Induced triangular norms and negations on bounded lattices

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***Abstract*—Some relevant notions in fuzzy set theory are those of triangular norm and conorm, and negation, which provide a systematic way of defining set-theoretic operations or, from other point of view, logical connectives. For instance, the majority of fuzzy implications are directly derived from these operators, so they play a prominent role in fuzzy control theory or in approximate reasoning. This incites the search of suitable t-norms, t-conorms and negations for solving each specific problem. In this paper we propose a procedure, that we call induction, for designing them on spaces of lattice-valued maps. Concretely, for each family of operators (t-norms, t-conorms or negations) indexed in the domain set, we may induce an operator of the same kind, so that our method offers a great flexibility in the design task. It may be applied to well-known fuzzy objects as interval-valued or type-2 fuzzy sets. Nevertheless, the theory is formally developed for arbitrary bounded lattices.**

***Index Terms*—triangular norm, triangular conorm, negation, De Morgan triple, lattice-valued map.**

I. Introduction

ALTHOUGH the notion of t-norm may be traced back to Menger's paper [1] in 1942, in the realm of probabilistic metric spaces, its present day definition may be find firstly in Schweizer and Sklar's paper [2]. In the context of fuzzy sets, t-norms and t-conorms are considered in order to generalize standard set-theoretic operations. This is done, actually, by Zadeh in his seminal paper [3], and mostly developed since early eighties. Later, the extensions of standard fuzzy sets to classes of objects in which the truth values belong to a lattice related to $[0, 1]$ (as, for instance, interval-valued, set-valued or type-2 fuzzy sets) originate the search of valuable t-norms, and design methodologies, on these lattices. See, for instance, [4], [5] for t-norms on type-2 fuzzy sets, [6]–[8] for t-norms on interval fuzzy sets, or [9]–[11]. The reader may also consult, for example, [12] for the ordinal sum construction, or [13, Chapter 3] for methods based on additive and multiplicative generators. These operators, in classical or in extended fuzzy sets, yield a wide and extensive variety of practical applications in computer sciences-related areas. For a concrete example, the reader may consult [14], where the Weber's parametric t-norms are employed for selecting a homogeneity measure for color image segmentation, or [15], where the standard t-conorm is replaced in the fuzzy expert system for internal medicine CADIAG-2 [16]. Concurrently, from the point of view of fuzzy logic, t-norms and negations (forming a De Morgan triple) have been used for different interpretations of logical formulas where t-norms, t-conorms and negations play the role of conjunction, disjunction and negation operators, respectively, for some truth assignment. Additionally, most of known implications are derived from them, allowing to make inferences in the so-called approximate reasoning, see the survey [17]. The investigation from this side has been stimulated by its applications in fields as control systems or artificial intelligence, among many others. See, for instance, the monograph [18].

In this paper we present a methodology for composing a t-norm (resp. t-conorm, negation) from a family of t-norms (resp. t-conorms, negations). Concretely, let us fix an arbitrary set X and a bounded lattice L (standardly, the lattice $[0, 1]$, although non necessarily this one). After a detailed recall of the needed notions and properties (Section II), in Section III, from a family $\{T_x\}_{x \in X}$ of t-norms (resp. t-conorms) on L indexed in X, we construct a t-norm $\mathbf{T}$ (resp. t-conorm) on the lattice of L-valued maps $\mathrm{Map}(X, L)$, see Proposition 2. We have named $\mathbf{T}$ the induced t-norm (resp. t-conorm) of the family. The procedure allows us then to obtain t-norms (resp. t-conorms) on well-known lattices as, for instance, the real unit square, that is $\mathrm{Map}(\{0, 1\}, [0, 1])$, see [19], or $\mathrm{Map}([0, 1], [0, 1])$ for defining set-theoretic operations in type-2 fuzzy sets. These induced objects are well-behaved from an algebraic perspective, since they heritage some properties from the original family. For instance, the property of being monotone. This yields the possibility of inducing t-norms (resp. t-conorms) on the sublattice $\mathrm{Hom}(X, L)$ formed by the monotone maps from X to L, see Proposition 13. The idea of induced t-norms on intervals in [0,1] is described in [6] and [20], and on n-dimensional intervals in [21].

Since this methodology is highly applicable to other structural maps, Section V is devoted to study the case of induced negations. In this case, the induction process is more flexible and it depends on a variable map $\sigma : X \to X$, yielding the σ-induced negations. Again, under a suitable map σ, the monotonicity is inhered by the induced negation, see Lemma 21, so we may induce negations on the space $\mathrm{Hom}(X, L)$, see [22] for the case of n-dimensional intervals.

Finally, in Section VI, we merge together the topics covered in the former sections and analyze when a triple formed by induced structures is, indeed, a De Morgan triple on $\mathrm{Map}(X, L)$, see Theorem 30.

The theory is illustrated by presenting possible scopes of application. Actually, in some of them, we generalize previously published studies about t-norms and negations over fuzzy sets.

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In particular, we analyze in depth the case of interval-valued fuzzy sets, i.e. t-norms and negations on $\mathrm{Hom}(\{0,1\},[0,1])$.

II. Preliminaires

In this section we introduce the notation and definitions concerning the theory developed in this paper. For self-completeness, we provide them in detail.

All along this work X denotes a set. A binary relation $\leq$ on X (that is, a subset of $X \times X$) is said to be a *partial order* if and only if it verifies the properties *reflexive*, *antisymmetric* and *transitive*. A *partially ordered set* (a *poset*, for short) is a pair $(P,\leq)$, where P is a set and $\leq$ is partial order on P. In general, if the context is clear, we simply denote it by P.

Let $(P,\leq)$ be a poset and $a,b \in P$, a *least upper bound* or *supremum* of a and b is an element $c \in P$ verifying that $a,b \leq c$ and, if $c' \in P$ with $a,b \leq c'$, then $c \leq c'$. Dually, a *greatest lower bound* or *infimum* of a and b is an element $d \in P$ verifying that $d \leq a,b$ and, if $d' \in P$ with $d' \leq a,b$, then $d' \leq d$. A poset $(L,\leq)$ is called a *lattice* if, for any $a,b \in L$, there exist a supremum and an infimum of them. We may, equivalently, say that a lattice is a set L with two operations $\wedge$ and $\vee$, called infimum and supremum, respectively, verifying the following properties:

$i)$ *associativity*, for any $a,b,c \in L$, $(a\wedge b)\wedge c = a \wedge (b \wedge c)$ and $(a \vee b)\vee c = a \vee (b \vee c)$.
$ii)$ *commutativity*, for any $a,b \in L$, $a\wedge b = b \wedge a$ and $a \vee b = b \vee a$.
$iii)$ *idempotency*, for any $a \in L$, $a \wedge a = a$ and $a \vee a = a$.
$iv)$ *absorption law*, for any $a,b \in L$, $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = b$.

Actually, the equivalence between both approaches is given by the relations $a \leq b \Longleftrightarrow a \vee b = b \Longleftrightarrow a \wedge b = a$.

Again, when the context is clear enough, we do not mention the operators $\wedge$ and $\vee$, and we simply denote by L the lattice $(L,\wedge,\vee)$.

A lattice $(L,\wedge,\vee)$ is said to be *bounded* if there exist a *maximum* and a *minimum* of L. That is, two elements in L, usually denoted by 1 and 0, respectively, such that, for every $a \in L$, $0 \leq a \leq 1$ or, equivalently, $a \wedge 0 = 0$, $a \wedge 1 = a$, $a \vee 0 = a$ and $a \vee 1 = 1$.

We recall, for instance from [23], that a *triangular norm* or *t-norm* on a bounded lattice L is a map $T : L \times L \longrightarrow L$ such that, for each $a,b,c \in L$, the following properties are satisfied:

$i)$ *commutativity*, $T(a,b) = T(b,a)$.
$ii)$ *associativity*, $T(a,T(b,c)) = T(T(a,b),c)$.
$iii)$ *monotonicity*, if $b \leq c$, then $T(a,b) \leq T(a,c)$.
$iv)$ *neutral element*, $T(a,1) = a$.

We shall denote by $\mathcal{T}(L)$ the set of all t-norms on the lattice L. If the neutral element property is changed by $T(a,0) = a$ for any $a \in L$, hence T is called a *triangular conorm* or *t-conorm*. As usual, we reserve the letter S for denoting a generic t-conorm. Analogously, we denote by $\mathcal{S}(L)$ the set of all t-conorms on L.

In [24] it is pointed out that the class $\mathcal{T}(L)$ of all t-norms on a lattice L is endowed with a poset structure provided by the following partial order: for each $T_1, T_2 \in \mathcal{T}(L)$,

$$T_1 \leq T_2 \text{ if and only if } T_1(a,b) \leq T_2(a,b)$$

for any $a,b \in L$. Actually, $\mathcal{T}(L)$ has a minimum for this order, which is the so-called *drastic t-norm*, defined as

$$T_D(a,b) = \begin{cases} 0, & \text{if } a \neq 1 \text{ and } b \neq 1, \\ a \wedge b, & \text{otherwise,} \end{cases}$$

and maximum, $T_\wedge$, given by $T_\wedge(a,b) = a \wedge b$ for any $a,b \in L$.

It is not difficult to check that, in general, $\mathcal{T}(L)$ is not a lattice with the structure induced from the former order. Indeed, the infimum and supremum of two t-norms T_1 and T_2 are defined by the rules $(T_1 \wedge T_2)(x,y) = T_1(x,y) \wedge T_2(x,y)$ and $(T_1 \vee T_2)(x,y) = T_1(x,y) \vee T_2(x,y)$, respectively, for any $x,y \in L$. Hence, $T_1 \wedge T_2$ fulfills the commutative, monotonic and neutral element properties, but, in general, not the associativity. For instance, set the t-norms on $[0,1]$

$$T^{nM}(a,b) = \begin{cases} \min\{a,b\}, & \text{if } a+b > 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $T_p(a,b) = a \cdot b$ for any $a,b \in [0,1]$. Fix now the real numbers $x = 0.8$, $y = 0.5$ and $z = 0.6$. Thus

$$(T^{nM} \wedge T_p)((T^{nM} \wedge T_p)(x,y),z) = 0, \text{ whilst}$$
$$(T^{nM} \wedge T_p)(x,(T^{nM} \wedge T_p)(y,z)) = 0.24.$$

The same can be said about the set of t-conorms on L, $\mathcal{S}(L)$.

A t-norm T in a lattice $(L,\wedge,\vee)$ is said to be $\wedge$-*distributive* if it satisfies

$$T(a \wedge b, c) = T(a,c) \wedge T(b,c), \text{ for every } a,b,c \in L,$$

and, $\vee$-*distributive*, if it satisfies

$$T(a \vee b, c) = T(a,c) \vee T(b,c), \text{ for every } a,b,c \in L.$$

Analogously, we may state the same properties for an arbitrary t-conorm S. Note that if L is a totally ordered set (that is, for each $a,b \in L$, $a \leq b$ or $b \leq a$), then $\wedge$-distributivity and $\vee$-distributivity are equivalent to T being monotone. In particular, every t-norm and every t-conorm on the unit interval $[0,1]$ is both $\wedge$-distributive and $\vee$-distributive.

An element e of a lattice L is said to be idempotent for a t-norm T (respectively, for a t-conorm S) if $T(e,e) = e$ (respectively, $S(e,e) = e$). Hence 0 and 1 are idempotent, which are called the trivial idempotents. Observe that, for the norm $T_\wedge$, all the elements are idempotent, whereas the norm T_D has no non-trivial idempotent.

Let n be a positive integer, an element a in the lattice L is said to be n-nilpotent for the t-norm T if $T^n(a) = 0$, where $T^1(x) = T(x,x)$ and $T^m(x) = T(T^{m-1}(x),x)$ for each $m > 1$. Finally, $a \in L$ is a zero divisor for T if there exists an element b such that $a \wedge b \neq 0$ and $T(a,b) = 0$.

A *negation* on a lattice L is a map $N : L \longrightarrow L$ such that $N(0) = 1$, $N(1) = 0$ and, for any $a,b \in L$, if $a \leq b$ then $N(b) \leq N(a)$ (that is, it is a decreasing map). If, in addition, $N(N(a)) = a$ for any $a \in L$, i. e. $N = N^{-1}$, the negation is called *involutive*. We denote by $\mathcal{N}(L)$ the set of all negations on the lattice L.

Let T be t-norm on a lattice L and N an involutive negation on L, the map $S : L \times L \to L$ defined by

$$S(a,b) = N(T(N(a),N(b))$$

for any $a, b \in L$ is a t-cornorm on L. The map S is known as the N-dual t-conorm of the T. Similarly, given an arbitrary t-conorm S on L and an involutive negation N on L, it may be defined the N-dual t-norm T of S as

$$T(a, b) = N(S(N(a), N(b))$$

for any $a, b \in L$. A *De Morgan triple* $\mathcal{D}$ on a lattice L is then a tuple (S, N, T), where T is a t-norm on L, N an involutive negation on L and S a t-conorm on L such that S and T are dual by N one to each other.

A De Morgan triple $\mathcal{D} = (S, N, T)$ on a lattice L can be used to obtain an alternative non distributive set operations in the class of all L-fuzzy sets over an universe X. Indeed, we may consider the binary operators $\cap_{\mathcal{D}}$ and $\cup_{\mathcal{D}}$, and the unary operator $\overline{(-)}^{\mathcal{D}}$ defined by

$$(A \cap_{\mathcal{D}} B)(x) = T(A(x), B(x)),$$

$$(A \cup_{\mathcal{D}} B)(x) = S(A(x), B(x)),$$

$$\overline{A}^{\mathcal{D}}(x) = N(A(x)),$$

for any L-fuzzy sets $A, B : X \longrightarrow L$ and any $x \in X$.

Given a map $f : X \to Y$ between two sets X and Y, and $A \subset X$ a subset of X, we denote by $f_{|A}$ the restriction of f to A. A subset A of a lattice L is said to be a sublattice of L if it is closed for the meet and joint operators. The proof of the following result is straightforward and left to the reader.

Lemma 1: Let L be a bounded lattice and A be a sublattice of L containing 0 and 1, then the following assertions hold:

- $a)$ If T is a t-norm on L and $T(A \times A) \subseteq A$, then $T_{|A}$ is a t-norm on A.
- $b)$ If S is a t-conorm on L and $S(A \times A) \subseteq A$, then $S_{|A}$ is a t-conorm on A.
- $c)$ If N is a (involutive) negation on L and $N(A) \subseteq A$, then $N_{|A}$ is a (involutive) negation on A.
- $d)$ If $\mathcal{D} = (S, N, T)$ is a De Morgan triple on L, $T(A \times A) \subseteq A$ and $N(A) \subseteq A$ then, $\mathcal{D}_{\mathcal{A}} = (S_{|A}, N_{|A}, T_{|A})$ is a De Morgan triple on A.

III. Induced t-norms

In practice, most of the lattices related to fuzzy sets have the unit interval $[0, 1]$ as sublattice. Then, an interesting problem, linked to Lemma 1, is the extension of t-norms on $[0, 1]$ to that superlattice. That is, in general, given t-norm (t-conorm, negation) on a lattice L, can we induce a t-norm (t-conorm, negation) on a superlattice A whose restriction to L is the original one? In this and the forecoming sections we deal with this question.

In [25] and [26] it is considered the case where the sublattice is given by a retraction. Given lattices L and M, M is said to be a *retract* in L if there exist lattice morphisms $r : L \longrightarrow M$ and $s : M \longrightarrow L$ such that $r \circ s = id_M$, the identity on M. In such a case, s has to be injective and therefore M is lattice-isomorphic to a sublattice $s(M)$ of L. Hence, if T is a t-norm on M, it can be extended to a t-norm $\overline{T}$ on L by considering

$$\overline{T}(x, y) = s(T(r(x), r(y))$$

for any $x, y \in L$. In this construction, the image by $\overline{T}$ of every pair of elements in L always belongs to $s(M)$. We aim to provide a more flexible way to expand t-norms. Our point of view has been considered before in [21] for t-norms in n-dimensional fuzzy sets and, in particular, in [27] for interval-valued fuzzy sets, see Section IV-B.

For a given bounded lattice L and an arbitrary set X, it is known that the class $\mathrm{Map}(X, L)$ of all set maps from X to L is endowed naturally with a structure of bounded lattice, induced from the one of L. Indeed, the induced partial order is given by, for any $f, g \in \mathrm{Map}(X, L)$,

$$f \leq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in X.$$

The lattice structure is defined similarly, that is,

$$(f \wedge g)(x) = f(x) \wedge g(x) \text{ and}$$

$$(f \vee g)(x) = f(x) \vee g(x)$$

for any $x \in X$. Then, the elements $\mathbf{0}$ and $\mathbf{1}$ of this lattice are given by the constant maps 0 and 1, respectively.

Observe that L can be seen as a sublattice of $\mathrm{Map}(X, L)$ via the identification that maps a $y \in L$ to the constant map $C_y : X \longrightarrow L$ given by $C_y(x) = y$ for all $x \in X$. In particular, $\mathbf{0} = C_0$ and $\mathbf{1} = C_1$. Then, L is a retract of $\mathrm{Map}(X, L)$ via the maps $s : L \to \mathrm{Map}(X, L)$, given by $s(y) = C_y$ for any $y \in L$, and $r : \mathrm{Map}(X, L) \to L$ given by $r(f) = f(0)$ for any $f \in \mathrm{Map}(X, L)$. Therefore, by using the techniques of [25], a t-norm T in L can be extended to

$$\overline{T}(f, g) = s(T(r(f), r(g)) = C_{T(f(0), g(0))},$$

for any $f, g \in \mathrm{Map}(X, L)$. In order to increase the diversity of the extensions, we introduce the following method for obtaining t-norms on $\mathrm{Map}(X, L)$.

Let L be a bounded lattice and X an arbitrary set. Let also $\{T_x\}_{x \in X}$ a family of t-norms on L indexed in X. We may consider the map

$$\mathbf{T} : \mathrm{Map}(X, L) \times \mathrm{Map}(X, L) \longrightarrow \mathrm{Map}(X, L)$$

defined by the formula

$$\mathbf{T}(f, g)(x) = T_x(f(x), g(x)) \tag{1}$$

for any $f, g \in \mathrm{Map}(X, L)$ and any $x \in X$.

Proposition 2: Let L be a bounded lattice, X an arbitrary set and $\{T_x\}_{x \in X}$ a family of t-norms on L indexed in X. Then the map $\mathbf{T}$ defined in (1) is a t-norm on $\mathrm{Map}(X, L)$.

Proof: We have to show that $\mathbf{T}$ holds the properties that define a t-norm. Let $f, g \in \mathrm{Map}(X, L)$. For the commutativity, note that, for any $x \in X$,

$$\begin{aligned} \mathbf{T}(f, g)(x) &= T_x(f(x), g(x)) \\ &= T_x(g(x), f(x)) \\ &= \mathbf{T}(g, f)(x), \end{aligned}$$

so $\mathbf{T}(f, g) = \mathbf{T}(g, f)$. For the associativity, for any $x \in X$,

$$\begin{aligned} \mathbf{T}(\mathbf{T}(f, g), h)(x) &= T_x(\mathbf{T}(f, g)(x), h(x)) \\ &= T_x(T_x(f(x), g(x)), h(x)) \\ &= T_x(f(x), T_x(g(x), h(x))) \\ &= \mathbf{T}(f, \mathbf{T}(g, h))(x), \end{aligned}$$

so $\mathbf{T}(\mathbf{T}(f,g),h)=\mathbf{T}(f,\mathbf{T}(g,h))$. Now, whenever $f\leq g$ then $f(x)\leq g(x)$ for any $x\in X$. Hence

$$\begin{aligned}\mathbf{T}(h,f)(x)&=T_x(h(x),f(x))\\&\leq T_x(h(x),g(x))\\&=\mathbf{T}(h,g)(x),\end{aligned}$$

and thus $\mathbf{T}(h,f)\leq\mathbf{T}(h,g)$. So the monotonicity holds. Finally, in $\mathrm{Map}(X,L)$, the neutral element is the constant map $\mathbf{1}$. Hence it is clear that $\mathbf{T}(f,\mathbf{1})=f$. ■

Observe that we may see a family $\beta=\{T_x\}_{x\in X}$ of t-norms on L as a map $\beta:X\to\mathcal{T}(L)$, mapping each $x\in X$ to T_x. We shall use both descriptions without distinction. Therefore, in virtue of Proposition 2, we may consider

$$\Phi:\mathrm{Map}(X,\mathcal{T}(L))\longrightarrow\mathcal{T}(\mathrm{Map}(X,L)) \tag{2}$$

defined by

$$\Phi(\beta)(f,g)(x)=\beta(x)(f(x),g(x))$$

for any $f,g\in\mathrm{Map}(X,L)$ and any $x\in X$. By using the powerset notation, Φ can be rewritten as

$$\Phi:\mathcal{T}(L)^X\longrightarrow\mathcal{T}(L^X) \tag{3}$$

showing clearer the relation between these two posets.

Proposition 3: Let L be a bounded lattice and X an arbitrary set, the map Φ defined in (2) is injective.

Proof: Let $\beta,\tau:X\longrightarrow\mathcal{T}(L)$ be two different maps, or, equivalently, two different families of t-norms on L indexed in X. Then we may choose an $x\in X$ such that $\beta(x)\neq\tau(x)$. Since both images provide us t-norms on L, there exists a pair $(t_1,t_2)\in L\times L$ such that $\beta(x)(t_1,t_2)\neq\tau(x)(t_1,t_2)$. Now, consider the constant maps $C_{t_1},C_{t_2}:X\to L$. Then

$$\Phi(\beta)(C_{t_1},C_{t_2})(x)=\beta(x)(C_{t_1}(x),C_{t_2}(x))=\beta(x)(t_1,t_2)$$

and

$$\Phi(\tau)(C_{t_1},C_{t_2})(x)=\tau(x)(C_{t_1}(x),C_{t_2}(x))=\tau(x)(t_1,t_2),$$

so $\Phi(\beta)(x)\neq\Phi(\tau)(x)$ and thus $\Phi(\beta)\neq\Phi(\tau)$. ■

Definition 4: A t-norm $\mathbf{T}$ on $\mathrm{Map}(X,L)$ is said to be *representable* if it belongs to the image of Φ, that is, if there exists a family $\beta=\{T_x\}_{x\in X}$ of t-norms on L such that $\mathbf{T}=\Phi(\beta)$.

Remark 5: As usual, t-conorms on $\mathrm{Map}(X,L)$ can be treated dually. That is, if $\beta=\{S_x\}_{x\in X}$ is a family of t-conorms on L indexed in X, analogously to (1) and Proposition 2, we may define the t-conorm

$$\mathbf{S}:\mathrm{Map}(X,L)\times\mathrm{Map}(X,L)\longrightarrow\mathrm{Map}(X,L)$$

given by

$$\mathbf{S}(f,g)(x)=S_x(f(x),g(x))$$

for any $f,g\in\mathrm{Map}(X,L)$ and any $x\in X$. Hence,

$$\Xi:\mathcal{S}(L)^X\longrightarrow\mathcal{S}(L^X) \tag{4}$$

given by $\Xi(\beta)=\mathbf{S}$ is well-defined, and a t-conorm on $\mathrm{Map}(X,L)$ is said to be representable if it belongs to the image of Ξ. Although we will not mention it explicitly, all the results obtained in this section can be stated and proved similarly in the realm of t-conorms.

Example 6: In general, Φ is no surjective, i.e. not all t-norms are representable. This is a consequence of the following fact: If a t-norm $\mathbf{T}$ on $\mathrm{Map}(X,L)$ is representable, then $\mathbf{T}(\chi_{x_0},\chi_{x_0})=\chi_{x_0}$, where $x_0\in X$ and $\chi_{x_0}:X\to L$ is the singleton map over x_0, that is, the map defined by

$$\chi_{x_0}(x)=\begin{cases}1, & \text{if } x=x_0,\\ 0, & \text{otherwise.}\end{cases}$$

Indeed, if $\mathbf{T}=\Phi(\{T_x\}_{x\in X})$ then

$$\mathbf{T}(\chi_{x_0},\chi_{x_0})(x_0)=T_{x_0}(\chi_{x_0}(x_0),\chi_{x_0}(x_0))=T_{x_0}(1,1)=1$$

and, for any other element $x\in X\setminus\{x_0\}$,

$$\mathbf{T}(\chi_{x_0},\chi_{x_0})(x)=T_x(\chi_{x_0}(x),\chi_{x_0}(x))=T_x(0,0)=0.$$

Hence, the drastic t-norm T_D on $\mathrm{Map}(X,L)$,

$$T_D(f,g)=\begin{cases}f, & \text{if } g=\mathbf{1},\\ g, & \text{if } f=\mathbf{1},\\ \mathbf{0}, & \text{otherwise}\end{cases}$$

is not representable, since $\chi_{x_0}\neq\mathbf{1}$. Thus $T_D(\chi_{x_0},\chi_{x_0})=\mathbf{0}$.

Example 7 (Representable t-norms and characteristic maps): By similarity with the latter example, given a set X and a subset $Y\subseteq X$, we denote by χ_Y the characteristic map of Y over X, that is, the map $\chi_Y:X\to L$ given by

$$\chi_Y(x)=\begin{cases}1, & \text{if } x\in Y,\\ 0, & \text{otherwise.}\end{cases}$$

Representable t-norms act over characteristic maps as the standard t-norm on the lattice of subsets. Indeed, let X be an arbitrary subset, L a bounded lattice, Y_1 and Y_2 subsets of X, $\beta=\{T_x\}_{x\in X}$ a family of t-norms on L and $\mathbf{T}=\Phi(\beta)$. Hence, for any $x\in X$,

$$\begin{aligned}\mathbf{T}(\chi_{Y_1},\chi_{Y_2})(x)&=T_x(\chi_{Y_1}(x),\chi_{Y_2}(x))\\&=\begin{cases}1, & \text{if } x\in Y_1\cap Y_2,\\ 0, & \text{otherwise}\end{cases}\\&=\chi_{Y_1\cap Y_2}(x)\end{aligned}$$

so $\mathbf{T}(\chi_{Y_1},\chi_{Y_2})=\chi_{Y_1\cap Y_2}$.

Proposition 8: Let L be a bounded lattice, X an arbitrary set and $\mathbf{T}$ a t-norm on $\mathrm{Map}(X,L)$. Then the following statements are equivalent:

a) $\mathbf{T}$ is representable.

b) $\mathbf{T}(f,g)(x)=\mathbf{T}(C_{f(x)},C_{g(x)})(x)$ for each maps $f,g\in\mathrm{Map}(X,L)$ and each $x\in X$.

c) Let $f,f',g,g'\in\mathrm{Map}(X,L)$ such that $f(x)=f'(x)$ and $g(x)=g'(x)$ for some $x\in X$, then $\mathbf{T}(f,g)(x)=\mathbf{T}(f',g')(x)$.

Proof:

a) $\Rightarrow$ *b)*. Let $\mathbf{T}=\Phi(\{T_x\}_{x\in X})$ be a representable t-norm on $\mathrm{Map}(X,L)$ and $f,g\in\mathrm{Map}(X,L)$. Then, for any $x\in X$,

$$\begin{aligned}\mathbf{T}(C_{f(x)},C_{g(x)})(x)&=T_x(C_{f(x)}(x),C_{g(x)}(x))\\&=T_x(f(x),g(x))\\&=\mathbf{T}(f,g)(x).\end{aligned}$$

b) $\Rightarrow$ *c)*. Let $f, f', g, g' \in \mathrm{Map}(X, L)$ such that $f(x) = f'(x)$ and $g(x) = g'(x)$ for certain $x \in X$. Then

$$\begin{aligned}\mathbf{T}(f,g)(x) &= \mathbf{T}(C_{f(x)}, C_{g(x)})(x)\\ &= \mathbf{T}(C_{f'(x)}, C_{g'(x)})(x)\\ &= \mathbf{T}(f', g')(x).\end{aligned}$$

c) $\Rightarrow$ *a)*. Given an $x \in X$, we define a map $T_x : L \times L \to L$ as $T_x(t_1, t_2) = \mathbf{T}(C_{t_1}, C_{t_2})(x)$ for any $t_1, t_2 \in L$. Let us first prove that T_x is a t-norm on L. Since,

$$T_x(t_1, t_2) = \mathbf{T}(C_{t_1}, C_{t_2})(x) = \mathbf{T}(C_{t_2}, C_{t_1})(x) = T_x(t_2, t_1),$$

the commutativity holds. Now,

$$\begin{aligned}T_x(t_1, T_x(t_2, t_3)) &= T_x(t_1, \mathbf{T}(C_{t_2}, C_{t_3})(x))\\ &= \mathbf{T}(C_{t_1}, C_{\mathbf{T}(C_{t_2}, C_{t_3})(x)})(x)\\ &\overset{\star}{=} \mathbf{T}(C_{t_1}, \mathbf{T}(C_{t_2}, C_{t_3}))(x)\\ &= \mathbf{T}(\mathbf{T}(C_{t_1}, C_{t_2}), C_{t_3})(x)\\ &= \mathbf{T}(C_{\mathbf{T}(C_{t_1}, C_{t_2})(x)}, C_{t_3})(x)\\ &= T_x(\mathbf{T}(C_{t_1}, C_{t_2})(x), t_3)\\ &= T_x(T_x(t_1, t_2), t_3),\end{aligned}$$

for any $t_1, t_2, t_2 \in L$, where $\star$ follows from the equality $C_{\mathbf{T}(C_{t_2}, C_{t_3})(x)}(x) = \mathbf{T}(C_{t_2}, C_{t_3})(x)$. Thus, T_x is associative. T_x is also monotone, since, for any $t, t_1, t_2 \in L$ with $t_1 \leq t_2$,

$$T_x(t, t_1) = \mathbf{T}(C_t, C_{t_1})(x) \leq \mathbf{T}(C_t, C_{t_2})(x) = T_x(t, t_2).$$

Finally, we may set the neutral element to the constant map $\mathbf{1} : X \to L$. Then,

$$T_x(t, 1) = \mathbf{T}(C_t, \mathbf{1})(x) = C_t(x) = t,$$

for all $t \in L$. Now, we have that

$$\begin{aligned}\Phi(\{T_x\}_{x\in X})(f,g)(y) &= T_y(f(y), g(y))\\ &= \mathbf{T}(C_{f(y)}, C_{g(y)})(y)\\ &= \mathbf{T}(f,g)(y)\end{aligned}$$

for any $f, g \in \mathrm{Map}(X, L)$ and $y \in L$. Hence $\mathbf{T}$ is representable. ■

As a particular case, we may consider a constant family of t-norms $\{T\}_{x\in X}$ on L, where $T_x = T$ for all $x \in X$ for certain t-norm T on L. Hence, a t-norm $\mathbf{T}$ on $\mathrm{Map}(X, L)$ is said to be *unirepresentable* if $\mathbf{T} = \Phi(\beta)$ for some constant map $\beta : X \to \mathcal{T}(L)$.

Corollary 9: Let L be a bounded lattice, X be an arbitrary set and $\mathbf{T}$ a representable t-norm on $\mathrm{Map}(X, L)$. Then $\mathbf{T}$ is unirepresentable if and only if, for each $x \in X$ and each $f, g \in \mathrm{Map}(X, L)$, $\mathbf{T}(C_{f(x)}, C_{g(x)}) = C_{\mathbf{T}(f,g)(x)}$.

Proof: Let $\mathbf{T}$ be unirepresentable with $\beta = \{T\}_{x\in X}$ such that $\Phi(\beta) = \mathbf{T}$. Let $x \in X$ and $f, g \in \mathrm{Map}(X, L)$,

$$\begin{aligned}\mathbf{T}(C_{f(x)}, C_{g(x)})(y) &= T(C_{f(x)}(y), C_{g(x)}(y))\\ &= T(f(x), g(x))\\ &= \mathbf{T}(f,g)(x),\end{aligned}$$

for all $y \in X$. Hence, $\mathbf{T}(C_{f(x)}, C_{g(x)}) = C_{\mathbf{T}(f,g)(x)}$.

Conversely, as in the proof of Proposition 8, for each $x \in X$, we may define the t-norm $T_x \in \mathcal{T}(L)$ given by $T_x(t_1, t_2) = \mathbf{T}(C_{t_1}, C_{t_2})(x)$ for any $t_1, t_2 \in L$, and $\mathbf{T} = \Phi(\{T_x\}_{x\in X})$. Obviously, $T_x = T_y$ for any $x, y \in X$. ■

As an application of Corollary 9, we may show that a representable t-norm do not need to be unirepresentable. Indeed, let $X = \{0, 1\}$ and $L = [0, 1]$. Consider the drastic t-norm on L,

$$T_D(r, s) = \begin{cases} r, & \text{if } s = 1,\\ s, & \text{if } r = 1,\\ 0, & \text{otherwise,}\end{cases}$$

and, the infimum t-norm $T_\wedge(r, s) = r \wedge s$ for any $r, s \in L$. Set $T_0 = T_D$ and $T_1 = T_\wedge$. Then the induced t-norm $\mathbf{T}$ in $\mathrm{Map}(X, L)$ is not unirepresentable. Choose different $r, s \neq 0, 1$ in $[0, 1]$, and let C_r and C_s the constant maps associated to r and s. Then $\mathbf{T}(C_r, C_s) : \{0, 1\} \to L$ is given by $\mathbf{T}(C_r, C_s)(0) = T_0(C_r(0), C_s(0)) = T_D(r, s) = 0$ and $\mathbf{T}(C_r, C_s)(1) = T_1(C_r(0), C_s(0)) = T_\wedge(r, s) = r \wedge s \neq 0$. So $\mathbf{T}(C_r, C_s)$ is not constant and, following Corollary 9, $\mathbf{T}$ is not unirepresentable.

We provide now some properties that are inherit from the family of t-norms by the induction process.

Proposition 10: Let L be a bounded lattice and X be an arbitrary set. Let $\beta = \{T_x\}_{x\in X}$ be a family of t-norms on L indexed in X and $\mathbf{T} = \Phi(\beta)$, where Φ is defined as in (2). Then the following statements are equivalent:

a) $\mathbf{T}$ is $\wedge$-distributive (resp. $\vee$-distributive).

b) For every $x \in X$, T_x is $\wedge$-distributive (resp. $\vee$-distributive).

Proof: a)$\Rightarrow$ b). First, observe that the constant map $C_t \wedge C_s = C_{t\wedge s}$ in $\mathrm{Map}(X, L)$ for any $t, s \in L$. Since $\mathbf{T}$ is $\wedge$-distributive, in particular, for any $t_1, t_2, t_3 \in L$,

$$\mathbf{T}(C_{t_1}, C_{t_2} \wedge C_{t_3}) = \mathbf{T}(C_{t_1}, C_{t_2}) \wedge \mathbf{T}(C_{t_1}, C_{t_3}).$$

Thus,

$$\begin{aligned}T_x(t_1, t_2 \wedge t_3) &= \mathbf{T}(C_{t_1}, C_{t_2} \wedge C_{t_3})(x)\\ &= \mathbf{T}(C_{t_1}, C_{t_2})(x) \wedge \mathbf{T}(C_{t_1}, C_{t_3})(x)\\ &= T_x(t_1, t_2) \wedge T_x(t_1, t_3)\end{aligned}$$

for any $x \in X$. The converse implication is straightforward. The distributivity with respect to $\vee$ can be proved analogously. ■

Corollary 11: If L is a totally ordered set then every representable t-norm on $\mathrm{Map}(X, L)$ is $\wedge$-distributive and $\vee$-distributive.

Proof: It follows directly from Proposition 10, since each t-norm on a totally ordered set is $\wedge$-distributive and $\vee$-distributive. ■

Proposition 12: Let L be a bounded lattice and X be an arbitrary set. Let $\beta = \{T_x\}_{x\in X}$ be a family of t-norms on L indexed in X and $\mathbf{T} = \Phi(\beta)$, where Φ is defined as in (2). The following statements hold.

a) A map $f \in \mathrm{Map}(X, L)$ is idempotent for $\mathbf{T}$ if and only if $f(x)$ is idempotent for T_x for any $x \in X$.

b) A map $f \in \mathrm{Map}(X, L)$ is n-nilpotent for $\mathbf{T}$ if and only if $f(x)$ is n-nilpotent for T_x for any $x \in X$.

c) If there exists $x \in X$ such that $f(x)$ is a zero divisor for T_x, then f is a zero divisor for $\mathbf{T}$.

Proof: $a)$ By definition, f is idempotent for $\mathbf{T}$ if and only if $\mathbf{T}(f,f)=f$ if and only if, for any $x\in X$,

$$\mathbf{T}(f,f)(x)=T_x(f(x),f(x))=f(x).$$

$b)$ can be proved similarly. Finally, in order to prove $c)$, if there exists certain $x_0\in X$ such that $f(x_0)$ is a nontrivial zero divisor for T_{x_0}, then there exists a nonzero $y_0\in L$ such that $T_{x_0}(f(x_0),y_0)=0$. We define $g:X\to L$ by $g(x)=y_0$ if $x=x_0$, and $g(x)=0$, otherwise, then

$$\mathbf{T}(f,g)(x_0)=T_{x_0}(f(x_0),g(x_0))=T_{x_0}(f(x_0),y_0)=0$$

and

$$\mathbf{T}(f,g)(x)=T_x(f(x),g(x))=T_x(f(x),0)=0$$

if $x\neq x_0$. So, $\mathbf{T}(f,g)=\mathbf{0}$. ∎

A. Application to fuzzy sets on the real unit square

The real unit square $[0,1]^2$ is bijective with $\mathrm{Map}(\mathbb{2},[0,1])$, where $\mathbb{2}=\{0,1\}$ is the two-element boolean algebra (viewed simply as a set), via the identification which maps $f\in\mathrm{Map}(\mathbb{2},[0,1])$ to the pair $(f(0),f(1))$. Actually, the product order on $[0,1]^2$ considered in [19] coincides via this identification with the order induced on $\mathrm{Map}(\mathbb{2},[0,1])$, so they are isomorphic as bounded lattices. Formula (1) gives in this case the map

$$\Phi:\mathrm{Map}(\mathbb{2},\mathcal{T}([0,1]))\longrightarrow\mathcal{T}(\mathrm{Map}(\mathbb{2},[0,1]))\qquad(5)$$

with associates to every pair of t-norms on $[0,1]$, T_0 and T_1, a t-norm $\mathbf{T}$ on $[0,1]^2$. Indeed, if $f,g\in\mathrm{Map}(\mathbb{2},[0,1])$ with $a=f(0)$, $b=f(1)$, $c=g(0)$ and $d=g(1)$, then $\mathbf{T}(f,g)(0)=T_0(a,c)$ and $\mathbf{T}(f,g)(1)=T_1(b,d)$, and therefore $\mathbf{T}((a,b),(c,d))=(T_0(a,c),T_1(b,d))$. So, the representable t-norms on $[0,1]^2$ are the product of two t-norms on $[0,1]$. As a consequence, following [19, Theorem 3.1], or [9, Theorem 7.1], being representable in $[0,1]^2$ is also equivalent to the fact that its partial mappings are meet-morphisms (or, equivalently, join-morphisms).

B. Application to type-2 fuzzy sets

We recall from [28] that a type-2 fuzzy set over X is a mapping $A:X\to[0,1]^{[0,1]}$. Therefore, type-2 fuzzy sets can be seen as L-fuzzy sets for the lattice $L=[0,1]^{[0,1]}=\mathrm{Map}([0,1],[0,1])$. It is of interest then to study t-norms on $\mathrm{Map}([0,1],[0,1])$ (see [5] or [29]). By means of the map Φ in (2), here we provide a t-norm that cannot be obtained by convolution as in [5], that uses the well-known Zadeh's extension principle.

Consider the family of Mayor-Torrens t-norms on $[0,1]$, see [13], $\{T^{MT}_\lambda\}_{\lambda\in[0,1]}$ given by

$$T^{MT}_\lambda(x,y)=\begin{cases}\max\{x+y-\lambda,0\}, & \lambda\in]0,1],\ x,y\in[0,\lambda]\\ \min\{x,y\}, & \text{otherwise.}\end{cases}$$

Then, by using the construction in (1), we obtain the t-norm $\mathbf{T}$ defined as

$$\mathbf{T}^{MT}(f,g)(x)=\begin{cases}\max\{f(x)+g(x)-x,0\},\\ \qquad\text{if } x\in]0,1],\ f(x),g(x)\in[0,x],\\ \min\{f(x),g(x)\},\ \text{otherwise.}\end{cases}$$

Observe that, from [5], a t-norm $\overline{T}$ constructed by convolution from a t-norm T in $[0,1]$ must verify that, for constant maps C_a and C_b with $a,b\in[0,1]$, $\overline{T}(C_a,C_b)(x)=a\wedge b$ for all $x\in[0,1]$. Nevertheless, this does not hold for $\mathbf{T}^{MT}$. For example, set $a=0.5$, $b=0.6$ and $x=1$. Then $\mathbf{T}^{MT}(C_{0.5},C_{0.6})(1)=0.1\neq0.5$.

IV. Monotone families of t-norms

Let us now suppose that X is also a partially ordered set. We may then consider the set $\mathrm{Hom}(X,L)\subseteq\mathrm{Map}(X,L)$ of all poset homomorphisms from X to L, i.e. the monotone maps from X to L. Observe that the supremum and infimum of two monotone maps are also monotone, and the constant maps $\mathbf{0}$ and $\mathbf{1}$ are monotone. Consequently, $\mathrm{Hom}(X,L)$ is a sublattice of $\mathrm{Map}(X,L)$ with the usual lattice structure.

We recall that, by Lemma 1, every t-norm T on $\mathrm{Map}(X,L)$ that is closed in $\mathrm{Hom}(X,L)$ gives a t-norm on $\mathrm{Hom}(X,L)$. Let us prove that this remains valid when considering an induced t-norm.

Proposition 13: Let L be a bounded lattice, X a partially ordered set, $\beta=\{T_x\}_{x\in X}$ a family of t-norms on L indexed in X and $\mathbf{T}=\Phi(\beta)$ the induced t-norm on $\mathrm{Map}(X,L)$. The following statements are equivalent.

a) The family β is monotone, that is, $x\leq y$ implies $T_x\leq T_y$ for any $x,y\in X$.

b) $\mathbf{T}$ is a t-norm on $\mathrm{Hom}(X,L)$.

Proof: $a)\Rightarrow b)$. By Proposition 2, $\mathbf{T}$ is a t-norm on $\mathrm{Map}(X,L)$, so we only need to prove that $\mathbf{T}(f,g)$ is a monotone map from X to L, whenever f and g are so. Let $x,y\in X$ with $x\leq y$, then $f(x)\leq f(y)$ and $g(x)\leq g(y)$. Now,

$$\begin{aligned}\mathbf{T}(f,g)(x)&=T_x(f(x),g(x))\\&\leq T_x(f(y),g(y))\\&\leq T_y(f(y),g(y))\\&=\mathbf{T}(f,g)(y),\end{aligned}$$

so we are done.

$b)\Rightarrow a)$. Let $x,y\in X$ with $x\leq y$ and $t_1,t_2\in L$, then

$$T_x(t_1,t_2)=\mathbf{T}(C_{t_1},C_{t_2})(x)\leq\mathbf{T}(C_{t_1},C_{t_2})(y)=T_y(t_1,t_2)$$

and thus $T_x\leq T_y$. ∎

Proposition 13 yields that a map β is in $\mathrm{Hom}(X,\mathcal{T}(L))$ if and only if $\Phi(\beta)$ is in $\mathcal{T}(\mathrm{Hom}(X,L))$. Therefore, the restriction of Φ to $\mathrm{Hom}(X,\mathcal{T}(L))$ provides a map

$$\bar{\Phi}:\mathrm{Hom}(X,\mathcal{T}(L))\longrightarrow\mathcal{T}(\mathrm{Hom}(X,L))\qquad(6)$$

Hence, the results described in this section are also valid whenever we consider the set of monotone maps. For instance, we may apply the theory to the class of all interval-valued fuzzy sets.

A. Illustrative toy application

A publisher handles five scientific journals within certain area of knowledge, say J_1, J_2, J_3, J_4 and J_5. The journals are inversely ordered by the handicap of publishing a manuscript,

This is the author's version of an article that has been published in this journal. Changes were made to this version by the publisher prior to publication.
The final version of record is available at http://dx.doi.org/10.1109/TFUZZ.2020.2985337

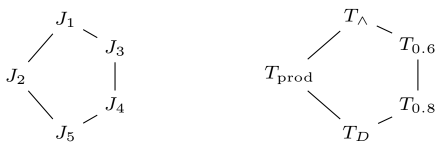

Fig. 1: Hasse diagram of the lattices of journals and t-norms

or their quality, forming a lattice L as described by the Hasse diagram in Figure 1 (left).

That is to say, J_5 is the flagship journal whilst J_1 publishes the lowest quality papers. In this model, J_2 is not comparable with J_3 and J_4, meaning that either cover different aspects of the topic, or, simply, it is not clear the order between them.

These journals share the same editorial board and the authors submit their manuscripts through a common system. Let us design a decision support system in order to choose the most suitable journal (if any) for publishing the paper. For simplicity, we assume that the manuscripts are sent to two reviewers that provide a value in $[0,1]$, meaning their opinion about it. For each journal, the intersection of both opinions are calculated by the following t-norms: $T_5 = T_D$, the drastic t-norm; $T_1 = T_\wedge$, the minimum t-norm; $T_2 = T_{\text{prod}}$, the product t-norm; $T_3 = T_{1.5}$ and $T_4 = T_{1.7}$, where

$$T_\alpha(x,y) = \begin{cases} \min(x,y), & \text{if } x+y \geq \alpha, \\ 0, & \text{otherwise,} \end{cases}$$

for any $\alpha \in [0,1]$ and $x, y \in [0,1]$. Then, these t-norms are also ordered following the lattice L, see Figure 1 (right). By Proposition 13, it is induced a t-norm $\mathbf{T}$ on $\operatorname{Hom}(L,[0,1])$, i.e. the set

$$\left\{ (x_1,x_2,x_3,x_4,x_5) \in [0,1]^5 \mid \begin{array}{c} x_1 \leq x_2 \leq x_5 \\ x_1 \leq x_3 \leq x_4 \leq x_5 \end{array} \right\}.$$

We may now describe the algorithm for deciding the most suitable journal of a submitted manuscript:

1) The reviewers return their opinions $x, y \in [0,1]$ about the manuscript.
2) Compute $\mathbf{T}(/x/,/y/)$, where $/a/ = (a,a,a,a,a)$ for any $a \in [0,1]$
3) Consider $D \subset L$, the elements corresponding to the coordinates that are greater or equal than 0.5.
4) If $D \neq \emptyset$, consider $D_{\min}$ the minimal elements of D. Otherwise, reject the paper.
5) Return a journal J in $D_{\min}$ whose associated coordinate has the greatest value.

For instance, suppose that the reviewers return the values $x = 0.8$ and $y = 0.7$ for a given manuscript. Hence $\mathbf{T}(/0.8/,/0.7/) = (0.7, 0.56, 0.7, 0, 0)$. So that, we may only consider the journals in $D = \{J_1, J_2, J_3\}$. Hence, $D_{\min} = \{J_2, J_3\}$ and the output of the algorithm is J_3.

Alternatively, the reviewers could also submit different values for each manuscript following the Hasse diagram of Figure 1. Observe that the method can be generalized to any measurement of the evaluation of the manuscript provided by the reviewers. For instance, they could provide not a single value but an interval in $[0,1]$, hence we should consider t-norms on interval-valued fuzzy sets, see the following section.

B. Application to interval-valued fuzzy sets

The class $\mathbb{I}$ of all subintervals of the unit interval $[0,1]$ may be endowed with the partial order

$$[a,b] \leq [c,d] \text{ if and only if } a \leq c \text{ and } b \leq d$$

for any interval $[a,b]$ and $[c,d]$. The induced lattice structure is then given by

$$[a,b] \wedge [c,d] = [a \wedge c, b \wedge d] \text{ and } [a,b] \vee [c,d] = [a \vee c, b \vee d]$$

for any $[a,b]$ and $[c,d]$ in $\mathbb{I}$. Therefore, $\mathbf{0} = [0,0]$ and $\mathbf{1} = [1,1]$ are the lowest and the greatest element, respectively. That is, $(\mathbb{I}, \wedge, \vee)$ is a bounded lattice. The literature about the properties and related structures to this lattice is extensive, since the set of maps $X \to \mathbb{I}$ is the well-known class of interval-valued fuzzy sets over the universe X.

Observe first that $\mathbb{I}$ is lattice-isomorphic to the lattice $\operatorname{Hom}(\mathbb{2},[0,1])$, where $\mathbb{2} = \{0,1\}$ is the two-element boolean algebra, via the identification that maps any $f \in \operatorname{Hom}(\mathbb{2},[0,1])$ to the interval $[f(0), f(1)]$ with $f(0) \leq f(1)$. Or, alternatively, it corresponds to the upper triangle of the unit square $[0,1]^2$, as it is explained in Figure 2. By Proposition

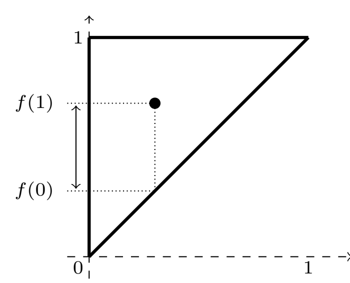

Fig. 2: A lattice-representation of $\mathbb{I}$ in $[0,1]^2$

13, a t-norm $\mathbf{T}$ on $\mathbb{I}$ is representable if and only if there exist two t-norms T_0 and T_1 on $[0,1]$ with $T_0 \leq T_1$ such that

$$\mathbf{T}([a,b],[c,d]) = [T_0(a,c), T_1(b,d)]$$

for any $[a,b],[c,d] \in \mathbb{I}$, see [27, Proposition 5.1]. Furthermore, in this case, by Corollary 11, $\mathbf{T}$ is $\wedge$-distributive and $\vee$-distributive. Actually, we may recover the description of a t-norm being unirepresentable given in [30], and [5] with a slightly different notation. Namely, $\mathbf{T}$ is unirepresentable if and only if there exist a t-norm T on $[0,1]$ such that, for every $[a,b],[c,d] \in \mathbb{I}$, $\mathbf{T}([a,b],[c,d]) = [T(a,c), T(b,d)]$.

The theory also covers the case of n-Dimensional Fuzzy Set (nDFS). We recall from [31] that an nDFS over a universe set X is an L_n-fuzzy set , where L_n is the set of n-dimensional intervals, that is,

$$L_n = \{(a_1,\ldots,a_n) \in [0,1]^n \text{ such that } a_1 \leq \cdots \leq a_n\}.$$

Now, $L_n = \operatorname{Hom}(\mathbb{n},[0,1])$, where $\mathbb{n} = \{1,\ldots,n\}$ the chain with n elements, using the standard order. Therefore, by Proposition 13, a representable t-norm $\mathbf{T}$ on n-dimensional intervals is constructed from n ordered t-norms on $[0,1]$, $T_1 \leq T_2 \leq \cdots \leq T_n$, by the rule $\mathbf{T}((x_i)_i,(y_i)_i) = (T_i(x_i,y_i))_i$, which is exactly [21, Theorem 1].

V. Induced negations

The nature of the process described in the latter section allows to explore the method for other structural maps. This section aims then to develop the theory of induced negations. Similar results in this sense may be found in [6], [32] for intervals on $[0,1]$ and [22] for n-dimensional intervals, see Section V-A for some details.

Let L be a bounded lattice and X an arbitrary set. Let $\{N_x\}_{x\in X}$ be a family of negations on L indexed in X, and $\sigma : X \longrightarrow X$ a map. We may consider the map

$$\mathbf{N}^\sigma : \mathrm{Map}(X,L) \longrightarrow \mathrm{Map}(X,L) \tag{7}$$

given by $\mathbf{N}^\sigma(f)(x) = N_x(f(\sigma(x)))$ for any $f \in \mathrm{Map}(X,L)$ and any $x \in X$.

Lemma 14: Let L be a bounded lattice, X an arbitrary set and $\sigma : X \longrightarrow X$ a map. If $\{N_x\}_{x\in X}$ is a family of negations on L indexed in X, then the map $\mathbf{N}^\sigma$ defined in (7) is a negation on $\mathrm{Map}(X,L)$.

Proof: Observe $\mathbf{N}^\sigma(\mathbf{0})(x) = N_x(\mathbf{0}(\sigma(x))) = N_x(0) = 1 = \mathbf{1}(x)$, for any $x \in X$, and therefore $\mathbf{N}^\sigma(\mathbf{0}) = \mathbf{1}$. $\mathbf{N}^\sigma(\mathbf{1}) = \mathbf{0}$ can be proved similarly. Now, let $f,g \in \mathrm{Map}(X,L)$ with $f \le g$. Then, for any $x \in X$, $f(\sigma(x)) \le g(\sigma(x))$, so $N_x(f(\sigma(x))) \ge N_x(g(\sigma(x)))$. Then $\mathbf{N}^\sigma(f)(x) \ge \mathbf{N}^\sigma(g)(x)$ for any $x \in X$. Thus $\mathbf{N}^\sigma(f) \ge \mathbf{N}^\sigma(g)$. ∎

Analogously to those explained in Section III, we may see a family $\{N_x\}_{x\in X}$ of negations on L as a set map $\beta : X \to \mathcal{N}(L)$, mapping each $x \in X$ to N_x, and we use both descriptions without distinction. Therefore, for any map $\sigma : X \to X$, by virtue of Lemma 14, we may consider

$$\Psi_\sigma : \mathrm{Map}(X,\mathcal{N}(L)) \to \mathcal{N}(\mathrm{Map}(X,L))$$

given by $\Psi_\sigma(\beta) = \mathbf{N}^\sigma$, defined as in (7). Or, by using the powerset notation, Ψ_σ can be rewritten as

$$\Psi_\sigma : \mathcal{N}(L)^X \to \mathcal{N}(L^X)$$

showing clearer the relation between these two sets.

Definition 15: Let X be a set, $\sigma : X \to X$ a map and L a bounded lattice. A negation $\mathbf{N}$ on $\mathrm{Map}(X,L)$ is said to be *σ-representable* if there exists a family $\beta = \{N_x\}_{x\in X}$ of negations on L indexed in X such that $\Psi_\sigma(\beta) = \mathbf{N}$. Additionally, if $N_x = N_y$ for any $x,y \in X$, $\mathbf{N}$ is said to be σ-unirepresentable.

Example 16 (Representable negations and characteristic maps): Analogously to Example 7, representable negations act over characteristic maps very closely to the standard negation on $[0,1]$. Let X be a set, L a bounded lattice and $Y \subseteq X$ and χ_Y its characteristic map. Let us suppose that $\mathbf{N}$ is a σ-representable negation for some map σ with $\Psi_\sigma(\{N_x\}_{x\in X}) = \mathbf{N}$. Then

$$\begin{aligned}\mathbf{N}(\chi_Y)(x) &= N_x(\chi_Y(\sigma(x)))\\ &= \begin{cases} N_x(1), & \text{if } \sigma(x) \in Y\\ N_x(0), & \text{if } \sigma(x) \notin Y\end{cases}\\ &= \begin{cases} 0, & \text{if } \sigma(x) \in Y\\ 1, & \text{if } \sigma(x) \notin Y\end{cases}\\ &= (1 - \chi_Y \circ \sigma)(x)\end{aligned}$$

for any $x \in X$. So $\mathbf{N}(\chi_Y) = 1 - \chi_Y \circ \sigma$.

The following result shows that the property of being representable can be characterized in terms of the constant maps. As usual in this paper, for any $a \in L$, we denote by C_a the constant map from X to L whose image is a.

Proposition 17: Let X be a set, $\sigma : X \to X$ a map and L a bounded lattice. A negation $\mathbf{N}$ in $\mathrm{Map}(X,L)$ is σ-representable if and only if

$$\mathbf{N}(f)(x) = \mathbf{N}(C_{f\sigma(x)})(x)$$

for any $f \in \mathrm{Map}(X,L)$ and any $x \in X$. As a consequence, if $\mathbf{N}$ is σ-representable, then it is σ-unirepresentable if and only if, for any $a \in L$, $\mathbf{N}(C_a)$ is a constant map.

Proof: Let us assume that $\mathbf{N}$ is σ-representable. Then there exists a family of negations $\beta = \{N_x\}_{x\in X}$ in L such that $\mathbf{N}(f)(x) = N_x(f(\sigma(x)))$ for any $f \in \mathrm{Map}(X,L)$ and any $x \in X$. Hence

$$N_x(f(\sigma(x))) = N_x(C_{f\sigma(x)}(\sigma(x))) = \mathbf{N}(C_{f\sigma(x)})(x).$$

Conversely, for any $x \in X$, let us define the map $N_x : L \to L$ given by $N_x(a) = \mathbf{N}(C_a)(x)$ for any $a \in L$. Firstly, observe that

$$N_x(0) = \mathbf{N}(C_0)(x) = \mathbf{N}(\mathbf{0})(x) = \mathbf{1}(x) = 1, \text{ and}$$

$$N_x(1) = \mathbf{N}(C_1)(x) = \mathbf{N}(\mathbf{1})(x) = \mathbf{0}(x) = 0.$$

Now, if $a,b \in L$ with $a \le b$, then $C_a \le C_b$ and then $\mathbf{N}(C_a) \ge \mathbf{N}(C_b)$. So

$$N_x(a) = \mathbf{N}(C_a)(x) \ge \mathbf{N}(C_b)(x) = N_x(b).$$

Thus $\beta = \{N_x\}_{x\in X}$ is a family of negations on L. Denote $\Psi_\sigma(\beta)$ by $\widehat{N}$. Then, for any $f \in \mathrm{Map}(X,L)$ and any $x \in X$, by hypothesis,

$$\widehat{N}(f)(x) = N_x(f\sigma(x)) = \mathbf{N}(C_{f\sigma(x)})(x) = \mathbf{N}(f)(x).$$

Thus $\mathbf{N} = \widehat{N}$.

For the last statement, if $\mathbf{N}$ is σ-unirepresentable then $\mathbf{N}(C_a)(x) = N(C_a(\sigma(x))) = N(a)$ for some negation N in L, so $\mathbf{N}(C_a)$ is constant for any $a \in L$. Conversely, for any $a \in L$ and any $x,y \in X$, $\mathbf{N}(C_a)(x) = \mathbf{N}(C_a)(y)$. Then $N_x(C_a(\sigma(x))) = N_y(C_a(\sigma(y)))$, so $N_x(a) = N_y(a)$. Thus $N_x = N_y$. ∎

Theorem 18: Let X be a set, $\sigma : X \to X$ a map and L a bounded lattice. Let $\beta = \{N_x\}_{x\in X}$ be a set of negations on L indexed in X. Then $\Psi_\sigma(\beta) = \mathbf{N}^\sigma$ is an involutive negation if and only if the following properties hold:

1) $N_x \circ N_{\sigma(x)} = \mathrm{id}_L$ for any $x \in X$.
2) σ is involutive, that is, $\sigma^2 = \mathrm{id}_X$.

Proof: Observe that, for any $f \in \mathrm{Map}(X,L)$ and $x \in X$,

$$\begin{aligned}\mathbf{N}^\sigma(\mathbf{N}^\sigma(f))(x) &= N_x(\mathbf{N}^\sigma(f)(\sigma(x)))\\ &= N_x \circ N_{\sigma(x)}(f(\sigma^2(x))).\end{aligned} \tag{8}$$

Then $\mathbf{N}^\sigma$ is an involutive negation on $\mathrm{Map}(X,L)$ if and only if $N_x \circ N_{\sigma(x)}(f(\sigma^2(x))) = f(x)$ for any $f \in \mathrm{Map}(X,L)$ and any $x \in X$. From (8), it is clear that, if the properties 1) and 2) hold, then $\mathbf{N}^\sigma$ is involutive.

Conversely, let $a \in L$ and C_a the constant map in $\mathrm{Map}(X, L)$, then $\mathbf{N}^\sigma(\mathbf{N}^\sigma(C_a))(x) = C_a(x) = a$ and, on the other hand, by (8),

$$\mathbf{N}^\sigma(\mathbf{N}^\sigma(C_a))(x) = N_x \circ N_{\sigma(x)}(C_a(\sigma^2(x))) = N_x \circ N_{\sigma(x)}(a),$$

so that, $N_x \circ N_{\sigma(x)} = \mathrm{id}_L$. Hence, by (8), $f(\sigma^2(x)) = f(x)$ for every $f \in \mathrm{Map}(X, L)$ and any $x \in X$. For every $x \in X$, let us then take $f_x : X \to L$ defined by

$$f_x(y) = \begin{cases} 0, & \text{if } y = x, \\ 1, & \text{otherwise.} \end{cases}$$

Hence $f_x(\sigma^2(x)) = f_x(x) = 0$ and, thus $\sigma^2(x) = x$. ∎

Corollary 19: Let σ be the identity map in X, $\mathbf{N}^\sigma$ is involutive if and only if N_x is involutive for each $x \in X$.

Example 20: Let us consider $L = [0, 1]$ with standard lattice structure, and the negations $N_0, N_1, N_2 : [0, 1] \to [0, 1]$ given by $N_0(x) = 1 - x$, $N_1(x) = \sqrt{1 - x}$, $N_2(x) = 1 - x^2$ for any $x \in [0, 1]$. Observe that N_0 is involutive, but $N_1^{-1} = N_2$. Let $X = [0, 1]$ and $\beta : X \to \{N_0, N_1, N_2\}$ the map defined by

$$\beta(x) = \begin{cases} N_1, & \text{if } x < 1/2, \\ N_0, & \text{if } x = 1/2, \\ N_2, & \text{if } x > 1/2, \end{cases}$$

for any $x \in [0, 1]$, and $\sigma : X \to X$ given by $\sigma(x) = 1 - x$. By Theorem 18, the induced negation $\mathbf{N}^\sigma$ in $\mathrm{Map}([0, 1], [0, 1])$ is involutive. Actually,

$$\mathbf{N}^\sigma(f)(x) = \begin{cases} \sqrt{1 - f(1 - x)}, & \text{if } x < 1/2, \\ 1 - f(1 - x), & \text{if } x = 1/2, \\ 1 - f(1 - x)^2, & \text{if } x > 1/2, \end{cases}$$

for any $f \in \mathrm{Map}([0, 1], [0, 1])$ and any $x \in [0, 1]$. Observe, as well, that for a constant map C_a for some $a \in [0, 1]$,

$$\mathbf{N}^\sigma(C_a)(x) = \begin{cases} \sqrt{1 - a}, & \text{if } x < 1/2, \\ 1 - a, & \text{if } x = 1/2, \\ 1 - a^2, & \text{if } x > 1/2, \end{cases}$$

is not constant, so, by virtue of Proposition 17, $\mathbf{N}^\sigma$ is not σ-unirepresentable, as expected.

Let us now suppose, additionally, that X has an ordered structure. Hence, under suitable conditions, the map (7) may provide a negation not only in $\mathrm{Map}(X, L)$, but also in $\mathrm{Hom}(X, L)$. That is to say, we may restrict (7) to a map

$$\mathbf{N}^\sigma : \mathrm{Hom}(X, L) \longrightarrow \mathrm{Hom}(X, L) \qquad (9)$$

Lemma 21: Let X and L be bounded lattices. If $\{N_x\}_{x \in X}$ is a monotone family of negations on L, that is $N_x \leq N_y$ if $x \leq y$, and σ is a decreasing map on X, then $\mathbf{N}^\sigma(\mathrm{Hom}(X, L)) \subseteq \mathrm{Hom}(X, L)$ and $\mathbf{N}^\sigma$ is a negation on $\mathrm{Hom}(X, L)$.

Proof: By Lemma 1, it suffices to prove that $\mathbf{N}^\sigma(\mathrm{Hom}(X, L)) \subseteq \mathrm{Hom}(X, L)$. Let $f \in \mathrm{Hom}(X, L)$ and $x \leq y \in X$. So $\sigma(x) \geq \sigma(y)$ and $f(\sigma(x)) \geq f(\sigma(y))$. Hence

$$\begin{aligned} \mathbf{N}^\sigma(f)(x) = N_x(f(\sigma(x))) &\leq N_x(f(\sigma(y))) \\ &\leq N_y(f(\sigma(y))) = \mathbf{N}^\sigma(f)(y), \end{aligned}$$

so $\mathbf{N}^\sigma(f)$ is an homomorphism of lattices. ∎

As a consequence of Theorem 18, the condition of being involutive for an induced negation on $\mathrm{Hom}(X, L)$ becomes very restrictive. We shall need the following technical result.

Lemma 22: Let N_1 and N_2 be negations on L such that $N_1 \leq N_2$ and $N_1^{-1} = N_2$, then N_1 is involutive. As a consequence, $N_1 = N_2$.

Proof: Let $a \in L$, by hypothesis, $N_1(a) \leq N_2(a)$. Then, by applying the negation N_1, $(N_1 \circ N_1)(a) \geq (N_1 \circ N_2)(a) = a$. On the other hand, $N_1(N_1(a)) \leq N_2(N_1(a)) = a$. Thus $N_1^2 = \mathrm{id}_L$. Now, since N_1 and N_2 are both inverses of N_1, $N_1 = N_2$. ∎

Corollary 23: Let X and L be bounded lattices, $\sigma : X \to X$ a decreasing map and $\beta = \{N_x\}_{x \in X}$ a monotone family of negations on L indexed in X. Then $\Psi_\sigma(\beta)$ is an involutive negation on $\mathrm{Hom}(X, L)$ if and only if

1) N_0 is involutive and, for each $x \in X$, $N_x = N_0$.
2) σ is involutive.

Consequently, a σ-representable involutive negation on $\mathrm{Hom}(X, L)$ is σ-unirepresentable.

Proof: Let us denote $\Psi_\sigma(\beta)$ by $\mathbf{N}$. If $\mathbf{N}$ is an involutive negation, by Theorem 18, $N_0 \circ N_{\sigma(0)} = \mathrm{id}_L$ and σ is involutive. Since, by monotonicity, $N_0 \leq N_{\sigma(0)}$, by Corollary 23, N_0 is involutive and $N_0 = N_{\sigma(0)}$. Now, for any $x \in X$, $0 \leq \sigma(x)$ and then, since σ is decreasing, $x = \sigma^2(x) \leq \sigma(0)$. So $N_0 \leq N_x = N_{\sigma^2(x)} \leq N_{\sigma(0)} = N_0$. Thus $N_0 = N_x$ for any $x \in X$. The converse implication follows directly from Theorem 18. ∎

A. Application to interval-valued fuzzy sets

We now deal with induced negations on the class all subintervals of the unit interval $\mathbb{I} = \mathrm{Hom}(\mathbb{2}, [0, 1])$, see Section IV-B, where $\mathbb{2}$ is the two-element boolean algebra. We firstly show the map described in (7) could not be a negation, if the map σ is not decreasing.

Example 24: Let $\sigma : \mathbb{2} \to \mathbb{2}$ be the identity map, and $\mathbf{n} : [0, 1] \to [0, 1]$ the negation defined as $\mathbf{n}(x) = 1 - x$ for any $x \in [0, 1]$. Let $\{N_0, N_1\}$ be the family of negations on $[0, 1]$ indexed in $\mathbb{2}$, where $N_0 = N_1 = \mathbf{n}$. Hence the induced map $\mathbf{N} = \Psi_\sigma(\{N_0, N_1\})$ is not a negation on $\mathbb{I}$. Indeed, given an interval $[a, b] \in \mathbb{I}$,

$$\mathbf{N}([a, b]) = [N_0(a), N_1(b)] = [1 - a, 1 - b],$$

which need not to be an interval.

In the lattice $\mathbb{2}$ there are only three decreasing maps: the map $\mathbf{n}$ defined as $\mathbf{n}(0) = 1$ and $\mathbf{n}(1) = 0$, which is, in addition, an involutive negation, and the constant maps $\mathbf{0}$ and $\mathbf{1}$.

Lemma 25: Let N_1 and N_2 be negations on $[0, 1]$ with $N_1 \leq N_2$, then the maps $\mathbf{N}^\mathbf{0}, \mathbf{N}^\mathbf{1}, \mathbf{N}^\mathbf{n} : \mathbb{I} \to \mathbb{I}$ defined by

$$\mathbf{N}^\mathbf{0}([a, b]) = [N_1(a), N_2(a)], \mathbf{N}^\mathbf{1}([a, b]) = [N_1(b), N_2(b)]$$
$$\text{and } \mathbf{N}^\mathbf{n}([a, b]) = [N_1(b), N_2(a)],$$

for any interval $[a, b] \in \mathbb{I}$, are negations on $\mathbb{I}$. Furthermore, $\mathbf{N}^\mathbf{0}$ and $\mathbf{N}^\mathbf{1}$ are not involutive, and $\mathbf{N}^\mathbf{n}$ is involutive if and only if N_1 is involutive and $N_1 = N_2$.

Proof: It follows by applying Lemma 21 for the decreasing maps $\mathbf{0}, \mathbf{1}$ and $\mathbf{n}$. The last statement follows from Corollary 23. ∎

Let us focus now on the conditions needed by a negation $\mathbf{N}$ on $\mathbb{I}$ for being σ-representable. Firstly, we fix some notation.

Let $[a,b] \in \mathbb{I}$, we denote by $\underline{[a,b]}$ and $\overline{[a,b]}$ the lower and upper bound of the interval $[a,b]$, that is, $\underline{[a,b]} = a$ and $\overline{[a,b]} = b$. Given a negation $\mathbf{N}$ on $\mathbb{I}$ we consider the maps $N_{(1)}, N_{(2)} : [0,1] \to [0,1]$ defined by

$$N_{(1)}(a) = \underline{\mathbf{N}([a,a])} \text{ and } N_{(2)}(a) = \overline{\mathbf{N}([a,a])}.$$

Both maps are negations on [0,1]. Indeed, clearly, for $i = 1,2$, $N_{(i)}(0) = 1$ and $N_{(i)}(1) = 0$, and, given $a,b \in [0,1]$ with $a \le b$, $[a,a] \le [b,b]$ and then $\mathbf{N}([a,a]) \ge \mathbf{N}([b,b])$ so $N_{(i)}(a) \ge N_{(i)}(b)$. The point here is when and how the negations $N_{(i)}$ on $[0,1]$ allow us to recover $\mathbf{N}$.

Lemma 26: Let $\mathbf{N} : \mathbb{I} \to \mathbb{I}$ be a negation on $\mathbb{I}$. Then

a) $\mathbf{N}$ is $\mathbf{0}$-representable if and only if $\mathbf{N}([a,a]) = \mathbf{N}([a,1])$ for every $a \in [0,1]$.

b) $\mathbf{N}$ is $\mathbf{1}$-representable if and only if $\mathbf{N}([b,b]) = \mathbf{N}([0,b])$ for every $b \in [0,1]$.

c) $\mathbf{N}$ is $\mathbf{n}$-representable if and only if $\underline{\mathbf{N}([b,b])} = \underline{\mathbf{N}([0,b])}$ and $\overline{\mathbf{N}([a,a])} = \overline{\mathbf{N}([a,1])}$ for every $a,b \in [0,1]$.

Proof: By Proposition 17, $\mathbf{N}$ is $\mathbf{0}$-representable if and only if $\mathbf{N}([a,b]) = \mathbf{N}([a,a])$ for any interval $[a,b]$ in $\mathbb{I}$. So that, if $\mathbf{N}$ is $\mathbf{0}$-representable, then $\mathbf{N}([a,a]) = \mathbf{N}([a,1])$ for every $a \in [0,1]$. Conversely, since $[a,a] \le [a,b] \le [a,1]$ for any interval $[a,b]$, then

$$\mathbf{N}([a,1]) = \mathbf{N}([a,a]) \ge \mathbf{N}([a,b]) \ge \mathbf{N}([a,1]).$$

b) and *c*) may be proved similarly. ■

As a consequence of Lemma 26, we may obtain [6, Proposition 4.1].

Corollary 27: [6, Proposition 4.1] Let $\mathbf{N} : \mathbb{I} \to \mathbb{I}$ be a negation on $\mathbb{I}$ verifying that $X \subseteq Y$ implies $\mathbf{N}(X) \subseteq \mathbf{N}(Y)$ for any intervals $X, Y \in \mathbb{I}$. Then $\mathbf{N}$ is $\mathbf{n}$-representable.

Proof: For any $b \in [0,1]$, $[0,b] \le [b,b]$ so $\mathbf{N}([0,b]) \ge \mathbf{N}([b,b])$ and then $\underline{\mathbf{N}([0,b])} \ge \underline{\mathbf{N}([b,b])}$. On the other hand, $[b,b] \subseteq [0,b]$ so, by hypothesis, $\mathbf{N}([b,b]) \subseteq \mathbf{N}([0,b])$ and then $\underline{\mathbf{N}([b,b])} \ge \underline{\mathbf{N}([0,b])}$ which yields the equality. Similarly, we may prove that $\overline{\mathbf{N}([a,a])} = \overline{\mathbf{N}([a,1])}$ for every $a,b \in [0,1]$. Then, by Lemma 26, the result follows. ■

Example 28: We show an example of a negation on $\mathbb{I}$ which is not σ-representable for any σ. Let $\mathbf{N} : \mathbb{I} \longrightarrow \mathbb{I}$ given by

$$\mathbf{N}([a,b]) = \begin{cases} [1,1], & \text{if } a < 0.5, \\ [1-b, 1-b], & \text{otherwise.} \end{cases}$$

This map is a negation on $\mathbb{I}$. Nevertheless, by Lemma 26, $\mathbf{N}$ is neither $\mathbf{0}$, nor $\mathbf{1}$, nor n-representable. Observe that $\mathbf{N}([1/2,1/2]) = [1/2,1/2]$, $\mathbf{N}([0,1/2]) = [1,1]$, $\mathbf{N}([2/3,2/3]) = [1/3,1/3]$ and $\mathbf{N}([2/3,1]) = [0,0]$.

Finally, we may recover [32, Corollary 3.4] as a consequence of Corollary 23. [32, Corollary 3.4] is written by using intuitionistic terminology, so we need a suitable translation. Each couple (x_1, x_2), as intuitionistic menbership couple (that is, $x_1, x_2 \in [0,1]$, $x_1 + x_2 \le 1$), corresponds with the interval $[x_1, 1-x_2] \subseteq [0,1]$. Conversely, each interval $[a,b] \in \mathbb{I}$ provides the intuitionistic menbership couple $[a, 1-b]$. Thus, the notation $pr_1(x_1,x_2)$ corresponds with $\underline{[x_1, 1-x_2]}$ and $pr_2(x_1,x_2)$, with $\overline{[x_1, 1-x_2]}$.

Corollary 29: [32, Corollary 3.4] Let $\mathbf{N} : \mathbb{I} \to \mathbb{I}$ be a negation on $\mathbb{I}$. $\mathbf{N}$ is involutive if and only if there exists an involutive negation N_1 in $[0,1]$ such that $\mathbf{N}([a,b]) = [N_1(b), N_1(a)]$ for every $[a,b] \in \mathbb{I}$.

Proof: Since the only decreasing map in $\mathbb{2}$ is $\mathbf{n}$, the statement is a direct consequence of Corollary 23. ■

We may apply Lemma 14 in the realm of n dimensional intervals. Then, an ordered negation $\mathbf{N}$ on L_n can be constructed from n ordered negations $N_1, \ldots, N_n$ on $[0,1]$ and a decreasing map $\sigma : \mathbb{n} \to \mathbb{n}$. Concretely,

$$\mathbf{N}(x_1, \ldots, x_n) = (N_1(x_{\sigma(1)}), \ldots, N_n(x_{\sigma(n)})).$$

Then, in this setting, Lemma 14 covers [22, Proposition 3.1] since, in [22], it is fixed the map $\sigma : \mathbb{n} \to \mathbb{n}$ given by $\sigma(i) = n+1-i$ for any $i = 1, \ldots, n$. Other maps could be considered in order to design negations in L_n. For instance, simply set $\sigma(i) = n+1-i$ if $i \ne 2$ and $\sigma(2) = n$. Unfortunately, as commented above, the property of being involutive becomes quite restrictive. In this case, [22, Theorem 3.3] is exactly Corollary 23, since no different map σ can be considered.

B. Illustrative example

Related to the latter paragraph, our theory allows us to extend [22, Proposition 3.1 and Theorem 3.3] to arbitrary bounded lattice. For instance, let us consider $X = \{0, a, b, 1\}$ the lattice with Hasse diagram in Figure 3. Set $\sigma : X \to X$

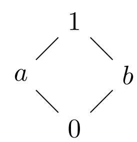

Fig. 3: Hasse diagram of the lattice X

as $\sigma(0) = 1$, $\sigma(1) = 0$, $\sigma(a) = a$ and $\sigma(b) = b$. Hence σ is an involutive negation on X, and we may consider the σ-unirepresentable negation $\mathbf{N} : L \to L$, where $L = [0,1]^X$, provided by the standard negation N on $[0,1]$. That is,

$$\mathbf{N}(x_0, x_a, x_b, x_1) = (1-x_1, 1-x_a, 1-x_b, 1-x_0).$$

By virtue of Corollary 23, $\mathbf{N}$ is an involutive negation on

$$L = \{(x_0, x_a, x_b, x_1) \in [0,1]^4 \text{ with } x_0 \le x_a, x_b \le x_1\}.$$

A possible application to multi-expert decision-making is provided in [22, Section V]. By using our theory, an interesting problem is to develop similar reasonings whenever the lattice under consideration is not $\mathbb{n}$, the chain with n elements. Obviously, the technical details are out of the scope of this paper and this problem is proposed for future work.

VI. Induced De Morgan triples

We finish this paper dealing with the development of De Morgan triples by the induction process. Let then $\{(T_x, N_x)\}_{x \in X}$ be a family of pairs indexed in a set X, where, for each $x \in X$, T_x is a t-norm on a bounded lattice L and N_x is an involutive negation on L. For each $x \in X$, we may associate a map $S_x : L \times L \to L$ defined as $S_x(a,b) = N_x(T_x(N_x(a)), N_x(b)))$ for any $a, b \in L$. Or, equivalently, since N_x is involutive, $N_x(S_x(a,b)) = T_x(N_x(a)), N_x(b))$ for any $a, b \in L$. Obviously, S_x is the dual t-conorm of the pair (T_x, N_x), so (T_x, N_x, S_x) is a De Morgan triple. We may then consider the induced t-norm $\mathbf{T}$ and t-conorm $\mathbf{S}$ from the families $\{T_x\}_{x\in X}$ and $\{S_x\}_{x\in X}$, respectively.

Let now $\sigma : X \to X$ be a map such that the negation induced from the family $\{N_x\}_{x\in X}$, say $\mathbf{N}^\sigma$, is involutive. Then, by Theorem 18, $N_x \circ N_{\sigma(x)} = \mathrm{id}_L$ and $\sigma^2 = id_X$ for any $x \in X$. Moreover, since we have assumed N_x to be involutive, hence $N_x = N_{\sigma(x)}$ for any $x \in X$. We may then guess if the triple $(\mathbf{T}, \mathbf{N}^\sigma, \mathbf{S})$ is a De Morgan triple, as well.

Theorem 30: Let L be a bounded lattice, X an arbitrary set, $\sigma : X \to X$ a map and $\{\mathcal{D}_x = (S_x, N_x, T_x)\}_{x\in X}$ a family of De Morgan triples on L indexed in X such that $\mathbf{N}^\sigma = \Psi_\sigma(\{N_x\}_{x\in X})$ is an involutive negation on $\mathrm{Map}(X, L)$. Let $\mathbf{T} = \Phi(\{T_x\}_{x\in X})$ and $\mathbf{S} = \Xi(\{S_x\}_{x\in X})$. Then $\mathbf{D} = (\mathbf{S}, \mathbf{N}^\sigma, \mathbf{T})$ is a De Morgan triple on $\mathrm{Map}(X, L)$ if and only if $T_{\sigma(x)} = T_x$ or, equivalently, $S_{\sigma(x)} = S_x$, for any $x \in X$.

Proof: Let us first suppose that $\mathbf{D}$ is a De Morgan triple on $\mathrm{Map}(X, L)$. So that $\mathbf{T}(f, g) = \mathbf{N}^\sigma(\mathbf{S}(\mathbf{N}^\sigma(f), \mathbf{N}^\sigma(g)))$ for any $f, g \in \mathrm{Map}(X, L)$. Let $a, b \in L$ and, C_a and C_b the constant maps, then $\mathbf{T}(C_a, C_b)(x) = T_x(C_a(x), C_b(x)) = T_x(a, b)$ for any $x \in X$. On the other hand,

$$
\begin{aligned}
&\mathbf{T}(C_a, C_b)(x) = \\
&= \mathbf{N}^\sigma(\mathbf{S}(\mathbf{N}^\sigma(C_a), \mathbf{N}^\sigma(C_b)))(x) \\
&= N_x(\mathbf{S}(\mathbf{N}^\sigma(C_a), \mathbf{N}^\sigma(C_b))(\sigma(x)) \\
&= N_x(S_{\sigma(x)}(\mathbf{N}^\sigma(C_a)(\sigma(x)), \mathbf{N}^\sigma(C_b)(\sigma(x))) \\
&= N_x(S_{\sigma(x)}(N_{\sigma(x)}(C_a(\sigma^2(x))), N_{\sigma(x)}(C_b(\sigma^2(x)))) \\
&\overset{*}{=} N_{\sigma(x)}(S_{\sigma(x)}(N_{\sigma(x)}(C_a)(\sigma^2(x)), N_{\sigma(x)}(C_b)(\sigma^2(x))) \\
&= T_{\sigma(x)}(C_a(\sigma^2(x)), C_b(\sigma^2(x))) \\
&= T_{\sigma(x)}(a, b),
\end{aligned}
$$

where $*$ is due to the equality $N_{\sigma(x)} = N_x$ for any $x \in X$, since $\mathbf{N}^\sigma$ is involutive and Theorem 18 applies. Conversely, we have to prove that $\mathbf{N}^\sigma(\mathbf{S}(f, g)) = \mathbf{T}(\mathbf{N}^\sigma(f), \mathbf{N}^\sigma(g)))$ for any $f, g \in \mathrm{Map}(X, L)$. Let $x \in X$,

$$
\begin{aligned}
\mathbf{N}^\sigma(\mathbf{S}(f, g))(x) &= N_x(\mathbf{S}(f, g)(\sigma(x))) \\
&= N_x(S_{\sigma(x)}(f(\sigma(x)), g(\sigma(x)))) \\
&= N_{\sigma(x)}(S_{\sigma(x)}(f(\sigma(x)), g(\sigma(x))) \\
&= T_{\sigma(x)}(N_{\sigma(x)}(f(\sigma(x)), N_{\sigma(x)}(g(\sigma(x))),
\end{aligned}
$$

since (S_x, N_x, T_x) is a De Morgan triple. On the other hand,

$$
\begin{aligned}
&\mathbf{T}(\mathbf{N}^\sigma(f), \mathbf{N}^\sigma(g)))(x) = \\
&= T_x(\mathbf{N}^\sigma(f)(x), \mathbf{N}^\sigma(g)(x))) \\
&\overset{*}{=} T_{\sigma(x)}(N_x(f(\sigma(x))), N_x(g(\sigma(x)))) \\
&= T_{\sigma(x)}(N_{\sigma(x)}(f(\sigma(x))), N_{\sigma(x)}(g(\sigma(x)))),
\end{aligned}
$$

where $*$ is due to $T_x = T_{\sigma(x)}$.

Since (S_x, N_x, T_x) is a De Morgan triple, it is clear that $T_x = T_{\sigma(x)}$ for any $x \in X$ if and only if $S_x = S_{\sigma(x)}$. ■

Finally, analogously the results proved in other sections, Theorem 30 may be restricted to De Morgan triples on $\mathrm{Hom}(X, L)$.

Corollary 31: Let L and X bounded lattice, $\sigma : X \to X$ a decreasing map with $\sigma^2 = \mathrm{id}_X$ and $\{\mathcal{D}_x = (S_x, N, T_x)\}_{x\in X}$ a family of De Morgan triples on L indexed on X such that $\{T_x\}_{x\in X}$ and $\{S_x\}_{x\in X}$ are monotone families of t-norms and t-conorms, respectively. Then $\mathbf{D} = (\mathbf{T}, \mathbf{N}^\sigma, \mathbf{S})$ is a De Morgan triple on $\mathrm{Hom}(X, L)$ if and only if $T_x = T_{\sigma(x)}$ for any $x \in X$, if and only if $S_x = S_{\sigma(x)}$ for any $x \in X$.

Example 32: Let $m : [0, 1] \to \{m_1, m_2, \ldots, m_n\} \subset [0, 1]$ be a map, we may refer to it as a "threshold map", and let $\tau = \{T_x\}_{x\in[0,1]}$ be a family of t-norms on $[0, 1]$, where

$$
T_x(a, b) = \begin{cases} 0, & \text{if } a < m(x) \text{ y } b < m(x), \\ a \wedge b, & \text{otherwise,} \end{cases}
$$

for any $a, b \in [0, 1]$.

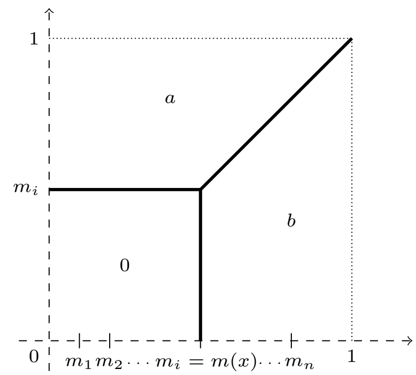

Fig. 4: Representation of the image of T_x

This family induces the t-norm $\mathbf{T}$ on $\mathrm{Map}([0, 1], [0, 1])$,

$$
\mathbf{T}(f, g)(x) = \begin{cases} 0, & \text{if } f(x), g(x) < m(x), \\ f(x) \wedge g(x), & \text{otherwise,} \end{cases}
$$

for any $f, g \in \mathrm{Map}([0, 1], [0, 1])$ and $x \in [0, 1]$. Let now $\mathbf{n} : [0, 1] \to [0, 1]$ be the involutive negation provided by $\mathbf{n}(x) = 1 - x$ for any $x \in [0, 1]$, and $\eta = \{\mathbf{n}\}_{x\in[0,1]}$ a constant family of negations and $\sigma = id_{[0,1]}$. Hence, following Section V, the induced involutive negation $\mathbf{N}$ on $\mathrm{Map}([0, 1], [0, 1])$ is given by $\mathbf{N}(f) = \mathbf{1} - f$ for any $f \in \mathrm{Map}([0, 1], [0, 1])$.

In this conditions we may get a family of De Morgan triples $\delta = \{(T_x, \mathbf{n}, S_x)\}_{x\in[0,1]}$, where, for any $a, b \in [0, 1]$,

$$
S_x(a, b) = \begin{cases} 1, & \text{if } a, b > 1 - m(x), \\ a \vee b, & \text{otherwise.} \end{cases}
$$

The corresponding induced t-conorm $\mathbf{S}$ is then defined as

$$
\mathbf{S}(f, g)(x) = \begin{cases} 1, & \text{if } f(x), g(x) > m(x), \\ f(x) \vee g(x), & \text{otherwise,} \end{cases}
$$

for any $f, g \in \mathrm{Map}([0, 1], [0, 1])$ and $x \in [0, 1]$. We find then a triple $\mathbf{D} = (\mathbf{T}, \mathbf{N}, \mathbf{S})$ in $\mathrm{Map}([0, 1], [0, 1])$. Actually, by Theorem 30, since $T_x = T_{\sigma(x)}$ for any $x \in X$, $\mathbf{D}$ is a De Morgan triple on $\mathrm{Map}([0, 1], [0, 1])$.

VII. Conclusion

We have presented a general scheme for designing set-theoretic operations (or logical connectives) on spaces of lattice-valued maps. Although the theory is particularized to t-norms, t-conorms and negations, it is feasible to apply the induction to another operators by a similar procedure. For instance, in this direction, it seems clear that aggregation operators are susceptible of being treated in a future work. Additionally, we have shown that it covers the cases of interval-valued and type-2 fuzzy sets, nevertheless, we have worked under quite mild assumptions so the results could be exported to another objects. The methodology offers a great flexibility for the design task, so we believe that it could exploited in practical applications.

REFERENCES

[1] K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. USA 8 (1942) 535-537.
[2] B. Schweizer and A. Sklar, "Espaces métriques aléatoires", C. R. Acad. Sci. Paris 247 (1958), 2092–2094.
[3] L. A. Zadeh, "Fuzzy Sets", Inform. and Control 8 (1965), 338–353.
[4] P. Hernández, S. Cubillo, and C. Torres-Blanc, "On t-norms for type-2 fuzzy sets", IEEE Trans. Fuzzy Syst. 23 (4) 2015, 1155–1163.
[5] C. L. Walker and E. A. Walker, "T-norms for type-2 fuzzy sets", in Proc. IEEE Int. Conf. Fuzzy Syst., Vancouver, Jul. 2006, pp. 1235–1239.
[6] B.C. Bedregal, A. Takahashi, "Interval valued versions of t-conorms, fuzzy negations and fuzzy implications", in: Proceedings of the IEEE International Conference on Fuzzy Systems, Vancouver, 2006, IEEE, Los Alamitos, 2006, pp. 19811987.
[7] B. C. Bedregal, "On interval fuzzy negations". Fuzzy Sets Syst. 161 (17) (2010), 2290–2313.
[8] H. Bustince, J. Montero, M. Pagola, E. Berrenechea, D. Gomez, "A survey of interval-valued fuzzy sets", in: Handbook of Granular Computing, John Wiley & Sons, West Sussex, 2008, pp. 491–515.
[9] B. De Baets and R. Mesiar, "Triangular norms on product lattices", Fuzzy Sets Syst. 104 (1999), 61–75.
[10] S. Saminger-Platz, E. P. Klement and R. Mesiar, "On extensions of triangular norms on bounded lattices", Indag. Mathern. 19 (1) (2008), 135–150.
[11] Ş. Yılmaz and O. Kazancı, "Constructions of triangular norms on lattices by means of irreducible elements", Inf. Sci. 397–398 (2017), 110–117.
[12] S. Saminger, On ordinal sums of triangular norms on bounded lattices, Fuzzy Sets and Systems 157 (2006), 1403–1416.
[13] E.P. Klement, R. Mesiar and E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.
[14] B. Prados-Suárez, J. Chamorro-Martínez, D. Sánchez and J. Abad, "Region-based fit of color homogeneity measures for fuzzy image segmentation", Fuzzy Sets Syst. 158 (2007), 215–229.
[15] M. Daniel, P. Hajek, P. H. Nguyen, "CADIAG-2 and MYCIN-like systems", Artif. Intell. Med. 9 (1997) 241–259.
[16] K. P. Adlassnig and G. Kolarz, "CADIAG-2: Computer-assisted medical diagnosis using fuzzy sets", In: Approximate Reasoning in Decision Analysis, M. M. Gupta and E. Sanchez (eds.), North-Holland Publishing Company, 1982. pp. 219–247.
[17] M. Mas, M. Monserrat, J. Torrens and E. Trillas, "A Survey on Fuzzy Implication Functions", IEEE Trans. Fuzzy Syst. 15 (2007), 1107–1121.
[18] T. J. Ross, Fuzzy Logic with Engineering Applications, Third Edition. 2010. John Wiley & Sons, Ltd.
[19] Baets, B. De, Mesiar, R., "Triangular norms on the real unit square". In: Proc. 1999 EUSFLAT-EST YLF Joint Conference, Palma de Mallorca 1999, pp. 351-354.
[20] G. Deschrijver and E. E. Kerre, "Triangular norms and related operators in L^*-fuzzy set theory". In: Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms. Edited by E.P. Klement and R. Mesiar. Elsevier, 2005, pp 231–259.
[21] B. Bedregal et al., "A characterization theorem for t-representable n-dimensional triangular norms". In: Proc. Eurofuse, 2011, pp. 103112.
[22] B. Bedregal, I. Mezzomo, and R. H. S. Reiser, "n-Dimensional fuzzy negations", IEEE Trans. Fuzzy Syst. 26 (6) (2018), 3660–3672.
[23] De Cooman, G. and Kerre, E., "Order norms on bounded partially ordered sets". J. Fuzzy Math. 2 (1994), 281–310.
[24] B. C. Bedregal, H. S. Santos and R. Callejas-Bedregal, "T-norms on bounded lattices: t-norm morphisms and operators", IEEE International Conference on Fuzzy Systems, Vancouver, BC, 2006, pp. 22–28.
[25] E.S. Palmeira, B.C. Bedregal. "Extension of fuzzy logic operators defined on bounded lattices via retractions" Comp. Math. with Appl. 63 (6) (2014), 1–21.
[26] E. S. Palmeira, B. C. Bedregal, R. Mesiar and J. Fernandez. "A new way to extend t-norms, t-conorms and negations". Fuzzy Sets Syst. 240 (2012), 1026–1038.
[27] B. C. Bedregal and A. Takahashi. "Interval t-norms as interval representations of t-norms". In: Proc. of IEEE International Conference on Fuzzy Systems (Fuzz-IEEE), Reno, Nevada, 2005, pp. 909–914.
[28] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning – 1", Inf. Sci. 8 (1975) 199–249.
[29] J. M. Mendel, "Advances in type-2 fuzzy sets and systems". Inf. Sci., 177 (1) (2007), pp. 84-110
[30] M. Gehrke, C.L. Walker and E.A. Walker, "Some comments on interval-valued fuzzy sets", Int J Intell Syst. 11 (10) (1996), 751–759.
[31] Y. Shang, X. Yuan, and E. S. Lee, "The n-dimensional fuzzy sets and Zadeh fuzzy sets based on the finite valued fuzzy sets", Comput. Math. Appl. 60 (2010), 442–463.
[32] G. Deschrijver, C. Cornelis, E.E. Kerre, "On the representation of intuitionistic fuzzy t-norms and t-conorms". IEEE Trans. Fuzzy Syst. 12 (1) (2004), 45–61.

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