,

$$\left|\left| \frac{x_m}{||x_m||} f_m(x) - \frac{y_m}{||y_m||} f_m(y) \right|\right| = |f_m(x) - f_m(y)| \leq \frac{r_m A_{m-1}}{r_{m-1}} ||x-y||,$$

so then

$$\begin{aligned} ||F_{m,m}(x) - F_{m,m}(y)|| &= \left|\left| P_{m-1}(x-y) + \left(\frac{x_m}{||x_m||} f_m(x) - \frac{y_m}{||y_m||} f_m(y) \right) \right|\right| \\ &\leq \left(1 + \frac{r_m A_{m-1}}{r_{m-1}} \right) ||x-y||. \end{aligned}$$

Finally, for the subcase when $\frac{x_m}{||x_m||} = -\frac{y_m}{||y_m||}$ we consider

$$t = \frac{f_m(x) + f_m(y)}{||x_m|| + ||y_m||} \in [0,1],$$

and again by convexity,

$$||F_{m,m}(x) - F_{m,m}(y)|| = ||P_{m-1}(x-y) + t(x_m - y_m)|| \leq ||x-y||.$$

■

Theorem 3.6. *For every $\delta > 0$ there exists a sequence $(q_n) \subset \mathbb{R}^+$ such that for every Banach space X with a monotone Schauder basis (e_n) and every decreasing sequence $(r_n) \subset \mathbb{R}^+$ satisfying $\frac{r_n}{r_{n-1}} \leq q_n$, the set*

$$K = \overline{\text{co}} \left(\bigcup_{n \in \mathbb{N}} r_n B_{\langle e_n \rangle} \right)$$

is a $(1+\delta)$-Lipschitz retract of X.

Proof. We begin by letting δ, δ_n, a_n, α_n, q_n and F_n be exactly as in the proof of Theorem 3.3. Suppose that (r_n) is as in the statement of the Theorem. We fix $n > 1$, so that making use of Lemma 3.1 and Proposition 3.5, the

following holds

$$\begin{aligned} m = 1 \quad &\Rightarrow ||F_{n,1}||_{Lip} \leq 1 + \sum_{k=2}^{n} \frac{r_k A_k}{r_{k-1}}. \\ m \in \{2, \dots, n-1\} \quad &\Rightarrow ||F_{n,m}||_{Lip} \leq 1 + \frac{r_m A_{m-1}}{r_{m-1}} + \sum_{k=m+1}^{n} \frac{r_k A_k}{r_{k-1}}. \\ m = n \quad &\Rightarrow ||F_{n,n}||_{Lip} \leq 1 + \frac{r_n A_{n-1}}{r_{n-1}}. \end{aligned}$$

Let us consider now the composed retraction

$$F_n = F_{n,1} \circ \cdots \circ F_{n,n} \circ P_n : X \to K_n.$$

As in Theorem 3.3 it is enough to show that $||F_n||_{Lip} \leq 1 + \delta$:

$$\begin{aligned} ||F_n||_{Lip} &\leq \prod_{m=1}^{n} ||F_{n,m}||_{Lip} \\ &\leq \left(1 + \sum_{k=2}^{n} \frac{r_k A_k}{r_{k-1}}\right) \left(\prod_{m=2}^{n-1} \left(1 + \frac{r_m A_{m-1}}{r_{m-1}} + \sum_{k=m+1}^{n} \frac{r_k A_k}{r_{k-1}}\right)\right) \\ &\quad \cdot \left(1 + \frac{r_n A_{n-1}}{r_{n-1}}\right) \\ &\leq (1 + \alpha_1) \left(\prod_{m=2}^{n-1} \left(1 + \frac{\delta_m}{2} + \alpha_m\right)\right) \left(1 + \frac{\delta_n}{2}\right) \leq \prod_{m=1}^{n} (1 + \delta_m) \leq 1 + \delta. \end{aligned}$$

■

4. The π-property

We pass to the results concerning the necessary conditions on the Banach space X so that X admits a GCCR $K \subset X$. Our methods require a certain quantitative "smallness" condition to be satisfied for K. Under such assumption we show that X must have the π-property. In fact, our argument makes no use of the convexity of K. The crucial condition is smallness. Our proof uses three main ingredients. The unpublished Milman lemma (communicated to us, with proof, by Bill Johnson) concerning the projection constant of a finite dimensional subspace of a Banach space, the finite dimensional "Lipschitzization" of coarse Lipschitz maps due to Bourgain (and streamlined by Begun), and the averaging of derivatives for finite dimensional Lipschitz maps. We start with a well-known fact.

Given $r \in \mathbb{R}^+$, we are going to denote $[r] = \max\{n \in \mathbb{N} \cup \{0\} \ : \ n \leq r\}$.

Lemma 4.1. *For every $n \in \mathbb{N}$ and $\varepsilon \in (0,1)$, if $(E, ||\cdot||)$ is a Banach space of dimension n, then there exists a renorming $|\cdot|$ of E such that $(E, |\cdot|)$*

embeds isometrically in ℓ_∞^N *where* $N = \left[(1+2/\varepsilon)^n\right]$ *and*

$$|x| \leq ||x|| \leq \frac{|x|}{1-\varepsilon}.$$

Proof. By Lemma 2.6 of [MS86], we know that there exists an ε-net in S_{X^*} consisting of N points, namely $\{x_1^*, \ldots, x_N^*\}$. Just consider the norm $|x| = \max_{i\in\{1,\ldots,N\}} x_i^*(x)$. ∎

If X is a Banach space and $E \subset X$ is a subspace, then the projection constant of E in X is defined as

$$\lambda(E,X) = \inf\left\{||P|| \ : \ P : X \to E \ , \ P\Big|_E = Id_E\right\}.$$

Lemma 4.2 (Vitali Milman-unpublished)**.** *Let* X *be a Banach space. For every* $\varepsilon \in (0,1)$ *and a subspace* $E \subset X$ *of dimension* $dim(E) = n$*, there is another subspace* $G_E \subset X$ *containing* E *such that* $dim(G_E) \leq \left(1+\frac{2}{\varepsilon}\right)^n$ *and*

$$\lambda(E,X) \leq \frac{2}{1-\varepsilon}\lambda(E,G_E).$$

Proof. (communicated to us by Bill Johnson) We follow the trace duality arguments set up in [Joh+79]. Pick $\varepsilon \in (0,1)$. For a given Banach space Y with a finite dimensional subspace $E \subset Y$ we define a pair of norms on the space of all linear operators $\mathcal{L}(E)$

$$||T||_Y = \inf\left\{||\widetilde{T}|| \ : \ \widetilde{T} \in \mathcal{L}(Y,E) \ , \ \widetilde{T}\Big|_E = T\right\},$$

$$||T||_{\Lambda Y} = ||i_E T||_\Lambda,$$

where $i_E : E \to Y$ is the inclusion map and $||\cdot||_\Lambda$ refers to the nuclear norm in $\mathcal{L}(E,Y)$, that is,

$$||T||_{\Lambda Y} = \inf\left\{\sum_{i=1}^n ||x_i^*|| \cdot ||y_i|| \ : \ n \in \mathbb{N}, T = \sum_{i=1}^n x_i^* \otimes y_i, x_i^* \in E^*, y_i \in Y\right\}.$$

We know from [Joh+79] pg. 377 that both norms are in trace duality. More precisely, we have a dual pairing $\langle \mathcal{L}(E), \mathcal{L}(E)\rangle$ given by $\langle T, S\rangle = tr(ST)$ for every $T, S \in \mathcal{L}(E)$, such that

$$(\mathcal{L}(E), ||\cdot||_Y)^* = (\mathcal{L}(E), ||\cdot||_{\Lambda Y}).$$

Thanks to this interpretation, we can compute

$$\begin{aligned}\lambda(E,Y) = ||Id_E||_Y &= \sup_{||T||_{\Lambda Y}=1} tr(TId_E) = \sup_{T\in\mathcal{L}(E)} tr\left(\frac{T}{||T||_{\Lambda Y}}\right) \\ &= \sup_{T\in\mathcal{L}(E)} \frac{1}{\left|\left|\frac{T}{tr(T)}\right|\right|_{\Lambda Y}} = \sup_{tr(T)=1} \frac{1}{||T||_{\Lambda Y}} = \frac{1}{\inf\limits_{tr(T)=1} ||T||_{\Lambda Y}}.\end{aligned}$$

Returning to the situation of our theorem, E is now a subspace of X. Let us first take $\mu \in (0, 1/2)$. Now we take, for $\delta = \frac{(1-\varepsilon)(1-\mu)-(1-\varepsilon)1/2}{\lambda(E,X)} > 0$, a trace one operator $S \in \mathcal{L}(E)$ such that

$$\inf_{tr(T)=1} ||T||_{\Lambda X} \geq ||S||_{\Lambda X} - \delta.$$

We also take the norm $|\cdot|$ given by Lemma 4.1 so that $(E, |\cdot|)$ is isometrically a subspace of $\ell_\infty^{\varphi(n)}$ where $\varphi(n) = \left[\left(1+\frac{2}{\varepsilon}\right)^n\right]$ and, denoting $|\cdot|_{\Lambda Y}$ the nuclear norm taking $(E, |\cdot|)$ as the domain of the operators instead of $(E, ||\cdot||)$, we have for every superspace $Y \supset E$ that

$$||S||_{\Lambda Y} \leq |S|_{\Lambda Y} \leq \frac{||S||_{\Lambda Y}}{1-\varepsilon}.$$

It is well-known (Proposition 47.6 in [Tre06]) that $i_E S$ admits an extension $\widetilde{S} : \ell_\infty^{\varphi(n)} \to X$ almost preserving the nuclear norm, that is $|S|_{\Lambda X} \geq (1-\mu)||\widetilde{S}||_\Lambda$. By Proposition 8.7 from [TJ89] we know that there exist $x_1, \ldots, x_{\varphi(n)} \in X$ such that $\widetilde{S} = \sum_{i=1}^{\varphi(n)} e_i^* \otimes x_i$ and

$$||\widetilde{S}||_\Lambda = \sum_{i=1}^{\varphi(n)} ||x_i||,$$

where $e_i^* \in \left(\ell_\infty^{\varphi(n)}\right)^*$ are the coordinate functionals. Just considering $G_E = [x_i]_{i=1}^{\varphi(n)}$, we can see that

$$\begin{aligned} ||S||_{\Lambda X} &\geq (1-\varepsilon)|S|_{\Lambda X} \geq (1-\varepsilon)(1-\mu)||\widetilde{S}||_\Lambda \\ &\geq (1-\varepsilon)(1-\mu)|S|_{\Lambda G_E} \geq ||S||_{\Lambda G_E}(1-\varepsilon)(1-\mu). \end{aligned}$$

Finally, taking into acount that $\lambda(E, G_E) \leq \lambda(E, X)$, we finish the proof because

$$\begin{aligned} \lambda(E,X) &\leq \frac{1}{||S||_{\Lambda X} - \delta} \leq \frac{1}{(1-\varepsilon)(1-\mu)||S||_{\Lambda G_E} - \delta} \\ &\leq \frac{1}{(1-\varepsilon)(1-\mu)\inf_{tr(T)=1} ||T||_{\Lambda G_E} - \delta} \\ &= \lambda(E,G_E)\frac{1}{(1-\varepsilon)(1-\mu) - \delta\lambda(E,G_E)} \leq \frac{2}{1-\varepsilon}\lambda(E,G_E). \end{aligned}$$

■

Definition 4.3. Given a separable Banach space X, we will say that $\beta = (e_n) \subset X$ is a fundamental sequence if $[e_n] = X$ and we will denote $E_n^\beta = [e_i]_{i=1}^n$.

Definition 4.4. Let X be a separable Banach space, $\beta = (e_n)$ a fundamental sequence and $K \subset X$ a bounded subset. We will define the following concepts:

- The sequence of inner radii (r_n^β) given by
$$r_n^\beta = \sup\left\{r \geq 0 \ : \ B_{E_n^\beta}(x,r) \subset K \cap E_n^\beta \ , \ x \in X\right\} \quad \forall n \in \mathbb{N}.$$
- The sequence of heights (h_n^β) given by
$$h_n^\beta = \sup\left\{d(x, E_n^\beta) \ : \ x \in K\right\} \quad \forall n \in \mathbb{N}.$$

Definition 4.5. We say that a bounded subset K of a separable Banach space X is small if there exist an $\varepsilon \in (0,1)$, a fundamental sequence $\beta = (e_n)$ in X and a strictly increasing $\sigma : \mathbb{N} \to \mathbb{N}$ such that
$$0 < \frac{h_{\sigma(n)}^\beta}{r_{\sigma(n)}^\beta} \leq \frac{1}{2\sigma(n)^2\left(\left(1+\frac{2}{\varepsilon}\right)^{\sigma(n)}+2\right)} \quad \forall n \in \mathbb{N}.$$

Note that such sets are necessarily relatively compact and generate X.

If $\beta = (e_n)$ is a monotone Schauder basis, $X_i = \langle e_i \rangle$ and we take $K = \overline{\text{co}}\left(\bigcup_{k\in\mathbb{N}} r_k B_{X_k}\right)$ as in Theorem 3.3, then it is easily seen that
$$\frac{h_n^\beta}{r_n^\beta} \leq \frac{2nr_{n+1}}{r_n},$$
so the constants r_n can be adjusted in such a way that K as defined is also small. Hence the following result follows from Theorem 3.3.

Proposition 4.6. *If a separable Banach space X has an FDD then X admits a small GCCR.*

We now pass to the promised opposite implication. Note that the convexity assumption on the generating compact set K is not needed.

Theorem 4.7. *Let X be a separable Banach space. If there exists a Lipschitz retraction from X onto a small compact subset, then X has the π-property.*

Proof. Assume that there is a Lipschitz retraction from X onto a small compact set K. Take $\varepsilon \in (0,1)$, $\beta = (e_n)$ a fundamental sequence of X and $\sigma : \mathbb{N} \to \mathbb{N}$ strictly increasing for which the inequality of Definition 4.5 holds true, and let $\varphi(n) = \left[\left(1+\frac{2}{\varepsilon}\right)^{\sigma(n)}\right]$, $E_n = E_{\sigma(n)}^\beta$, $h_n = h_{\sigma(n)}^\beta$ and $r_n = r_{\sigma(n)}^\beta$. Lemma 4.2 guarantees that for every $n \in \mathbb{N}$ there is a finite dimensional subspace $G_n \subset X$ of dimension $dim(G_n) = \varphi(n)$ such that for every projection $P : G_n \to E_n$, the inequality $||P|| \geq (1-\varepsilon)\frac{\lambda(E_n,X)}{2}$ holds. Assume that $R : X \to K$ is the Lipschitz retraction, then taking $C_n : K \to E_n$ a nearest point map (it may not be unique), we define $\widetilde{R}_n = (C_n \circ R)\big|_{G_n} : G_n \to E_n$ for every $n \in \mathbb{N}$. Now,
$$||\widetilde{R}_n(x) - \widetilde{R}_n(y)|| \leq ||R||_{Lip}\left(||x-y|| + \frac{2h_n}{||R||_{Lip}}\right) \quad \forall x, y \in G_n,$$

so by the Proposition of [Beg99], for every $\tau > 0$, there is a Lipschitz mapping

$$R_{n,\tau} : G_n \to E_n$$

such that

$$||R_{n,\tau}||_{Lip} \leq ||R||_{Lip}\left(1 + \frac{\varphi(n)h_n}{||R||_{Lip}\tau}\right),$$

$$||R_{n,\tau}(x) - \widetilde{R}_n(x)|| \leq ||R||_{Lip}\left(\tau + \frac{2h_n}{||R||_{Lip}}\right) \quad \forall x \in G_n.$$

For the rest of the argument we fix $x_n \in K_n := K \cap E_n$ such that $B_{E_n}(x_n, r_n) \subset K_n$ (it exists by the definition of the inner radius and the compactness). Now we choose $\tau_n = \frac{\varphi(n)h_n}{||R||_{Lip}}$, and define $R_n : G_n \to E_n$ by $R_n(x) = R_{n,\tau_n}(x + x_n) - x_n$. If $x + x_n \in K_n$ then $\widetilde{R}_n(x + x_n) = x + x_n$ acts as an identity. Hence we have that for every $x \in K_n + \{-x_n\}$

$$||R_n(x) - x|| = ||R_{n,\tau_n}(x + x_n) - \widetilde{R}_n(x + x_n)|| \leq h_n(\varphi(n) + 2) =: \rho_n.$$

Now, let $(a_i, a_i^*)_{i=1}^{\varphi(n)}$ be a normalized linear basis for G_n with projections $(S_i)_{i=1}^{\varphi(n)}$ such that $(a_i, a_i^*)_{i=1}^{\sigma(n)}$ is an Auerbach basis for E_n. Then,

$$B_n = r_n \operatorname{co}\left(\{\pm a_i \ , \ i = 1, \ldots, \sigma(n)\}\right) \subset K_n + \{-x_n\}.$$

Fix a sequence (δ_k) of positive numbers converging to zero. Now we define for every $k \in \mathbb{N}$ the compact set

$$B_{n,k} = B_n + \delta_k \sum_{i=\sigma(n)+1}^{\varphi(n)} [-a_i, a_i] \subset G_n.$$

The geometrical shape of this set can be described as a r_n-multiple of the unit ball of $\ell_1^{\sigma(n)}$ in E_n, which form the base of a hypercylinder, times a hypercube of side length $2\delta_k$ occupying the remaining dimensions of G_n. Letting δ_k go to zero of course means that this set gets squashed down to its base in E_n. For any $i \in \{1, \ldots, \sigma(n)\}$ we denote $(B_{n,k})_i = (Id - a_i^* \otimes a_i)(B_{n,k})$. This set is just a one-codimensional section (or a projection of rank $\varphi(n) - 1$) of $B_{n,k}$ which reduces the base by one coordinate.

In order to recover the shape $B_{n,k}$ from its section $(B_{n,k})_i$ we pass from any point $x^i \in (B_{n,k})_i$ to a boundary point of $B_{n,k}$ which got projected onto it. The newly acquired coordinate vector will then be denoted by $x_i(x^i)$ and given by the formula

$$x_i(x^i) = \left(r_n - \sum_{\substack{j=1 \\ j \neq i}}^{\sigma(n)} |a_j^*(x^i)|\right) a_i, \quad x^i \in (B_{n,k})_i.$$

For convenience in our computations, we also introduce the quantity

$$z_i(x^i) = R_n(x^i + x_i(x^i)) - R_n(x^i - x_i(x^i)) - 2x_i(x^i).$$

As $S_{\sigma(n)}$ is a linear projection, we know that $-S_{\sigma(n)}\big(x^i + x_i(x^i)\big) + S_{\sigma(n)}\big(x^i - x_i(x^i)\big) = -2x_i(x^i)$. Using this and a triangle inequality with four terms we have that

$$\begin{aligned}||z_i(x^i)|| \leq & \big|\big|R_n\big(x^i + x_i(x^i)\big) - R_n\big(S_{\sigma(n)}\big(x^i + x_i(x^i)\big)\big)\big|\big| \\ & + \big|\big|R_n\big(S_{\sigma(n)}\big(x^i + x_i(x^i)\big)\big) - S_{\sigma(n)}\big(x^i + x_i(x^i)\big)\big|\big| \\ & + \big|\big|R_n\big(S_{\sigma(n)}\big(x^i - x_i(x^i)\big)\big) - R_n\big(x^i - x_i(x^i)\big)\big|\big| \\ & + \big|\big|S_{\sigma(n)}\big(x^i - x_i(x^i)\big) - R_n\big(S_{\sigma(n)}\big(x^i - x_i(x^i)\big)\big)\big|\big| \\ \leq & 2||R_n||_{Lip}\varphi(n)\delta_k + 2\rho_n.\end{aligned}$$

For a set $J \subset \{1, \dots, \varphi(n)\}$ with $\#J = m \geq 1$, we define the measure in $[a_i]_{i\in J}$ as

$$\lambda_J^m(A) = \lambda_m\bigg(\Big(\big(a_i^*\big)_{i\in J}\Big)(A)\bigg) \quad \forall A \in \mathcal{M}_J^m,$$

where λ_m is the Lebesgue measure in $\mathbb{R}^m$ and $\mathcal{M}_J^m = \Big\{A \subset [a_i]_{i\in J} \ : \ \Big(\big(a_i^*\big)_{i\in J}\Big)(A)$ is a Lebesgue measurable subset of $\mathbb{R}^m\Big\}$. If $J = \{1, \dots, \varphi(n)\}$ we denote $\lambda^{\varphi(n)} = \lambda_J^{\varphi(n)}$ and for every $i \in \{1, \dots, \varphi(n)\}$, if $J = \{1, \dots, \varphi(n)\}\setminus \{i\}$, we denote $\lambda_i^{\varphi(n)-1} = \lambda_J^{\varphi(n)-1}$. Then, we are ready to define the linear operators $P_{n,k} : G_n \to E_n$, for every $k \in \mathbb{N}$, as

$$P_{n,k}(v) = \frac{1}{\lambda^{\varphi(n)}(B_{n,k})}\int_{B_{n,k}} dR_n(x)[v]d\lambda^{\varphi(n)}(x).$$

In [Bra+14] pg. 47 the volumes of ℓ_p^n balls B_p^n have been computed as $|B_p^n| = \frac{2^n\Gamma(\frac{1}{p}+1)^n}{\Gamma(\frac{n}{p}+1)}$. Using this result for $p = 1$ (for the base part of our set $B_{n,k}$) and the standard properties of Lebesgue measure we obtain the following values for our sets for arbitrary $i \in \{1, \dots, \sigma(n)\}$

$$\lambda_i^{\varphi(n)-1}\big((B_{n,k})_i\big) = \frac{2^{\varphi(n)-1}r_n^{\sigma(n)-1}\delta_k^{\varphi(n)-\sigma(n)}}{(\sigma(n)-1)!},$$

$$\lambda^{\varphi(n)}\big(B_{n,k}\big) = \frac{2^{\varphi(n)}r_n^{\sigma(n)}\delta_k^{\varphi(n)-\sigma(n)}}{\sigma(n)!},$$

so the quotient is

$$\frac{\lambda_i^{\varphi(n)-1}((B_{n,k})_i)}{\lambda^{\varphi(n)}(B_{n,k})} = \frac{\sigma(n)}{2r_n}.$$

Note that the expression

$$R_n(x^i + x_i(x^i)) - R_n(x^i - x_i(x^i)) = z_i(x^i) + 2x_i(x^i)$$

represents the difference of the values of the operator R_n between the endpoints of a segment cutting through $B_{n,k}$, which passes through the point x^i with direction a_i. As $B_{n,k} = \{x^i + x_i \in G_n \ : \ x^i \in (B_{n,k})_i \ , \ x_i \in$

$[-x_i(x^i), x_i(x^i)]\}$, thanks to Fubini's Theorem and the Fundamental Theorem of Calculus applied to the i-th coordinate, we can compute for each $i \in \{1, \dots, \sigma(n)\}$

$$\begin{aligned}
\big|\big|a_i - P_{n,k}(a_i)\big|\big| &= \left|\left|a_i - \frac{1}{\lambda^{\varphi(n)}(B_{n,k})} \int_{B_{n,k}} dR_n(x)[a_i] d\lambda^{\varphi(n)}(x)\right|\right| \\
&= \left|\left|a_i - \frac{1}{\lambda^{\varphi(n)}(B_{n,k})} \int_{(B_{n,k})_i} z_i(x^i) + 2x_i(x^i) d\lambda_i^{\varphi(n)-1}(x^i)\right|\right| \\
&= \left|\left|\frac{1}{\lambda^{\varphi(n)}(B_{n,k})} \int_{(B_{n,k})_i} z_i(x^i) d\lambda_i^{\varphi(n)-1}(x^i)\right|\right| \\
&\leq \frac{\lambda_i^{\varphi(n)-1}((B_{n,k})_i)}{\lambda^{\varphi(n)}(B_{n,k})} \big(2||R_n||_{Lip}\varphi(n)\delta_k + 2\rho_n\big) \\
&= \frac{\sigma(n)||R_n||_{Lip}\varphi(n)\delta_k + \sigma(n)\rho_n}{r_n}.
\end{aligned}$$

We may assume that $P_{n,k}$ pointwise converge in k and define $P_n(x) = \lim_{k\to\infty} P_{n,k}(x)$ for every $x \in G_n$, which is a linear operator from G_n to E_n satisfying

$$||P_n(a_i) - a_i|| \leq \frac{\sigma(n)h_n(\varphi(n)+2)}{r_n} \quad \forall i \in \{1, \dots, \sigma(n)\}.$$

Using the fact that K is small and $||a_i^*|| = 1$, $i \in \{1, \dots, \sigma(n)\}$, we obtain that for every $x \in E_n$

$$||P_n(x) - x|| = \left|\left|\sum_{i=1}^{\sigma(n)} a_i^*(x)(P_n(a_i) - a_i)\right|\right| \leq \frac{h_n\sigma(n)^2(\varphi(n)+2)}{r_n}||x|| \leq \frac{1}{2}||x||.$$

Finally, we construct the projection $\widetilde{P}_n = \left(P_n\big|_{E_n}\right)^{-1} \circ P_n : G_n \to E_n$ of norm

$$||\widetilde{P}_n|| \leq \left|\left|\left(P_n\big|_{E_n}\right)^{-1}\right|\right| \cdot ||P_n|| \leq 2 \cdot 2||R||_{Lip} = 4||R||_{Lip} \quad \forall n \in \mathbb{N}.$$

This implies that X has the π-property since

$$\lambda(E_n, X) \leq \frac{2}{1-\varepsilon}\lambda(E_n, G_n) \leq \frac{2}{1-\varepsilon}||\widetilde{P}_n|| \leq \frac{8||R||_{Lip}}{1-\varepsilon} \quad \forall n \in \mathbb{N}.$$

∎

Recall that thanks to Theorem 1.3 in [JRZ71], dual Banach spaces have an FDD if and only if they enjoy the π-property. Hence we have the next characterization.

Corollary 4.8. *Let X be a separable dual Banach space. Then X has an FDD if and only if X admits a small GCCR, if and only if X admits a small subset which is a Lipschitz retract of X.*

In Section 3 we found a sequence $(q_n) \subset \mathbb{R}^+$ such that for every sequence $r = (r_n) \subset \mathbb{R}^+$ satisfying that $\frac{r_n}{r_{n-1}} \leq q_n$ there is a λ-Lipschitz retraction $R(r) : X \to K(r)$, where $K(r) = \operatorname{co}\Big(\bigcup_{k\in\mathbb{N}} r_k B_{X_k} \Big)$ for some FDD $(X_n)_{n\in\mathbb{N}}$. Let $r = (r_n)$ be such a sequence and denote for every $k, m \in \mathbb{N}$ the sequence $r^{k,m} = (r_1, \dots, r_m, r_{m+1}/k, \dots, r_n/k, \dots)$. Taking subsequences, we may assume that for every $x \in X$ and every $m \in \mathbb{N}$ there exists $R_m(x) = \lim_{k\to\infty} R(r^{k,m})(x)$ which define retractions onto increasing finite dimensional compact sets. This leads to the π-property for Lipschitz retractions.

Definition 4.9. Let X be a separable Banach space and $\lambda > 0$. We say that X has the Lipschitz π_λ-property if there exists an increasing sequence of finite dimensional convex subsets (C_n) of X such that $X = \overline{\bigcup_{n\in\mathbb{N}} span(C_n)}$ and there exists a λ-Lipschitz retraction $R_n : X \to C_n$ for every $n \in \mathbb{N}$.

Analogously to the result of Godefroy and Kalton on the Lipschitz bounded approximation property (Theorem 2.9), we are going to prove that this new property is nothing else but the well-known π-property. This result is a direct consequence of the next Theorem, which is mainly based on an ultraproduct technique similar to the result of Lindenstrauss in [Lin64], see Corollary 7.3 of [BL00].

Theorem 4.10. *Let X be a Banach space, $\lambda_1, \lambda_2 > 0$ and $Y \subset X$ a subspace λ_1-complemented in its bidual. If there is a λ_2-Lipschitz retraction from X onto a convex subset K containing 0 such that $\overline{span}(K) = Y$ then Y is $\lambda_1\lambda_2$-complemented in X.*

Proof. Let $R : X \to K \subset Y$ be such a retraction. Then for every $n \in \mathbb{N}$ we define the λ_2-Lipschitz retraction $R_n : X \to nK$ given by $R_n(x) = nR(x/n)$, for every $x \in X$. As in the proof of Theorem 7.2 of [BL00] we let $\mathcal{U}$ to be a free ultrafilter on $\mathbb{N}$, and we put

$$S(x) = \lim_{\mathcal{U}} R_n(x) \quad \forall x \in X.$$

It is standard to check that $S : X \to Y^{**}$ is a λ_2-Lipschitz mapping, which is the identity on Y. Now, if $L : Y^{**} \to Y$ is a bounded linear projection, we just define $\widetilde{R} : X \to Y$ by

$$\widetilde{R}(x) = \big(L \circ S\big)(x) \quad \forall x \in X.$$

Finally $\widetilde{R}$ is a $\lambda_1\lambda_2$-Lipschitz retraction from X onto Y so by Corollary 7.3 of [BL00] we are done. ∎

Corollary 4.11. *Let X be a separable Banach space and $\lambda > 0$. Then X has the Lipschitz π_λ-property if and only if it has the π_λ-property.*

Proof. This is a straightforward consequence of Proposition 2.3 and Theorem 4.10. ∎

5. Compact sets without Lipschitz retractions

We proceed by constructing an example of a small convex and compact K in $C[0,1]$, which is contained in a small GCCR, contains a small GCCR, and yet there is no Lipschitz retraction onto K. The idea behind the construction can be described as follows. The GCCR constructed at the beginning of our note are well "aligned" with the FDD on X, and in the proof that the smallness condition of GCCR implies the π-property the projections are also aligned with the structure of the compact set. So our strategy is to employ badly complemented finite dimensional subspaces (in fact, Hilbert spaces) of $C[0,1]$ as the sections of the sought compact set K. In order to glue the decreasing sequence of these pieces together, we use the $\mathcal{L}_\infty$-FDD in $C[0,1]$. We start with a standard argument.

Lemma 5.1. *For any $\varepsilon > 0$ there is an FDD in $C[0,1]$ with associated projections P_n, and a sequence $(a_n) \subset C[0,1]$, $(1+\varepsilon)$-equivalent to the ℓ_2 basis such that*

- $\sup_{n\in\mathbb{N}} d(P_n(C[0,1]), \ell_\infty^{d(n)}) < \infty$ *where* $d(n) = dim(P_n(C[0,1]))$.
- $a_n \in (P_n - P_{n-1})(C[0,1])$ *for every* $n \in \mathbb{N}$.

Proof. By Remark 5.2 of [JRZ71] there is an FDD (X_n) in $C[0,1]$ such that

$$\sup_{n\in\mathbb{N}} d(\tilde{Q}_n(C[0,1]), \ell_\infty^{d(n)}) = d < \infty, \tag{5.1}$$

where the $\tilde{Q}_n$ are the natural projections of the FDD given by $\tilde{Q}_n(\sum_{i=1}^\infty x_i) = \sum_{i=1}^n x_i$. Take an ℓ_2-basis $(b_n)_{n\in\mathbb{N}} \subset C[0,1]$, which is certainly a w^*-null normalized sequence. For a given $\varepsilon > 0$ apply the standard Bessaga-Pelczynski blocking principle (e.g. [Fab+11] p. 194) to obtain a subsequence $(b_{k(n)})$ of the original ℓ_2 basis, and its perturbed companion (a_n), $\sum_{n=1}^\infty ||a_n - b_{k(n)}|| < \varepsilon$, such that $a_n \in (\tilde{Q}_{\eta(n)} - \tilde{Q}_{\eta(n-1)})(C[0,1])$, for some increasing sequence of indices $(\eta(n))$. To finish the proof it remains to let $\big((P_n - P_{n-1})(C[0,1])\big)$ be the desired FDD, where $P_n = \tilde{Q}_{\eta(n)}$. ∎

To simplify the notation, let us put $X = C[0,1]$ and let P_n and (a_n) be as in Lemma 5.1 for some fixed $\varepsilon \in (0,1)$. Now define the function $\sigma : \mathbb{N} \to \mathbb{N}$, so that $\sigma(1) = 1$ and $\sigma(n) = \sigma(n-1) + n$ for every $n \geq 2$. Then, letting $Q_n = P_{\sigma(n)}$ and $X_n = (Q_n - Q_{n-1})(X)$ we have that $Y_n = span(\{a_{\sigma(n-1)+1}, \ldots, a_{\sigma(n)}\})$ is a subspace of X_n, $(1+\varepsilon)$-isometric to ℓ_2^n. From now on in this section we are going to denote $E_n = Q_n(X)$.

We define for every $n \in \mathbb{N}$ and $\delta > 0$ the set

$$B_n^\delta = \Big\{ x \in X_n \ : \ d\big(x, B_{Y_n}\big) \leq \delta \Big\}.$$

It is clear that B_n^δ is a compact convex subset of $(1+\delta)B_{X_n}$ that generates X_n. We will need the following results to continue with our construction.

Lemma 5.2. *Let d be as in (5.1). For every $n \in \mathbb{N}$, if there is a Lipschitz retraction $\varphi : B_{E_n} \to B_{Y_n}$, then there is a Lipschitz retraction $\psi : \ell_\infty \to \ell_2^n$ satisfying the following inequality*

$$||\psi||_{Lip} \leq \frac{2d||\varphi||_{Lip}}{1-\varepsilon}.$$

Proof. By composing φ with the norm 2 radial retraction from E_n onto B_{E_n} we have a retraction $\widehat{\varphi} : E_n \to B_{Y_n}$ with $||\widehat{\varphi}||_{Lip} \leq 2||\varphi||_{Lip}$. Using the technique in the proof of Theorem 4.10, since Y_n is finite dimensional, there exists even a linear projection $\widetilde{\varphi} : E_n \to Y_n$ of norm $||\widetilde{\varphi}|| \leq 2||\varphi||_{Lip}$. Since E_n is d-isometric to $\ell_\infty^{\dim(E_n)}$, and Y_n is $(1+\varepsilon)$ isometric to ℓ_2^n the desired estimate follows. ∎

Lemma 5.3 (Lindenstrauss,'64)**.** *For every $n \in \mathbb{N}$, if $\psi : \ell_\infty \to \ell_2^n$ is a Lipschitz retraction then*

$$||\psi||_{Lip} \geq \frac{n^{1/4}}{3}.$$

Proof. Just use Lemma 1.28 of [BL00] with $r = n^{1/4}$ and $\varepsilon = 1$. ∎

Proposition 5.4. *There exist sequences $(M_n), (\delta_n) \subset \mathbb{R}^+$, $M_n \to \infty$, $\delta_n \to 0$, such that for every $n \in \mathbb{N}$ there is no M_n-Lipschitz retraction from B_{E_n} onto $B_n^{\delta_n}$.*

Proof. It suffices to let

$$M_n = \frac{n^{1/4}(1-\varepsilon)}{25d} \quad \forall n \in \mathbb{N}.$$

Now, for a fixed $n \in \mathbb{N}$ we are going to prove the existence of $\delta_n > 0$ satisfying the statement of the Proposition by contradiction. Suppose that for every $\delta > 0$ there exists a retraction $\phi^\delta : B_{E_n} \to B_n^\delta$ satisfying $||\phi^\delta||_{Lip} \leq M_n$. Then we define $N : E_n \to B_{E_n}$ the radial projection, and $C^\delta : B_n^\delta \to B_{Y_n}$ a nearest point map. The map $\psi^\delta = C^\delta \circ \phi^\delta \circ N : E_n \to B_{Y_n}$ satisfies the following inequality

$$||\psi^\delta(x) - \psi^\delta(y)|| \leq 2M_n\left(||x-y|| + \frac{\delta}{M_n}\right),$$

so we are allowed to use the Proposition of [Beg99]. For every $\tau > 0$ there is a Lipschitz map $\varphi^{\delta,\tau} : B_{E_n} \to B_{Y_n}$ satisfying

$$||\varphi^{\delta,\tau}(x) - \psi^\delta(x)|| \leq 2M_n\left(\tau + \frac{\delta}{M_n}\right) \quad \forall x \in B_{E_n},$$

$$||\varphi^{\delta,\tau}||_{Lip} \leq 2M_n\left(1 + \frac{\delta N_n}{2\tau M_n}\right).$$

Let us take a sequence $(\delta_k) \subset \mathbb{R}^+$ decreasing to 0, and put $\tau_k = \frac{\delta_k N_n}{2M_n}$ for every $k \in \mathbb{N}$. Denoting $\varphi_k = \varphi^{\delta_k,\tau_k}$ we have that a subsequence of

(φ_k) converge pointwise to a retraction $\varphi : B_{E_n} \to B_{Y_n}$ with Lipschitz norm $||\varphi||_{Lip} \leq 4M_n$. By Lemma 5.2 there exists a Lipschitz retraction $\psi : \ell_\infty \to \ell_p^n$ such that

$$||\psi||_{Lip} \leq \frac{2d||\varphi||_{Lip}}{1-\varepsilon} \leq \frac{8dM_n}{1-\varepsilon} = \frac{8}{25}n^{1/4} < \frac{n^{1/4}}{3}.$$

This contradicts Lindenstrauss' Lemma 5.3. ∎

Finally, we are ready to construct nonretractable convex subsets of $C[0,1]$. Note that if λ_n below are decreasing sufficiently fast then the resulting set K will be small.

Theorem 5.5. *For every sequence $(\lambda_n) \subset \mathbb{R}^+$, the subset of $C[0,1]$ given by*

$$K = \overline{\text{co}}\Big(\bigcup_{n\in\mathbb{N}} \lambda_n B_n^{\delta_n} \Big)$$

is not Lipschitz retractable.

Proof. Suppose there exists a Lipschitz retraction $\phi : C[0,1] \to K$ and let $||\phi||_{Lip} = L$. Then, there is an $n \in \mathbb{N}$ such that $L < \frac{M_n}{2M}$ where M is the constant of the FDD (Q_n), that is, $||Q_n|| \leq M$ for every $n \in \mathbb{N}$. Now, the retraction $R_n = (Q_n - Q_{n-1}) \circ \phi\big|_{E_n} : E_n \to \lambda_n B_n^{\delta_n}$ has Lipschitz norm $||R_n||_{Lip} \leq 2ML < M_n$. Finally, the retraction $F_n : E_n \to B_n^{\delta_n}$ given by $F_n(x) = \lambda_n^{-1} R_n(\lambda_n x)$ for every $x \in E_n$ has Lipschitz norm $||F_n||_{Lip} \leq ||R_n||_{Lip} < M_n$ which contradicts Proposition 5.4. ∎

6. Nearest point maps

The radial projection mapping from a Banach space X onto B_X can be shown to have a Lipschitz constant at most 2, [DW64].

Figueiredo and Karlovitz [FK67] proved that a real normed linear space of dimension 3 or higher is an inner product space if and only if the radial projection onto the unit ball is nonexpansive (see also [Phe58], [Sch65]).

For more general sets C a natural candidate for the retraction mapping is the nearest point map (sometimes also called the proximity map, of which the radial projection is a special case), provided the nearest point is unique. Again, it turns out that such a mapping is nonexpansive for every closed convex set if and only if the space is Hilbertian [Phe58]. In fact, in the papers [Bru74] [FK70], the following characterization is shown. Let X be a strictly convex Banach space with $3 \leq dim(X)$; then there exists a nonexpansive retraction onto the unit ball if and only if X is a Hilbert space. The restriction to dimension at least 3 is necessary. Karlovitz [Kar72] showed that if X is 2-dimensional then a non-expansive retraction onto any closed convex set always exists and it can be realized as a proximity mapping with respect to a new norm.

Let us begin our last section with a new characterization of the Hilbert space in the spirit of the above results.

Theorem 6.1. *Let X be a Banach space. Then X is isomorphic to a Hilbert space if and only if every convex and compact subset $K \subset X$ is a Lipschitz retract of X.*

Proof. Of course, it suffices to deal with infinite dimensional spaces X, and in light of the Phelps [Phe58] characterization, it suffices to prove only one implication (passing from retractions to Hilbert space). By inspection of the proof of the Lindenstrauss-Tzafriri characterization of the Hilbert space [LT71], it is clear that a Banach space is isomorphically Hilbert if and only if there exists some $\lambda > 0$ such that every finite-dimensional subspace $E \subset X$ is λ-complemented in X. Proceeding by contradiction, suppose that every convex compact set is a Lipschitz retract of X, but the complementation constants $\lambda(E, X)$, where E runs through all finite-dimensional subspaces of X, are not uniformly bounded. It is easy to see that this also means that for any finite-codimensional subspace $Y \subset X$, the complementation constants $\lambda(E, X)$, where $E \subset Y$ is finite-dimensional, are not uniformly bounded. We construct inductively a sequence (E_n) of finite-dimensional subspaces of X, such that this sequence forms an FDD with the projection constant at most 2 of its closed span (a subspace of X), so that $\lambda(E_n, X) > n$.

Let us describe the inductive step. Suppose we have constructed the initial sequence $(E_n)_1^N$ so that

$$\|y\| \leq (1 + \varepsilon_n)\|y + z\|, \text{ for all } y \in E_1 \oplus \cdots \oplus E_n,\ z \in E_{n+1},\ n < N.$$

There exists a finite set of functionals $\{f_i\}_1^M$ from X^* that is $(1 + \varepsilon_{N+1})$-norming for $E_1 \oplus \cdots \oplus E_N$. Let $Y = \cap_1^M \mathrm{Ker} f_i$ be a finite-codimensional subspace of X. Now, it suffices to choose a finite-dimensional subspace E_{N+1} of Y such that $\lambda(E_{N+1}, X) > N + 1$. The projection constant for the new FDD is under control.

By the same argument as in the proof of Theorem 4.10 it follows that the Lipschitz norm of any Lipschitz retraction from X onto B_E cannot be less than $\lambda(E, X)$. Let us pass to the construction of K. Fix a sequence (τ_n) of positive numbers such that $\sum \tau_n$ is finite. Finally, set K to be the closed convex hull of $\bigcup \tau_n B_{E_n}$. Supposing that $\phi : X \to K$ is a λ-Lipschitz retraction, for some $\lambda > 0$, we compose ϕ with the canonical retraction r_n of K onto $\tau_n B_{E_n}$ (which exists due to the structure of K, and has a Lipschitz constant at most 8), for $n \geq 8\lambda$. Then $r_n \circ \phi : X \to \tau_n B_{E_n}$ is a Lipschitz retraction of Lipschitz norm at most 8λ, hence $\lambda(E_n, X) \leq 8\lambda \leq n$ which is a contradiction.

■

This leaves us with the natural question whether at least uniformly continuous retractions onto convex compact subsets of a general Banach space are always possible. We give a strong positive answer to this problem, showing

that in fact under the URED renorming every convex and compact subset K is a uniform retract, from any bounded set B containing K, by means of the nearest point map. Without loss of generality, it suffices to deal with the case when $B = B_X$ is the unit ball of X.

Definition 6.2. Given a subset K of a Banach space X, we will say that X is K-URED if it is uniformly rotund in the direction z for every $z \in span(K)$.

Definition 6.3. Given a subset K of a Banach space X, we will say that X is K-UR if for every pair of sequences $(x_n), (y_n) \in S_X$ such that $x_n - y_n \in K$ and $||x_n + y_n|| \to 2$ we have that $||x_n - y_n|| \to 0$.

Lemma 6.4. *Let X be a Banach space that is uniformly rotund in the direction $z \in X \setminus \{0\}$. If there are $v_n, w_n \in S_X$ such that $v_n - w_n \to z$, then (by taking subsequence if necessary) there exist $\widetilde{v_n}, \widetilde{w_n} \in S_X$ such that $v_n - \widetilde{v_n}, w_n - \widetilde{w_n} \to 0$ and $\widetilde{v_n} - \widetilde{w_n} = \lambda_n z$ for some $\lambda_n \in \mathbb{R}$.*

Proof. First take for every $n \in \mathbb{N}$

$$t_n = \max\{t \geq 0 \ : \ w_n + tz \in S_X\}.$$

Clearly, $t_n||z|| - 1 \leq ||w_n + t_n z|| = 1$ so that $t_n \leq 2/||z||$. Hence, we may assume (taking subsequence if necessary) that t_n converge to some $t \in [0, 2/||z||]$. Let us prove that $t \leq 1$. If we suppose that $t > 1$, then taking $\lambda_n = 1/t_n$ we have that

$$w_n + z = (1 - \lambda_n)w_n + \lambda_n(w_n + t_n z),$$

where $w_n, w_n + t_n z \in S_X$ and $w_n - (w_n + t_n z) = -t_n z$. In particular, $w_n + z \in B_X$ and we have that

$$1 \geq ||w_n + z|| \geq |\, ||v_n|| - ||z - (v_n - w_n)||\,| \to 1.$$

Now, from Lemma 1, Chapter 1 in [Ziz71] we have that $||1/2w_n + 1/2(w_n + t_n z)|| \to 1$ so that by the assumption that the norm is uniformly rotund in the direction z we get that $||w_n + t_n z - w_n|| = t_n \|z\| \to 0$ which leads to a contradiction because $t_n \to t > 1$. This means we can assume that $t \leq 1$.

If $t = 1$ then we define $\widetilde{w_n} = w_n$ and $\widetilde{v_n} = w_n + t_n z$ and we are done. If $t < 1$ we can assume that $t_n < 1$ for every $n \in \mathbb{N}$. In this case, we define $\widetilde{v_n} = \frac{w_n + z}{||w_n + z||}$ and

$$s_n = \max\{s \geq 0 \ : \ \widetilde{v_n} + s(-z) \in S_X\}.$$

Following the same argument as above, we know that $s = \lim s_n \leq 1$. As $t < 1$ we know that $||w_n + z|| > 1$, so $\frac{w_n}{||w_n + z||} = \widetilde{v_n} - \frac{z}{||w_n + z||} \in B_X$. Hence $s_n \geq \frac{1}{||w_n + z||}$. Just notice that $||w_n + z|| \to 1$ because

$$1 < ||w_n + z|| \leq ||v_n|| + ||z - (v_n - w_n)|| \to 1,$$

so

$$1 \geq s = \lim s_n \geq \lim \frac{1}{||w_n + z||} = 1 \quad \Rightarrow \quad s = 1.$$

Finally, we are able to define $\widetilde{w_n} = \widetilde{v_n} + s_n(-z)$. ∎

Proposition 6.5. *Let X be a Banach space and $z \in S_X$, the following assertions are equivalent:*

- *X is uniformly rotund in the direction z.*
- *For every pair of sequences $x_n, y_n \in S_X$ such that $x_n - y_n \to \lambda z$, for some $\lambda \in \mathbb{R}$, if $||x_n + y_n|| \to 2$ then $\lambda = 0$.*

Proof. If X is uniformly rotund in the direction z and we suppose by contradiction that the second assertion does not hold, then there exist $x_n, y_n \in S_X$ such that $x_n - y_n \to \lambda z$ for some $\lambda \neq 0$ with $||x_n + y_n|| \to 2$. Then using Lemma 6.4 we get (by taking subsequence if necessary) $\widetilde{x_n}, \widetilde{y_n} \in S_X$ such that $\widetilde{x_n} - x_n, \widetilde{y_n} - y_n \to 0$ and $\widetilde{x_n} - \widetilde{y_n} = \lambda_n z$. Now, we have

$$2 \geq ||\widetilde{x_n} + \widetilde{y_n}|| \geq ||x_n + y_n|| - (||x_n - \widetilde{x_n}|| + ||y_n - \widetilde{y_n}||) \to 2,$$

so $\lambda_n \to 0$ which is impossible because $\lambda_n z = \widetilde{x_n} - \widetilde{y_n} \to \lambda z \neq 0$. The other implication is straightforward. ∎

Proposition 6.6. *Let X be a Banach space and $K \subset X$. If K is compact, then X is K-URED if and only if it is K-UR.*

Proof. If X is K-URED, let us take $x_n, y_n \in S_X$ such that $x_n - y_n \in K$ and $||x_n + y_n|| \to 2$. Then, as K is compact, $x_n - y_n \to z \in K \subset span(K)$ so $||x_n - y_n|| \to 0$ just by Proposition 6.5.
If X is K-UR then it is trivially $K - URED$. ∎

Proposition 6.7. *Let X be a Banach space and $K \subset B_X$ be a compact convex subset. If X is $K - URED$ then the nearest point map from B_X onto K is uniformly continuous.*

Proof. We are going to argue by contradiction. Assuming that the nearest point map $R : B_X \to K$ is not uniformly continuous, then there exists an $\varepsilon > 0$ and a pair of sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset B_X$ such that $||x_n - y_n|| \to 0$ and $||Rx_n - Ry_n|| > \varepsilon$ for every $n \in \mathbb{N}$. By compactness, we may assume that the sequences $(||Rx_n - x_n||)$, $(||Ry_n - y_n||)$, (Rx_n) and (Ry_n) are all convergent. In particular, we claim that $\lim ||Rx_n - x_n|| = \lim ||Ry_n - y_n|| = d \in \mathbb{R}^+$. Let us first prove that $\lim ||Rx_n - x_n|| = \lim ||Ry_n - y_n||$. Otherwise, we may assume that $\lim ||Rx_n - x_n|| > \lim ||Ry_n - y_n|| + \rho$ for some $\rho > 0$. Then there is an $n \in \mathbb{N}$ such that $||Rx_n - x_n|| > ||Ry_n - y_n|| + \rho/2$ and $||x_n - y_n|| \leq \rho/2$, so

$$||Ry_n - x_n|| \leq ||Ry_n - y_n|| + ||x_n - y_n|| \leq ||Ry_n - y_n|| + \rho/2 < ||Rx_n - x_n||.$$

This contradicts the definition of R. Now, if $d = 0$ then $Rx_n, x_n \to p \in K$ and $Ry_n, y_n \to q \in K$ so $p = q$ because $x_n - y_n \to 0$. This means that $\lim ||Rx_n - Ry_n|| \to 0$ which is impossible.

Assuming that $d > 0$, we are going to apply Lemma 6.4 to $v_n = \frac{Rx_n - x_n}{||Rx_n - x_n||}$, $w_n = \frac{Ry_n - y_n}{||Ry_n - y_n||}$ and $z = \frac{p-q}{d}$ where $p = \lim Rx_n$ and $q = \lim Ry_n$. Indeed,

$$v_n - w_n = \frac{Rx_n}{||Rx_n - x_n||} - \frac{Ry_n}{||Ry_n - y_n||} + \frac{y_n||Rx_n - x_n|| - x_n||Ry_n - y_n||}{||Rx_n - x_n||\,||Ry_n - y_n||},$$

so $v_n - w_n \to z = \frac{p-q}{d}$. Let us take $\widetilde{v_n}, \widetilde{w_n} \in S_X$ given by Lemma 6.4. As

$$||\widetilde{v_n} + \widetilde{w_n}|| \geq ||v_n + w_n|| - (||\widetilde{v_n} - v_n|| + ||\widetilde{w_n} - w_n||),$$

we have (by taking subsequence if necessary) that $2 \geq \lim ||\widetilde{v_n} + \widetilde{w_n}|| \geq \lim ||v_n + w_n||$. For this reason, in order to prove that $||\widetilde{v_n} + \widetilde{w_n}|| \to 2$, it is enough to prove that $\left|\left|\frac{v_n + w_n}{2}\right|\right| \to 1$.

Equivalently, we are going to prove that $\left|\left|\frac{(Rx_n - x_n) + (Ry_n - y_n)}{2}\right|\right| \to d$. If that does not hold, then there exists $\rho > 0$ such that

$$d = \lim ||Rx_n - x_n|| > \lim \left|\left|\frac{(Rx_n - x_n) + (Ry_n - y_n)}{2}\right|\right| + \rho.$$

Now, there exists an $n \in \mathbb{N}$ such that

$$||Rx_n - x_n|| > \left|\left|\frac{(Rx_n - x_n) + (Ry_n - y_n)}{2}\right|\right| + \rho/2 \quad , \quad ||x_n - y_n|| < \rho.$$

So taking into account the convexity of K, the next inequality is a contradiction with the definition of R:

$$\begin{aligned}\left|\left|\frac{Rx_n + Ry_n}{2} - x_n\right|\right| &\leq \left|\left|\frac{(Rx_n - x_n) + (Ry_n - y_n)}{2}\right|\right| + \left|\left|\frac{x_n + y_n}{2} - x_n\right|\right| \\ &< \left|\left|\frac{(Rx_n - x_n) + (Ry_n - y_n)}{2}\right|\right| + \rho/2 < ||Rx_n - x_n||.\end{aligned}$$

Finally, we know that $||\widetilde{v_n} + \widetilde{w_n}|| \to 2$ and $\widetilde{v_n} - \widetilde{w_n} = \lambda_n(p - q)$ with $\lambda_n \in \mathbb{R}$. As the norm of X is K-URED, we get that $\lambda_n \to 0$. This is in fact impossible because

$$||\widetilde{v_n} - \widetilde{w_n}|| \geq ||v_n - w_n|| - (||\widetilde{v_n} - v_n|| + ||\widetilde{w_n} - w_n||) \to ||z|| > 0.$$

■

Recall that thanks to Zizler's result ([Ziz71]) every separable Banach space has an equivalent URED renorming.

Corollary 6.8. *If X is a separable Banach space, then it can be renormed so that the nearest point map from B_X onto any compact and convex subset of B_X is uniformly continuous.*

The next proof is a slight modification of the proof given by V. Zizler to renorm separable spaces with URED norm ([Ziz71]).

Proposition 6.9. *For every compact subset K of a Banach space X, there is an equivalent K-URED norm on X.*

Proof. Let us take $V = \overline{span(K)}$ which has to be a separable Banach space. Now, there is a countable set $\{f_n\}_{n\in\mathbb{N}} \subset S_{X^*}$ separating the points of V. We define the linear operator $T : X \to \ell_2$ as $Tx = \left(\frac{f_n(x)}{2^n}\right)_{n\in\mathbb{N}}$. It is easily seen that T is bounded and its restriction to V is injective. Our new equivalent norm is going to be

$$|||x||| = \sqrt{||x||_X^2 + ||Tx||_2^2},$$

where $||\cdot||_X$ is the initial norm on X and $||\cdot||_2$ is the usual norm on ℓ_2. If we suppose that there exists a bounded sequence $(x_n) \subset X$ such that

$$2(|||x_n + z|||^2 + |||x_n|||^2) - |||2x_n + z|||^2 \to 0, \tag{6.1}$$

for some $z \in V$, then

$$2(||x_n+z||^2+||x_n||^2)-||2x_n+z||^2+2(||Tx_n+Tz||_2^2+||Tx_n||_2^2)-||2Tx_n+Tz||_2^2 \to 0.$$

In particular, $2(||Tx_n + Tz||_2^2 + ||Tx_n||_2^2) - ||2Tx_n + Tz||_2^2 \to 0$. Making use of Proposition 1 of [Ziz71], the equivalence $1 \Leftrightarrow 7$ means that ℓ_2 is not uniformly rotund in the direction Tz, leading to a contradiction because ℓ_2 is in fact UR. Then, (6.1) does not hold for any bounded sequence $(x_n) \subset X$ which again by the equivalence $1 \Leftrightarrow 7$ means that X is uniformly rotund in the direction z. ∎

Corollary 6.10. *For every compact and convex subset $K \subset B_X$ of a Banach space X, there is an equivalent norm on X such that the nearest point map from B_X onto K is well defined and uniformly continuous.*

The following corollaries were recently established in [CCW21].

Corollary 6.11. *For every compact and convex subset K of a Banach space X there is a uniformly continuous retraction from X onto K.*

Proof. We may assume without loss of generality that $K \subset B_X$. It suffices to compose the 2-Lipschitz retraction of X onto B_X with the uniformly continuous retraction from B_X onto K obtained earlier. ∎

Corollary 6.12. *Compact convex subsets of Banach spaces are absolute uniform retracts.*

Proof. Just embed the metric space into $\ell_\infty(\Gamma)$ for some set Γ and use Corollary 6.11. ∎

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Chapter 4

Retractions and the bounded approximation property in Banach spaces

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RETRACTIONS AND THE BOUNDED APPROXIMATION PROPERTY IN BANACH SPACES

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Abstract. In the present paper we prove that a necessary condition for a Banach space X to admit a generating compact Lipschitz retract K, which satisfies an additional mild assumption on its shape, is that X enjoys the Bounded Approximation Property. This is a partial solution to a question raised by Godefroy and Ozawa.

1. Introduction

In this note, we focus on the problem of the existence of Lipschitz retractions onto compact and generating subsets of a Banach space X. A subset C of a Banach space X is called generating if the closed linear span of C is the whole space X.

In [13] (and subsequently in [10], [11], [12], and [8]) Godefroy and Ozawa asked whether every separable Banach space X admits a generating convex and compact Lipschitz retract (GCCR for short) K. The main result of our paper (Theorem 2.7) hints towards a negative answer, since we prove that the existence of a generating compact Lipschitz retract K (not necessarily convex) satisfying some additional (not exceedingly restrictive) assumptions implies that X has the Bounded Approximation Property. If we moreover assume that K is convex, the additional assumptions take on a simpler form (Corollary 2.9).

Of course, thanks to Enflo we know that BAP is not true for every separable Banach space [6].

Our argument is a variant of that in [14], where a much stronger flatness assumption on K was used to imply the π-property of X. Roughly speaking, the flatness assumption is a quantitative condition on the sequence of distances of the given compact set K to its finite-dimensional sections. The compactness condition itself implies that this sequence tends to zero. In our result, we assume that it is controlled by a geometric sequence. This hints

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to the possibility of the existence of a counterexample to the question of Godefroy and Ozawa.

It should be noted that in [14] we construct a GCCR in every Banach space with an FDD, but their existence in every Banach space with the weaker property BAP is not clear to us.

The proof of the main Theorem 2.7 is based on several deep ingredients and follows the same scheme developed in Section 4 of [14]. We produce the finite rank operators approximating the identity by means of the Lipschitz retraction in a non-constructive way, only proving their existence. This is done in several steps. First, we deal with coarse Lipschitz maps given by the composition of the retraction with compressions onto the n-dimensional sections of the space (given by the nearest point map, not necessarily unique). Since these maps may not be Lipschitz, we need to "Lipschitzize" them making use of a technique by Bourgain [3], streamlined later by Begun [1] (see [9] for a very good exposition of this technique). This technique works only in the finite-dimensional setting so the crucial part here is to prove that we can restrict each map to some finite-dimensional subspace whose dimension is under control. To do so we present a generalization of Lemma 4.2 in [14] originally due to Vitali Milman. Finally, once we have the Lipschitz maps, it is a matter of using the differentiation theory and averaging in the finite-dimensional setting to obtain the uniformly bounded finite rank operators pointwise converging to the identity.

Let us now review the basic definitions that will be used in the next section.

Definition 1.1. Let X be a Banach space. If there is a uniformly bounded net of finite rank operators (T_α) on X tending strongly to the identity on X, then we say that X has the bounded approximation property (BAP for short). If $\lambda \geq 1$ is a uniform bound for the net then we say that X has the λ-bounded approximation property (λ-BAP for short).

In the separable case we may assume that the net is actually a sequence.

Definition 1.2. A map T from a metric space M into another metric space N is said to be Lipschitz if there exists some $\lambda > 0$ such that

$$d(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in M.$$

We say that λ is a Lipschitz constant for T and we call the infimum of all Lipschitz constants for T the Lipschitz norm of T, that is,

$$||T||_{Lip} = \inf\big\{\lambda > 0 : \lambda \text{ is a Lipschitz constant for } T\big\} = \sup_{x,y\in M, x\neq y} \frac{d\big(T(x), T(y)\big)}{d(x, y)}.$$

If $\lambda > 0$ is a Lipschitz constant for T then we say that T is λ-Lipschitz.

A retraction from a metric space M onto its subset N is a map $R : M \to N$ whose restriction to N is the identity on N. Given a finite subset $\{x_1, \ldots, x_n\}$ of a Banach space X we denote $[x_i]_{i=1}^n = span\big(\{x_1, \ldots, x_n\}\big)$, the linear span of the set in X.

For the background in nonlinear analysis we refer the reader to the authoritative monograph [2]. For the remaining general concepts and results of Banach space theory we refer to [7] and the background on the approximation properties can be found in [5].

2. Main result

In this section we introduce the key concept of flatness for a compact set and prove the main result of this paper. Namely, every Banach space with a Lipschitz retraction onto a flat and generating set with nonempty local interior has the Bounded Approximation Property (BAP for short). To do so, we follow the ideas in [14] Section 4. In particular we need to generalize Lemma 4.2 of [14], a crucial result about projection constants due to Vitali Milman, to the setting of BAP type operators.

For that purpose, we will need the next well known result.

Lemma 2.1. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$. If $(F, ||\cdot||)$ is a Banach space of dimension n, then there exists a renorming $|\cdot|$ of F such that $(F, |\cdot|)$ embeds isometrically in ℓ_∞^N where $N = (3+\varepsilon)^n$, and*

$$|x| \leq ||x|| \leq |x| \frac{(2+\varepsilon)}{\varepsilon}.$$

Proof. By Lemma 2.6 of [16], we know that there exists an $\frac{2}{2+\varepsilon}$-net $\{x_1^*, \dots, x_N^*\}$ in S_{X^*}. Just consider the norm $|x| = \max_{i \in \{1,\dots,N\}} x_i^*(x)$. ∎

For a triplet X, F, E of Banach spaces such that $X \supset F \supset E$ and both F, E are finite-dimensional, we define the BAP constant of (X, F, E) as

$$\lambda(X, F, E) = \inf\{||T|| \ : \ T \in \mathcal{L}(X, F), \ T\big|_E = Id\}.$$

The next theorem is a generalization of Milman's Lemma 4.2 in [14].

Theorem 2.2. *Let X be a separable Banach space, and E, F be finite-dimensional subspaces of X such that $E \subset F$. Then, for every $\varepsilon > 0$ and $\rho > 1$ there is $G \subset X$, a subspace containing E, such that $dim(G) \leq (3+\varepsilon)^{dim(F)}$ and*

$$\lambda(X, F, E) \leq \rho \frac{2+\varepsilon}{\varepsilon} \lambda(G, F, E).$$

To prove Theorem 2.2, we rely on a variant of the trace duality introduced in [15]. Given $T \in \mathcal{L}(E, F)$ we define the norm

$$||T||_X = \inf\{||\widetilde{T}|| \ : \ \widetilde{T} \in \mathcal{L}(X, F), \ \widetilde{T}\Big|_E = T\}.$$

Similarly, if $S \in \mathcal{L}(F, E)$ we define the norm

$$||S||_{\Lambda X} = ||i_{E,X} S||_\Lambda$$

where $||\cdot||_\Lambda$ refers to the nuclear norm in $\mathcal{N}(F,X)$ and $i_{Y,Z}: Y \to Z$ is the inclusion whenever $Y \subset Z$. Let us recall that for every $S \in \mathcal{N}(F,X)$, the nuclear norm of S is given by

$$||S||_\Lambda = \inf\Big\{ \sum_{n=1}^{\infty} ||f_n^*||\,||x_n|| \ : \ f_n^* \in F^*,\ x_n \in X,\ S = \sum f_n^* \otimes x_n \Big\}.$$

We have the following duality.

Proposition 2.3. *Let X be a Banach space and $E \subset F$ finite-dimensional subspaces of X. Then,*

$$(\mathcal{L}(E,F), ||\cdot||_X)^* \equiv (\mathcal{L}(F,E), ||\cdot||_{\Lambda X}),$$

where the duality is given by the trace, that is for $T \in \mathcal{L}(E,F)$

$$||T||_X = \sup_{\substack{S\in\mathcal{L}(F,E)\\ ||S||_{\Lambda X}\leq 1}} tr(ST).$$

Proof. If $S \in \mathcal{L}(F,E)$ then by the trace duality of $\mathcal{L}(X,F)$ and $\mathcal{N}(F,X)$ we know that

$$||S||_{\Lambda X} = ||i_{E,X}S||_\Lambda = \sup_{\substack{T\in\mathcal{L}(X,F)\\ ||T||\leq 1}} tr(Ti_{E,X}S) = \sup_{\substack{T\in\mathcal{L}(E,F)\\ ||T||_X\leq 1}} tr(TS).$$

Our claim follows now directly. ∎

For the following proof we will need to define the ceiling of a real number $r \in \mathbb{R}$ as

$$[r] = \min\{n \in \mathbb{Z} \ : \ n > r\}.$$

Proof of Theorem 2.2. By Proposition 2.3, for every superspace $Y \supset F$,

$$\begin{aligned} \lambda(Y,F,E) =& ||i_{E,F}||_Y = \sup_{\substack{T\in\mathcal{L}(F,E)\\ ||T||_{\Lambda Y}=1}} tr(T\big|_E) = \sup_{T\in\mathcal{L}(F,E)} tr\left(\frac{T\big|_E}{||T||_{\Lambda Y}}\right) \\ &= \sup_{T\in\mathcal{L}(F,E)} \frac{1}{\left|\left|\frac{T}{tr(T\big|_E)}\right|\right|_{\Lambda Y}} = \frac{1}{\inf\limits_{\substack{T\in\mathcal{L}(F,E)\\ tr(T\big|_E)=1}} ||T||_{\Lambda Y}}. \end{aligned} \tag{2.1}$$

Let us first take $\mu \in (1/\rho, 1)$ and set $n = dim(F)$. Now we choose

$$\delta = \frac{\varepsilon}{\lambda(X,F,E)(2+\varepsilon)}(\mu - 1/\rho) > 0,$$

and an operator $S \in \mathcal{L}(F,E)$ such that

$$\inf_{\substack{T\in\mathcal{L}(F,E)\\ tr(T\big|_E)=1}} ||T||_{\Lambda X} \geq ||S||_{\Lambda X} - \delta \quad \text{and} \quad tr(S\big|_E) = 1.$$

We also take the norm $|\cdot|$ given by Lemma 2.1 so that $(F,|\cdot|)$ is isometrically a subspace of $\ell_\infty^{\varphi(n)}$ where $\varphi(n)=\left[\left(3+\varepsilon\right)^n\right]$. Then, for every superspace $Y\supset F$,

$$||S||_{\Lambda Y}\leq |S|_{\Lambda Y}\leq \frac{2+\varepsilon}{\varepsilon}||S||_{\Lambda Y},$$

where $|\cdot|_{\Lambda Y}$ denotes the norm $||\cdot||_{\Lambda Y}$ but taking $(F,|\cdot|)$ as the domain of the operators instead of $(F,||\cdot||)$. It is well-known (Proposition 47.6 in [18]) that $i_{E,X}S$ admits an extension $\widetilde{S}:\ell_\infty^{\varphi(n)}\to X$ almost preserving the nuclear norm, that is $|S|_{\Lambda X}\geq \mu||\widetilde{S}||_\Lambda$. By Proposition 8.7 in [17] we know that there exist $x_1,\dots,x_{\varphi(n)}\in X$ such that $\widetilde{S}=\sum_{i=1}^{\varphi(n)} e_i^*\otimes x_i$ and

$$||\widetilde{S}||_\Lambda=\sum_{i=1}^{\varphi(n)}||x_i||,$$

where $e_i^*\in\left(\ell_\infty^{\varphi(n)}\right)^*$ are the coordinate functionals. Just considering $G=[x_i]_{i=1}^{\varphi(n)}$, we may see that

$$||S||_{\Lambda X}\geq\frac{\varepsilon}{2+\varepsilon}|S|_{\Lambda X}\geq\frac{\varepsilon\mu}{2+\varepsilon}||\widetilde{S}||_\Lambda\geq\frac{\varepsilon\mu}{2+\varepsilon}|S|_{\Lambda G}\geq\frac{\varepsilon\mu}{2+\varepsilon}||S||_{\Lambda G}.$$

Finally, taking into account that $\lambda(G,F,E)\leq\lambda(X,F,E)$ and using (2.1) with $Y=X$ and $Y=G$,

$$\begin{aligned}\lambda(X,F,E)&\leq\frac{1}{||S||_{\Lambda X}-\delta}\leq\frac{1}{\frac{\varepsilon\mu}{2+\varepsilon}||S||_{\Lambda G}-\delta}\leq\frac{1}{\frac{\varepsilon\mu}{2+\varepsilon}\inf\limits_{tr(T\big|_E)=1}||T||_{\Lambda G}-\delta}\\&=\lambda(G,F,E)\frac{1}{\frac{\varepsilon\mu}{2+\varepsilon}-\delta\lambda(G,F,E)}\leq\rho\frac{2+\varepsilon}{\varepsilon}\lambda(G,F,E).\end{aligned}$$

■

Definition 2.4. Let X be a separable Banach space, $K\subset X$ be a bounded generating subset and $\beta=(e_n)_{n\in\mathbb{N}}\subset span(K)$ a fundamental sequence of X. We define the sequence of heights $(h_n^\beta)_{n\in\mathbb{N}}$ of K relative to β as

$$h_n^\beta=\sup\left\{d(x,[e_i]_{i=1}^n)\ :\ x\in K\right\}\quad\forall n\in\mathbb{N}.$$

If there exists a fundamental sequence β and $\varepsilon>0$ satisfying that

$$0=\liminf_n h_n^\beta(3+\varepsilon)^n,$$

then we say that K is β-flat. If $0=\lim h_n^\beta(3+\varepsilon)^n$ then we say that K is basically β-flat. We will say that K is flat (resp. basically flat) whenever it is β-flat (resp. basically β-flat) for some fundamental sequence of X $\beta\subset span(K)$.

Let us point out that K is flat if and only if there is (possibly a different) $\varepsilon > 0$ and $\sigma : \mathbb{N} \to \mathbb{N}$ strictly increasing such that $h^{\beta}_{\sigma(n)} \leq (3+\varepsilon)^{-\sigma(n)}$. In the case when K is basically flat we may just take $\sigma(n) = n$ for every $n \in \mathbb{N}$.

Let K be a compact set in an infinite-dimensional Banach space X. For every fundamental sequence β in X the sequence of heights relative to β necessarily converges to zero. The above definition of flat set requires one of these sequences of heights to decrease faster than a geometric sequence of ratio smaller than $1/3$.

Definition 2.5. Let X be a separable Banach space, a subset $C \subset X$ and a fundamental sequence $\beta = (e_n)$ of X. We say that C has nonempty local β-interior whenever

$$int(C \cap [e_i]_{i=1}^n) \neq \emptyset \quad \forall n \in \mathbb{N},$$

where the interior is taken in the topology relative to $[e_i]_{i=1}^n$.

For the sake of completeness, we state now Bourgain's discretization theorem [3] in the same way that Begun [1, Proposition] did it in his paper. For that purpose, let us set up first some notation. Given a subset K of a normed space X, we denote by $\chi_K : X \to \mathbb{R}$ the indicator function on K and for $\tau > 0$ we consider the τ- neighbourhood of K as $K_\tau = \{x \in X \ : \ d(x, K) < \tau\}$. The symbol $*$ will denote the usual convolution of functions.

Proposition 2.6 (Begun, [1])**.** *Let $0 < \delta, L < \infty$. Let Y be a normed space and K be a convex subset of an n-dimensional normed space E. Assume there is a map $f : K_\tau \to Y$ satisfying*

$$||f(x) - f(y)|| \leq L(||x - y|| + \delta) \quad \forall x, y \in K_\tau.$$

Then the map $g = f * \chi_{B(0,\tau)} : K \to Y$, where $B(0,\tau)$ is the open unit ball in E of radius τ satisfies

$$||g||_{Lip} \leq L\Big(1 + \frac{\delta n}{2\tau}\Big).$$

Theorem 2.7. *Let X be a Banach space, K a generating bounded subset of X and $\beta = (e_n) \subset span(K)$ a fundamental sequence of X such that K is β-flat and has nonempty local β-interior. Then a necessary condition for the existence of a Lipschitz retraction from X onto K is that X has the BAP.*

Proof. Assume that there is a Lipschitz retraction from X onto a generating subset $K \subset X$ which is basically β-flat and has a nonempty local β-interior. By continuity, we may assume that K is closed (extending R to the closure of K if necessary) Take $\varepsilon > 0$ such that $h_n(3+\varepsilon)^n \to 0$ where the h_n's are the heights of K relative to β, and let $\phi(n) = \Big[\big(3+\varepsilon\big)^n\Big]$ and $E_n = [e_i]_{i=1}^n$. We define the inner radius relative to β as

$$r_n = \sup\big\{r \geq 0 \ : \ B_{E_n}(x, r) \subset K \cap E_n \ , \ x \in X\big\} \quad \forall n \in \mathbb{N}.$$

As K has nonempty β-interior $r_n > 0$ for every $n \in \mathbb{N}$ so there must exist a strictly increasing $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$h_{\gamma(n)}(2 + \phi(\gamma(n)) \leq \frac{r_n}{n^3} \quad \forall n \in \mathbb{N}.$$

Indeed, since the sequences $h_m(3+\varepsilon)^m$ and h_m are converging to 0 we deduce that $h_m(2 + \phi(m)) \to 0$. Hence, for every $n \in \mathbb{N}$ there must be $m \in \mathbb{N}$ big enough satisfying that $h_m(2 + \phi(m)) \leq r_n/n^3$. Just take $\gamma(n) = m$.

Theorem 2.2 guarantees that for every $n \in \mathbb{N}$ there is a finite-dimensional subspace $G_n \subset X$ of dimension $dim(G_n) = \varphi(n) = \phi(\gamma(n))$ such that for every linear operator $T : G_n \to E_{\gamma(n)} = F_n$ with $T\big|_{E_n} = Id_{E_n}$, the inequality $||T|| \geq \lambda(X, F_n, E_n)\frac{\varepsilon}{2(2+\varepsilon)}$ holds. Assume that $R : X \to K$ is the Lipschitz retraction. Then taking the nearest point map $C_n : K \to F_n$ (it may not be unique), we define $\widetilde{R}_n = (C_n \circ R)\big|_{G_n} : G_n \to F_n$ for every $n \in \mathbb{N}$. Now,

$$||\widetilde{R}_n(x) - \widetilde{R}_n(y)|| \leq ||R||_{Lip}\left(||x - y|| + \frac{2h_{\gamma(n)}}{||R||_{Lip}}\right) \quad \forall x, y \in G_n.$$

So by Proposition 2.6 with $K, E = G_n$, for every $\tau > 0$, there is a Lipschitz mapping

$$R_{n,\tau} : G_n \to F_n,$$

such that

$$||R_{n,\tau}||_{Lip} \leq ||R||_{Lip}\left(1 + \frac{\varphi(n)h_{\gamma(n)}}{\tau||R||_{Lip}}\right),$$

$$||R_{n,\tau}(x) - \widetilde{R}_n(x)|| \leq ||R||_{Lip}\left(\tau + \frac{2h_{\gamma(n)}}{||R||_{Lip}}\right) \quad \forall x \in G_n.$$

For the rest of the argument we fix $n \in \mathbb{N}$ and $x_n \in K_n = K \cap E_n$ such that $B_{E_n}(x_n, r_n) \subset K_n$ (it exists by the definition of the inner radius and the compactness of K_n). Now we let $\tau_n = \frac{\varphi(n)h_{\gamma(n)}}{||R||_{Lip}}$, and define $R_n : G_n \to F_n$ by $R_n(x) = R_{n,\tau_n}(x + x_n) - x_n$. Then, by the previous inequality it holds that

$$||R_n||_{Lip} = ||R_{n,\tau_n}||_{Lip} \leq 2||R||_{Lip}. \tag{2.2}$$

If $x + x_n \in K_n$ then $\widetilde{R}_n(x + x_n) = x + x_n$ acts as an identity. Hence, we have that for every $x \in K_n - x_n$

$$||R_n(x) - x|| = ||R_{n,\tau_n}(x + x_n) - \widetilde{R}_n(x + x_n)|| \leq h_{\gamma(n)}(\varphi(n) + 2) = \rho_n.$$

Now, let $(a_i, a_i^*)_{i=1}^{\varphi(n)}$ be a normalized linear basis for G_n with projections $(S_i)_{i=1}^{\varphi(n)}$ such that $(a_i, a_i^*)_{i=1}^{n}$ is an Auerbach basis for E_n. Then,

$$B_n = r_n \operatorname{co}\left(\{\pm a_i \ , \ i = 1, \ldots, n\}\right) \subset K_n - x_n.$$

Fix a sequence (δ_k) of positive numbers converging to zero. Now we define for every $k \in \mathbb{N}$ the compact

$$B_{n,k} = B_n + \delta_k \sum_{i=n+1}^{\varphi(n)} \mathrm{co}\big(\{\pm a_i\}\big) \subset G_n.$$

The geometrical shape of this set can be described as a r_n-multiple of the unit ball of ℓ_1^n located in E_n, a sort of a base of a hypercylinder, times a hypercube of side length $2\delta_k$ protruding into the remaining dimensions of G_n. Letting δ_k go to zero of course means that this set gets squashed down to its base in E_n. For any $i \in \{1, \dots, n\}$ we denote $(B_{n,k})_i = (Id - a_i^* \otimes a_i)(B_{n,k})$. This set is just a one-codimensional section (or a projection of rank $\varphi(n)-1$) of $B_{n,k}$ which reduces the base by one coordinate.

In order to recover the shape $B_{n,k}$ from its section $(B_{n,k})_i$ we pass from any point $x^i \in (B_{n,k})_i$ to the boundary point of $B_{n,k}$ which got projected onto it. The newly acquired coordinate vector will then be denoted by $x_i(x^i)$ and given by the formula

$$x_i(x^i) = \left(r_n - \sum_{\substack{j=1 \\ j\neq i}}^{n} \big|a_j^*(x^i)\big|\right) a_i, \quad x^i \in (B_{n,k})_i.$$

In fact, the previous formula arises when considering $x_i(x^i) \in [a_i]$ to be such that the point $x^i + x_i(x^i)$ lies in the boundary of $B_{n,k}$. In particular, the truncation to the first n coordinates belong to B_n and therefore $||x_i(x^i)|| + \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_j^*(x^i)| = r_n$.

For convenience in our computations, we also introduce the quantity

$$z_i(x^i) = R_n(x^i + x_i(x^i)) - R_n(x^i - x_i(x^i)) - 2x_i(x^i).$$

As S_n is a linear projection, we know that $-S_n\big(x^i + x_i(x^i)\big) + S_n\big(x^i - x_i(x^i)\big) = -2x_i(x^i)$. Using this and the triangle inequality with four terms we have that

$$\begin{aligned} ||z_i(x^i)|| \leq & \big|\big|R_n\big(x^i + x_i(x^i)\big) - R_n\big(S_n\big(x^i + x_i(x^i)\big)\big)\big|\big| \\ & + \big|\big|R_n\big(S_n\big(x^i + x_i(x^i)\big)\big) - S_n\big(x^i + x_i(x^i)\big)\big|\big| \\ & + \big|\big|R_n\big(S_n\big(x^i - x_i(x^i)\big)\big) - R_n\big(x^i - x_i(x^i)\big)\big|\big| \\ & + \big|\big|S_n\big(x^i - x_i(x^i)\big) - R_n\big(S_n\big(x^i - x_i(x^i)\big)\big)\big|\big| \\ \leq & 2||R_n||_{Lip}\varphi(n)\delta_k + 2\rho_n. \end{aligned}$$

For a set $J \subset \{1, \dots, \varphi(n)\}$ with $\#J = m \geq 1$, we define the measure in $[a_i]_{i\in J}$ as

$$\lambda_J^m(A) = \lambda^m\left(\Big(\big(a_i^*\big)_{i\in J}\Big)(A)\right) \quad \forall A \in \mathcal{M}_J^m,$$

where λ^m is the Lebesgue measure in $\mathbb{R}^m$ and $\mathcal{M}_J^m = \Big\{ A \subset [a_i]_{i\in J} \ : \ \Big(\big(a_i^*\big)_{i\in J}\Big)\big(A\big)$ is a Lebesgue measurable subset of $\mathbb{R}^m \Big\}$. If $J = \{1, \dots, \varphi(n)\}$ we denote $\lambda^{\varphi(n)} = \lambda_J^{\varphi(n)}$. For every $i \in \{1, \dots, \varphi(n)\}$, if $J = \{1, \dots, \varphi(n)\} \setminus \{i\}$ then we denote $\lambda_i^{\varphi(n)-1} = \lambda_J^{\varphi(n)-1}$. We are ready to define the linear operators $T_{n,k} : G_n \to F_n$, for every $k \in \mathbb{N}$, as

$$T_{n,k}(v) = \frac{1}{\lambda^{\varphi(n)}(B_{n,k})} \int_{B_{n,k}} dR_n(x)[v] d\lambda^{\varphi(n)}(x),$$

where $dR_n(x)[v]$ is the derivative of R_n at x evaluated in v, which exists almost everywhere.

In [4] pg. 47 the volumes of ℓ_p^n balls B_p^n have been computed as $|B_p^n| = \frac{2^n \Gamma(\frac{1}{p}+1)^n}{\Gamma(\frac{n}{p}+1)}$. Using this result for $p = 1$ (for the base part of our set $B_{n,k}$) and standard properties of the Lebesgue measure we obtain the following values for our sets for arbitrary $i \in \{1, \dots, n\}$

$$\lambda_i^{\varphi(n)-1}\big((B_{n,k})_i\big) = \frac{2^{\varphi(n)-1} r_n^{n-1} \delta_k^{\varphi(n)-n}}{(n-1)!},$$

$$\lambda^{\varphi(n)}\big(B_{n,k}\big) = \frac{2^{\varphi(n)} r_n^{n} \delta_k^{\varphi(n)-n}}{n!},$$

so the quotient is equal to

$$\frac{\lambda_i^{\varphi(n)-1}((B_{n,k})_i)}{\lambda^{\varphi(n)}(B_{n,k})} = \frac{n}{2r_n}.$$

Note that the expression

$$R_n(x^i + x_i(x^i)) - R_n(x^i - x_i(x^i)) = z_i(x^i) + 2x_i(x^i)$$

represents the difference of the values of the operator R_n between the endpoints of a segment cutting through $B_{n,k}$, which passes through the point x^i with direction a_i. As $B_{n,k} = \{x^i + x_i \in G_n \ : \ x^i \in (B_{n,k})_i \ , \ x_i \in [-x_i(x^i), x_i(x^i)]\}$, thanks to Fubini's Theorem and the Fundamental Theorem of Calculus applied to the i-th coordinate, we can compute for each

$i \in \{1, \dots, n\}$

$$\begin{aligned}
\big|\big|a_i - T_{n,k}(a_i)\big|\big| &= \left|\left|a_i - \frac{1}{\lambda^{\varphi(n)}(B_{n,k})}\int_{B_{n,k}} dR_n(x)[a_i]d\lambda^{\varphi(n)}(x)\right|\right| \\
&= \left|\left|a_i - \frac{1}{\lambda^{\varphi(n)}(B_{n,k})}\int_{(B_{n,k})_i} \big(z_i(x^i) + 2x_i(x^i)\big)\, d\lambda_i^{\varphi(n)-1}(x^i)\right|\right| \\
&= \left|\left|\frac{1}{\lambda^{\varphi(n)}(B_{n,k})}\int_{(B_{n,k})_i} z_i(x^i)d\lambda_i^{\varphi(n)-1}(x^i)\right|\right| \\
&\leq \frac{\lambda_i^{\varphi(n)-1}((B_{n,k})_i)}{\lambda^{\varphi(n)}(B_{n,k})}\big(2||R_n||_{Lip}\varphi(n)\delta_k + 2\rho_n\big) \\
&= \frac{n||R_n||_{Lip}\varphi(n)\delta_k + n\rho_n}{r_n}.
\end{aligned}$$

We may assume that $T_{n,k}$ pointwise converge in k and define $T_n(x) = \lim_{k\to\infty} T_{n,k}(x)$ for every $x \in G_n$. It is a linear operator from G_n to F_n satisfying

$$||T_n(a_i) - a_i|| \leq \frac{nh_{\gamma(n)}(\varphi(n)+2)}{r_n} \quad \forall i \in \{1, \dots, n\}.$$

As $||a_i^*|| = 1$, $i \in \{1, \dots, n\}$, we obtain that for every $x \in E_n$,

(2.3)

$$||T_n(x) - x|| = \left|\left|\sum_{i=1}^n a_i^*(x)(T_n(a_i) - a_i)\right|\right| \leq \frac{h_{\gamma(n)}(\varphi(n)+2)n^2}{r_n}||x|| \leq \frac{||x||}{n}.$$

Hence, $T(E_n) = Y_n$ is an n-dimensional subspace of F_n and so there is a projection $P_n : F_n \to Y_n$ of norm n. Finaly, we construct as in [5] Lemma 3.2 the linear mapping

$$\widetilde{T}_n = \Big(\big(T_n\big|_{E_n}\big)^{-1} \circ P_n + Id_{F_n} - P_n\Big) \circ T_n : G_n \to F_n.$$

Replacing $x = \big(T_n\big|_{E_n}\big)^{-1}(y)$ for arbitrary $y \in Y_n$ we get

$$\left|\left|\left(\big(T_n\big|_{E_n}\big)^{-1} - Id_{Y_n}\right)(y)\right|\right| = \left|\left|\big(T_n\big|_{E_n}\big)^{-1}(y) - y\right|\right| \leq \frac{\left|\left|\big(T_n\big|_{E_n}\big)^{-1}(y)\right|\right|}{n}.$$

This implies that $\big|\big|\big(T_n\big|_{E_n}\big)^{-1}\big|\big| \leq \frac{n}{n-1}$ and therefore the previous inequality yields

$$\left|\left|\big(T_n\big|_{E_n}\big)^{-1} - Id_{Y_n}\right|\right| \leq \frac{1}{n-1}.$$

Thus, taking into account (2.2),

$$||\widetilde{T}_n - T_n|| \leq \left|\left|\big(T_n\big|_{E_n}\big)^{-1} - Id_{Y_n}\right|\right| \cdot ||P_n|| \cdot ||T_n|| \leq \frac{n}{n-1}||R_n||_{Lip} \leq \frac{2n}{n-1}||R||_{Lip}.$$

This implies that X has the BAP since

$$\lambda(X, F_n, E_n) \leq \frac{2(2+\varepsilon)}{\varepsilon}||\widetilde{T}_n|| \leq \frac{2(2+\varepsilon)(4n-2)}{\varepsilon(n-1)}||R||_{\text{Lip}} \quad \forall n \in \mathbb{N}.$$

The proof in the case of basically flat sets is completed. The general case of a flat set is essentially the same, considering just the sequence $\sigma(n)$ in place of that of all integers n. ∎

Remark 2.8. We didn't strive for the best possible bound on $\lambda(X, F_n, E_n)$. In fact, using the same argument but with sharper estimates it is possible to prove that X has the λ^+-BAP (meaning that X has the $(\lambda+\delta)$-BAP for every $\delta > 0$) where

$$\lambda = \frac{2+\varepsilon}{\varepsilon}||R||_{\text{Lip}}.$$

However, this is the best possible bound we are able to find using this approach.

Corollary 2.9. *Let X be a separable Banach space. If there exists a Lipschitz retraction from X onto a generating convex bounded flat subset K with $0 \in K$ then X has the BAP.*

Proof. Let $\beta = (e_n) \subset span(K)$ be a fundamental sequence of X relative to which K is β-flat. During this proof we set $E_n = [e_i]_{i=1}^n$ for every $n \in \mathbb{N}$. Our aim here is to prove that there is a Lipschitz retraction onto a subset satisfying the assumptions of Theorem 2.7. In particular, we will see that there is $y \in K$ such that $K - y$ is β-flat and has nonempty local β-interior.

Let us first show that if K is β-flat then $K - z$ is β-flat whenever $z \in K$. To see this, let h_n be the sequence of heights for K and $\widetilde{h}_n$ the sequence of heights for $K - z$. Given $x \in K$ we take $y_x, y_z \in E_n$ such that $d(x, E_n) = ||x - y_x||$ and $d(z, E_n) = ||z - y_z||$. Clearly, $z + (y_x - y_z) \in z + E_n$. Hence,

$$d(x-z, E_n) = d(x, z+E_n) \leq ||x-(z+y_x-y_z)|| \leq ||x-y_x||+||z-y_z|| \leq 2h_n.$$

This implies that $\widetilde{h}_n \leq 2h_n$. Let us take $\varepsilon > 0$ such that $\liminf h_n(3+\varepsilon)^n = 0$. Then,

$$0 \leq \liminf \widetilde{h}_n(3+\varepsilon)^n \leq 2\liminf h_n(3+\varepsilon)^n = 0,$$

which proves that $K - z$ is β-flat.

We pass on to find $y \in K$ such that $K - y$ has nonempty local β-interior. For every $n \in \mathbb{N}$ there are elements $x_1, \ldots, x_{k_n} \in K$ such that $E_n \subset span(x_1, \ldots, x_{k_n})$. We consider then

$$y_n = \frac{1}{k_n+1}\sum_{i=1}^{k_n} x_i,$$

which is again in K (since $0 \in K$). Now, we claim that

$$int\big((K - y_n) \cap E_n\big) \neq \emptyset \quad \forall n \in \mathbb{N}. \tag{2.4}$$

To show (2.4) it is enough to prove that for every $z \in span(x_1, \ldots, x_{k_n})$ there is $\varepsilon > 0$ such that $y_n + \varepsilon z \in K$. Let us take $z = \sum_{i=1}^{k_n} \lambda_i x_i \in span(x_1, \ldots, x_{k_n})$

and $\varepsilon > 0$ satisfying that

$$\frac{1}{k_n+1} + \varepsilon \min_{i=1,\dots,k_n}(\lambda_i) > 0 \quad \text{and} \quad \frac{k_n}{k_n+1} + \varepsilon \sum_{i=1}^{k_n} \lambda_i \leq 1.$$

The claim follows immediately since

$$y_n + \varepsilon z = \sum_{i=1}^{k_n} \Big(\frac{1}{k_n+1} + \varepsilon\lambda_i \Big) x_i \in K.$$

The statement (2.4) is now proven. Let us consider the element

$$y = \sum_{k\in\mathbb{N}} 2^{-k} y_k.$$

We finish the proof by showing that $(K-y) \cap E_n$ has nonempty interior for every $n \in \mathbb{N}$. Again, it is enough to see that for every $z \in E_n$ there is $\varepsilon > 0$ such that $y + \varepsilon z \in K$. Given $n \in \mathbb{N}$ we take $z \in E_n$ and set $z_n = \sum_{k\neq n} \frac{2^{-k}}{1-2^{-n}} y_k \in K$ so that $y = 2^{-n} y_n + (1-2^{-n}) z_n$. By (2.4) there is $\varepsilon > 0$ such that $y_n + \varepsilon z \in K$. Hence, letting $\varepsilon' = 2^{-n}\varepsilon$ we are done since

$$y + \varepsilon' z = 2^{-n}(y_n + \varepsilon z) + (1-2^{-n}) z_n \in K.$$

■

Data availability statement. The manuscript has no associated data.

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Chapter 5

On the constant of Lipschitz approximability

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ON THE CONSTANT OF LIPSCHITZ APPROXIMABILITY

RUBÉN MEDINA

ABSTRACT. In this note we find $\lambda > 1$ and give an explicit construction of a separable Banach space X such that there is no λ-Lipschitz retraction from X onto any compact convex subset of X whose closed linear span is X. This is closely related to a well-known open problem raised by Godefroy and Ozawa in 2014 and represents the first known example of a Banach space with such a property.

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1. INTRODUCTION

1.1. **Motivation and background.** This paper is motivated by a natural question asked by Godefroy and Ozawa in [GO14], and then subsequently in [God15b], [God15a], [GMZ16], [GLP19] and [God20]. They wonder whether for every separable Banach space there is a Lipschitz retraction onto a compact convex subset whose closed linear span is the whole space. Recall that this question only makes sense for separable spaces since any space with such a property must be generated by a compact set. In [HM23a, Theorem 3.3], the latter question is solved in the positive for Banach spaces with a finite dimensional decomposition. In fact, for every $\varepsilon > 0$ the retraction can be constructed to be $(1 + \varepsilon)$-Lipschitz whenever the space has a monotone Schauder basis. However, the general problem is still open although a Hölder version of the problem was solved positively in [Med23, Theorem 2.1].

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We prove that there exists a separable Banach space X such that there is no λ-Lipschitz retraction onto any compact convex subset of X whose closed linear span is the whole space for some $\lambda > 1$. However, our example does have such a retraction with larger Lipschitz constant (see Corollary 2.11).

The Godefroy-Ozawa question is closely related to the theory of nonlinear approximation properties. Indeed, if there is a λ-Lipschitz retraction R from a Banach space X onto a generating compact convex subset K of X then there is a sequence of λ-Lipschitz retractions $R_n : X \to X$ with compact range and pointwise converging to the identity on X. Specifically, if we fix $k_0 \in K$, the mapping R_n may be defined as $R_n(x) = nR\big(\frac{x}{n} + k_0\big) - nk_0$ which is a translation of R (so that 0 is in the image) composed with an expansion. Notice that $R_n(x) \to x$ for all $x \in X$ since $\overline{\bigcup_n R_n(X)} = \overline{\bigcup_n n(K - k_0)} = X$. Therefore, Godefroy-Ozawa question is strongly related to Kalton's Problem 1 in [Kal12, page 1260], namely, whether in every separable Banach space X there is a sequence of equi-uniformly continuous mappings with compact range pointwise converging to the indentity on X. It is worth mentioning that a counterexample to Kalton's problem would provide a renorming of ℓ_1 without the metric approximation property, solving a classical open problem in the theory of approximation properties.

Nonlinear approximation properties are intimately connected to the linear ones (for instance, see [GK03, Theorem 5.3], [HM23a, Corollary 4.8] and [HM23b, Corollary 2.9]). For this exact reason, in our desire of solving Godefroy-Ozawa question, we have decided to follow the path traced by Enflo to construct a separable Banach space without a basis ([Enf73b]). He first tackled the problem of finding a Banach space with basis constant greater than 1 (see [Enf73a]) and so have we done in our analogous nonlinear setting. Moreover, Enflo himself mentions in [Enf73a, Second paragraph of page 309] that the approach developed in [Enf73b] is similar to that of [Enf73a], which we find highly encouraging.

Section 2 is divided into two subsections. In order to prevent the reader from losing the intuition among the computations developed in our proof, we have added a short subsection explaining the general ideas behind the construction. In the last subsection, both the construction and the proof of our main result Theorem 2.1 are carried out.

1.2. **Definitions and notation.** Let (M, d_M) and (N, d_N) be two metric spaces and let $f : M \to N$ be an arbitrary mapping. We will say that f is *Lipschitz* if there is $\lambda \geq 0$ such that $d_N(f(x), f(y)) \leq \lambda d_M(x, y)$ for every $x, y \in M$. We may specify that some constant $\lambda \geq 0$ satisfy the latter inequality saying that f is λ-Lipschitz. The infimum over all $\lambda > 0$ such that f is λ-Lipschitz is called the *Lipschitz constant* of f.

A *retraction* from a metric space (M, d) onto a subset $N \subset M$ is a map $R : M \to N$ satisfying that $R(x) = x$ for every $x \in N$. The image of a retraction is called a *retract*. We say that R is a *Lipschitz retraction* or equivalently that $R(M)$ is a *Lipschitz retract* whenever the retraction R is Lipschitz.

If X is a Banach space, we say that a subset S of X is a *generating* subset whenever the closed linear span of S is X. We denote the closed linear span of S as $\overline{\text{span}}(S)$.

Given some countable set Γ, we denote $\ell_\infty(\Gamma)$ the space of bounded functions $x : \Gamma \to \mathbb{R}$ endowed with the supremum norm $\|x\|_\infty = \sup_{\gamma\in\Gamma} |x(\gamma)|$. Whenever we are working with an element x belonging to some $\ell_\infty(\Gamma)$ we will denote the supremum norm of x simply by $\|x\|$. We also denote $\ell_1(\Gamma)$ the space of summable functions $x : \Gamma \to \mathbb{R}$ endowed with the norm $\|x\|_1 = \sum_{\gamma\in\Gamma} |x(\gamma)|$. For $N \in \mathbb{N}$ we will refer as ℓ_∞^N and ℓ_1^N to the space $\ell_\infty(\Gamma)$ and $\ell_1(\Gamma)$ respectively for $\Gamma = \{1, \dots, N\}$. We say that a sequence $(e_n)_{n\in\mathbb{N}}$ of normalised vectors from a Banach space E is *equivalent to the* $\ell_1(\mathbb{N})$ *basis* if for every sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ with finitely many nonzero elements it holds that

$$\left\| \sum_{n\in\mathbb{N}} \lambda_n e_n \right\| = \sum_{n\in\mathbb{N}} |\lambda_n|.$$

It is worth mentioning that the given definition is not the usual concept of basic sequences being equivalent.

Given a Banach space E, two nonempty subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of E and an element $v \in E$, we say that v is a *midpoint* of $\mathcal{S}_1$ and $\mathcal{S}_2$ whenever for every $z \in \mathcal{S}_1 \cup \mathcal{S}_2$, $\|v - z\| = d(\mathcal{S}_1, \mathcal{S}_2)/2$.

Throughout the entire note, we will consider $\mathbb{N}$ the natural numbers starting from 1 and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Throughout the entire note we will follow the terminology and notation used in [FHH+11]. For the background on Lipschitz and uniformly continuous retractions we refer the reader to the first two chapters of the authoritative monograph [BL00].

2. Main result

2.1. **Sketch of the proof.** Let us begin with a somehow imprecise but useful formulation of the approach used in subsection 2.2. We will try to construct X such that if R is a $(1+\varepsilon)$-Lipschitz retraction with convex image K (where $0 \in K$) for some small enough $\varepsilon > 0$ almost preserving some fixed vectors $\pm U_1, \pm U_2, \pm V_{1,1}, \pm V_{2,1} \in X$ then there must be a sequence in K without any Cauchy subsequence. The reasoning is going to follow an iterative argument. More precisely, out of the eight vectors $\pm U_1, \pm U_2, \pm V_{1,1}, \pm V_{2,1}$ we will be able to find another four vectors $\pm V_{1,2}, \pm V_{2,2}$ distant from the rest which are also almost preserved by R. Then, using the fact that $\pm U_1, \pm U_2, \pm V_{1,2}, \pm V_{2,2}$ are almost preserved by R we will find another four vectors $\pm V_{1,3}, \pm V_{2,3}$ distant from the rest which are again almost preserved by R and so on. The sought sequence without Cauchy subsequences is going to be $(R(V_{1,n}))_{n\in\mathbb{N}}$. The unique problem being how to produce the new four almost preserved vectors $\pm V_{1,n+1}, \pm V_{2,n+1}$ out of the previous almost preserved vectors $\pm U_1, \pm U_2, (\pm V_{1,m})_{m=1}^n, (\pm V_{2,m})_{m=1}^n$:

For technical reasons, the argument depends on whether n is even or odd. Assume first that n is odd and $\pm U_1, \pm U_2, \pm V_{1,n}, \pm V_{2,n}$ are almost preserved by R. We need to consider two subsets of vectors $\mathcal{S}_n^+, \mathcal{S}_n^- \subset X$ such that

$$\mathcal{S}_n^\pm \subset \mathrm{co}\{R(\pm U_1), R(\pm U_2), R(\pm V_{1,n}), R(\pm V_{2,n})\} \subset K.$$

More precisely, the sets $\mathcal{S}_n^\pm$ contain 8 vectors $\mathcal{S}_n^\pm = \{Z_{1,n}^\pm, \dots, Z_{8,n}^\pm\}$ chosen such that the elements $Z_{1,n}^+, \dots, Z_{8,n}^+$ are close to $\frac{U_1 + \frac{V_{1,n}+V_{2,n}}{2}}{2}$ and the elements $Z_{1,n}^-, \dots, Z_{8,n}^-$ are close to $-\frac{U_1 + \frac{V_{1,n}+V_{2,n}}{2}}{2}$. The space X is constructed so that $V_{1,n+1}$ is the unique midpoint of $\{0\}$ and $\mathcal{S}_n^+ \setminus \{Z_{2,n}^+\}$ and the vector $V_{2,n+1}$ is the unique midpoint of $\{0\}$ and $\mathcal{S}_n^- \setminus \{Z_{3,n}^-\}$. Since all the latter sets are in K their elements are preserved by R and since there are no other midpoints of those sets and $\{0\}$ rather than $V_{1,n+1}, V_{2,n+1}$, these vectors must also be almost preserved by the $(1+\varepsilon)$-Lipschitz retraction R.

We may define $U_1, U_2, V_{1,n}, V_{2,n}$ satisfying $\|U_1\|, \|U_2\| = 1$, $\|V_{1,n}\|, \|V_{2,n}\| \approx 1/3$ and

$$d(\{0\}, \mathcal{S}_n^\pm) \approx \frac{\|U_1\| + \frac{\|V_{1,n}\|+\|V_{2,n}\|}{2}}{2} \approx 2/3 \quad \text{for } i = 2, 3.$$

Hence, the vectors $V_{1,n+1}, V_{2,n+1}$ must satisfy that

$$\|V_{1,n+1}\|, \|V_{2,n+1}\| = d(\{0\}, \mathcal{S}_n^\pm)/2 \approx 1/3 \approx \|V_{1,n}\|, \|V_{2,n}\|.$$

This is the key ingredient which prevents the sequence $(V_{1,n})_{n\in\mathbb{N}}$ from converging to 0. Therefore, it is possible to define the vectors distant from each other. If, on the contrary, n is even, we proceed analogously but using U_2 instead of U_1.

Let us now introduce the purpose that our main lemmas below serve. Lemma 2.6 shows that $\pm V_{i,n+1}$ is the midpoint of $\{0\}$ and its respective set $\mathcal{S}$ described above whereas Lemma 2.7 proves that $\pm V_{i,n+1}$ is the unique element with such a property. Moreover, since the argument must work up to $\varepsilon > 0$, we show in Lemma 2.7 that if a vector B is almost the midpoint (up to ε) of $\{0\}$ and $\mathcal{S}$ then B is almost $\pm V_{i,n+1}$ (up to a constant $C > 0$ times ε). Finally, Lemmas 2.6 and 2.7 are used in the proof of Theorem 2.8 (a more precise formulation of Theorem 2.1) to finish the section.

2.2. **Final construction.** We will give an explicit definition of a separable Banach space X and a proof of the fact that there is no λ-Lipschitz retraction onto any generating compact and convex subset of X for some $\lambda > 1$. That is, we will be proving the following result.

Theorem 2.1. There is $\lambda > 1$ and a separable Banach space X such that no retraction onto any generating compact and convex subset of X is λ-Lipschitz.

The space X will arise as the closed linear span of countably many carefully chosen vectors from $\ell_\infty(\mathbb{N}^2)$.

Let us then proceed with the definition of X. The construction will depend on some constants $\delta, M, \Delta > 0$ which will be specified later. We set for every $1 \leq k \leq 8$ the following quantities,

$$x_k = \frac{\frac{1}{2} + \frac{k}{M}}{1 - \frac{2k}{M}} \quad , \quad t(k) = \frac{1}{2} - \frac{k}{M}$$
$$\alpha = 1 - x_8 \quad , \quad \beta = 1 + x_1^2,$$

We now start defining functions $f, g : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \alpha + x \quad \text{and} \quad g(x) = \beta - x^2 \qquad (x \in \mathbb{R}).$$

We also define three vectors $u, v, c \in \mathbb{R}^8$ given by $u(k) = f(x_k)$, $v(k) = g(x_k)$ and $c(k) = 1$ for $k = 1, \ldots, 8$.

Fact 2.2. The vector $(x_k)_{k=1}^8$ is strictly increasing in $k = 1, \ldots, 8$. Moreover, if $M > 16$ and we set $\delta_M := x_8^2 - x_1^2$ then $\delta_M > x_8 - x_1 > 0$, $\delta_M \xrightarrow{M\to\infty} 0$ and

$$\max_{k=1,\ldots,8} u(k) = u(8) = 1 \quad , \quad \min_{k=1,\ldots,8} u(k) = u(1) = 1 - (x_8 - x_1),$$
$$\max_{k=1,\ldots,8} v(k) = v(1) = 1 \quad , \quad \min_{k=1,\ldots,8} v(k) = v(8) = 1 - (x_8^2 - x_1^2).$$

Proof. $M > 16$ implies $x_k > \frac{1}{2}$ for every $1 \leq k \leq 8$. Then, $x_8 + x_1 > 1$ and hence

$$\delta_M = (x_8 + x_1)(x_8 - x_1) > x_8 - x_1.$$

The rest of the proof is straightforward. □

From now on, we will consider a sequence $(e_n) \subset S_{\ell_\infty(\mathbb{N})}$ equivalent to the $\ell_1(\mathbb{N})$ basis in $\ell_\infty(\mathbb{N})$. This is possible since $\ell_\infty(\mathbb{N})$ is isometrically universal for separable spaces. Also, let us consider $\delta_i \in \mathbb{R}^8$ for $i = 1, \ldots, 8$ given by $\delta_i(k) = 0$ if $i \neq k$ and $\delta_i(i) = \Delta$.

We are finally ready to present the vectors from $\ell_\infty(\mathbb{N}^2)$ that will span our space. All the vectors will be defined following the same approach, namely, each vector will be defined in $\{m\} \times \mathbb{N}$ in a specific way for each $m \in \mathbb{N}$. In the case $m = 1$, the definition will be splitted into blocks of 8 elements in a row, that is, the first block will be formed by $(1,1), \ldots, (1,8)$, the second will be formed by $(1, 8{+}1), \ldots, (1, 2{\cdot}8)$ and the n^{th} block will be formed by $(1, (n-1)8+1), \ldots, (1, n8)$ (see Figure 1). We denote as *BLOCK* m the elements $(1, m8+1), \ldots, (1, (m+1)8)$ for every $m \in \mathbb{N}_0$.

Consider now the elements $U_1, U_2 \in \ell_\infty(\mathbb{N}^2)$ given by

$$U_i(x) = \begin{cases} u(k) & \text{if } x = (1, 8(2m + j - 1) + k) \text{ for } m \in \mathbb{N}_0,\ k = 1, \ldots, 8, \\ \delta_8(k) & \text{if } x = (1, 8(2m - j) + k) \text{ for } m \in \mathbb{N},\ k = 1, \ldots, 8, \\ \frac{1}{10} e_1(k) & \text{if } x = (p, k) \text{ for } p, k \in \mathbb{N},\ p \geq 2. \end{cases}$$

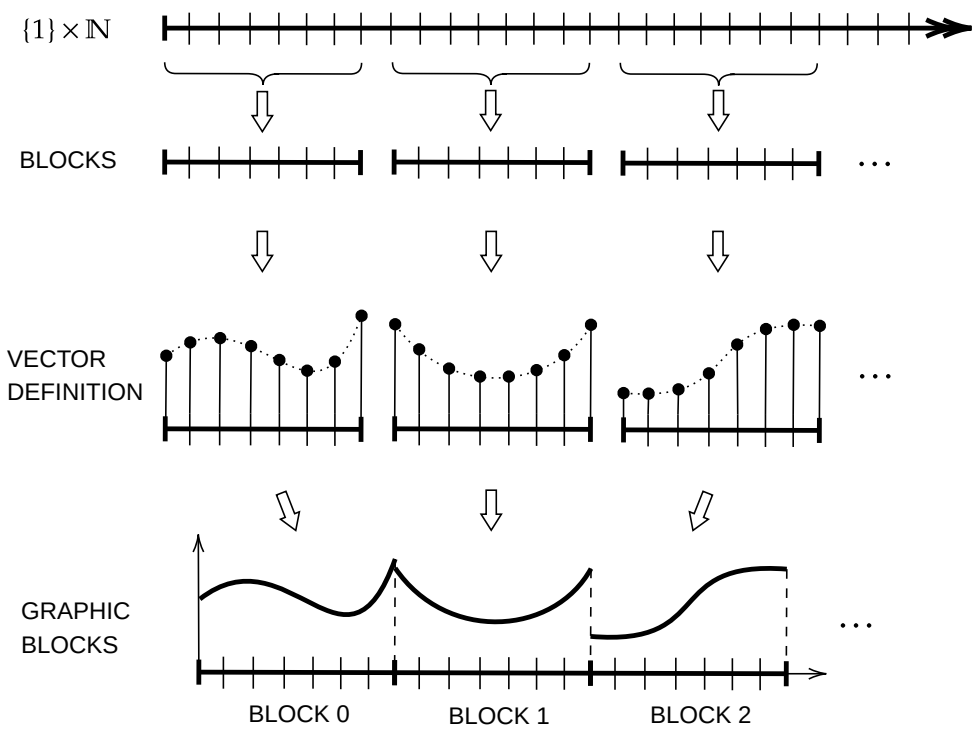

FIGURE 1. This image is originally published in [HM24].

Then, we define for every $n \in \mathbb{N}$ the elements $V_{1,n}, V_{2,n} \in \ell_\infty(\mathbb{N}^2)$ as

$$V_{i,n}(x) = \begin{cases} (-1)^{i-1}(\frac{1-\delta}{3} - \delta_{i+1}(k)) & \text{if } x = (1, n8+k),\ k = 1,\dots,8 \\ v(k) - \frac{2+2\delta}{3} + (-1)^i \delta_7(k) & \text{if } x = (1,(n+1)8+k),\ k = 1,\dots,8 \\ (-1)^{i-1}(\frac{1-2\delta}{3} - \delta_6(k)) & \text{if } x = (1, m8+k),\ k = 1,\dots,8,\ m \in 2\mathbb{N}_0 + \frac{1-(-1)^n}{2} \setminus \{n\}, \\ (-1)^{i-1}\frac{1-\delta}{3} e_1(k) & \text{if } x = (2n+i-1, k),\ k \in \mathbb{N}, \\ \frac{1}{10} e_{2n+i-1}(k) & \text{if } x = (m,k),\ \begin{array}{l} m,k \in \mathbb{N},\ m \geq 2, \\ m \neq 2n+i-1, 2(n+1), 2(n+1)+1, \end{array} \\ 0 & \text{elsewhere.} \end{cases}$$

Our sought space is

$$X = \overline{\operatorname{span}}\Big(\{V_{i,n}\}_{\substack{i=1,2 \\ n\in\mathbb{N}}} \cup \{U_1, U_2\}\Big).$$

See Figure 2 and Figure 3 to have a conceptual idea of what the vectors $U_i, V_{i,n}$ look like when restricted to $\{1\} \times \mathbb{N}$ (we present the picture by giving the graph of the blocks arranged like in the last step of Figure 1). Notice that, when restricted to $\{m\} \times \mathbb{N}$ for $m \neq 1$, the previous vectors are nothing but multiples of some vector e_j from the ℓ_1 basis. We have included in Figure 3 a description of $\frac{V_{1,n}+V_{2,n}}{2}\Big|_{\{1\}\times\mathbb{N}}$ because it is also going to be central in further discussions.

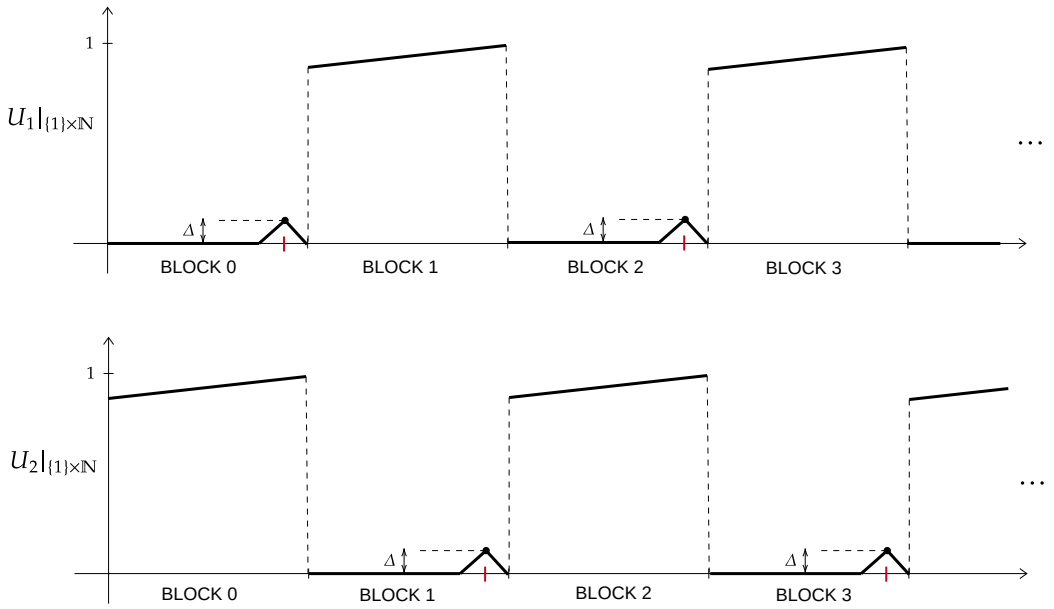

FIGURE 2. This image is originally published in [HM24].

Now, we turn our attention to the proof of Theorem 2.1. For that purpose, we first need to state and prove some technical auxiliary results. We start with the following simple but handy lemma.

Lemma 2.3. Let $N, n \in \mathbb{N}$ with $N > n$, $\{v_1, \ldots, v_n\} \in \ell_\infty^N \setminus \{0\}$ and $c = (1, \ldots, 1) \in \ell_\infty^N$. If $v_1, \ldots, v_n, c$ are linearly independent then there exists $K > 0$ such that for every $a \in \operatorname{span}\{v_1, \ldots, v_n, c\}$ with $a = \lambda c + \sum_{i=1}^{n} \lambda_i v_i$ it follows that

$$|\lambda - 1|, |\lambda_i| \leq K\|a - c\| \,, \quad i = 1, \ldots, n.$$

Proof. It follows immediately from the fact that the sequence $\{v_1, \ldots, v_n, c\}$ is a block basis but, for the convienence of the reader, we include here a complete proof.

Since $v_1, \ldots, v_n, c$ are linearly independent, we know that there exist $f_0, f_1, \ldots, f_n \in (\ell_\infty^N)^* = \ell_1^N$ such that $z = f_0(z)c + \sum_{i=1}^{n} f_i(z)v_i$ for every $z \in \ell_\infty^N$. Moreover, we know that $f_0(c) = 1$ and $f_i(c) = 0$ for $i = 1, \ldots, n$. Let us consider $K = \max\{\|f_0\|, \|f_1\|, \ldots, \|f_n\|\} < \infty$. It is clear that for every $1 \leq i \leq n$,

$$|\lambda_i| = |f_i(a)| = |f_i(a - c)| \leq \|f_i\| \cdot \|a - c\| \leq K\|a - c\|.$$

Finally, it is immediate that

$$|\lambda - 1| = \|f_0(a - c)\| \leq \|f_0\|\|a - c\| \leq K\|a - c\|.$$

□

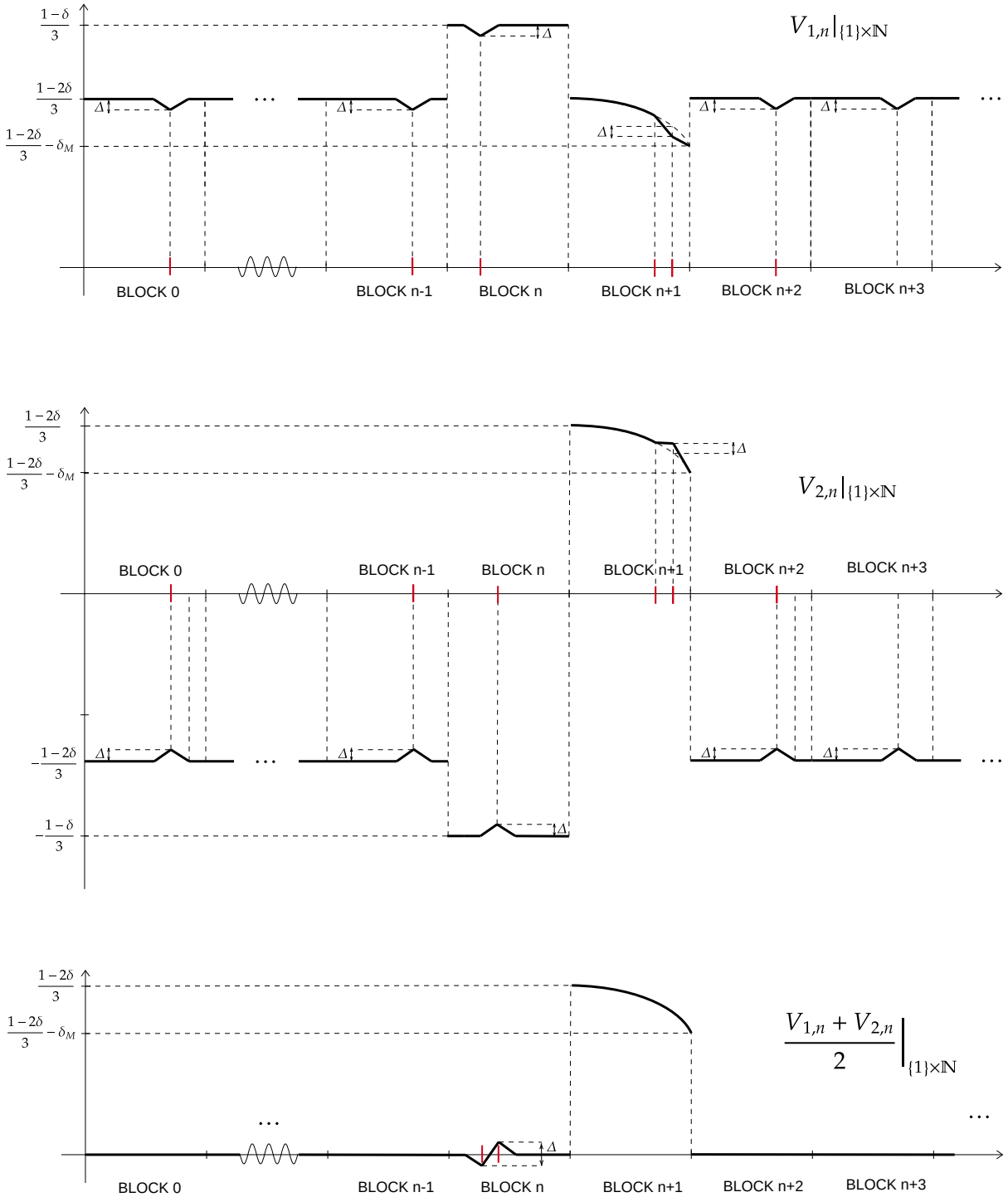

FIGURE 3. This image is originally published in [HM24].

We will make use of the following quantity which depends exclusively on $M > 0$,

$$\mu_M = \min_{\substack{j,k=1,\dots,8 \\ j\neq k}} \{(1-t(k))(u(k)-u(j)) + t(k)(v(k)-v(j))\}. \tag{2.1}$$

Fact 2.4. For every $M > 16$ it holds that $\mu_M > 0$ and $\mu_M \xrightarrow{M\to\infty} 0$.

Proof. Let us consider for $1 \leq k \leq 8$ the function $h_k : \mathbb{R} \to \mathbb{R}$ given by

$$h_k(x) = (1-t(k))f(x) + t(k)g(x) \qquad (x \in \mathbb{R}).$$

It is immediate that

$$\mu_M = \min_{\substack{j,k=1,\dots,8 \\ j\neq k}} \{h_k(x_k) - h_k(x_j)\}.$$

For every choice of $M > 16$, the function h_k is a quadratic polynomial attaining a unique maximum at x_k. Therefore, if $j \neq k$ then $h_k(x_k) > h_k(x_j)$ and thus $\mu_M > 0$. Now, since h_k is contiuous and $x_k, x_j \xrightarrow{M\to\infty} 1/2$, we clearly have

$$h_k(x_k) - h_k(x_j) \xrightarrow{M\to\infty} h_k\Big(\frac{1}{2}\Big) - h_k\Big(\frac{1}{2}\Big) = 0$$

and we are done. □

Let us finally pick appropriate $\delta, M, \Delta > 0$ for which X will satisfy the statement of Theorem 2.1 for some $\lambda > 1$.

Choice of δ, M and Δ:
We take some $\delta > 0$ satisfying

$$\delta < \frac{1}{10}, \tag{2.2}$$

and find M large enough so that the following inequalities hold,

$$\frac{8}{M} + \delta_M + \mu_M < \frac{1-8\delta}{6}, \tag{2.3}$$

$$\delta_M < \frac{\delta}{3}. \tag{2.4}$$

Finally, we pick $\Delta > 0$ so that

$$\Delta < \frac{\mu_M}{2}, \ \frac{\delta}{6}, \ \frac{1-2\delta}{3} - \delta_M, \delta_M, \ \frac{1-\delta_M}{2}. \tag{2.5}$$

We denote now for $n \in \mathbb{N}$ and $1 \leq k \leq 8$ the element $S_{k,n} \in X$ given by

$$S_{k,n} = (1-t(k))U_{\frac{3+(-1)^n}{2}} + t(k)\frac{V_{1,n}+V_{2,n}}{2}.$$

The element $S_{k,n} \in X$ will be very relevant for the proof of Theorem 2.1. See Figure 4 to have a rough idea of what it looks like when restricted to $\{1\} \times \mathbb{N}$.

For the latter choices of δ, M and Δ, we have the following proposition.

Proposition 2.5. The following properties are satisfied for every $n \in \mathbb{N}$,

(1) $\|V_{i,n}\| = \frac{1-\delta}{3}$ and $\|U_i\| = 1$ for $i = 1, 2$.
(2) $|(V_{1,n} + V_{2,n})(1, m)| \leq \Delta$ for every $m \in \mathbb{N} \setminus \{(n+1)8+1, \dots, (n+1)8+8\}$.

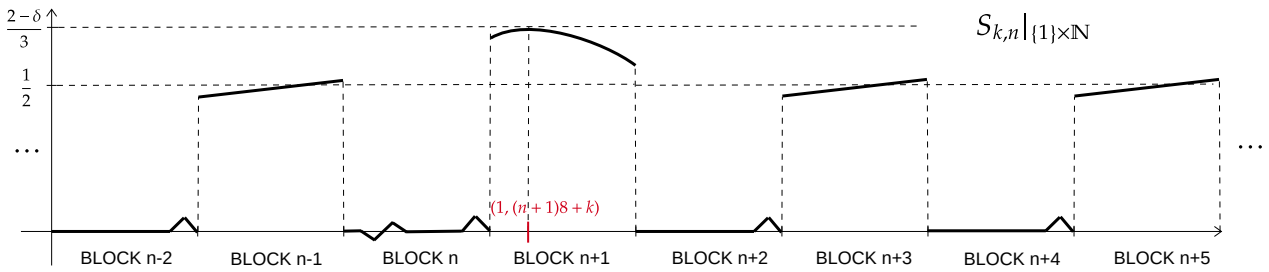

FIGURE 4. This image is originally published in [HM24].

(3) For every $k = 1, \dots, 8$,

$$S_{k,n}(1,(n+1)8+k) > \frac{2-\delta}{3} - \delta_M + \frac{\Delta}{2}.$$

(4) For every $x \in \mathbb{N}^2 \setminus \{(1,(n+1)8+k)\}$,

$$|S_{k,n}(x)| \leq S_{k,n}(1,(n+1)8+k) - \mu_M.$$

Proof.

(1): It is clear that $\|U_i\| = u(8) = 1$. Now, taking into account the constraints of δ, M and Δ, a straightforward case by case check yields that $\|V_{i,n}\| = \frac{1-\delta}{3}$.

(2): The proof is straightforward from the definition.

(3): By Fact 2.2,

$$u(1) + v(1) = 2 - (x_8 - x_1) > 2 - \delta_M. \tag{2.6}$$

We also deduce that

$$u(1) = 1 - (x_8 - x_1) \overset{\text{Fact 2.2}}{>} 1 - \delta_M \overset{(2.4)}{>} \frac{1-2\delta}{3} = v(1) - \frac{2+2\delta}{3}. \tag{2.7}$$

This, in addition to the fact that $t(k) < 1/2$, yields

$$\begin{aligned} 0 &\overset{(2.7)}{<} \Big(\frac{1}{2} - t(k)\Big)\Big(u(1) - \Big(v(1) - \frac{2+2\delta}{3}\Big)\Big) \\ &= (1-t(k))u(1) + t(k)\Big(v(1) - \frac{2+2\delta}{3}\Big) - \frac{u(1)+v(1)-\frac{2+2\delta}{3}}{2}. \end{aligned}$$

Equivalently,

$$(1-t(k))u(1) + t(k)\Big(v(1) - \frac{2+2\delta}{3}\Big) > \frac{u(1)+v(1)-\frac{2+2\delta}{3}}{2}. \tag{2.8}$$

Finally, since $\mu_M > 0$ we know that $(1-t(k))u(k) + t(k)(v(k)) \geq (1-t(k))u(1) + t(k)v(1)$ and therefore

$$\begin{aligned} S_{k,n}(1,(n+1)8+k) &= (1-t(k))u(k) + t(k)\Big(v(k) - \frac{2+2\delta}{3}\Big) \\ &\geq (1-t(k))u(1) + t(k)\Big(v(1) - \frac{2+2\delta}{3}\Big) \\ &\overset{(2.8)}{>} \frac{u(1)+v(1)-\frac{2+2\delta}{3}}{2} \\ &\overset{(2.6)}{>} \frac{2-\delta_M-\frac{2+2\delta}{3}}{2} \\ &= \frac{2-\delta}{3} - \frac{\delta_M}{2} \overset{\delta_M>\Delta}{>} \frac{2-\delta}{3} - \delta_M + \frac{\Delta}{2}. \end{aligned}$$

(4): We distinguish here between different cases depending on the element $x \in \mathbb{N}^2 \setminus \{(1,(n+1)8+k)\}$:

Case 1, $x = (1,(n+1)8+j)$ for $j = 1,\dots,8$ and $j \neq k$. In this case we have

$$\begin{aligned} |S_{k,n}(x)| =& (1-t(k))u(j) + t(k)\Big(v(j) - \frac{2+2\delta}{3}\Big) \\ =& (1-t(k))u(k) + t(k)\Big(v(k) - \frac{2+2\delta}{3}\Big) \\ & - \big((1-t(k))(u(k)-u(j)) + t(k)(v(k)-v(j))\big) \\ \overset{(2.1)}{\leq}& S_{k,n}(1,(n+1)8+k) - \mu_M. \end{aligned}$$

Case 2, $x = (1,q)$ with $q \neq (n+1)8+j$ for $j = 1,\dots,8$. From property (2) we know that $\left|\frac{V_{1,n}+V_{2,n}}{2}(x)\right| \leq \Delta/2$ and thus

$$\begin{aligned} |S_{k,n}(x)| \leq& (1-t(k))\left|U_{\frac{3+(-1)^n}{2}}(x)\right| + \Delta/2 \\ \overset{(1)}{\leq}& 1 - t(k) + \Delta/2 \overset{(2.3)}{\leq} \frac{2-\delta}{3} - \delta_M - \mu_M + \Delta/2 \\ \overset{(3)}{<}& S_{k,n}(1,(n+1)8+k) - \mu_M. \end{aligned}$$

Case 3, $x = (p,q)$ for some $p,q \in \mathbb{N}$ with $p \geq 2$. In this case

$$|U_{\frac{3+(-1)^n}{2}}(x)| \leq \frac{1}{10} \quad \text{and} \quad \frac{1}{3} \overset{(2.3)}{\leq} \frac{2-\delta}{3} - \delta_M - \mu_M \overset{(3)}{<} S_{k,n}(1,(n+1)8+k) - \mu_M,$$

so then

$$\begin{aligned} |S_{k,n}(x)| \leq& (1-t(k))\left|U_{\frac{3+(-1)^n}{2}}(x)\right| + t(k)\frac{|V_{1,n}(x)| + |V_{2,n}(x)|}{2} \\ \overset{(1)}{\leq}& (1-t(k))\frac{1}{10} + t(k)\frac{1-\delta}{3} < \frac{1}{3} < S_{k,n}(1,(n+1)8+k) - \mu_M. \end{aligned}$$

□

The following lemma is crucial for our later purposes. Despite being large and seemingly involved, its proof is elementary and the computations needed are not so hard. However, we are forced to work distinguishing cases due to the nature of the definition of the vectors $U_i, V_{i,n} \in X$.

Lemma 2.6. There exists $\varepsilon_0 > 0$ such that if $E_1, E_2, W_1, W_2 \in X$ satisfy $\|W_i - V_{i,n}\|, \|E_i - U_i\| \leq \varepsilon_0$ for $i = 1, 2$ and some $n \in \mathbb{N}$ then for $k = 1, \dots, 8$ we have that $\|Z_{k,n}\| \geq \frac{2-2\delta}{3}$ and

$$\|V_{i,n+1}\|, \left\|(-1)^{i+1}V_{i,n+1} - \frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\right\| = \frac{1-\delta}{3} \quad \forall k \neq i+1, \tag{2.9}$$

where $Z_{k,n} = (1 - t(k))E_{\frac{3+(-1)^n}{2}} + t(k)\frac{W_1+W_2}{2}$.

Proof. We first choose ε_0 satisfying some inequalities which will be used throughout the proof. Let us take $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < \frac{\mu_M - 2\Delta}{2}, \ \frac{1}{40}, \ \frac{\delta}{3} - \delta_M, \ \frac{\delta}{3} - 2\Delta, \frac{1-8\delta}{6} - \frac{8}{M}, \frac{1-10\delta}{60}. \tag{2.10}$$

Notice that thanks to the appropriate choice of δ, M and Δ, the quantities in the right-hand side are strictly positive and hence such an ε_0 exists. Let us show that this ε_0 satisfies the statement of Lemma 2.6. From Properties (3) and (4) of Proposition 2.5 we get that for $k = 1, \dots, 8$,

$$\|S_{k,n}\| = S_{k,n}(1, (n+1)8 + k) > \frac{2-\delta}{3} - \delta_M. \tag{2.11}$$

Hence, for every $k = 1, \dots, 8$ we have that

$$\begin{aligned} \|Z_{k,n}\| \geq & \|S_{k,n}\| - \varepsilon_0 \overset{(2.11)}{>} \frac{2-\delta}{3} - \delta_M - \varepsilon_0 \\ = & \frac{2-2\delta}{3} + (\delta/3 - \delta_M - \varepsilon_0) \overset{(2.10)}{>} \frac{2-2\delta}{3}. \end{aligned} \tag{2.12}$$

We turn our attention to the proof of (2.9). We already know from Proposition 2.5 that $\|V_{i,n+1}\| = \frac{1-\delta}{3}$ so that we only need to prove the second equality of (2.9). It is immediate that

$$\left\|(-1)^{i+1}V_{i,n+1} - \frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\right\| \geq \left\|\frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\right\| - \|V_{i,n+1}\| \overset{(1)}{=} \frac{1-\delta}{3}.$$

Therefore, it is enough to show that for every $x \in \mathbb{N}^2$ and $k \neq i+1$,

$$\left|\left((-1)^{i+1}V_{i,n+1} - \frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\right)(x)\right| \leq \frac{1-\delta}{3}. \tag{2.13}$$

We proceed by splitting the proof of inequality (2.13) into different cases.

Case 1, $x = (1, (n+1)8 + k)$. In this case

$$V_{i,n+1}(x) = (-1)^{i-1}\left(\frac{1-\delta}{3} - \delta_{i+1}(k)\right) \overset{k\neq i+1}{=} (-1)^{i-1}\frac{1-\delta}{3},$$

and since by (2.10) we know that $2\varepsilon_0 < \mu_M$ then for every $z \in \mathbb{N}^2 \setminus \{x\}$,

$$\begin{aligned} |Z_{k,n}(z)| \leq |S_{k,n}(z)| + \varepsilon_0 &\overset{(4)}{\leq} S_{k,n}(x) - \mu_M + \varepsilon_0 \\ &\leq Z_{k,n}(x) - \mu_M + 2\varepsilon_0 \\ &< Z_{k,n}(x). \end{aligned}$$

This shows that

$$\|Z_{k,n}\| = Z_{k,n}(1,(n+1)8+k). \tag{2.14}$$

Therefore,

$$\left|\left((-1)^{i+1}V_{i,n+1} - \frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\right)(x)\right| \overset{(2.14)}{=} \left|\frac{1-\delta}{3} - \frac{2-2\delta}{3}\right| = \frac{1-\delta}{3}.$$

Case 2, $x = (1,(n+1)8+j)$ for $j \in \{1,\dots,8\} \setminus \{k\}$. In this case,

$$V_{i,n+1}(x) = (-1)^{i-1}\Big(\frac{1-\delta}{3} - \delta_{i+1}(j)\Big), \tag{2.15}$$

$$S_{k,n}(x) = (1-t(k))u(j) + t(k)\Big(v(j) - \frac{2+2\delta}{3}\Big). \tag{2.16}$$

Now, since $\delta_M \overset{(2.4)}{<} \frac{\delta}{3} \overset{(2.2)}{<} \frac{1}{30}$ it holds that

$$v(j) - \frac{2+2\delta}{3} \overset{\text{Fact 2.2}}{\geq} 1 - \delta_M - \frac{2+2\delta}{3} > 0. \tag{2.17}$$

Therefore, taking into account that $1 - t(k) \geq 1/2$ we get that

$$\begin{aligned} Z_{k,n}(x) \geq S_{k,n}(x) - \varepsilon_0 &\overset{(2.16)}{=} (1-t(k))u(j) + t(k)\Big(v(j) - \frac{2+2\delta}{3}\Big) - \varepsilon_0 \\ &\overset{(2.17)}{\geq} \frac{u(j)}{2} - \varepsilon_0 \overset{\text{Fact 2.2}}{\geq} \frac{1-\delta_M}{2} - \varepsilon_0 > 0. \end{aligned} \tag{2.18}$$

The last inequality holds since $\delta_M < \frac{1}{30}$ and $\varepsilon_0 \overset{(2.10)}{<} \frac{1}{40}$. Combining previous inequalities we get

$$\Big((-1)^{i+1}V_{i,n+1} - \frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\Big)(x) \overset{(2.15)(2.18)}{<} \frac{1-\delta}{3}. \tag{2.19}$$

Since

$$\|S_{k,n}\| \leq (1-t(k))\|U_{\frac{3+(-1)^n}{2}}\| + t(k)\frac{\|V_{1,n}\| + \|V_{2,n}\|}{2} \overset{(1)}{<} 1, \tag{2.20}$$

and $\varepsilon_0 \overset{(2.10)}{<} \frac{1-4\delta}{3}$ we have that

$$\|Z_{k,n}\| \leq \|S_{k,n}\| + \varepsilon_0 \overset{(2.20)}{<} 1 + \frac{1-4\delta}{3} = \frac{4-4\delta}{3}. \tag{2.21}$$

Also,

$$|Z_{k,n}(x)| \leq |S_{k,n}(x)| + \varepsilon_0 \overset{(4)}{\leq} \|S_{k,n}\| - \mu_M + \varepsilon_0 \leq \|Z_{k,n}\| - \mu_M + 2\varepsilon_0. \tag{2.22}$$

Using the above inequalities we deduce that

$$\begin{aligned}\Big(\frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}-(-1)^{i+1}V_{i,n+1}\Big)(x) &\overset{(2.15)}{\leq} \frac{2-2\delta}{3\|Z_{k,n}\|}|Z_{k,n}(x)|-|V_{i,n+1}(x)|\\ &\overset{(2.22)}{\leq}\Big(\frac{2-2\delta}{3}-\frac{2-2\delta}{3\|Z_{k,n}\|}(\mu_M-2\varepsilon_0)\Big)-\Big(\frac{1-\delta}{3}-\Delta\Big)\\ &\overset{(2.21)}{<}\frac{1-\delta}{3}-\frac{1}{2}(\mu_M-2\varepsilon_0-2\Delta)\overset{(2.10)}{<}\frac{1-\delta}{3}.\end{aligned} \tag{2.23}$$

This case is hence done by combining (2.19) and (2.23).

Case 3, $x=(1,(n+2)8+j)$ with $1\leq j\leq 8$. In this case we have

$$V_{i,n+1}(x)=v(j)-\frac{2+2\delta}{3}+(-1)^i\delta_7(j),$$

which taking into account that $v(j)\overset{\text{Fact 2.2}}{\leq}1$ implies that

$$|V_{i,n+1}(x)|\leq\frac{1-2\delta}{3}+\Delta. \tag{2.24}$$

We now compute $|S_{k,n}(x)|$. Since we have that $|(V_{1,n}+V_{2,n})(x)|\overset{(2)}{\leq}\Delta$ and $U_{\frac{3+(-1)^n}{2}}(x)=\delta_8(j)$, it clearly follows that

$$|S_{k,n}(x)|\leq(1-t(k))|\delta_8(j)|+t(k)\Delta\leq\Delta. \tag{2.25}$$

Therefore,

$$\begin{aligned}\Big|\Big((-1)^{i+1}V_{i,n+1}-\frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\Big)(x)\Big| &\overset{(2.12)}{\leq}|V_{i,n+1}(x)|+|Z_{k,n}(x)|\\ &\leq|V_{i,n+1}(x)|+|S_{k,n}(x)|+\varepsilon_0\\ &\overset{(2.24)(2.25)}{\leq}\frac{1-2\delta}{3}+2\Delta+\varepsilon_0\\ &=\frac{1-\delta}{3}-\Big(\frac{\delta}{3}-2\Delta-\varepsilon_0\Big)\overset{(2.10)}{<}\frac{1-\delta}{3}.\end{aligned}$$

Case 4, $x=(1,m8+j)$ where $m\in 2\mathbb{N}_0+\frac{1-(-1)^n}{2}\setminus\{n+2\}$ and $1\leq j\leq 8$. In this case by definition we get that

$$V_{i,n+1}(x)=0\ \ ,\ \ |(V_{1,n}+V_{2,n})(x)|\overset{(2)}{\leq}\Delta\ \ ,\ \ U_{\frac{3+(-1)^n}{2}}(x)=\delta_8(j).$$

Hence, $|S_{k,n}(x)|\leq(1-t(k))\Delta+t(k)\Delta=\Delta$ and we conclude

$$\Big|\Big((-1)^{i+1}V_{i,n+1}-\frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\Big)(x)\Big|\overset{(2.12)}{\leq}|Z_{k,n}(x)|\leq|S_{k,n}(x)|+\varepsilon_0\leq\Delta+\varepsilon_0\leq\frac{1-\delta}{3}.$$

The very last inequality follows from the constraints established for Δ and ε_0 in (2.5) and (2.10).

Case 5, $x = (1, m8 + j)$ with m distinct from the previous cases and $1 \leq j \leq 8$, that is, m is any number with the opposite parity to n except $n+1$:

$$m \in 2\mathbb{N}_0 + \frac{1-(-1)^{n+1}}{2} \setminus \{n+1\}.$$

In this case,

$$V_{i,n+1}(x) = (-1)^{i-1}\Big(\frac{1-2\delta}{3} - \delta_6(j)\Big) \quad , \quad U_{\frac{3+(-1)^n}{2}}(x) = u(j). \tag{2.26}$$

Then, since $\delta_M \overset{(2.4)}{<} \delta/3 < 1/30$ and $\Delta \overset{(2.5)}{<} \delta/6 \overset{(2.2)}{<} 1/60$ we have that

$$\begin{aligned} S_{k,n}(x) &\overset{(2.26)}{=} (1-t(k))u(j) + t(k)\frac{(V_{1,n}+V_{2,n})(x)}{2} \\ &\overset{(2)}{\geq} (1-t(k))u(j) - \Delta \\ &\overset{\text{Fact 2.2}}{\geq} (1-t(k))(1-\delta_M) - \Delta \\ &\geq \frac{1-\delta_M}{2} - \Delta > 0. \end{aligned} \tag{2.27}$$

Then,

$$\begin{aligned} \Big((-1)^{i+1}V_{i,n+1} - \frac{2-2\delta}{3\|Z_{k,n}\|}S_{k,n}\Big)(x) &\overset{(2.27)}{\leq} (-1)^{i+1}V_{i,n+1}(x) \\ &\overset{(2.26)}{=} \frac{1-2\delta}{3} - \delta_6(j) \\ &\overset{\varepsilon_0<\delta/3}{<} \frac{1-\delta}{3} - \varepsilon_0. \end{aligned} \tag{2.28}$$

Also, we have that

$$S_{k,n}(x) \overset{(2.26)}{=} (1-t(k))u(j) + t(k)\frac{(V_{1,n}+V_{2,n})(x)}{2} \overset{(2)}{\leq} (1-t(k))u(j) + \Delta, \tag{2.29}$$

and so,

$$\begin{aligned} \Big(\frac{2-2\delta}{3\|Z_{k,n}\|}S_{k,n} - (-1)^{i+1}V_{i,n+1}\Big)(x) &\overset{(2.12)}{\leq} S_{k,n}(x) - (-1)^{i+1}V_{i,n+1}(x) \\ &\overset{(2.26)(2.29)}{\leq} (1-t(k))u(j) + \Delta - \frac{1-2\delta}{3} + \delta_6(j) \\ &\overset{u(j)\leq 1}{\leq} \frac{1}{2} + \frac{k}{M} - \frac{1-2\delta}{3} + 2\Delta \\ &\overset{\Delta<\delta/6}{<} \frac{1}{6} + \frac{8}{M} + \delta \leq \frac{1-\delta}{3} - \varepsilon_0. \end{aligned} \tag{2.30}$$

Notice that $u(j) \leq 1$ follows from Fact 2.2 and $\Delta < \delta/6$ is one of the inequalities in (2.5).

Inequalities (2.28) and (2.30) yield

$$\Big|\Big((-1)^{i+1}V_{i,n+1} - \frac{2-2\delta}{3\|Z_{k,n}\|}S_{k,n}\Big)(x)\Big| < \frac{1-\delta}{3} - \varepsilon_0, \tag{2.31}$$

and therefore we finish this case using the triangle inequality

$$\begin{aligned}\left|\left((-1)^{i+1}V_{i,n+1}-\frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\right)(x)\right| &\leq \left|\left((-1)^{i+1}V_{i,n+1}-\frac{2-2\delta}{3\|Z_{k,n}\|}S_{k,n}\right)(x)\right| \\ &\quad+\frac{2-2\delta}{3\|Z_{k,n}\|}\|S_{k,n}(x)-Z_{k,n}(x)\| \\ &\overset{(2.12)(2.31)}{<} \frac{1-\delta}{3}.\end{aligned}$$

Case 6, $x=(2(n+1)+i-1,j)$ where $j\in\mathbb{N}$. In this case,

$$\begin{aligned}V_{i,n+1}(x)=(-1)^{i-1}\frac{1-\delta}{3}e_1(j) \quad &, \quad V_{1,n}(x)=V_{2,n}(x)=0,\\ U_{\frac{3+(-1)^n}{2}}(x)=\frac{1}{10}e_1(j) \quad &, \quad S_{k,n}(x)=(1-t(k))\frac{1}{10}e_1(j).\end{aligned} \tag{2.32}$$

First, we compute

$$\frac{2-2\delta}{3\|Z_{k,n}\|}\cdot\frac{1}{10}(1-t(k)) \overset{(2.12)}{<} \frac{1}{10} \overset{(2.2)}{<} \frac{1-\delta}{3}. \tag{2.33}$$

Now, taking into account that $1-t(k)\geq\frac{1}{2}$ we have

$$\begin{aligned}\left|\left((-1)^{i+1}V_{i,n+1}-\frac{2-2\delta}{3\|Z_{k,n}\|}S_{k,n}\right)(x)\right| &\overset{(2.32)}{=} \left|\left(\frac{1-\delta}{3}-\frac{2-2\delta}{3\|Z_{k,n}\|}\cdot\frac{1}{10}(1-t(k))\right)e_1(j)\right| \\ &\overset{(2.33)}{=} \left(\frac{1-\delta}{3}-\frac{2-2\delta}{3\|Z_{k,n}\|}\cdot\frac{1}{10}(1-t(k))\right)|e_1(j)| \\ &\overset{(2.21)}{\leq} \frac{1-\delta}{3}-\frac{1}{40} \overset{(2.10)}{<} \frac{1-\delta}{3}-\varepsilon_0.\end{aligned}$$

Therefore, by the triangle inequality and the last shown equation we get

$$\begin{aligned}\left|\left((-1)^{i+1}V_{i,n+1}-\frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\right)(x)\right| &\leq \left|\left((-1)^{i+1}V_{i,n+1}-\frac{2-2\delta}{3\|Z_{k,n}\|}S_{k,n}\right)(x)\right|+\varepsilon_0 \\ &< \frac{1-\delta}{3}.\end{aligned}$$

Case 7, $x=(p,j)\in\mathbb{N}^2$ different from the previous cases. In this case, $p\geq 2$ and hence we know that $V_{i,n}(x)\in\left\{(-1)^{i-1}\frac{1-\delta}{3}e_1(j),\frac{1}{10}e_{2n+i-1}(j),0\right\}$ where it cannot happen that $V_{1,n}(x)=V_{2,n}(x)=(-1)^{i-1}\frac{1-\delta}{3}e_1(j)$. Then,

$$\left|\frac{V_{1,n}(x)+V_{2,n}(x)}{2}\right|\leq\frac{\frac{1-\delta}{3}+\frac{1}{10}}{2}=\frac{13-10\delta}{60}. \tag{2.34}$$

Also, since x is different from that of Case 6, necessarily $p\neq 2(n+1)+i-1$ so that $V_{i,n+1}(x)\in\left\{0,\frac{1}{10}e_{2(n+1)+i-1}(j)\right\}$. Hence,

$$|V_{i,n+1}|\leq\frac{1}{10} \quad\text{and}\quad U_{\frac{3+(-1)^n}{2}}(x)=\frac{1}{10}e_1(j). \tag{2.35}$$

This, together with the fact that $\frac{1}{10} \overset{(2.2)}{<} \frac{13-10\delta}{60}$ yields

$$\begin{aligned} |S_{k,n}(x)| &\leq (1-t(k))\big|U_{\frac{3+(-1)^n}{2}}(x)\big| + t(k)\Big|\frac{V_{1,n}(x)+V_{2,n}(x)}{2}\Big| \\ &\overset{(2.34)(2.35)}{\leq} (1-t(k))\frac{1}{10} + t(k)\frac{13-10\delta}{60} < \frac{13-10\delta}{60}. \end{aligned} \tag{2.36}$$

Therefore, taking into account that by (2.10), $\varepsilon_0 \leq \frac{1-10\delta}{60}$ we finally conclude

$$\begin{aligned} \Big|\Big((-1)^{i+1}V_{i,n+1} - \frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\Big)(x)\Big| &\overset{(2.12)}{\leq} |V_{i,n+1}(x)| + |S_{k,n}(x)| + \varepsilon_0 \\ &\overset{(2.35)(2.36)}{<} \frac{1}{10} + \frac{13-10\delta}{60} + \varepsilon_0 \leq \frac{1-\delta}{3}. \end{aligned}$$

□

Our next objective is to prove that $(-1)^{i+1}V^i_{n+1}$ is the unique vector (up to ε) satisfying the thesis of Lemma 2.6 (up to ε). The proof relies on the ℓ_1-like behaviour of our previously defined vectors.

Lemma 2.7. There exists $C > 0$ such that if $Z_{1,n}, \ldots, Z_{8,n}$ are like in Lemma 2.6 for some $n \in \mathbb{N}$ then whenever there is $B \in X$ satisfying

$$\|B\|, \Big\|B - \frac{2-2\delta}{3\|Z_{k,n}\|}Z_{k,n}\Big\| \leq \frac{1-\delta}{3} + \varepsilon, \quad \forall k \neq i_0+1,$$

for some $i_0 \in \{1,2\}$ and $\varepsilon > 0$ it follows that

$$\|B - (-1)^{i_0+1}V_{i_0,n+1}\| \leq C\varepsilon.$$

Proof. We may assume without loss of generality that

$$B \in \operatorname{span}\Big(\{V_{i,n}\}_{\substack{i=1,2\\ n\in\mathbb{N}}} \cup \{U_1, U_2\}\Big).$$

Let us consider $y_k = (1, (n+1)8+k)$ for $k \in \{1,\ldots,8\}$ and $\Gamma = \{y_k\}_{\substack{k=1,\ldots,8\\ k\neq i_0+1}}$. Now, we consider the vectors $w, c, b, u_1, u_2, v_{i,m} \in \ell_\infty(\Gamma) \equiv \ell_\infty^7$ for $i = 1,2$ and $m \in \mathbb{N}$ given by

$$w(y_k) = \frac{1-2\delta}{3} - \delta_6(k) \ , \ c(y_k) = 1 \ , \ b(y_k) = B(y_k),$$
$$u_1(y_k) = U_1(y_k) \ , \ u_2(y_k) = U_2(y_k) \ , \ v_{i,m}(y_k) = V_{i,m}(y_k).$$

Claim. If $i_1 \in \{1,2\} \setminus \{i_0\}$ then the vectors $c, v_{i_1,n+1}, v_{1,n}, v_{2,n}, u_1, u_2, w$ are linearly independent vectors from $\ell_\infty(\Gamma)$.

Proof of the Claim. Let us first define the vectors $p, q \in \ell_\infty(\Gamma)$ as

$$p(y_k) = g(x_k) \quad , \quad q(y_k) = f(x_k) \qquad \forall k \in \{1,\ldots,8\} \setminus \{i_0+1\}.$$

Consider also $e_s \in \ell_\infty(\Gamma)$ for $s \in \mathbb{N}$ as

$$e_s(y_k) = \begin{cases} 1 & \text{if } k = s \\ 0 & \text{if } K \neq s. \end{cases}$$

If one takes into account the immediate fact that p, q and c are respectively the evaluation in $\{x_k\}_{\substack{k=1,\dots,8\\ k\neq i_0+1}}$ of a polynomial of degree exactly 2, a polynomial of degree exactly 1 and a non-zero constant polynomial, it follows that the vectors from $\beta = \{p, q, c, e_{i_1+1}, e_6, e_7, e_8\}$ are linearly independent. Hence, β is a basis for $\ell_\infty(\Gamma)$ and we just need to compute the determinant of the matrix M whose rows are the coordinates of the vectors $c, v_{i_1,n+1}, v_{1,n}, v_{2,n}, u_1, u_2, w$ for the basis β. It is easy to check that

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & (-1)^{i_1-1}\frac{1-\delta}{3} & (-1)^{i_1}\Delta & 0 & 0 & 0 \\ 1 & 0 & -\frac{2+2\delta}{3} & 0 & 0 & -\Delta & 0 \\ 1 & 0 & -\frac{2+2\delta}{3} & 0 & 0 & \Delta & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta \\ 0 & 0 & \frac{1-2\delta}{3} & 0 & -\Delta & 0 & 0 \end{pmatrix}.$$

To show that $det(M) \neq 0$ it suffices to repeatedly use the Laplace expansion of the determinant. □

Let us continue now the proof of Lemma 2.7. We consider $i_1 \in \{1, 2\}$ distinct from i_0. There must exist unique $\lambda, \lambda_1, \dots, \lambda_5, \rho_{i,m} \in \mathbb{R}$ for $i = 1, 2$ and $m \in \mathbb{N} \setminus \{n, n+1\}$ such that just finitely many of them are nonzero and satisfy that

(2.37)
$$B = \lambda(-1)^{i_0+1}V_{i_0,n+1} + \lambda_1 V_{i_1,n+1} + \lambda_2 V_{1,n} + \lambda_3 V_{2,n} + \lambda_4 U_1 + \lambda_5 U_2 + \sum_{\substack{i=1,2\\ m\in\mathbb{N}\setminus\{n,n+1\}}} \rho_{i,m} V_{i,m}.$$

If $m \in \mathbb{N} \setminus \{n, n+1\}$ and $i = 1, 2$ then

$$v_{i,m}(y_k) = \begin{cases} (-1)^{i-1}\big(\frac{1-2\delta}{3} - \delta_6(k)\big) = (-1)^{i-1}w(y_k) & \text{if } m \neq n \bmod 2, \\ 0 & \text{if } m = n \bmod 2. \end{cases}$$

Therefore, since $\frac{1-\delta}{3}c = (-1)^{i_0-1}v_{i_0,n+1}$ we have that

$$b = \lambda\frac{1-\delta}{3}c + \lambda_1 v_{i_1,n+1} + \lambda_2 v_{1,n} + \lambda_3 v_{2,n} + \lambda_4 u_1 + \lambda_5 u_2 + \lambda_6 w, \tag{2.38}$$

where $\lambda_6 = \sum_{\substack{m\in\mathbb{N}\setminus\{n,n+1\}\\ m\neq n \bmod 2}} \rho_{1,m} - \rho_{2,m}$. Also, for $i = 1, 2$,

$$v_{i,n}(y_k) = v(k) - \frac{2+2\delta}{3} + (-1)^i\delta_7(k) \quad , \quad v_{i,n+1}(y_k) = (-1)^{i-1}\Big(\frac{1-\delta}{3} - \delta_{i+1}(k)\Big),$$

$$u_{\frac{3+(-1)^n}{2}}(y_k) = u(k) \quad , \quad u_{\frac{3-(-1)^n}{2}}(y_k) = \delta_8(k).$$

From the previous Claim we know that the vectors $c, v_{i_1,n+1}, v_{1,n}, v_{2,n}, u_1, u_2, w$ are linearly independent vectors from $\ell_\infty(\Gamma)$. Hence, we may use Lemma 2.3 with $a = \frac{3}{1-\delta}b$ which provides us with some $K > 0$ independent of ε and n such that $|\lambda -$

$1|, \left|\lambda_s \frac{3}{1-\delta}\right| \leq K\|a-c\|$ for $s=1,\dots,6$. Thus,

$$|\lambda-1|, |\lambda_s| \leq 4K\left\|b-\frac{1-\delta}{3}c\right\| \quad \text{for} \quad s=1,\dots,5. \tag{2.39}$$

From the hypothesis we deduce that for $k \neq i_0+1$, $B(y_k) \leq \frac{1-\delta}{3}+\varepsilon$ and

$$\frac{2-2\delta}{3} - B(y_k) \overset{(2.14)}{=} \frac{2-2\delta}{3\|Z_{k,n}\|} Z_{k,n}(y_k) - B(y_k) \leq \frac{1-\delta}{3}+\varepsilon.$$

This implies that $|B(y_k) - \frac{1-\delta}{3}| \leq \varepsilon$ for $k \in \{1,\dots,8\}\setminus\{i_0+1\}$ and hence

$$\left\|b-\frac{1-\delta}{3}c\right\| \leq \varepsilon. \tag{2.40}$$

Taking into account inequalities (2.39) and (2.40) we deduce that

$$|\lambda-1|, |\lambda_s| \leq 4K\varepsilon \quad \text{for} \quad s=1,\dots,5. \tag{2.41}$$

Finally, we consider

$$\widetilde{b} = B|_{\{2(n+1)+i_0-1\}\times\mathbb{N}} \quad \text{and} \quad \widetilde{d} = \widetilde{b} - \sum_{\substack{i=1,2\\ m\in\mathbb{N}\setminus\{n,n+1\}}} \rho_{i,m}\, V_{i,m}|_{\{2(n+1)+i_0-1\}\times\mathbb{N}}.$$

Notice that for every $1\leq i\leq 2$ and $m\in\mathbb{N}\setminus\{n,n+1\}$ it holds that $V_{i,m}|_{\{2(n+1)+i_0-1\}\times\mathbb{N}} = \frac{1}{10}e_{2m+i-1}$. Also, by (2.37) and (2.41) we have that

$$\left\|\widetilde{d}-\frac{1-\delta}{3}e_1\right\| = \|\widetilde{d}-(-1)^{i_0-1}\, V_{i_0,n+1}|_{\{2(n+1)+i_0-1\}\times\mathbb{N}}\| \leq |\lambda-1|+\sum_{s=1}^{5}|\lambda_s| \overset{(2.41)}{\leq} 24K\varepsilon.$$

Hence, by the triangle inequality,

$$\begin{aligned}
\|\widetilde{b}\| &= \left\|\widetilde{d} + \sum_{\substack{i=1,2\\ m\in\mathbb{N}\setminus\{n,n+1\}}} \rho_{i,m}\frac{1}{10}e_{2m+i-1}\right\| \\
&= \left\|\left(\widetilde{d}-\frac{1-\delta}{3}e_1\right) + \left(\frac{1-\delta}{3}e_1 + \frac{1}{10}\cdot \sum_{\substack{i=1,2\\ m\in\mathbb{N}\setminus\{n,n+1\}}} \rho_{i,m}e_{2m+i-1}\right)\right\| \\
&\geq \left(\frac{1-\delta}{3} + \frac{1}{10}\cdot \sum_{\substack{i=1,2\\ m\in\mathbb{N}\setminus\{n,n+1\}}} |\rho_{i,m}|\right) - 24K\varepsilon.
\end{aligned}$$

On the other hand $\|\widetilde{b}\| \leq \|B\| \leq \frac{1-\delta}{3}+\varepsilon$. Therefore,

$$\frac{1}{10}\cdot \sum_{\substack{i=1,2\\ m\in\mathbb{N}\setminus\{n,n+1\}}} |\rho_{i,m}| \leq (24K+1)\varepsilon.$$

This finishes the proof since

$$\|B-(-1)^{i_0+1}V_{i_0,n+1}\|$$
$$\overset{(2.37)}{=}\left\|(\lambda-1)(-1)^{i_0+1}V_{i_0,n+1}+\lambda_1V_{i_1,n+1}+\lambda_2V_{1,n}+\lambda_3V_{2,n}+\lambda_4U_1+\lambda_5U_2+\sum_{\substack{i=1,2\\ m\in\mathbb{N}\setminus\{n,n+1\}}}\rho_{i,m}V_{i,m}\right\|$$
$$\leq|\lambda-1|+\sum_{s=1}^{5}|\lambda_s|+\sum_{\substack{i=1,2\\ m\in\mathbb{N}\setminus\{n,n+1\}}}|\rho_{i,m}|\overset{(2.41)}{\leq}(264K+10)\varepsilon.$$

□

We are finally ready to prove Theorem 2.1, which we state here in a more precise way.

Theorem 2.8. There are $\varepsilon_0, C>0$ such that if D is a convex subset of X with $0\in D$ and $R:X\to X$ is a $(1+\varepsilon_0/C)$-Lipschitz retraction onto D satisfying for $i,j=1,2$,

$$\|R((-1)^jV_{i,1})-(-1)^jV_{i,1}\|,\|R((-1)^jU_i)-(-1)^jU_i\|\leq\varepsilon_0,$$

then for every $i,j=1,2$ and $n\in\mathbb{N}$,

$$\|R((-1)^jV_{i,n})-(-1)^jV_{i,n}\|\leq\varepsilon_0.$$

Proof. We will consider $\varepsilon_0, C>0$ given respectively by Lemma 2.6 and Lemma 2.7. Let us proceed by induction, showing that for every $n\in\mathbb{N}$ we have that $\|R((-1)^jV_{i,n})-(-1)^jV_{i,n}\|\leq\varepsilon_0$. The case $n=1$ follows directly from the assumptions above. Now, if we assume that $\|R((-1)^jV_{i,n})-(-1)^jV_{i,n}\|\leq\varepsilon_0$ for some $n\in\mathbb{N}$, let us prove that

$$\|R((-1)^jV_{i,n+1})-(-1)^jV_{i,n+1}\|\leq\varepsilon_0. \tag{2.42}$$

We first denote for $i,j=1,2$ and $k=1,\dots,8$,

$$W_i^j:=(-1)^jR((-1)^jV_{i,n})\quad,\quad E_i^j:=(-1)^jR((-1)^jU_i)\quad,\quad B_i^j=(-1)^jR((-1)^{i+j+1}V_{i,n+1}),$$

$$Z_{k,n}^j:=(1-t(k))E^j_{\frac{3+(-1)^n}{2}}+t(k)\frac{W_1^j+W_2^j}{2}.$$

By property (1) from Proposition 2.5,

$$\|B_i^j\|=\|R((-1)^{i+j+1}V_{i,n+1})-R(0)\|\leq(1+\varepsilon_0/C)\frac{1-\delta}{3}. \tag{2.43}$$

It is immediate that $(-1)^jE_i^j,(-1)^jW_i^j\in D$ and hence $\frac{2-2\delta}{3\|Z_{k,n}^j\|}(-1)^jZ_{k,n}^j\in D$ for $j\in\{1,2\}$ since $\frac{2-2\delta}{3\|Z_{k,n}^j\|}\leq1$ by (2.12). Notice also that by the induction hypothesis

we know that $E_1^j, E_2^j, W_1^j, W_2^j$ satisfy the requirements of Lemma 2.6. Therefore, by Lemma 2.6, if $k \neq i+1$,

$$\begin{aligned}\Big\|B_i^j - \frac{2-2\delta}{3\|Z_{k,n}^j\|}Z_{k,n}^j\Big\| =& \Big\|R((-1)^{i+j+1}V_{i,n+1}) - R\Big(\frac{2-2\delta}{3\|Z_{k,n}^j\|}(-1)^j Z_{k,n}^j\Big)\Big\| \\ \leq& (1+\varepsilon_0/C)\Big\|(-1)^{i+1}V_{i,n+1} - \frac{2-2\delta}{3\|Z_{k,n}^j\|}Z_{k,n}^j\Big\| \\ =& (1+\varepsilon_0/C)\frac{1-\delta}{3}.\end{aligned} \tag{2.44}$$

Putting together inequalities (2.43) and (2.44) we have that whenever $k \neq i+1$,

$$\|B_i^j\|, \Big\|B_i^j - \frac{2-2\delta}{3\|Z_{k,n}^j\|}Z_{k,n}^j\Big\| \leq \frac{1-\delta}{3} + \varepsilon_0/C.$$

Clearly, from the induction hypothesis we have $\|W_i^j - V_{i,n}\| \leq \varepsilon_0$ and $\|E_i^j - U_i\| \leq \varepsilon_0$ for $i,j \in \{1,2\}$. Therefore, we are allowed to use Lemma 2.7 in this situation. Then, for every $1 \leq i,j \leq 2$,

$$\|B_i^j - (-1)^{i+1}V_{i,n+1}\| \leq \varepsilon_0.$$

This proves (2.42) and hence finishes the induction since for every $1 \leq i,j \leq 2$,

$$\begin{aligned}\|R((-1)^{i+j+1}V_{i,n+1}) - (-1)^{i+j+1}V_{i,n+1}\| =& \|(-1)^j R((-1)^{i+j+1}V_{i,n+1}) - (-1)^{i+1}V_{i,n+1}\| \\ =& \|B_i^j - (-1)^{i+1}V_{i,n+1}\| \leq \varepsilon_0,\end{aligned}$$

□

The following Lemma 2.9 may be considered as part of the folklore of convex geometry but, as it is related to generating convex subsets, we state and prove it here.

Lemma 2.9. Let Y be any Banach space. For every convex set $K \subset Y$ with $0 \in K$ there is $x_0 \in K$ such that

$$\overline{\operatorname{span}(K)} = \overline{\bigcup_{k\in\mathbb{N}} k(K - x_0)}.$$

Proof. We follow an idea developed in [HM23b, Corollary 2.8] but we add the proof here for the sake of completeness.

Let us denote $Z = \overline{\operatorname{span}(K)}$. If Z is finite dimensional then the result is immediate by taking any x_0 in the interior of K with respect to the norm topology of Z. If, on the contrary, Z is a separable infinite dimesional Banach space, we consider a sequence $\beta = (e_n) \subset Z$ such that $\overline{span}\{e_n\}_{n\in\mathbb{N}} = Z$ and denote for every $n \in \mathbb{N}$ the space $E_n = \operatorname{span}\{e_1, \ldots, e_n\}$.

For every $n \in \mathbb{N}$ there are elements $x_1, \ldots, x_{k_n} \in K$ such that $E_n \subset \operatorname{span}(x_1, \ldots, x_{k_n})$. We consider then

$$y_n = \frac{1}{k_n+1}\sum_{i=1}^{k_n} x_i,$$

which is again in K (since $0 \in K$). Now, we claim that

$$\forall n \in \mathbb{N} \quad \exists \varepsilon > 0 \quad : \quad y_n + \varepsilon B_{E_n} \subset K. \tag{2.45}$$

To show (2.45) it is enough to prove that for every $z \in \operatorname{span}(x_1, \dots, x_{k_n})$ there is $\varepsilon > 0$ such that $y_n + \varepsilon z \in K$. Let us take $z = \sum_{i=1}^{k_n} \lambda_i x_i \in \operatorname{span}(x_1, \dots, x_{k_n})$ and $\varepsilon > 0$ satisfying that

$$\frac{1}{k_n + 1} + \varepsilon \min_{i=1,\dots,k_n} (\lambda_i) > 0 \quad \text{and} \quad \frac{k_n}{k_n + 1} + \varepsilon \sum_{i=1}^{k_n} \lambda_i \leq 1.$$

The claim follows immediately since

$$y_n + \varepsilon z = \sum_{i=1}^{k_n} \Big(\frac{1}{k_n + 1} + \varepsilon \lambda_i \Big) x_i \in K.$$

The statement (2.45) is now proven. Let us consider the element

$$x_0 = \sum_{k \in \mathbb{N}} 2^{-k} y_k.$$

We finish the proof by showing that

$$\forall n \in \mathbb{N} \quad \exists \varepsilon > 0 \quad : \quad x_0 + \varepsilon B_{E_n} \subset K. \tag{2.46}$$

Again, it is enough to see that for every $z \in E_n$ there is $\varepsilon > 0$ such that $x_0 + \varepsilon z \in K$. Given $n \in \mathbb{N}$ we take $z \in E_n$ and set $z_n = \sum_{k \neq n} \frac{2^{-k}}{1 - 2^{-n}} y_k \in K$ so that $x_0 = 2^{-n} y_n + (1 - 2^{-n}) z_n$. By (2.45) there is $\varepsilon > 0$ such that $y_n + \varepsilon z \in K$. Hence, letting $\varepsilon' = 2^{-n} \varepsilon$ we are done since

$$x_0 + \varepsilon' z = 2^{-n}(y_n + \varepsilon z) + (1 - 2^{-n}) z_n \in K.$$

□

Proof of Theorem 2.1. We take $\lambda = 1 + \varepsilon_0 / C > 1$. Assume that there is a λ-Lipschitz retraction R from X onto a generating compact and convex subset K of X. Then, for every $k \in \mathbb{N}$ we may shift R by an element $x_0 \in X$ and dilate it with ratio k to obtain

$$R_k(x) = k R\Big(\frac{x}{k} + x_0 \Big) - k x_0.$$

Notice that R_k is a λ-Lipschitz retraction onto $k(K - x_0)$. Picking $x_0 \in K$ as in Lemma 2.9 we get from the fact that K is generating that $R_k(x) \xrightarrow[k \to \infty]{\|\cdot\|} x$. Therefore, for large enough $k \in \mathbb{N}$, R_k meets the hypothesis of Theorem 2.8 and hence

$$\|R_k(V_{1,n}) - V_{1,n}\| \leq \varepsilon_0 \quad \forall n \in \mathbb{N}.$$

Clearly, $\|V_{1,m} - V_{1,n}\| \geq 1/5$ for every $n, m \in \mathbb{N}$ distinct. Hence, for every $m, n \in \mathbb{N}$ distinct, $\|R_k(V_{1,m}) - R_k(V_{1,n})\| \geq 1/5 - 2\varepsilon_0 > 0$. Therefore we find the contradiction since the compact set $k(K - x_0)$ contains the sequence $\big(R_k(V_{1,n})\big)_{n \in \mathbb{N}}$ which has no Cauchy subsequence. □

It is worth mentioning that X does enjoy both linear and nonlinear approximation properties. In fact, we have the following straightforward result.

Proposition 2.10. The space

$$X = \overline{\operatorname{span}}\Big(\{V_{i,n}\}_{\substack{i=1,2\\ n\in\mathbb{N}}} \cup \{U_1, U_2\}\Big)$$

is isomorphic to ℓ_1.

Proof. We know that $X = \operatorname{span}(\{V_{1,1}, V_{2,1}, U_1, U_2\}) \oplus \overline{\operatorname{span}}\Big(\{V_{i,n}\}_{\substack{i=1,2\\ n\geq 2}}\Big)$. Therefore, it is enough to show that $Y = \overline{\operatorname{span}}\Big(\{V_{i,n}\}_{\substack{i=1,2\\ n\geq 2}}\Big)$ is isomorphic to ℓ_1. Clearly $V_{i,n}|_{\{2\}\times\mathbb{N}} = \frac{1}{10} e_{2n+i-1}$ for every $1 \leq i \leq 2$, $n \geq 2$ and so for every sequence $(\lambda_{i,n})_{\substack{i=1,2\\ n\geq 2}} \subset \mathbb{R}$ with finitely many nonzero elements it holds that

$$\Big\| \sum_{\substack{i=1,2\\ n\geq 2}} \lambda_{i,n} V_{i,n} \Big\| \geq \Big\| \sum_{\substack{i=1,2\\ n\geq 2}} \lambda_{i,n}\, V_{i,n}|_{\{2\}\times\mathbb{N}} \Big\| = \Big\| \sum_{\substack{i=1,2\\ n\geq 2}} \lambda_{i,n} \frac{1}{10} e_{2n+i-1} \Big\| = \frac{1}{10} \sum_{\substack{i=1,2\\ n\geq 2}} |\lambda_{i,n}|.$$

□

Corollary 2.11. There is a Lipschitz retraction from $X = \overline{\operatorname{span}}\Big(\{V_{i,n}\}_{\substack{i=1,2\\ n\in\mathbb{N}}} \cup \{U_1, U_2\}\Big)$ onto a generating compact and convex subset of X.

Proof. Since ℓ_1 has a Schauder basis, necessarily X has a basis, too. Then, the statement follows from Theorem 3.3 in [HM23a]. □

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Chapter 6

Compact Hölder retractions and nearest point maps

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COMPACT HÖLDER RETRACTIONS AND NEAREST POINT MAPS

RUBÉN MEDINA

Abstract. In this paper, an α-Hölder retraction from any separable Banach space onto a compact convex subset whose closed linear span is the whole space is constructed for every positive $\alpha < 1$. This constitutes a positive solution to a Hölder version of a question raised by Godefroy and Ozawa. In fact, compact convex sets are found to be absolute α-Hölder retracts under certain assumption of flatness.

1. Introduction

In this note we study retractions onto compact convex subsets of Banach spaces. A retraction is the natural nonlinear counterpart to the concept of projection for subsets of Banach spaces. In particular, Lipschitz and uniformly continuous retractions have been widely studied (see [Phe58], [Lin64], [Bjo79]), often in the search of a way to extend Lipschitz and uniformly continuous maps (see [Whi34] ,[JLS86], [Bou87], [LN05], [God15b], [BDMS21]).

The main result in this paper is motivated by the natural question asked by Godefroy and Ozawa in [GO14], and then subsequently in [God15a], [God15b], [GMZ16], [GLP19] and [God20]. They wonder whether for every separable Banach space there is a Lipschitz retraction onto a compact convex subset whose closed linear span is the whole space. In [HM23a], the latter question is solved in the positive for Banach spaces with a finite dimensional decomposition but the general problem is still open. The main result of this work is a positive solution to a Hölder version of the problem. Indeed, an α-Hölder retraction onto such a subset is constructed for every positive $\alpha < 1$. In [HM23b], the bounded approximation property is proved to be necessary for a separable Banach space to have a Lipschitz retraction onto a generating compact convex subset satisfying an additional mild assumption on its shape. This hints to the possibility of a negative solution for the case $\alpha = 1$ (Lipschitz case).

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It is worth mentioning that in [CCW21] and later but independently in [HM23a], compact convex subsets of Banach spaces are proven to be absolute uniform retracts. In fact, for separable Banach spaces it is enough to consider the nearest point map respect to a URED renorming of the space. This means that the modulus of continuity of the retraction depends on the modulus of convexity of the given URED norm and it is independent of the shape of the retract. Here, a completely different approach is used, allowing us to tune up the shape of the retract and obtain a much better estimate on the modulus of continuity of the retraction.

In section 2 we analyse the compact convex subsets of Banach spaces in terms of flatness. That is, we are interested in the heights of a compact set respect to its finite dimensional sections (see Definition 2.2 below). In particular, for every $\alpha \in (0,1)$ we find a flatness condition such that every compact convex set under this condition is an absolute α-Hölder retract (Theorem 2.10). This leads to the main result of the paper Theorem 2.1, namely, every separable Banach space X has an α-Hölder generating compact convex retract. Moreover, we prove that the provided generating compact convex subset of X is not a Lipschitz retract whenever X does not enjoy the bounded approximation property.

1.1. **Definitions and notation.** Let (M, d_M) and (N, d_N) be two metric spaces and let $f : M \to N$ be an arbitrary mapping. We consider the modulus of continuity of f as the function $\omega_f : [0,\infty) \to [0,\infty]$ given by

$$\omega_f(t) = \sup\big\{d_N\big(f(x), f(y)\big) \ : \ d_M(x,y) \leq t\big\},$$

where the supremum is considered to be infinite whenever it does not exist. We will say that f is uniformly continuous if ω_f is continuous at $t = 0$. Given some $\alpha \in (0,1]$, we will say that f is α-Hölder whenever there exists $C > 0$ such that $\omega_f(t) \leq Ct^\alpha$ for every t. In the particular case of $\alpha = 1$ we say that f is Lipschitz.

A retraction from a metric space (M, d) onto a subset $N \subset M$ is a map $R : M \to N$ satisfying that $R(x) = x$ for every $x \in N$. The image of a retraction is called a retract. If the retraction is uniformly continuous (resp. α-Hölder or Lipschitz) then we say that its image is a uniform (resp. α-Hölder or Lipschitz) retract. We say that N is an absolute uniform (resp. α-Hölder or Lipschitz) retract if it is a uniform (resp. α-Hölder or Lipschitz) retract of every metric space containing it.

We will refer as a net of a metric space (M, d) to any subset $N \subset M$ such that there is $a, b \in \mathbb{R}^+$ satisfying the next two properties:

- N is a-separated, that is, $d(x,y) \geq a$ for every $x \neq y \in N$.
- N is b-dense, that is, for every $x \in M$ there is $y \in N$ such that $d(x,y) \leq b$.

In this case we say that N is an (a,b)-net of M. For every $\varepsilon > 0$ there is an $(\varepsilon,\varepsilon)$-net of M. In fact, it is enough to take a maximal ε-separated subset of M.

If X is a Banach space, we say that a subset S of X is a generating subset whenever the closed linear span of S is X. We denote the closed linear span as $[S]$. A fundamental sequence of X is any generating countable subset of X.

For the remaining general concepts and results of Banach space theory we refer to [Fab+11]. Throughout the entire note we will follow the terminology and notation used in [Fab+11]. For the background on Lipschitz and uniformly continuous retractions we refer the reader to the first two chapters of the authoritative monograph [BL00].

2. Hölder retractions onto compact sets

In this section we will prove the following result.

Theorem 2.1. *For every separable Banach space X and every $\alpha \in (0,1)$ there is a generating compact convex subset K of X which is an α-Hölder retract of X.*

In order to keep a clean notation, if $(x_n)_{n\in\mathbb{N}}$ is any sequence of a Banach space, we agree that $\sum_{i=n}^{m} x_i = 0$ and $[x_i]_{i=n}^{m} = \{0\}$ whenever $n > m$.

2.1. **The compact convex set.** Let us first focus on the compact set that will be the target of our retraction. The following concept was previously considered in [HM23a] and [HM23b] and has proved to be very useful to measure the 'asymptotic size' of a compact subset of a Banach space.

Definition 2.2. Given a compact subset K of a Banach space X with $0 \in K$ and a fundamental sequence $\beta = (e_n)$ of $[K]$, we define the sequence of heights $(h_n^\beta)_{n\in\mathbb{N}\cup\{0\}}$ of K relative to β as

$$h_n^\beta = \sup\{d(x, K \cap [e_i]_{i=1}^n) \ : \ x \in K\}.$$

Notice that the Banach space X plays no role in Definition 2.2.

Let us mention now that since K is compact it holds that $\lim_n h_n^\beta = 0$ for every choice of β. We will say that K is (r_n)-flat for some null sequence of positive real numbers (r_n) whenever there exists a fundamental sequence β of $[K]$ such that $h_n^\beta \leq r_n$ for every $n \in \mathbb{N}\cup\{0\}$. We will say that K is Flat if it is $(3+\varepsilon)^{-n}$-flat for some $\varepsilon > 0$. See [HM23b] for some results on Flat sets, although our definition here is slightly stronger than the one given in [HM23b].

For those readers familiarised with the concept of entropy of a metric space M, $m(\varepsilon)$ (logarithm of the size of a smallest ε-net of M), let us briefly mention a connection between this concept and Definition 2.2. If K and $\beta = (e_n)$ are like in Definition 2.2 and we denote $m(\varepsilon)$ and $m_n(\varepsilon)$ the metric entropy of K and $K \cap [e_i]_{i=1}^n$ respectively, then

$$m(2h_n^\beta) \leq m_n(h_n^\beta).$$

Indeed, if N is a h_n^β-net of $K \cap [e_i]_{i=1}^n$ then clearly $m(2h_n^\beta) \leq \log(\#N)$.

Example 2.3. For every decreasing null sequence of positive real numbers (r_n) and every separable Banach space X, there is an (r_n)-flat generating compact convex subset of X. In fact, consider $(e_n) \subset S_X$ a fundamental sequence of X and let $E_n = [e_i]_{i=1}^n$ for every $n \in \mathbb{N} \cup \{0\}$. Then, we claim that both

$$K_1 = \overline{\bigcup_{n\in\mathbb{N}} \sum_{k=1}^{n} 2^{-k}[-r_{k-1}, r_{k-1}]e_k} \quad , \quad K_2 = \overline{\text{co}}\Big(\bigcup_{n\in\mathbb{N}} [-r_{n-1}, r_{n-1}]e_n \Big)$$

are (r_n)-flat respect to the fundamental sequence $\beta = (e_n)$.

Proof. We prove the result for K_1, the proof for K_2 is essentially the same. Let us take $x = \sum_{k=1}^{m} \lambda_{k-1}e_k$ for some $m \in \mathbb{N}$ where $|\lambda_{k-1}| \leq 2^{-k}r_{k-1}$ for $k = 1, \ldots, m$. Let $n \in \mathbb{N} \cup \{0\}$ be arbitrary. If $n \geq m$ then $x \in K_1 \cap E_n$ and so $d(x, K_1 \cap E_n) = 0 < r_n$. Otherwise,

$$\begin{aligned} d(x, K_1 \cap E_n) &\leq \Big|\Big| x - \sum_{k=1}^{n} \lambda_{k-1}e_k \Big|\Big| = \Big|\Big| \sum_{k=n+1}^{m} \lambda_{k-1}e_k \Big|\Big| \leq \sum_{k=n}^{m-1} |\lambda_k| \\ &\leq \sum_{k=n}^{m-1} 2^{-k-1}r_k \leq r_n \sum_{k=n}^{m-1} 2^{-k-1} \leq r_n. \end{aligned}$$

Hence, $d(x, K_1 \cap E_n) \leq r_n$ for every $n \in \mathbb{N} \cup \{0\}$ and $x \in K_1$. Therefore,

$$h_n^\beta = \sup\{d(x, K_1 \cap [e_i]_{i=1}^n) \ : \ x \in K\} \leq r_n \quad \forall n \in \mathbb{N} \cup \{0\}$$

and we are done. ∎

Let us point out that under the same circumstances provided in Example 2.3 we can always construct a compact and convex subset of X which is not Flat. Indeed, it easily follows from the next proposition.

Proposition 2.4. *Let (r_n), X and $\beta = (e_n)$ be as in Example 2.3. If $K = \overline{\text{co}}\big(\bigcup_{n\in\mathbb{N}} [-r_{n-1}, r_{n-1}]e_n \big)$ then for every $n \in \mathbb{N}$,*

$$h_n^\beta \geq \begin{cases} h_{n+1}^\beta & \text{if } e_{n+1} \in [e_i]_{i=1}^n, \\ \frac{r_n}{1+||P_n||} & \text{otherwise,} \end{cases}$$

where $P_n : [e_i]_{i=1}^{n+1} \to [e_i]_{i=1}^n$ is the canonical projection given by $P_n(\sum_{i=1}^{n+1} \mu_i e_i) = \sum_{i=1}^{n} \mu_i e_i$ for every $\sum_{i=1}^{n+1} \mu_i e_i \in [e_i]_{i=1}^{n+1}$.

Proof. If $e_{n+1} \in [e_i]_{i=1}^n$ then for every $x \in K$ we have that $d(x, K \cap [e_i]_{i=1}^n) = d(x, K \cap [e_i]_{i=1}^{n+1})$ and therefore $h_n^\beta = h_{n+1}^\beta$. Otherwise, if we consider $x = r_n e_{n+1} \in K$ and $y \in [e_i]_{i=1}^n$ we clearly have that $P_n(x) = 0$ and $P_n(y) = y$. Hence, $||y|| = ||P_n(x) - P_n(y)|| \leq ||P_n||\,||x - y||$ and by the triangle inequality

$$||x - y|| \geq \frac{||x||}{1 + ||P_n||} = \frac{r_n}{1 + ||P_n||}.$$

Since $y \in [e_i]_{i=1}^n$ was arbitrary, we conclude that

$$h_n^\beta \geq d(x, K \cap [e_i]_{i=1}^n) \geq \frac{r_n}{1 + ||P_n||}.$$

■

From now on during this section K is an arbitrary but fixed (r_n)-flat convex subset of a Banach space X where $(r_n) \subset \mathbb{R}^+$ is an arbitrary decreasing null sequence. Let us consider $\beta = (e_n)$ the fundamental sequence of $[K]$ satisfying that $h_n^\beta \leq r_n$ for every $n \in \mathbb{N}$. We will denote $E_n = [e_i]_{i=1}^n$ for $n \in \mathbb{N}$. Given $\varepsilon > 0$ we define here

$$n(\varepsilon) = \min\{n \in \mathbb{N} \cup \{0\} \ : \ r_n \leq \varepsilon\}.$$

Now, for every $\varepsilon > 0$ we consider an $(\varepsilon, \varepsilon)$-net $N_\varepsilon = (x_i^\varepsilon)_{i=1}^{m_\varepsilon}$ of $K \cap E_{n(\varepsilon)}$. Clearly, N_ε is a $(\varepsilon, 2\varepsilon)$-net of K for every $\varepsilon > 0$.

In the next subsections we are going to construct a retraction from X onto K.

2.2. **Whitney-type partitions of unity.** In this subsection we are going to follow the construction of a partition of unity in $X \setminus K$ given in [BDMS21], which follow the path traced by Whitney in [Whi34] and was intended to provide a simpler proof of a celebrated result of Lee and Naor [LN05]. Let us recall that their construction is done for doubling metric spaces which may not be the case of K. This partition of unity will be crucial later for the definition of the retraction.

We set $\varepsilon_n = 2^{-n}$ for every $n \in \mathbb{Z}$ and rename N_{ε_n} as N_n relabeling its elements as $x_i^n := x_i^{\varepsilon_n}$ for every $i \in \{1, \dots, m_{\varepsilon_n} = m_n\}$. Then, let us consider for each $n \in \mathbb{Z}$ and $i \leq m_n$ the sets

$$\widetilde{V}_i^n = \{x \in X \ : \ \varepsilon_n \leq d(x, K) < \varepsilon_{n-1} \ , \ d(x, x_i^n) = \min_j d(x, x_j^n)\}$$

and

$$V_i^n = \{x \in X \ : \ d(x, \widetilde{V}_i^n) \leq \varepsilon_{n+1}\}.$$

Lemma 2.5. *The family $\mathcal{F} = \{V_i^n \ : \ n \in \mathbb{Z}, \ i \in \{1, \dots, m_n\}\}$ is a locally finite cover of $X \setminus K$. Moreover, if $x \in V_i^n \in \mathcal{F}$ then,*

(1) $d(x, K)/5 \leq ||x - x_i^n|| \leq 9d(x, K)$.
(2) $\#\{V \in \mathcal{F} \ : \ x \in V\} \leq 5 \cdot 20^{n(d(x,K)/10)}$.
(3) $d(x, K)/4 \leq \max_{V \in F} d(x, V^c) \leq d(x, K)$.

Proof. Let us take $x \in V_i^n$. The computations proving (1) and (3) are already done in [BDMS21] but we include them here for the sake of completeness. We start proving (1). An easy application of the triangular inequality yields

$$d(x, K) \leq d(x, \widetilde{V}_i^n) + \sup_{y \in \widetilde{V}_i^n} d(y, K) \leq \varepsilon_{n+1} + \varepsilon_{n-1} = 5\varepsilon_{n+1}. \tag{2.1}$$

Now, take $\delta > 0$, $y \in \widetilde{V}_i^n$ and $k \in K$ such that $||x-y|| \leq \varepsilon_{n+1} + \delta$ and $d(y,K) \geq ||y-k|| - \delta$. Consider also $j \leq m_n$ such that $||k - x_j^n|| \leq 2\varepsilon_n = \varepsilon_{n-1}$. Then,

$$||x - x_i^n|| \geq d(x,K) \geq d(y,K) - ||x-y|| \geq \varepsilon_n - \varepsilon_{n+1} - \delta = \varepsilon_{n+1} - \delta \tag{2.2}$$

and

$$||y - x_i^n|| \leq ||y - x_j^n|| \leq ||y-k|| + ||k - x_j^n|| \leq d(y,K) + \delta + \varepsilon_{n-1} \leq 2\varepsilon_{n-1} + \delta.$$

Therefore,

$$||x - x_i^n|| \leq ||x-y|| + ||y - x_i^n|| \leq \varepsilon_{n+1} + 2\varepsilon_{n-1} + 2\delta = 9\varepsilon_{n+1} + 2\delta. \tag{2.3}$$

By (2.2), $\varepsilon_{n+1} \leq d(x,K) + \delta$ so the previous inequality yields

$$||x - x_i^n|| \leq 9d(x,K) + 11\delta. \tag{2.4}$$

Since $\delta > 0$ was arbitrary (1) follows from (2.1), (2.2) and (2.4). In particular from (2.3) the sharper estimate $||x - x_i^n|| \leq 9\varepsilon_{n+1}$ is deduced. We will use this last estimate to prove (2). Let us call $A_k = \{j \in \mathbb{N} \,:\, j \leq m_k \,,\, x \in V_j^k\}$ for every $k \in \mathbb{Z}$. We will first obtain an upper bound for $\#A_k$. Let us fix a pair j,k such that $x \in V_j^k$. If $x \in V_l^k$ for some other $l \leq m_k$ then

$$||x_j^k - x_l^k|| \leq ||x_j^k - x|| + ||x_l^k - x|| \leq 18\varepsilon_{k+1} = 9\varepsilon_k.$$

In particular, if for each $y \in E_{n(\varepsilon_k)}$ and $r \in \mathbb{R}^+$ we denote the ball in $E_{n(\varepsilon_k)}$ of radius r centered at y as $B(y,r)$, then

$$B(x_l^k, \varepsilon_k/2) \subset B(x_j^k, 10\varepsilon_k),$$

for every $l \leq m_k$ such that $x \in V_l^k$. In other words,

$$A_k \subset \{l \in \mathbb{N} \,:\, l \leq m_k,\ B(x_l^k, \varepsilon_k/2) \subset B(x_j^k, 10\varepsilon_k)\} =: B_k.$$

Now, if λ stands for the Lebesgue measure in $E_{n(\varepsilon_k)}$ then by the change of variables theorem,

$$\lambda\big(B(x_{j_1}^k, 10\varepsilon_k)\big) = 20^{n(\varepsilon_k)} \lambda\big(B(x_{j_2}^k, \varepsilon_k/2)\big) \quad \forall j_1, j_2 \in B_k. \tag{2.5}$$

Also, since N_k is ε_k-separated we have that

$$\lambda\big(B(x_{j_1}^k, \varepsilon_k/2) \cap B(x_{j_2}^k, \varepsilon_k/2)\big) = 0 \quad \forall j_1, j_2 \in B_k. \tag{2.6}$$

From (2.5) and (2.6) we deduce that

$$\begin{aligned}\lambda\big(B(x_j^k, 10\varepsilon_k)\big) &\geq \lambda\Big(\bigcup_{l \in B_k} B(x_l^k, \varepsilon_k/2)\Big) \\ &= \sum_{l \in B_k} \lambda\big(B(x_l^k, \varepsilon_k/2)\big) = \#B_k \frac{\lambda\big(B(x_j^k, 10\varepsilon_k)\big)}{20^{n(\varepsilon_k)}}.\end{aligned}$$

Multiplying by $\frac{20^{n(\varepsilon_k)}}{\lambda\left(B(x_j^k, 10\varepsilon_k)\right)}$ on both sides of the previous inequality we get $\#B_k \leq 20^{n(\varepsilon_k)}$ so that we ultimately obtain

$$\#A_k \leq 20^{n(\varepsilon_k)}.$$

Now, if j,k are such that $x \in V_j^k$ then clearly $|k-n| \leq 2$. Hence,

$$\#\{V \in \mathcal{F} \,:\, x \in V\} \leq \sum_{\substack{k\in\mathbb{N}\\ |k-n|\leq 2}} \#A_k \leq 5 \cdot 20^{n(\varepsilon_{n+2})}.$$

The inequality (2) follows now from the fact that $n(\varepsilon)$ is a decreasing function and that by (2.1), $\varepsilon_{n+2} = \varepsilon_{n+1}/2 \geq d(x,K)/10$. Finally, we pass to the proof of (3). The inequality $\max_{V\in\mathcal{F}} d(x,V^c) \leq d(x,K)$ is true since $K \subset V^c$ for all $V \in \mathcal{F}$. To prove the other inequality, let us take j,k such that $x \in \widetilde{V}_j^k$. Then $d(x,(V_j^k)^c) \geq \varepsilon_{k+1}$ and thus

$$d(x,K)/4 < \varepsilon_{k-1}/4 = \varepsilon_{k+1} \leq d(x,(V_j^k)^c) \leq \max_{V\in\mathcal{F}} d(x,V^c).$$

■

Once we have defined the locally finite cover of $X \setminus K$, it is time to define the associated functions that will form our partition of unity. As an additional property, we are going to prove that these functions are $L(\varepsilon)$-Lipschitz when restricted to $\{x \in X \,:\, d(x,K) \geq \varepsilon\}$ for every $\varepsilon > 0$. The difference between our case and the case of doubling metric spaces arises when considering small values of ε since we will see that $\sup_{\varepsilon>0} L(\varepsilon) = \infty$.

The partition of unity is now defined in a very natural way. For every $n \in \mathbb{Z}$, $i \in \{1,\dots,m_n\}$ and $x \in X \setminus K$, we set

$$\varphi_i^n(x) = \frac{d(x,(V_i^n)^c)}{\sum\limits_{k,j} d(x,(V_j^k)^c)}.$$

It is clear that the pair $\big(\mathcal{F},(\varphi_i^n)_{i,n}\big)$ is a partition of unity. Let us quantify its Lipschitz behaviour.

Lemma 2.6. *For every $x,y \in X \setminus K$,*

$$\sum_{i,n} |\varphi_i^n(x) - \varphi_i^n(y)| \leq \frac{40\big(20^{n(d(x,k)/10)} + 20^{n(d(y,k)/10)}\big)}{\max\{d(x,K),d(y,K)\}}||x-y||.$$

Proof. For aesthetic purposes, we rename and relabel the family of functions $\{d(x,(V_i^n)^c)\}_{\substack{n\in\mathbb{Z}\\ i\in\{1,\dots,m_n\}}}$ as $\{d_i(x)\}_{i\in\mathbb{N}}$. Clearly, $|d_i(x) - d_i(y)| \leq ||x-y||$ for

every $i \in \mathbb{N}$. Hence, we compute

$$\begin{aligned}\left|\frac{d_i(x)}{\sum\limits_j d_j(x)} - \frac{d_i(y)}{\sum\limits_j d_j(y)}\right| \leq & \left|\frac{d_i(x)}{\sum\limits_j d_j(x)} - \frac{d_i(y)}{\sum\limits_j d_j(x)}\right| + \left|\frac{d_i(y)}{\sum\limits_j d_j(x)} - \frac{d_i(y)}{\sum\limits_j d_j(y)}\right| \\ = & \frac{|d_i(x) - d_i(y)|}{\sum\limits_j d_j(x)} + d_i(y)\frac{\Big|\sum\limits_j d_j(x) - d_j(y)\Big|}{\Big(\sum\limits_j d_j(x)\Big)\Big(\sum\limits_j d_j(y)\Big)} \\ \leq & \left(\frac{1}{\sum\limits_j d_j(x)} + \frac{d_i(y)\#\{V \in \mathcal{F} \,:\, \{x,y\} \cap V \neq \emptyset\}}{\Big(\sum\limits_j d_j(x)\Big)\Big(\sum\limits_j d_j(y)\Big)}\right)||x-y||.\end{aligned}$$

Taking into account the previous inequality and the fact that $\varphi_i^n(x) - \varphi_i^n(y) = 0$ if $\{x,y\} \cap V_i^n = \emptyset$ we deduce that

$$\sum_{i,n} |\varphi_i^n(x) - \varphi_i^n(y)| \leq \frac{2\#\{V \in \mathcal{F} \,:\, \{x,y\} \cap V \neq \emptyset\}}{\sum\limits_j d_j(x)}||x-y||.$$

Clearly, by (2) of Lemma 2.5,

$$\begin{aligned}\#\{V \in \mathcal{F} \,:\, \{x,y\} \cap V \neq \emptyset\} \leq & \#\{V \in \mathcal{F} \,:\, x \in V\} + \#\{V \in \mathcal{F} \,:\, y \in V\} \\ \leq & 5 \cdot \big(20^{n(d(x,K)/10)} + 20^{n(d(y,K)/10)}\big),\end{aligned}$$

and by (3) of Lemma 2.5,

$$\sum_j d_j(x) \geq \max_{V \in \mathcal{F}} d(x, V^c) \geq d(x,K)/4.$$

Hence,

$$\sum_{i,n} |\varphi_i^n(x) - \varphi_i^n(y)| \leq \frac{40\big(20^{n(d(x,k)/10)} + 20^{n(d(y,k)/10)}\big)}{d(x,K)}||x-y||.$$

Interchanging the roles of x and y shows that

$$\sum_{i,n} |\varphi_i^n(x) - \varphi_i^n(y)| \leq \frac{40\big(20^{n(d(x,k)/10)} + 20^{n(d(y,k)/10)}\big)}{d(y,K)}||x-y||,$$

which finishes the proof. ∎

2.3. **The retraction.** Now, we are ready to define the retraction onto K. Let us consider the map $R : X \to K$ given by

$$R(x) = \begin{cases} \sum\limits_{i,n} \varphi_i^n(x) x_i^n & \text{if } x \in X \setminus K, \\ x & \text{if } x \in K. \end{cases}$$

Clearly, if $x \in X \setminus K$ then by (1) of Lemma 2.5,

$$||R(x) - x|| \leq \sum_{i,n} \varphi_i^n(x)||x_i^n - x|| \leq 9d(x, K).$$

Hence, we have the following Fact 2.1.

Fact 2.1. *For every $x \in X$,*

$$||R(x) - x|| \leq 9d(x, K).$$

Proposition 2.7. *For every $x, y \in X \setminus K$,*

$$||R(x) - R(y)|| \leq 760\big(20^{n(d(x,K)/10)} + 20^{n(d(y,K)/10)}\big)||x - y||.$$

Proof. Let us distinguish two different cases. The first case is when $||x-y|| \geq \max\{d(x, K), d(y, K)\}$. In this case, from Fact 2.1 we deduce that

$$||R(x) - x|| \leq 9d(x, K) \leq 9 \max\{d(x, K), d(y, K)\} \leq 9||x - y||.$$

Similarly, $||R(y) - y|| \leq 9||x - y||$ so that

$$||R(x) - R(y)|| \leq ||R(x) - x|| + ||R(y) - y|| + ||x - y|| \leq 19||x - y||.$$

Now, we may assume that $||x - y|| \leq \max\{d(x, K), d(y, K)\}$. In this case we fix i_0, n_0 such that $x \in V_{i_0}^{n_0}$. If $x \in V_i^n$ for some other pair i, n then by (1) in Lemma 2.5 we obtain

$$||x_i^n - x_{i_0}^{n_0}|| \leq ||x_i^n - x|| + ||x_{i_0}^{n_0} - x|| \leq 18 \max\{d(x, K), d(y, K)\}.$$

On the other hand, if the pair i, n is such that $y \in V_i^n$ then

$$||x_i^n - x_{i_0}^{n_0}|| \leq ||x_i^n - y|| + ||x_{i_0}^{n_0} - x|| + ||x - y|| \leq 19 \max\{d(x, K), d(y, K)\}.$$

Hence, by Lemma 2.6,

$$\begin{aligned} ||R(x) - R(y)|| =& \Big|\Big| \sum_{i,n} \big(\varphi_i^n(x) - \varphi_i^n(y)\big)(x_i^n - x_{i_0}^{n_0})\Big|\Big| \\ &\leq \sum_{i,n} |\varphi_i^n(x) - \varphi_i^n(y)| \cdot 19 \max\{d(x, K), d(y, K)\} \\ &\leq 760\big(20^{n(d(x,k)/10)} + 20^{n(d(y,k)/10)}\big)||x - y||. \end{aligned}$$

■

This result is showing how the retraction looses its Lipschitz constant when the points get close to K. In fact, the slower $n(d(x, K)/10)$ grows to infinity when $d(x, K)$ tends to zero, the slower the Lipschitz constant is lost when the points are approaching K. The good news are that the sequence (r_n) controls how fast $n(d(x, K)/10)$ grows. This means that the sequence (r_n) is monitoring how fast the Lipschitz constant is lost.

In order to present the latter in a precise way, we compute an upper bound for the modulus of continuity of R.

Theorem 2.8. *For every $t \in \mathbb{R}^+$,*

$$\omega_R(t) \leq 1520 \cdot 20^{n(t/20)}t.$$

Proof. Let us take $y \in X$. We are going to distinguish between several different cases.

Case 1. The case when $x \in K$. Clearly $||x-y|| \geq d(y,K)$. Therefore, by Fact (2.1),

$$\begin{aligned}||R(x)-R(y)|| \leq & ||R(x)-x|| + ||R(y)-y|| + ||x-y|| \\ \leq & 9d(y,K) + ||x-y|| \leq 10||x-y||.\end{aligned}$$

Case 2. The case when $x \in X \setminus K$ and $d(y,K) < d(x,K)/2$. By the triangle inequality $||x-y|| \geq d(x,K) - d(y,K) \geq d(x,K)/2$. Hence, using again Fact (2.1),

$$\begin{aligned}||R(x)-R(y)|| \leq & ||R(x)-x|| + ||R(y)-y|| + ||x-y|| \\ \leq & 9(d(x,K)+d(y,K)) + ||x-y|| \leq 28||x-y||.\end{aligned}$$

Case 3. The case when $x \in X \setminus K$, $d(y,K) \geq d(x,K)/2$ and $||x-y|| \geq d(x,K)$. Here, $d(y,K) \leq d(x,K) + ||x-y|| \leq 2||x-y||$. Hence,

$$\begin{aligned}||R(x)-R(y)|| \leq & ||R(x)-x|| + ||R(y)-y|| + ||x-y|| \\ \leq & 9(d(x,K)+d(y,K)) + ||x-y|| \leq 55||x-y||.\end{aligned}$$

Case 4. The case when $x \in X \setminus K$, $d(y,K) \geq d(x,K)/2$ and $||x-y|| < d(x,K)$. By Proposition 2.7,

$$\begin{aligned}||R(x)-R(y)|| \leq & 760\big(20^{n(d(x,K)/10)} + 20^{n(d(y,K)/10)}\big)||x-y|| \\ \leq & 1520 \cdot 20^{n(d(x,K)/20)}||x-y|| \leq 1520 \cdot 20^{n(||x-y||/20)}||x-y||.\end{aligned}$$

■

Remark 2.9. We did not strive during this section to obtain the best estimate for $\omega_R(t)$ in Theorem 2.8. Very probably sharper estimates can be derived.

Since all the previous constructions and results are proven for an arbitrary (r_n)-flat convex subset of an arbitrary Banach space, we may state and prove the following theorem.

Theorem 2.10. *If $\alpha \in (0,1)$ then every $\big(20^{\frac{n}{\alpha-1}}\big)$-flat convex subset of a Banach space X is an absolute α-Hölder retract.*

Proof. Since every metric space is isometrically embeddable into a Banach space, it is enough to find an α-Hölder retraction from X onto K. By Theorem 2.8, it is enough to show that for $r_n = 20^{\frac{n}{\alpha-1}}$ there is a constant $C > 0$ such that for every $t > 0$,

$$n(t/20) \leq \log_{20}(t^{\alpha-1}) + C.$$

It is immediate that in this case,

$$n(t) \leq \min\{n \in \mathbb{N} \,:\, n \geq \log_{20}(t^{\alpha-1})\} \leq \log_{20}(t^{\alpha-1}) + 1.$$

Therefore,

$$n(t/20) \leq \log_{20}\big((t/20)^{\alpha-1}\big) + 1 = \log_{20}(t^{\alpha-1}) + 2 - \alpha.$$

■

Theorem 2.1 follows now as a consequence of Theorem 2.10 and Example 2.3. We state it here in the form of a corollary.

Corollary 2.11. *For every separable Banach space X and every $\alpha \in (0,1)$ there is a generating compact convex set K of X which is an α-Hölder retract of X. Moreover, K can be chosen to be Flat.*

This result is to some extent optimal and underlines the differences between Lipschitz and Hölder retractions. More precisely,

Corollary 2.12. *For every separable Banach space X without the bounded approximation property and for every $\alpha \in (0,1)$ there is a generating compact convex subset of X which is an α-Hölder retract of X but not a Lipschitz retract of X.*

Proof. From [HM23b] we know that a separable Banach space admitting a Lipschitz retraction onto a Flat generating convex subset has the bounded approximation property. Hence, if X does not enjoy the bounded approximation property then it is enough to consider the Flat generating compact convex subset given by Corollary 2.11. ■

Let us remind that by a famous example of Enflo [Enf73] we know of the existence of separable Banach spaces without the bounded approximation property.

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Chapter 7

Schauder basis in Lipschitz free spaces over nets in Banach spaces

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SCHAUDER BASES IN LIPSCHITZ FREE SPACES OVER NETS IN BANACH SPACES

PETR HÁJEK AND RUBÉN MEDINA

ABSTRACT. In the present note we give two explicit constructions (based on a retractional argument) of a Schauder basis for the Lipschitz free space $\mathcal{F}(N)$, over certain uniformly discrete metric spaces N. The first one applies to every net N in a finite dimensional Banach space, leading to the basis constant independent of the dimension. The second one applies to grids in Banach spaces with an FDD.

As a corollary, we obtain a retractional Schauder basis for the Lipschitz free space $\mathcal{F}(N)$ over a net N in every Banach space X with a Schauder basis containing a copy of c_0, as well as in every Banach space with a c_0-like FDD.

1. INTRODUCTION

The structural properties of metric spaces, in terms of the various extensional or retractional properties of their subsets and Lipschitz functions thereon, are a very active field of research with many applications in surrounding areas of Banach space geometry, theoretical computer science, geometric group theory and so on. A convenient conceptual framework in this respect is provided by the notion of a Lipschitz free space $\mathcal{F}(M)$ over a metric space M (also known as the Arens-Eells space). It is a Banach space containing an isometric copy of the original metric space M, which allows the natural linearization of Lipschitz mappings between metric spaces. The structural properties of Lipschitz free spaces $\mathcal{F}(M)$ are still very far from being understood, except for some very special cases of M such as e.g. the doubling metric spaces, or (separable) uniformly discrete spaces M. In the former case it was shown by Lancien and Pernecká [LP13] that $\mathcal{F}(M)$ has a λ-BAP for any $M \subset \mathbb{R}^n$, where λ is of the order $\sqrt{n}$ and independent of M. It remains an important open problem whether λ can be chosen a constant independent of n (see e.g. [PS15]).

In the latter case, Kalton [Kal04] has shown that $\mathcal{F}(M)$ is a Schur space with the RNP property and the approximation property (AP for short).

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Kalton has posed in [Kal12] an important question whether these spaces have the bounded approximation property (BAP) as well. In fact, in [Kal12] it is shown that in order to solve this problem it suffices to consider the special case when M is a net in a Banach space X. A positive answer in the separable case would imply that every separable Banach space is approximable in the sense defined in [Kal12] while a counterexample would yield a renorming of ℓ_1 without the metric approximation property (MAP), answering a classical open question.

Kalton went on proving that for a Banach space X with a separable dual (or a separable dual space itself) the answer is positive. So for every net N in such X the Lipschitz free space $\mathcal{F}(N)$ has the BAP. The case of a general separable Banach space X still remains open. Let us add the remark that nets in metric spaces are quite useful (e.g. in theoretical computer science) and their study is an ongoing topic in the last years (see [HN17], [Dil+08], [Kal12]).

Our note contains two main results which considerably strengthen the above mentioned results in some special cases.

We construct an explicit family of Lipschitz retractions for nets in finite dimensional Banach spaces, which yields a Schauder basis for their Lipschitz free spaces whose constant is independent of the dimension. Based on this ingredient, we proceed with the construction of an explicit family of Lipschitz retractions (again leading to a Schauder basis for the free space) for grids in Banach spaces with an FDD. Our construction can be applied to obtain a retractional Schauder basis for $\mathcal{F}(N)$, where N is a net in several types of Banach spaces X. In particular, X can be chosen to be a separable Banach space with a Schauder basis, which contains a copy of c_0, or a Banach space admitting a c_0-like FDD. Let us now proceed with the necessary background.

Given $a, b \in \mathbb{R}^+$ we will say that a subset N of a metric space M is an (a, b)-net of M whenever it is a-separated and b-dense. More precisely, N is considered to be a-separated whenever $d(x, y) \geq a$ for every $x, y \in N$, $x \neq y$, and b-dense if for every $x \in M$ there is $y \in N$ such that $d(x, y) \leq b$. We will say that $N \subset M$ is a net whenever there are $a, b \in \mathbb{R}^+$ such that N is an (a, b)-net. It is worth to mention here that in the case N is an (a, b)-net of a Banach space then $a \leq 2b$.

A map T from a metric space M into another metric space N is said to be Lipschitz if there exists some $\lambda > 0$ such that

$$d(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in M.$$

We say that λ is a Lipschitz constant for T and we call the infimum of all Lipschitz constants for T the Lipschitz norm of T, that is,

$$||T||_{\mathrm{Lip}} = \sup_{x,y \in M, x \neq y} \frac{d\big(T(x), T(y)\big)}{d(x, y)}.$$

If $\lambda > 0$ is a Lipschitz constant for T then we say that T is λ-Lipschitz.

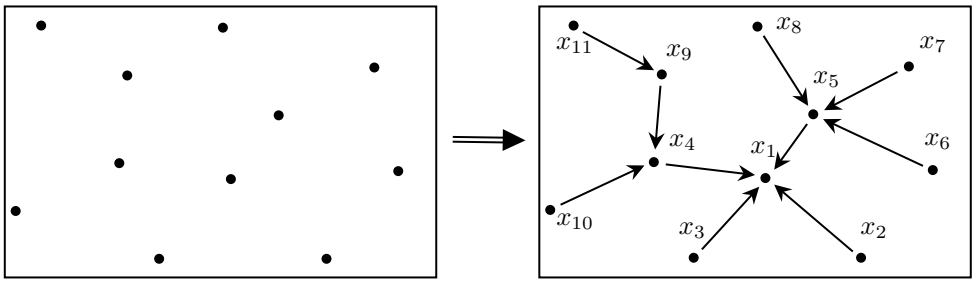

FIGURE 1. Retractional basis in a net. This image is originally published in [HM22].

Definition 1.1. Let N_1 and N_2 be metric spaces. We say that N_1 and N_2 are Lipschitz equivalent whenever there is a bijection $F : N_1 \to N_2$ such that both F and F^{-1} are Lipschitz. We call $\|F\|_{\mathrm{Lip}}\|F^{-1}\|_{\mathrm{Lip}}$ the Lipschitz distortion of F.

In this note, we are mainly interested in finding a retractional structure in nets. To this end, we are going to order the elements of the net into a sequence and introduce K-Lipschitz retractions onto the first n points of the sequence with a commuting behaviour, as shown in Figure 1. More precisely,

Definition 1.2. Let M be a countable metric space and $K \geq 1$. A K-retractional basis of M is a sequence of retractions $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n : M \to M$ satisfying,

(1) $\varphi_n(M) = M_n := \bigcup_{j=1}^{n}\{x_j\}$,
(2) $\bigcup_{j=1}^{\infty}\{x_j\} = M$,
(3) φ_n is K-Lipschitz for every $n \in \mathbb{N}$,
(4) $\varphi_m \circ \varphi_n = \varphi_{\min(m,n)}$ for every $m, n \in \mathbb{N}$.

We will say that M has a retractional basis whenever it has a K-retractional basis for some $K \geq 1$.

It is clear that these retractions can be used to extend Lipschitz maps from a finite number of points $\{x_1, \ldots, x_n\}$ to the whole metric space by composition, keeping the Lipschitz constant under control independently of the number of points from which the extension is made. By contrast, as shown by Naor and Rabani [NR17] for every $n \in \mathbb{N}$ there is an n-point subset S of a metric space M and a 1-Lipschitz function f from S into a certain Banach space Z such that the Lipschitz constant of every extension of f to the domain M is bounded below by a multiple of $\sqrt{\log n}$.

The existence of retractional bases has been studied previously (see [HN17] and [Nov20]). Let us recall that by [BK98] nets of a finite dimenisional space are not necessarily Lipschitz equivalent, so the problem is non-trivial even in the case of a fixed normed space (and varying nets).

Retractional bases in countable metric spaces are clearly preserved under Lipschitz equivalences. More precisely,

Proposition 1.3. *Let N_1 and N_2 be countable Lipschitz equivalent metric spaces, with distortion D. If N_1 has a K-retractional basis for some $K > 0$ then N_2 has a DK-retractional basis.*

A Schauder basis for a real Banach space X is a sequence $(x_n) \subset X$ with the property that for every $x \in X$, there exists a unique sequence $(\alpha_n) \subset \mathbb{R}$ such that

$$\left|\left|x - \sum_{i=1}^{n} \alpha_i x_i\right|\right| \xrightarrow{n\to\infty} 0.$$

If (x_n) is a Schauder basis then there is a $K > 0$ such that $\left|\left| \sum_{i=1}^{n} \alpha_i x_i\right|\right| \leq K||x||$ for every $x \in X$ and $n \in \mathbb{N}$. In this case we say that (x_n) is a K-Schauder basis. A more general concept, also used in our note is the following.

Definition 1.4. A sequence (X_n) of finite dimensional subspaces of a Banach space X is called the finite dimensional decomposition (FDD for short) if for every $x \in X$ there is a unique sequence $x_n \in X_n$ so that

$$\left|\left|x - \sum_{i=1}^{n} x_i\right|\right| \xrightarrow{n\to\infty} 0.$$

In this case we say that X has an FDD and we call the projections $P_n(x) = \sum_{i=1}^{n} x_i$, which are again uniformly bounded, the natural projections of the FDD.

A retractional basis for a net implies that the Lipschitz free space over the net has a Schauder basis, made out of the linearization of the retractions.

Theorem 1.5 ([HN17])**.** *Let M be a countable metric space with a K-retractional basis $\{\varphi_n\}_{n=1}^{\infty}$. Then the Lipschitz free space over M has a K-Schauder basis.*

In [HN17] a retractional basis in grids of spaces with an unconditional basis (Theorem 13) was constructed. In Section 3 we generalize these results to spaces with a Schauder basis using a completely different approach.

The paper is divided into two main sections. In Section 2 we tackle the problem in finite dimensional spaces. We first define a family of nets in $\mathbb{R}^n$ (we call those nets spiderwebs) such that every net of $\mathbb{R}^n$ is Lipschitz equivalent to a spiderweb with a distortion independent of n. Then, we construct a K-retractional basis for an arbitrary (a, b)-net of the family with K depending only on $\frac{b}{a}$, leading to our first main result Theorem 2.6. In Section 3 we focus on grids of spaces with an FDD. We prove our main technical result Theorem 3.9 showing that if X is a space with an FDD, then some grids of the FDD have a retractional basis (a precise definition of this kind of grids is given in Section 3). This result, combined with the coefficient quantization technique in [Dil+08] yields our second main result,

that every net of a Banach space with a Schauder basis containing a copy of c_0, or a Banach space with a c_0-like FDD has a retractional basis (Corollary 3.12).

Let us remark that the importance of the mentioned coefficient quantization approach can be gleaned from [Dil+08], [Cas+08] and [DD03]. In particular, in [DD03] Fourier series of bandlimited functions (functions whose Fourier transform has a bounded support) are approximated replacing Fourier series coefficients by quantized coefficients, obtaining a net using a two-sided version of the $\Sigma - \Delta$ quantization algorithm. It is proved in the Main Theorem of [Dil+08] that there is a Schauder basis of a space such that every grid with respect to that basis is a net if and only if the space has a Schauder basis and contains a copy of c_0.

For background on the approximation properties of Banach spaces we refer to the Handbook article [Cas01]. We will use the standard notation and terminology of the Banach spaces theory, as in [Fab+11]. For background on Lipschitz free spaces and their main properties we refer to [GK03].

2. Finite dimensional nets

Let X be a finite dimensional Banach space. Our aim in this section is to prove that every (a, b)-net of X has a retractional basis with a bound depending only on the fraction $\frac{b}{a}$ (and independent of the dimension of X). To this end, we will reduce the problem to the special case of nets described below.

Definition 2.1. Let $a, b \in \mathbb{R}^+$. We say that $N \subset X$ is a base of an (a, b)-spiderweb of X whenever there exists a sequence of nonempty subsets $S_n \subset nS_X$ with $n \in \mathbb{N} \cup \{0\}$ such that

$$\begin{cases} 2S_{2^{n-1}} \subset S_{2^n} \ \ \forall n \in \mathbb{N}, \\ S_n \text{ is } (a, b)\text{-net of } nS_X \quad \forall n \in \mathbb{N}, \\ S_m = \frac{m}{2^n} S_{2^n} \ \ \forall m \in \{2^n + 1, \dots, 2^{n+1} - 1\} \ \ \forall n \in \mathbb{N}, \end{cases}$$

where $N = \bigcup_{n=0}^{\infty} S_n$.

The existence in X of a base of an $(a, 2a)$-spiderweb for every $a \in (0, 2)$ is proved by induction, taking S_1 an (a, a)-net of S_X and finding S_{2^n} as a maximal a-separated subset of $2^n S_X$ containing $2S_{2^{n-1}}$.

Definition 2.2. Let $a, b \in \mathbb{R}^+$. We say that $S \subset X$ is an (a, b)-spiderweb of X if it is an (a, b)-net containing as a subset a base of an (a, b)-spiderweb of X.

Lemma 2.3. *Let N be an (a,b)-net of X and C be a c-separated subset of X for some $a, b, c \in \mathbb{R}^+$. Then, there exists an $(\frac{ac}{6b}, \frac{2c}{3})$-net M of X containing C and Lipschitz equivalent to N with Lipschitz distortion at most $(\frac{2b}{a} + 1)(\frac{4b}{a} + 1)$.*

Proof. Clearly, every (a,b)-net is bi-Lipschitz equivalent, with a distortion 1, to an $(\widetilde{a},\widetilde{b})$-net with $\widetilde{a}=\frac{ac}{3b}$ and $\widetilde{b}=c/3$ (multiplying the net by $\frac{c}{3b}$). Hence, we may assume that $N\subset X$ is an $(\widetilde{a},\widetilde{b})$-net for such $\widetilde{a}$ and $\widetilde{b}$. We consider $T: C\to N$ the nearest point map (which is not uniquely determined). First, let us see that T is injective. By contradiction, if $x,y\in C$ such that $T(x)=T(y)=z\in N$ then $||x-y||\leq ||x-z||+||y-z||\leq 2\widetilde{b}=2c/3$. Since C is c-separated it follows that $x=y$.

We define the set $M=C\cup(N\setminus T(C))$ and the map $F:N\to M$ as

$$F(x)=\begin{cases} x & \text{if } x\in N\setminus T(C),\\ T^{-1}(x) & \text{if } x\in T(C).\end{cases}$$

F is well-defined and it is a bijection satisfying $||F(x)-x||\leq\widetilde{b}=c/3$ for every $x\in N$. We are going to prove that M is an $(\widetilde{a}/2,2\widetilde{b})$-net. Let us first prove that M is $\widetilde{a}/2$-separated. Take $x,y\in M$. If $x,y\in N\setminus T(C)$ then obviously $||x-y||\geq\widetilde{a}>\widetilde{a}/2$ and if $x,y\in C$ then clearly $||x-y||\geq 3\widetilde{b}\geq\widetilde{a}/2$. It remains to deal with the case when $x\in N\setminus T(C)$ and $y\in C$. In this case, let us assume that $||x-y||<\widetilde{a}/2$ and find a contradiction. By definition of T it holds that $||T(y)-y||=\min_{z\in N}||z-y||\leq||x-y||<\widetilde{a}/2$. Hence, we have $||T(y)-x||\leq||T(y)-y||+||y-x||<\widetilde{a}/2+\widetilde{a}/2=\widetilde{a}$ which is impossible because N is $\widetilde{a}$-separated.

Finally, let us prove that M is $2\widetilde{b}$-dense in X. For every $x\in X$ there exists a $y\in M$ such that $||x-F^{-1}(y)||\leq\widetilde{b}$. As N is $\widetilde{b}$-dense we have that $||x-y||\leq||x-F^{-1}(y)||+||F^{-1}(y)-y||\leq\widetilde{b}+\widetilde{b}=2\widetilde{b}=2c/3$. It remains to prove that F is bi-Lipschitz and compute its distortion.

$$\begin{aligned}||F(x)-F(y)||&\leq||F(x)-x||+||F(y)-y||+||x-y||\leq 2\widetilde{b}+||x-y||\\ &\leq\left(\frac{2\widetilde{b}}{\widetilde{a}}+1\right)||x-y||\ \ \forall x,y\in N,\end{aligned}$$

and

$$\begin{aligned}||F^{-1}(x)-F^{-1}(y)||&\leq||F^{-1}(x)-x||+||F^{-1}(y)-y||+||x-y||\\ &\leq 2\widetilde{b}+||x-y||\leq\left(\frac{4\widetilde{b}}{\widetilde{a}}+1\right)||x-y||\ \ \forall x,y\in M.\end{aligned}$$

We are done since $\frac{\widetilde{b}}{\widetilde{a}}=\frac{b}{a}$. ∎

From now on we are going to deal with a fixed (a,b)-spiderweb S throughout all this section, with $a,b\in\mathbb{R}^+$ arbitrary. By assumption there is a base of an (a,b)-spiderweb $N\subset S$. We will denote $S_n=\big(nS_X\big)\cap N$ and $B_n=\big(nB_X\big)\cap S$ for every $n\in\mathbb{N}\cup\{0\}$. The set B_n is a discrete ball of radius n onto which we want to retract S. Let $\rho_n: nS_X\to\frac{n}{n-1}S_{n-1}$ be the nearest point map when $n\geq 2$ and let $\rho_1:S_X\to\{0\}$ be constant.

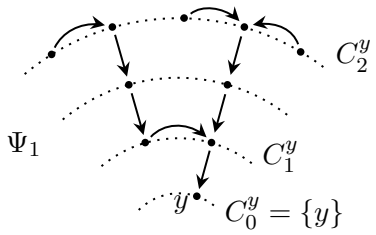

FIGURE 2. A base of a spiderweb in a 2-dimensional space. This image is originally published in [HM22].

For each $n \in \mathbb{N}$ we define the "local" retraction $\psi_n : B_n \to B_{n-1}$ as

$$\psi_n(x) = \begin{cases} \frac{n-1}{n}\rho_n\left(n\frac{x}{||x||}\right) & \text{if } x \in B_n \setminus B_{n-1}, \\ x & \text{if } x \in B_{n-1}. \end{cases}$$

Then, we define $n(x) = \min\{n \in \mathbb{N} \,:\, ||x|| \leq n\}$ for every $x \in S$. Finally, the global retractions are defined for every $n \in \mathbb{N} \cup \{0\}$ as $\Psi_n : S \to B_n$ given by

$$\Psi_n(x) = \begin{cases} \psi_{n+1} \circ \cdots \circ \psi_{n(x)}(x) & \text{if } x \in S \setminus B_n, \\ x & \text{if } x \in B_n. \end{cases}$$

See Figure 2 for an example on how Ψ works, where the picture refers to the different concentric spheres in X and the points of the base of the spiderweb only.

Before we go on with the proof of the main result of the section, we should mention some properties of the maps ψ_n and Ψ_n which will be used later.

(1) If $n \leq m \in \mathbb{N}$ then $\psi_m \circ \psi_n = \psi_n$.
(2) If $n \in \mathbb{N}$ and $x \in B_n \setminus B_{n-1}$ then $\psi_n(x) \in S_{n-1}$ and moreover, if $x \in S_n$ then

$$\psi_n(x) = \begin{cases} \frac{n-1}{n}x & \text{if } n \neq 2^m \text{ for any } m \in \mathbb{N}, \\ \frac{n-1}{n}\rho_n(x) & \text{if } n = 2^m \text{ for some } m \in \mathbb{N}. \end{cases}$$

(3) For every $n, m \in \mathbb{N} \cup \{0\}$ it holds that $\Psi_n \circ \Psi_m = \Psi_{\min\{n,m\}}$.
(4) If $n \in \mathbb{N} \cup \{0\}$ and $x \in S \setminus B_n$ then $\Psi_n(x) \in S_n$.

Theorem 2.4. *If $n \in \mathbb{N}$ then*

$$||\Psi_n(x) - R_n(x)|| \leq 6b \quad \forall x \in S, \tag{2.1}$$

where $R_n : S \to nB_X$ is the radial projection.

Proof. Before focusing on the statement of the theorem, we will show some properties of the retraction Ψ_n. Given a point $y \in S_{2^m}$ for some $m \in \mathbb{N} \cup \{0\}$, we want to locate the points from $\bigcup_{n=0}^{\infty} S_{2^n}$ that are mapped into $\{y\}$ by Ψ_{2^m}.

To this end, we first define $C_0^y = \{y\}$ and proceed inductively for $k \geq 1$, letting

$$C_k^y = \big\{x \in S_{2^{m+k}} \ : \ \rho_{2^{m+k}}(x) \in 2C_{k-1}^y\big\}.$$

One may see in Figure 2 a particular example where the sets C_k^y are related to the behaviour of ψ_{2^0} for some $y \in S_{2^0}$.

We claim that for every $k \geq 1$,

$$S_{2^{m+k}} \cap \Psi_{2^m}^{-1}(y) = C_k^y. \tag{2.2}$$

We prove (2.2) by induction in k. From property (2) it follows that for every $x \in S_{2^{m+1}}$,

$$\Psi_{2^m}(x) = \psi_{2^m+1} \circ \cdots \circ \psi_{2^{m+1}}(x) = \Big(\prod_{j=2^m+1}^{2^{m+1}} \frac{j-1}{j}\Big)\rho_{2^{m+1}}(x) = \frac{1}{2}\rho_{2^{m+1}}(x).$$

This proves the case $k = 1$. For $k \geq 2$ by inductive assumption we have that $C_{k-1}^y = S_{2^{m+k-1}} \cap \Psi_{2^m}^{-1}(y)$. Since $\Psi_{2^m} = \Psi_{2^m} \circ \Psi_{2^{m+k-1}}$, we have for $x \in S_{2^{m+k}}$,

$$x \in \Psi_{2^m}^{-1}(y) \ \Leftrightarrow \ \Psi_{2^{m+k-1}}(x) \in \Psi_{2^m}^{-1}(y) \ \Leftrightarrow \ \Psi_{2^{m+k-1}}(x) \in C_{k-1}^y.$$

So it is enough to see that $\Psi_{2^{m+k-1}}(x) \in C_{k-1}^y$ if and only if $x \in C_k^y$. Indeed,

$$\begin{aligned}\Psi_{2^{m+k-1}}(x) =& \psi_{2^{m+k-1}+1} \circ \cdots \circ \psi_{2^{m+k}}(x)\\ =& \Big(\prod_{j=2^{m+k-1}+1}^{2^{m+k}} \frac{j-1}{j}\Big)\rho_{2^{m+k}}(x) = \frac{1}{2}\rho_{2^{m+k}}(x),\end{aligned}$$

which finally proves (2.2). Denoting $C^y = \bigcup_{k=0}^{\infty} C_k^y$, the following is satisfied.

$$C^y = \Big(\bigcup_{n=1}^{\infty} S_{2^n}\Big) \cap \Psi_{2^m}^{-1}(y). \tag{2.3}$$

Our goal now is to study the properties of C^y in order to control the preimages of a point $y \in S_{2^m}$. In particular, if $F^y = \{x \in S_{||y||} \ : \ ||y - x|| \leq 2b\}$ and $D^y = [1, \infty)F^y$, we are going to show that

$$C^y \subset D^y \qquad \forall y \in \bigcup_{n=0}^{\infty} S_{2^n}. \tag{2.4}$$

To prove (2.4) we first prove the following equation for $k \in \mathbb{N} \cup \{0\}$.

$$C_{k+1}^y \subset 2C_k^y + 2bB_X \quad \forall y \in \bigcup_{n=0}^{\infty} S_{2^n}. \tag{2.5}$$

We proceed by taking the complements in (2.5) for a fixed $y \in \bigcup_{n=0}^{\infty} S_{2^n}$. If $x \in S_{||y||2^{k+1}} \setminus \big(2C_k^y + 2bB_X\big)$ then $d(x, 2C_k^y) > 2b$. As $2S_{||y||2^k}$ is $2b$-dense in $||y||2^{k+1}S_X$, there exists a $z \in 2S_{||y||2^k} \setminus 2C_y^k$ such that $||x - z|| \leq 2b$ and

therefore $\rho_{||y||2^{k+1}}(x) \notin 2C_k^y$. This means that $x \in S_{||y||2^{k+1}} \setminus C_{k+1}^y$ and (2.5) is proved.

We return to the proof of (2.4). Our goal is to prove that $C^y \subset D^y$ for $y \in \bigcup_{n\in\mathbb{N}} S_{2^n}$. To this end we are going to see that $\big(C_k^y + 2bB_X\big) \cap S_{||y||2^k} \subset 2^k F^y$ for every $k \in \mathbb{N} \cup \{0\}$. We proceed by induction in k. For $k = 0$ obviously $\big(C_0^y + 2bB_X\big) \cap S_{||y||} = F^y$. Let us consider the general case when $k+1 \in \mathbb{N}$, assuming by induction that $\big(C_k^y + 2bB_X\big) \cap S_{||y||2^k} \subset 2^k F^y$. Using (2.5) we have

$$\begin{aligned}\big(C_{k+1}^y + 2bB_X\big) \cap S_{||y||2^{k+1}} &\subset \big(2C_k^y + 4bB_X\big) \cap S_{||y||2^{k+1}} \\ &\subset 2\Big(\big(C_k^y + 2bB_X\big) \cap S_{||y||2^k}\Big) \subset 2^{k+1}F^y,\end{aligned}$$

which ends the proof of (2.4).

We are now ready to prove the statement of the theorem in two steps. First, we prove (2.1) for points in N. Let us consider $n \in \mathbb{N}$ and $x \in N \setminus B_n$ (clearly the statement of the theorem is trivial for $x \in B_n$). Then there are $k_1, k_2 \in \mathbb{N} \cup \{0\}$ with $k_1 \leq k_2$ such that

$$n \in \{2^{k_1}, \dots, 2^{k_1+1} - 1\} \quad , \quad ||x|| \in \{2^{k_2}, \dots, 2^{k_2+1} - 1\}.$$

Let $y = \Psi_{2^{k_1}}(x)$ and $z = R_{2^{k_2}}(x)$. From (2.3) and $\Psi_{2^{k_1}}(z) = \Psi_{2^{k_1}}(x) = y$ we see that $z \in C^y$. So thanks to (2.4), we may conclude that $z \in D^y$. Then, $R_{2^{k_1}}(x) = R_{2^{k_1}}(z) = R_{||y||}(z) \in R_{||y||}(D^y) = F^y$ which means that

$$||\Psi_{2^{k_1}}(x) - R_{2^{k_1}}(x)|| = ||y - R_{2^{k_1}}(x)|| \leq 2b.$$

Consequently,

$$||\Psi_n(x) - R_n(x)|| = \frac{n}{2^{k_1}}||\Psi_{2^{k_1}}(x) - R_{2^{k_1}}(x)|| < 4b,$$

so we conclude that

$$||\Psi_n(x) - R_n(x)|| \leq 4b \qquad \forall x \in N. \tag{2.6}$$

Next, we consider the general case when $x \in S \setminus B_n$. Then $n(x) \geq n$ and

$$\begin{aligned}\left|\left|R_n\left(\rho_{n(x)}\left(\frac{n(x)x}{||x||}\right)\right) - R_n(x)\right|\right| &= \left|\left|\frac{n}{n(x)}\rho_{n(x)}\left(\frac{n(x)x}{||x||}\right) - \frac{n}{n(x)}\left(\frac{n(x)x}{||x||}\right)\right|\right| \\ &= \frac{n}{n(x)}\left|\left|\rho_{n(x)}\left(\frac{n(x)x}{||x||}\right) - \frac{n(x)x}{||x||}\right|\right| \leq 2b.\end{aligned}$$

By definition we have that $\Psi_n(x) = \Psi_n\big(\rho_{n(x)}\big(\frac{n(x)x}{||x||}\big)\big)$ where $\rho_{n(x)}\big(\frac{n(x)x}{||x||}\big) \in N$. So taking into account (2.6) and the previous inequality we have that

$$\begin{aligned}||\Psi_n(x) - R_n(x)|| \leq &\left|\left|\Psi_n\left(\rho_{n(x)}\left(\frac{n(x)x}{||x||}\right)\right) - R_n\left(\rho_{n(x)}\left(\frac{n(x)x}{||x||}\right)\right)\right|\right| \\ &+ \left|\left|R_n\left(\rho_{n(x)}\left(\frac{n(x)x}{||x||}\right)\right) - R_n(x)\right|\right| \leq 4b + 2b.\end{aligned}$$

■

Corollary 2.5. *Every (a,b)-spiderweb S of a finite dimensional Banach space X has a K-retractional basis where*

$$K = \frac{12b+2}{a} + 2.$$

Proof. Denote $B_{-1} = \emptyset$ and $k_n = \#(B_n \setminus B_{n-1})$ for every $n \in \mathbb{N} \cup \{0\}$. Recall that

$$S = \{x_{(n,i)}\}_{\substack{n\in\mathbb{N}\cup\{0\}\\ i=1,\dots,k_n}},$$

where $\{x_{(n,1)},\dots,x_{(n,k_n)}\} = B_n \setminus B_{n-1}$ is an arbitrary order of $B_n \setminus B_{n-1}$ for every $n \in \mathbb{N}\cup\{0\}$. Order S lexicographically according to the previous notation and define the retractions of the retractional basis $\varphi_{(n,i)} : S \to S_{(n,i)} := \{x_{(m,j)}\}_{(m,j)\leq(n,i)}$ as

$$\varphi_{(n,i)}(x) = \begin{cases} \Psi_n(x) & \text{if } \Psi_n(x) \in S_{(n,i)}, \\ \Psi_{n-1}(x) & \text{if } \Psi_n(x) \notin S_{(n,i)}. \end{cases}$$

It is easy to see that $\varphi_{(n,i)}$ is a well-defined retraction. Also, a straightforward computation gives $\varphi_{(n,i)} \circ \varphi_{(m,j)} = \varphi_{\min((n,i),(m,j))}$ so that it only remains to prove that $\varphi_{(n,i)}$ is K-Lipschitz. Indeed, by Lemma 2.4, if $x \in S$ then

$$||\varphi_{(n,i)}(x) - R_n(x)|| \leq \max\{||\Psi_n(x) - R_n(x)||, ||\Psi_{n-1}(x) - R_n(x)||\} \leq 6b+1.$$

Then for every $x, y \in S$,

$$||\varphi_{(n,i)}(x) - \varphi_{(n,i)}(y)|| \leq 2(6b+1) + ||R_n(x) - R_n(y)|| \leq \left(\frac{12b+2}{a} + 2\right)||x-y||.$$

∎

Theorem 2.6. *Every (a,b)-net of a finite dimensional Banach space X has a K-retractional basis where*

$$K = \left(\frac{2b}{a}+1\right)\left(\frac{4b}{a}+1\right)\left(\frac{156b}{a}+2\right).$$

Proof. Let N be an arbitrary (a,b)-net of X. By Lemma 2.3 (with $c=1$ and C a base of a $(1,2)$-spiderweb) N is Lipschitz equivalent to a $\left(\frac{a}{6b},2\right)$-spiderweb S of X, with distortion $D = \left(\frac{2b}{a}+1\right)\left(\frac{4b}{a}+1\right)$. By corollary 2.5, S has a $\widetilde{K}$-retractional basis where

$$\widetilde{K} = \frac{12\cdot 2+2}{\frac{a}{6b}} + 2 = \frac{156b}{a} + 2.$$

Finally, thanks to Proposition 1.3 we know that N has a K-retractional basis where

$$K = D\widetilde{K} = \left(\frac{2b}{a}+1\right)\left(\frac{4b}{a}+1\right)\left(\frac{156b}{a}+2\right).$$

∎

Remark 2.7. It is worth mentioning that b/a may serve as a measure of homogeneity of an (a,b)-net. We have not strived to obtain an optimal value of K in the previous theorem.

3. FDD grids

In the present section we will study the retractional basis when N is a certain grid in a Banach space X admitting an FDD. In general, such grids are not necessarily nets in X (i.e. they may not be sufficiently dense in X), unless X has some additional c_0-like structure. Our retractional technique is a spin-off of our methods in [HM23]. In the mentioned paper we have constructed a diamond-shaped generating convex compact set in X, with fast decreasing sequence of "heights", which served as a target for a Lipschitz retraction from X. In the present paper we proceed on the contrary by blowing up the generating diamond, creating an inverse-limit type of situation based on diamond-like (but discrete, in fact finite) subsets of the grid of a growing size. The diamond shapes are precisely tuned to match the grid in X, in the sense that every element of the grid is in some sense a "boundary" point of a certain diamond and then it becomes an "interior" point for all larger diamonds. The main mechanism of the construction of our retractions consists roughly speaking of reducing the last non-zero coordinate (whose norm is an integer) of an element of the grid to a coordinate of norm smaller by one. In this way, the element is moved from its boundary position, with respect to a specific diamond, to an interior position thereof. A crucial ingredient is also the independence of the retractional constant from the previous finite dimensional case on the dimension.

Definition 3.1. Let X be a Banach space with an FDD (X_n) and $a,b\in\mathbb{R}^+$. A subset $S\subset X$ is an (a,b)-spiderweb grid with respect to the FDD if

$$S=\bigcup_{n\in\mathbb{N}}\sum_{m=1}^{n}S^m,$$

where S^m is a base of an (a,b)-spiderweb of X_m, for every $m\in\mathbb{N}$.

From now on X will be a Banach space with an FDD (X_n) and $S=\bigcup_{n\in\mathbb{N}}\sum_{m=1}^{n}S^m$ an (a,b)-spiderweb grid with respect to this FDD. We denote the n^{th} coordinate of $x\in X$ by $x_n=\big(P_n-P_{n-1}\big)(x)$, where (P_n) is the sequence of the natural projections of the FDD. We proceed by defining the diamond-shaped sets needed for our construction.

For every $s\in\mathbb{N}\cup\{0\}$ we let

$$D^s=\overline{\mathrm{co}}\Big(\bigcup_{k\in\mathbb{N}}r_k^sB_{X_k}\Big),$$

where the sequence $(r_k^s)_{k\in\mathbb{N}}$ satisfies

$$\begin{cases} r_1^s = s, \\ r_{k+1}^s = \frac{1}{k2^{k+2}} r_k^s. \end{cases}$$

Notice that $D^0 = \{0\}$ and every D^s is compact since $r_n^s \xrightarrow{n\to\infty} 0$. Throughout we will agree that $\sum_{i=n}^{m} a_i = 0$ for every sequence (a_i) and $n, m \in \mathbb{Z}$ such that $n < m$. From the definition of the sequences $(r_n^s)_{n\in\mathbb{N}}$ it follows easily that

$$\frac{r_n^s}{r_m^s} = \frac{r_n^t}{r_m^t} \in \mathbb{N}\,, \qquad \frac{r_n^s}{r_n^t} = \frac{s}{t}\,, \qquad \forall n, m, s, t \in \mathbb{N}\,, \;\; n \le m. \tag{3.1}$$

We begin the construction of the retractional basis for S by defining an increasing sequence of finite subsets $(N_s)_{s\in\mathbb{N}\cup\{0\}}$ of S as

$$N_s = D^s \cap S \quad \forall s \in \mathbb{N} \cup \{0\}.$$

The set N_s is a discrete diamond whose highest length is $s \in \mathbb{N} \cup \{0\}$. Obviously, $N_0 = \{0\}$ and for aesthetic purposes we set $N_{-1} = \emptyset$. For each $n \in \mathbb{N}$ we consider the sequence of retractions $(\Psi_s^n)_{s\in\mathbb{N}\cup\{0\}}$ where $\Psi_s^n : S^n \to sB_{X_n} \cap S^n$ is the retraction defined in Section 2, but now the radius of the discrete ball onto which we retract is given by $s \in \mathbb{N} \cup \{0\}$ (in section 2 we used n). Recall that

$$||\Psi_s^n(x) - R_s(x)|| \le 6b \quad \forall x \in S^n, \tag{3.2}$$

where $R_s : X \to sB_X$ is the radial projection. To simplify the notation we will abbreviate Ψ_s^n as Ψ_s since the superindex n only indicates the changing domain. For every $x \in \bigcup_{n\in\mathbb{N}} P_n(X) \setminus \{0\}$, we are going to denote $n(x) = \max\{n \in \mathbb{N} \,:\, x_n \neq 0\}$. Next, we define the "local" retractions $\varphi_s : N_s \to N_{s-1}$ for $s \in \mathbb{N}$ as

$$\varphi_s(x) = \begin{cases} x & \text{if } x \in N_{s-1}, \\ P_{n(x)-1}(x) + \Psi_{||x_{n(x)}||-1}(x_{n(x)}) & \text{if } x \in N_s \setminus N_{s-1}. \end{cases}$$

Before getting into the definition of the global retractions onto the blocks (N_s), we should prove that φ_s is well-defined, in particular, we may prove that $\varphi_s(x) \in N_{s-1}$ for every $x \in N_s$. For this purpose we will prove the following Lemma 3.2.

For each $x \in S$ let us define $s(x)$ as the unique $s \in \mathbb{N} \cup \{0\}$ such that $x \in N_s \setminus N_{s-1}$. Given $n \in \mathbb{N}$ we are going to denote $q_n = \frac{1}{r_n^1} = \frac{r_1^s}{r_n^s} \in \mathbb{N}$ and $q_0 = 0$.

Lemma 3.2. *For every $x \in S \setminus \{0\}$ the following properties are satisfied.*

(1)

$$s(x) = \sum_{i=1}^{n(x)} ||x_i||q_i,$$

(2)

$$\sum_{i=1}^{n(x)} \frac{||x_i||}{r_i^{s(x)}} = 1,$$

(3)

$$s(\varphi_{s(x)}(x)) = s(x) - q_{n(x)}.$$

Proof. Let us prove both (1) and (2) together by induction in $n(x)$. If $n(x) = 1$ it is trivially satisfied. We assume that it is true for every $x \in S$ such that $n(x) \leq k-1$ for a fixed $k \in \mathbb{N} \setminus \{1\}$. Now, if $x \in S$ is such that $n(x) = k$ then $s(P_{k-1}(x)) = \sum_{i=1}^{k-1} ||x_i||q_i$ and $\sum_{i=1}^{k-1} \frac{||x_i||}{r_i^t} = 1$, where $t = s(P_{k-1}(x))$. It is then enough to prove that (2) is satisfied for $s(x) = t + ||x_k||q_k$. By (3.1) we know that $r_k^{t+||x_k||q_k} = (t + ||x_k||q_k)r_k^1$, so

$$\frac{1}{r_k^{t+||x_k||q_k}} = \frac{q_k}{t + ||x_k||q_k}.$$

If $t \neq 0$ then by (3.1) and the previous equality,

$$\begin{aligned}\sum_{i=1}^{k} \frac{||x_i||}{r_i^{t+||x_k||q_k}} &= \left(\frac{t}{t + ||x_k||q_k} \sum_{i=1}^{k-1} \frac{||x_i||}{r_i^t}\right) + \frac{||x_k||}{r_k^{t+||x_k||q_k}} \\ &= \frac{t}{t + ||x_k||q_k} + \frac{||x_k||q_k}{t + ||x_k||q_k} = 1.\end{aligned}$$

Otherwise, if $t = 0$ then

$$\sum_{i=1}^{k} \frac{||x_i||}{r_i^{t+||x_k||q_k}} = \frac{||x_k||}{r_k^{||x_k||+q_k}} = \frac{||x_k||q_k}{||x_k||q_k} = 1,$$

finishing the proof of (1) and (2). Finally, to prove (3) we let $y = \varphi_{s(x)}(x)$. We split the proof in two cases, $s(x) = q_{n(x)}$ and $s(x) - q_{n(x)} \geq 1$ which are exhaustive thanks to (1). For the case when $s(x) = q_{n(x)}$ it is enough to show that $y = 0$ since then $y \in N_0 = N_{s(x)-q_{n(x)}} \setminus N_{s(x)-q_{n(x)}-1}$. If $s(x) = q_{n(x)}$ then by (1) we have that $x = x_{n(x)}$ with $||x_{n(x)}|| = 1$. Hence, $\varphi_{s(x)}(x) = P_{n(x)-1}(x) + \Psi_{||x_{n(x)}||-1}(x_{n(n)}) = 0 + \Psi_0(x_{n(x)})$=0 and this case is proven.

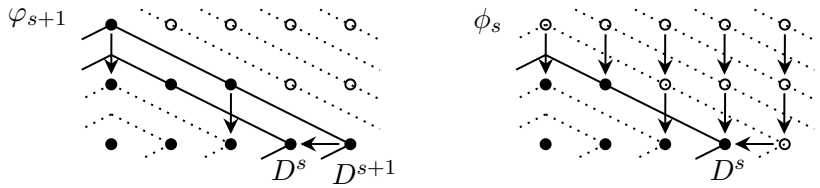

FIGURE 3. The lines of the picture refer to the different diamond-shaped sets covering the space. This image is originally published in [HM22].

Otherwise, $s(x) - q_{n(x)} > 0$ so similarly to the argument used to prove (1) and (2),

$$\begin{aligned}\sum_{i=1}^{n(y)} \frac{||y_i||}{r_i^{s(x)-q_{n(x)}}} &= \left(\sum_{i=1}^{n(x)-1} \frac{||x_i||}{r_i^{s(x)-q_{n(x)}}} \right) + \frac{||x_{n(x)}|| - 1}{r_{n(x)}^{s(x)-q_{n(x)}}} \\ &= \frac{s(x)}{s(x)-q_{n(x)}} \left(\sum_{i=1}^{n(x)} \frac{||x_i||}{r_i^{s(x)}} \right) - \frac{1}{r_{n(x)}^{s(x)-q_{n(x)}}} \\ &= \frac{s(x)}{s(x)-q_{n(x)}} - \frac{q_{n(x)}}{s(x)-q_{n(x)}} = 1,\end{aligned}$$

and we are done. ∎

Let us point out that (2) in Lemma 3.2 means that every point of $S \setminus \{0\}$ lies on the boundary of a diamond D^s for some suitable $s \in \mathbb{N}$ (in fact the suitable s for x is $s(x)$).

From (3) clearly follows that $\varphi_s(x) \in N_{s-1}$ for every $x \in N_s$. In fact, $x \in N_s$ implies $s(x) \leq s$. If $s(x) \leq s-1$ then $x \in N_{s-1}$ and we are done since in this case $\varphi_s(x) = x$. Otherwise $s(x) = s$ and by (3) we have $s(\varphi_s(x)) = s - q_{n(x)} \leq s-1$ so $\varphi_s(x) \in N_{s-1}$.

Finally, we are ready to define the global retractions $\phi_s : S \to N_s$ for every $s \in \mathbb{N} \cup \{0\}$ as

$$\phi_s(x) = \begin{cases} \varphi_{s+1} \circ \cdots \circ \varphi_{s(x)}(x) & \text{if } x \in S \setminus N_s, \\ x & \text{if } x \in N_s. \end{cases}$$

One may see in Figure 3 the action of φ_{s+1} and ϕ_s.

It follows from the definition that

$$\phi_m \circ \phi_n = \phi_{\min(m,n)} \qquad \forall m, n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Every point $x \in S$ gets iteratively mapped by the local retractions when it is retracted by some ϕ_s. Some of these local retractions act like an identity on x and others (which we will call nontrivial local retractions) do not. In order to obtain the precise information on how a point gets mapped throughout the

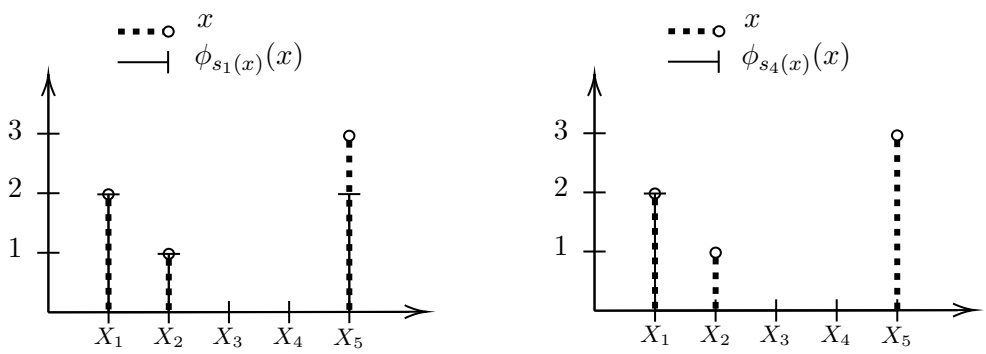

FIGURE 4. The difference between the graph of x and $\phi_{s_k(x)}(x)$ is shown for $k = 1$ and $k = 4$. This image is originally published in [HM22].

nontrivial local retractions we define for each $x \in S$ the sequence $(s_n(x))_{n=0}^{\infty}$ by

$$\begin{cases} s_0(x) = s(x), \\ s_n(x) = s(\phi_{s_{n-1}(x)-1}(x)) \quad \forall n \in \mathbb{N}, \end{cases}$$

where we set $\phi_{-1} : S \to \{0\}$ to be constant.

Notice that the retractions $\phi_{s_k(x)}(x)$ k-times iteratively subtract 1 from the norm of the last nonzero coordinate. See the examples of Figure 4, where the graph refers to the norms of the coordinates. It is clear that the sequence $(s_n(x)) \subset \mathbb{N} \cup \{0\}$ is strictly decreasing until it reaches 0 and becomes constant. See (4) of the next Lemma 3.3 for a detailed analysis of this behaviour.

Lemma 3.3. *Let $x \in S$ and $k \in \mathbb{N} \cup \{0\}$. Then the following equations are satisfied:*

(1)

$$\phi_{s_k(x)-1}(x) = \phi_{s_{k+1}(x)}(x),$$

(2)

$$s_j(\phi_{s_{k-j}(x)}(x)) = s_k(x) \quad \forall j \in \{0, \dots, k\},$$

(3)

$$\big(\varphi_s \circ \cdots \circ \varphi_{s_k(x)}\big)\big(\phi_{s_k(x)}(x)\big) = \varphi_{s_k(x)}\big(\phi_{s_k(x)}(x)\big) = \phi_{s_{k+1}(x)}(x) \quad \forall s \geq s_{k+1}(x)+1,$$

(4)

$$s_{k+1}(x) = s_k(x) - q_{n(\phi_{s_k(x)}(x))}.$$

Proof. By the definition of $s_{k+1}(x)$ we know that $\phi_{s_k(x)-1}(x) \in N_{s_{k+1}(x)}$. So

$$\begin{aligned}\phi_{s_k(x)-1}(x) &= \big(\varphi_{s_{k+1}(x)+1} \circ \cdots \circ \varphi_{s_k(x)}\big) \circ \phi_{s_k(x)-1}(x) \\ &= \big(\varphi_{s_{k+1}(x)+1} \circ \cdots \circ \varphi_{s_k(x)}\big) \circ \big(\varphi_{s_k(x)} \circ \cdots \circ \varphi_{s(x)}\big)(x) = \phi_{s_{k+1}(x)}(x),\end{aligned}$$

which proves (1). To prove (2) we proceed by induction in k. If $k = 0$ it is straightforward to see that $s_0(\phi_{s_0(x)}(x)) = s_0(x)$. We proceed with the inductive step from $k-1$ to k. If $j \in \{1, \dots, k\}$ then we let $y = \phi_{s_{k-j}(x)}(x)$, so by the inductive assumption we know that

$$s_{j-1}(y) = s_{j-1}\big(\phi_{s_{k-1-(j-1)}(x)}(x)\big) = s_{k-1}(x).$$

In this case we compute

$$\begin{aligned}s_j(\phi_{s_{k-j}(x)}(x)) &= s_j(y) = s(\phi_{s_{j-1}(y)-1}(y)) = s(\phi_{s_{k-1}(x)-1}(\phi_{s_{k-j}(x)}(x))) \\ &= s(\phi_{s_{k-1}(x)-1}(x)) = s_k(x).\end{aligned}$$

Otherwise, if $j = 0$ then by (1)

$$s_0(\phi_{s_k(x)}(x)) = s(\phi_{s_{k-1}(x)-1}(x)) = s_k(x),$$

which finishes the proof of (2).

We turn our attention to the point (3). This shows the amount of trivial local retractions that act in between the nontrivial ones.

Consider $s \geq s_1(x)+1$. By the definition of $s_1(x)$ we have that $\varphi_{s(x)}(x) = \phi_{s(x)-1}(x) \in N_{s_1(x)}$. Hence,

$$\big(\varphi_s \circ \cdots \circ \varphi_{s(x)}\big)(x) = \varphi_{s(x)}(x) = \big(\varphi_{s_1(x)+1} \circ \cdots \circ \varphi_{s(x)}(x)\big) = \phi_{s_1(x)}(x),$$

that is,

$$\big(\varphi_s \circ \cdots \circ \varphi_{s(x)}\big)(x) = \varphi_{s(x)}(x) = \phi_{s_1(x)}(x) \quad \forall s \geq s_1(x) + 1. \tag{3.4}$$

This is exactly the equation (3) for $k = 0$. Now, for an arbitrary $k \in \mathbb{N}$, just take $y = \phi_{s_{k-1}(x)-1}(x) = \phi_{s_k(x)}(x)$ so that by definition $s_k(x) = s(y)$ and by (2) it follows that $s_1(y) = s_{k+1}(x)$. Using (3.4) with y instead of x we finish the proof of (3).

Next, let us prove (4). If $y = \phi_{s_{k-1}(x)-1}(x) = \phi_{s_k(x)}(x) \neq 0$ then using property (3) of Lemma 3.2 we have that

$$\begin{aligned}s_{k+1}(x) &= s\big(\phi_{s_k(x)-1}(x)\big) = s\big(\phi_{s_k(x)-1}(y)\big) = s\big(\phi_{s(y)-1}(y)\big) \\ &= s(y) - q_{n(y)} = s_k(x) - q_{n(\phi_{s_k(x)}(x))}.\end{aligned}$$

In the case when $y = 0$ (4) is trivially true as $s_k(x) = s(y) = 0$ and $q_0 = 0$. ∎

The main feature of the definition of $\phi_s : S \to S$ is that it is a discretization of the Lipschitz retraction defined in [HM23] (Section 3). To be more precise, we need to recall some definitions of [HM23]:

For every $m, s \in \mathbb{N}$ we define the function $f_m^s : X \to \mathbb{R}$ as

$$f_m^s(x) = r_m^s \left(1 - \sum_{i=1}^{m-1} \frac{||x_i||}{r_i^s}\right).$$

For a given $x \in \bigcup_{n\in\mathbb{N}} P_n(X)$ and $s \in \mathbb{N}$ we set $m(x,s) = \max\left\{k \in \{1,\dots,n(x)+1\} : \sum_{i=1}^{k-1} \frac{||x_i||}{r_i^s} \leq 1\right\}$. Next, we define the constant mapping $F^0 : X \to \{0\}$, and we also define the map $F^s : X \to D^s$ as the unique extension of the map $\widetilde{F^s} : \bigcup_{n\in\mathbb{N}} P_n(X) \to D^s$ given by

$$\widetilde{F^s}(x) = \begin{cases} P_{m(x,s)-1}(x) + \frac{x_{m(x,s)}}{||x_{m(x,s)}||} f^s_{m(x,s)}(x) & \text{if } m(x,s) \leq n(x), \\ x & \text{if } m(x,s) = n(x)+1. \end{cases}$$

In [HM23] it is proved that $\widetilde{F^s}$ is well-defined and Lipschitz, with a Lipschitz norm independent of s. We would like to point out that F^s is defined in [HM23] as a limit of retractions but the definition given here is equivalent (see Section 3 of [HM23]). Also, it is important to mention that

$$F^s \circ F^t = F^{\min(s,t)} \quad \forall s,t \in \mathbb{N}.$$

Lemma 3.4. *For every $k \in \mathbb{N}$ and $x \in S$,*

$$||F^{s_k(x)}(x) - \phi_{s_k(x)}(x)|| \leq 6b. \tag{3.5}$$

Proof. If $x = 0$ then the above is trivially true, so we may assume that $x \in S \setminus \{0\}$. We proceed by giving first an alternative definition of $\phi_{s_k(x)}$ and $F^{s_k(x)}$ which will be later convenient for the proof of the Lemma. One has to view $k \in \mathbb{N}$ as the number of nontrivial local retractions that act on x when it is mapped by $\phi_{s_k(x)}$. In fact $\sum_{i=1}^{n(x)} ||x_i||$ is the amount of nontrivial local retractions needed to map x to the origin, see Figure 4. We need a couple of definitions before we get on to the equivalent definitions of $\phi_{s_k(x)}$ and $F^{s_k(x)}$.

For every $x \in S \setminus \{0\}$ and $k \in \mathbb{N}$ such that $k \leq \sum_{i=1}^{n(x)} ||x_i||$ we claim that there is only one pair $I, J \in \mathbb{N} \cup \{0\}$ satisfying that $1 \leq J \leq ||x_{n(x)-I}||$ and $k = J + \sum_{i=1}^{I} ||x_{n(x)-i+1}||$. We will use the notation $I = i(k,x)$ and $J = j(k,x)$, respectively. Indeed, if I_1, J_1 and I_2, J_2 were two suitable pairs for $x \in S \setminus \{0\}$ and $k \in \{1,\dots,\sum_{i=1}^{n(x)} ||x_i||\}$, then assuming $I_1 + 1 \leq I_2$ we

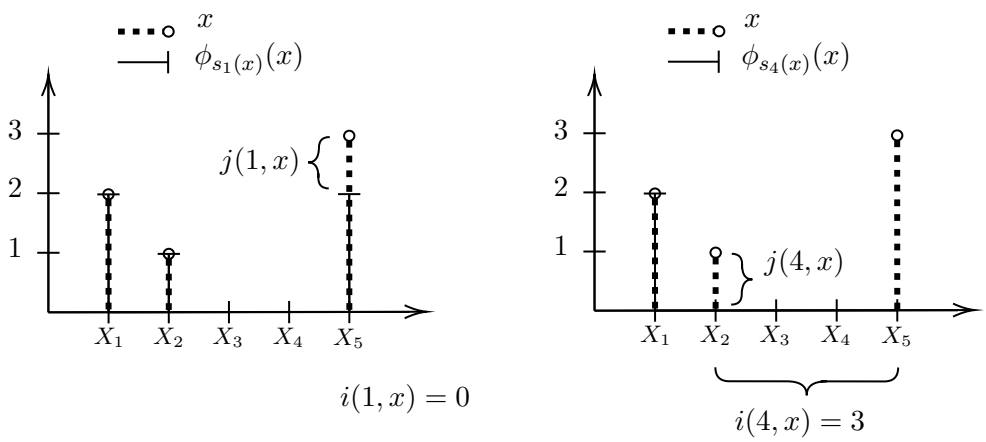

FIGURE 5. The pair $i(k,x), j(k,x)$ with $k = 1, 4$ and x the same as in Figure 4. This image is originally published in [HM22].

obtain

$$\begin{aligned} k =& J_2 + \sum_{i=1}^{I_2} ||x_{n(x)-i+1}|| \geq J_2 + \sum_{i=1}^{I_1+1} ||x_{n(x)-i+1}|| \\ \geq& J_2 + J_1 + \sum_{i=1}^{I_1} ||x_{n(x)-i+1}|| = k + J_2 > k, \end{aligned}$$

which is a contradiction. So $I_1 = I_2$ and therefore $J_1 = J_2$, proving the uniqueness of the pair I, J. Let us now prove that such pair exists. The set

$$\Big\{ I \in \mathbb{N} \cup \{0\} \ : \ k \leq \sum_{i=0}^{I} ||x_{n(x)-i}|| \Big\}$$

is nonempty because $n(x) - 1$ is an element. It is easy to see that the pair

$$i(k,x) = \min \Big\{ I \in \mathbb{N} \cup \{0\} \ : \ k \leq \sum_{i=0}^{I} ||x_{n(x)-i}|| \Big\},$$

$$j(k,x) = k - \sum_{i=1}^{i(k,x)} ||x_{n(x)-i+1}||,$$

meet the required conditions.

The meaning of $i(k,x)$ is the count of how many x_i's are retracted to zero by $\phi_{s_k(x)}$, while $j(k,x)$ tells us the difference between the norm of x and $\phi_{s_k(x)}(x)$ in their first different coordinate. See the examples shown in Figure 5.

With these two new quantities we obtain a more convenient definition of both $\phi_{s_k(x)}$ and $F^{s_k(x)}$.

Claim 3.5. *For every $k \in \mathbb{N}$ and $x \in S$ such that $k \leq \sum_{i=1}^{n(x)} ||x_i||$ the equalities*

$$\phi_{s_k(x)}(x) = P_{n(x)-i(k,x)-1}(x) + \Psi_{||x_{n(x)-i(k,x)}||-j(k,x)}(x_{n(x)-i(k,x)}), \tag{3.6}$$

$$F^{s_k(x)}(x) = P_{n(x)-i(k,x)-1}(x) + R_{||x_{n(x)-i(k,x)}||-j(k,x)}(x_{n(x)-i(k,x)}), \tag{3.7}$$

are satisfied.

Proof. Equation (3.6) for $k = 1$ follows inmediately after realizing that $i(1,x) = 0$ and $j(1,x) = 1$ for every $x \in S \setminus \{0\}$. Now we prove (3.7) for $k = 1$. Since $s\big(\varphi_{s(x)}(x)\big) = s_1(x)$, using (2) of Lemma 3.2 with $\varphi_{s(x)}(x)$ instead of x we obtain that

$$\sum_{i=1}^{n(x)-1} \frac{||x_i||}{r_i^{s_1(x)}} + \frac{||x_{n(x)}|| - 1}{r_{n(x)}^{s_1(x)}} = 1.$$

Then,

$$\sum_{i=1}^{n(x)-1} \frac{||x_i||}{r_i^{s_1(x)}} \leq \sum_{i=1}^{n(x)-1} \frac{||x_i||}{r_i^{s_1(x)}} + \frac{||x_{n(x)}|| - 1}{r_{n(x)}^{s_1(x)}} = 1.$$

Also, using now (2) of Lemma 3.2 with x we get that

$$\sum_{i=1}^{n(x)} \frac{||x_i||}{r_i^{s_1(x)}} > \sum_{i=1}^{n(x)} \frac{||x_i||}{r_i^{s(x)}} = 1.$$

This means that $m(x, s_1(x)) = n(x)$. Now, as

$$\sum_{i=1}^{n(x)-1} \frac{||x_i||}{r_i^{s_1(x)}} + \frac{||x_{n(x)}|| - 1}{r_{n(x)}^{s_1(x)}} = 1 \;\; \Rightarrow \;\; \frac{||x_{n(x)}|| - 1}{r_{n(x)}^{s_1(x)}} = 1 - \sum_{i=1}^{n(x)-1} \frac{||x_i||}{r_i^{s_1(x)}},$$

it follows that

$$\begin{aligned}
F^{s_1(x)}(x) &= P_{n(x)-1}(x) + \frac{x_{n(x)}}{||x_{n(x)}||} f_{n(x)}^{s_1(x)}(x) \\
&= P_{n(x)-1}(x) + \frac{x_{n(x)}}{||x_{n(x)}||} r_{n(x)}^{s_1(x)} \left(1 - \sum_{i=1}^{n(x)-1} \frac{||x_i||}{r_i^{s_1(x)}}\right) \\
&= P_{n(x)-1}(x) + \frac{x_{n(x)}}{||x_{n(x)}||} r_{n(x)}^{s_1(x)} \left(\frac{||x_{n(x)}|| - 1}{r_{n(x)}^{s_1(x)}}\right) \\
&= P_{n(x)-1}(x) + \frac{||x_{n(x)}|| - 1}{||x_{n(x)}||} x_{n(x)} = P_{n(x)-1}(x) + R_{||x_{n(x)}||-1}(x_{n(x)}).
\end{aligned}$$

Hence, both (3.6) and (3.7) are established for $k = 1$. We proceed with the inductive step from $k-1$ to $k \geq 2$. We focus on (3.6) since the proof of (3.7) is essentially the same. Let $y = \phi_{s_{k-1}(x)}(x)$ so that $\phi_{s_k(x)}(x) = \phi_{s_1(y)}(y)$. If

$y = 0$ then both sides of the equation (3.6) are 0 so we may assume that $y \neq 0$. Then,

$$y = P_{n(x)-i(k-1,x)-1}(x) + \Psi_{||x_{n(x)-i(k-1,x)}||-j(k-1,x)}(x_{n(x)-i(k-1,x)}).$$

We distinguish two cases. First, let us assume that $||x_{n(x)-i(k-1,x)}|| = j(k-1,x)$. In this case $y = P_{n(x)-i(k-1,x)-1}(x)$ meaning that if there is $i \in \mathbb{N}$ such that $n(y)+1 \leq i \leq n(x)-i(k-1,x)-1$ then $||x_i|| = 0$. Equivalently, if there is $i \in \mathbb{N}$ such that $i(k-1,x)+2 \leq i \leq n(x)-n(y)$ then $||x_{n(x)-i+1}|| = 0$. With that in mind we deduce that

$$k-1 = \sum_{i=1}^{i(k-1,x)+1} ||x_{n(x)-i+1}|| \;\Rightarrow\; k = 1 + \sum_{i=1}^{n(x)-n(y)} ||x_{n(x)-i+1}||,$$

with $1 \leq n(y) = n(x)-(n(x)-n(y))$. It follows that $i(k,x) = n(x)-n(y)$ and $j(k,x) = 1$ since $||x_{n(x)-(n(x)-n(y))}|| = ||x_{n(y)}|| = ||y_{n(y)}|| \geq 1$. Computing $\phi_{s_k(x)}(x)$ we have that

$$\begin{aligned}\phi_{s_k(x)}(x) =& \phi_{s_1(y)}(y) = P_{n(y)-1}(y) + \Psi_{||y_{n(y)}||-1}(y_{n(y)}) \\ =& P_{n(x)-i(k,x)-1}(x) + \Psi_{||x_{n(x)-i(k,x)}||-j(k,x)}(x_{n(x)-i(k,x)}).\end{aligned}$$

In the second case $1 \leq j(k-1,x) \leq ||x_{n(x)-i(k-1,x)}|| - 1$. We claim that $i(k,x) = i(k-1,x)$ and $j(k,x) = j(k-1,x)+1$. Indeed,

$$k-1 = j(k-1,x) + \sum_{i=1}^{i(k-1,x)} ||x_{n(x)-i+1}|| \;\Rightarrow\; k = (j(k-1,x)+1) + \sum_{i=1}^{i(k-1,x)} ||x_{n(x)-i+1}||,$$

with $j(k-1,x)+1 \leq ||x_{n(x)-i(k-1,x)}||$. As $y = P_{n(x)-i(k-1,x)-1}(x) + \Psi_{||x_{n(x)-i(k-1,x)}||-j(k-1,x)}(x_{n(x)-i(k-1,x)})$ with $||x_{n(x)-i(k-1,x)}||-j(k-1,x) \geq 1$ it follows that $n(y) = n(x) - i(k-1,x) = n(x) - i(k,x)$ and

$$\begin{aligned}||y_{n(y)}|| =& \left|\left|\Psi_{||x_{n(x)-i(k-1,x)}||-j(k-1,x)}(x_{n(x)-i(k-1,x)})\right|\right| \\ =& ||x_{n(x)-i(k-1,x)}|| - j(k-1,x) = ||x_{n(x)-i(k,x)}|| - j(k,x) + 1.\end{aligned}$$

Now, it is only a matter of computing $\phi_{s_k(x)}(x)$ again,

$$\begin{aligned}\phi_{s_k(x)}(x) =& \phi_{s_1(y)}(y) = P_{n(y)-1}(y) + \Psi_{||y_{n(y)}||-1}(y_{n(y)}) \\ =& P_{n(x)-i(k,x)-1}(y) \\ &+ \Psi_{||x_{n(x)-i(k,x)}||-j(k,x)}(\Psi_{||x_{n(x)-i(k,x)}||-j(k,x)+1}(x_{n(x)-i(k,x)})) \\ =& P_{n(x)-i(k,x)-1}(x) + \Psi_{||x_{n(x)-i(k,x)}||-j(k,x)}(x_{n(x)-i(k,x)}).\end{aligned}$$

■

We return to the proof of (3.5). In fact, taking into account Claim 3.5, if $k > \sum_{i=1}^{n(x)} ||x_i|| = t$ then $i(t,x) = n(x)-1$ and $j(t,x) = ||x_{n(x)}||$. So $\phi_{s_t(x)}(x) = 0 = F^{s_t(x)}(x)$ and we are done as $\phi_{s_k(x)}(x) = \phi_{s_k(x)}(\phi_{s_t(x)}(x)) =$

$\phi_{s_k(x)}(0) = 0 = F^{s_k(x)}(F^{s_t(x)}(x)) = F^{s_k(x)}(x)$. Otherwise, if $k \leq \sum_{i=1}^{n(x)} ||x_i||$ then by Claim 3.5 and equality (3.2),

$$\begin{aligned}&\big|\big|\phi_{s_k(x)}(x) - F^{s_k(x)}(x)\big|\big| \\ &\quad = \big|\big|\big(\Psi_{||x_{n(x)-i(k,x)}||-j(k,x)} - R_{||x_{n(x)-i(k,x)}||-j(k,x)}\big)(x_{n(x)-i(k,x)})\big|\big| \leq 6b.\end{aligned}$$

■

Remark 3.6. If the blocks X_n of the FDD are 1-dimensional (in which case X has a Schauder basis), then one may see from Claim 3.5 that $\phi_{s_k(x)}(x) = F^{s_k(x)}(x)$ for every $k \in \mathbb{N}$.

The following lemma is needed for the case when $s \neq s_k(x)$ in the statement of Lemma 3.4.

Lemma 3.7. *Suppose that $n \in \mathbb{N}$ and $t, s \in \mathbb{N} \cup \{0\}$ are such that $t \leq s$. If $x - F^t(x) = \frac{x_n}{||x_n||}$ for some $x \in X$ with $||x_n|| > 0$, then there exists $\lambda \in [0,1]$ such that $\big(F^s - F^t\big)(x) = \lambda \frac{x_n}{||x_n||}$.*

Proof. By density we may assume that $x \in \bigcup_{n\in\mathbb{N}} P_n(X) \setminus \{0\}$, so there exists an $n(x) \in \mathbb{N}$. Let us assume first that $t \geq 1$. Obviously, as $F^t(x) \neq x$ we deduce that $x \notin D^t$ so then $m(x,t) \leq n(x) \in \mathbb{N}$. Now, looking at the definition of F^t it is clear that

$$\frac{x_n}{||x_n||} = x - F^t(x) = \left(\sum_{i=m(x,t)+1}^{n(x)} x_i\right) + x_{m(x,t)}\left(1 - \frac{f^t_{m(x,t)}(x)}{||x_{m(x,t)}||}\right).$$

Since $m(x,t) \leq n(x)$ by the definition of $m(x,t)$ it is deduced that $P_{m(x,t)}(x) \notin D^t$ so $1 - \frac{f^t_{m(x,t)}(x)}{||x_{m(x,t)}||} > 0$. Then, $m(x,t) = n(x) = n$ and

$$x_n - \frac{x_n}{||x_n||} f^t_n(x) = \frac{x_n}{||x_n||}, \tag{3.8}$$

by the linear independence of the blocks of the FDD. Clearly $m(x,s) \geq m(x,t) = n$. As $n(x) = n$ there are only two possibilities, namely $m(x,s) = n$ or $m(x,s) = n+1$, equivalently $x \in D^s$. In the case when $m(x,s) = n$ we have that

$$F^s(x) - F^t(x) = F^s_k(x) - F^t_k(x) = \frac{x_n}{||x_n||}\big(f^s_n(x) - f^t_n(x)\big).$$

Here just looking at the definitions and taking into account that $m(x,s) = n$ implies $\sum_{i=1}^{n} \frac{||x_i||}{r^s_i} > 1$, it is easy to see that $0 \leq f^t_n(x) \leq f^s_n(x) < ||x_n||$. Now by (3.8) we have that $||x_n|| - f^t_n(x) = 1$. This finishes the argument in this case since

$$0 \leq f^s_n(x) - f^t_n(x) = f^s_n(x) - ||x_n|| + 1 < 1.$$

In the remaining case when $x \in D^s$, $F_k^s(x) = x$ so we have $F^s(x) - F^t(x) = x - F^t(x) = \frac{x_n}{||x_n||}$ and we are done.
Finally, if $t = 0$ then $x - F^t(x) = x = \frac{x_n}{||x_n||}$. In particular, $||x_n|| = 1$ and we may assume $s \geq 1$ because $s = 0$ is trivially true with $\lambda = 0$. Again, $m(x, s) = n$ or $x \in D^s$. If $m(x, s) = n$ then $0 \leq f_n^s(x) < ||x_n|| = 1$ and

$$F^s(x) - F^t(x) = F^s(x) = f_n^s(x)\frac{x_n}{||x_n||},$$

so the statement of the Lemma is satisfied. Otherwise, if $x \in D^s$ then again $F^s(x) - F^t(x) = x = \frac{x_n}{||x_n||}$. ∎

Lemma 3.8. *For every $s \in \mathbb{N}$,*

$$||\phi_s(x) - F^s(x)|| \leq 1 + 6b \quad \forall x \in S.$$

Proof. Take $x \in S \setminus N_s$. It is clear that $s \in \{1, \dots, s_0(x) - 1\}$ so there exists $k \in \mathbb{N} \cup \{0\}$ such that $s \in \{s_{k+1}(x), \dots, s_k(x) - 1\}$. Then, by Lemma 3.3, equation (3),

$$\phi_s(x) = \varphi_{s+1} \circ \cdots \circ \varphi_{s(x)}(x) = \big(\varphi_{s+1} \circ \cdots \circ \varphi_{s_k(x)}\big)\big(\phi_{s_k(x)}(x)\big) = \phi_{s_{k+1}(x)}(x).$$

Thanks to equations (3.6) and (3.7) we know that for every $x \in S$, $s\big(\phi_{s_k(x)}(x)\big) = s\big(F^{s_k(x)}(x)\big)$. Now, if we denote $y = F^{s_k(x)}(x)$ then, again by equation (3.7) we have that

$$\begin{aligned} y - F^{s_1(y)}(y) =& y - \big(P_{n(y)-1}(y) + R_{||y_{n(y)}||-1}(y_{n(y)})\big) \\ =& y_{n(y)} - R_{||y_{n(y)}||-1}(y_{n(y)}) = \frac{y_{n(y)}}{||y_{n(y)}||} \quad . \end{aligned}$$

Then, thanks to Lemma 3.7 with $t = s_1(y)$ we know that there is a $\lambda \in [0, 1]$ such that

$$F^s(x) - F^{s_{k+1}(x)}(x) = F^s(y) - F^{s_1(y)}(y) = \lambda\frac{y_{n(y)}}{||y_{n(y)}||}.$$

Finally,

$$\begin{aligned} ||\phi_s(x) - F^s(x)|| =& ||\phi_{s_{k+1}(x)}(x) - F^s(x)|| \\ \leq& ||\phi_{s_{k+1}(x)}(x) - F^{s_{k+1}(x)}(x)|| + ||F^{s_{k+1}(x)}(x) - F^s(x)|| \\ \leq& 6b + \lambda. \end{aligned}$$

∎

Theorem 3.9. *If X is a Banach space with an FDD, then every spiderweb grid respect to the FDD has a retractional basis.*

Proof. First we claim that for every $x \in X$ and $s \in \mathbb{N}$

$$||F^{s+1}(x) - F^s(x)|| \leq 1. \tag{3.9}$$

In fact, it is enough to prove that $||x - F^s(x)|| \leq 1$ for every $x \in D^{s+1}$ by the conmutativity of F^s. If $x \in D^s$ then the inequality is trivial since

$x = F^s(x)$. Then, let us assume that $x \in D^{s+1} \setminus D^s$. As $x \in D^{s+1}$ we obtain the following,

$$\begin{aligned}\sum_{i=m(x,s)}^{n(x)} ||x_i|| &\leq r_{m(x,s)}^{s+1} \sum_{i=m(x,s)}^{n(x)} \frac{||x_i||}{r_i^{s+1}} \\ &= r_{m(x,s)}^{s+1}\left(\sum_{i=1}^{n(x)} \frac{||x_i||}{r_i^{s+1}} - \sum_{i=1}^{m(x,s)-1} \frac{||x_i||}{r_i^{s+1}}\right) \\ &\leq r_{m(x,s)}^{s+1}\left(1 - \sum_{i=1}^{m(x,s)-1} \frac{||x_i||}{r_i^{s+1}}\right) = f_{m(x,s)}^{s+1}.\end{aligned} \tag{3.10}$$

Next, since $x \notin D^s$ we know that $1 - \frac{f^s_{m(x,s)}(x)}{||x_{m(x,s)}||} > 0$ so that we may compute using (3.10),

$$\begin{aligned}||x - F^s(x)|| &= \left|\left|\left(\sum_{i=m(x,s)}^{n(x)} x_i\right) - \frac{x_{m(x,s)}}{||x_{m(x,s)}||} f^s_{m(x,s)}(x)\right|\right| \\ &\leq \left(\sum_{i=m(x,s)+1}^{n(x)} ||x_i||\right) + \left|\left|x_{m(x,s)}\left(1 - \frac{f^s_{m(x,s)}(x)}{||x_{m(x,s)}||}\right)\right|\right| \\ &= \left(\sum_{i=m(x,s)}^{n(x)} ||x_i||\right) - f^s_{m(x,s)}(x) \leq f^{s+1}_{m(x,s)}(x) - f^s_{m(x,s)}(x) \\ &= r^{s+1}_{m(x,s)} - r^s_{m(x,s)}.\end{aligned}$$

By (3.1) we know $r^{s+1}_{m(x,s)} = \frac{s+1}{s} r^s_{m(x,s)}$ so then

$$||x - F^s(x)|| \leq r^{s+1}_{m(x,s)} - r^s_{m(x,s)} = \frac{r^s_{m(x,s)}}{s} = r^1_{m(x,s)} \leq 1$$

and (3.9) is proved. Now, we are going to proceed as in Corollary 2.5. Let us consider S to be the (a,b)-spiderweb grid fixed at the beginning of the section. Denote $N_{-1} = \emptyset$ and $k_s = \#(N_s \setminus N_{s-1})$ for every $s \in \mathbb{N} \cup \{0\}$. Recall that

$$S = \{x_{(s,i)}\}_{\substack{n \in \mathbb{N}\cup\{0\} \\ i=1,\ldots,k_s}},$$

where $\{x_{(s,1)}, \ldots, x_{(s,k_s)}\} = N_s \setminus N_{s-1}$ is an arbitrary order of $N_s \setminus N_{s-1}$ for every $s \in \mathbb{N} \cup \{0\}$. Order S lexicographically according to the previous notation and define the retractions of the retractional basis $\varphi_{(s,i)} : S \to S_{(s,i)} := \{x_{(m,j)}\}_{(m,j)\leq(s,i)}$ as

$$\varphi_{(s,i)}(x) = \begin{cases} \phi_s(x) & \text{if } \phi_s(x) \in S_{(s,i)}, \\ \phi_{s-1}(x) & \text{if } \phi_s(x) \notin S_{(s,i)}. \end{cases}$$

It is easy to see that $\varphi_{(s,i)}$ is a well defined retraction. Also, a straightforward computation gives $\varphi_{(s,i)} \circ \varphi_{(m,j)} = \varphi_{\min((s,i),(m,j))}$ so that it only remains to prove that $\varphi_{(s,i)}$ is Lipschitz. Indeed, by Lemma 3.8 and equation (3.9), if $x \in S$ then

$$||\phi_{s-1}(x) - F^s(x)|| \leq ||\phi_{s-1}(x) - F^{s-1}(x)|| + ||F^{s-1}(x) - F^s(x)|| \leq 6b + 2$$

so it follows that

$$||\varphi_{(s,i)}(x) - F^s(x)|| \leq \max\{||\phi_s(x) - F^s(x)||, ||\phi_{s-1}(x) - F^s(x)||\} \leq 6b + 2.$$

Then for every $x, y \in S$,

$$||\varphi_{(s,i)}(x) - \varphi_{(s,i)}(y)|| \leq 2(6b+2) + ||F^s(x) - F^s(y)|| \leq \left(\frac{12b+4}{a} + ||F^s||_{\text{Lip}}\right)||x-y||.$$

We are done since $||F^s||_{\text{Lip}}$ is independent of s and S was fixed but arbitrary. ∎

Corollary 3.10. *Every grid respect to a Schauder basis of a Banach space has a retractional basis.*

Definition 3.11. A Banach space X is said to be griddable respect to a sequence of finite dimensional subspaces (X_n) of X whenever (X_n) is an FDD of X and for every sequence of (a,b)-nets $G_n \subset X_n$ the set $G = \bigcup_{n\in\mathbb{N}} \sum_{k=1}^{n} G_k$ is a net of X.

We will say that X is griddable if it is griddable for some sequence (X_n).

If a Banach space X has a c_0-like FDD (X_n), i.e. $X = \oplus_{c_0} \sum X_n$, then it is clearly griddable. If X has a Schauder basis and contains a copy of $c_0 \hookrightarrow X$ then it is griddable (with respect to possibly different Schauder basis) by Theorem 4.1 of [Dil+08].

Corollary 3.12. *If X is griddable, then every net N of X has a retractional basis. In particular, if X is a Banach space with a Schauder basis containing a copy of c_0, or if X has a c_0-like FDD then N has a retractional basis.*

Proof. Let N be an arbitrary net of X. If X is griddable respect to an FDD we consider a spiderweb grid S in X respect to the same FDD. Now, S is a net in X and by Theorem 3.9 it has a retractional basis. By [LMP00] every two nets of an infinite dimensional Banach space are Lipschitz equivalent so N is Lipschitz equivalent to S. Hence, by Proposition 1.3 N has a retractional basis. ∎

Remark 3.13. Let us point out that separable L_1-preduals, as well as subspaces of c_0 admitting a shrinking FDD, meet the conditions of Corollary 3.12. In fact, by Theorem 1 of [JZ72] every subspace of a quotient of $\left(\sum G_n\right)_{c_0}$ with shrinking FDD, where the G_n are finite dimensional spaces, admits a c_0-like FDD.

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Chapter 8

Schauder basis in Lipschitz free spaces over nets of $\mathcal{L}_\infty$-spaces

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SCHAUDER BASES IN LIPSCHITZ FREE SPACES OVER NETS OF $\mathcal{L}_\infty$-SPACES

PETR HÁJEK AND RUBÉN MEDINA

Abstract. In the present note we give a construction (based on a retractional argument) of a Schauder basis for the Lipschitz free space $\mathcal{F}(N)$ over a net N in any separable infinite dimensional $\mathcal{L}_\infty$-space X. In particular, this yields the first example of an infinite dimensional Banach space X not containing c_0 with such a property.

1. Introduction

The topic of our note was motivated by the important and well-known question of Kalton [Kal12] whether every (separable) Banach space is approximable, or equivalently whether the Lipschitz free space $\mathcal{F}(N)$ over a net N in any separable Banach space has the Bounded Approximation Property (BAP for short). Kalton proved that for a Banach space X with a separable dual (or a separable dual space itself), as well as for Banach spaces X with the BAP, the answer is positive.

In the present note we study the retractional structure of nets in Banach spaces X of the type $\mathcal{L}_\infty$. As a corollary of our construction we obtain the result that the Lipschitz free space $\mathcal{F}(N)$ over a net in any separable $\mathcal{L}_\infty$-space has a Schauder basis. This is a much stronger condition than the BAP, guaranteed by Kalton's results, and moreover our proof is entirely constructive. It should be noted that our main result extends our previous papers [HM21], resp. [HN17], where a retractional structure of nets in Banach spaces with a Schauder basis (resp. FDD) and containing a copy of c_0 has been investigated (for the same purpose). One of the key ingredients in our previous results was the coefficient quantization theorem of [Dil+08], which requires that the Banach space X contains a copy of c_0. Our present results are a genuine extension, as there exist (by a famous result of Bourgain and Delbaen [BD80]) separable $\mathcal{L}_\infty$-spaces without a copy of c_0. Although the retractional approach for the problem is the same as in

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our previous papers, the construction is rather different and actually relies on the explicit description of the abstract $\mathcal{L}_\infty$-spaces as Bourgain-Delbaen spaces obtained in [BD80], and an alternate quantization which crucially uses the ℓ_∞-space ambient. It should be added that the class of $\mathcal{L}_\infty$-spaces is very significant and in some sense large, and extremely useful for constructing (counter)examples ([BD80], [AH11]). For instance, every Banach space with a separable dual is contained in an isomorphic ℓ_1-predual (of the type $\mathcal{L}_\infty$) [FOS11].

Let us now proceed with some technical background.

Given $a, b \in \mathbb{R}^+$ we will say that a subset N of a metric space M is an (a, b)-net of M whenever it is a-separated and b-dense. More precisely, N is considered to be a-separated whenever $d(x, y) \geq a$ for every $x, y \in N$, $x \neq y$, and b-dense if for every $x \in M$ there is $y \in N$ such that $d(x, y) \leq b$. We will say that $N \subset M$ is a net whenever there are $a, b \in \mathbb{R}^+$ such that N is an (a, b)-net.

A map T from a metric space M into another metric space N is said to be Lipschitz if there exists some $\lambda > 0$ such that

$$d(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in M.$$

We say that λ is a Lipschitz constant for T and we call the infimum of all Lipschitz constants for T the Lipschitz norm of T, that is,

$$||T||_{\text{Lip}} = \sup_{x,y \in M, x \neq y} \frac{d\big(T(x), T(y)\big)}{d(x, y)}.$$

If $\lambda > 0$ is a Lipschitz constant for T then we say that T is λ-Lipschitz.

Definition 1.1. Let N_1 and N_2 be metric spaces. We say that N_1 and N_2 are Lipschitz equivalent whenever there is a bijection $F : N_1 \to N_2$ such that both F and F^{-1} are Lipschitz. We define the *distorsion* of F by $\text{dist}(F) = \|F\|_{\text{Lip}} \|F^{-1}\|_{\text{Lip}}$.

In this note, we are mainly interested in finding a retractional structure in nets. To this end, we are going to order the elements of the net into a sequence and introduce K-Lipschitz retractions onto the first n points of the sequence with a commuting behaviour. More precisely,

Definition 1.2. Let M be a countable metric space and $K \geq 1$, a K-retractional basis of M is a sequence of retractions $\{\varphi_n\}_{n=1}^\infty$, $\varphi_n : M \to M$ satisfying,

(1) $\varphi_n(M) = M_n := \bigcup_{j=1}^n \{x_j\}$,
(2) $\bigcup_{j=1}^\infty \{x_j\} = M$,
(3) φ_n is K-Lipschitz for every $n \in \mathbb{N}$,
(4) $\varphi_m \circ \varphi_n = \varphi_{\min(m,n)}$ for every $m, n \in \mathbb{N}$.

We will say that M has a retractional basis whenever it has a K-retractional basis for some $K \geq 1$.

It is clear that these retractions can be used to extend Lipschitz maps from a finite number of points $\{x_1, \dots, x_n\}$ to the whole net by composition, keeping the Lipschitz constant under control independently of the number of points from which the extension is made.

The existence of retractional bases has been studied previously (see [HN17], [HM21] and [Nov20]). It is easy to see that retractional bases in nets are clearly preserved under Lipschitz equivalences. More precisely,

Proposition 1.3. *Let N_1 and N_2 be countable Lipschitz equivalent metric spaces, with distorsion D. If N_1 has a K-retractional basis for some $K > 0$ then N_2 has a DK-retractional basis.*

A Schauder basis for a real Banach space X is a sequence $(x_n) \subset X$ with the property that for every $x \in X$, there exists a unique sequence $(\alpha_n) \subset \mathbb{R}$ such that

$$\left|\left| x - \sum_{i=1}^{n} \alpha_i x_i \right|\right| \xrightarrow{n\to\infty} 0.$$

If (x_n) is a Schauder basis then there is a $K > 0$ such that $\left|\left| \sum_{i=1}^{n} \alpha_i x_i \right|\right| \leq K||x||$ for every $x \in X$ and $n \in \mathbb{N}$. In this case we say that (x_n) is a K-Schauder basis.

A retractional basis for a net implies that the Lipschitz-free space over the net has a Schauder basis, made out of the linearization of the retractions.

Theorem 1.4 ([HN17])**.** *Let M be a countable metric space with a K-retractional basis $\{\varphi_n\}_{n=1}^{\infty}$. Then the Lipschitz-free space over M has a K-Schauder basis.*

It might be worth giving a short explanation of what a Lipschitz free space is, despite not being necessary for the proofs below: Given a pointed metric space $(M, d, 0)$, we consider the Banach space $\mathrm{Lip}_0(M)$ as the space of all real-valued Lipschitz functions that vanish at the base point, endowed with the norm given by the Lipschitz constant,

$$||f||_{\mathrm{Lip}} = \sup\left\{ \frac{|f(p) - f(q)|}{d(p,q)} \ : \ p, q \in M,\ p \neq q \right\},$$

for every $f \in \mathrm{Lip}_0(M)$. The space $\mathrm{Lip}_0(M)$ is a dual space with unique predual. This predual is known as the Lipschitz free space associated to M, denoted by $\mathcal{F}(M)$. It is given by $\mathcal{F}(M) = \overline{\mathrm{span}}\{\delta_M(p) \ : \ p \in M\} \subset \mathrm{Lip}_0(M)^*$, where $\delta_M : M \to \mathrm{Lip}_0(M)^*$ is the operator sending points of p to their Dirac deltas. Its name comes from the fact that for any Lipschitz map $F : M \to N$ between two pointed metric spaces with $F(0) = 0$, there exists a linear and bounded operator $\widehat{F} : \mathcal{F}(M) \to \mathcal{F}(N)$ such that $||\widehat{F}|| = ||F||_{\mathrm{Lip}}$ and $\widehat{F} \circ \delta_M = \delta_N \circ F$.

In [HM22] we obtained a retractional basis for nets in Banach spaces with a Schauder basis and containing a copy of c_0, resp. having a c_0-like FDD.

It is known [JRZ71] that $\mathcal{L}_\infty$-spaces have a Schauder basis, but as shown in [BD80] they do not necessarily contain a copy of c_0.

For background on the approximation properties of Banach spaces we refer to the Handbook article [Cas01]. We will use the standard notation and terminology of the Banach spaces theory, as in [Fab+11]. For background on Lipschitz-free spaces and their main properties we refer to [GK03].

2. Nets in $\mathcal{L}_\infty$-spaces

The $\mathcal{L}_\infty$-spaces were introduced by Lindenstrauss and Pełczynski in [LP68]. They have played an important role in Banach space theory, and their structure theory is a very actively pursued topic of research (see for example [LR69],[JRZ71],[BD80],[BP83],[AH11] and [AGM16]). A crucial step in this theory was provided by Bourgain and Delbaen [BD80] who introduced an inductive construction scheme of such spaces with various prescribed properties. In particular, it was shown that $\mathcal{L}_\infty$-spaces need not necessarily contain a copy of c_0. It turned out recently [AGM16], that in fact the scheme can be applied to all separable $\mathcal{L}_\infty$-spaces. We will rely on this approach in our paper.

Definition 2.1. A Banach space X is called an $\mathcal{L}_{\infty,\lambda}$-space, for some $\lambda \geq 1$, if for every finite dimensional subspace F of X there is a finite dimensional subspace G of X, containing F and λ-isomorphic to ℓ_∞^n for $n = \dim G$. X is said to be an $\mathcal{L}_\infty$-space if it is an $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda \geq 1$.

Throughout this section, we are going to follow the notation, terminology and concepts used in [AGM16]. In particular, we are going to use the equivalent definition of $\mathcal{L}_\infty$-spaces given in [AGM16].

Definition 2.2. Let Γ_1, Γ be non-empty sets with $\Gamma_1 \subset \Gamma$. A linear operator $i : \ell_\infty(\Gamma_1) \to \ell_\infty(\Gamma)$ is said to be an extension operator if for every $x \in \ell_\infty(\Gamma_1)$ and $\gamma \in \Gamma_1$ we have that $x(\gamma) = i(x)(\gamma)$.

Definition 2.3. Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of non-empty sets and $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$. A sequence of extension operators $(i_n)_{n\in\mathbb{N}}$ with $i_n : \ell_\infty(\Gamma_n) \to \ell_\infty(\Gamma)$ for every $n \in \mathbb{N}$ is said to be compatible if for every $n, m \in \mathbb{N}$ with $n < m$ it holds that $i_n = i_m \circ r_m \circ i_n$, where $r_n : \ell_\infty(\Gamma) \to \ell_\infty(\Gamma_n)$ denotes the natural restriction operator.

Definition 2.4. A Banach space X is said to be a Bourgain-Delbaen space if there is a strictly increasing sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of non-empty finite sets and a sequence of compatible extension operators $(i_n)_{n\in\mathbb{N}}$ with $i_n : \ell_\infty(\Gamma_n) \to \ell_\infty(\Gamma)$ where $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ such that $\lambda = \sup_n ||i_n|| < \infty$ and

$$X = \overline{\bigcup_{n\in\mathbb{N}} i_n(\ell_\infty(\Gamma_n))}.$$

It is proved in Theorem 3.6 of [AGM16] that every separable infinite dimensional $\mathcal{L}_\infty$-space is isomorphic to a Bourgain-Delbaen space. The reverse

result is clearly satisfied. From now on, let X be a Bourgain-Delbaen space. Let us denote by (Γ_n) and (i_n) the sequences given in definition 2.4, and $\lambda = \sup_n ||i_n||$. We will also denote $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$.

For every $r \in \mathbb{R}$ we define the entire part of r as

$$[r] = \frac{r}{|r|} \max\big\{s \in \mathbb{N} \cup \{0\} \ : \ s \leq |r|\big\}$$

if $r \neq 0$ and $[0] = 0$. Given a subset S of Γ, we define the quantization of the elements of $\ell_\infty(S)$ by $f : \ell_\infty(S) \to \ell_\infty(S)$ where

$$f(x)(\gamma) = [x(\gamma)],$$

for every $x \in \ell_\infty(S)$ and $\gamma \in S$. Also, we define for every $s > 0$ the truncation of radius $s > 0$ in $\ell_\infty(S)$ as $T_s : \ell_\infty(S) \to sB_{\ell_\infty(S)}$ given by

$$T_s(x)(\gamma) = \begin{cases} x(\gamma) & \text{if } |x(\gamma)| \leq s, \\ s\frac{x(\gamma)}{|x(\gamma)|} & \text{if } |x(\gamma)| > s. \end{cases}$$

If $\Gamma_n \subset S$ for $n \in \mathbb{N}$ we will denote by $r_n : \ell_\infty(S) \to \ell_\infty(\Gamma_n)$ the natural restriction operator. It is worth mentioning that r_n and T_s are nonexpansive. When dealing with f, r_n and T_s we simplify the notation by not mentioning the set S, since it will be clear from the context.

Our intention is to prove that every net in X has a retractional basis. The proof is split into six steps. In the first two steps we define a separated set $M \subset \ell_\infty(\Gamma)$ and retractions onto some finite subsets of M. In the steps 3 to 5 we construct the retractional basis for M. Finally, in the last step we prove that there is a net in X that is Lipschitz equivalent to M. This finishes the proof since any two nets of an infinite-dimensional Banach space are necessarily Lipschitz equivalent [LMP00].

2.1. **Step 1.** In the first step we are going to define the set M by describing an increasing sequence of finite sets $(M_n)_{n\in\mathbb{N}}$ whose union will be M. We will also construct a sequence of commuting Lipschitz retractions (ϕ_n) from M onto M_n.

First, for every $n \in \mathbb{N}$ we set $\widetilde{M_n} = i_n \circ f\big(\lambda^n B_{\ell_\infty(\Gamma_n)}\big)$. Then, we define inductively an increasing sequence of sets $(M_n) \subset \ell_\infty(\Gamma)$ by taking $M_1 = f\big(\widetilde{M_1}\big)$ and for every $n \in \mathbb{N}$,

$$M_n = M_{n-1} \cup f \circ i_n\Big(f\big(\lambda^n B_{\ell_\infty(\Gamma_n)}\big) \setminus r_n\big(M_{n-1}\big)\Big).$$

Now we define M as $\bigcup\limits_{n\in\mathbb{N}} M_n$, which is clearly 1-separated. In order to construct the Lipschitz retractions onto the blocks M_n we need the following Lemmas.

Lemma 2.5. *For every $n \in \mathbb{N}$ and $x \in M_n$ the following inequality holds*

$$\big|\big|i_n \circ r_n(x) - x\big|\big| \leq \lambda + 1.$$

Proof. We let $\widetilde{x} = i_n \circ r_n(x) \in \widetilde{M_n}$. Now, let us take $m \leq n$ such that $x \in M_m \backslash M_{m-1}$ (considering $M_0 = \emptyset$). Then, by the definition of $M_m \backslash M_{m-1}$ it follows that

$$x = f \circ i_m \circ r_m(x).$$

Hence, it is clear that $||x - i_m \circ r_m(x)|| \leq 1$. Finally,

$$\begin{aligned} ||\widetilde{x} - x|| =& ||\widetilde{x} - i_m \circ r_m(x)|| + ||i_m \circ r_m(x) - x|| \\ \leq& ||i_n \circ r_n(x) - i_n \circ r_n(i_m \circ r_m(x))|| + 1 \\ \leq& ||i_n \circ r_n||_{\text{Lip}} ||x - i_m \circ r_m(x)|| + 1 \leq \lambda + 1. \end{aligned}$$

■

Lemma 2.6. *Given $n \in \mathbb{N}$, the map $r_n\big|_{M_n}$ is an injection. Moreover, $r_n(M_n) = f(\lambda^n B_{\ell_\infty(\Gamma_n)})$ and $\big(r_n\big|_{M_n}\big)^{-1} : f(\lambda^n B_{\ell_\infty(\Gamma_n)}) \to M_n$ is $(3\lambda + 2)$-Lipschitz.*

Proof. Let us first prove that $r_n(M_n) = f(\lambda^n B_{\ell_\infty(\Gamma_n)})$. Clearly, the inclusion $r_n(M_n) \subset f(\lambda^n B_{\ell_\infty(\Gamma_n)})$ is satisfied. Let us prove the other inclusion. In fact, for $n = 1$ it is trivially true and if $n \geq 2$ then

$$\begin{aligned} r_n(M_n) =& r_n\Big(M_{n-1} \cup f \circ i_n\Big(f\big(\lambda^n B_{\ell_\infty(\Gamma_n)}\big) \setminus r_n\big(M_{n-1}\big)\Big)\Big) \\ \supset& r_n\big(M_{n-1}\big) \cup \Big(r_n \circ f \circ i_n \circ f\big(\lambda^n B_{\ell_\infty(\Gamma_n)}\big) \setminus r_n \circ f \circ i_n \circ r_n\big(M_{n-1}\big)\Big) \\ =& r_n\big(M_{n-1}\big) \cup \Big(f\big(\lambda^n B_{\ell_\infty(\Gamma_n)}\big) \setminus r_n\big(M_{n-1}\big)\Big) = f\big(\lambda^n B_{\ell_\infty(\Gamma_n)}\big). \end{aligned}$$

We continue by proving that $r_n\big|_{M_n}$ is injective. We proceed by induction in $n \in \mathbb{N}$. Clearly, $r_1\big|_{M_1}$ is injective. Assuming that for arbitrary $n \geq 2$ the retraction $r_{n-1}\big|_{M_{n-1}}$ is injective, it suffices to prove that $r_n\big|_{M_n}$ is also injective. Let us take $x, y \in M_n$ such that $r_n(x) = r_n(y)$. From the assumption it follows that $r_n\big|_{M_{n-1}}$ is injective so we may restrict to the case when $x \notin M_{n-1}$. In this case, we claim that $y \notin M_{n-1}$. In fact, $r_n(y) = r_n(x) \in r_n(M_n \setminus M_{n-1})$ with

$$\begin{aligned} r_n(M_n \setminus M_{n-1}) =& r_n\big(f \circ i_n\big(f(\lambda^n B_{\ell_\infty(\Gamma_n)}) \setminus r_n(M_{n-1})\big)\big) \\ =& r_n\big(f \circ i_n\big(r_n(M_n) \setminus r_n(M_{n-1})\big)\big) \\ =& r_n(M_n) \setminus r_n(M_{n-1}). \end{aligned} \tag{2.1}$$

This means that $r_n(y) \notin r_n(M_{n-1})$ so $y \notin M_{n-1}$. Then, it is enough to prove the case when $x, y \in M_n \setminus M_{n-1}$. In this case,

$$x = f \circ i_n \circ r_n(x) = f \circ i_n \circ r_n(y) = y,$$

and we have shown that $r_n\big|_{M_n}$ is injective. Finally, let us prove that $\big(r_n\big|_{M_n}\big)^{-1}$ is $(3\lambda+2)$-Lipschitz. We take two points $x, y \in M_n$ with $r_n(x) \neq$

$r_n(y)$. Then, by Lemma 2.5 and the fact that $||r_n(x) - r_n(y)|| \geq 1$,

$$\begin{aligned}\big|\big|\big(r_n\big|_{M_n}\big)^{-1}(r_n(x)) - \big(r_n\big|_{M_n}\big)^{-1}(r_n(y))\big|\big| &= ||x-y|| \\ &\leq \big|\big|i_n \circ r_n(x) - x\big|\big| + \big|\big|i_n \circ r_n(y) - y\big|\big| + ||i_n \circ r_n(x) - i_n \circ r_n(y)|| \\ &\leq 2\lambda + 2 + \lambda||r_n(x) - r_n(y)|| \leq (3\lambda+2)||r_n(x) - r_n(y)||,\end{aligned}$$

which was to be proved. ∎

We may assume without loss of generality that $\lambda \in \mathbb{N}$. Take $s_n = \lambda^n$ for every $n \in \mathbb{N}$ and define the retraction $\phi_n : M \to M_n$ as $\phi_n = \big(r_n\big|_{M_n}\big)^{-1} \circ T_{s_n} \circ r_n$. Let us establish some useful properties of these retractions.

Proposition 2.7. *For every $m, n \in \mathbb{N}$ it holds that $\phi_n \circ \phi_m = \phi_{\min(m,n)}$ and $||\phi_n||_{Lip} \leq 3\lambda + 2$.*

Proof. The case when $n \geq m$ is clearly satisfied by the definition of retraction. Let us assume that $n < m$. It is easy to check that in this case $r_n \circ r_m = r_n$ and $\big(T_{s_n} \circ r_n\big) \circ \big(T_{s_m} \circ r_m\big) = T_{s_n} \circ r_n$. Hence,

$$\begin{aligned}\phi_n \circ \phi_m =& \big(r_n\big|_{M_n}\big)^{-1} \circ T_{s_n} \circ r_n \circ \big(r_m\big|_{M_m}\big)^{-1} \circ T_{s_m} \circ r_m \\ =& \big(r_n\big|_{M_n}\big)^{-1} \circ T_{s_n} \circ r_n \circ r_m \circ \big(r_m\big|_{M_m}\big)^{-1} \circ T_{s_m} \circ r_m \\ =& \big(r_n\big|_{M_n}\big)^{-1} \circ \big(T_{s_n} \circ r_n\big) \circ \big(T_{s_m} \circ r_m\big) = \big(r_n\big|_{M_n}\big)^{-1} \circ T_{s_n} \circ r_n = \phi_n.\end{aligned}$$

Finally, by Lemma 2.6 we conclude that $||\phi_n||_{\text{Lip}} \leq 3\lambda + 2$. ∎

This finishes the first step. The only difference between the constructions made in this step and a retractional basis is that the number of points of $M_n \setminus M_{n-1}$ is not one in general. In the next steps we will sharpen this result by defining blocks between M_n and M_{n-1} whose differences consist only of one element, and the corresponding commutative Lipschitz retractions.

2.2. **Step 2.** In this step we are going to define intermediate blocks D_n such that $M_n \subset D_n \subset M_{n+1}$ in order to retract the elements of M one by one.

To this end, we define an increasing sequence of sets (C_n) as

$$C_n = f \circ i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n\big(f(s_{n+1}B_{\ell_\infty(\Gamma_n)}) \setminus f(s_n B_{\ell_\infty(\Gamma_n)})\big),$$

for every $n \in \mathbb{N}$.

Lemma 2.8. *For every $n \in \mathbb{N}$, $C_n \subset M_{n+1} \setminus M_n$.*

Proof. We denote by $A = f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n\big(f(s_{n+1}B_{\ell_\infty(\Gamma_n)}) \setminus f(s_n B_{\ell_\infty(\Gamma_n)})\big)$. We claim that for every $n \in \mathbb{N}$,

$$A \subset r_{n+1}(M_{n+1}) \setminus r_{n+1}(M_n). \tag{2.2}$$

Clearly $A \subset f(s_{n+1}B_{\ell_\infty(\Gamma_{n+1})}) = r_{n+1}(M_{n+1})$ so it only remains to prove that $A \cap r_{n+1}(M_n) = \emptyset$. Indeed, $r_n(A) = f(s_{n+1}B_{\ell_\infty(\Gamma_n)}) \setminus f(s_n B_{\ell_\infty(\Gamma_n)}) = f(s_{n+1}B_{\ell_\infty(\Gamma_n)}) \setminus r_n(M_n)$ meaning that $r_n(A) \cap r_n(M_n) = \emptyset$. Hence,

$$r_n(A \cap r_{n+1}(M_n)) \subset r_n(A) \cap r_n(M_n) = \emptyset,$$

so $A \cap r_{n+1}(M_n) = \emptyset$. This proves (2.2) and it follows that

$$C_n = f \circ i_{n+1}(A) \subset f \circ i_{n+1}\big(r_{n+1}(M_{n+1}) \setminus r_{n+1}(M_n)\big) = M_{n+1} \setminus M_n.$$

■

It now makes sense to define $D_n = M_n \cup C_n$ for every $n \in \mathbb{N}$. Then, we have the followig increasing chain of subsets of M

$$M_1 \subset D_1 \subset M_2 \subset D_2 \subset \cdots \subset M_n \subset D_n \subset \cdots$$

We are going to make use of the following Lemma 2.9 which is an analogue of Lemma 2.6 in this new setting.

Lemma 2.9. *For every $n \in \mathbb{N}$ the map $r_n\big|_{D_n}$ is injective. Moreover, $r_n(D_n) = r_n(M_{n+1}) = f(s_{n+1}B_{\ell_\infty(\Gamma_n)})$ and $\big(r_n\big|_{D_n}\big)^{-1} : f(s_{n+1}B_{\ell_\infty(\Gamma_n)}) \to D_n$ is $(\lambda^2 + 4\lambda + 4)$-Lipschitz.*

Proof. Let us first show that $r_n(D_n) = r_n(M_{n+1}) = f(s_{n+1}B_{\ell_\infty(\Gamma_n)})$. In fact,

$$\begin{aligned} r_n(D_n) =& r_n(M_n) \cup r_n(C_n) \\ =& f(s_nB_{\ell_\infty(\Gamma_n)}) \cup \big(f(s_{n+1}B_{\ell_\infty(\Gamma_n)}) \setminus f(s_nB_{\ell_\infty(\Gamma_n)})\big) \\ =& f(s_{n+1}B_{\ell_\infty(\Gamma_n)}). \end{aligned}$$

By Lemma 2.6,

$$r_n(M_{n+1}) = r_n \circ r_{n+1}(M_{n+1}) = r_n\big(f(s_{n+1}B_{\ell_\infty(\Gamma_{n+1})})\big) = f(s_{n+1}B_{\ell_\infty(\Gamma_n)}).$$

We proceed by proving that $r_n\big|_{D_n}$ is injective. Let us take $x, y \in D_n$ such that $r_n(x) = r_n(y)$. We may assume that $x \notin M_n$ because the case when $x, y \in M_n$ was proven in Lemma 2.6. As $x \in C_n$, there is $\gamma \in \Gamma_n$ such that $|x(\gamma)| > s_n$. Therefore, $y \notin M_n$ since $|y(\gamma)| = |x(\gamma)| > s_n$ and so $x, y \in C_n$. Then, $x = f \circ i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n(r_n(x)) = f \circ i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n(r_n(y)) = y$, which proves that $r_n\big|_{D_n}$ is injective. Finally, to prove that $\Big|\Big|\Big(r_n\big|_{D_n}\Big)^{-1}\Big|\Big|_{\mathrm{Lip}} \leq \lambda^2 + 4\lambda + 4$ we take $x, y \in D_n$ with $r_n(x) \neq r_n(y)$. If $x, y \in M_n$ then by Lemma 2.6,

$$||x - y|| \leq (3\lambda + 2)||r_n(x) - r_n(y)||.$$

Now, if $x \in D_n \setminus M_n$ and $y \in M_n$ then

$$x = f \circ i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n(r_n(x)),$$

and by Lemma 2.5, $||y - y_1|| \leq \lambda + 1$ where $y_1 = i_n \circ r_n(y)$. Clearly $||y_1|| \leq \lambda||r_n(y)|| \leq \lambda^{n+1} = s_{n+1}$ so that $T_{s_{n+1}}(y_1) = y_1$. Also, by the compatibility condition of the extensions we have that $y_1 = i_{n+1} \circ r_{n+1}(y_1)$. Hence,

$$y_1 = i_{n+1} \circ r_{n+1} \circ T_{s_{n+1}} \circ i_n \circ r_n(y) = i_{n+1} \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(y).$$

We let $y_2 = i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(y)$ so that $||y_2 - y_1|| \leq \lambda$. Now, since $||f(u) - f(v)|| \leq 2 + ||u - v||$ for every $u, v \in \ell_\infty(\Gamma)$, we carry out the following computation

(2.3)

$$\begin{aligned}
&||f \circ i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(x) - f \circ i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(x)|| \\
&\quad \leq 2 + ||i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(x) - i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(y)|| \\
&\quad \leq 2 + \lambda ||f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(x) - f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(y)|| \\
&\quad \leq 2 + \lambda(2 + ||T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(x) - T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(y)||) \\
&\quad \leq 2 + \lambda(2 + \lambda ||x - y||) \leq (2 + 2\lambda + \lambda^2)||r_n(x) - r_n(y)||.
\end{aligned}$$

Using (2.3) we finish with this case since

$$\begin{aligned}
&||x - y|| \leq ||x - f(y_2)|| + ||f(y_2) - y_2|| + ||y_2 - y_1|| + ||y_1 - y|| \\
&\leq ||f \circ i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(x) - f \circ i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n \circ r_n(x)|| \\
&\quad + 2\lambda + 2 \leq (\lambda^2 + 4\lambda + 4)||r_n(x) - r_n(y)||.
\end{aligned}$$

The last case is when $x, y \in D_n \setminus M_n$ so that

$$\begin{aligned}
x &= f \circ i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n(r_n(x)), \\
y &= f \circ i_{n+1} \circ f \circ T_{s_{n+1}} \circ r_{n+1} \circ i_n(r_n(y)).
\end{aligned}$$

Therefore by (2.3),

$$||x - y|| \leq (\lambda^2 + 2\lambda + 2)||r_n(x) - r_n(y)||.$$

■

Lemma 2.9 allows us to define the intermediate retractions $\Psi_n : M_n \to D_{n-1}$ as $\Psi_n = \left(r_{n-1}\big|_{D_{n-1}}\right)^{-1} \circ r_{n-1}$ for every $n \geq 2$. Clearly, $\Psi_n \circ \Psi_m = \Psi_m$ for every $n > m \geq 2$. These retractions factorize $\phi_{n-1} : M \to M_{n-1}$ making use of D_{n-1} and M_n. More precisely,

$$\phi_{n-1} = \phi_{n-1}\big|_{D_{n-1}} \circ \Psi_n \circ \phi_n. \tag{2.4}$$

In fact, $\Psi_n\big|_{D_{n-1}} = Id_{D_{n-1}}$. The plan for the third step is to factorize Ψ_n by certain retractions that only retract one element while in the forth step we will do the same for $\phi_{n-1}\big|_{D_{n-1}}$.

2.3. **Step 3.** This is the longest and most technical step.

Our goal is to relabel the points of $M_n \setminus D_{n-1}$ as a finite sequence $(x_i^n)_{i=1}^{i(n)}$ and define Lipschitz retractions from M_n onto $D_{n-1} \cup (x_l^n)_{l \leq i}$ for every $i \in \{1, \ldots, i(n)\}$ which factorize Ψ_n. More precisely, in this step we prove the following result.

Proposition 2.10. *For every $n \in \mathbb{N}$, $n \geq 2$ there exists an order $(x_i^n)_{i=1}^{i(n)} = M_n \setminus D_{n-1}$ and for every $i \in \{0, \ldots, i(n) - 1\}$ there exists a retraction $\Psi_{n,i} : M_n \to D_{n-1} \cup (x_l^n)_{l \leq i}$ such that for every $i, i_1, i_2 \in \{0, \ldots, i(n) - 1\}$,*

(1) $\Psi_{n,i_1} \circ \Psi_{n,i_2} = \Psi_{n,\min(i_1,i_2)}$.

(2) $\Psi_{n,0} = \Psi_n$.
(3) $\Psi_{n,i}$ *is* K*-Lipschitz for some* K *independent of* i.

Throughout this step we fix some $n \in \mathbb{N}$ with $n \geq 2$. We will make use of the following notation.

If $A, B \subset \Gamma$ are disjoint subsets, we define the operation $\oplus : \ell_\infty(A) \times \ell_\infty(B) \to \ell_\infty(A \cup B)$ as

$$x \oplus y(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \in A, \\ y(\gamma) & \text{if } \gamma \in B. \end{cases}$$

Given $C_A \subset \ell_\infty(A)$ and $C_B \subset \ell_\infty(B)$ we denote

$$C_A \oplus C_B = \{x \oplus y \in \ell_\infty(A \cup B) \,:\, x \in C_A,\ y \in C_B\}.$$

With this terminology, we may identify M_n with $r_n(M_n)$ by Lemma 2.6 and it is easy to see that $r_n(M_n) = r_{n-1}(D_{n-1}) \oplus f(s_n B_{\ell_\infty(\Delta_n)})$ where $\Delta_n = \Gamma_n \setminus \Gamma_{n-1}$. We need to find a way to retract the points of $f(s_n B_{\ell_\infty(\Delta_n)})$ to an arbitrary $d \in D_{n-1}\big|_{\Delta_n}$ one by one in a Lipschitz manner. Our approach is to first define a retractional basis in $f(\ell_\infty(\Delta_n))$ whose first element in the order is 0 and then translate its retractions by the element d.

Denote $N^n = f(\ell_\infty(\Delta_n))$ and for every $m \in \mathbb{N} \cup \{0\}$ we also denote $N^n_m = f\big(mB_{\ell_\infty(\Delta_n)}\big)$. It is clear that $N^n_m \setminus N^n_{m-1} = f\big(mS_{\ell_\infty(\Delta_n)}\big)$ for every $m \in \mathbb{N}$. Let us denote the points of $N^n_m \setminus N^n_{m-1}$ as $(y^n_{m,1}, \dots y^n_{m,p_{n,m}})$ where $p_{n,m} = \#\big(N^n_m \setminus N^n_{m-1}\big)$ for every $m \in \mathbb{N}$. We consider the set of indices $A(n) = \bigcup_{m\in\mathbb{N}} \bigcup_{p=1}^{p_{n,m}} (m,p)$. Let us order the set $A(n) = (a^n_j)_{j\in\mathbb{N}}$ using the lexicographic order. Then, $N^n = (y^n_{a^n_j})_{j\in\mathbb{N}\cup\{0\}}$ where $y^n_{a^n_0} = 0$. To simplify the notation we rename $y^n_j = y^n_{a^n_j}$ for every $j \in \mathbb{N}\cup\{0\}$. It is immediate that the order respects the norm, that is, if $j_1 \leq j_2 \in \mathbb{N}\cup\{0\}$ then $||y^n_{j_1}|| \leq ||y^n_{j_2}||$.

Let us denote $N^{n,j} = (y^n_l)_{l\leq j}$. Then, we may define the 'local' retractions of N^n, $T^n_j : N^{n,j} \to N^{n,j-1}$ as the retraction such that $T^n_j(y^n_j) = T_{||y^n_j||-1}(y^n_j)$ for every $j \in \mathbb{N}$. It is straightforward from the definition of retraction that

$$T^n_{j_1} \circ T^n_{j_2} = T^n_{j_2} \quad \forall j_1 \geq j_2. \tag{2.5}$$

Given a pair $j_1, j_2 \in \mathbb{N}\cup\{0\}$ with $j_1 < j_2$, we define the value $m(j_2, j_1) \in \mathbb{N}\cup\{0\}$ as the unique element of $\mathbb{N}\cup\{0\}$ such that

$$T^n_{j_1+1} \circ \cdots \circ T^n_{j_2}(y^n_{j_2}) = T_{m(j_2,j_1)}(y^n_{j_2}).$$

Let us point out that $m(j_2, j_1)$ is necessarily unique since $T^n_{j_1+1} \circ \cdots \circ T^n_{j_2}(y^n_{j_2}) \neq y^n_{j_2}$. We start the proof of Proposition 2.10 which will be splitted into a series of auxiliary lemmas.

Lemma 2.11. *Let* $j_1, j_2 \in \mathbb{N}\cup\{0\}$ *be such that* $j_1 < j_2$. *Then* $||y^n_{j_1}|| - 1 \leq m(j_2, j_1) \leq ||y^n_{j_1}||$.

Proof. Clearly, if $y \in N^{n,j}$ then

$$||T_j^n(y)|| \geq \min\{||y||, ||y_j^n|| - 1\}. \tag{2.6}$$

Also, by induction in $j_2 - j_1$ using (2.6) it is easy to prove that for every $y \in N^{n,j_2}$,

$$||T_{j_1+1}^n \circ \cdots \circ T_{j_2}^n(y)|| \geq \min\{||y||, ||y_{j_1}^n|| - 1\}.$$

Then, it is straightforward to see that $m(j_2, j_1) = ||T_{j_1+1}^n \circ \cdots \circ T_{j_2}^n(y_{j_2}^n)|| \geq \min\{||y_{j_2}^n||, ||y_{j_1}^n|| - 1\} = ||y_{j_1}^n|| - 1$. Finally, since $T_{j_1+1}^n \circ \cdots \circ T_{j_2}^n(N^{n,j_2}) = N^{n,j_1}$ we know that $m(j_2, j_1) \leq ||y_{j_1}^n||$. ∎

Remark 2.12. Clearly $m(j_2, j_1)$ depends on $n \in \mathbb{N}$ but since $n \geq 2$ is fixed throughout the step 3 and the value $m(j_2, j_1)$ is used only within this step we do not express the dependence on n.

We are now ready to use the above constructions in our setting. Let us pick an arbitrary order in D_{n-1} given by $(d_1^n, \ldots, d_{k(n)}^n)$ where $k(n) = \#\big(D_{n-1}\big)$. For every $j \in \mathbb{N} \cup \{0\}$ we consider the sets of indices $K(j,n) = \{k \in \{1, \ldots, k(n)\} \,:\, d_k^n\big|_{\Delta_n} + y_j^n \in N_{s_n}^n\}$ and $J(n) = \{j \in \mathbb{N} \,:\, K(j,n) \neq \emptyset\}$. It is easy to see that $J(n)$ is finite. Finally, let us define the set of indices $E(n) = \{(j,k) \,:\, j \in J(n),\ k \in K(j,n)\}$, which is also a finite set. Now, we rename $E(n)$ as $\{e_1^n, \ldots, e_{i(n)}^n\}$ following the lexicographic order. If $i \in \{1, \ldots, i(n)\}$ with $e_i^n = (j,k)$ then we denote $e_i^n(1) = j$ and $e_i^n(2) = k$. The fact that $(e_i)_{i=1}^{i(n)}$ is ordered lexicographically may be understood as follows. $i_1, i_2 \in \{1, \ldots, i(n)\}$ with $i_1 < i_2$ means that $e_{i_1}^n(1) \leq e_{i_2}^n(1)$ and $e_{i_1}^n(1) = e_{i_2}^n(1)$ implies $e_{i_1}^n(2) < e_{i_2}^n(2)$.

Lemma 2.13. *The following equivalences hold:*

- $x \in M_n \setminus D_{n-1}$ *if, and only if, there is a unique* $e_i^n = (j,k) \in E(n)$ *such that* $x = \big(r_n\big|_{M_n}\big)^{-1}\big(r_{n-1}(d_k^n) \oplus \big(d_k^n\big|_{\Delta_n} + y_j^n\big)\big)$.
- $x \in D_{n-1}$ *if and only if there is a unique* $k \in \{1, \ldots, k(n)\} = K(0,n)$ *such that* $x = \big(r_n\big|_{M_n}\big)^{-1}\big(r_{n-1}(d_k^n) \oplus \big(d_k^n\big|_{\Delta_n} + y_0^n\big)\big) = d_k^n$.

Proof. It follows from the fact that $\big(r_n\big|_{M_n}\big)^{-1} : r_n(M_n) \to M_n$ is a bijection and the fact that $r_n(M_n) = r_{n-1}(D_{n-1}) \oplus N_{s_n}^n$. ∎

Following Lemma 2.13, we are able to establish an order of $M_n \setminus D_{n-1}$ by $(x_i^n)_{i=1}^{i(n)}$ where $x_i^n = \big(r_n\big|_{M_n}\big)^{-1}\big(r_{n-1}(d_k^n) \oplus \big(d_k^n\big|_{\Delta_n} + y_j^n\big)\big)$ whenever $e_i^n = (j,k) \in E(n)$. Now for every $i \in \{1, \ldots, i(n)\}$ we consider the set $D_{n-1,i} = D_{n-1} \cup (x_l^n)_{l \leq i}$ and $D_{n-1,0} = D_{n-1}$. This allows us to define the local intermediate retractions $\psi_{n,i} : D_{n-1,i} \to D_{n-1,i-1}$ for every $i \in \{1, \ldots i(n)\}$ by

$$\psi_{n,i}(x) = \begin{cases} \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d_k^n) \oplus \big(d_k^n\big|_{\Delta_n} + T_{||y_j^n||-1}(y_j^n)\big)\Big) & \text{if } x = x_i^n, \\ x & \text{otherwise,} \end{cases}$$

where $e_i^n = (j,k)$. Finally, for every $i \in \{0,\dots,i(n)-1\}$ we define the global intermediate retractions as

$$\Psi_{n,i} = \psi_{n,i+1} \circ \cdots \circ \psi_{n,i(n)} : M_n \to D_{n-1,i}.$$

As we pointed out in (2.5), in this setting we also want to mention that

$$\psi_{n,i_1} \circ \psi_{n,i_2} = \psi_{n,i_2} \quad \text{if } \ i_1 \geq i_2 \in \{1,\dots,i(n)\}, \tag{2.7}$$

and

$$\Psi_{n,i_1} \circ \Psi_{n,i_2} = \Psi_{n,\min(i_1,i_2)} \quad \text{if } \ i_1, i_2 \in \{0,\dots,i(n)-1\}. \tag{2.8}$$

Lemma 2.14. *Let $i_1, i_2 \in \{1,\dots,i(n)\}$ be such that $i_1 < i_2$. Then*

$$\Psi_{n,i_1}(x_{i_2}^n) = \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d_{k_2}^n) \oplus \Big(d_{k_2}^n\big|_{\Delta_n} + T_{M(i_1,i_2)}(y_{j_2}^n)\Big)\Big), \tag{2.9}$$

where $e_{i_1}^n = (j_1,k_1), e_{i_2}^n = (j_2,k_2) \in E(n)$ and

$$M(i_1,i_2) = \begin{cases} m(j_2,j_1) & \text{if } k_2 \leq k_1, \\ m(j_2,j_1-1) & \text{if } k_2 > k_1. \end{cases}$$

Remark 2.15. $M(i_1,i_2)$ is well defined whenever $i_1 < i_2$ since in this case $j_1 < j_2$ provided $k_1 \geq k_2$. Also $j_1 - 1 < j_2$ provided $k_1 < k_2$.

Proof. We may assume that $\psi_{n,i_2}(x_{i_2}^n) \notin D_{n-1}$ because if $\psi_{n,i_2}(x_{i_2}^n) \in D_{n-1}$ then $\Psi_{n,i_1}(x_{i_2}^n) = \psi_{n,i_2}(x_{i_2}^n) \in D_{n-1}$ and the statement is satisfied with $M(i_1,i_2) = 0$. Throughout all the proof we will denote $i_3 \in \{1,\dots,i_2-1\}$ such that $x_{i_3}^n = \psi_{n,i_2}(x_{i_2}^n)$ and $e_{i_3}^n = (j_3,k_2)$ for some $j_3 < j_2$. It holds that

$$T_{j_2}^n(y_{j_2}^n) = y_{j_3}^n. \tag{2.10}$$

Case 1, $k_2 \leq k_1$. In this case we prove (2.9) for $M(i_1,i_2) = m(j_2,j_1)$ by induction in $i_2 - i_1 \in \mathbb{N}$. Let us recall that, as mentioned in Remark 2.15, $j_1 < j_2$ throughout this case.

If $i_2 - i_1 = 1$ then $i_3 \leq i_2 - 1 = i_1$. Hence,

$$\Psi_{n,i_1}(x_{i_2}^n) = \psi_{n,i_2}(x_{i_2}^n) = \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d_{k_2}^n) \oplus \Big(d_{k_2}^n\big|_{\Delta_n} + T_{j_2}^n(y_{j_2}^n)\Big)\Big).$$

It is then enough to show that $T_{j_2}^n(y_{j_2}^n) = T_{m(j_2,j_1)}(y_{j_2}^n)$. Following the lexicographic order $j_3 = e_{i_3}^n(1) \leq e_{i_1}^n(1) = j_1 < j_2$, equivalently $j_3 < j_1 + 1 \leq j_2$. There are two possible cases, namely $j_2 = j_1 + 1$ or $j_2 - 1 \geq j_1 + 1$. If $j_2 = j_1 + 1$ then by definition $T_{j_2}^n(y_{j_2}^n) = T_{m(j_2,j_1)}(y_{j_2}^n)$. Otherwise, if $j_2 - 1 \geq j_1 + 1$ then

$$T_{j_2}^n(y_{j_2}^n) = y_{j_3}^n = T_{j_1+1}^n \circ \cdots \circ T_{j_2-1}^n(y_{j_3}^n) = T_{j_1+1}^n \circ \cdots \circ T_{j_2}^n(y_{j_2}^n) = T_{m(j_2,j_1)}(y_{j_2}^n).$$

This proves the first step of the induction.

Now we go on and prove the inductive step. Let us assume that $i_2 - i_1 \geq 2$ and that

$$\Psi_{n,i_1}(x_i^n) = \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d_k^n) \oplus \Big(d_k^n\big|_{\Delta_n} + T_{m(j,j_1)}(y_j^n)\Big)\Big) \tag{2.11}$$

holds for every index $i \in \{i_1+1, \cdots, i_2-1\}$ with $e_i^n = (j,k)$ and $k \leq k_1$. We now split the proof into two different cases, namely $i_1+1 \leq i_3$ and $i_1+1 > i_3$. If $i_1+1 \leq i_3$, then we claim that $j_1 < j_3 < j_2$. In fact, $i_1 < i_3$ and $e_{i_1}^n(2) = k_1 \geq k_2 = e_{i_3}^n(2)$ implies that $j_1 = e_{i_1}^n(1) < e_{i_3}^n(1) = j_3$. Also, $i_3 < i_2$ and $e_{i_3}^n(2) = k_2 = e_{i_2}^n(2)$ implies that $j_3 = e_{i_3}^n(1) < e_{i_2}^n(1) = j_2$ so we have shown that $j_1 < j_3 < j_2$. Therefore, by (2.5),

$$\begin{aligned} T_{m(j_3,j_1)}(y_{j_3}^n) =& T_{j_1+1}^n \circ \cdots \circ T_{j_3}^n\big(y_{j_3}^n\big) \\ =& T_{j_1+1}^n \circ \cdots \circ T_{j_3}^n\big(T_{j_3+1}^n \circ \cdots \circ T_{j_2+1}^n(y_{j_3}^n)\big) \\ =& T_{j_1+1}^n \circ \cdots \circ T_{j_2}^n\big(y_{j_2}^n\big) = T_{m(j_2,j_1)}(y_{j_2}^n). \end{aligned}$$

Hence, by equality (2.7) and the assumption (2.11) with $i = i_3$ we have that

$$\begin{aligned} \Psi_{n,i_1}(x_{i_2}^n) =& \Psi_{n,i_1}(x_{i_3}^n) = \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d_{k_2}^n) \oplus \Big(d_{k_2}^n\big|_{\Delta_n} + T_{m(j_3,j_1)}(y_{j_3}^n)\Big)\Big) \\ =& \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d_{k_2}^n) \oplus \Big(d_{k_2}^n\big|_{\Delta_n} + T_{m(j_2,j_1)}(y_{j_2}^n)\Big)\Big). \end{aligned}$$

Otherwise, if $i_1+1 > i_3$ then $i_2 - 1 \geq i_1 + 1 > i_3$ so by (2.7),

$$\begin{aligned} \Psi_{n,i_1}(x_{i_2}^n) =& \psi_{n,i_1+1} \circ \cdots \circ \psi_{n,i_2-1}(x_{i_3}^n) = x_{i_3}^n \\ =& \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d_{k_2}^n) \oplus \Big(d_{k_2}^n\big|_{\Delta_n} + T_{j_2}^n(y_{j_2}^n)\Big)\Big). \end{aligned}$$

It only remains to see that $T_{j_2}^n(y_{j_2}^n) = T_{m(j_2,j_1)}(y_{j_2}^n)$. Since $i_3 \leq i_1$, we have $j_3 \leq j_1 < j_2$. If $j_1 + 1 = j_2$ then $T_{j_2}^n(y_{j_2}^n) = T_{m(j_2,j_1)}(y_{j_2}^n)$ by definition. Otherwise, $j_3 < j_1 + 1 \leq j_2 - 1$ so we finish the proof with

$$T_{j_2}^n(y_{j_2}^n) = y_{j_3}^n = T_{j_1+1}^n \circ \cdots \circ T_{j_2-1}^n(y_{j_3}^n) = T_{j_1+1}^n \circ \cdots \circ T_{j_2}^n(y_{j_2}^n) = T_{m(j_2,j_1)}(y_{j_2}^n).$$

Case 2, $k_2 > k_1$**.** In this case we prove (2.9) with $M(i_1,i_2) = m(j_2, j_1 - 1)$. Again, let us prove this case by induction in $i_2 - i_1 \in \mathbb{N}$. If $i_2 - i_1 = 1$ then

$$\Psi_{n,i_1}(x_{i_2}^n) = \psi_{n,i_2}(x_{i_2}^n) = \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d_{k_2}^n) \oplus \Big(d_{k_2}^n\big|_{\Delta_n} + T_{j_2}^n(y_{j_2}^n)\Big)\Big).$$

Thus, it is enough to show that $T_{j_2}^n(y_{j_2}^n) = T_{m(j_2,j_1-1)}(y_{j_2}^n)$. Since $i_3 \leq i_2 - 1 = i_1$ with $e_{i_3}^n(2) = k_2 > k_1 = e_{i_1}^n(2)$ we deduce that $j_3 = e_{i_3}^n(1) < e_{i_1}^n(1) = j_1$. That is, $j_3 < j_1 \leq j_2$ and we distinguish two cases, namely $j_1 = j_2$ and $j_1 \leq j_2 - 1$. If $j_1 = j_2$ then $T_{j_2}^n(y_{j_2}^n) = T_{m(j_2,j_1-1)}(y_{j_2}^n)$ by definition. Otherwise $j_3 < j_1 \leq j_2 - 1$ and therefore,

$$T_{j_2}^n(y_{j_2}^n) = y_{j_3}^n = T_{j_1}^n \circ \cdots \circ T_{j_2-1}^n(y_{j_3}^n) = T_{j_1}^n \circ \cdots \circ T_{j_2}^n(y_{j_2}^n) = T_{m(j_2,j_1-1)}(y_{j_2}^n).$$

Now that we have proven the first step of the induction we pass to the inductive step when $i_2 - i_1 \geq 2$. We assume here that

$$\Psi_{n,i_1}(x_i^n) = \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d_k^n) \oplus \Big(d_k^n\big|_{\Delta_n} + T_{m(j,j_1-1)}(y_j^n)\Big)\Big) \tag{2.12}$$

holds for every index $i \in \{i_1+1, \ldots, i_2-1\}$ with $e_i^n = (j,k) \in E(n)$. We again split the proof of the inductive step into two different cases, namely,

$i_1+1\leq i_3$ and $i_1+1>i_3$. If $i_1+1\leq i_3$ then $j_1\leq j_3<j_2$ so by (2.5),

$$\begin{aligned}T_{m(j_3,j_1-1)}(y^n_{j_3}) =&T^n_{j_1}\circ\cdots\circ T^n_{j_3}(y^n_{j_3}) = T^n_{j_1}\circ\cdots\circ T^n_{j_3}\big(T^n_{j_3+1}\circ\cdots\circ T^n_{j_2}(y^n_{j_3})\big)\\ =&T^n_{j_1}\circ\cdots\circ T^n_{j_2}(y^n_{j_2}) = T_{m(j_2,j_1-1)}(y^n_{j_2}).\end{aligned}$$

Hence, using (2.12) with $i=i_3$,

$$\begin{aligned}\Psi_{n,i_1}(x^n_{i_2}) =&\Psi_{n,i_1}(x^n_{i_3}) = \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d^n_{k_2})\oplus\Big(d^n_{k_2}\big|_{\Delta_n}+T_{m(j_3,j_1-1)}(y^n_{j_3})\Big)\Big)\\ =&\Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d^n_{k_2})\oplus\Big(d^n_{k_2}\big|_{\Delta_n}+T_{m(j_2,j_1-1)}(y^n_{j_2})\Big)\Big).\end{aligned}$$

It only remains to treat the case when $i_1+1>i_3$, equivalently $i_3\leq i_1$. We first claim that

$$T^n_{j_2}(y^n_{j_2}) = T_{m(j_2,j_1-1)}(y^n_{j_2}). \tag{2.13}$$

Since $e^n_{i_3}(2)=k_2>k_1=e^n_{i_1}(2)$, by means of the lexicographic order it must hold that $j_3=e^n_{i_3}(1)<e^n_{i_1}(1)=j_1\leq j_2$. We split the proof of (2.13) into two different cases, $j_1=j_2$ and $j_1\leq j_2-1$. If $j_1=j_2$ then (2.13) holds by definition. Otherwise, $j_3<j_1\leq j_2-1$ so that

$$T^n_{j_2}(y^n_{j_2})=y^n_{j_3}=T^n_{j_1}\circ\cdots\circ T^n_{j_2-1}(y^n_{j_3}) = T^n_{j_1}\circ\cdots\circ T^n_{j_2}(y^n_{j_2}) = T_{m(j_2,j_1-1)}(y^n_{j_2}).$$

This finishes the proof since using (2.7) it follows that

$$\begin{aligned}\Psi_{n,i_1}(x^n_{i_2}) =&\Psi_{n,i_1}(x^n_{i_3}) = \psi_{n,i_1+1}\circ\cdots\circ\psi_{n,i_2-1}(x^n_{i_3}) = x^n_{i_3}\\ =&\Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d^n_{k_2})\oplus\Big(d^n_{k_2}\big|_{\Delta_n}+T^n_{j_2}(y^n_{j_2})\Big)\Big)\\ =&\Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d^n_{k_2})\oplus\Big(d^n_{k_2}\big|_{\Delta_n}+T_{m(j_2,j_1-1)}(y^n_{j_2})\Big)\Big).\end{aligned}$$

■

Lemma 2.16. *For every $i_1<i_2\in\{1,\dots,i(n)\}$,*

$$||y^n_{j_1}||-2\leq M(i_1,i_2)\leq||y^n_{j_1}||,$$

where $j_1=e^n_{i_1}(1)$.

Proof. Set $j_2=e^n_{i_2}(1)$, by Lemma 2.11 it follows that

$$M(i_1,i_2)\leq\max\{m(j_2,j_1),m(j_2,j_1-1)\}\leq\max\{||y^n_{j_1}||,||y^n_{j_1-1}||\}=||y^n_{j_1}||.$$

Also, it is straightforward to check that $||y^n_{j_1-1}||\geq||y^n_{j_1}||-1$ so again using Lemma 2.11 we obtain that

$$M(i_1,i_2)\geq\min\{m(j_2,j_1),m(j_2,j_1-1)\}\geq\min\{||y^n_{j_1}||-1,||y^n_{j_1-1}||-1\}\geq||y^n_{j_1}||-2.$$

■

Proposition 2.17. *It holds that $\Psi_{n,0}=\Psi_n$.*

Proof. Let us take $i \in \{1, \dots, i(n)\}$. We are denoting for this proof $e_i^n = (j,k)$ and $e_1^n = (j_1, k_1)$. We claim that $||y_{j_1}^n|| = 1$.

As $x_1^n \notin D_{n-1}$ we know that $||y_{j_1}^n|| \neq 0$ so we only have to show that $||y_{j_1}^n|| \leq 1$. If $d_{k_1}^n\big|_{\Delta_n} = 0$ then clearly $d_{k_1}^n\big|_{\Delta_n} + y_1^n = y_1^n \in f(B_{\ell_\infty(\Delta_n)}) \subset N_{s_n}^n$ so $1 = \min\{j \in J(n) \ : \ k_1 \in K(j,n)\} = j_1$. Hence, $||y_{j_1}^n|| = ||y_1^n|| = 1$. Otherwise, if $||d_{k_1}^n\big|_{\Delta_n}|| \geq 1$ then clearly $d_{k_1}^n\big|_{\Delta_n} - f\left(\frac{d_{k_1}^n\big|_{\Delta_n}}{||d_{k_1}^n\big|_{\Delta_n}||}\right) \in N_{s_n}^n$. This means that there must exist a $j_2 \in J(n)$ such that $y_{j_2}^n = -f\left(\frac{d_{k_1}^n\big|_{\Delta_n}}{||d_{k_1}^n\big|_{\Delta_n}||}\right)$ and $k_1 \in K(j_2, n)$. Hence, $j_2 \geq \min\{j \in J(n) \ : \ k_1 \in K(j,n)\} = j_1$ so $||y_{j_1}^n|| \leq ||y_{j_2}^n|| = 1$ and the claim is proven.

Now, we pass to the proof that $\Psi_{n,0} = \Psi_n$ distinguishing two different cases. The first case is when $\Psi_{n,1}(x_i^n) = x_1^n$. In this case,

$$\begin{aligned}
\Psi_{n,0}(x_i^n) =& \psi_{n,1} \circ \Psi_{n,1}(x_i^n) = \psi_{n,1}(x_1^n) \\
=& \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d_{k_1}^n) \oplus \Big(d_{k_1}^n\big|_{\Delta_n} + T_{||y_{j_1}^n||-1}(y_{j_1}^n)\Big)\Big) \\
=& \Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_n(d_k^n)\Big) \\
=& d_k^n = \Big(r_{n-1}\big|_{D_{n-1}}\Big)^{-1}\Big(r_{n-1}(x_i^n)\Big) = \Psi_n(x_i^n).
\end{aligned}$$

The second case is when $\Psi_{n,1}(x_i^n) \neq x_1^n$ or equivalently $\Psi_{n,1}(x_i^n) \in D_{n-1,1} \setminus \{x_1^n\} = D_{n-1}$. In this case $i > 1$ so by Lemma 2.14,

$$\begin{aligned}
r_{n-1}\big(\Psi_{n,1}(x_i^n)\big) =& r_{n-1}\Bigg(\Big(r_n\big|_{M_n}\Big)^{-1}\Big(r_{n-1}(d_k^n) \oplus \Big(d_k^n\big|_{\Delta_n} + T_{M(1,i)}(y_j^n)\Big)\Big)\Bigg) \\
=& r_{n-1}(d_k^n).
\end{aligned}$$

Finally, since $r_{n-1}\big|_{D_{n-1}}$ is an injection and $\Psi_{n,1}(x_i^n) \in D_{n-1}$ we have that

$$\begin{aligned}
\Psi_{n,0}(x_i^n) =& \psi_{n,1} \circ \Psi_{n,1}(x_i^n) = \Psi_{n,1}(x_i^n) \\
=& \Big(r_{n-1}\big|_{D_{n-1}}\Big)^{-1}\Big(r_{n-1}\big(\Psi_{n,1}(x_i^n)\big)\Big) \\
=& \Big(r_{n-1}\big|_{D_{n-1}}\Big)^{-1}\Big(r_{n-1}\big(d_k^n\big)\Big) \\
=& d_k^n = \Big(r_{n-1}\big|_{D_{n-1}}\Big)^{-1}\Big(r_{n-1}(x_i^n)\Big) = \Psi_n(x_i^n).
\end{aligned}$$

■

Proposition 2.18. *For every $i \in \{1, \dots, i(n)\}$ the retraction $\Psi_{n,i}$ is $(3\lambda + 2)(2\lambda^2 + 8\lambda + 12)$-Lipschitz.*

Proof. Let us take distinct points $x_1, x_2 \in M_n$, i.e. $||x_1 - x_2|| \geq 1$. We may assume without loss of generality that $x_1 \in M_n \setminus D_{n-1,i}$ because otherwise $\Psi_{n,i}$ acts as an identity. Hence, there is $i_1 \in \{i+1, \dots, i(n)\}$ with $e_{i_1}^n =$

$(j_1,k_1)\in E(n)$ such that $x_1=x^n_{i_1}$. There are two possibilities, namely either $x_2\in D_{n-1,i}$ or $x_2\in M_n\setminus D_{n-1,i}$. If $x_2\in D_{n-1,i}$ then by Lemma 2.13 there are unique $j_2\in J(n)\cup\{0\}$ and $k_2\in K(j_2,n)$ such that $(j_2,k_2)\leq(j,k)=e^n_i$ (considering the lexicographic order) and $x_2=\big(r_n\big|_{M_n}\big)^{-1}\big(r_{n-1}(d^n_{k_2})\oplus\big(d^n_{k_2}\big|_{\Delta_n}+y^n_{j_2}\big)\big)$. As $j_2\leq j\leq j_1$, by Lemma 2.16 we have $||y^n_{j_2}||\leq||y^n_j||\leq M(i,i_1)+2$ and so $\max\{0,||y^n_{j_2}||-M(i,i_1)\}\leq 2$. Hence,

$$\begin{aligned}||T_{M(i,i_1)}(y^n_{j_1})-y^n_{j_2}||\leq&||T_{M(i,i_1)}(y^n_{j_1})-T_{M(i,i_1)}(y^n_{j_2})||+||T_{M(i,i_1)}(y^n_{j_2})-y^n_{j_2}||\\=&||T_{M(i,i_1)}(y^n_{j_1})-T_{M(i,i_1)}(y^n_{j_2})||+\max\{0,||y^n_{j_2}||-M(i,i_1)\}\\\leq&||y^n_{j_1}-y^n_{j_2}||+2.\end{aligned}$$

Also, by Lemma 2.9, it follows that

$$\begin{aligned}||d^n_{k_1}\big|_{\Delta_n}-d^n_{k_2}\big|_{\Delta_n}||\leq&||d^n_{k_1}-d^n_{k_2}||\\=&\Big|\Big|\Big(r_{n-1}\big|_{D_{n-1}}\Big)^{-1}(r_{n-1}(d^n_{k_1}))-\Big(r_{n-1}\big|_{D_{n-1}}\Big)^{-1}(r_{n-1}(d^n_{k_2}))\Big|\Big|\\=&\Big|\Big|\Big(r_{n-1}\big|_{D_{n-1}}\Big)^{-1}(r_{n-1}(x_1))-\Big(r_{n-1}\big|_{D_{n-1}}\Big)^{-1}(r_{n-1}(x_2))\Big|\Big|\\\leq&(\lambda^2+4\lambda+4)||x_1-x_2||.\end{aligned}$$

Therefore,

$$\begin{aligned}||y^n_{j_1}-y^n_{j_2}||\leq&\big|\big|\big(d^n_{k_1}\big|_{\Delta_n}+y^n_{j_1}\big)-\big(d^n_{k_2}\big|_{\Delta_n}+y^n_{j_2}\big)\big|\big|+\big|\big|d^n_{k_1}\big|_{\Delta_n}-d^n_{k_2}\big|_{\Delta_n}\big|\big|\\=&||(x_1-x_2)\big|_{\Delta_n}||+\big|\big|d^n_{k_1}\big|_{\Delta_n}-d^n_{k_2}\big|_{\Delta_n}\big|\big|\leq(\lambda^2+4\lambda+5)||x_1-x_2||.\end{aligned}$$

By Lemma 2.6 and Lemma 2.14 we compute

$$\begin{aligned}||\Psi_{n,i}(x_1)-\Psi_{n,i}(x_2)||\leq&\Big|\Big|\Big(r_n\big|_{M_n}\Big)^{-1}\Big|\Big|_{\mathrm{Lip}}\Big(||r_{n-1}(d^n_{k_1}-d^n_{k_2})||\\&+||d^n_{k_1}\big|_{\Delta_n}-d^n_{k_2}\big|_{\Delta_n}||+||T_{M(i,i_1)}(y^n_{j_1})-y^n_{j_2}||\Big)\\\leq&(3\lambda+2)(2\lambda^2+8\lambda+12)||x_1-x_2||.\end{aligned}$$

Finally, if $x_2\in M_n\setminus D_{n-1,i}$ then $x_2=x^n_{i_2}$, for some $i_2\in\{i+1,\dots,i(n)\}$. If we denote $e^n_{i_2}=(j_2,k_2)$, then by Lemma 2.16,

$$|M(i,i_1)-M(i,i_2)|\leq\big|\,||y^n_j||-||y^n_j||\,\big|+2=2.$$

Therefore,

$$\begin{aligned}||T_{M(i,i_1)}(y^n_{j_1})-T_{M(i,i_2)}(y^n_{j_2})||\leq&||T_{M(i,i_1)}(y^n_{j_1})-T_{M(i,i_1)}(y^n_{j_2})||\\&+||T_{M(i,i_1)}(y^n_{j_2})-T_{M(i,i_2)}(y^n_{j_2})||\\\leq&||y^n_{j_1}-y^n_{j_2}||+|M(i,i_1)-M(i,i_2)|\\\leq&||y^n_{j_1}-y^n_{j_2}||+2.\end{aligned}$$

Following the same computation as in the previous case we have that

$$||d^n_{k_1}\big|_{\Delta_n}-d^n_{k_2}\big|_{\Delta_n}||\leq(\lambda^2+4\lambda+4)||x_1-x_2||,$$

and

$$||y^n_{j_1}-y^n_{j_2}||\leq(\lambda^2+4\lambda+5)||x_1-x_2||.$$

By Lemma 2.6 and Lemma 2.14 we finally conclude that

$$\begin{aligned}||\Psi_{n,i}(x_1) - \Psi_{n,i}(x_2)|| \leq & \left|\left|\left(r_n\big|_{M_n}\right)^{-1}\right|\right|_{\mathrm{Lip}}\Big(||r_{n-1}(d^n_{k_1} - d^n_{k_2})|| \\ & + ||d^n_{k_1}\big|_{\Delta_n} - d^n_{k_2}\big|_{\Delta_n}|| + ||T_{M(i,i_1)}(y^n_{j_1}) - T_{M(i,i_2)}(y^n_{j_2})||\Big) \\ & \leq (3\lambda+2)(2\lambda^2+8\lambda+12)||x_1 - x_2||.\end{aligned}$$

■

This finishes the proof of Proposition 2.10 with $K = (3\lambda+2)(2\lambda^2+8\lambda+12)$.

2.4. **Step 4.** In this step we focus on the points that are in $D_n \setminus M_n = C_n$. Similarly to what we have done in the third step, we will prove the following Proposition 2.19.

Proposition 2.19. *For every $n \in \mathbb{N}$, there exists an order $(c^n_i)_{i=1}^{c(n)} = D_n \setminus M_n = C_n$ and for every $i \in \{0, \dots, c(n)-1\}$ there exists a retraction $\phi_{n,i} : D_n \to M_n \cup (c^n_l)_{l\leq i}$ such that for every $i, i_1, i_2 \in \{0, \dots, c(n)-1\}$, we have*

(1) $\phi_{n,i_1} \circ \phi_{n,i_2} = \phi_{n,\min(i_1,i_2)}$.
(2) $\phi_{n,0} = \phi_n\big|_{D_n}$.
(3) $\phi_{n,i}$ is L-Lipschitz for some L independent of i.

Again, let us fix an $n \in \mathbb{N}$. Thanks to Lemma 2.9 we may identify C_n with $r_n(C_n) = f(s_{n+1}B_{\ell_\infty(\Gamma_n)}) \setminus f(s_nB_{\ell_\infty(\Gamma_n)}) = \bigcup_{k=s_n+1}^{s_{n+1}} f(kS_{\ell_\infty(\Gamma_n)})$. Now, it is a matter of ordering $\bigcup_{k=s_n+1}^{s_{n+1}} f(kS_{\ell_\infty(\Gamma_n)})$. To do so, we give an arbitrary order to $f(kS_{\ell_\infty(\Gamma_n)}) = N_k$ for $k \in \mathbb{N} \cap [s_n+1, s_{n+1}]$ as $N_k = (z_{k,1}, \dots, z_{k,l_{n,k}})$. Here we consider the set of indices $G(n) = \bigcup_{k=s_n+1}^{s_{n+1}} \bigcup_{l=1}^{l_{n,k}} (k,l)$. We rename the points of $G(n)$ using the lexicographic order by $G(n) = (g^n_i)_{i=1}^{c(n)}$ where $c(n) = \#G(n) = \#C_n$. Let us denote $z^n_i = z_{g^n_i}$ for every $i \in \{1, \dots, c(n)\}$. The key property of this order is that if $i_1, i_2 \in \{1, \dots, c(n)\}$ with $i_1 \leq i_2$, then $||z^n_{i_1}|| \leq ||z^n_{i_2}||$. This allows us to list the elements of C_n as $c^n_i = \left(r_n\big|_{C_n}\right)^{-1}(z^n_i)$. Considering $C_n = (c^n_i)_{i=1}^{c(n)}$ as an ordered set we obtain

$$||r_n(c^n_{i_1})|| = ||z^n_{i_1}|| \leq ||z^n_{i_2}|| = ||r_n(c^n_{i_2})|| \quad \text{whenever } i_1 \leq i_2 \in \{1, \dots, c(n)\}.$$

Let $M_{n,i} = M_n \cup (c^n_l)_{l\leq i}$ for $i \in \{1, \dots, c(n)\}$ and $M_{n,0} = M_n$. Then for every $i \in \{1, \dots, c(n)\}$ we define the local retraction $T_{n,i} : M_{n,i} \to M_{n,i-1}$ by

$$T_{n,i}(x) = \begin{cases} \left(r_n\big|_{D_n}\right)^{-1}\left(T_{||r_n(x)||-1}(r_n(x))\right) & \text{if } x = c^n_i, \\ x & \text{otherwise.} \end{cases}$$

We are finally ready to define the retractions of Proposition 2.19. We define for each $i \in \{0, \dots, c(n)-1\}$ the retraction $\phi_{n,i} : D_n \to M_{n,i}$ by

$$\phi_{n,i} = T_{n,i+1} \circ \cdots \circ T_{n,c(n)}.$$

Lemma 2.20. *For every $i \in \{0, \dots, c(n)-1\}$ and $x \in D_n$,*

$$\phi_{n,i}(x) = \big(r_n\big|_{D_n}\big)^{-1}\big(T_{||r_n \circ \phi_{n,i}(x)||}(r_n(x))\big).$$

Proof. We proceed by induction in $c(n) - i \in \mathbb{N}$. In case when $c(n) - i = 1$ we distinguish two cases. If $x = c^n_{c(n)}$ then

$$\phi_{n,c(n)-1}(x) = T_{n,c(n)}(x) = \big(r_n\big|_{D_n}\big)^{-1}\big(T_{||r_n(x)||-1}(r_n(x))\big),$$

so that we only need to show that $||r_n(x)|| - 1 = ||r_n \circ \phi_{n,c(n)-1}(x)||$. In fact,

$$||r_n \circ \phi_{n,c(n)-1}(x)|| = ||T_{||r_n(x)||-1}(r_n(x))|| = ||r_n(x)|| - 1.$$

If $x \neq c^n_{c(n)}$ then

$$\phi_{n,c(n)-1}(x) = T_{n,c(n)}(x) = x = \big(r_n\big|_{D_n}\big)^{-1}\big(T_{||r_n(x)||}(r_n(x))\big),$$

which ends the first step of the induction since $||r_n(x)|| = ||r_n \circ \phi_{n,c(n)-1}(x)||$. Finally, let us prove the induction step. If $c(n) - i \geq 2$ we assume that

$$\phi_{n,i+1}(x) = \big(r_n\big|_{D_n}\big)^{-1}\big(T_{||r_n \circ \phi_{n,i+1}(x)||}(r_n(x))\big).$$

Again, we just split this step into two cases. If $\phi_{n,i+1}(x) = c^n_{i+1}$ then

$$\begin{aligned}
\phi_{n,i}(x) =& T_{n,i+1} \circ \phi_{n,i+1}(x) = \big(r_n\big|_{D_n}\big)^{-1}\big(T_{||r_n \circ \phi_{n,i+1}(x)||-1}(r_n \circ \phi_{n,i+1}(x))\big) \\
=& \big(r_n\big|_{D_n}\big)^{-1}\big(T_{||r_n \circ \phi_{n,i+1}(x)||-1}(T_{||r_n \circ \phi_{n,i+1}(x)||}(r_n(x)))\big) \\
=& \big(r_n\big|_{D_n}\big)^{-1}\big(T_{||r_n \circ \phi_{n,i+1}(x)||-1}(r_n(x))\big) \\
=& \big(r_n\big|_{D_n}\big)^{-1}\big(T_{||r_n \circ \phi_{n,i}(x)||}(r_n(x))\big).
\end{aligned}$$

Otherwise, if $\phi_{n,i+1}(x) \neq c^n_{i+1}$ then

$$\begin{aligned}
\phi_{n,i}(x) =& T_{n,i+1} \circ \phi_{n,i+1}(x) = \phi_{n,i+1}(x) = \big(r_n\big|_{D_n}\big)^{-1}\big(T_{||r_n \circ \phi_{n,i+1}(x)||}(r_n(x))\big) \\
=& \big(r_n\big|_{D_n}\big)^{-1}\big(T_{||r_n \circ \phi_{n,i}(x)||}(r_n(x))\big).
\end{aligned}$$

■

We are finally ready to prove the properties stated in Proposition 2.19. Property (1) of Proposition 2.19 follows from the definition of $\phi_{n,i}$. We prove properties (2) and (3) separately in the next Propositions.

Proposition 2.21. *The equality $\phi_{n,0} = \phi_n\big|_{D_n}$ holds true.*

Proof. If $x \in M_n$ then $\phi_{n,0}(x) = x = \phi_n(x)$. Let us take $c^n_i \in C_n$. Clearly, if some $x \in M_{n,i}$ is such that $||r_n(x)|| \geq s_n$ then $||r_n \circ T_{n,i}(x)|| \geq s_n$ so that iductively it follows that $||r_n \circ \phi_{n,0}(c^n_i)|| = ||r_n \circ T_{n,1} \circ \cdots \circ T_{n,c(n)}(c^n_i)|| \geq s_n$.

Also, we know that $r_n \circ \phi_{n,0}(c_i^n) \in r_n(M_n) = f(s_n B_{\ell_\infty(\Gamma_n)})$ so necessarily $||r_n \circ \phi_{n,0}(c_i^n)|| = s_n$. We are done since Lemma 2.20 yields that

$$\begin{aligned}\phi_{n,0}(c_i^n) =& \big(r_n\big|_{D_n}\big)^{-1}\big(T_{||r_n\circ\phi_{n,0}(c_i^n)||}(r_n(c_i^n))\big)\\ =&\big(r_n\big|_{D_n}\big)^{-1}\big(T_{s_n}(r_n(c_i^n))\big) = \big(r_n\big|_{M_n}\big)^{-1}\big(T_{s_n}(r_n(c_i^n))\big) = \phi_n(c_i^n).\end{aligned}$$

■

Proposition 2.22. *For every $i \in \{1,\dots,c(n)-1\}$, the retraction $\phi_{n,i}$ is $(2\lambda^2+8\lambda+8)$-Lipschitz.*

Proof. Clearly, if $i \in \{0,\dots,c(n)-1\}$ and $x \in M_{n,i+1}$ then

$$||r_n \circ T_{n,i+1}(x)|| \geq \min\{||r_n(x)||, ||z_i^n|| - 1\}. \tag{2.14}$$

We claim that for every $x \in D_n$ and every $i \in \{0,\dots,c(n)-1\}$,

$$||z_i^n|| \geq ||r_n \circ \phi_{n,i}(x)|| \geq \min\{||r_n(x)||, ||z_i^n|| - 1\}. \tag{2.15}$$

The first inequality follows from the fact that $r_n \circ \phi_{n,i}(x) = z_{\widetilde{i}}^n \in r_n(M_{n,i})$ for some $\widetilde{i} \leq i$ whenever $\phi_{n,i}(x) \in C_n$; and $||r_n \circ \phi_{n,i}(x)|| \leq s_n < ||z_i^n||$ whenever $\phi_{n,i}(x) \in M_n$. To prove the second inequality, we distinguish two cases. If $x \in M_{n,i}$ then (2.15) is satisfied since $r_n \circ \phi_{n,i}(x) = r_n(x)$ and clearly $||r_n(x)|| \geq \min\{||r_n(x)||, ||z_i^n|| - 1\}$. Otherwise, if $x \in D_n \setminus M_{n,i}$ we proceed by induction in $c(n) - i \in \mathbb{N}$. For the first step, $i = c(n)-1$ and $x \in D_n \setminus M_{n,c(n)-1}$ so that $x = c_{c(n)}^n$. Then,

$$\begin{aligned}||r_n \circ \phi_{n,c(n)-1}(x)|| =& ||r_n \circ T_{n,c(n)}(c_{c(n)}^n)|| = ||r_n(c_{c(n)}^n)|| - 1\\ =& ||z_{c(n)-1}^n|| - 1 \geq \min\{||r_n(x)||, ||z_{c(n)-1}^n|| - 1\}.\end{aligned}$$

To prove the inductive step we take $i \in \{0,\dots,c(n)-2\}$ and assume by induction hypothesis that

$$||r_n \circ \phi_{n,i+1}(x)|| \geq \min\{||r_n(x)||, ||z_{i+1}^n|| - 1\}. \tag{2.16}$$

Now, (2.15) follows from (2.14) and (2.16) since

$$\begin{aligned}||r_n \circ \phi_{n,i}(x)|| =& ||r_n \circ T_{n,i+1} \circ \phi_{n,i+1}(x)|| \geq \min\{||r_n \circ \phi_{n,i+1}(x)||, ||z_i^n|| - 1\}\\ &\geq \min\{||r_n(x)||, ||z_{i+1}^n|| - 1, ||z_i^n|| - 1\} = \min\{||r_n(x)||, ||z_i^n|| - 1\}.\end{aligned}$$

Now that we know (2.15) holds true, we may compute the Lipschitz norm of $\phi_{n,i}$. Let us take distinct points $x, y \in D_n$ (meaning that $r_n(x) \neq r_n(y)$), assuming without loss of generality that $x \notin M_{n,i}$ (otherwise the statement is trivially satisfied). Since $x \notin M_{n,i}$ we know that $||r_n(x)|| \geq ||z_i^n||$. By (2.15) it follows that

$$||r_n\circ\phi_{n,i}(y)|| - ||r_n\circ\phi_{n,i}(x)|| \leq ||z_i^n|| - \min\{||r_n(x)||, ||z_i^n|| - 1\} = 1 \leq ||x-y||.$$

Also, taking again into account that $||z_i^n|| \leq ||r_n(x)||$,

$$||r_n \circ \phi_{n,i}(x)|| - ||r_n \circ \phi_{n,i}(y)|| \leq ||z_i^n|| - \min\{||r_n(y)||, ||z_i^n|| - 1\} \leq ||x - y||.$$

Therefore, using Lemma 2.20 we obtain that

$$\begin{aligned}||r_n\circ\phi_{n,i}(x)-r_n\circ\phi_{n,i}(y)||=&||T_{||r_n\circ\phi_{n,i}(x)||}(r_n(x))-T_{||r_n\circ\phi_{n,i}(y)||}(r_n(y))||\\ \leq&||T_{||r_n\circ\phi_{n,i}(x)||}(r_n(x))-T_{||r_n\circ\phi_{n,i}(y)||}(r_n(x))||\\ &+||T_{||r_n\circ\phi_{n,i}(y)||}(r_n(x))-T_{||r_n\circ\phi_{n,i}(y)||}(r_n(y))||\\ \leq&\big|\,||r_n\circ\phi_{n,i}(x)||-||r_n\circ\phi_{n,i}(y)||\,\big|\\ &+||r_n(x)-r_n(y)||\\ \leq&2||x-y||.\end{aligned}$$

Hence,

$$\begin{aligned}||\phi_{n,i}(x)-\phi_{n,i}(y)||\leq&||\big(r_n\big|_{D_n}\big)^{-1}\big(r_n\circ\phi_{n,i}(x)\big)-\big(r_n\big|_{D_n}\big)^{-1}\big(r_n\circ\phi_{n,i}(y)\big)||\\ \leq&(\lambda^2+2\lambda+2)||r_n\circ\phi_{n,i}(x)-r_n\circ\phi_{n,i}(y)||\\ \leq&(\lambda^2+4\lambda+4)2||x-y||.\end{aligned}$$

■

We are now done with the fourth step, proving Proposition 2.19 with $L=2\lambda^2+8\lambda+8$.

2.5. **Step 5.** In this step we are finally constructing a retractional basis for M. It is just a matter of amalgamating the retractions of Proposition 2.10 and Proposition 2.19 together.

Theorem 2.23. *M has a retractional basis.*

Proof. We list the elements of M_1 as $(x_1^1,\dots,x_{i(1)}^1)$ where $i(1)=\#M_1$ and $x_1^1=0\in M_1$. Let us define the mapping $I:M\to\mathbb{N}$ as

$$I(x)=\begin{cases}i & \text{if } x=x_i^1\in M_1,\\ i+\#M_n & \text{if } x=c_i^n\in C_n=D_n\setminus M_n \text{ for } n\in\mathbb{N},\\ i+\#D_n & \text{if } x=x_i^n\in M_n\setminus D_{n-1} \text{ for } n\geq 2.\end{cases}$$

I is clearly well defined and bijective. It is straightforward to see that

- $I(x)\in\{1,\dots,i(1)\}$ if, and only if, $x=x_{I(x)}^1\in M_1$.
- $I(x)\in\{\#M_n+1,\dots,\#D_n\}$ if, and only if, $x=c_{I(x)-\#M_n}^n\in D_n\setminus M_n=C_n$.
- $I(x)\in\{\#D_{n-1}+1,\dots,\#M_n\}$ if, and only if, $x=x_{I(x)-\#D_{n-1}}^n\in M_n\setminus D_{n-1}$.

We denote $M^i=(I^{-1}(l))_{l\leq i}$ for every $i\in\mathbb{N}$. For each $i\in\mathbb{N}$ the retraction of the retractional basis onto M^i $\varphi_i:M\to M^i$ is given by

$$\varphi_i(x)=\begin{cases}x & \text{if } x\in M^i,\\ \phi_{n,i-\#M_n}\circ\Psi_{n+1}\circ\phi_{n+1}(x) & \text{if } I^{-1}(i)\in D_n\setminus M_n \text{ for } n\in\mathbb{N},\\ \Psi_{n,i-\#D_{n-1}}\circ\phi_n(x) & \text{if } I^{-1}(i)\in M_n\setminus D_{n-1} \text{ for } n\geq 2,\\ 0 & \text{if } I^{-1}(i)\in M_1,\ x\notin M^i.\end{cases}$$

An easy case by case computation shows that φ_i is a well defined retraction onto M^i satisfying $\varphi_{i_1}\circ\varphi_{i_2}=\varphi_{\min(i_1,i_2)}$ for every $i_1,i_2\in\mathbb{N}$. Indeed, if $i_2<i_1$ then the definition of retraction yields the wished equality. Otherwise, if $i_1<i_2$ then the only nontrivial cases are those cases when $I^{-1}(i_1),I^{-1}(i_2)$ are in $D_n\setminus M_n$ or $M_n\setminus D_{n-1}$ for (possibly) different n's. Hence, there are only 4 cases to treat, which are deduced from Proposition 2.10, Proposition 2.19 and the following easy to check equations,

(1) $\phi_{n_1}=\phi_{n_1}\circ\Psi_{n_2,i_2}\circ\phi_{n_2}$ whenever $n_1<n_2$, $n_2\geq 2$, $i_2\in\{0,\dots,i(n_2)-1\}$,
(2) $\phi_{n_1}\circ\phi_{n_2,i_2}=\phi_{n_1}$, whenever $n_1\leq n_2$, $i_2\in\{0,\dots,c(n_2)-1\}$.

Let us prove two cases as an example:
First case. $I^{-1}(i_1)\in D_{n_1}\setminus M_{n_1}$ and $I^{-1}(1_2)\in D_{n_2}\setminus M_{n_2}$ for some $n_1,n_2\in\mathbb{N}$. Clearly $n_1\leq n_1$. If $n_1=n_2$ then ψ_{n_1+1} and ϕ_{n_1+1} act as an identity over the image of $\phi_{n_1,i_2-\#M_{n_1}}$ and therefore

$$\begin{aligned}\varphi_{i_1}\circ\varphi_{i_2}=&\phi_{n_1,i_1-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\phi_{n_1+1}\circ\phi_{n_1,i_2-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\phi_{n_1+1}\\=&\phi_{n_1,i_1-\#M_{n_1}}\circ\phi_{n_1,i_2-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\phi_{n_1+1}\\=&\phi_{n_1,i_1-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\phi_{n_1+1}=\varphi_{i_1}.\end{aligned}$$

Whereas if $n_1<n_2$, then

$$\begin{aligned}\varphi_{i_1}\circ\varphi_{i_2}=&\phi_{n_1,i_1-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\phi_{n_1+1}\circ\phi_{n_2,i_2-\#M_{n_2}}\circ\Psi_{n_2+1}\circ\phi_{n_2+1}\\\overset{(2)}{=}&\phi_{n_1,i_1-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\phi_{n_1+1}\circ\Psi_{n_2+1}\circ\phi_{n_2+1}\\\overset{(1)}{=}&\phi_{n_1,i_1-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\phi_{n_1+1}=\varphi_{i_1}.\end{aligned}$$

Second case. $I^{-1}(i_1)\in D_{n_1}\setminus M_{n_1}$ and $I^{-1}(i_2)\in M_{n_2}\setminus D_{n_2-1}$ for some $n_1,n_2\in\mathbb{N}$ with $n_2\geq 2$. In this case, it is straightforward to see that $n_2>n_1$ and so we distinguish between two different cases, namely, the case when $n_2=n_1+1$ and the case when $n_2>n_1+1$. If $n_2=n_1+1$ then ϕ_{n_1+1} acts as an identity over the image of $\Psi_{n_1+1,i_2-\#D_{n_1}}$ and so

$$\begin{aligned}\varphi_{i_1}\circ\varphi_{i_2}=&\phi_{n_1,i_1-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\phi_{n_1+1}\circ\Psi_{n_1+1,i_2-\#D_{n_1}}\circ\phi_{n_1+1}\\=&\phi_{n_1,i_1-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\Psi_{n_1+1,i_2-\#D_{n_1}}\circ\phi_{n_1+1}\\=&\phi_{n_1,i_1-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\phi_{n_1+1}=\varphi_{i_1}.\end{aligned}$$

On the other hand, if $n_2>n_1+1$ then

$$\begin{aligned}\varphi_{i_1}\circ\varphi_{i_2}=&\phi_{n_1,i_1-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\phi_{n_1+1}\circ\Psi_{n_2,i_2-\#D_{n_2-1}}\circ\phi_{n_2}\\\overset{(1)}{=}&\phi_{n_1,i_1-\#M_{n_1}}\circ\Psi_{n_1+1}\circ\phi_{n_1+1}=\varphi_{i_1}.\end{aligned}$$

The remaining two cases are straightforward using (1) and (2).

It only remains to prove that φ_i is Lipschitz. Let us take distinct $x,y\in M$. The case $x,y\in M^i$ is trivial since $\varphi_i(x)-\varphi_i(y)=x-y$. Hence, we may assume that $x\notin M^i$. If $I^{-1}(i)\in D_n\setminus M_n=C_n$ then $\varphi_i(x)=\phi_{n,i-\#M_n}\circ$

$\Psi_{n+1}\circ\phi_{n+1}(x)$ and $\varphi_i(y)=\phi_{n,i-\#M_n}\circ\Psi_{n+1}\circ\phi_{n+1}(y)$. Therefore,

$$\begin{aligned}||\varphi_i(x)-\varphi_i(y)||\leq&||\phi_{n,i-\#M_n}||_{\mathrm{Lip}}||\Psi_{n+1}||_{\mathrm{Lip}}||\phi_{n+1}||_{\mathrm{Lip}}||x-y||\\ \leq&(2\lambda^2+8\lambda+8)(\lambda^2+4\lambda+4)(3\lambda+2)||x-y||.\end{aligned}$$

If $I^{-1}(i)\in M_n\setminus D_{n-1}$ then $\phi_i(x)=\Psi_{n,i-\#D_{n-1}}\circ\phi_n(x)$ and $\phi_i(y)=\Psi_{n,i-\#D_{n-1}}\circ\phi_n(y)$. Therefore,

$$\begin{aligned}||\varphi_i(x)-\varphi_i(y)||\leq&||\Psi_{n,i-\#D_{n-1}}||_{\mathrm{Lip}}||\phi_n||_{\mathrm{Lip}}||x-y||\\ \leq&(3\lambda+2)^2(2\lambda^2+8\lambda+12)||x-y||.\end{aligned}$$

Finally, we focus on the case when $I^{-1}(i)\in M_1$. If $y\in M^i$ then $y\in M^1=f\circ i_1\circ f(\lambda B_{\ell_\infty(\Gamma_1)})$ so $||y||\leq\lambda^2$. Hence, $||\varphi_i(x)-\varphi_i(y)||=||y||\leq\lambda^2\leq\lambda^2||x-y||$. Otherwise, if $y\notin M^i$ the Lipschitz inequality is trivially satisfied since $\phi_i(x)=\phi_i(y)=0$.

∎

In general M is not a net of X because it may not be contained in X. Replacing it with a net in X with the same essential properties is the aim of the last step.

2.6. **Step 6.** We first prove the next simple lemma.

Lemma 2.24. *For every $x\in X$ there is $y\in M$ such that $||x-y||\leq 2\lambda+2$.*

Proof. Clearly, for any $x\in X$ there exists $n\in\mathbb{N}$ such that $f(r_n(x))\in\lambda^nB_{\ell_\infty(\Gamma_n)}$ and $||x-i_n\circ r_n(x)||\leq 1$ because $(i_n\circ r_n)_{n\in\mathbb{N}}$ are the natural projections of a Schauder Decomposition of X (Remark 2.9 of [AGM16]). Therefore, $\widetilde{x}:=i_n\circ r_n\circ f(x)\in\widetilde{M_n}$ satisfy that $||x-\widetilde{x}||\leq||x-i_n\circ r_n(x)||+||i_n\circ r_n(x)-i_n\circ r_n\circ f(x)||\leq 1+\lambda$. We let $y=\left(r_n\big|_{M_n}\right)^{-1}(r_n(\widetilde{x}))$ so that $i_n\circ r_n(y)=i_n\circ r_n(\widetilde{x})=\widetilde{x}$. By Lemma 2.5 we have that

$$||\widetilde{x}-y||=||i_n\circ r_n(y)-y||\leq\lambda+1.$$

Finally,

$$||x-y||\leq||x-\widetilde{x}||+||\widetilde{x}-y||\leq 2\lambda+2,$$

and we are done. ∎

Proposition 2.25. *There is a net N in X which is Lipschitz equivalent to M.*

Proof. For every $m\in M$ there is a unique $n(m)\in\mathbb{N}$ such that $m\in M_{n(m)}\setminus M_{n(m)-1}$ (where $M_0=\emptyset$). We define the map $\rho:M\to X$ as $\rho(m)=i_{n(m)}\circ r_{n(m)}(m)$ for every $m\in M$. If $m\in M$, it follows from the definition of $M_n\setminus M_{n-1}$ that $m=f\circ\rho(m)$ so $||m-\rho(m)||\leq 1$. The problem here is that ρ may not be injective and $\rho(M)$ may not be uniformly discrete so we are going to force it. First, we order $\rho(M)$ (which is countable) as $(m_n)_{n\in\mathbb{N}}$ and produce a sequence of points $(\widetilde{m}_n)\subset\rho(M)$ together with a sequence of disjoint non-empty subsets $N_n\subset\rho(M)$ inductively. Let us take $a>1$,

we set $\widetilde{m}_1 = m_1$ and $N_1 = \{m \in \rho(M) \ : \ ||m - \widetilde{m}_1|| \leq a\}$ and if $\widetilde{m}_k$ and N_k are known for every $k < n$ we set $\widetilde{m}_n = \min \rho(M) \setminus \big(N_1 \cup \cdots \cup N_{n-1}\big)$ and $N_n = \{m \in \rho(M) \setminus \big(N_1 \cup \cdots \cup N_{n-1}\big) \ : \ ||m - \widetilde{m}_n|| \leq a\}$. Obviously, $\widetilde{N} := (\widetilde{m}_n)$ is an (a, a)-net of $\rho(M)$. Hence, it is an (a, b)-net of X with $b = a + 2\lambda + 3$, since by Lemma 2.24, $\rho(M)$ is $(2\lambda + 3)$-dense in X. Also, $\rho(M) = \bigcup_{k\in\mathbb{N}} N_k$ where $N_i \cap N_j = \emptyset$ for every $i, j \in \mathbb{N}$ with $i \neq j$.

Now, since X is infinite dimensional, there is a sequence of points $(x_n) \subset \frac{3}{4}B_X$ such that $||x_m - x_n|| \geq 1/2$ for every $m, n \in \mathbb{N}$ with $m \neq n$. To simplify the next definition, we are going to order the points of $\rho^{-1}(N_k) = (m_k^n)_n$ for every $k \in \mathbb{N}$ (this is possible because M is countable). Finally, since $M = \bigcup_{k\in\mathbb{N}} \rho^{-1}(N_k)$, we define the injection $\mu : M \to X$ as

$$\mu(m_k^n) = \widetilde{m}_k + \frac{a}{2}x_n.$$

We claim that $N = \mu(M)$ is a net of X Lipschitz equivalent to M. Clearly, N is $(b + 3a/8)$-dense in X since $\widetilde{N}$ is b-dense in X. It is also easy to see that

$$||\mu(m_{k_1}^{n_1}) - \mu(m_{k_2}^{n_2})|| \geq ||\widetilde{m}_{k_1} - \widetilde{m}_{k_2}|| - \frac{a}{2}\big(||x_{n_1}|| + ||x_{n_2}||\big) \geq a - \frac{a}{2}\left(\frac{3}{4} + \frac{3}{4}\right) = a/4$$

whenever $k_1 \neq k_2$. If $k_1 = k_2$ then obviously $||\mu(m_{k_1}^{n_1}) - \mu(m_{k_2}^{n_2})|| = a/2||x_{n_1} - x_{n_2}|| \geq a/4$. This proves that N is a $(a/4, b + 3a/8)$-net in X. It only remains to prove that μ and μ^{-1} (seeing μ^{-1} restricted to N) are Lipschitz. This follows immediately from the fact that for every $m_k^n \in M$,

$$\begin{aligned} ||\mu(m_k^n) - m_k^n|| =& ||\widetilde{m}_k - m_k^n + x_n a/2|| \\ \leq& ||\widetilde{m}_k - \rho(m_k^n)|| + ||\rho(m_k^n) - m_k^n|| + ||x_n a/2|| \\ \leq& a + 1 + 3a/8. \end{aligned}$$

■

Theorem 2.26 (Main Theorem)**.** *A net in any separable infinite dimensional $\mathcal{L}_\infty$-space has a retractional basis.*

Corollary 2.27. *If N is a net in a separable infinite dimensional $\mathcal{L}_\infty$-space then $\mathcal{F}(N)$ has a Schauder basis.*

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Chapter 9

A characterisation of the Daugavet property in spaces of vector-valued Lipschitz functions

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A CHARACTERISATION OF THE DAUGAVET PROPERTY IN SPACES OF VECTOR-VALUED LIPSCHITZ FUNCTIONS

RUBÉN MEDINA AND ABRAHAM RUEDA ZOCA

Abstract. Let M be a metric space and X be a Banach space. In this paper we address several questions about the structure of $\mathcal{F}(M)\widehat{\otimes}_\pi X$ and $\mathrm{Lip}_0(M,X)$. Our results are the following:

(1) We prove that if M is a length metric space then $\mathrm{Lip}_0(M,X)$ has the Daugavet property. As a consequence, if M is length we obtain that $\mathcal{F}(M)\widehat{\otimes}_\pi X$ has the Daugavet property. This gives an affirmative answer to [13, Question 1] (also asked in [24, Remark 3.8]).
(2) We prove that if M is a non-uniformly discrete metric space or an unbounded metric space then the norm of $\mathcal{F}(M)\widehat{\otimes}_\pi X$ is octahedral, which solves [6, Question 3.2 (1)].
(3) We characterise all the Banach spaces X such that $L(X,Y)$ is octahedral for every Banach space Y, which solves a question by Johann Langemets.

1. Introduction

In this note, we will focus mainly on the study of Lipschitz mappings from a metric space M to a Banach space X, that is, we will focus on the space $\mathrm{Lip}_0(M,X)$. In particular, we are mainly interested in the Daugavet property on these spaces since it is known that $\mathrm{Lip}_0(M)$ enjoys it if, and only if, the metric space M is length. The natural question that arises in this setting is therefore whether the same metric characterisation works in the vector valued case, that is, whether the space $\mathrm{Lip}_0(M,X)$ being Daugavet is equivalent to M being length regardless which X is used as codomain. This problem has been around since the latter characterisation of Lipschitz spaces was found in [13, Theorem 3.5] and it has been explicitly asked in [13, Question 1] and [24, Remark 3.8]. We give here a complete positive solution.

1.1. Motivation and background.

A Banach space X is said to have the Daugavet property if every rank-one operator $T : X \longrightarrow X$ satisfies the

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equality

$$\|T + I\| = 1 + \|T\|, \tag{1.1}$$

where I denotes the identity operator. The previous equality is known as *Daugavet equation* because I. Daugavet proved in [10] that every compact operator on $\mathcal{C}([0,1])$ satisfies (1.1). Since then, many examples of Banach spaces enjoying the Daugavet property have appeared. E.g. $\mathcal{C}(K)$ for a perfect compact Hausdorff space K; $L_1(\mu)$ and $L_\infty(\mu)$ for a non-atomic measure μ; or preduals of Banach spaces with the Daugavet property (see [18, 19, 34] and references therein for a detailed treatment of the Daugavet property).

The question of determining when the space of Lipschitz functions $\mathrm{Lip}_0(M)$ has the Daugavet property has attracted the attention of many researchers since in [34, Section 6, Question 1] D. Werner asked whether $\mathrm{Lip}_0([0,1]^2)$ has the Daugavet property. As a consequence of succesive works [17, 13, 4] the following characterisation was obtained: Given a complete metric space M, then $\mathrm{Lip}_0(M)$ has the Daugavet property if, and only if, M is length if, and only if, $\mathcal{F}(M)$ has the Daugavet property, where $\mathcal{F}(M)$ stands for the Lipschitz-free space over M (also known as transportation cost space). All the above are equivalent to the absence of strongly exposed points of the unit ball of $\mathcal{F}(M)$.

In the above line, the first result appeared in [13, Proposition 3.11] where, as a main consequence, it is obtained that $\mathrm{Lip}_0(M,H)$ has the Daugavet property when H is a Hilbert space. This result is obtained as a consequence of a well known result of extension of Lipschitz functions between Hilbert spaces due to Kirszbraun (see [7, Theorem 1.12]). More general results were obtained in [31], where it was proved that $\mathrm{Lip}_0(M,X)$ has the Daugavet property for every Banach space X if M is a convex subset of a Hilbert space or if M is a Banach space whose dual is isometrically $L_1(\mu)$. In all of the above cases, it is heavily used that given any finite subset N of M and any Lipschitz function $\varphi : N \longrightarrow M$, there exists an extension $\phi : M \longrightarrow M$ which almost preserves the Lipschitz constant. That is, a (demanding) property of extension of Lipschitz functions is required as a tool for making perturbations of given Lipschitz functions $f : M \longrightarrow X^*$ almost preserving their Lipschitz constant.

A feature which explains the difficulty of the aforementioned problem arises when one considers dual Banach targets. In such case, $\mathrm{Lip}_0(M, X^*) = L(\mathcal{F}(M), X^*) = (\mathcal{F}(M)\widehat{\otimes}_\pi X)^*$, where $\mathcal{F}(M)\widehat{\otimes}_\pi X$ stands for the projective tensor product of $\mathcal{F}(M)$ and X. Taking into account the immediate fact that the Daugavet property passes from a Banach space to its predual, the above mentioned open question is related to the question of how the Daugavet property is inherited by taking projective tensor products, which is a long standing open problem from [34, Section 6].

In this paper we explore new ideas for the perturbation of Lipschitz functions taking values on a Banach space in order to solve many open questions. Indeed, we obtain the following main result.

Theorem 1. *If M is a complete length metric space and X is a Banach space then $\mathrm{Lip}_0(M,X)$ has the Daugavet property.*

As a corollary (Corollary 3.1) we get that if M is a complete length metric space and X is a Banach space then $\mathcal{F}(M)\widehat{\otimes}_\pi X$ has the Daugavet property (see the definition below). This solve [13, Question 1] (also asked in [24, Remark 3.8]) in complete generality.

One of the new tools behind the proof of the above mentioned Theorem 1 is Lemma 2.4, which informally says that given any Banach space X there is an almost 1-Lipschitz perturbation of the identity mapping which vanishes at a ball centred at 0 (we thank Richard Smith who pointed us out the existence of such functions). We also considered in Lemma 2.5 the "dual construction", that is, a function which is the identity on a neighbourhood of 0 which vanishes out of a bounded subset of X.

Next we continue using the ideas of perturbation of vector-valued Lipschitz functions to solve another open problem on the space $\mathcal{F}(M)\widehat{\otimes}_\pi X$.

In order to put the problem into context [6, Question 3.2 (1)], recall that a Banach space X is said to be *octahedral* if for every $x_1,\ldots,x_n \in S_X$ and $\varepsilon>0$ there is $y\in S_X$ such that $\|x_i+y\|\geq 2-\varepsilon$ for every $i=1,\ldots,n$ (see Definition 1.2 and the surrounding paragraphs for background). With the terminology of octahedral norms in mind, [6, Question 3.2 (1)] asks whether $\mathcal{F}(M)\widehat{\otimes}_\pi X$ is octahedral whenever M is a complete metric space which is not uniformly discrete and bounded. In Section 4 we give a positive answer to this question.

Theorem 2. *Let M be a metric space. If M is not a bounded and uniformly discrete space then $\mathcal{F}(M)\widehat{\otimes}_\pi X$ is octahedral for every Banach space X.*

Recall that a metric space is said to be uniformly discrete if $\inf\{d(x,y)\ ;\ x,y\in M,\ x\neq y\}>0$. Observe that this result constitutes a step forward for the open question [22, Question 4.4], where it is asked whether $X\widehat{\otimes}_\pi Y$ is octahedral provided X is octahedral.

We finish Section 4 with Theorem 4.4 where we prove that, when M is a metric space with infinitely many cluster points, then we get the stronger condition that $(\mathcal{F}(M)\widehat{\otimes}_\pi X)^{**}$ is octahedral.

We end the paper with Section 5, which is of independent interest, where we answer the following question: when does a Banach space X satisfies that $L(X,Y)$ is octahedral for every Banach space Y? This question was posed by J. Langemets in the congress International Workshop on Operator Theory and its Applications (IWOTA 2022), Krakow, Poland (Sep 2022)).

1.2. **Terminology.** We will consider only real Banach spaces. Given a Banach space X, B_X (resp. S_X) stands for the closed unit ball (resp. the unit sphere).

We recall that the *projective tensor product of two Banach spaces X and Y*, denoted by $X\widehat{\otimes}_\pi Y$, is the completion of $X\otimes Y$ under the norm given by

$$\|u\| := \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\| \ : \ n\in\mathbb{N}, x_i\in X, y_i\in Y, u=\sum_{i=1}^n x_i\otimes y_i\right\},$$

for every $u\in X\otimes Y$.

We recall that the space $L(X,Y^*)$ is linearly isometric to the topological dual of $X\widehat{\otimes}_\pi Y$ by the action $T(x\otimes y):=T(x)(y)$. As an immediate consequence of the above identification we get that $B_{X\widehat{\otimes}_\pi Y}=\overline{\text{co}}(B_X\otimes B_Y)$. We refer the reader to [32] for background on tensor product spaces.

Also, since we are mainly interested in the case when one of the factors is a Lipschitz free space, let us introduce that notion from scratch. A pointed metric space $(M,d,0)$ is a metric space (M,d) with a selected distinguished point 0 in M, called the base point. For a pointed metric space M, we consider the Banach space $\text{Lip}_0(M)$ formed by all real-valued Lipschitz functions that vanish at 0 endowed with the norm given by the least Lipschitz constant, that is,

$$\|f\|_{Lip} = \inf\{C>0 \ : \ |f(x)-f(y)|\leq Cd(x,y)\ ,\ \forall x,y\in M\},$$

for every $f\in\text{Lip}_0(M)$. The Banach space $\text{Lip}_0(M)$ is a dual space and its canonical predual is our sought space, often called the *Lipschitz free space* associated to M, denoted by $\mathcal{F}(M)$. It is given by $\mathcal{F}(M)=\overline{\text{span}}\{\delta_p\}_{p\in M}\subseteq \text{Lip}_0(M)^*$, where $\delta_p:\text{Lip}_0(M)\to\mathbb{R}$ is the linear functional defined by $\delta_p(f)=f(p)$ for all $f\in\text{Lip}_0(M)$. Now, if X is a Banach space we denote $\text{Lip}_0(M,X)$ to the space of Lipschitz mappings from M to X vanishing at 0 endowed with the Lipschitz norm. We are interested in vector-valued Lipschitz mappings since the isometric identification $\text{Lip}_0(M,X^*)=L(\mathcal{F}(M),X^*)$ leads a structure of dual Banach space and its canonical predual is our object of study $\mathcal{F}(M)\widehat{\otimes}_\pi X$.

We must define now the metric spaces M such that $\text{Lip}_0(M)$ is Daugavet. These spaces are known as length spaces (see e.g. [8]). Specifically, a metric space (M,d) is a *length* space whenever it is arc-connected and for every $x,y\in M$,

$$d(x,y)=\inf\{\text{l}(\alpha) \ : \ \alpha:[0,1]\to M,\ \alpha(0)=x,\ \alpha(1)=y\},$$

where $\text{l}(\alpha)$ stands for the length of the curve α. In the proof of the equivalence between M length and $\mathcal{F}(M)$ Daugavet the notion of spreadingly local metric spaces is used and we will need it, too. According to [17], a metric space is said to be *spreadingly local* whenever

$$A_{\varepsilon,f}=\{m\in M \ : \ \inf_{r>0}\|f\|_{Lip}>1-\varepsilon\}$$

is infinite for every $\varepsilon>0$ and $f\in\text{Lip}_0(M)$. Following the proof of Proposition 2.3 in [17] it is easy to see that every length space is spreadingly local, regardless it is complete or not.

In the setting of metric spaces, we are going to make use of the well-known Mcshane extension of Lipschitz functions. Given a metric space M, a subset N of M and $f \in \mathrm{Lip}_0(N)$ there is always an extension $g \in \mathrm{Lip}_0(M)$ of f to M, often called the McShane extension, such that $\|g\|_{Lip} = \|f\|_{Lip}$. In general, there is no McShane extension for vector-valued Lipschitz mappings, which is the main problem that we face.

Let us define now the main geometric properties of Banach spaces we are going to work with throughout the entire note. A slice S of the unit ball B_X of a Banach space X is a subset of B_X of the form

$$S = S(B_X, f, \alpha) = \{z \in B_X \ : \ f(z) > 1 - \alpha\},$$

For some $f \in X^*$ and $\alpha > 0$. If X is a dual space and f belongs to its predual we call the previous set a w^*-slice.

Definition 1.1. Given X a Banach space, we say that X has the *Daugavet property* (or simply that X is Daugavet) whenever for every $x \in S_X$, $\varepsilon > 0$ and S slice of B_X there is $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$. If X is a dual space and the latter holds at least for w^*-slices we say that X has the w^*-Daugavet property (or simply X is w^*-Daugavet).

Notice that this definition of the Daugavet property is equivalent to the one given at the beginning of the paper [19, Lemmata 2.1 and 2.2].

Now we will present two more definitions closely related to the Daugavet property.

Definition 1.2. Given X a Banach space, we say that X is *octahedral* if for every $x_1, \ldots, x_n \in S_X$ and $\varepsilon > 0$ there is $y \in S_X$ such that $\|x_i - y\| \geq 2 - \varepsilon$ for every $i = 1, \ldots, n$.

Definition 1.3. Given X a Banach space, we say that X enjoys the *strong diameter 2 property* (SD2P for short) whenever every convex combination of slices has diameter 2.

It is worth mentioning that by [5, Theorem 2.1], for a Banach space X it is equivalent being octahedral to its dual X^* having the w^*-SD2P. Moreover, it follows that if X is a Daugavet space then both X and X^* have octahedral norms [19, Lemmata 2.8 and 2.12]. For background on diameter two properties, octahedral norms and their interrelations with the Daugavet property we refer the reader to [21, 28, 30].

2. Lipschitz functions on Banach spaces

Let us start with the following straigthforward lemmata which will shorten future arguments.

Lemma 2.1. *Let X be any Banach space, $\lambda \geqslant 0$ and $f : X \to X$. If f is λ-Lipschitz when restricted to segments then f is λ-Lipschitz.*

Lemma 2.2. *Let X be any Banach space, $a < b < c$ in $\mathbb{R}$ and $f : [a, c] \to X$. If $f|_{[a,b]}$ and $f|_{[b,c]}$ are λ-Lipschitz for some $\lambda \geqslant 0$ then f is λ-Lipschitz.*

Proof. Let $t, s \in [a, c]$ distinct. We may assume that $t \in [a, b]$ and $s \in [b, c]$. Then,

$$\|f(t) - f(s)\| \leqslant \|f(t) - f(b)\| + \|f(b) - f(s)\| \leqslant \lambda(|t - b| + |b - s|) = \lambda|t - s|.$$

∎

Lemma 2.3. *Let X be a Banach space, let $0 < r < R$ and consider a function $f : X \longrightarrow X$. Assume that there are $a, b, c > 0$ such that $\sup_{x,y \in B(0,r), x \neq y} \frac{\|f(x)-f(y)\|}{\|x-y\|} \leqslant a$, $\sup_{x,y \in C(0,r,R), x \neq y} \frac{\|f(x)-f(y)\|}{\|x-y\|} \leqslant b$ and $\sup_{x,y \in X \setminus B(0,R), x \neq y} \frac{\|f(x)-f(y)\|}{\|x-y\|} \leqslant c$. Then f is Lipschitz and $\|f\| \leqslant \max\{a, b, c\}$.*

Proof. A combination of Lemma 2.1 and Lemma 2.2 does the trick. ∎

In the following result we will construct a Lipschitz function on a Banach space which coincides with the identity map out of a ball centred at the origin and that it is flat at a smaller ball contained. This function, which to our knownledge appeared in the thesis of Filip Talimdjiosk, was provided to the authors by his thesis advisor R. Smith. The authors are deeply grateful to R. Smith for providing them the following proposition and its proof.

Proposition 2.4. *Let X be a Banach space and let $0 < a < b$. Then the function $f : X \longrightarrow X$ defined by*

$$f(x) := \begin{cases} 0 & \text{if } \|x\| \leqslant a, \\ \frac{b}{b-a}\left(1 - \frac{a}{\|x\|}\right) x & \text{if } a \leqslant \|x\| \leqslant b, \\ x & \text{if } b \leqslant \|x\|, \end{cases}$$

is Lipschitz and $\|f\| \leqslant \frac{b}{b-a}$.

In particular, it follows that for every $x_0 \in X$ and every pair of positive numbers R, ε there exist $\delta > 0$ and a Lipschitz-mapping $\psi : X \longrightarrow X$ such that $\psi(x) = x$ holds for every $x \in X \setminus B(x_0, R)$, $\|\psi\| \leqslant 1 + \varepsilon$ and $\psi(z) = x_0$ holds for every $z \in B(x_0, \delta)$.

Proof. Let $x, y \in X$ with $x \neq y$. In virtue of Lemma 2.3, and since f is clearly Lipschitz at $B(a, r)$ and on $X \setminus B(0, b)$, let us assume that $x, y \in C(0, a, b)$. We assume with no loss of generality that $\|x\| \leqslant \|y\|$. Now we

compute $\|f(x)-f(y)\|$.

$$\begin{aligned}
\|f(x)-f(y)\| &= \frac{b}{b-a}\left\|y-x+\frac{a}{\|x\|}x-\frac{a}{\|y\|}y\right\| \\
&= \frac{b}{b-a}\left\|y-x+\frac{a}{\|x\|}x-\frac{a}{\|y\|}y+\frac{a}{\|x\|}y-\frac{a}{\|x\|}y\right\| \\
&\leqslant \frac{b}{b-a}\left(\left(1-\frac{a}{\|x\|}\right)\|x-y\|+\left|\frac{a}{\|x\|}-\frac{a}{\|y\|}\right|\|y\|\right) \\
&= \frac{b}{b-a}\left(\left(1-\frac{a}{\|x\|}\right)\|x-y\|+\frac{a|\|x\|-\|y\||}{\|x\|\|y\|}\|y\|\right) \\
&\leqslant \frac{b}{b-a}\left(\left(1-\frac{a}{\|x\|}\right)\|x-y\|+\frac{a}{\|x\|}\|x-y\|\right) \\
&= \frac{b}{b-a}\|x-y\|,
\end{aligned}$$

as desired.

For the second part of the theorem, given $x_0 \in X$ and $R, \varepsilon > 0$, take $\delta > 0$ such that $\frac{R}{R-\delta} < 1+\varepsilon$, consider the function above $\varphi : X \longrightarrow X$, then the function $\psi(z) := x_0 + \varphi(x - x_0)$ does the trick. ∎

In the following result we will construct a function whose behaviour is the opposite, i.e., a function which is the identity in a neighbourhood of 0 and that is constantly 0 out of a certain ball centred at 0.

Proposition 2.5. *Let X be a Banach space and $0 < a < b$. The function $f : X \longrightarrow X$ defined by*

$$f(x) := \begin{cases} x & \text{if } x \in B(0,a), \\ \frac{b-\|x\|}{b-a}x & \text{if } x \in C(0,a,b), \\ 0 & \text{if } x \in X \setminus B(0,R), \end{cases}$$

is Lipschitz and $\|f\| \leqslant \frac{b}{b-a}$.

Proof. Again by an application of Lemma 2.3 and because f is clearly Lipschitz on $B(0,a)$ and on $X \setminus B(0,b)$, it is enough to prove that, given $x, y \in C(0,a,b)$ with $x \neq y$, we get $\|f(x)-f(y)\| \leqslant \frac{b}{b-a}\|x-y\|$. Take such $x, y \in C(0,a,b)$ and assume with no loss of generality that $\|x\| \leqslant \|y\|$.

Then

$$\begin{aligned}\|f(x)-f(y)\| &= \frac{1}{b-a}\,\|(b-\|x\|)x-(b-\|y\|)y\| \\ &= \frac{1}{b-a}\,\|(b-\|y\|)x+(\|y\|-\|x\|)x-(b-\|y\|)y\| \\ &\leqslant \frac{1}{b-a}((b-\|y\|)\|x-y\|+|\|x\|-\|y\||\|x\|) \\ &\leqslant \frac{1}{b-a}((b-\|x\|)\|x-y\|+\|x-y\|\|x\|) \\ &= \frac{b}{b-a}\|x-y\|.\end{aligned}$$

■

The following lemma shows that the set of Lipschitz functions which is injective on a given sequence is norm-dense. We establish and prove the result because we will use it several times throughout the text.

Lemma 2.6. *Let M be a metric space and X be a Banach space. Consider a sequence of pairwise-disjoint balls $\{B(x_n, r_n)\}$ in M. Then for every Lipschitz function $F : M \longrightarrow X$ and $\varepsilon > 0$ there exists a Lipschitz function $G : M \longrightarrow X$ with the following properties:*

(1) *$\|F - G\| < \varepsilon$ and,*
(2) *$G(x_n) \neq G(x_m)$ holds for every $n \neq m$.*

Proof. Given $n \in \mathbb{N}$ we can take, by McShane extension theorem, a Lispchitz function $\varphi_n : M \longrightarrow \mathbb{R}$ such that $\varphi_n(x_n) \neq 0$ for every $n \in \mathbb{N}$, $\|\varphi_n\| = 1$ and $\varphi_n = 0$ on $M \setminus B(x_n, r_n)$. Let F and ε as in the hypothesis, and consider a sequence $\{\varepsilon_n\}$ of strictly positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$. Take also $x \in S_X$.

We will construct by induction a sequence $\{\delta_n\}$ of positive numbers such that, for every $n \in \mathbb{N}$, the following conditions hold:

(1) $\delta_n \leqslant \varepsilon_n$ holds for every $n \in \mathbb{N}$ and;
(2) $F(x_i) + \delta_i \varphi(x_i)x \neq F(x_n) + \delta_n \varphi(x_n)x$ holds for $1 \leqslant i \leqslant n-1$.

For $n = 1$ take $\delta_1 = \varepsilon_1$. Now assume $\delta_1 \ldots, \delta_n$ have been constructed and let us construct δ_{n+1}. In order to do so, observe that the set

$$\{F(x_{n+1}) + \delta\varphi_{n+1}(x_{n+1})x : 0 < \delta < \varepsilon_{n+1}\}$$

is an infinite set since x is a non-zero vector. Since the set

$$\{F(x_i) + \delta_i\varphi_i(x_i)x : 1 \leqslant i \leqslant n\}$$

is a finite set we can find $0 < \delta_{n+1} < \varepsilon_{n+1}$ such that

$$F(x_{n+1}) + \delta\varphi_{n+1}(x_{n+1})x \notin \{F(x_i) + \delta_i\varphi_i(x_i)x : 1 \leqslant i \leqslant n\}.$$

Now $G = F + \sum_{n=1}^{\infty} \delta_n \varphi_n \otimes x$ satisfies our requirements. To begin with, the inequality $\|F - G\| \leqslant \sum_{n=1}^{\infty} \delta_n \|\varphi_n\|\|x\| \leqslant \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$ holds. On the other hand, the construction of δ_n together with the fact that $\operatorname{supp}(\varphi_n) \subseteq$

$B(x_n, r_n)$ implies that $G(x_n) = F(x_n) + \delta_n \varphi_n(x_n)x$, from where the proof follows. ∎

Now we introduce the following result, whose proof is easy, and which will save us a lot of notation in the following. The proof is straightforward, but let us include it for the sake of completeness.

Lemma 2.7. *Let M be a complete metric space and X be a Banach space. Let $f, g : M \longrightarrow X$ be two Lipschitz functions. Assume that there exists $m \in M$ and $0 < \delta < R$ so that*

(1) *g is constant on $B(m, R)$.*
(2) *$f(x) = f(m)$ holds for every $x \in M \setminus B(m, \delta)$.*

Then $\|f + g\| \leqslant \max\{\|f\|, \|g\|\}(1 + \frac{2\delta}{R-\delta})$.

Proof. Call $M := \max\{\|f\|, \|g\|\}$. Let $x, y \in M$ with $x \neq y$. Let us estimate

$$A := \frac{\|(f(x) + g(x)) - (f(y) + g(y))\|}{d(x, y)} = \frac{\|f(x) - f(y) + g(x) - g(y)\|}{d(x, y)}.$$

Let us observe that if $f(x) - f(y) = 0$ or $g(x) - g(y) = 0$ then $A \leqslant M$.

The unique possibility for the previous condition not to hold is that, up to relabeling, $x \notin B(m, R)$ and $y \in B(m, \delta)$. In that case $f(x) = f(m)$ and $g(y) = g(m)$. Consequently

$$A \leqslant \frac{\|f(y) - f(m)\| + \|g(x) - g(m)\|}{d(x, y)} \leqslant M \frac{d(y, m) + d(x, m)}{d(x, y)}.$$

Since $d(x, m) \leqslant d(x, y) + d(y, m)$ the above inequality yields

$$A \leqslant M \frac{d(x, y) + 2d(y, m)}{d(x, y)} = M \left(1 + \frac{2d(y, m)}{d(x, y)}\right).$$

Now using $d(x, y) \geqslant d(x, m) - d(y, m) \geqslant R - \delta$ we get

$$A \leqslant M \left(1 + \frac{2\delta}{R - \delta}\right),$$

as desired. ∎

Let us end by recalling the following criterion of weakly null sequences on $\mathrm{Lip}_0(M, X)$ from [9], which will be used several times throughout the text and whose proof can be found in [9, Lemma 1.5].

Lemma 2.8. *Let M be a pointed metric space, let X be a Banach space, and let $\{f_n\}$ be a sequence of functions in the unit ball of $\mathrm{Lip}_0(M, X)$. For each $n \in \mathbb{N}$, we write $U_n := \{x \in M \colon f_n(x) \neq 0\}$ for the support of f_n. If $U_n \cap U_m = \emptyset$ for every $n \neq m$, then the sequence $\{f_n\}$ is weakly null.*

3. Daugavet property in $\mathrm{Lip}_0(M,X)$

In this section we will make use of the previous results to give the prove of Theorem 1.

Proof of Theorem 1. Let $f,g \in S_{\mathrm{Lip}_0(M,X)}$, and, in order to prove that $\mathrm{Lip}_0(M,X)$ has the Daugavet property, let us prove that for every $\varepsilon>0$ there exists a sequence $(f_n)\subseteq (1+\varepsilon)B_{\mathrm{Lip}_0(M,X)}$ such that $(f_n)\longrightarrow g$ weakly and $\|f+f_n\|\geqslant 2-\varepsilon$ holds for every $n\in\mathbb{N}$. This is enough thanks to [31, Theorem 2.1. (5)].

Since $\|f\|_{Lip}=1$ we can find $y^*\in S_{X^*}$ such that $y^*\circ f: M\longrightarrow\mathbb{R}$ given by $y^*\circ f(m):=y^*(f(m))$ satisfies $\|y^*\circ f\|_{Lip}>1-\frac{\varepsilon}{2}$.

Since M is length, it is spreadingly local (see the terminology section), that is, the set

$$A=\left\{m\in M \ : \ \inf_{r>0}\|y^*\circ f\restriction B(m,r)\|_{Lip}>1-\varepsilon/4\right\}$$

is infinite. Hence we can take a sequence of pairwise distinct points $(x_n)\subseteq A$. Up to an application of Lemma 2.6 we can assume, up to a norm-perturbation argument, that $g(x_n)\neq g(x_m)$ if $n\neq m$. Hence we can find, for every $n\in\mathbb{N}$, an element $\alpha_n>0$ such that $\{B(g(x_n),\alpha_n) : n\in\mathbb{N}\}$ is pairwise disjoint (observe that since $\|g\|\leqslant 1$ it is clear that $\{B(x_n,\alpha_n)\}$ is also pairwise disjoint).

Consider $0<\beta_n<\alpha_n$ for every $n\in\mathbb{N}$ such that $\frac{\alpha_n}{\alpha_n-\beta_n}\to 1$ and consider, in virtue of Proposition 2.4, a Lipschitz function $\varphi_n: X\longrightarrow X$ such that $\|\varphi_n\|\leqslant\frac{\alpha_n}{\alpha_n-\beta_n}$ holds for every $n\in\mathbb{N}$, that $\varphi_n(x)=x$ holds for every $x\in X\setminus B(g(x_n),\alpha_n)$ and $\varphi_n(x)=g(x_n)$ holds for every $x\in B(g(x_n),\beta_n)$.

Now for every $n\in\mathbb{N}$ write $h_n:=\varphi_n\circ g: M\longrightarrow X$. It is immediate that $\|h_n\|\to 1$ (since $\|h_n\|\leqslant\|\varphi_n\|\|g\|\leqslant\frac{\alpha_n}{\alpha_n-\beta_n}$).

Now we claim that (h_n-g) is a sequence of mappings with pariwise disjoint support. Indeed, it is immediate that, given $n\in\mathbb{N}$, it follows that $h_n(x)-g(x)\neq 0$ implies $g(x)\in B(g(x_n),\alpha_n)$ which, in other words, mean that $\mathrm{supp}(h_n-g)\subseteq g^{-1}(B(g(x_n),\alpha_n))$ for every $n\in\mathbb{N}$. The fact that $\mathrm{supp}(h_n-g)$ is pairwise disjoint is immediate now since $B(g(x_n),\alpha_n)$ is pairwise disjoint. Consequently h_n-g is weakly null in virtue of Lemma 2.8 or, equivalently, $(h_n)\to g$ weakly.

On the other hand observe that $h_n=\varphi_n\circ g$ takes the value $g(x_n)$ at $B(x_n,\beta_n)$. Indeed, given $z\in B(x_n,\beta_n)$ it follows that $\|g(z)-g(x_n)\|\leqslant\|g\|d(z,x_n)<\beta_n$, which implies $\varphi_n(g(z))=g(x_n)$ by the very definition of φ_n.

Now consider a sequence (r_n) of strictly positive numbers such that $\frac{r_n}{\beta_n-2r_n}\to 0$.

Since $\{x_n : n\in\mathbb{N}\}\subseteq A$ we can find $y_n\in M$ with $0<d(x_n,y_n)<r_n$ such that

$$\frac{y^*(f(y_n))-y^*(f(x_n))}{d(y_n,x_n)}>1-\frac{\varepsilon}{4}.$$

Now define a function $s_n : M \longrightarrow \mathbb{R}$ with $\|s_n\| \leqslant 1$, $s_n(z) = 0$ if $z \in M \setminus B(x_n, 2r_n)$, $s_n(x_n) = 0$ and $s_n(y_n) := d(x_n, y_n)$. This function may be constructed as the McShane extension to M of the function

$$\widetilde{s}_n : M \setminus B(x_n, 2r_n) \cup \{x_n, y_n\} \to \mathbb{R}$$

defined as it was previously stated which is clearly 1-Lipschitz.

Now since $\|y^*\| = 1$ find $y \in S_X$ such that $y^*(y) > 1 - \frac{\varepsilon}{4}$. Consider $S_n := s_n \otimes y : M \longrightarrow X$. Observe that $\|S_n\| \leqslant 1$. Moreover S_n is clearly a sequence of pairwise disjoint Lipschitz functions, since the support of S_n is contained in $B(x_n, r_n) \subseteq B(x_n, \alpha_n)$. Consequently by Lemma 2.8 we get $(S_n) \to 0$ weakly.

Now define $g_n := h_n + S_n$, and we claim that the sequence $\{g_n\}$ does the trick. On the one hand, the convergence conditions on h_n and S_n imply that $g_n \to g$ weakly. On the other hand, given $n \in \mathbb{N}$, an appeal to Lemma 2.7 for $m = x_n$, $R = \beta_n$ and $\delta = 2r_n$ implies

$$\|g_n\| \leqslant \frac{\alpha_n}{\alpha_n - \beta_n}(1 + \frac{4r_n}{\beta_n - 2r_n}) \to 1.$$

Consequently there exists $m \in \mathbb{N}$ such that $n \geqslant m$ implies $g_n \in (1 + \varepsilon)B_{\mathrm{Lip}_0(M,X)}$. Finally, given $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|f + g_n\| &\geqslant y^* \left(\frac{(f + g_n)(y_n) - (f + g_n)(x_n)}{d(y_n, x_n)} \right) \\
&= \frac{y^*(f(y_n)) - y^*(f(x_n))}{d(y_n, x_n)} + \frac{y^*(S_n)(y_n) - y^*(S_n)(x_n)}{d(y_n, x_n)} \\
&> 1 - \frac{\varepsilon}{4} + \frac{s_n(y_n) - s_n(x_n)}{d(y_n, x_n)} y^*(y) > 1 - \frac{\varepsilon}{4} + 1 - \frac{\varepsilon}{4} > 2 - \varepsilon.
\end{aligned}$$

This finishes the proof. ∎

As a particular case of the above theorem we have the following corollary.

Corollary 3.1. *Let M be a metric space and let X be a non-zero Banach space. If M is length then $(\mathcal{F}(M)\widehat{\otimes}_\pi X)^*$ has the Daugavet property. In particular, $\mathcal{F}(M)\widehat{\otimes}_\pi X$ has the Daugavet property.*

4. Octahedrality of $\mathcal{F}(M)\widehat{\otimes}_\pi X$

In this section we will focus on the proof of Theorem 2. The proof is splited in two theorems: We first prove in Theorem 4.1 the case in which M is not uniformly discrete and we then prove in Theorem 4.3 the case when M is not bounded.

Given a metric space M, for a given point $x \in M$ define $d_x = 0$ if x is a cluster point, otherwise define

$$d_x := \sup\{r > 0 : B(x, r) = \{x\}\}.$$

Now we have the following result.

Theorem 4.1. *Let M be a metric space which is not uniformly discrete and let X be a Banach space. Then the norm of $\mathcal{F}(M)\widehat{\otimes}_\pi X$ is octahedral.*

Proof. As before let $\mu_1, \dots, \mu_q \in \mathcal{F}(M)\widehat{\otimes}_\pi X$ and $\varepsilon > 0$, and let us find $\mu \in S_{\mathcal{F}(M)\widehat{\otimes}_\pi X}$ such that $\|\mu_i + \mu\| \geqslant \frac{(1-\varepsilon)^2}{1+\varepsilon}(\|\mu_i\| + 1)$.

By a density argument we can assume that, for every $1 \leqslant i \leqslant q$, we can write $\mu_i := \sum_{j=1}^{n_i} \lambda_{ij}\delta_{m_{ij}} \otimes x_{ij}$, for certain $n_i \in \mathbb{N}$, $\lambda_{ij} \in \mathbb{R}\backslash\{0\}, m_{ij} \in M\backslash\{0\}$ and $x_{ij} \in X \setminus \{0\}$.

By Hahn-Banach theorem we can find, for every $i \in \{1, \dots, q\}$, an element $f_i \in (\mathcal{F}(M)\widehat{\otimes}_\pi X)^* = \mathrm{Lip}_0(M, X^*)$ such that $\|f_i\| < 1$ and $f_i(\mu_i) > (1 - \varepsilon)\|\mu_i\|$.

Since M is discrete but not uniformly discrete we can find a sequence $\{x_n\} \subseteq M$ such that $d_{x_n} \to 0$.

Given $1 \leqslant i \leqslant q$ we have two possibilities, depending on whether or not $f_i(x_{n_k})$ has a Cauchy subsequence:

(1) There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $f_i(x_{n_k})$ is norm convergent.
(2) There exists $\delta_i > 0$ such that $\|f_i(x_{n_k}) - f_i(x_{n_p})\| \geqslant \delta_i$ if $k \neq p$.

Thus we will assume that, up to relabeling the sequence $\{x_n\}$, there exists $\delta > 0$ with the following property: given $i \in \{1, \dots, q\}$ then either $f_i(x_n)$ is norm-convergent or $\|f_i(x_n) - f_i(x_m)\| \geqslant \delta$ if $n \neq m$.

The above allows us, in any case of i, to assume up to a perturbation argument calling Lemma 2.6 that there exists some $R > 0$ such that $\|f_i(x_n) - f_i(m_{ij})\| \geqslant R$ holds for every $n \in \mathbb{N}$, every $1 \leqslant i \leqslant q$ and every $1 \leqslant j \leqslant n_i$.

Take $r > 0$ such that $\frac{R}{R-r}\|f_i\| < 1$ holds for every $1 \leqslant i \leqslant q$ and define $\varphi_n^i : X^* \longrightarrow X^*$ a $\frac{R}{R-r}$-Lipschitz function such that $\varphi_n^i(x^*) = x^*$ holds for every $x^* \in X^* \setminus B(f_i(x_n), R)$ and $\varphi_n^i(z) = f_i(x_n)$ if $z \in B(f_i(x_n), r)$. If we define $g_n^i := \varphi_n^i \circ f_i : M \longrightarrow X^*$ we get that $\|g_n^i\| < 1$, that $g_n^i(m_{ij}) = f_i(m_{ij})$ (in particular $g_n^i(\mu_i) > (1 - \varepsilon)\|\mu_i\|$) and moreover $g_n^i(z) = f_i(x_n)$ for every $z \in B(x_n, r)$.

Now select $\alpha > 0$ small enough so that $\frac{2\alpha}{r-\alpha} < \varepsilon$ and find n large enough so that $d_{x_n} < \frac{\alpha}{2}$. Now we consider $y_n \neq x_n$ with $d(x_n, y_n) < \frac{\alpha}{2}$ and define, using the McShane extension theorem and functions of the form $s \otimes v^*$ for $s : M \longrightarrow \mathbb{R}$, a Lipschitz function $S_n : M \longrightarrow X^*$ so that $S_n = 0$ on $M \setminus B(x_n, \alpha) \cup \{x_n\}$, $S_n(y_n) = d(x_n, y_n)v^*$ for some $v^* \in S_{X^*}$ and $\|S_n\| = 1$.

Then, consider $g_i := g_n^i + S_n$, whose norm is smaller than $1 + \frac{2\alpha_n}{r-\alpha_n} < 1 + \varepsilon$ by applying Lemma 2.7 for $m = x_n$.

Finally, taking $v \in S_X$ with $v^*(v) > 1-\varepsilon$ and putting $\mu = \frac{\delta_{y_n}-\delta_{x_n}}{d(y_n,x_n)} \otimes v$ we get that

$$\begin{aligned}(1+\varepsilon)\|\mu_i+\mu\| \geqslant g_i(\mu_i+\mu) &= g_i(\mu_i)+g_i(\mu)\\ &= g_n^i(\mu_i)+S_n(\mu)\\ &> (1-\varepsilon)\|\mu_i\|+(1-\varepsilon)^2\\ &\geqslant (1-\varepsilon)^2(\|\mu_i\|+1),\end{aligned}$$

which implies $\|\mu_i+\mu\| \geqslant \frac{(1-\varepsilon)^2}{1+\varepsilon}(\|\mu_i\|+1)$, and the proof is finished.

∎

Now, we move on to the unbounded case. For that, we first need the following auxiliary result.

Lemma 4.2. *Let M be any metric space and let X be a Banach space. Let $F \subseteq M$ be a finite set and $f : M \longrightarrow X$ be a Lipschitz function with $\|f\| < 1$. Then there exists $g : M \longrightarrow X$ with $\|g\| < 1$ such that $g = f$ on F and $g = 0$ on $M \setminus B(0,R)$, for certain $R > 0$.*

Proof. By Proposition 2.5 we may assume that $f(M) \subset B_X(0,r)$ and $F \subset B_M(0,r)$ for some $r > 0$. Since $\|f\| < 1$ there must be $R > 0$ satisfying $\frac{r}{R-r} + \|f\| < 1$. Let us then define $\widetilde{\lambda} : [0,\infty) \to [0,1]$ as

$$\widetilde{\lambda}(t) = \begin{cases} 1\ , & \text{if } t \leqslant r,\\ \frac{R-t}{R-r}\ , & \text{if } r \leqslant t \leqslant R,\\ 0\ , & \text{elsewhere.}\end{cases}$$

Consider now $\lambda : M \to [0,1]$ as $\lambda = \widetilde{\lambda} \circ d(0,\cdot)$ and $g = \lambda \cdot f$. It is clear that $g = f$ on $B_M(0,r) \supset F$ and $g = 0$ on $M \setminus B_M(0,R)$ so that it suffices to compute

$$\|g\| \leqslant \|\lambda\| \sup f + \|f\| \sup \lambda \leq \frac{r}{R-r} + \|f\| < 1.$$

The last computation holds since clearly $\|\lambda\| \leq \|\widetilde{\lambda}\| \leq \frac{1}{R-r}$ by Lemma 2.2.

∎

Now we are ready to prove the following result.

Theorem 4.3. *Let M be a uniformly discrete unbounded metric space and let X be a non-zero Banach space. Then the norm of $\mathcal{F}(M)\widehat{\otimes}_\pi X$ is octahedral.*

Proof. As before let $\mu_1, \ldots, \mu_q \in \mathcal{F}(M)\widehat{\otimes}_\pi X$ and $\varepsilon > 0$, and let us find $\mu \in S_{\mathcal{F}(M)\widehat{\otimes}_\pi X}$ such that $\|\mu_i+\mu\| > \frac{(1-\varepsilon)(\|\mu_i\|+1)-2\varepsilon}{1+\varepsilon}$.

By a density argument we can assume that, for every $1 \leqslant i \leqslant q$, we can write $\mu_i := \sum_{j=1}^{n_i} \lambda_{ij}\delta_{m_{ij}} \otimes x_{ij}$, for certain $n_i \in \mathbb{N}$, $\lambda_{ij} \in \mathbb{R}\setminus\{0\}, m_{ij} \in M\setminus\{0\}$ and $x_{ij} \in X \setminus \{0\}$.

By Hahn-Banach theorem we can find, for every $i \in \{1, \dots, q\}$, an element $f_i \in (\mathcal{F}(M)\widehat{\otimes}_\pi X)^* = \mathrm{Lip}_0(M, X^*)$ such that $\|f_i\| < 1$ and $f_i(\mu_i) > (1-\varepsilon)\|\mu_i\|$.

By Lemma 4.2 we can find, for every $1 \leqslant i \leqslant q$, a Lipschitz function $g_i \in \mathrm{Lip}_0(M, X^*)$ with $\|g_i\| < 1$, such that $g_i = f_i$ on $\{m_{ij} : 1 \leqslant j \leqslant n_i\}$ (in particular $g_i(\mu_i) = f_i(\mu_i)$) and $g_i = 0$ on $M \setminus B(0,R)$ for $R > 0$ big enough for every $1 \leqslant i \leqslant q$.

Select $x \in M$ satisfying that $\frac{R}{d(x,0)-R} < \varepsilon$, which is possible since M is unbounded. Moreover, since M is uniformly discrete there exists a 1-Lipschitz function h such that $h(z) = 0$ if $z \neq x$.

We claim that $\|g_i \pm h\| \leqslant 1+\varepsilon$ holds for $1 \leqslant i \leqslant q$. Indeed, select $1 \leqslant i \leqslant q$ and take $u, v \in M$ with $u \neq v$. Let us analyse $A := \frac{\|g_i(u) \pm h(u) - (g_i(v) \pm h(v))\|}{d(u,v)}$.

Observe that if $u \neq x$ and $v \neq x$ then $A = \frac{\|g_i(u)-g_i(v)\|}{d(u,v)} < 1$. Otherwise assume, up to relabeling, that $v = x$, which implies $g_i(v) = 0$. If $g_i(u) = 0$ then it is immediate that $A = \frac{\|h(u)-h(v)\|}{d(u,v)} \leqslant \|h\| \leqslant 1$. Finally, if $g_i(u) \neq 0$ this implies $u \in B(0,R)$, from where we get

$$A \leqslant \frac{\|f_i(u)\| + \|h(u)-h(x)\|}{d(u,x)} \leqslant 1 + \frac{\|f_i(u)-f_i(0)\|}{d(u,x)} \leqslant 1 + \frac{d(u,0)}{d(u,x)}.$$

Now $d(u,x) = d(x,0) - d(u,0) \geqslant d(x,0) - R$. Moreover since $d(u,0) \leqslant R$ we infer that

$$1 + \frac{d(u,0)}{d(u,x)} \leqslant 1 + \varepsilon,$$

from where $\|g_i \pm h\| \leqslant 1+\varepsilon$ holds for $1 \leqslant i \leqslant q$.

Now consider $\mu \in S_{\mathcal{F}(M)\widehat{\otimes}_\pi X}$ such that $h(\mu) > 1-\varepsilon$. We claim now that $\|g_i(\mu)\| < 2\varepsilon$ holds for every i. In fact, select $\sigma \in \{-1, 1\}$ such that $|g_i(\mu) + \sigma h(\mu)\| = |g_i(\mu)| + |h(\mu)|$. Hence

$$1 + \varepsilon \geqslant \|g_i + \sigma h\| \geqslant |g_i(\mu) + \sigma h(\mu)| = |g_i(\mu)| + |h(\mu)| > 1 - \varepsilon + |g_i(\mu)|,$$

so $|g_i(\mu)| \leqslant 2\varepsilon$ holds for $1 \leqslant i \leqslant q$. On the other hand, since h vanishes on $B(0,R)$ we infer that $h(\mu_i) = 0$ holds for $1 \leqslant i \leqslant q$. Hence

$$\begin{aligned}(1+\varepsilon)\|\mu_i + \mu\| \geqslant (g_i + h)(\mu_i + \mu) &= g_i(\mu_i) + h(\mu) + g_i(\mu) \\ &\geqslant (1-\varepsilon)\mu_i + 1 - 3\varepsilon \\ &\geqslant (1-\varepsilon)(\|\mu_i\| + 1) - 2\varepsilon,\end{aligned}$$

so $\|\mu_i + \mu\| > \frac{(1-\varepsilon)(\|\mu_i\|+1)-2\varepsilon}{1+\varepsilon}$, as desired.

■

To finish this section we will give a stronger property for the case in which M' is infinite.

Theorem 4.4. *Let M be a metric space such that M' is infinite and let X be a non-zero Banach space. Then the norm of $(\mathcal{F}(M)\widehat{\otimes}_\pi X)^{**}$ is octahedral.*

For the proof we will use that the norm of $(\mathcal{F}(M)\widehat{\otimes}_\pi X)^{**}$ is octahedral if, and only if, $(\mathcal{F}(M)\widehat{\otimes}_\pi X)^* = \mathrm{Lip}_0(M, X^*)$ has the SD2P (see Definition 1.3). We will indeed prove something stronger, which is the symmetric strong diameter two property. Even though we are not explicitly using this property, let us point out its definition for completeness: A Banach space X is said to have the *symmetric strong diameter two property (SSD2P)* if, for every $k \in \mathbb{N}$, every finite family of slices $S_1, \ldots, S_k$ of B_X, and every $\varepsilon > 0$, there are $x_i \in S_i$ and there exists $\varphi \in B_X$ with $\|\varphi\| > 1 - \varepsilon$ such that $x_i \pm \varphi \in S_i$ for every $i \in \{1, \ldots, k\}$.

Observe that the SSD2P implies the SD2P, and the converse does not hold (see [3]). In spite of being a stronger property, SSD2P is sometimes easier to check than the SD2P. This happens in the case of infinite-dimensional uniform algebras or in Banach spaces with an infinite-dimensional centralizer (c.f. e.g. [15] and references therein). It is particularly useful in Banach spaces for which there is not a good description of the dual Banach space (as spaces of Lipschitz functions), because [15, Theorem 2.1] gives a criterion for the SSD2P, which only makes use of weakly convergent nets. This idea was used in [24, Lemma 2.3] where, combining this idea with the Lemma 2.8, a criterion for the SSD2P on $\mathrm{Lip}_0(M, X)$ is given, and this criterion will be the key to proving the following theorem, which will give us Theorem 4.4 by simply taking dual targets.

Theorem 4.5. *Let M be a metric space such that M' is infinite and let X be a non-zero Banach space. Then $\mathrm{Lip}_0(M, X)$ has the SSD2P.*

Proof. Let $f_1, \ldots, f_k \in S_{\mathrm{Lip}_0(M,X)}$. Let $\{x_n\} \subseteq M'$ be a sequence of different cluster points. We can assume with no loss of generality that $f_i(x_n) \neq f_i(x_m)$ for every $n, m \in \mathbb{N}$ with $n \neq m$ and every $1 \leqslant i \leqslant k$.

Consequently, for every $1 \leqslant i \leqslant k$ and every $n \neq m$ we can find $r_n^i > 0$ such that $\{B(f_i(x_n), r_n^i) : n \in \mathbb{N}\}$ is a sequence of pairwise disjoint balls for every $n \in \mathbb{N}$. Now given $1 \leqslant i \leqslant k$ and given $n \in \mathbb{N}$ we can find Lipschitz functions $\varphi_n^i : X \longrightarrow X$ such that $\|\varphi_n^i\| \to 1$ $(n \to \infty)$ for every i, such that $\varphi_n^i(z) = z$ holds for every $z \in X \setminus B(f_i(x_n), r_n^i)$ and such that $\varphi_n^i(z) = f_i(x_n)$ holds for z on a certain ball centred at $f_i(x_n)$.

Define $g_{n,i} := \varphi_n^i \circ f_i : M \longrightarrow X$. Observe that $\lim \|g_{n,i}\| = 1$. On the other hand observe that $\mathrm{supp}(g_{n,i} - f_i) \cap \mathrm{supp}(g_{m,i} - g) = \emptyset$ if $n \neq m$ for every $1 \leqslant i \leqslant k$.

Now observe that $g_{n,i}$ is constant at an open ball centred at x_n so we can find $r_n > 0$ such that $g_{n,i}(x) = g_{n,i}(x_n)$ holds for every $x \in B(x_n, r_n)$ for every $n \in \mathbb{N}$ and every $1 \leqslant i \leqslant k$.

Now given $n \in \mathbb{N}$ select $0 < \beta_n < r_n$ such that $\frac{\beta_n}{r_n - \beta_n} \to 0$ and we construct a Lipschitz function $h_n : M \longrightarrow X$ with $\|h_n\| = 1$ and $h_n(x) = 0$ for every $x \in \{x_n\} \cup X \setminus B(x_n, \beta_n)$.

It is immediate that $\mathrm{supp}(h_n) \cap \mathrm{supp}(h_m) = \emptyset$ if $n \neq m$. Now we claim that $\|g_n^i \pm h_n\| \to 1$ for $1 \leqslant i \leqslant k$.

Indeed, given $1 \leqslant i \leqslant k$ and given $n \in \mathbb{N}$, we get by applying Lemma 2.7 to $m = x_n$, $R = r_n$ and $\delta = \beta_n$ that $\|g_n^i \pm h_n\| \leqslant \|g_{n,i}\|(1 + \frac{2\beta_n}{r_n - \beta_n}) \to 1$.

Now we have that the sequences $(g_{n,i})$ and (h_n) satisfy the requirements of [24, Lemma 2.4], so applying the above result we get that $\mathrm{Lip}_0(M, X)$ has the SSD2P, and the proof is finished. ∎

5. Universally octahedral domains

In [29] the question of when a Banach space X satisfies that $L(Y, X)$ is octahedral for every Banach space Y is addressed (these spaces are called *universally octahedral*, [29, Definition 3.1]). The following result is proved.

Theorem 5.1. *Let X be a Banach space. The following are equivalent:*

1. *For every Banach space Y and every subspace $H \subseteq L(Y, X)$ containing the finite-rank operators, the space H is octahedral.*
2. *X is universally octahedral.*
3. *For every finite dimensional Banach space Y, the space $L(Y, X)$ is octahedral.*
4. *For every finite dimensional uniformly convex Banach space Y, the space $L(Y, X)$ is octahedral.*
5. *For every $1 < p < \infty$ and every $n \in \mathbb{N}$ the space $L(\ell_p^n, X)$ is octahedral.*
6. *For every $\varepsilon > 0$, for every finite dimensional subspace Z of X and for every $n \in \mathbb{N}$, there exists an element $T : \ell_\infty^n \longrightarrow X$ with $\|T\| \leqslant 1$ and such that*
$$\|z + T(y)\| \geqslant (1 - \varepsilon)(\|z\| + \|y\|)$$
holds for every $y \in \ell_\infty^n$ and every $z \in Z$.

A natural question in view of the previous result is to ask for "universal octahedral domains" in the following sense: when a Banach space X satisfies that $L(X, Y)$ is octahedral for every Banach space Y?

The following gives a complete characterisation.

Theorem 5.2. *Let X be a Banach space. The following assertions are equivalent:*

1. *For every Banach space Y, H is octahedral for every subspace $H \subseteq L(X, Y)$ with $\mathcal{F}(X, Y) \subseteq H$.*
2. *For every Banach space Y, $L(X, Y)$ is octahedral.*
3. *X^* is universally octahedral, i.e. for every Banach space Y, $L(Y, X^*)$ is octahedral.*
4. *For every $\varepsilon > 0$, for every finite dimensional subspace Z of X^* and for every $n \in \mathbb{N}$, there exists an operator $T : \ell_\infty^n \longrightarrow X^*$ with $\|T\| \leqslant 1$ and such that*
$$\|z + T(y)\| \geqslant (1 - \varepsilon)(\|z\| + \|y\|)$$
holds for every $y \in \ell_\infty^n$ and every $z \in Z$.

Proof. (1)⇒(2) is immediate, and (3)⇔(4) is precisely Theorem 5.1. To prove (2)⇒(3), take a Banach space Y. We want to prove that $L(Y, X^*)$ is octahedral. But this is immediate since the identification $L(Y, X^*) = L(X, Y^*)$ is isometric (c.f. e.g. [32, pp. 24]), so the conclussion is immediate from (2).

Let us prove (3)⇒(1). To this end take Y an arbitrary Banach space and $H \subseteq L(X, Y)$ containing $\mathcal{F}(X, Y)$. In order to prove that H is octahedral, take $T_1, \dots T_n \in S_H$ and $\varepsilon > 0$, and let us find $T \in S_H$ such that

$$\|T_i + T\| > 2 - \varepsilon.$$

To this end, consider adjoint operators $T_i^* : Y^* \longrightarrow X^*$ for every $1 \leqslant i \leqslant n$, which are norm-one operators since taking adjoint is an isometric action. Since X^* is universally octahedral, an inspection in the proof of [29, Theorem 3.6] allows to find a finite rank operator $G : Y^* \longrightarrow X^*$ with $\|G\| = 1$ and $\|T_i^* + G\| > 2 - \varepsilon$ for $1 \leqslant i \leqslant n$. For every $i \in \{1, \dots, n\}$, find an element $y_i^* \in S_{Y^*}$ such that

$$\|T_i^*(y_i^*) + G(y_i^*)\| > 2 - \varepsilon \ \forall 1 \leqslant i \leqslant n.$$

Let $\eta > 0$. Applying [27, Theorem 2.5] for $F := \operatorname{span}\{y_1^*, \dots, y_n^*\} \subseteq Y^*$ and η, there exists an operator $T \in \mathcal{F}(X, Y) \subseteq H$ such that $|\|T\| - \|G\|| < \eta$ and

$$T^*(y_i^*) = G(y_i^*)$$

holds for every $1 \leqslant i \leqslant n$. In particular, given $1 \leqslant i \leqslant n$, we have

$$2 - \varepsilon < \|T_i^*(y_i^*) + G(y_i^*)\| = \|T_i^*(y_i^*) + T^*(y_i^*)\| \leqslant \|T_i^* + T^*\| = \|T_i + T\|.$$

Since $1 - \eta < \|T\| < 1 + \eta$ we have $T/\|T\| \in S_H$ and

$$\begin{aligned}\left\|T_i + \frac{T}{\|T\|}\right\| \geqslant \|T_i + T\| - \left\|T - \frac{T}{\|T\|}\right\| &> 2 - \varepsilon - \|T\|\frac{|\|T\| - 1|}{\|T\|} \\ &> 2 - \varepsilon - (1 + \eta)\frac{\eta}{1 - \eta},\end{aligned}$$

and the conclusion follows from the arbitrariness of ε and η. ∎

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Chapter 10

A characterization of the weak topology in the unit ball of purely atomic L_1-preduals

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A CHARACTERIZATION OF THE WEAK TOPOLOGY IN THE UNIT BALL OF PURELY ATOMIC L_1 PREDUALS

GINÉS LÓPEZ-PÉREZ AND RUBÉN MEDINA

Abstract. We study Banach spaces with a weak stable unit ball, that is, Banach spaces where every convex combination of relatively weakly open subsets in its unit ball is again a relatively weakly open subset in its unit ball. It is proved that the class of L_1 preduals with a weak stable unit ball agree with those L_1 preduals which are purely atomic, that is preduals of $\ell_1(\Gamma)$ for some set Γ, getting in this way a complete geometrical characterization of purely atomic preduals of L_1, which answers a setting problem. As a consequence, we prove the equivalence for L_1 preduals of different properties previously studied by other authors, in terms of slices around weak stability. Also we get the weak stability of the unit ball of $C_0(K, X)$ whenever K is a Hausdorff and scattered locally compact space and X has a norm stable and weak stable unit ball. This gives a characterization of weak stability of the unit ball in $C_0(K, X)$ for finite-dimensional X. Finally we prove that Banach spaces with a weak stable unit ball satisfy a very strong new version of diameter two property.

1. Introduction

A convex subset C of a topological vector space X is said to be stable if the midpoint map $C \times C \to C$ given by $(x, y) \to \frac{x+y}{2}$ is open for the restricted topology on C. Thus, C is stable if, and only if, every convex combination of relatively open subsets in C is again a relatively open subset in C [20] (not to confuse with the concept of weakly stable Banach spaces in the Krivine-Maurey sense). In the past, the stability has been a relevant tool in different contexts. For example, in [20] it was proved, in the case when C is compact, that the openness of the above midpoint map is equivalent to the openness of the barycentre map (see [23]), which gives the continuity of the convex envelope for every continuous function on C [26]. Also, stability implies regularity (see [9]), which gives solution to the abstract Dirichlet problem [8]. Stability is a key to the study of extremal operators, since for

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a compact and Hausdorff space K, a norm one element $f \in C(K,X)$, the space of continuous functions on K into a Banach space X with stable unit ball, is an extreme point of the unit ball if, and only if, every value of f is an extreme point in the unit ball of X [12]. More recently, the stability of the convex set of positive trace-class operators on a separable Hilbert space with trace one has been proved in [24]. This set is known as the set of quantum states or density operators and this fact is the key to study the continuity of Von Neumann entropy, relevant in theoretical quantum information.

The prototype example of a stable compact convex set is a Bauer simplex, that is the positive face of the unit sphere in the dual of a $C(K)$ space or equivalently the set of Radon probability measures for its weak-star topology [20]. Examples of Banach spaces with a norm stable unit ball are strictly convex spaces or spaces with 3.2 intersection property [9]. Also stable convex compact sets in finite-dimensional spaces are completely characterized. Indeed, a compact and convex subset K in $\mathbb{R}^n$ is stable if, and only if, the p-skeleton $\{x \in K : \dim(\text{face}(x)) \leqslant p\}$ is closed for $0 \leqslant p \leqslant n-1$ (face(x) denotes the minimal face in K containing x, and dim means the dimension of the affine span). Then K is always stable if $n = 2$, K is stable if, and only if, the set of extreme points of K is closed, in the case $n = 3$, and for $n > 3$, there are compact and convex subsets K with a closed set of extreme points, failing to be stable. As a consequence of the above, every unit ball of a finite-dimensional Banach space with a finite set of extreme points, or such that every point in the unit sphere is an extreme point, is stable [22].

As the above cited example, most results about stability in the past, have been obtained in the compact case, or concern with norm stability. More recently, motivated by the fact that the positive face of the unit sphere of the space $L_1[0,1]$ is weak stable [11, Remark IV.5], in relation with other geometrical properties in Banach spaces as diameter two properties [19] (also octahedrality, Daugavet property, etc [6]), the weak stability has attracted the attention of researchers in geometry of Banach spaces, and new examples of weak stable unit balls of Banach spaces have emerged. For example, in [1] and [13] it is proved that the unit ball of $C_0(K)$, for a Hausdorff and locally compact space K, is weak stable if, and only if, K is scattered (see also [2] and [5]). As it can be seen in the above references, the current interest is to know how big the class of Banach spaces with a weak stable unit ball is, and the natural questions in the setting of the above papers is to know what preduals of L_1 or $C_0(K,X)$ spaces have a weak stable unit ball.

From now, we will say that a Banach space X has a weak stable unit ball B_X, if it is stable for the inherited weak topology (other names have been used to mention stability in recent papers, but it seems natural keeping the used notation in the past). In the case B_X is stable for the norm topology, we will say that X has a stable unit ball.

Observe that a Banach space with a weak stable unit ball satisfies the strong diameter two property [19]. Then X^* is octahedral from [6] and so the class of L_1 preduals is a natural class to looking for weak stable unit

balls. Our main result in Section 2 proves that a L_1 predual X is weak stable if, and only if, X^* is purely atomic, that is, $X^* = \ell_1(\Gamma)$ for some set Γ, which gives a complete geometric characterization of purely atomic L_1 preduals. Taking into account the existence of preduals of ℓ_1 which are not isomorphic to complemented subspaces of $C(K)$ spaces [7], the above fact improves the aforementioned results around weak stability, known up to now. Our main result uses continuous selections of multivalued maps techniques in the setting of L_1 preduals. In Section 3, after introduce some easy consequences of Banach spaces with a stable unit ball, we prove using again multivalued maps techniques the weak stability of the unit ball of $C_0(K, X)$, whenever K is a Hausdorff and scattered locally compact space and X has a stable and weak stable unit ball, which improves the known results up to now. Observe for example that in [2], the above result is obtained for finite-dimensional X satisfying an hypothesis which we will prove to be equivalent to norm stability of the unit ball in X. We finish the paper by showing that Banach spaces with a weak stable unit ball satisfy a stronger new version among the known diameter two properties.

We use standard terminology for Banach spaces. S_X and B_X denote the unit sphere and the closed unit ball, respectively, of a Banach space X. w and w^* denote the weak and weak-star topologies and X^* is the topological dual of X. A slice in B_X is the intersection of B_X with an open semi-space. A face of a subset C in a Banach space X is a subset $F \subset C$ satisfying that $x, y \in F$ whenever $x, y \in C$ and $tx + (1-t)y \in F$ for some $t \in [0, 1]$. An extreme point of C is a point $x \in C$ such that the singleton $\{x\}$ is a face of C. $\text{co}(A)$ denotes the convex hull of A. A L_1 predual is a Banach space X so that X^* is linearly isometric to $L_1(\mu)$ for some measure space (Ω, Σ, μ). If Γ is a set, $\ell_1(\Gamma, \mathbb{K})$ is the classical Banach space of (absolutely) summable families of scalars in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Given a Hausdorff locally compact space K, $C_0(K, X)$ stands for the Banach space of X-valued continuous maps defined over the one-point compactification of K that are null at infinity.

Now we pass to introduce some notation and known results, which will be used in the following. For X and Y topological spaces, we denote by $\mathcal{P}(Y)$ the power set of Y. A map defined on X with values into $\mathcal{P}(Y)$ will be called a multivalued map $\phi : X \to Y$ so that $\phi(x)$ is a nonempty subset of Y for every $x \in X$. We recall that ϕ is called lower semicontinuous (l.s.c) or hemicontinuous at $x \in X$ if for every open subset U of Y such that $\phi(x) \cap U \neq \emptyset$, there is an open subset V of X with $x \in V$ such that $\phi(z) \cap U \neq \emptyset$ whenever $z \in V$. ϕ is called lower semicontinuous if ϕ is lower semicontinuous at x, for every $x \in X$. Also, ϕ is lower semicontinuous if, and only if, the set $\phi^\ell(U) := \{x \in X : \phi(x) \cap U \neq \emptyset\}$ is an open subset of X for every open subset U of Y. Then a map $f : X \to Y$ between topological spaces X and Y is open if, and only if, the multivalued map $f^{-1} : Y \to X$ given by $f^{-1}(y) = \{x \in X : f(x) = y\}$ is l.s.c. A multivalued map $\phi : X \to Y$, with X and Y convex subsets of real or complex vector spaces, is called convex if $t\phi(x) + (1-t)\phi(y) \subset \phi(tx + (1-t)y)$ for every

$t \in [0,1]$ and $x, y \in X$. ϕ is called symmetric if $\phi(zx) = z\phi(x)$ for every $x \in X$ and z a modulus one complex scalar. A continuous selection of a multivalued map $\phi : X \to Y$ will be a continuous map $f : X \to Y$ such that $f(x) \in \phi(x)$ for every $x \in X$. The well known Michael selection principle asserts that the above ϕ has a continuous selection whenever X is a paracompact topological space, Y is a Fréchet space, ϕ is l.s.c and $\phi(x)$ is a closed and convex subset of Y for every $x \in X$. Finally, we recall a useful criteria for l.s.c. multivalued maps (see [4, Chapter 17] for background about the above topics). We will use the above freely in the sequel.

Lemma 1.1. *([4, Theorem 17.21]) For X and Y topological spaces, and $\phi : X \to Y$ a multivalued map, the following assertions are equivalent:*

i) *ϕ is l.s.c.*
ii) *For every net $\{x_\alpha\}$ in X converging to $x \in X$, and for every $y \in \phi(x)$, there is a subnet $\{x_{\alpha_\lambda}\}$ of $\{x_\alpha\}$ and a net $\{y_\lambda\}$ in Y such that $y_\lambda \in \phi(x_{\alpha_\lambda})$ for every λ and $\{y_\lambda\}$ converges to y.*

Moreover, net and subnet can be changed in ii) by sequence and subsequence, whenever X and Y are first countable.

2. Purely atomic L_1-preduals

As we said in the introduction, our goal is to enlarge the known class of Banach spaces with a weak stable unit ball. It is known that $C(K)$ spaces, with K a scattered, Hausdorff and compact topological space, are weak stable [1], [13]. Also, other properties around weak stability has been considered by different authors (see [1, 2, 5, 13, 19]), like the weak relative openess of convex combination of slices in the unit ball, nonempty relative weak interior of convex combinations of slices in the unit ball, nonempty relative weak interior of convex combinations of relative weak open subsets in the unit ball, or property $(\overline{P1})$. A Banach space X is said to verify the property $\overline{\text{P1}}$ if for every convex combination of slices in B_X, C, and for every $x \in C$ there is a weak open subset relative to B_X, U, so that $x \in U$ and $U \subset \overline{C}$. Clearly weak stability implies all the above properties. With the above in mind, it is natural trying to characterize the weak stability for the class of preduals of L_1, improving the above known results. This will be our first goal, and for that, our main tool will be the next well known result about continuous selections in the setting of real or complex L_1 preduals.

Theorem 2.1. *([18, Theorem 2.2], [21]) Let X be a real or complex Banach space with X^* isometric to $L_1(\mu)$ space and E be a Fréchet space. Consider a convex, symmetric and w^*-l.s.c. multivalued map $\phi : B_{X^*} \to E$ such that $\phi(x^*)$ is a convex and closed subset of E for every $x^* \in B_{X^*}$. Then ϕ admits an affine, symmetric and w^*-continuous selection $h : B_{X^*} \to E$. Moreover, if F is a face of B_{X^*} with $H := \text{co}(F \cup -F)$ w^*-closed and $g : H \to E$ is an affine, symmetric and w^*-continuous selection of $\phi_{|H}$, then h can be chosen so that $h_{|H} = g$.*

The above result was proved in [18, Theorem 2.2] in the real case. The complex case appears in [21], assumming that H is the intersection of B_{X^*} with an L-ideal. This intersection agree with the concept of biface in [18] in the case of real preduals of L_1, as said in [21] (see also [3, Pag. 168]).

Before showing our main result we need a simple lemma.

Lemma 2.2. *Let Γ a nonempty set and $E := \{e_\gamma : \gamma \in \Gamma\}$ the set of basis vectors in $\ell_1(\Gamma, \mathbb{K})$. Then the linear span of E is norm dense in $\ell_1(\Gamma, \mathbb{K})$, and for a finite subset C of E, one has that $\mathrm{co}(C)$ is a w^*-closed face of $B_{\ell_1(\Gamma,\mathbb{K})}$ and $\mathrm{co}(C)$ is a simplex, that is, every element in $\mathrm{co}(C)$ has a unique expression as a convex combination of elements in C.*

Proof. Lets take $G = \{\gamma_1, \ldots, \gamma_k\} \subset \Gamma$ and define $C = \{e_\gamma,\ \gamma \in G\} \subset E$. In order to show that $\mathrm{co}(C)$ is a face of $B_{\ell_1(\Gamma,\mathbb{K})}$, let $\lambda_1, \ldots, \lambda_k \in (0, 1)$ be such that $\sum_{i=1}^{k} \lambda_i = 1$ and take the point $x = \sum_{i=1}^{k} \lambda_i e_{\gamma_i}$. Given $a, b \in B_{\ell_1(\Gamma,\mathbb{K})}$ such that $\frac{a+b}{2} = x$ then we claim that $a, b \in co(C)$:

First of all, if we fix $\gamma \in \Gamma \setminus G$, then $b(\gamma) = -a(\gamma)$ so we have

$$2 \geq ||a|| + ||b|| \geq 2|a(\gamma)| + \sum_{i=1}^{k} |a(\gamma_i)| + |b(\gamma_i)| \geq 2|a(\gamma)| + \sum_{i=1}^{k} |a(\gamma_i) + b(\gamma_i)|$$

$$= 2|a(\gamma)| + 2\sum_{i=1}^{k} \lambda_i = 2|a(\gamma)| + 2,$$

which yields that $a(\gamma) = b(\gamma) = 0$ for every $\gamma \in \Gamma \setminus G$. The real case is now done. Lets consider $\mathbb{K} = \mathbb{C}$ for the rest of the proof. If we suppose that there exists $j \in \{1, \ldots, k\}$ such that $a(\gamma_j) \notin \mathbb{R}$, then

$$||a|| + ||b|| = \sum_{i=1}^{k} |a(\gamma_i)| + |b(\gamma_i)| > \sum_{i=1}^{k} |Re(a(\gamma_i))| + |Re(b(\gamma_i))|$$

$$\geq \sum_{i=1}^{k} |Re(a(\gamma_i) + b(\gamma_i))| = 2,$$

which is a contradiction with the fact that $a, b \in B_{\ell_1(\Gamma,\mathbb{K})}$.
The rest of the proof is straightforward. ∎

Now we pass to get the characterization of weak stable L_1 predual unit balls.

Theorem 2.3. *Let X be a real or complex isometric predual of a $L_1(\mu)$ space for some measure μ. Then B_X is weak stable if, and only if, μ is purely atomic.*

Proof. If X is a real Banach space with a weak stable unit ball, in particular every convex combination of slices in B_X has nonempty relative weak interior in B_X. Then we deduce that μ is purely atomic, from [19, Theorem 4.7] (note

that [19, Theorem 4.7] is deduced from [19, Theorem 4.1] and a localizable measure space is used, however $L_1(\mu)$ is linearly isometric to a ℓ_1-sum of spaces $L_1(\mu_i)$ with μ_i localizable from [16, Pag. 136], see also [10, Pag. 501]). The above is valid in the real case, but it can be easily adapted too for the complex case.

Assume now that μ is purely atomic, so $X^* = \ell_1(\Gamma)$ for some set Γ. Pick O_1, O_2 relatively weak open subsets of B_X and $x \in \frac{O_1+O_2}{2}$. Put $x = \frac{z_1+z_2}{2}$, with $z_i \in O_i$, $i = 1, 2$. Then there is $1 > \rho > 0$ such that $s_i := (1-\rho)z_i + \rho x \in O_i$, $i = 1, 2$. Now

$$|x^*(s_i)| \leqslant 1 - \rho + \rho|x^*(x)| \ \forall x^* \in B_{X^*}. \tag{2.1}$$

As the subspace generated by the set E of basis vectors of B_{X^*} is dense in X^* for the norm topology from Lemma 2.2, we can choose weak neighborhoods U_i of s_i, relative to B_X, satisfying $U_i \subset O_i$, given by

$$U_i = \{z \in B_X : |x_j^*(z - s_i)| < \varepsilon, \ j \in 1, \dots, k\},$$

where $\varepsilon > 0$, $k \in \mathbb{N}$ and $x_j^* \in E$ for $1 \leqslant j \leqslant k$. Now, our goal will be to prove that $\frac{U_1+U_2}{2}$ is a weak neighborhood of x relative to B_X. In order to do that, define $C := \{x_j^*\}_{j=1}^k$ and $\delta := \min\limits_{\substack{x^* \in C \\ |x^*(x)|<1}} \{1 - |x^*(x)|\} > 0$, in the case $A_x^C := \{x^* \in C : |x^*(x)| < 1\} \neq \emptyset$, otherwise we put $\delta := 1$ (observe that $|x^*(x)| \leqslant 1 - \delta$ whenever $x^* \in A_x^C$). Also we consider

$$U := \{z \in B_X \, |x^*(z - x)| < \mu \ \forall x^* \in C\},$$

for $0 < \mu < \min\{\rho\delta, \varepsilon\}$. As U is a neighborhood of x relative to B_X, it is enough to show that $U \subset \frac{U_1+U_2}{2}$. In order to prove that, fix $y \in U$ and put $F = \text{co}(C)$. As C is a finite subset of points in E, we have that F is a w^*-closed face of B_{X^*} from Lemma 2.2, and so $H := \text{co}(S_{\mathbb{K}}F)$ is w^*-closed (in the complex case, even in the real one, we have that H agree with the intersection of an L- ideal, the linear span of C, with B_{X^*}). In order to apply Theorem 2.1, we define $g_1, g_2 : C \to \mathbb{K}$ by

$$g_i(x^*) = \begin{cases} x^*(s_i + y - x) & \text{if } \ x^* \in A_x^C \text{ and so } |x^*(x)| \leq 1 - \delta \\ x^*(y) & \text{if } \ |x^*(x)| = 1 \end{cases}$$

It is clear that g_1 and g_2 admit unique affine and symmetric extensions to H from Lemma 2.2, which we will call again $g_1, g_2 : H \to \mathbb{K}$. In order to prove that $g_i(x^*) \in B_{\mathbb{K}}$ for $i = 1, 2$, $x^* \in H$, it is enough to see that $g_i(x^*) \in B_{\mathbb{K}}$ for every $x^* \in C$. Indeed, in the case $x^* \in C$ with $|x^*(x)| \leq 1 - \delta$ we have, applying (2.1) and taking into account that $y \in U$ with $\mu < \rho\delta$,

$$\begin{aligned} |g_i(x^*)| =& |x^*(s_i + y - x)| \leq |x^*(s_i)| + |x^*(y - x)| \\ \leq& 1 - \rho + \rho|x^*(x)| + \mu < 1 - \rho(1 - |x^*(x)| - \delta) \leq 1, \end{aligned}$$

while in the case $x^* \in C$ with $|x^*(x)| = 1$, we have $|g_i(x^*)| = |x^*(y)| \leq 1$.

Now we define the multivalued map $\phi : B_{X^*} \to \mathbb{K} \times \mathbb{K}$ by

$$\phi(x^*) = \left\{ (r_1, r_2) \in B_{\mathbb{K}} \times B_{\mathbb{K}} : \frac{r_1 + r_2}{2} = x^*(y) \right\} \ \forall x^* \in B_{X^*}.$$

It is easy to see that ϕ is an convex and symmetric multivalued map, with $\phi(x^*)$ a closed and convex subset of $\mathbb{K} \times \mathbb{K}$, for every $x^* \in B_{X^*}$. The fact that ϕ is a w^*-l.s.c. multivalued map is consequence that $\phi = f^{-1} \circ \hat{y}$, where $f : B_{\mathbb{K}} \times B_{\mathbb{K}} \to B_{\mathbb{K}}$ is the midpoint map, and $\hat{y}$ is the natural injection of y in X^{**} (observe that $\hat{y}$ is w^*-continuous as an element of X^{**} and f^{-1} is a l.s.c multivalued map since $B_{\mathbb{K}}$ is stable, that is since f is open).

We will check now that $g := (g_1, g_2)$ is a symmetric, affine and w^*-continuous selection map of $\phi_{|H}$. It is clear that g is symmetric, affine and w^*-continuous. Since ϕ is symmetric and convex, it is enough to show that $g(x^*) \in \phi(x^*)$ for every $x^* \in C$ in order to prove that $g(x^*) \in \phi(x^*)$ for every $x^* \in H$. In fact, in the case $x^* \in C$ with $|x^*(x)| \leq 1 - \delta$ we have

$$\begin{aligned} \frac{g_1(x^*) + g_2(x^*)}{2} =& \frac{x^*(s_1 + y - x + s_2 + y - x)}{2} \\ =& x^* \left(\frac{s_1 + s_2}{2} \right) + x^*(y - x) = x^*(x) + x^*(y - x) = x^*(y), \end{aligned}$$

while in the case $x^* \in C$ with $|x^*(x)| = 1$ we have

$$\frac{g_1(x^*) + g_2(x^*)}{2} = \frac{x^*(y) + x^*(y)}{2} = x^*(y).$$

Then we have proved that g is a symmetric, affine and w^*-continuous selection map of $\phi_{|H}$. We apply now Theorem 2.1 to find out $h := (h_1, h_2)$, a symmetric, affine and w^*-continuous selection map of ϕ, satisfying that $h_{|H} = g$. Define, for $i = 1, 2$

$$x_i : X^* \to \mathbb{K}$$

by

$$x_i(x^*) = \begin{cases} 0 & \text{if} \quad x^* = 0 \\ ||x^*|| h_i \left(\frac{x^*}{||x^*||} \right) & \text{if} \quad x^* \in X^* \setminus \{0\} \end{cases}$$

From Krein-Smulyan Theorem, x_1, x_2 are w^*-continuous linear functionals on X^* such that $x_i(B_{X^*}) \subset B_{\mathbb{K}}$. Hence, $x_1, x_2 \in B_X$ and for $0 \neq x^*$, as (h_1, h_2) is a selection of ϕ, we have that

$$\begin{aligned} x^* \left(\frac{x_1 + x_2}{2} \right) =& \frac{x^*(x_1) + x^*(x_2)}{2} = \frac{||x^*|| h_1 \left(\frac{x^*}{||x^*||} \right) + ||x^*|| h_2 \left(\frac{x^*}{||x^*||} \right)}{2} \\ =& ||x^*|| \left(\frac{h_1 + h_2}{2} \right) \left(\frac{x^*}{||x^*||} \right) = ||x^*|| \left(\frac{x^*}{||x^*||} \right) (y) = x^*(y). \end{aligned}$$

So we deduce that $\frac{x_1 + x_2}{2} = y$.

Finally, we show that $x_i \in U_i$ for $i \in \{1,2\}$. In the case $x^* \in C$ with $|x^*(x)| \leq 1-\delta$, one has

$$\begin{aligned}|x^*(x_i - s_i)| &= \left| ||x^*|| h_i\left(\frac{x^*}{||x^*||}\right) - x^*(s_i)\right| = |g_i(x^*) - x^*(s_i)| \\ &= |x^*(s_i + y - x) - x^*(s_i)| = |x^*(y-x)| < \mu < \varepsilon,\end{aligned}$$

while in the case $x^* \in C$ with $|x^*(x)| = 1$ we deduce from $\frac{x^*(s_1)+x^*(s_2)}{2} = x^*(x)$ and $|x^*(s_i)| \leqslant 1$, that $x^*(x) = x^*(s_1) = x^*(s_2)$. Then

$$\begin{aligned}|x^*(x_i - s_i)| &= \left| ||x^*|| h_i\left(\frac{x^*}{||x^*||}\right) - x^*(s_i)\right| = |g_i(x^*) - x^*(s_i)| \\ &= |x^*(y) - x^*(s_i)| = |x^*(y-x)| < \mu < \varepsilon.\end{aligned}$$

So $x_i \in U_i$ for $i \in \{1,2\}$ with $\frac{x_1+x_2}{2} = y$ and we are done ∎

It is worth to mention that inside the class of isometric preduals of ℓ_1, there are Banach spaces not isomorphic to complemented subspaces of any $C(K)$ space [7, Theorem 2.13], so that Theorem 2.3 definitively enlarges the class of spaces with a weak stable unit ball far away, even isomorphically speaking, to $C(K)$ spaces known up to now, in a natural way. Also, as a consequence of the above result, we get the equivalence between a number of properties considered up to now by different authors, around weak stability, in the natural setting of L_1 preduals.

Corollary 2.4. *Let X be an isometric predual of some $L_1(\mu)$ space. Then the following assertions are equivalent:*

i) *B_X is weak stable.*
ii) *Every convex combination of slices in B_X is a relatively weakly open subset in B_X.*
iii) *Every convex combination of slices in B_X has nonempty relatively weak interior in B_X.*
iv) *Every convex combination of relatively weakly open subsets in B_X has nonempty relatively weak interior in B_X.*
v) *X satisfies the property $\overline{\text{P1}}$.*

Moreover, one of the above assertions holds if, and only if, μ is purely atomic, that is X^* is isometrically isomorphic to $\ell_1(\Gamma)$ for some set Γ.

Proof. i) ⇒ ii) ⇒ iii), i) ⇒ iv) ⇒ iii) and i) ⇒ iv) are clear. iii) ⇒ i) is a consequence of [19, Theorem 4.7] and Theorem 2.3. v) ⇒ i) is a consequence of [5, Theorem 2.13], [14] and Theorem 2.3. ∎

We don't know the exact relation between the assertions i) to v) in Corollary 2.4 for general Banach spaces. In particular, we don't know if assertions i) and ii) in Corollary 2.4 are equivalent for general Banach spaces.

Thanks to Theorem 2.3, we can partially answer a question posed in [2], wether the injective tensor product of 2 Banach spaces with weak stable

unit ball has a weak stable unit ball.

Given $\ell_1(\Gamma_1)$ and $\ell_1(\Gamma_2)$, we are able to completely identify $\ell_1(\Gamma_1, \ell_1(\Gamma_2))$ and $\ell_1(\Gamma_1 \times \Gamma_2)$ by the identification

$$T : \ell_1(\Gamma_1, \ell_1(\Gamma_2)) \to \ell_1(\Gamma_1 \times \Gamma_2)$$

where $T(x)(\gamma_1, \gamma_2) = x(\gamma_1)(\gamma_2)$ for every $\gamma_i \in \Gamma_i$ with $i = 1, 2$. It is well known that $\ell_1(\Gamma_1, \ell_1(\Gamma_2)) = \ell_1(\Gamma_1)\widehat{\otimes}_\pi \ell_1(\Gamma_2)$ so the next result arises as a direct consequence of this and Theorem 2.3.

Corollary 2.5. *Let X and Y be L_1 preduals, then the injective tensor product of X and Y has a weak stable unit ball if and only if both X and Y have weak stable unit balls.*

Let us now get into another special case of injective tensor product, the one in which one of the factors is a $C(K)$ space.

3. Weak stability in the unit ball of $C_0(K, X)$

First, we are going to remind the concept of stable unit ball for the norm topology.

Definition 3.1. Let X be a Banach space. We say that B_X is stable if it is stable for the norm topology, that is, if every convex combination of relatively norm open subsets of B_X is relatively open in B_X for the norm topology.

Stability and weak stability seem to have a very strong connection. In fact, some of the main known properties for weak stable unit balls (see [1] and [13]) are also true for (norm) stable unit balls, as the following results show.

Proposition 3.2. *Let X and Y be Banach spaces with stable unit balls, then:*

i) *If Z is a 1-complemented subspace of X, B_Z is stable.*
ii) *If $W = X \oplus_\infty Y$ then B_W is stable.*

Proof. i) Let $P : X \to Z$ be the norm 1 projection. If X is stable, let $O_1, O_2 \subset B_Z$ relatively norm open subsets, and $\lambda \in (0, 1/2]$. We call $O_Z = (1-\lambda)O_1 + \lambda O_2$, so that $O_X = (1-\lambda)(P^{-1}(O_1) \cap B_X) + \lambda(P^{-1}(O_2) \cap B_X)$ satisfy $O_Z = O_X \cap B_Z$ and i) is proved.

ii) Let us take $w_i = (x_i, y_i) \in B_W$ for $i = 1, 2$ and consider $\lambda \in (0, 1/2]$ and $\delta > 0$. Considering now $B_\lambda = (1-\lambda)B_1 + \lambda B_2$ where $B_i = B_W(w_i, \delta) \cap B_W$ for $i = 1, 2$ we prove ii) since

$$B_\lambda = ((1-\lambda)B_1^X + \lambda B_2^X) \times ((1-\lambda)B_1^Y + \lambda B_2^Y)$$

where $B_i^X = B_X(x_i.\delta)$ y $B_i^Y = B_Y(y_i, \delta)$ for $i = 1, 2$. ∎

Proposition 3.3. *If (X_n) is a sequence of Banach spaces with stable unit ball, then $X = c_0(X_n)$ has a stable unit ball.*

Proof. Let $x, s^1, s^2 \in c_0(X_n)$ and $\lambda > 0$ be such that $(1-\lambda)s^1 + \lambda s^2 = x$ where $x = (x_n)$, $s^1 = (s^1_n)$ and $s^2 = (s^2_n)$. We define for arbitrary $\varepsilon \in (0,1)$ the set

$$B_x = (1-\lambda)B_{c_0(X_n)}(s^1, \varepsilon) \cap B_{c_0(X_n)} + \lambda B_{c_0(X_n)}(s^2, \varepsilon) \cap B_{c_0(X_n)}.$$

Let us now take $N \in \mathbb{N}$ such that $\forall n > N$, $||s^i_n|| < 1 - \varepsilon$ for $i = 1, 2$. For every $n \leq N$, we first consider $\delta_n > 0$ satisfying

$$B_{X_n}(x_n, \delta_n) \cap B_{X_n} \subset (1-\lambda)B_{X_n}(s^1_n, \varepsilon) \cap B_{X_n} + \lambda B_{X_n}(s^2_n, \varepsilon) \cap B_{X_n} := B_{x_n}.$$

Now, taking $0 < \delta < \min_{n=1,\ldots,N}\{\delta_n, \varepsilon\}$, we claim that $B_{c_0(X_n)}(x, \delta) \cap B_{c_0(X_n)} \subset B_x$:

If $y = (y_n) \in B_{c_0(X_n)}(x, \delta) \cap B_{c_0(X_n)}$, for $n \leq N$, then $y_n \in B_{X_n}(x_n, \delta) \cap B_{X_n} \subset B_{x_n}$. Otherwise, if $n > N$ then $y_n = (1-\lambda)y^1_n + \lambda y^2_n$ where

$$y^1_n = s^1_n + y_n - x_n \in B_{X_n}(s^1_n, \varepsilon) \cap B_{X_n} \quad , \quad y^2_n = s^2_n + y_n - x_n \in B_{X_n}(s^2_n, \varepsilon) \cap B_{X_n}$$

and we are done. ∎

It can be easily shown that there exist Banach spaces with stable but not weak stable unit balls such as strictly convex spaces [9]. However, all the examples that we have of weak stable unit balls of Banach spaces are also norm stable. We don't know if weak stability implies norm stability.

As we said in the introduction, our main goal is to get conditions to have (weak) stability in $C_0(K, X)$. For the weak stability, we will use strongly multivalued maps techniques. So, we begin by showing an easy consequence of these techniques, which will be useful to get our main result.

Lemma 3.4. *Let C be a stable convex subset of a metrizable topological vector space X. Consider a sequence $\{x_n\}$ in C and $x \in C$ such that $\{x_n\}$ converges to x. If $p_1, p_2, q_1, q_2 \in C$ satisfying that $x = \frac{p_1+p_2}{2} = \frac{q_1+q_2}{2}$, then there are $\{p^n_1\}$, $\{p^n_2\}$, $\{q^n_1\}$ and $\{q^n_2\}$ sequences in C, and there is $\{x_{\sigma(n)}\}$ a subsequence of $\{x_n\}$, such that $x_{\sigma(n)} = \frac{p^n_1+p^n_2}{2} = \frac{q^n_1+q^n_2}{2}$ for every $n \in \mathbb{N}$, with $\{p^n_i\}$ converging to p_i and $\{q^n_i\}$ converging to q_i, for $i = 1, 2$.*

Proof. Consider $\phi : C \times C \to C$ the midpoint map on C, given by $\phi(x, y) = \frac{x+y}{2}$. As C is stable, we have that ϕ is open, so the map $\phi \times \phi : (C \times C) \times (C \times C) \to (C \times C)$ given by $(\phi \times \phi)(x, y, z, w) = (\phi(x, y), \phi(z, w))$ is also open. Then the multivalued map $(\phi \times \phi)^{-1}$ is l.s.c. Since $\{x_n, x_n\}$ converges to (x, x), we get, from Lemma 1.1, the desired conclusion. ∎

We obtain now our result about stability of $C_0(K, X)$.

Theorem 3.5. *Let X be a stable and weak stable Banach space and K a scattered Hausdorff topological space. Then $C_0(K,X)$ is weak stable.*

Proof. Consider O_1, O_2 nonempty relative weakly open subsets of $B_{C_0(K,X)}$. We are going to prove that $\frac{O_1+O_2}{2}$ is again a relative weakly open subset of $B_{C_0(K,X)}$. Pick $x \in \frac{O_1+O_2}{2}$ and put $x = \frac{s_1+s_2}{2}$, with $s_i \in O_i$, $i = 1, 2$.

We can assume that, for $i = 1, 2$, $O_i = \{y \in B_{C_0(K,X)} : |f_{i,j}(y) - \alpha_{i,j}| < \varepsilon_{i,j}, 1 \leqslant j \leqslant k_i\}$, where $k_i \in \mathbb{N}$, $f_{i,j} \in S_{C_0(K,X)^*}$ and $\alpha_{i,j} \in [-1, 1]$ for $1 \leqslant j \leqslant k_i$. Define $\mu_{i,j} := \varepsilon_{i,j} - |f_{i,j}(s_i) - \alpha_{i,j}|$ and $\varepsilon := \min_{1\leqslant i\leqslant 2, 1\leqslant j\leqslant k_i} \mu_{i,j} > 0$.

Note now that $B_{C_0(K,X)^*} = \overline{\text{co}}\{\delta_t \otimes x^* : t \in K, x^* \in S_{X^*}\}$ (δ_t denotes the evaluation functional at t as an element in $C_0(K)^*$). Indeed, as $C_0(K,X)^* = C_0(K)^* \hat{\otimes}_\pi X^*$ and $B_{C_0(K,X)^*} = \overline{\text{co}}(S_{C_0(K)^*} \otimes S_{X^*})$, we get that $B_{C_0(K,X)^*} = \overline{\text{co}}\{\delta_t \otimes x^* : t \in K, x^* \in S_{X^*}\}$, since K is scattered and so $C_0(K)^* = \ell_1(K)$. Then, for every $i \in \{1, 2\}$ and every $1 \leqslant j \leqslant k_i$ we can find out $N_{ij} \in \mathbb{N}$ and finite subsets $E \subset K$ and $A \subset S_{X^*}$, both with N_{ij} elements, so that $\|g_{ij} - f_{ij}\| < \varepsilon/3$, where $g_{ij} \in \text{co}(\delta_E \otimes A)$ being $\delta_E = \{\delta_t : t \in E\}$.

Now we get that

(3.1)

$$U_i := \{y \in B_{C_0(K,X)} : |g_{ij}(y - s_i)| < \varepsilon/3, 1 \leqslant j \leqslant k_i\} \subset O_i \ \forall i \in \{1, 2\}.$$

For every $t \in E$ and for every $i \in \{1, 2\}$ we define

$$B_i^t = \{z \in B_X : |x^*(z - s_i(t))| < \varepsilon/4 \ \forall x^* \in A\}.$$

It is clear that B_i^t is a nonempty, convex and relatively weak open subset of B_X. Furthermore, as X is weak stable, we have that $O_t := \frac{B_1^t + B_2^t}{2}$ is a relatively weak open subset of B_X containing $x(t)$, for every $t \in E$. So there are $\delta > 0$ and $x_1^*, \ldots, x_m^* \in B_{X^*}$ such that

$$\{z \in B_X : |x_i^*(z - x(t))| < 2\delta, \ 1 \leqslant i \leqslant m\}\} \subset O_t.$$

Put

$$U = \{y \in B_{C_0(K,X)} : |x^*(y(t) - x(t))| < \delta \ \forall t \in E, \ \forall x^* \in A \cup \{x_1^*, \ldots, x_m^*\}\}.$$

It is clear that U is a relatively weak open subset of $B_{C_0(K,X)}$ containing x. Now our goal will be to show that $U \subset \frac{O_1+O_2}{2}$, which is enough to finish the proof.

Take $y \in U$. Then we may choose a finite family of disjoint open subsets $\{V_t : t \in E\}$ of K such that $t \in V_t$ for every $t \in E$ and satisfying

$$|x^*(y(s) - y(t))| < \delta \ \forall s \in V_t, \ \forall x^* \in A \cup \{x_1^*, \ldots, x_m^*\}$$

for every $t \in E$. Now it is clear that $y(s) \in O_t$ for every $s \in V_t$.

Our next step will be to define for $i = 1, 2$ a function $s_i : K \to X$. If $s \in K \setminus \bigcup_{t\in E} V_t$ we define $\widetilde{s}_i(s) = y(s)$. Given $t \in E$, before define $s_i(s)$ for $s \in V_t$, we consider the multivalued map $F_t : O_t \twoheadrightarrow X^2$ given by

$$F_t(z) = \left\{(z_1, z_2) \in \overline{B_1^t} \times \overline{B_2^t} : \frac{z_1 + z_2}{2} = z\right\} \ \forall z \in O_t.$$

Note that O_t is paracompact in norm, since X is. Also, X^2 is a Fréchet space and $F_t(z)$ is a nonempty, closed and convex subset for every $z \in O_t$. Then, in order to apply the Michael selection principle, it is enough to check that F_t is a norm lower semicontinuous multivalued map. To do this we will apply Lema 1.1, so pick $\{z_n\}$ a sequence in O_t converging in the norm topology to some point $z \in O_t$. Also we pick $(p_1, p_2) \in \overline{B_1^t} \times \overline{B_2^t}$ such that $\frac{p_1+p_2}{2} = z$. As $O_t = \frac{B_1^t+B_2^t}{2}$, we can take $(q_1, q_2) \in B_1^t \times B_2^t$ such that $\frac{q_1+q_2}{2} = z$. From Lemma 3.4 there are a strictly increasing map $\sigma : \mathbb{N} \to \mathbb{N}$ and sequences $\{(p_1^n, p_2^n)\}$, $\{(q_1^n, q_2^n)\}$ in $B_X \times B_X$ so that for every $i = 1, 2$ $\{p_i^n\}$ converges to p_i, $\{q_i^n\}$ converges to q_i and

$$\frac{p_1^n + p_2^n}{2} = \frac{q_1^n + q_2^n}{2} = z_{\sigma(n)} \ \forall n \in \mathbb{N}.$$

For $i = 1, 2$, $\lambda \in [0, 1]$ and $n \in \mathbb{N}$ define $v_i^n(\lambda) = (1-\lambda)q_i^n + \lambda p_i^n$ so that $\{v_i^n\}$ converges $v_i(\lambda) := (1 - \lambda)q_i + \lambda p_i$. As $p_i \in \overline{B_i^t}$, $q_i \in B_i^t$ and B_i^t is convex we get that $v_i(\lambda) \in B_i^t$ for every $\lambda \in [0, 1)$. Take now a sequence $(t_r)_{r\in\mathbb{N}} \subset [0, 1)$ converging to 1. As B_i^t is open, we can choose a strictly increasing sequence $\{N_r\} \in \mathbb{N}$ such that $v_i^{N_r}(t_r) \in B_i^t$. Finally we put $y_r := (v_1^{N_r}(t_r), v_2^{N_r}(t_r))$. It is clear that $y_r \in F_t(z_{\sigma(N_r)})$ for every r, and $\{y_r\}$ converges to (p_1, p_2), which proves that F_t is norm l.s.c.

Appliyng the Michael selection principle to F_t, we can take a continuous selection $f_t = (f_1^t, f_2^t)$ of F_t. From Urysohn lemma there is a continuous function $n_t : K \to [0, 1]$ such that $n_t(t) = 1$ and $n_t(s) = 0$ for every $s \in K \setminus V_t$. Now we define for $i = 1, 2$

$$\widetilde{s_i}(s) = n_t(s)f_i^t(y(s)) + (1 - n_t(s))y(s) \ \forall s \in V_t. \tag{3.2}$$

Recall that we had defined before $\widetilde{s_i}(s) = y(s)$ for every $s \in K \setminus \bigcup_{t\in E} V_t$. It is clear that $\widetilde{s_i} \in B_{C_0(K,X)}$ for $i = 1, 2$ and $y = \frac{\widetilde{s_1}+\widetilde{s_2}}{2}$. Therefore, we will be done by proving that $\widetilde{s_i} \in U_i \subset O_i$ for $i = 1, 2$. In fact, if we take $t \in E$ and $x^* \in A$ then taking into account that $f_i^t(y(t)) \in \overline{B_i^t}$ we have by (3.2) that

$$|(\delta_t \otimes x^*)(\widetilde{s_i} - s_i)| = |x^*(f_i^t(y(t)) - s_i(t))| \leq \varepsilon/4 < \varepsilon/3.$$

Since $g_{ij} \in \text{co}(\delta_E \otimes A)$ we get that $|g_{i,j}(\widetilde{s_i} - s_i)| < \varepsilon/3$ for $i = 1, 2$ and $j = 1, \dots k_i$. Thus, from (3.1), $\widetilde{s_i} \in U_i \subset O_i$. ∎

It is worth to mention that this result generalizes a known result [2, Th. 2.5 b)] for finite dimensional X. In fact, in [2] it is considered the property (co):

Definition 3.6. A Banach space X is said to be (co) if for every $x, s_1, s_2 \in B_X$, $\lambda \in (0, 1/2]$ satisfying that $x = (1-\lambda)s_1 + \lambda s_2$ and $\varepsilon > 0$, there is $\delta > 0$ such that there exist two continuous functions

$$v_i : B_X(x, \delta) \cap B_X \to B_X(s_i, \varepsilon) \cap B_X \quad \forall i \in \{1, 2\} \tag{3.3}$$

satisfying $y = (1 - \lambda)v_1(y) + \lambda v_2(y)$ for every $y \in B(x, \delta) \cap B_X$.

Using our selection technique, we prove that property (co) is equivalent to norm stability of the unit ball for arbitrary Banach spaces.

Proposition 3.7. *Let X be a Banach space. X is (co) if and only if B_X is stable.*

Proof. Property (co) implies the stability of B_X by Proposition 2.2 of [2]. Now, if B_X is stable and we have $x, s_1, s_2 \in B_X$, $\lambda \in (0, 1/2]$ and $\varepsilon > 0$ such that $x = (1-\lambda)s_1 + \lambda s_2$, then we consider the multivalued map given by

$$\phi : B_X(x,\delta) \cap B_X \to \big(B_X(s_1,\varepsilon) \cap B_X\big) \times \big(B_X(s_2,\varepsilon) \cap B_X\big)$$

$$\phi(y) = \big\{(z_1,z_2) \in B_X \times B_X \ : \ (1-\lambda)z_1 + \lambda z_2 = y, \ z_i \in \overline{B_X(s_i,\varepsilon/2)}, \ i = 1,2\big\}$$

where $\delta > 0$ is such that $B_X(x,\delta) \cap B_X \subset (1-\lambda)B_X(s_1,\varepsilon/2) \cap B_X + \lambda B_X(s_2,\varepsilon/2) \cap B_X$ (which exists because B_X is stable). By the same argument as in Theorem 3.5 we are able to prove that ϕ is norm l.s.c. so applying the Michael Selection Principle to ϕ we are done. ∎

Proposition 3.7 turns Theorem 2.5 (b) of [2] into a complete characterization since weak and norm stability agree on finite dimensional spaces. Observe that the weak stability of the unit ball in $C_0(K,X)$ implies the weak stability of the unit ball in both $C_0(K)$ and X [2], since $C_0(K)$ and X are 1-complemented in $C_0(K,X)$. We don't know if the converse is true.

4. Relationship with diameter two properties

As Theorem 2.3 is a geometric characterization of purely atomic preduals of $L_1(\mu)$ spaces, one can get, as a consequence, geometric implications of such spaces in relation with other well known geometric properties, as diameter two properties (see [6] and references there for background). In [25] a very strong diameter 2 property is defined, called the symmetric strong diameter 2 property (SSD2P for short).

Definition 4.1. Let X be a Banach space. We will say that X has the symmetric strong diameter 2 property (SSD2P for short) whenever for every finite family $\{S_i\}_{i=1}^n$ of slices of B_X and $\varepsilon > 0$, there exist $x_i \in S_i$ for $i = 1, \ldots, n$ and $y \in B_X$ with $||y|| > 1 - \varepsilon$, such that $x_i \pm y \in S_i$ for every $i = 1, \ldots, n$.

In view of this, we define the next property, which will be proved to be satisfied by Banach spaces with stable unit balls.

Definition 4.2. Let X be a Banach space. We will say that X has the attaining symmetric strong diameter 2 property (ASSD2P for short) whenever for every finite family $\{S_i\}_{i=1}^n$ of slices of B_X, there exist $x_i \in S_i$ for $i = 1, \ldots, n$ and $y \in S_X$, such that $x_i \pm y \in S_i$ for every $i = 1, \ldots, n$.

Obviously, the ASSD2P implies the SSD2P which in turn implies the strong diameter 2 property (see [25]). In order to obtain a characterization

of the ASSD2P, we need the concept of symmetric slice. A symmetric slice of a Banach space X is nothing but a subset $C \subset B_X$ such that

$$C = \frac{S - S}{2}$$

where S is slice of B_X.

Proposition 4.3. *Let X be an infinite-dimensional Banach space. Then,*

i) *X has the ASSD2P if, and only if, every finite intersection of symmetric slices intersects with the sphere.*
ii) *X has the SSD2P if, and only if, every finite intersection of symmetric slices has diameter 2.*

Proof. i) Let $\{S_i\}_{i=1}^n$ be a finite family of Slices of the unit ball and consider the set

$$C = \bigcap_{i=1}^{n} \frac{S_i - S_i}{2}.$$

If we assume that intersection of symmetric slices reaches the sphere then there is $y \in S_X \cap C$. Then, for every $i \in \{1, \ldots, n\}$ there are two points $s_i, z_i \in S_i$ such that $y = \frac{s_i - z_i}{2}$. Considering $x_i = \frac{s_i + z_i}{2}$ we are done since $x_i - y = z_i \in S_i$ and $x_i + y = s_i \in S_i$. To prove the other implication, consider for every $i \in \{1, \ldots, n\}$ a symmetric slice $\frac{S_i - S_i}{2}$ where S_i is an arbitrary slice in X. By definition of ASSD2P, we may take for every $i \in \{1, \ldots, n\}$ a point $x_i \in S_i$ and a point $y \in S_X$ such that $x_i \pm y \in S_i$ for every $i \in \{1, \ldots, n\}$. We are now done since

$$y = \frac{(x_i + y) - (x_i - y)}{2} \in \frac{S_i - S_i}{2} \quad \forall i \in \{1, \ldots, n\}.$$

The assertion ii) of the Proposition is proved in the same way. ∎

Corollary 4.4. *Infinite dimensional Banach spaces with a weak stable unit ball satisfy ASSD2P.*

Proof. By Proposition 4.3, it is enough to show that finite intersections of symmetric slices intersects with the sphere. If the space has weak stable unit ball these finite intersections are nonempty relatively weakly open subsets of the unit ball so clearly intersect with the sphere. ∎

The question now is wether the reverse result is true or not. For that purpose, we are going to give an example of a space with the ASSD2P whose unit ball is not weak stable. The space $C[0,1]$ does not have a weak stable unit ball as it is a non purely atomic L_1-predual. Let us prove that $C[0,1]$ is the example that we are looking for.

Proposition 4.5. *The space $C[0,1]$ enjoy the ASSD2P.*

Proof. Let $\varepsilon \in (0,1)$ and $\mu_1, \ldots, \mu_n \in S_{\mathcal{M}[0,1]}$, so that we consider the slices S_i given by

$$S_i = \{f \in B_{C[0,1]} \ : \ |\mu_i(f)| > 1 - 2\varepsilon\} \ \ \forall i \in \{1, \ldots, n\}$$

and we take for every $i \in \{1, \ldots, n\}$ an element $f_i \in S_i$ such that $|\mu_i(f_i)| > 1 - \varepsilon$. First, we take $r \in (0,1)$ and $\delta > 0$ such that

$$\mu_i(r - \delta, r + \delta) < \frac{\varepsilon}{2} \ \ \forall i \in \{1, \ldots, n\}.$$

Let $g : [0,1] \to [0,1]$ be a continuous function such that $g(t) = 1$ for every $t \in [0,1] \setminus (r - \delta, r + \delta)$ and $g(t) = 0$ for every $t \in [r - \delta/2, r + \delta/2]$. If we define $g_i = f_i g$ it is straightforward that $g_i \in B_{C[0,1]}$ is such that

$$\begin{aligned} |\mu_i(g_i - f_i)| &= \left| \int_{[0,1]} g_i - f_i \, d\mu_i \right| \\ &\leq \int_{(r-\delta, r+\delta)} |g_i - f_i| \, d\mu_i + \int_{[0,1]\setminus(r-\delta, r+\delta)} |g_i - f_i| \, d\mu_i < \varepsilon, \end{aligned}$$

so $g_i \in S_i$. Now, it is enough to consider a continuous function $h : [0,1] \to [0,1]$ such that $h(t) = 0$ for $t \in [0,1] \setminus (r - \delta/2, r + \delta/2)$ and $g(r) = 1$. It is easy to check that $g_i + h, g_i - h \in S_i$ as g_i and h have disjoint support, where $h \in S_{C[0,1]}$. ∎

Thanks to Proposition 4.5 we have that

$$w\text{-stable} \ \not\Leftarrow \ \text{ASSD2P}.$$

Now, let us prove that the ASSD2P is strictly stronger than the SSD2P.

From [27] page 168 we know there exists a Banach space X which is both Almost Square (ASQ) and strictly convex. This means that X has the Symmetric Strong Diameter 2 Property (SSD2P) as it is an Almost Square Banach space [25]. Since the ASSD2P implies the Attaining Strong Diameter 2 Property (see Theorem 3.4 of [19]), which is incompatible with strictly convex spaces (Proposition 2.3 of [19]), it follows that X does not have the ASSD2P. That is,

$$\text{ASSD2P} \ \not\Leftarrow \ \text{SSD2P}.$$

As purely atomic L_1 preduals satisfy ASSD2P, it is natural wondering what is the class of L_1 preduals with ASSD2P.

We have shown that stability arises a lot of geometric consequences related with diameter 2 properties. Finally, let us now stablish the next Proposition 4.6 focusing on some geometric consequences relating extreme points.

Proposition 4.6. *Let X be a Banach space with a weak stable unit ball. Then:*

i) *Every face of B_X has a weakly closed set of extreme points and so $Ext(B_X)$ is weakly closed.*

ii) *$Ext(B_X)$ is nowhere weakly dense in B_X, that is, for every $U \subset B_X$ nonempty relatively weakly open subset, there exists another $V \subset U$ nonempty relatively weakly open subset such that $V \cap Ext(B_X) = \emptyset$.*

Proof. From [22, Proposition 1.1], we know that the multivalued map $T : C \to C$ which gives for every $x \in C$ the face generated by x, that is

$$T(x) = \{y \in C \ : \ \exists z \in C, \ \exists \lambda \in (0,1) \ \text{ with } \ x = \lambda y + (1-\lambda)z\}$$

is l.s.c. when C is a stable convex set. In particular, it says that $Ext(C)$ is closed when C is stable.

The first assertion is due to that and the fact that every face of a stable convex set is again stable. In fact, if C is a stable convex set and F is one of its faces, then, taking $U_1, U_2 \subset C$ relatively open sets and making use of the definition of face, we have that

$$\frac{U_1 \cap F + U_2 \cap F}{2} = \left(\frac{U_1 + U_2}{2}\right) \cap F,$$

so F is stable.

For the second assertion, let us consider U a nonempty relatively weakly open subset of B_X. Then by Bourgain's Lemma [11, Lemma II.1] we know there is a nontrivial convex combination of disjoint slices V inside U. As slices are relatively weakly open subsets of B_X and B_X is weak stable, we know that V is a nonempty relatively weakly open. It is straightforward to see that V cannot have any extreme point of B_X, as it is a convex combination of more than one nonempty disjoint subsets of B_X. ∎

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Chapter 11

Conclusions

11.1 Summary of the advances made

In this thesis we have tackled several problems of completely different nature and achieved some progress on them, including solutions to some questions risen in previously published papers.

We worked on Goal (G_1) and were able to modify Schachermayer's result. This gave us a completely new approach to the RNP VS KMP problem, where the construction of a 'tricky' topology for certain c.c.b. sets is shown to be sufficient for the equivalence between the latter properties to hold.

We also dealt with (G_2),(G_4) and (G_5). In fact, not only have we found large families of uniformly discrete metric spaces whose Lipschitz free space has a Schauder basis but also provided several connections between linear approximation properties and nonlinear ones, including a nonlinear characterization of the FDD property for dual spaces in terms of one specific Lipschitz retraction.

Even though we have not found a complete solution to the Godefroy-Ozawa problem, an extensive study has been made in order to get closer to goal (G_3). On the one hand, the existence of an FDD in a Banach space is shown to be sufficient in order to admit a GCCR. Furthermore, an α-Hölder version of the Godefroy-Ozawa problem is solved in the positive in full generality for every $\alpha < 1$. On the other hand, the BAP is proven to be a necessary condition for the space to admit a GCCR with an additional 'mild' assumption on its asymptotic shape. Additionally, we have constructed a separable Banach space with no λ-GCCR's for some $\lambda > 1$ which hints towards a negative solution to the problem.

Regarding goal (G_6) we have completely characterized length metric spaces M as those satisfying that $\mathrm{Lip}_0(M, X)$ is Daugavet for every Banach space X. Also, all metric spaces but uniformly discrete bounded ones are proven to satisfy that $\mathcal{F}(M)\widehat{\otimes}_\pi X$ is octahedral for every Banach space X. These results give completely positive solutions to questions risen in several published papers.

Finally, in respect to goal (G_7), we have covered a gap that was present in the bibliography, giving a complete characterization of which L_1-preduals have a weakly stable unit ball. This accidentally lead us to a geometric characterization of purely atomic L_1-preduals, a task that had been quite elusive since Lindenstrauss joined the project of characterizing important families of L_1-preduals among all L_1-preduals (see for instance [LL71, Section 4]).

11.2 Open questions

Besides all the open lines commented in goals (G_1) to (G_7), we would like to finish this section asking more specific questions that arise in this setting.

We were able to show in Theorem 1.3.5 that a Banach space X with an FDD admits a Small GCCR whereas in Theorem 1.3.6 we proved that having a Small GCCR is enough for the space to have an FDD (at least for dual spaces). We then defined Flat GCCR's and prove that the existence of these more general GCCR's assures that the space enjoys the BAP. Therefore, the following question naturally arises.

Question 11.2.1. *Does the converse of Theorem 1.3.11 hold? That is, if X has the BAP, does it admit a Flat GCCR?*

At the moment this thesis is written it is even unknown whether BAP spaces admit a GCCR.

As it has been mentioned in the Introdution, an important open problem introduced by Kalton may be restated as whether the Lipschitz free space over any net of a separable space enjoys the BAP. Our approach to that problem has always required the construction of a retractional basis in the underlying net so, following Kalton's spirit, we wonder the following.

Question 11.2.2. *Is there a separable infinite dimensional Banach space whose net does not have a retractional basis? Does the net of a Banach space with a basis necessarily admit a retractional basis?*

Along the same line we also wonder:

Question 11.2.3. *Is there a separable Banach space without Schauder basis such that the Lipschitz free space over its net does have a basis? Moreover, can this basis come from a retractional basis of the net?*

It is worth mentioning that by Theorem 1.3.23, it would be enough to find a sequence of finite dimensional Banach spaces whose c_0 sum does not have a basis.

Motivated by the characterization of metric spaces whose Lipschitz free space is octahedral given in [PRZ18] we wonder if the same characterization works in the vector-valued case, that is:

Question 11.2.4. *Give a characterization of the metric spaces M such that $\mathcal{F}(M)\widehat{\otimes}_\pi X$ is octahedral for every Banach space X. Does the characterization known for the case $X = \mathbb{R}$ work for every general Banach space X?*

Now, even though goals (G_1) and (G_7) might seem disconnected at first glance, there are strong links between them since weakly stable unit balls are extremely non RNP. Therefore, we find the next question interesting.

Question 11.2.5. *Let C be a c.c.b. subset of a separable Banach space. Does weak stability of C imply that C does not enjoy the KMP? Moreover, does it imply that there is a locally convex topology τ finer than the weak topology such that C is τ-DδP for some $\delta > 0$ (possibly $\delta = 2$) and C is τ-SRO?*

This question might be related to Question (c) in [AGH$^+$20].

Question 11.2.6 (Question (c) in [AGH$^+$20])**.** *Is there any dual Banach space with a weak stable unit ball?*

Indeed, if in the search of a solution to Question 11.2.5 one finds out that a weak stable c.c.b. set does not have enough extreme points then a negative solution to Question 11.2.6 will be automatically delivered.

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Appendix

Other academic achievements

In addition, 4 more articles have been produced by the candidate, 3 of them in Mathematics:

1. S. Dantas, R. Medina, A. Quilis and Ó. Roldán, "On isometric embeddings into the set of strongly norm-attaining Lipschitz functions". In: Nonlinear Anal. 232 (2023), Paper No. 113287 doi:10.1016/j.na.2023.113287.

2. R. Medina and A. Quilis, "Non-uniformly continuous nearest point maps". ArXiv: 2402.04747.

3. S. Dantas and R. Medina, "On holomorphic functions attaining their weighted norms". ArXiv: 2206.11206.

Also, the author co-directed a multidisciplinary project.
Title: Estudio de la estimación conjunta de Direction-of-Arrival (DoA) y Angle-of-Arrival (AoA) para geometrías genéricas en el rango de ondas milimétricas.
Reference: PPJIB2022-05.
Entity: Plan Propio de la Universidad de Granada.
Duration: 01/01/2023 - 31/12/2023.
As a byproduct of the aforementioned project, an article related to signal processing was published:

4. A. Ramírez-Arroyo, A. Alex-Amor, R. Medina, P. Padilla and J. F. Valenzuela-Valdés, "Joint Ultra-wideband Characterization of Azimuth, Elevation and Time of Arrival with Toric Arrays". In: IEEE Transactions on Wireless Communications. Doi: 10.1109/TWC.2024.3377539.