This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2022.3217400

A lattice structure on hesitant fuzzy sets

Pascual Jara, Luis Merino, Gabriel Navarro, and Evangelina Santos

***Abstract*—In this paper we deal with the lattice-compatibility between several classes of extended fuzzy sets. Concretely, we treat the problem of finding a lattice structure on set-valued fuzzy sets (SVFSs) whose restriction to interval-valued fuzzy sets (IVFSs) and (type-1) fuzzy sets (FSs) match Zadeh's classical lattice operations. A prominent approach to this problem was given by Torra by means of the so-called hesitant fuzzy sets (HFSs). Nevertheless, despite their usefulness in group decision making problems, it is well-known that Torra's operations do not produce a lattice. Here, we mend partially this handicap by giving two lattice orders. Each of them preserves one of the Torra's operations and, additionally, reduces to Zadeh's orders on FSs and on IVFSs. As a counterpart, they cannot be defined on the whole class of HFSs, or SVFSs. Finally, we provide a full answer combining both orders. We define a partial order, that we call the symmetric order, on the whole class of non-empty subsets of $[0,1]$. This order extends the usual ones on $[0,1]$ and on closed intervals of $[0,1]$. As a consequence, we find a lattice structure on HFSs whose restriction to FSs and IVFSs reduces to Zadeh's operations.**

***Index Terms*—Fuzzy sets, interval-valued fuzzy sets, set-valued fuzzy sets, hesitant fuzzy sets, order, lattice.**

I. Introduction

The estimation of the membership degrees of the fuzzy sets comprising a system is a widely-known obstacle when applying Fuzzy Set Theory in practice. Indeed, this was pointed out first by Zadeh himself in [1], motivating the use of the so-called type-2 fuzzy sets (T2FSs). In order to ease this issue, and make it more feasible, a significant number of extensions of the original notion of fuzzy set (FS) [2] have been defined. For instance, concerning this paper, in 1971, Sambuc [3] proposed to assign closed intervals, yielding the concept of interval-valued fuzzy sets (IVFSs), although under the name of Φ-Flou sets. The purpose there was to establish lower and upper bounds of the membership degrees aiming to handle the inaccuracies, or uncertainty, when computing them. Later, in 1976, Grattan-Guinness [4] introduces the notion of set-valued fuzzy sets (SVFSs), as mappings whose codomain is the class of non-empty subsets in $[0,1]$, expressing the membership degrees not necessarily as a closed neighborhood, but also as a collection, finite or infinite, of possible candidates.

Whatever the nature of these, an important point to discern is how membership degrees of the same kind can be compared. That is to say, mathematically speaking, how the set of possible degrees can be endowed with a partial order. Although this is quite clear when dealing with the set $[0,1]$, and consequently with Zadeh's fuzzy sets, the question might be thorny in certain cases. Mainly, because of the class under consideration cannot be undoubtedly ordered as a chain, as it can be done with $[0,1]$. By way of illustration one simply has to think about how to order closed intervals or subsets in $[0,1]$, or mappings in $[0,1]^{[0,1]}$. The lack of such a total ordering may lead to certain computational problems when working with non-related objects. For example, how to establish the minimal requisites, with respect to the membership degrees, of a system composed by extended fuzzy objects. A mathematical formalism that attends to solve these questions is the notion of lattice, something more restrictive than a poset, since it allows to establish relations between non-linked objects through common subobjects or superobjects. In this sense, several papers have dealt with this task. For instance, in [5], the standard lattice operations for IVFSs are defined, see also [6] and [3]. With respect to SVFSs, in [4], lattice operations derived from the set-theoretical union and intersection are given.

Nevertheless, certain compatibility between the lattice structures is desirable since, mostly, the extended fuzzy sets are naturally contained one to each other. This fit ensures a proper use of a class of fuzzy sets inside a superclass, allowing the treatment of membership degrees of different nature simultaneously. In this context, note that the lattice operations on SVFSs derived from set intersection and union do not extend Zadeh's operations on FSs. In [7], see also [8] and [9], Torra intents to mend this gap by defining the class of the so-called hesitant fuzzy sets (HFSs). Formally, these are SVFSs, where the empty set is allowed as membership degree, with different operations that, indeed, extend the usual lattice structure for FSs. In despite their relatively recent appearance, HFSs have attracted the attention of a great number of researchers with hundred of papers developing both theoretical and practical applications, mainly in information fusion and decision-making. Just to mention a few examples, we may point out the decomposition of typical hesitant fuzzy sets by means of families of HFSs given in [10], obtaining a representation theorem and extension principles for HFSs; the three-way decision method by means of HFSs and canonical soft sets explained in [11]; or the aggregation of hesitant fuzzy information for decision-making developed in [9].

Unfortunately, Torra's operations on HFSs do not give a lattice structure and, up to our knowledge, only a few studies treat this problem. Mostly, they deal with typical hesitant fuzzy sets (THFSs). For instance, in [12] a poset structure is provided on the class $\mathbb{H}$ formed by all finite subsets of $[0,1]$. It simply compares one by one the ordered elements of both subsets, which have previously been converted into multisets

The authors are listed in alphabetical order. All the authors contributed equally to this work. P. Jara, L. Merino and E. Santos are with IMAG and the Department of Algebra, University of Granada. G. Navarro is with CITIC and the Department of Computer Science and Artificial Intelligence, University of Granada.

This work is supported by grant A-FQM-394-UGR20 from Programa Operativo FEDER 2014-2020 and Consejería de Economía, Conocimiento, Empresas y Universidad de la Junta de Andalucía (Spain) and by the "María de Maeztu" Excellence Unit IMAG, reference CEX2020-001105-M, funded by MCIN/AEI/10.13039/501100011033/.

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2022.3217400

by means of an adequate repetition of their elements. This obviously preserves the usual ordering on $[0,1]$, although, as a counterpart, it changes the original data information and only gives a lattice structure on certain subclasses of $\mathbb{H}$. Something more sophisticated can be found in [13] by using the so-called α and β normalization processes. In this case, the resulting order is a lattice order. In [14], see also [15] and [16], lexicographical-based orders on $\mathbb{H}$ are defined. These are total orders, so they produce a complete lattice. In addition, they reduce to the standard operations on $[0,1]$ when restricting to singletons. With respect to orders on intervals, the reader may consult [17] for a method to build total orders in terms of two continuous aggregation functions. These orders do not match the usual order on intervals.

Despite all these proposals, the question of extending min-max operations on FSs to the whole class of HFSs remained open. In this paper we solve this problem. Concretely, we define an order that produces a lattice structure on HFSs (or SVFSs) extending the Zadeh's operations for FSs and IVFSs. This also answer [18, Challenge 5.3] and helps to treat some of the other challenges described in [18] and [19] for HFSs.

The structure of the paper is simple. After recalling the needed concepts in Section II, in Section III we define two orders aiming to obtain two lattice structures on HFSs whose join or meet, obviously not both simultaneously, coincides with the union or intersection, respectively, provided by Torra in [7]. As a counterpart, the orders cannot be defined in the whole class of HFSs but rather in the subclasses formed by those whose image has maximum or minimum, respectively. In Section IV we prove the main result of the paper. We define an order, that we call the symmetric order, on $\mathcal{P}^*([0,1])$, the class of non-empty subsets in $[0,1]$. This provides a lattice structure on SVFSs (or HFSs) compatible with the classical Zadeh's operations on FSs and the standard operations on IVFSs. In particular, this fully answers the open problem stated in [20, Remark 2]: '*An interesting open problem is to provide a lattice structure on* SVFS*s in such a way that Zadeh's union and intersection are preserved when restricted to* FS*s.*".

II. Preliminaires

The mathematical background needed to follow our results is quite basic. We recall that a partially ordered set (*poset*) is a set P endowed with a relation $\leq$ satisfying reflexivity, antisymmetry and transitivity. The poset P is said to be *bounded* if there exist elements $0,1 \in P$ such that $0 \leq p$ and $p \leq 1$ for all $p \in P$.

Given a poset $(P, \leq)$ and $a,b \in P$, a least upper bound or *supremum* of a and b is an element $c \in P$ verifying that $a,b \leq c$ and, if $c' \in P$ with $a,b \leq c'$, then $c \leq c'$. Dually, a greatest lower bound or *infimum* of a and b is an element $d \in P$ verifying that $d \leq a,b$ and, if $d' \in P$ with $d' \leq a,b$, then $d' \leq d$. A poset L is said to be a *lattice* if, for any $a,b \in L$, there exists a supremum and an infimum. Equivalently, we may say that a lattice is a set L with two operations $\wedge$ and $\vee$, called infimum and supremum, respectively, verifying associativity, commutativity, idempotency and absorption law. The relationship between both approaches is given by the relations $a \leq b$ if and only if $a \vee b = b$ if and only if $a \wedge b = a$. Note then that every lattice is a poset with the former order. Nevertheless, not every poset defines a lattice.

A lattice $(L, \wedge, \vee)$ is said to be *bounded* if the underlying poset is bounded, or, equivalently, there exist two elements in L, usually denoted by 1 and 0 such that, for every $a \in L$, $a \wedge 0 = 0$, $a \wedge 1 = a$, $a \vee 0 = a$ and $a \vee 1 = 1$.

Let L be a bounded lattice, given a universe set X, an *L-valued fuzzy subset* (or L-fuzzy subset) of X, see [21], is a mapping $A : X \longrightarrow L$, meaning that, for any element $x \in X$, $A(x)$ is the membership degree of x to X. For instance, whenever $L = [0,1]$, we obtain the usual FSs of X defined by Zadeh [2]. Or, if $L = \mathbb{I}$, the set of closed intervals in $[0,1]$, it yields the well-known class of IVFSs of X [1], [3], [4], [6].

Given two sets A and B, we shall denote by $\mathrm{Map}(A,B)$ the class of all set mappings from A to B. Therefore the class of all L-fuzzy subsets of X is $\mathrm{Map}(X,L)$.

Observe that, when L is a bounded lattice with operations $\wedge_L$ and $\vee_L$, and underlying order $\leq_L$, the class of all L-fuzzy subsets of X becomes also a bounded lattice with point-wise operations $\sqcap$ and $\sqcup$ defined, for each $f,g \in \mathrm{Map}(X,L)$, $(f \sqcap g)(x) = f(x) \wedge_L g(x)$ and $(f \sqcup g)(x) = f(x) \vee_L g(x)$ for all $x \in X$. The point-wise order here is given as follows: for any $f,g \in \mathrm{Map}(X,L)$, $f \sqsubseteq g$ if and only if $f(x) \leq_L g(x)$ for all $x \in X$.

For instance, if $L = [0,1]$ is endowed with the usual order, FSs are endowed with the standard min-max lattice operations provided by Zadeh in [2]. Analogously, suppose that $\mathbb{I}$ is endowed with the classical order on $\mathbb{I}$, i. e, given $A = [a,c], B = [b,d] \in \mathbb{I}$, $A \leq B$ if and only if $a \leq b$ and $c \leq d$. This order provides a lattice structure on $\mathbb{I}$ with operations $\vee$ and $\wedge$ given by $A \vee B = [\max(a,b), \max(c,d)]$ and $A \wedge B = [\min(a,b), \min(c,d)]$. Then, the point-wise lattice operations on IVFSs coincide with the usual ones, see for instance [1] and [22].

We also recall that a hesitant fuzzy subset (HFS) [7] of X is a mapping $A : X \longrightarrow \mathcal{P}([0,1])$, where $\mathcal{P}([0,1])$ denotes the powerset of $[0,1]$. Torra considers in [7] the operations

$$(f \sqcup_H g)(x) = \{t \in f(x) \cup g(x) \mid t \geq \mathrm{Inf}\, f(x) \vee \mathrm{Inf}\, g(x)\}$$

and

$$(f \sqcap_H g)(x) = \{t \in f(x) \cup g(x) \mid t \leq \mathrm{Sup}\, f(x) \wedge \mathrm{Sup}\, g(x)\},$$

for any HFSs f and g, and any $x \in X$. These are derived point-wise from the operations on $\mathcal{P}([0,1])$ given by $A \cup_H B = \{t \in A \cup B \mid t \geq \mathrm{Inf}(A) \vee \mathrm{Inf}(B)\}$ and $A \cap_H B = \{t \in A \cup B \mid t \leq \mathrm{Sup}(A) \wedge \mathrm{Sup}(B)\}$ for any subsets $A,B \subseteq [0,1]$. Unfortunately, it is well-known, see [20] or [23], that they do not provide a structure of lattice neither on HFSs nor on $\mathcal{P}([0,1])$, respectively. In the next section we show a partial solution to this problem.

III. The right and left orders

Given a set $A \in \mathcal{P}([0,1])$, for simplicity, we shall denote by A^- and A^+ the infimum and supremum of A, respectively. Obviously, in case A^+ belongs to A, then A^+ is the maximum

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2022.3217400

of A and we say that A is *right-closed*. We denote by $\mathcal{P}^+([0,1])$ the class of all right-closed subsets in $[0,1]$. Note that $\mathcal{P}^+([0,1])$ contains any finite subset in $[0,1]$, any closed interval in $[0,1]$ and any finite union of them. The empty set is not right-closed since it does not contain its supremum, the element 0. Analogously, we denote the infimum of a subset A by A^- and we say that A is *left-closed* if A^- belongs to A, i. e., it is a minimum. We denote by $\mathcal{P}^-([0,1])$ the class of all left-closed subsets in $[0,1]$.

For simplicity, given two subsets $X, Y \in \mathcal{P}([0,1])$ we say that $X < Y$ if, for all $x \in X$ and all $y \in Y$, $x < y$. Observe that, whenever X or Y are the empty set, $X < Y$ holds trivially.

Let us now define the relation $\leq^r$ on $\mathcal{P}^+([0,1])$.

Definition 1. Given two subsets $A, B \in \mathcal{P}^+([0,1])$, we say that $A \leq^r B$ if and only if the following two conditions hold:

$a)$ $A^+ \leq B^+$, and we refer to it as *the supremum property*.

$b)$ $A < B \setminus A$, that is to say, any element in A is lower than all the elements in B except, possibly, the elements in the intersection of A and B. We refer to it as *the difference property*.

Let us prove that this relation endows $\mathcal{P}^+([0,1])$ with a structure of bounded poset.

Lemma 2. *The relation $\leq^r$ is an order on $\mathcal{P}^+([0,1])$, whose minimum and maximum are $\{0\}$ and $\{1\}$, respectively.*

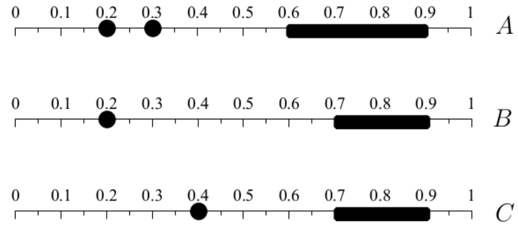
Proof. Let $A, B, C \in \mathcal{P}^+([0,1])$. Since $A \setminus A = \emptyset$, reflexivity holds trivially. For the antisymmetric property, suppose $A \leq^r B$ and $B \leq^r A$. Hence, by the supremum property, $A^+ = B^+$. Let us suppose that $b \in B \setminus A$. By the difference property, $A^+ < b$ and, on the other hand, $b \leq B^+ = A^+$, which yields a contradiction. Then $B \setminus A = \emptyset$ and thus $B \subseteq A$. Analogously, one may obtain that $A \subseteq B$. Thus $A = B$.

Let us prove transitivity. Suppose $A \leq^r B$ y $B \leq^r C$. Then $A^+ \leq B^+ \leq C^+$. Now, given $c \in C \setminus A$, either $c \in B$ or $c \notin B$. In the first case, $c \in B \setminus A$ and, by the difference property for $A \leq^r B$, it holds that $a < c$ for any $a \in A$. In the second case, $c \in C \setminus B$ and, by the difference property for $B \leq^r C$, one may obtain that, for any $a \in A$, $a \leq A^+ \leq B^+ < c$.

Denote $\mathbf{0} = \{0\}$. Since $\mathbf{0}^+ = 0$, for any $A \in \mathcal{P}^+([0,1])$, $\mathbf{0}^+ \leq A^+$. Given $a \in A \setminus \mathbf{0}$, then $a > 0$, so the difference property is satisfied. Then $\mathbf{0} \leq^r A$. Analogously, it can be proved that $A \leq^r \mathbf{1}$, where $\mathbf{1}$ denotes the set $\{1\}$. $\square$

Remark 3. *The interval $[0,1]$ can be identified with the class of subsets in $\mathcal{P}^+([0,1])$ containing a single element, i. e. the singletons. It is easy to see that, by means of this identification, the right order on $[0,1]$ reduces to the usual one.*

Example 4. *Let us illustrate the description of this order with an example. Observe that the following sets are membership degrees of interval-valued hesitant fuzzy sets (unions of closed intervals) as defined in [24] and, therefore, belong to $\mathcal{P}^+([0,1])$. Consider $A = \{0.2\} \cup \{0.3\} \cup [0.6, 0.9]$, $B = \{0.2\} \cup [0.7, 0.9]$ and $C = \{0.4\} \cup [0.7, 0.9]$, then $A \leq^r B$ and $A \not\leq^r C$.*



Remark 5. *1) Observe that, if $A \cap B = \emptyset$, $A \leq^r B$ if and only if $A < B$.*

2) If $B \not\subseteq A$, the supremum property follows from the difference property. Hence, in such a case, $A \leq^r B$ if and only if $A < B \setminus A$.

Given $A, B \in \mathcal{P}^+([0,1])$ such that $A^+ = B^+$, $A \leq^r B$ if and only if $B \subseteq A$. Although, at first sight, this fact may seem paradoxical, it is reasonable, since both sets are just as close to $\mathbf{1}$, albeit B is farther from $\mathbf{0}$. Actually, it holds the following property, that relates the right order to the set inclusion.

Lemma 6. *Let $A, B \in \mathcal{P}^+([0,1])$. Consider the following conditions:*

(a) $A \leq^r B$

(b) $B \subseteq A$

(c) $A^+ = B^+$

Any two of the above conditions implies the third.

Proof. Suppose (a) and (b). Since $B \subseteq A$, it follows $B^+ \leq A^+$. Then $A^+ = B^+$. If (b) and (c) hold, by (c) the supremum property is satisfied. Now, $B \setminus A = \emptyset$, so the difference property is trivially satisfied. Finally, suppose (a) and (c). If $B \not\subseteq A$, hence there exists $b \in B \smallsetminus A$. Then $b \leq B^+$, because of $b \in B$, and $b > A^+ = B^+$, since $A^+ \in A$; a contradiction. $\square$

Next, we prove that the order $\leq^r$ restricted to the class $\mathbb{I}$ of all closed intervals in $[0,1]$ equals the usual order on $\mathbb{I}$.

Lemma 7. *Given two closed intervals in $[0,1]$, $A = [A^-, A^+]$ and $B = [B^-, B^+]$, $A \leq^r B$ if and only if $A^- \leq B^-$ and $A^+ \leq B^+$.*

Proof. Suppose that $A^- \leq B^-$ and $A^+ \leq B^+$. Then, the supremum property is satisfied. Now, given $b \in B \setminus A$, either $b > A^+$ or $b < A^-$. In the former case, the difference property holds, whilst, in the latter, $b < A^- \leq B^-$, a contradiction. Thus $A \leq^r B$.

Conversely, suppose that $A \leq^r B$. Then, by the supremum property, $A^+ \leq B^+$. If $A^- > B^-$, then $B^- \in B \setminus A$ and, by the difference property, $B^- > A^- \in A$, a contradiction. Thus $A^- \leq B^-$. $\square$

We prove now that the right order is a lattice order by describing the associated join and meet operators.

Proposition 8. *Given $A, B \in \mathcal{P}^+([0,1])$, the set*

$$A \wedge_r B = \{x \in A \cup B \mid x \leq A^+ \wedge B^+\}$$

is the infimum of A and B for the order $\leq^r$. Additionally, $(A \wedge_r B)^+ = A^+ \wedge B^+$ and $(A \wedge_r B)^- = A^- \wedge B^-$, and, if $A^+ \leq B^+$, then $A \subseteq A \wedge_r B$.

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2022.3217400

Proof. Without loss of generality, assume that $A^+ \leq B^+$. Let us denote by X the set $\{x \in A \cup B \mid x \leq A^+\}$. Then, it is clear that $A \subseteq X$. Observe that $X^+ = A^+$. Indeed, since A^+ is an upper bound of X, then $X^+ \leq A^+$. Now, since $A \subseteq X$, it follows that $A^+ \leq X^+$. By Lemma 6, it follows $A \leq^r X$.

Let us prove that $X \leq^r B$. We know that $X^+ = A^+ \leq B^+$, so the supremum property holds. Let $b \in B \setminus X$, by definition of X, $b > A^+$ so, for each $x \in X$, $x \leq X^+ = A^+ < b$, and the difference property follows.

Suppose that Z is another lower bound of A and B, that is to say, $Z \leq^r A$ and $Z \leq^r B$. Hence $Z^+ \leq A^+ = X^+$. Now, let $x \in X \setminus Z$. If $x \in A \setminus Z$ then $z < x$ for any $z \in Z$, by the difference property for the relation $Z \leq^r A$. Otherwise, $x \in B \setminus Z$, so $z < x$ for each $z \in Z$ by the difference property for the relation $Z \leq^r B$.

Finally, since $X \subseteq A \cup B$, $X^- \geq (A \cup B)^- = A^- \wedge B^-$. Clearly $A^- \wedge B^- \in X$, so we obtain the other inequality, and the result follows. □

Remark 9. *The relation $\leq^r$ is the only order relation on $\mathcal{P}^+([0,1])$ for which the infimum of two subsets A and B on $\mathcal{P}^+([0,1])$ coincides with $A \cap_H B$, the operation that yields the "infimum" defined by Torra in [7] for hesitant fuzzy sets.*

Proposition 10. *Given $A, B \in \mathcal{P}^+([0,1])$, the set*

$$A \vee_r B = (A \cap B) \cup \{x \in A \cup B \mid x > A^+ \wedge B^+\}$$

is the supremum of A and B for the order $\leq^r$. Additionally, $(A \vee_r B)^+ = A^+ \vee B^+$ and, if $A^+ \leq B^+$, then $A \vee_r B \subseteq B$.

Proof. The proof is similar to the one of Proposition 8. Simply observe that, if $A^+ \leq B^+$,

$$\begin{aligned} A \vee_r B &= (A \cap B) \cup \{x \in A \cup B \mid x > A^+\} \\ &= (A \cap B) \cup \{x \in B \mid x > A^+\} \\ &= \{x \in B \mid x \in A \text{ or } x > A^+\}. \end{aligned}$$

□

Remark 11. *From Lemma 7 it follows that the operations $\wedge_r$ and $\vee_r$, when restricted to the class of closed intervals, correspond to the usual infimum and supremum, respectively. That is to say, given two closed intervals $A = [A^-, A^+]$ and $B = [B^-, B^+]$, $A \wedge_r B = [A^- \wedge B^-, A^+ \wedge B^+]$ and $A \vee_r B = [A^- \vee B^-, A^+ \vee B^+]$.*

We summarize the former results in the following theorem.

Theorem 12. *The class $\mathcal{P}^+([0,1])$ is endowed with a structure of bounded lattice for the order $\leq^r$ with infimum and supremum given by the operations $\wedge_r$ and $\vee_r$, respectively.*

Example 13. *Let us consider the subsets described in Example 4. Then,*

$$A \wedge_r C = \{0.2, 0.3, 0.4\} \cup [0.6, 0.9]$$

and

$$A \vee_r C = [0.7, 0.9].$$

Observe that $A \vee_r C \subset A \wedge_r C$, which agrees with Lemma 6.

Consider now the class $\mathcal{P}^-([0,1])$ formed by all left-closed subsets in $[0,1]$. We may define the relation $\leq^l$.

Definition 14. Let $A, B \in \mathcal{P}^-([0,1])$, we say that $A \leq^l B$ if and only if the following two conditions hold:

a) $A^- \leq B^-$, and we refer to it as *the infimum property*.

b) $A \setminus B < B$, that is to say, any element in B is greater than all the elements in A except, possibly, the elements in the intersection of A and B. We refer to it as the *the difference property*.

For any $A \in \mathcal{P}^-([0,1])$, we may consider the set $1 - A = \{1 - a \mid a \in A\}$, i. e. the image of A under the standard negation on $[0,1]$. Clearly, if $A \in \mathcal{P}^-([0,1])$, then $1 - A \in \mathcal{P}^+([0,1])$.

Proposition 15. *Given $A, B \in \mathcal{P}^-([0,1])$, $A \leq^l B$ if and only if $1 - B \leq^r 1 - A$.*

Proof. For any $X \subseteq [0,1]$, we have $(1-X)^+ = 1 - X^-$, so $A^- \leq B^-$ if and only if $(1-B)^+ \leq (1-A)^+$. Now, suppose that $1 - A \leq^r 1 - B$. Let $a \in A \setminus B$ and $b \in B$, then $1 - a \in (1-A) \setminus (1-B)$ and $1 - b \in 1 - B$. Since $1 - B \leq^r 1 - A$, then $1 - b < 1 - a$, and thus $a < b$. The converse result can be proved similarly. □

It follows from Proposition 15 that the relation $\leq^l$ is an order on $\mathcal{P}^-([0,1])$, that we shall call the left order. The left order yields a bounded lattice structure on $\mathcal{P}^-([0,1])$. In the following theorem we describe the operations associated to this lattice.

Theorem 16. *The class $\mathcal{P}^-([0,1])$ is endowed with a structure of bounded lattice for the order $\leq^l$ with infimum and supremum given by the operations*

$$A \wedge_l B = (A \cap B) \cup \{x \in A \cup B \mid x < A^- \vee B^-\}, \text{ and}$$

$$A \vee_l B = \{x \in A \cup B \mid x \geq A^- \vee B^-\} = A \cup_H B$$

for any $A, B \in \mathcal{P}^-([0,1])$.

Proof. It follows from Lemma 2 and Proposition 15 that $\leq^l$ is an order with minimum $\mathbf{0}$ and maximum $\mathbf{1}$. Now, by Proposition 15, $A \vee_l B = 1 - ((1-A) \wedge_r (1-B))$. Hence,

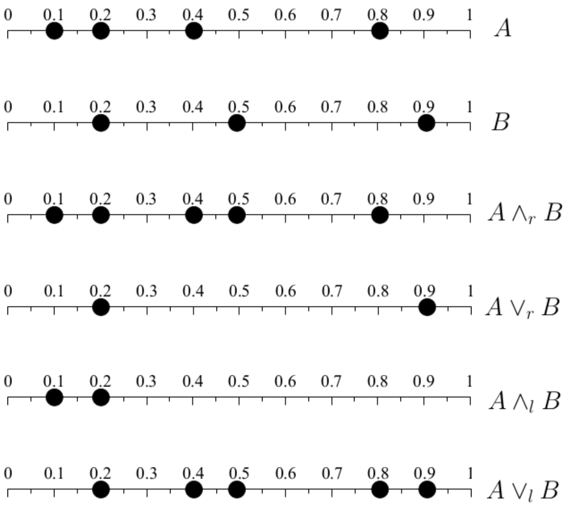
$$\begin{aligned} & (1-A) \wedge_r (1-B) \\ = \; & \{y \in (1-A) \cup (1-B) \mid y \leq (1-A)^+ \wedge (1-B)^+\} \\ = \; & \{y \in 1 - (A \cup B) \mid y \leq 1 - (A^- \vee B^-)\} \\ = \; & 1 - \{x \in A \cup B \mid 1 - x \leq 1 - (A^- \vee B^-)\} \\ = \; & 1 - \{x \in A \cup B \mid x \geq A^- \vee B^-\}. \end{aligned}$$

The remaining facts can be proved similarly. □

Example 17. *Let us show an example of calculation for finite sets, the membership degrees of typical hesitant fuzzy sets. Consider $A = \{0.1, 0.2, 0.4, 0.8\}$ and $B = \{0.2, 0.5, 0.9\}$, then $A \wedge_r B = \{0.1, 0.2, 0.4, 0.5, 0.8\} = A \cap_H B$ and $A \vee_r$*

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2022.3217400

$B = \{0.2, 0.9\}$, *and* $A \wedge_l B = \{0.1, 0.2\}$ *and* $A \vee_l B = \{0.2, 0.4, 0.5, 0.8, 0.9\} = A \cup_H B$.



Observe, in particular, that neither $A \leq^r B$ *nor* $B \leq^r A$. *Therefore the right order is not a chain when restricted to finite subsets. Thus it does not match any of the admissible total orders described in [14]–[16]. The same may be said about the left order.*

Both orders are related in the following sense. Let us denote by $\mathcal{P}^{\pm}([0,1])$ the intersection $\mathcal{P}^{+}([0,1]) \cap \mathcal{P}^{-}([0,1])$, that is, the class of all right-closed and left-closed subsets in $[0,1]$.

Corollary 18. *Given* $A, B \in \mathcal{P}^{\pm}([0,1])$, *the following statements hold:*

(a) $A \wedge_l B \subseteq A \wedge_r B$
(b) $A \vee_r B \subseteq A \vee_l B$.

Proof. (a). Let us first assume that $A^- \vee B^- \leq A^+ \wedge B^+$. Given $x \in A \wedge_l B$, then either $x \in A \cap B$, or $x \in A \cup B$ and $x < A^- \vee B^-$. In the first case, $x \leq A^+$ and $x \leq B^+$, and thus $x \leq A^+ \wedge B^+$ and $x \in A \wedge_r B$. In the second cade $x < A^- \vee B^- \leq A^+ \wedge B^+$, and then $x \in A \wedge_r B$.

Now, assume that $A^- \vee B^- > A^+ \wedge B^+$. Without loss of generality, suppose that $A^- \leq B^-$. Then $B^- > A^+ \wedge B^+$ and, since $B^- \leq B^+$, $B^- > A^+$. This implies that $A \cap B = \emptyset$ and, for any $a \in A$ and $b \in B$, $a \leq A^+ < B^- \leq b$. Thus $A \leq^r B$ and $A \leq^l B$. Consequently, $A \wedge_r B = A = A \wedge_l B$.

(b) follows from (a) and Proposition 15. □

Corollary 19. *Let* $A, B \in \mathcal{P}^{\pm}([0,1])$, *the following assertions are equivalent:*

(a) $A \wedge_l B = A \wedge_r B$.
(b) *For all* $x \in A \cup B$ *such that* $A^- \vee B^- \leq x \leq A^+ \wedge B^+$, $x \in A \cap B$.
(c) $A \vee_r B = A \vee_l B$.

Proof. Observe that $A \wedge_l B \neq A \wedge_r B$ if and only if there exists an $x \in A \wedge_r B$ such that $x \notin A \wedge_l B$, or, equivalently, if there exists an $x \in A \cup B$ such that $x \leq A^+ \wedge B^+$, $x \notin A \cap B$ and $x \geq A^- \vee B^-$. That is, if (b) does not hold.

Similarly, $A \vee_r B \neq A \vee_l B$ if and only if there exists an $x \in A \vee_l B$ such that $x \notin A \vee_r B$, or, equivalently, if there exists an $x \in A \cup B$ such that $x \geq A^- \vee B^-$, $x \notin A \cap B$ and $x \leq A^+ \wedge B^+$. That is, if (b) does not hold. □

Remark 20. *Given two closed intervals* $A = [A^-, A^+]$ *and* $B = [B^-, B^+]$, *they satisfy the conditions of Corollary 19. Consequently,* $A \wedge_l B = A \wedge_r B = [A^- \wedge B^-, A^+ \wedge B^+]$ *and* $A \vee_l B = A \vee_r B = [A^- \vee B^-, A^+ \vee B^+]$

The left order matches the order defined in [23] on $\mathcal{P}^c([0,1])$, the class of closed subsets of $[0,1]$. We recall from [23] that, given $A, B \in \mathcal{P}^c([0,1])$, we say $A \leq_S B$ if and only if $A^- \leq B^-$, $A^+ \leq B^+$ and $A \cap [B^-, A^+] \subseteq B$.

Proposition 21. *Let* $A, B \in \mathcal{P}^c([0,1])$, $A \leq_S B$ *if and only if* $A \leq^l B$. *Consequently, the lattice structures are the same for these orders.*

Proof. Suppose that $A \leq_S B$, we only need to see that $A \backslash B < B$. But, given $a \in A \setminus B$, since $A \cap [B^-, A^+] \subseteq B$, hence $a < B^-$ and thus $a < b$ for all $b \in B$.

Conversely, let $a \in A \cap [B^-, A^+]$. If $a \notin B$, by hypothesis, $a < b$ for any $b \in B$ and, in particular, $a < B^-$, which yields a contradiction. Thus $a \in B$. That is to say, $A \cap [B^-, A^+] \subseteq B$. We need also to prove that $A^+ \leq B^+$. Indeed, if $A^+ \notin B$, by the difference property, $A^+ < b$ for all $b \in B$, and then $A^+ < B^+$. Otherwise, if $A^+ \in B$, A^+ must be less or equal than the supremum of B. □

Remark 22. *The results of this section allow us to define lattice structures, by point-wise extensions, on* L*-fuzzy sets, where* L *is the lattice* $\mathcal{P}^{-}([0,1])$, $\mathcal{P}^{+}([0,1])$ *or* $\mathcal{P}^{\pm}([0,1])$. *These extend Zadeh's operations on* FS*s and on* IVFS*s. These also provide lattice operations on* IVHFS*s [24].*

IV. The symmetric order

In this section we prove the main result of the paper. We provide a non-trivial order in the class of non-empty subsets of $[0,1]$. This yields a lattice structure compatible with the standard lattice structures on $[0,1]$ and $\mathbb{I}$. As a consequence, the class of SVFSs is endowed with a lattice structure whose restriction to FSs reduces to the Zadeh's lattice defined in [2], answering an open question proposed in [20].

A. Definition and first properties

Let us denote by $\mathcal{P}^{*}([0,1])$ the class of all non-empty subsets of $[0,1]$. We define the relation $\leq^0$ on $\mathcal{P}^{*}([0,1])$ as follows.

Definition 23. Given $A, B \in \mathcal{P}^{*}([0,1])$, we say $A \leq^0 B$ if and only if the following two conditions hold.

a) $A < B \backslash A$, and we refer to it as the difference property 1, **DP1** for short, and
b) $A \backslash B < B$ and we refer to it as the difference property 2, **DP2** for short.

As commented above, if $B \backslash A = \emptyset$, then **DP1** holds trivially, and, similarly, if $A \setminus B = \emptyset$, **DP2** is satisfied trivially.

Example 24. *Let us consider the subsets described in Example 4. Observe that* $B \setminus A = \emptyset$, *so* ***DP1*** *trivially holds.*

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2022.3217400

Nevertheless, $A \setminus B = \{0.3\}$ *and* $0.2 \in B$*, so* ***DP2*** *fails. Thus* $A \nleq^0 B$.

Fix now $D = [0.7, 0.9] \cup \{1\}$*. Hence,* $D \setminus C = \{1\}$*. Since* $1 \notin C$*,* $C < D \setminus C$ *and* ***DP1*** *holds. On the other hand,* $C \setminus D = \{0.4\}$ *and* $0.4 < d$ *for any* $d \in D$*, so* ***DP2*** *holds. Thus* $C \leq^0 D$.

Proposition 25. *The relation* $\leq^0$ *is a partial order on* $\mathcal{P}^*([0,1])$ *with minimum* $\mathbf{0} = \{0\}$ *and maximum* $\mathbf{1} = \{1\}$.

Proof. Reflexivity holds trivially since $A \setminus A = \emptyset$ for any $A \in \mathcal{P}^*([0,1])$. In order to prove the antisymmetric property, let $A, B \in \mathcal{P}^*([0,1])$ such that $A \leq^0 B$ and $B \leq^0 A$. Let $b \in B \setminus A$, by **DP1** of the relation $A \leq^0 B$, $a < b$ for any $a \in A$, and, by **DP2** of the relation $B \leq^0 A$, $b < a$ for any $a \in A$. This is a contradiction. Thus $B \setminus A = \emptyset$. Similarly it may be proved that $A \setminus B = \emptyset$. Then $A = B$.

Let us prove transitivity. Let $A, B, C \in \mathcal{P}^*([0,1])$ such that $A \leq^0 B$ and $B \leq^0 C$. Given $c \in C \setminus A$, either $c \in B$ or $c \notin B$. If $c \in B$, then $c \in B \setminus A$ so $a < c$ for any $a \in A$, by **DP1** for $A \leq^0 B$. If $c \notin B$, then $c \in C \setminus B$. Given $a \in A$, if $a \in B$ hence, by **DP1** for $B \leq^0 C$, $a < c$. If $a \notin B$, $a \in A \setminus B$ so, by **DP2** for $A \leq^0 B$, $a < b$ for any $b \in B$. But, for any $b \in B$, by **DP1** for $B \leq^0 C$, $b < c$. This yields that $a < c$. Hence $A < C \setminus A$.

Let now $a \in A \setminus C$. If $a \in B$, then $a \in B \setminus C$, so, by **DP2** for $B \leq^0 C$, $a < c$ for any $c \in C$. Otherwise, $a \in A \setminus B$, so, by **DP2** for $A \leq^0 B$, $a < b$ for any $b \in B$. Therefore, given $c \in C$, if $c \in B$, $a < c$. If not, $c \in C \setminus B$ and, by **DP1** for $B \leq^0 C$, $b < c$ for any $b \in B$. Hence $a < c$, and thus $A \setminus C < C$.

Finally, it is clear that $\mathbf{0}$ and $\mathbf{1}$ are the minimum and maximum, respectively. ☐

We shall name the order $\leq^0$ as the *symmetric order*.

Proposition 26. *For any subsets* $A, B \in \mathcal{P}^*([0,1])$*,* $A \leq^0 B$ *if and only if* $1 - B \leq^0 1 - A$.

Proof. The proof is similar to the one of Proposition 15. ☐

Lemma 27. *If* $A \leq^0 B$*, then* $A^- \leq B^-$ *and* $A^+ \leq B^+$.

Proof. Suppose that $A^- > B^-$. Hence, there exists some $b \in B$ such that $b < a$ for all $a \in A$. In particular, $b \notin A$. Hence, $A \setminus B \nless B$, and **DP1** is not satisfied. On the other hand, suppose that $A^+ > B^+$. Hence, there exists some $a \in A$ such that $a > b$ for all $b \in B$. In particular, $a \notin B$. Hence $A \setminus B \nless B$, and **DP2** is not satisfied. ☐

As a direct consequence, we may state the following result.

Corollary 28. *If* $A, B \in \mathcal{P}^{\pm}([0,1])$*, then* $A \leq^0 B$ *if and only if* $A \leq^l B$ *and* $A \leq^r B$.

Remark 29. *Given* $a, b \in [0,1]$ *with* $a < b$*, by Corollary 28, note that* $[a,b[\leq^0 [a,b] \leq^0]a,b]$ *and* $[a,b[\leq^0]a,b[\leq^0]a,b]$*. But neither* $[a,b] \leq^0]a,b[$*, nor* $]a,b[\leq^0 [a,b]$*. Therefore the symmetric order is not a total order. Furthermore, from Example 17 and Corollary 28, we may deduce that the symmetric order restricted to finite sets is not a chain either, unlike the total orders in [14]–[16].*

We shall need the following result.

Proposition 30. *Let* $A \in \mathcal{P}^*([0,1])$*, then* $\{A^-\} \leq^0 A$.

Proof. If $A^- \in A$, the statement trivially holds. Otherwise, $\{A^-\} \cap A = \emptyset$, so **DP1** and **DP2** follows from the fact that $\{A^-\} < A$. ☐

B. The infimum

Let us now prove that the symmetric order can be endowed a with meet operation for any subsets $A, B \in \mathcal{P}^*([0,1])$. We need the following notation. Given $A, B \in \mathcal{P}^*([0,1])$, we shall denote by $T_{A,B}$ the subset

$$T_{A,B} = \{t \in A \cup B \setminus A \cap B \text{ such that } t > A^- \vee B^-\},$$

and by $t_{A,B}$ the infimum of the subset $T_{A,B}$. As we will prove, the description of the infimum of A and B depends on the relation between the values A^-, B^- and $t_{A,B}$, and the subsets A and B.

Remark 31. *The set* $T_{A,B}$ *could be the empty set even though* $A \neq B$ *and, in this case,* $t_{A,B} = 1$*. For instance, let* $A = [0.2, 0.5[\cup \{0.6, 0.7, 0.8\}$ *and* $B = \{0.5, 0.6, 0.7, 0.8\}$*. Hence,* $A^- \vee B^- = 0.5$*. Observe that* $A \cup B \setminus A \cap B = [0.2, 0.5]$*, so* $T_{A,B} = \emptyset$*. Even though* $T_{A,B} \neq \emptyset$*,* $t_{A,B}$ *could be 1. For instance, the reader may consider*

$$A = [0.2, 0.5[\cup \left\{ \frac{n}{n+1} \mid n \geq 2 \right\} \cup \{1\}$$

and $B = \left\{ \frac{n}{n+1} \mid n \geq 1 \right\}$*. Hence* $T_{A,B} = \{1\}$.

We shall need the following technical lemma.

Lemma 32. *Let* $A, B \in \mathcal{P}^*([0,1])$*, for each* $Z \in \mathcal{P}^*([0,1])$ *such that* $Z \leq^0 A$ *and* $Z \leq^0 B$*, it holds* $z \leq t_{A,B}$ *for any* $z \in Z$.

Proof. If $T_{A,B} = \emptyset$, $t_{A,B} = 1$ and the statement holds trivially, so assume that $T_{A,B} \neq \emptyset$. Let $t \in T_{A,B}$ and $z \in Z$. Suppose that $t \in Z$. Since $t \notin A \cap B$, either $t \in Z \setminus A < A$ by **DP2** for $Z \leq^0 A$, and then $t \leq A^-$, or $t \in Z \setminus B < B$ by **DP2** for $Z \leq^0 B$, and then $t \leq B^-$. In both cases $t \leq A^- \vee B^-$, a contradiction. Therefore $t \notin Z$. Now, $t \in A \cup B$ so that either $t \in A \setminus Z > Z$ by **DP1** for $Z \leq^0 A$, or $t \in B \setminus Z > Z$ by **DP2** for $Z \leq^0 B$. In both cases $z < t$. Thus $z \leq t_{A,B}$. ☐

Remark 33. *Note the former proof shows that, for any lower bound* Z *of* A *and* B*,* $T_{A,B} \cap Z = \emptyset$ *and* $Z < T_{A,B}$.

Given $A, B \in \mathcal{P}^*([0,1])$, we consider the subsets $X_{A,B}$ and $\overline{X}_{A,B}$ defined as

$$X_{A,B} = \{x \in A \cup B \mid x < t_{A,B}\}, \text{ and}$$

$$\overline{X}_{A,B} = \{x \in A \cup B \mid x \leq t_{A,B}\}.$$

As we will see, under certain conditions, these sets play the role of the infimum. They are related in the following way.

Lemma 34. $X_{A,B} \leq^0 \overline{X}_{A,B}$

Proof. Clearly, either $\overline{X}_{A,B} = X_{A,B}$ or $\overline{X}_{A,B} = X_{A,B} \cup \{t_{A,B}\}$. In the first case, the result holds by reflexivity. In the

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2022.3217400

second case, $X_{A,B} \setminus \overline{X}_{A,B} = \emptyset$, so trivially $X_{A,B} \setminus \overline{X}_{A,B} < \overline{X}_{A,B}$. On the other hand, since $x < t_{A,B}$ for all $x \in X_{A,B}$ and $\overline{X}_{A,B} \setminus X_{A,B} = \{t_{A,B}\}$, then $X_{A,B} < \overline{X}_{A,B} \setminus X_{A,B}$. This proves the statement. □

The subset $X_{A,B}$ could be the empty set. For instance, set $A = \{\frac{1}{2n} \mid n \geq 1\}$ and $B = \{\frac{1}{2n+1} \mid n \geq 1\}$. Then $A^- = B^- = 0$ and $T_{A,B} = A \cup B$, so $t_{A,B} = 0$. Since $0 \notin A \cup B$ then $X_{A,B} = \emptyset$. Actually, in this case, also $\overline{X}_{A,B} = \emptyset$.

In the following statement we characterize whether or not $X_{A,B}$ is the empty set.

Lemma 35. *Let $A, B \in \mathcal{P}^*([0,1])$, the following conditions are equivalent:*
(a) $X_{A,B} = \emptyset$.
(b) $t_{A,B} = A^- = B^-$.

Proof. The implication $b) \Rightarrow (a)$ is clear. Let us prove $(a) \Rightarrow (b)$. Since $X_{A,B} = \emptyset$, if $x \in A \cup B$, then $x \geq t_{A,B}$. Hence, for any $a \in A$, $a \geq t_{A,B}$. This implies $A^- \geq t_{A,B}$. The same reasoning for the elements in B yields $B^- \geq t_{A,B}$. Then $A^- \wedge B^- \geq t_{A,B}$. Therefore $A^- \wedge B^- \geq t_{A,B} \geq A^- \vee B^-$. Thus $t_{A,B} = A^- = B^-$. □

For simplicity, in what follows, we shall assume that the infimum of A is lower or equal than the infimum of B.

Lemma 36. *Let $A, B \in \mathcal{P}^*([0,1])$, if $A^- \leq B^-$ then the following statements hold:*
(a) $X_{A,B} \subseteq A \cup \{B^-\}$.
(b) $\overline{X}_{A,B} \subseteq A \cup \{B^-, t_{A,B}\}$.
(c) $X_{A,B} < A \setminus X_{A,B}$.
(d) $\overline{X}_{A,B} < A \setminus \overline{X}_{A,B}$.
(e) $\overline{X}_{A,B} < B \setminus \overline{X}_{A,B}$.

Proof. Let us prove (a). If $X_{A,B} \setminus A = \emptyset$, we are done. Otherwise, let $x \in X_{A,B}$ such that $x \notin A$. Then $x \in B$, so $x \geq B^-$. Now, since $x \in X_{A,B}$, then $x < t_{A,B}$, and then $x \notin T_{A,B}$. This forces that $x \leq A^- \vee B^- = B^-$. Thus $x = B^-$. (b) follows immediately from (a).

In order to prove (c), let $x \in X_{A,B}$. Then $x < t_{A,B}$. Now, for any $a \in A \setminus X_{A,B}$, $t_{A,B} \leq a$. Thus $x < a$. (d) and (e) can be proved similarly. □

The following two results are the key-tools in order to specify the infimum of two subsets on the symmetric order.

Proposition 37. *Let $A, B \in \mathcal{P}^*([0,1])$ such that $A^- \leq B^-$ and $X_{A,B} \neq \emptyset$. Then:*
(a) *$X_{A,B} \leq^0 A$ if and only if at least one the following assertions hold:*
  $i)$ $A^- = B^-$.
  $ii)$ $B^- = t_{A,B}$.
  $iii)$ $B^- \notin B \setminus A$.
(b) *$\overline{X}_{A,B} \leq^0 A$ if and only if $t_{A,B} \notin B \setminus A$ and $X_{A,B} \leq^0 A$.*
(c) *$X_{A,B} \leq^0 B$.*
(d) *$\overline{X}_{A,B} \leq^0 B$ if and only if $t_{A,B} = B^- \notin B$ or $t_{A,B} \notin A \setminus B$.*
(e) *Let $Z \in \mathcal{P}^*([0,1])$, if $Z \leq^0 A$ and $Z \leq^0 B$, $Z \leq^0 \overline{X}_{A,B}$.*

Proof. (a) Suppose $X_{A,B} \leq^0 A$. If $X_{A,B} \setminus A \neq \emptyset$, by Lemma 36$(a)$, $X_{A,B} \setminus A = \{B^-\}$. Since $X_{A,B} \setminus A < A$, then $B^- \leq A^-$, and then $A^- = B^-$. If $X_{A,B} \setminus A = \emptyset$, we have the following options:

- If $B^- \in A$, hence $B^- \notin B \setminus A$.
- If $B^- \notin A$, hence $B^- \notin X_{A,B}$. This happens if, either $B^- \notin B$, or $B^- \geq t_{A,B}$. In the first case, $B^- \notin B \setminus A$. In the second case, $B^- = t_{A,B}$.

Conversely, by Lemma 36(c), $X_{A,B} < A \setminus X_{A,B}$ so **DP1** holds. Let us see that $X_{A,B} \setminus A < A$ (**DP2**). We recall that, by Lemma 36(a), $X_{A,B} \setminus A \subseteq \{B^-\}$. Hence:

- If $A^- = B^-$, then $B^- \leq a$ for any $a \in A$. Now, if $B^- \in A$, then $X_{A,B} \setminus A = \emptyset$ so the property is satisfied trivially. If $B^- \notin A$, hence $B^- < a$ for all $a \in A$.
- If $B^- = t_{A,B}$, then $B^- \notin X_{A,B}$, so $X_{A,B} \setminus A = \emptyset$ and we are done.
- If $B^- \notin B \setminus A$, then either $B^- \in A$ or $B^- \notin A \cup B$. If $B^- \in A$, $X_{A,B} \setminus A = \emptyset$. Otherwise, $B^- \notin A \cup B$, and then $B^- \notin X_{A,B}$, so, again, $X_{A,B} \setminus A = \emptyset$.

(b) Suppose that $\overline{X}_{A,B} \leq^0 A$. Since, by Lemma 34, $X_{A,B} \leq^0 \overline{X}_{A,B}$, by transitivity, $X_{A,B} \leq^0 A$. Now, if $t_{A,B} \notin \overline{X}_{A,B} \setminus A$. Hence, $t_{A,B} \notin \overline{X}_{A,B}$, and then $t_{A,B} \notin A \cup B$, or $t_{A,B} \in A$. In both cases, $t_{A,B} \notin B \setminus A$. Otherwise, if $t_{A,B} \in \overline{X}_{A,B} \setminus A$, by **DP2**, $t_{A,B} < a$ for all $a \in A$. Hence, $t_{A,B} \leq A^- \leq B^- \leq t_{A,B}$. So $t_{A,B} = A^- = B^-$. By Lemma 35, $X_{A,B} = \emptyset$, a contradiction.
Conversely, by Lemma 36(d), $\overline{X}_{A,B} < A \setminus \overline{X}_{A,B}$. Now, observe that $\overline{X}_{A,B} \subseteq X_{A,B} \cup \{t_{A,B}\}$. Then, if $t_{A,B} \in A$, $\overline{X}_{A,B} \setminus A = X_{A,B} \setminus A < A$, by hypothesis. If $t_{A,B} \notin A$ then $t_{A,B} \notin A \cup B$, therefore $t_{A,B} \notin \overline{X}_{A,B}$. This means that $\overline{X}_{A,B} = X_{A,B}$, so that $\overline{X}_{A,B} \setminus A = X_{A,B} \setminus A < A$, again by **DP2**.

(c) Let us see first that $X_{A,B} < B \setminus X_{A,B}$. Let $x \in X_{A,B}$, then $x < t_{A,B}$. Now, for any $b \in B \setminus X_{A,B}$, $b \geq t_{A,B}$, so $x < t_{A,B} \leq b$, yielding **DP1**. Let us prove **DP2**. Let $x \in X_{A,B} \setminus B$ then $x \in A$, $x \notin B$ and $x < t_{A,B}$. Therefore $x \notin T_{A,B}$. In particular, since $x \in A \cup B \setminus A \cap B$, $x \leq B^-$. If $x = B^- \notin B$, then $x < b$ for any $b \in B$. Analogously, if $x < B^-$, then $x < b$ for any $b \in B$.

(d) Suppose that $\overline{X}_{A,B} \leq^0 B$. If $t_{A,B} \notin \overline{X}_{A,B}$ then $t_{A,B} \notin A \cup B$, and then $t_{A,B} \notin A \setminus B$. Otherwise, $t_{A,B} \in \overline{X}_{A,B} \leq^0 B$. If $t_{A,B} \in B$, then, again, $t_{A,B} \notin A \setminus B$. If not, $t_{A,B} \in \overline{X}_{A,B} \setminus B < B$, by **DP2**, so $t_{A,B} < b$ for all $b \in B$ and then $t_{A,B} \leq B^-$. Therefore, since $t_{A,B} \geq A^- \vee B^- = B^-$, we obtain $t_{A,B} = B^-$.
Let us prove the converse. By Lemma 36(e), $\overline{X}_{A,B} < B \setminus \overline{X}_{A,B}$, that is, **DP1** holds. Now, by (c), $X_{A,B} \leq^0 B$ so $X_{A,B} \setminus B < B$. If $t_{A,B} \notin \overline{X}_{A,B} \setminus B$, we conclude that **DP2** holds. Let us suppose then that $t_{A,B} \in \overline{X}_{A,B} \setminus B$. On the one hand, if $t_{A,B} = B^- \notin B$, then $t_{A,B} < b$ for any $b \in B$ and **DP2** is satisfied. On the other hand, if $t_{A,B} \notin A \setminus B$, then either $t_{A,B} \notin A \cup B$ or $t_{A,B} \in B$. The first case implies $t_{A,B} \notin \overline{X}_{A,B}$, a contradiction; whilst the second implies that $t_{A,B} \notin \overline{X}_{A,B} \setminus B$, also a contradiction.

(e) If there exists some $x \in \overline{X}_{A,B} \setminus Z$, then $x \in (A \cup B) \setminus Z$. Then, either $x \in A \setminus Z$, or $x \in B \setminus Z$. Therefore, since, by hypothesis, $Z < A \setminus Z$ and $Z < B \setminus Z$, $z < x$ for all $z \in Z$, and we get **DP1**.

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2022.3217400

Let $z \in Z$, by Lemma 32, $z \leq t_{A,B}$. If $z \in Z \setminus \overline{X}_{A,B}$ then $z \notin A \cup B$, so $z \in Z \setminus (A \cup B) = (Z \setminus A) \cap (Z \setminus B)$. Since we have the inequalities $Z \setminus A < A$ and $Z \setminus B < B$, it holds that, simultaneously, $z < a$ for all $a \in A$ and $z < b$ for all $b \in B$. Now, since $\overline{X}_{A,B} \subseteq A \cup B$ we get that $z < x$ all $x \in \overline{X}_{A,B}$, and **DP2** holds.

□

Theorem 38. *Let $A, B \in \mathcal{P}^*([0,1])$ such that $A^- \leq B^-$, $X_{A,B} \neq \emptyset$ and $t_{A,B} \in \overline{X}_{A,B}$. Suppose that $A^- = B^-$ or $B^- \notin B \setminus A$, then the following conditions are equivalent:*

(a) *For any $Z \in \mathcal{P}^*([0,1])$ such that $Z \leq^0 A$ and $Z \leq^0 B$, $Z \leq^0 X_{A,B}$.*

(b) *$\overline{X}_{A,B}$ is not a lower bound of A and B.*

(c) *One of the following conditions is satisfied:*

  $i)$ $t_{A,B} \in B \setminus A$

  $ii)$ $t_{A,B} \in A \setminus B$ *and* $B^- < t_{A,B}$

Proof. We first prove $(a) \Rightarrow (b)$. By Lemma 34, $X_{A,B} \leq^0 \overline{X}_{A,B}$. Then $\overline{X}_{A,B}$ cannot be a lower bound of A and B. Indeed, in such a case, for hypothesis, $\overline{X}_{A,B} \leq^0 X_{A,B}$. Then $\overline{X}_{A,B} = X_{A,B}$. This is a contradiction with the assumption $t_{A,B} \in \overline{X}_{A,B}$.

Let us show $(b) \Rightarrow (c)$. By hypothesis, by Proposition 37(a), $X \leq^0 A$. Suppose, contrary to (c), that $t_{A,B} \notin B \setminus A$ and $t_{A,B} \notin A \setminus B$ or $B^- = t_{A,B}$. Since $t_{A,B} \notin B \setminus A$, by Proposition 37$(b)$, we have that $\overline{X}_{A,B} \leq^0 A$.

Now, if $t_{A,B} \notin A \setminus B$, by Proposition 37$(d)$, $\overline{X}_{A,B} \leq^0 B$, which contradicts (b). Suppose then that $t_{A,B} = B^-$. If $B^- \notin B$, again, by Proposition 37(d), $\overline{X}_{A,B} \leq^0 B$, a contradiction. Finally, if $B^- \in B$ then $B^- = t_{A,B} \notin A \setminus B$ which has been considered above.

Finally, we prove $(c) \Rightarrow (a)$. Let $Z \in \mathcal{P}^*([0,1])$ such that $Z \leq^0 A$ and $Z \leq^0 B$. We know that $Z < X_{A,B} \setminus Z$ (**DP1**) since $X_{A,B} \subseteq A \cup B$. In order to prove $Z \setminus X_{A,B} < X_{A,B}$ (**DP2**), consider $z \in Z \setminus X_{A,B}$. Whenever $z \notin A \cup B$, then $z \in Z \setminus A < A$ and $z \in Z \setminus B < B$, therefore $z < x$ for all $x \in X_{A,B} \subseteq A \cup B$. If $z \in A \cup B$, since $z \notin X_{A,B}$, $z \geq t_{A,B}$. But, by Lemma 32, $z \leq t_{A,B}$. Thus $z = t_{A,B}$. Now,

$i)$ if $t_{A,B} \in B \setminus A$, then $t_{A,B} \in Z \setminus A < A$, then $t_{A,B} \leq A^-$. But, by definition, $t_{A,B} \geq A^- \vee B^-$, then $t_{A,B} = A^-$. Hence $A^- = B^- = t_{A,B}$ and, by Lemma 35, $X_{A,B} = \emptyset$, a contradiction.

$ii)$ if $t_{A,B} \in A \setminus B$ and $B^- < t_{A,B}$, then $t_{A,B} \in Z \setminus B < B$ and $B^- < t_{A,B}$, also a contradiction.

Therefore, the case $t_{A,B} \in (Z \cap (A \cup B)) \setminus X_{A,B}$ cannot occur, and this finishes the proof. □

We are now in conditions to describe the infimum of two non-empty subsets of $[0,1]$. We shall denote by $A \wedge_0 B$ the infimum of $A, B \in \mathcal{P}^*([0,1])$ with respect to the symmetric order.

Theorem 39. *Let $A, B \in \mathcal{P}^*([0,1])$ such that $A^- \leq B^-$.*

A. *If $A^- = B^- = t_{A,B}$, then $A \wedge_0 B = \{B^-\}$.*

B. *If $A^- = B^- < t_{A,B}$, then*

$$A \wedge_0 B = \begin{cases} \overline{X}_{A,B}, & \text{if } t_{A,B} \in A \cap B, \\ X_{A,B}, & \text{if } t_{A,B} \notin A \cap B. \end{cases}$$

C. *If $A^- < B^-$ and $B^- \in B \setminus A$, then*

$$A \wedge_0 B = \{a \in A \mid x < B^-\}$$

D. *If $A^- < B^-$, $B^- \notin B \setminus A$ and $B^- < t_{A,B}$, then*

$$A \wedge_0 B = \begin{cases} \overline{X}_{A,B}, & \text{if } t_{A,B} \in A \cap B, \\ X_{A,B}, & \text{if } t_{A,B} \notin A \cap B. \end{cases}$$

E. *If $A^- < B^-$, $B^- \notin B \setminus A$ and $B^- \notin A \setminus B$, then*

$$A \wedge_0 B = \begin{cases} \overline{X}_{A,B}, & \text{if } t_{A,B} \in A \cap B, \\ X_{A,B}, & \text{if } t_{A,B} \notin A \cap B. \end{cases}$$

F. *If $A^- < B^-$, $B^- \notin B \setminus A$ and $B^- = t_{A,B} \in A \setminus B$, then $A \wedge_0 B = \overline{X}_{A,B}$.*

Proof. **A.** It is immediate that $\{A^-\} \leq^0 A$ and $\{B^-\} \leq^0 B$, so it is a lower bound. Now, suppose there exists $Z \in \mathcal{P}^*([0,1])$ such that $Z \leq^0 A$ and $Z \leq^0 B$. By Lemma 32, for any $z \in Z$, $z \leq t_{A,B} = A^- = B^-$. Then $Z \setminus \{A^-\} < \{A^-\}$ and $Z < \{A^-\} \setminus Z$. Thus $Z \leq^0 \{A^-\}$.

B. Let us first suppose that $t_{A,B} \notin A \cup B$. Then $X_{A,B} = \overline{X}_{A,B}$. By Proposition 37$(a)$ and (c), $X_{A,B} \leq^0 A$ and $X_{A,B} \leq^0 B$. Finally, Proposition 37(e) asserts that $X_{A,B}$ is the greatest lower bound.

If $t_{A,B} \in A \cup B$, then $t_{A,B} \in \overline{X}_{A,B}$. There are two possibilities:

1) $t_{A,B} \in A \cap B$, and then $t_{A,B} \notin B \setminus A$ and $t_{A,B} \notin A \setminus B$. Since $A^- = B^-$, we are under the conditions of Proposition 37(b), so $\overline{X}_{A,B} \leq^0 A$. Also, by Proposition 37(d), $\overline{X}_{A,B} \leq^0 B$. Proposition 37$(e)$ ensures that $\overline{X}_{A,B}$ is the greatest lower bound.
2) If $t_{A,B} \notin A \cap B$, hence either $t_{A,B} \in B \setminus A$ or $t_{A,B} \in A \setminus B$. Firstly, by Proposition 37(a) and (c), $X_{A,B}$ is a lower bound of A and B. Now, in the former both cases, the condition in Theorem 38(c) is satisfied, that is, equivalently, Theorem 38(a) applies. So $X_{A,B}$ is the infimum.

C. Let us denote $Y = \{y \in A \,|\, y < B^-\} \subseteq A$. Hence $Y \setminus A = \emptyset$, so **DP2** holds. Also, by its definition, $Y < A \setminus Y$ (**DP1**), therefore $Y \leq^0 A$. Observe also that $Y < B$, so $Y \leq^0 B$. Let now $Z \in \mathcal{P}^*([0,1])$ such that $Z \leq^0 A$ and $Z \leq^0 B$. Since $Z < A \setminus Z$ and $Y \setminus Z \subseteq A \setminus Z$, then $Z < Y \setminus Z$ (**DP1**).

In order to prove **DP2**, if $Z \setminus Y = \emptyset$, this holds trivially. So assume $Z \setminus Y \neq \emptyset$. Observe that $B^- \notin Z$. Indeed, by hypothesis, $A^- < B^-$. Then there exists some $a \in A$ such that $a < B^-$. Since $Z \setminus A < A$ by **DP1** for $Z \leq^0 A$, then $B^- \notin Z \setminus A$. Now, $B^- \in B \setminus A$, so, in particular, $B^- \notin A$. Hence, $B^- \notin Z$.

Consequently, $B^- \in B \setminus Z$. By **DP1** for $Z \leq^0 B$, $Z < B \setminus Z$, so $z < B^-$ for any $z \in Z$. Then $Z \cap A \subseteq Y$, which implies that $Z \setminus Y = Z \setminus A < A$ (**DP2**). In particular, $Z \setminus Y < Y$.

D. Firstly, the condition $A^- < B^-$ ensures that $X_{A,B} \neq \emptyset$. Then:

- $t_{A,B} \notin (A \cup B) \setminus (A \cap B)$. In this case $t_{A,B} \notin A \setminus B$, so Proposition 37$(d)$ applies and $\overline{X}_{A,B} \leq^0 B$. Analogously, $t_{A,B} \notin B \setminus A$ and, since $B^- \notin B \setminus A$, by Proposition 37(a) and (b), $\overline{X}_{A,B} \leq^0 A$. Proposition 37$(e)$ finishes the proof. Observe that, if $t_{A,B} \notin A \cup B$, then $\overline{X}_{A,B} = X_{A,B}$.

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2022.3217400

- $t_{A,B} \in (A \cup B) \setminus (A \cap B)$. In this case $t_{A,B} \in \overline{X}_{A,B}$. By Proposition 37$(c)$, $X_{A,B} \leq^0 B$. The hypothesis $B^- \notin B \setminus A$, by Proposition 37(a), yields $X_{A,B} \leq^0 A$. Now, $(A \cup B) \setminus A \cap B = (A \setminus B) \cup (B \setminus A)$. Both if $t_{A,B} \in B \setminus A$ or $t_{A,B} \in A \setminus B$, since $B^- < t_{A,B}$ by hypothesis, condition (c) in Theorem 38 is verified. Then the condition (a) in Theorem 38 is also satisfied, so $X_{A,B}$ is the greatest lower bound.

E. It can be proved almost as **D**. Simply observe that when $t_{A,B} \in (A \cup B) \setminus (A \cap B)$, then $t_{A,B} \in A \setminus B$ or $t_{A,B} \in B \setminus A$. Then, since $B^- \notin B \setminus A$ and $B^- \notin A \setminus B$, $B^- < t_{A,B}$ and **D** applies.

F. Since $t_{A,B} = B^- \notin B \setminus A$, the result follows from Proposition 37(a), (b), (d) and (e). □

We sum up the above discussion in the decision tree contained in Figure 1. Simply observe that, in the leaf node that belongs to cases **E-F**, if $t_{A,B} \in A \setminus B$, **F** applies; whilst, if $t_{A,B} \in A \cap B$, $B^- = t_{A,B} \notin A \setminus B$, and then **E** applies. Note also that we are assuming $A^- \leq B^-$. Otherwise, simply exchange A by B and vice versa.

Example 40. *Let us consider the subsets*

$$A = \left\{ \frac{1}{n} \text{ such that } n \in \mathbb{N},\ n > 0 \right\}$$

and

$$B = \{0.2\} \cup [0.5, 0.7].$$

*Then $A^- = 0$, $B^- = 0.2$ and $T_{A,B} = (0.5, 0.7] \cup \{0.25, \frac{1}{3}, 1\}$. We apply the decision tree in Figure 1. Observe $A^- \neq B^-$ and $B^- \in A \cap B$. Also $B^- \neq t_{A,B} = 0.25$ and $t_{A,B} \notin A \cap B$. Therefore, we are in the case **D** and*

$$A \wedge_0 B = X_{A,B} = \left\{ \frac{1}{n} \text{ such that } n \in \mathbb{N},\ n > 4 \right\}.$$

C. The supremum

The supremum for this order can be computed through the calculation of the infimum by virtue of Proposition 26.

Proposition 41. *Given $A, B \in \mathcal{P}^*([0,1])$, the supremum for the symmetric order $A \vee_0 B = 1 - ((1 - A) \wedge_0 (1 - B))$.*

Proof. Since $(1 - A) \wedge_0 (1 - B) \leq^0 (1 - A)$, by Proposition 26, $A \leq^0 A \vee_0 B$, and similarly, $A \vee_0 B$ is an upper bound of B. Now, if Z is an upper bound of A and B, $1 - Z$ is a lower bound of $1 - A$ and $1 - B$, then $1 - Z \leq^0 (1 - A) \wedge_0 (1 - B)$. By Proposition 26, $A \vee_0 B = 1 - ((1 - A) \wedge_0 (1 - B)) \leq^0 Z$. □

Nevertheless, we may provided an explicit description. We only have to consider the dual version of the procedure for computing the minimum. Given $A, B \in \mathcal{P}^*([0,1])$, let

$$S_{A,B} = \{s \in A \cup B \setminus A \cap B \text{ such that } s < A^+ \wedge B^+\},$$

and $s_{A,B}$ the supremum of $S_{A,B}$. Let also

$$Y_{A,B} = \{y \in A \cup B \mid s_{A,B} < y\}, \text{ and}$$
$$\overline{Y}_{A,B} = \{y \in A \cup B \mid s_{A,B} \leq y\}.$$

Theorem 42. *Let $A, B \in \mathcal{P}^*([0,1])$ such that $A^+ \leq B^+$.*

A*. *If $A^+ = B^+ = s_{A,B}$, then $A \vee_0 B = \{A^+\}$.*

B*. *If $s_{A,B} < A^+ = B^+$, then*

$$A \vee_0 B = \begin{cases} \overline{Y}_{A,B}, & \text{if } t_{A,B} \in A \cap B, \\ Y_{A,B}, & \text{if } t_{A,B} \notin A \cap B. \end{cases}$$

C*. *If $A^+ < B^+$ and $A^+ \in A \setminus B$, then*

$$A \vee_0 B = \{b \in B \mid A^+ < b\}.$$

D*. *If $A^+ < B^+$, $A^+ \notin A \setminus B$ and $s_{A,B} < A^+$, then*

$$A \vee_0 B = \begin{cases} \overline{Y}_{A,B}, & \text{if } s_{A,B} \in A \cap B, \\ Y_{A,B}, & \text{if } s_{A,B} \notin A \cap B. \end{cases}$$

E*. *If $A^+ < B^+$, $A^+ \notin A \setminus B$ and $A^+ \notin B \setminus A$, then*

$$A \vee_0 B = \begin{cases} \overline{Y}_{A,B}, & \text{if } s_{A,B} \in A \cap B, \\ Y_{A,B}, & \text{if } s_{A,B} \notin A \cap B. \end{cases}$$

F*. *If $A^+ < B^+$, $A^+ \notin A \setminus B$ and $A^+ = s_{A,B} \in B \setminus A$, then $A \vee_0 B = \overline{Y}_{A,B}$.*

Proof. The proof is dual to the one of Theorem 39, using the dual results to the ones developed in this section, and it is left to the reader. □

A decision tree for computing the supremum $A \vee_0 B$ is described in Figure 2.

Example 43. *Let us consider the subsets in Example 40. We exchange A by B in order to ensure that $A^+ \leq B^+$. In that case, $A^+ = 0.7$ and $B^+ = 1$. We have then that $A^+ \neq B^+$ and $A^+ \in A \setminus B$. Thus we are in case **C***, so that*

$$A \vee_0 B = \{b \in B \mid 0.7 < b\} = \{1\}.$$

D. Embeddings between classes of fuzzy sets

We finally extend the lattice associated to the symmetric order to the class of SVFSs. We also describe its relations with other well-known classes of extended fuzzy sets. First, we pay attention to the class $\mathbb{I}$ of closed intervals of $[0,1]$, due to its importance in practical applications.

Theorem 44. *The partially ordered set $(\mathcal{P}^*([0,1]), \leq^0)$ restricted to the class $\mathbb{I}$ matches $(\mathbb{I}, \leq)$. As a consequence, $(\mathbb{I}, \wedge, \vee)$ is a sublattice of $(\mathcal{P}^*([0,1]), \wedge_0, \vee_0)$.*

Proof. It follows from Corollary 19 and Corollary 28. □

In order to illustrate the use of the decision tree displayed in Figure 1, we check that the operation $\wedge_0$ on $\mathbb{I}$ equals $\wedge$.

Example 45. *Let $A = [A^-, A^+]$ and $B = [B^-, B^+]$ be two different intervals. Assume that $A^- \leq B^-$.*

*If $A^- = B^-$ and $A^+ < B^+$, then $T_{A,B} =]A^+, B^+]$ and $t_{A,B} = A^+$ Now, if $A^+ = A^-$, we are under the conditions of **A**. Then $A \wedge_0 B = \{A^-\} = A \wedge B$. If $A^- < A^+$, we are under the conditions of **B**. Since $A^+ \in A \cap B$, $A \wedge_0 B = \overline{X}_{A,B} = [A^-, A^+] = A \wedge B$, as expected, since $A \leq B$ and, equivalently, $A \leq^0 B$. If $B^+ < A^+$, we may exchange A by B and obtain the same result.*

If $A^- < B^-$ we may consider several options:

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2022.3217400

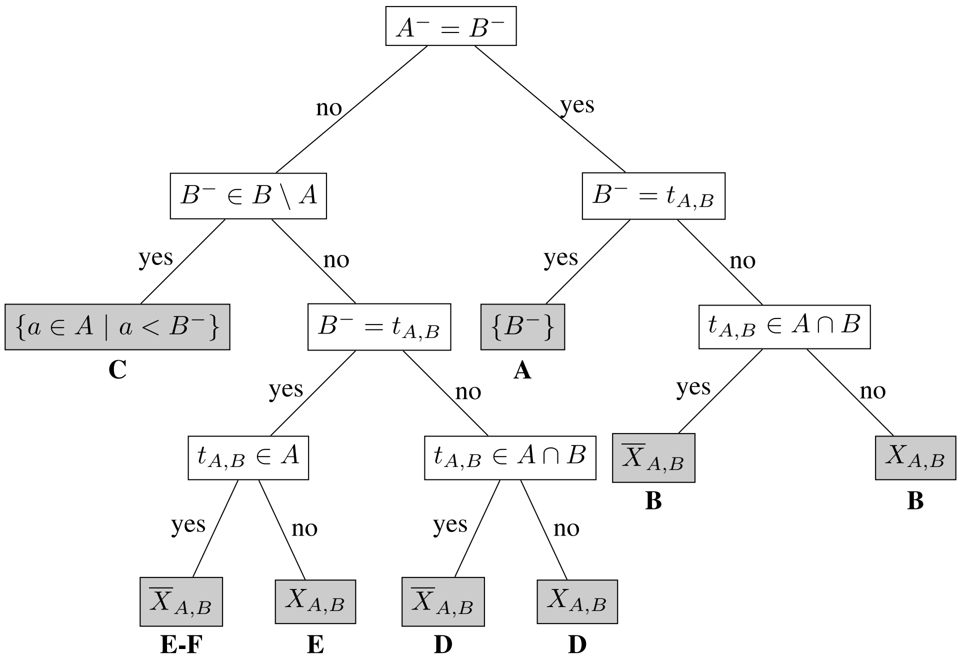


Fig. 1: A decision tree for computing the infimum for the symmetric order

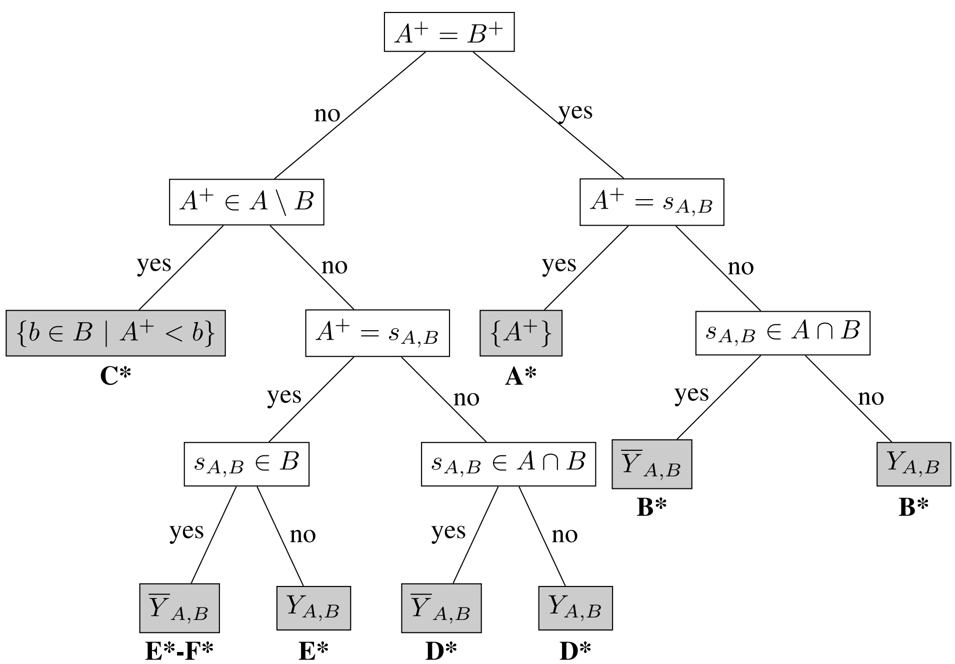


Fig. 2: A decision tree for computing the supremum for the symmetric order

- $A^+ < B^-$. *Then* $T_{A,B} = (B^-, B^+]$ *and* $t_{A,B} = B^-$. *Since* $B^- \in B \setminus A$, *we are now under the conditions of* ***C***. *Then* $A \wedge_0 B = \{a \in A \mid a < B^-\} = A = A \wedge B$.
- $B^- = A^+ = B^+$. *Hence* $T_{A,B} = \emptyset$, *so* $t_{A,B} = 1$. *Observe then that* $B^- \notin B \setminus A$. *Now,* $B^- \neq t_{A,B}$ *if and only if* $B^- < 1$. *In this case,* $t_{A,B} \notin A \cap B$, *so we apply* ***E*** *and* $A \wedge_0 B = X_{A,B} = [A^-, A^+] = A \wedge B$.
  If $B^- = 1$, *since* $t_{A,B} \in A$, ***D*** *applies, so* $A \wedge_0 B = \overline{X} = [A^-, 1] = A \wedge B$.
- $B^- \leq A^+ < B^+$. *In this case* $T_{A,B} = (A^+, B^+]$ *and* $t_{A,B} = A^+$. *Also* $B^- \notin B \setminus A$ *and* $t_{A,B} \in A \cap B$. *Therefore, if* $B^- < A^+$, $t_{A,B} \neq B^-$ *and* ***D*** *applies, so* $A \wedge_0 B = \overline{X}_{A,B} = [A^-, A^+] = A = A \wedge B$.
  If $B^- = A^+$, *we are under the conditions of* ***E-F***, *obtaining the same result.*
- $B^- < B^+ = A^+$. *Then* $T_{A,B} = \emptyset$ *and* $t_{A,B} = 1$. *Following the decision tree of Figure 1,* $B^- \notin B \setminus A$ *and* $B^- \neq t_{A,B}$, *so we are under the conditions of* ***D***. *Now, if* $A^+ = B^+ = 1$, $t_{A,B} \in A \cap B$, *so* $A \wedge_0 B = \overline{X}_{A,B} = [A^-, 1] = A \wedge B$. *Otherwise,*

$t_{A,B} \notin A \cap B$, *so* $A \wedge_0 B = X_{A,B} = [A^-, A^+] = A \wedge B$.

- $B^- \leq B^+ < A^+$. *Then* $T_{A,B} = (B^+, A^+]$ *and* $t_{A,B} = B^+$. *Observe that* $B^- \notin B \setminus A$. *Now, if* $B^- = B^+ = t_{A,B}$, *since* $t_{A,B} \in A$, ***E-F*** *in Figure 1 applies. If not,* ***D*** *applies. In both cases, the infimum is* $A \wedge_0 B = \overline{X}_{A,B} = [A^-, B^+] = A \wedge B$.

Following [23], the interval $[0,1]$, endowed with the max-min lattice structure, can be embedded into $(\mathbb{I}, \wedge, \vee)$ in several ways:

$i)$ Each element $x \in [0,1]$ is mapped to $[0,x]$, or
$ii)$ each element $x \in [0,1]$ is mapped to $[x,1]$, or
$iii)$ each element $x \in [0,1]$ is mapped to the singleton $[x,x] = \{x\}$.

We may then compose each of these mappings with the natural assignment from $\mathbb{I}$ to $\mathcal{P}^*([0,1])$, given by considering an interval as a subset in $[0,1]$, in order to get lattice mappings from $([0,1], \wedge, \vee)$ to $(\mathcal{P}^*([0,1]), \wedge_0, \vee_0)$.

Corollary 46. *The partially ordered set* $(\mathcal{P}^*([0,1]), \leq^0)$ *restricted to the class of:*

$a)$ *intervals of the form* $[0,x]$ *for some* $x \in [0,1]$*, or*
$b)$ *intervals of the form* $[x,1]$ *for some* $x \in [0,1]$*, or*
$c)$ *singleton sets,*

coincides with $([0,1], \leq)$ *via the mappingss* $i)$*,* $ii)$ *and* $iii)$*, respectively. As a consequence,* $([0,1], \wedge, \vee)$ *is a sublattice of* $(\mathcal{P}^*([0,1]), \wedge_0, \vee_0)$ *via any of the mappings* $i)$*,* $ii)$ *and* $iii)$*.*

Corollary 47. *The lattices* $([0,1], \wedge, \vee)$*,* $(\mathbb{I}, \wedge, \vee)$ *and* $(\mathcal{P}^*([0,1]), \wedge_0, \vee_0)$ *form a chain of lattices via any of the mappings* $i)$*,* $ii)$ *and* $iii)$*.*

Finally, this last statement can be broadened to the associated classes of extended fuzzy sets. Given a universe set X, let us denote by $\mathrm{FS}(X)$ the class of all FSs on X, by $\mathrm{IVFS}(X)$ the class of all IVFSs on X, and by $\mathrm{SVFS}(X)$ the class of all SVFSs on X. We may then consider the point-wise derived orders on $\mathrm{FS}(X)$, $\mathrm{IVFS}(X)$ and $\mathrm{SVFS}(X)$ from the posets $([0,1], \leq)$, $(\mathbb{I}, \leq)$ and $(\mathcal{P}^*([0,1]), \leq^0)$, respectively. We shall denote these partially ordered sets as $(\mathrm{FS}(X), \sqsubseteq)$, $(\mathrm{IVFS}(X), \sqsubseteq)$ and $(\mathrm{SVFS}(X), \sqsubseteq^0)$. Additionally, we may obtain the point-wise lattice structures $(\mathrm{FS}(X), \min, \max)$, $(\mathrm{IVFS}(X), \sqcap, \sqcup)$ and $(\mathrm{SVFS}(X), \sqcap_0, \sqcup_0)$, respectively.

As a final consequence of the above discussion, we may state the following result, that answers the open problem formulated in [20, Remark 2]: *given a fixed universe set* X*, is there a lattice structure on* $\mathrm{SVFS}(X)$ *extending Zadeh's union and intersection on* $\mathrm{FS}(X)$*?*

Theorem 48. *The restrictions of the lattice* $(\mathrm{SVFS}(X), \sqcap_0, \sqcup_0)$ *to the subclasses* $\mathrm{FS}(X)$ *and* $\mathrm{IVFS}(X)$ *reduce to the lattices* $(\mathrm{FS}(X), \min, \max)$ *and* $(\mathrm{IVFS}(X), \sqcap, \sqcup)$*, respectively.*

Proof. It follows immediately from Corollary 47. □

V. Conclusion

The necessity of dealing with uncertainty in real world problems has originated several concepts that extend the Zadeh's classical fuzzy sets. Nevertheless, handling simultaneously different sources of vagueness requires certain compatibility between the operations associated to them. Despite the recognized importance of HFSs, a lattice order on HFSs extending the usual structures on FSs and IVFSs had not been defined, as stated in [20, Remark 2]. In this paper we have mended this gap by describing the symmetric order. This lattice order allows then to work with different treatments of the uncertainty (as single values, intervals, union of intervals, etc) simply considering all of them as subsets of $[0,1]$. As a counterpart, the symmetric join and meet do not coincide with Torra's operations [7]. For this reason, we have proposed two additional lattice orders that preserve, at least, one of these operations. The right order is the only one preserving Torra's meet, whilst the left order does likewise with respect to Torra's join. The drawback here is both orders cannot be defined in the whole class of HFS's, although they are valid in the most important subclasses, as IVFSs or THFSs.

Our results also give an answer to [18, Challenge 5.3], and open the door to tackle other challenges explained there. Actually, as a future work, we may consider the design of certain fuzzy operators as t-norms, negations, implications or aggregation functions with respect to the symmetric order; the definition of an appropriate notion of continuity for functions on HFSs; or, in general, the development of the foundations of the theory of HFSs under this order. For instance, the design of suitable aggregation operators might offer some potential applications in the realm of decision-making [25], where HFSs are mostly applied, fuzzy logic and rule-based systems, or image processing [26].

Notwithstanding the symmetric order solves the problem proposed in [20], it is far from being a chain. Actually, many pair of subsets are not comparable. This may produce a handicap in some decision-making problems, since they could need to set a ranking of alternatives. Therefore, the search of a total order on HFSs, if it exists, should be a key aspect to be studied in the future.

References

[1] L. A. Zadeh, Quantitative fuzzy semantics, Information Sciences 3 (1971), 159–176.
[2] L. A. Zadeh, Fuzzy sets, Information and Control 3 (1965), 338–353.
[3] R. Sambuc, Function Φ-Flous, Application a l'aide au Diagnostic en Pathologie Thyroidienne, Thése de Doctorat en Médicine, Univ. Marseille, Marseille, France, 1975.
[4] I. Grattan-Guinness, Fuzzy membership mapped onto interval and many-valued quantities, Zeitschrift für mathematische Logik und Grundladen der Mathematik 22 (1976), 149–160.
[5] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning – 1, Information Sciences 8 (1975) 199–249.
[6] K.U. Jahn, Intervall-wertige Mengen, Mathematische Nachrichten 68 (1975) 115–132.
[7] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems 25 (2010), 529–539.
[8] V. Torra and Y. Narukawa, On hesitant fuzzy sets and decision, 2009 IEEE International Conference on Fuzzy Systems, 1378–1382.
[9] M. Xia and Z. Xu, Hesitant fuzzy information aggregation in decision making, International Journal of Approximate Reasoning 52 (3) (2011), 395–407.
[10] J. C. R. Alcantud and V. Torra, Decomposition theorems and extension principles for hesitant fuzzy sets, Information Fusion 41 (2018), 48–56.
[11] F. Feng, Z. Wan, J. C. R. Alcantud and H. Garg, Three-way decision based on canonical soft sets of hesitant fuzzy sets, Mathematics 7(2) (2022), 2061–2083.

[12] L. Garmendia, R. González del Campo and J. Recasens, Partial orderings for hesitant fuzzy sets, International Journal of Approximate Reasoning 84 (2017), 159–167.

[13] B. Bedregal, R. H. S. Reiser, H. Bustince, H. López-Molina and V. Torra, Aggregation functions for typical hesitant fuzzy elements and the action of automorphism, Information Sciences 255 (1) (2014), 82–99.

[14] H. Wang and Z. Xu, Admissible orders of typical hesitant fuzzy elements and their application in ordered information fusion in multi-criteria decision making, Information Fusion 29 (2016), 98–104.

[15] M. Matzenauer, R. Reiser, H. Santos, B. Bedregal, and H. Bustince, Strategies on admissible total orders over typical hesitant fuzzy implications applied to decision making problems, International Journal of Intelligent Systems 36(5) (2021), 2144–2182.

[16] M. Matzenauer, H. Santos, B. Bedregal, H. Bustince and R. Reiser, On admissible total orders for typical hesitant fuzzy consensus measures, International Journal of Intelligence Systems 37 (1) (2022), 264-286.

[17] H. Bustince, J. Fernandez, A. Kolesárová and R. Mesiar, Generation of linear orders for intervals by means of aggregation functions, Fuzzy Sets and Systems 220 (2013), 69–77.

[18] R.M. Rodríguez, B. Bedregal, H. Bustince, Y.C. Dong, B. Farhadinia, C. Kahraman, L. Martínez, V. Torra, Y.J. Xu, Z.S. Xu, F. Herrera, A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress, Information Fusion 29 (2016) 89–97.

[19] R. M. Rodríguez, L. Martínez, V. Torra, Z. S. Xu, F. Herrera, Hesitant Fuzzy Sets: State of the Art and Future Directions, International Journal of Intelligent Systems 29 (2014), 495-524.

[20] H. Bustince, E. Barrenechea, M. Pagola, J. Fernandez, Z. Xu, B. Bedregal, J. Montero, H. Hagras, F. Herrera, and B. De Baets, A historical account of types of fuzzy sets and their relationships, IEEE Transactions on Fuzzy Systems 24 (1) (2016), 179–194.

[21] J.A Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1) (1967), 145–174.

[22] M. B. Gorzalczany, A method of inference in approximate reasoning based on the interval-valued fuzzy sets, Fuzzy Sets and Systems 21 (1987), 1–17.

[23] F.J. Lobillo, L. Merino, G. Navarro, E. Santos, Embeddings between lattices of fuzzy sets: An application of closed-valued fuzzy sets, Fuzzy Sets and Systems 352 (2018), 56–72.

[24] N. Chen, Z. Xuand M. Xia, Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis, Applied Mathematical Modeling 37 (4) (2013), 2197–2211.

[25] J. C. R. Alcantud and A. Giarlotta, Necessary and possible hesitant fuzzy sets: A novel model for group decision making, Information Fusion 46 (2019) 63–76.

[26] G. Kaur and H. A. Garg, A new method for image processing using generalized linguistic neutrosophic cubic aggregation operator, Complex and Intelligence Systems 2022, https://doi.org/10.1007/s40747-022-00718-5.

Pascual Jara received the Ph.D. degree in mathematics from the University of Granada, Granada, Spain, in 1984. He is Professor in the Department of Algebra of the University of Granada since 2004. He has published more than sixty papers in international journals. His current research interests are fuzzy sets, the algebraic treatment of uncertainty and ring theory. He is head of the research project *A categorical approach to fuzzy sets*.

Luis Merino received the Ph. D. degree in mathematics from the University of Granada, Spain, in 1991. He is Associated Professor of the University of Granada since 1992. Currently, he is head of the Department of Algebra. He is member of the Royal Spanish Mathematical Society (RSME). His main research interest include ring and module theory, the algebraic treatment of uncertainty, fuzzy sets, rough sets and soft sets.

Gabriel Navarro is an Associated Professor with the Department of Computer Sciences and Artificial Intelligence at the University of Granada. He is also member of the Research Centre for Information and Communications Technologies of the University of Granada (CITIC-UGR). He received its Ph.D. degree in mathematics (2006) from the University of Granada. His current research interests are coding theory, computational algebra and fuzzy and rough sets.

Evangelina Santos received the Ph. D. degree in mathematics from the University of Granada, Spain, in 1993. She is Associated Professor with the Department of Algebra at the University of Granada since 1995. She is member of the Royal Spanish Mathematical Society (RSME) and of THALES society of Andalusian teachers of Mathematics. Her research interests comprise commutative algebra, fuzzy sets and rough sets.