A Markov chains prognostics framework for complex degradation processes

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Abstract

This paper presents a prognostics methodology to deal with complex degradation processes for which condition monitoring data constitute the only available source of physical information. The proposed methodology is general, but here it is illustrated and tested using a case study about fatigue crack propagation in metallic structures. The prognostics method relies on a stochastic damage model based on Markov chains, which is embedded within a sequential state estimation framework for remaining useful life prediction and time-dependent reliability estimation. As key contribution, the resulting prediction and updating equations are shown to be obtained as closed-form expressions, whereby analytical equations for the remaining useful life and time-dependent reliability are obtained as by-product. Original contributions to the model parameter inference and the minimum required amount of data for prognostics are also provided.

Keywords: Damage prognostics, Markov chains, Sequential prediction, Time-dependent reliability, Remaining useful life

1. Introduction

The degradation of structures and infrastructures is a key problem with profound implications in safety and cost. Material degradation processes such as fatigue and corrosion become main ageing factors in many critical infrastructures and engineering assets such as bridges, offshore structures, and power plants. The onset and posterior evolution of these degradation modes is a complex and partially-understood process highly influenced by the uncertainty in the loadings, material properties and environmental conditions. Anticipating the future stages of degradation of such critical structures and infrastructures, and moreover, estimating their *remaining useful life* (RUL) with quantified uncertainty, is a key piece of information for safety, maintenance cost reduction, and improved asset availability.

Over the last decades, numerous material degradation models have been proposed in the literature for degradation modes such as fatigue, creep, and corrosion, and a large amount of degradation data have been

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derived from extensive experimental programs. A thorough overview about these methods can be found in a number of research works such as [1–3], to cite but a few. In general, the vast majority of these modelling approaches are deterministic physics-based or semi-empirical formulations calibrated for a particular damage mode under specific laboratory testing conditions; therefore, their fit to reality becomes limited in many cases. In this sense, a number of stochastic degradation models have been proposed in the literature aiming at representing the large variability observed in real-life degradation data. Stochastic modelling approaches such as Lèvy processes [4] (specially the Wiener process [5] and the Gamma process [6]), Markov models [7, 8], autoregressive models [9, 10], and Inverse Gaussian processes [11, 12], among others, have shown efficiency in representing complex degradation data for different engineering applications ranging from rotating machinery health assessment [13] to degradation and maintenance modelling of wind power systems [14], to cite but any. As a general comment, the *state transition probabilities* and other parameters that characterise the stochastic nature of these models are typically derived from historical data collected from large populations of identical components; therefore, the resulting stochastic models are able to represent the observed damage process in a coherent statistical sense, but once the process has ended. This poses an important limitation when the interest is in accurately predicting the future stages of degradation and the remaining useful life of a particular component or system *during* service, based on its actual health state. In this context, we find advanced Prognostics and Health Management (PHM) methodologies of major interest here since they enable an in-situ assessment of the state of degradation and the estimation of RUL during the process, as long as condition monitoring data are collected [15]. Recent literature reviews on this topic can be found in [16, 17].

In this work, an efficient computational framework is proposed for condition-based prognostics of complex degradation processes for which a physics-based model of such a degradation is seldom known in practice. Our assumption is that if condition data about the degraded system are available in real time, this information can be used to reduce the initial modelling uncertainty associated with the deterioration process as observed from historical data. In this context, the key contribution of this paper is a rigorous Bayesian prognostics framework where both sources of information, namely, a stochastic degradation model and condition monitoring data taken in an on-line fashion, are fused in a rigorous and principled way. This is carried out by embedding a stochastic model for damage evolution into a condition-based prognostics framework for sequential damage state estimation and damage prognostics [15]. This is typically known as *filtering*, denoting that the received condition monitoring data are processed sequentially rather than as a *batch*, which is particularly useful in very long term degradation processes such as fatigue and corrosion. For the problem of stochastic modelling of damage propagation, a non-stationary Markov chains damage model is adopted as *state transition equation* within the proposed filtering-based prognostics framework. This model has shown efficiency in representing highly scattered sequences of degradation using a reduced set of parameters due to a low-dimensional parameterisation proposed in [20]. Given the low dimensional and discrete state space assumptions prescribed by the adopted Markov chains model, the *prediction* and *updating* recursive equations constituting the basis for prognostics are shown to reduce to closed-form expressions. This is a relevant result for the PHM literature which enables a genuine real-time prognostics estimate of the degradation process, thus avoiding the computational complexities typically associated to prognostics algorithms such as *particle filters* [16], required when approaching non-linear non-Gaussian degradation processes. Moreover, closed-form expressions for the RUL and the time-dependent reliability of the system are obtained as by-product from the elementary sequential state prediction and updating equations. A scheme of the proposed methodology is depicted in Figure 1.

The suitability and efficiency of the proposed methodology is demonstrated using a case study about stochastic fatigue damage evolution in metals. In particular, the Virkler's *et al.* fatigue dataset [22] is adopted, consisting of 68 crack growth histories from 68 nominally identical center-cracked aluminium panels subjected to identical loading conditions. One arbitrary subset of these sequences of damage is used to fit the Markov chains model parameters (*calibration dataset*), while the rest is used for prognostics purposes (*prognostics dataset*). To rigorously estimate the minimum required amount of calibration data, a methodology is proposed consisting on computing the probability of the model calibrated with a particular subset of the data to reproduce the prognostics dataset. This probability was referred to as the *prospective evidence* of the model in [23]. The results show that the prospective evidence tends to reach an asymptote by increasing the size of the calibration dataset, denoting that the model has extracted enough information from the calibration dataset to make accurate predictions about the unobserved prognostics dataset. For the estimation of model parameters, a rigorous Bayesian inverse problem approach, such as the one proposed by Chiachío *et al.* in [20], can be adopted to provide plausible values of the model parameters according to the modelling and data uncertainties. However, for the sake of simplicity and better extensibility to practical scenarios, a novel and simple approach for parameter estimation is proposed herein based on minimising the distance between the empirical and modelled cumulative distribution functions (CDF) of damage, using a genetic algorithm (GA) [24] for this purpose. Once the Markov chains model is calibrated, then it is embedded within the recursive filtering equations for sequential damage state estimation and RUL prediction using data coming in an on-line fashion. The results demonstrate good agreement with the prognostics data with very low computational cost.

The paper is organised as follows. In Section 2, the mathematical description of the Markov chains damage model together with the methodology for model parameter inference, are presented. The methodology for filtering-based prognostics is described in Section 3. Section 4 is devoted to the validation of the proposed methodology using experimental data. A discussion about the advantages and limitations of the proposed methodology is provided in Section 5. Finally, Section 6 provides concluding remarks.

2. Stochastic degradation modelling by Markov chains

The degradation of structures, and in general, of physical assets, is regarded here as an irreversible memoryless stochastic damage process. Based on such observation, the temporal evolution of damage is proposed to be represented using a parameterised non-stationary Markov chains model. The reader is referred to [20] for specific details about this stochastic model, but for the sake of clarity and better readability, the key formulation is reproduced here with an uniform notation.

2.1. Modelling assumptions and key formulation

The following modelling assumptions are adopted to mathematically describe the temporal evolution of damage:

a) Damage is represented by a nondecreasing stochastic process $\{x_n, n \in \mathbb{N}\}$ taking values in $\mathcal{X} = [0,1] \subset \mathbb{R}$, with n being the discrete times (e.g., load cycles, hours, days, etc.) in which damage can be accumulated;

b) Damage passes monotonically through a finite set of discrete damage states, $\mathcal{X}_i$, where $i \in \mathcal{S} = \{0,\ldots,s\} \subset \mathbb{N}$, until the *absorbing* (failure) state s is reached. Each damage state $\mathcal{X}_i \subset \mathcal{X}$ is a subinterval represented by its centre $\bar{x}_n^i = {}^{(2i+1)}/_{(2s+2)}$ such that $p_n^i \triangleq p(\bar{x}_n^i) \triangleq P[x_n \in \mathcal{X}_i] \geqslant 0$. Thus, damage at a particular time n can be represented by the probability density function (PDF) $p(x_n) = \sum_{i=0}^{s} p_n^i \delta(x_n - \bar{x}_n^i)$, where δ is the Dirac delta function which makes unity when $x_n = \bar{x}_n^i$ and 0 otherwise;

c) The evolution of damage is governed by the *Markov property*, which states that the future of the process is conditionally independent of the past states, given the present [25]. In mathematical terms: $p(\bar{x}_{n+1}^k|\bar{x}_0^i,\ldots,\bar{x}_n^j) = p(\bar{x}_{n+1}^k|\bar{x}_n^j) \triangleq p_n^{j,k};\;\; i,j,k \in \mathcal{S}$. In the more general case, the transition probabilities may change with time, so the process is referred to as *non-stationary*;

d) During time or load cycle n, damage may only go from a certain damage state $\bar{x}_n^i$ to the next $\bar{x}_{n+1}^{i+1}$ with probability $p(\bar{x}_{n+1}^{i+1}|\bar{x}_n^i) \triangleq p_n^{i,i+1}$, and remain in the same state with a probability $p(\bar{x}_{n+1}^i|\bar{x}_n^i) \triangleq p_n^{i,i} = 1 - p_n^{i,i+1}$. Therefore, damage is described as a *cumulative process*;

Under these assumptions, the evolution of damage from time n to $m > n \in \mathbb{N}$ can be modelled using Markov chains theory as [7]:

$$\mathbf{p}_m = \mathbf{p}_n \cdot \prod_{j=n}^{m-1} \mathbf{P}_j\,, \tag{1}$$

where $\mathbf{p}_n = \left(p_n^0, p_n^1, \ldots, p_n^s\right)$, and $\{\mathbf{P}_j\}_{j=n}^{m-1}$ is the set of *probability transition matrices* from time n to $m-1$. For a certain time or load cycle n, the transition probability matrix stores in its $(i+1, j+1)$ element the probability of transition $p(\bar{x}_{n+1}^j|\bar{x}_n^i) \triangleq p_n^{i,j}$, with $i,j = 0,\ldots,s$, so that the whole process is completely defined when all possible transition probabilities are known for every time $m > n$. Note that this model definition would imply determining the value of $s \cdot (m-n)$ unknown parameters to completely characterise the proposed Markov chains damage model. To avoid such a high dimensional inference problem while preserving the non-stationarity of the model, a model reduction strategy was proposed by Chiachío *et al.* in [20] consisting on introducing an artificial alteration of the time scale of the stochastic process so that the transition probabilities remain equal and invariant along the process. To avoid repetition of literature, the reader is referred to [20] for details about this methodology. Under such a transformation, Equation (1) rewrites as:

$$\mathbf{p}_m = \mathbf{p}_n \cdot \mathbf{Q}^{g'(m/N)-g'(n)}, \tag{2}$$

where $\mathbf{Q}$ is a $(s+1)\times(s+1)$ bi-diagonal transition probability matrix given by

$$\mathbf{Q} = \begin{pmatrix} 1-p & p & & & \\ & 1-p & p & & \\ & & \ddots & \ddots & \\ & & & 1-p & p \\ & & & & 1 \end{pmatrix}, \tag{3}$$

with $p \in [0,1]$ being the probability of transition between consecutive damage states, which remain invariant along the process. In Equation (2), $g'(n)$ is a nonlinear function of the time n defined as $g'(n) = N \cdot g(n/N)$, where $g : [0,1] \to [0,1]$ is the monotonic cubic Hermite interpolant [26, 27] of the set of coordinates $\{(0,0),(\theta_1,\theta'_1)\ldots,(\theta_j,\theta'_j),\ldots,(1,1)\}$, with $0<\theta_j<\theta_{j+1}<1$, $0<\theta'_j<\theta'_{j+1}<1$, and N is a sufficiently large amount of time units after which the absorbing state is completely developed. The reduced set of parameters $\boldsymbol{\theta} = \{p,\theta_1,\theta'_1,\ldots,\theta_j,\theta'_j\}$, taking values in $\boldsymbol{\Theta} = [0,1]^{N_p}$ with $N_p = \dim(\boldsymbol{\theta})$, allows for a complete description of the non-stationary Markov chains damage model.

2.2. Parameter estimation method

As stated before, a complete characterisation of the Markov chains damage model introduced in Section 2.1 implies having the value of model parameters $\boldsymbol{\theta} = \{p,\theta_1,\theta'_1,\ldots,\theta_j,\theta'_j\}$ as known. A rigorous method to infer such model parameters is by Bayesian updating, which provides the *posterior* PDF of the plausible values of the parameters for a given set of data [20]. Besides its rigorousness and correctness, the Bayesian inference method is subjected to a number of mathematical and computational complexities which may bound the practical scope of the proposed methodology in real life engineering applications. In this section, a novel and simple approach for parameter estimation is proposed based on minimising the distance between the empirical and modelled cumulative distribution functions of damage. To this end, we assume that a set of K experimental sequences of damage $\mathcal{D} = \{y^{(1)}_{1:N},\ldots,y^{(j)}_{1:N},\ldots,y^{(K)}_{1:N}\}$ from K nominally-identical components or physical assets is available, where $y^{(j)}_{1:N} = \left[y^{(j)}_1,\ldots,y^{(j)}_n,\ldots,y^{(j)}_N\right]$ is the j-th sequence of damage measured at the set of times or load cycles[1] $\mathcal{T}_{\mathcal{D}} = \{1,\ldots,n,\ldots,N\}$. From this dataset, the empirical CDF

[1]The same methodology applies when the data are available at non-regularly staggered times $\mathcal{T} = \{n_1,\ldots,n_k,\ldots,n_N\}$.

of damage at a certain time $n \in \mathcal{T}_{\mathcal{D}}$ can be obtained as:

$$F(y_n) = \frac{1}{K} \sum_{j=1}^{K} \mathbb{I}_{[0,y_n]}(y_n^{(j)}) \,, \tag{4}$$

with y_n taking values in $\mathcal{X}$, and $\mathbb{I}_{[0,y_n]}$ being an indicator function which makes unity if $y_n^{(j)} \in [0, y_n]$, and 0 otherwise. Next, let $F(x_n|\boldsymbol{\theta})$ be the CDF of damage at time n, given by the Markov chains damage model parameterised by $\boldsymbol{\theta}$, defined as:

$$F(x_n|\boldsymbol{\theta}) = \sum_{i=0}^{s} \sum_{j=0}^{i} p_n^j \cdot \delta(x_n - \bar{x}_n^i) \,. \tag{5}$$

Then, a reasonable estimator of $\boldsymbol{\theta}$ is a value $\hat{\boldsymbol{\theta}} \in \boldsymbol{\Theta}$ that minimises any function $\mathcal{C} : \boldsymbol{\Theta} \to \mathbb{R}$ of the distance between the CDFs (4) and (5) for every n in $\mathcal{T}_{\mathcal{D}}$, given by [28, 29]:

$$d_n(\boldsymbol{\theta}) = \int_{\mathcal{X}} \Big(F(y_n) - F(x_n|\boldsymbol{\theta}) \Big)^2 \, dx_n \,. \tag{6}$$

The function $\mathcal{C}$ can be suitably defined as the ℓ_2-norm of the vector $(d_1(\boldsymbol{\theta}), \ldots, d_n(\boldsymbol{\theta}), \ldots, d_N(\boldsymbol{\theta}))$, i.e.,

$$\mathcal{C}(\boldsymbol{\theta}) = \| \, (d_1(\boldsymbol{\theta}), \ldots, d_n(\boldsymbol{\theta}), \ldots, d_N(\boldsymbol{\theta})) \, \|_2 \,, \tag{7}$$

therefore the inference problem is reduced to the problem of finding $\hat{\boldsymbol{\theta}} \in \boldsymbol{\Theta}$ that satisfies

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \mathcal{C}(\boldsymbol{\theta}) \,. \tag{8}$$

The optimisation problem in Equation (8) can be computationally solved using numerical search methods such as gradient based methods, random search, or advanced optimisation algorithm such as genetic algorithms (GA) [24, 30], or ant/bee colony algorithms [18, 19], among others. In this work, GA are selected for their ability to search in large parameter spaces avoiding local minima and for their ease of implementation and use. Genetic algorithms assume that the potential solution of any optimization problem is an individual or *chromosome* $\boldsymbol{\theta}_i$ belonging to a *population* $\boldsymbol{\Psi}_j = \{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{N_{pop}}\} \subset \boldsymbol{\Theta}$, where each succeeding population within the sequence $\{\boldsymbol{\Psi}_1, \ldots, \boldsymbol{\Psi}_{N_{gen}}\}$ forms a *generation*. GA then replicates the idea of *natural selection* in the sense that the stronger individuals in a population are more likely to survive and reproduce under several "genetic" transformations along the generations. For the GA-based optimisation problem presented here, the goodness of each *chromosome* $\boldsymbol{\theta}_i$ is measured by its "fitness" value given by $\mathcal{C}(\boldsymbol{\theta}_i)$. More details about GA optimisation are provided in Section 4 in the context of a case study.

3. Damage prognostics and reliability

Damage prognostics is concerned with predicting the future stages of degradation of engineering systems or components based on the current degree of wear or damage, and, subsequently, estimating the remaining time in which the system is expected to continue performing its intended function [15]. Having defined a stochastic model for the evolution of damage forward in time, as given in Section 2, the next required step for prognostics is the sequential prediction and updating of the plausible damage states of the system as long as condition monitoring data are collected, referred to as *filtering*. In this section, a closed-form formulation for filtering based prognostics is shown to be obtained by embedding the Markov chains damage model presented in Section 2 within the recursive filtering equations for damage prediction and updating. As a by-product, closed-form expressions for the time-dependent reliability and the remaining useful life of the system are also obtained.

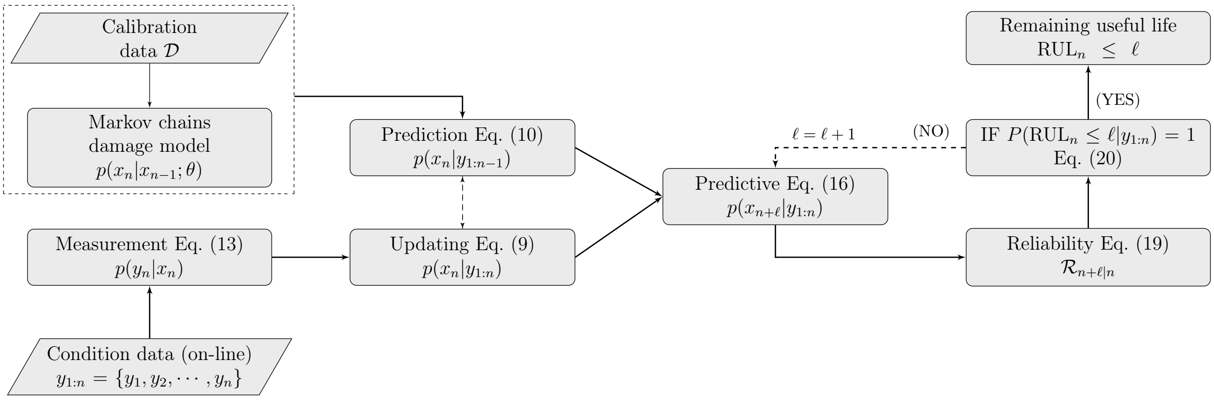


Figure 1: Schematic overview of the proposed prognostics framework

3.1. Sequential damage state estimation

The goal of sequential state estimation is to obtain an updated probability-based estimation of the current damage state of the system every time a new measurement is received. In the Bayesian approach to sequential state estimation, the required output is the *posterior* PDF of the damage state x_n conditional to the available data, namely $p(x_n|y_{1:n})$, where $y_{1:n} \triangleq \{y_1, y_2, \ldots, y_n\}$ is the sequence of damage measurements collected up to current time or cycle n. This posterior PDF represents the most up-to-date information about the damage state given by both the degradation model and the available data, and it is obtained by updating the *prior* PDF $p(x_n|y_{1:n-1})$ using the most recent measurement y_n at current time n. This is achieved using Bayes' Theorem, as

$$p(x_n|y_{1:n}) = \frac{\overbrace{p(x_n|y_{1:n-1})}^{\text{Eq. (10)}}\, p(y_n|x_n)}{\int_{\mathcal{X}} p(x_n|y_{1:n-1}) p(y_n|x_n) dx_n}\,, \tag{9}$$

where the PDF $p(y_n|x_n)$ is the *measurement equation* of the system, which relates the (noisy) measurement y_n to its corresponding damage state x_n based on a particular probability model. The prior PDF $p(x_n|y_{1:n-1})$

in Equation (9) is given by Total Probability Theorem as:

$$p(x_n|y_{1:n-1}) = \int_{\mathcal{X}} p(x_n|x_{n-1}, \hat{\boldsymbol{\theta}}) \underbrace{p(x_{n-1}|y_{1:n-1})}_{\text{Eq. (9) at } n-1} dx_{n-1} \,, \tag{10}$$

where the term $p(x_{n-1}|y_{1:n-1})$ is the posterior PDF at previous time $n-1$, and $p(x_n|x_{n-1}, \hat{\boldsymbol{\theta}})$ is the *state transition equation* prescribed by a particular model parameterised by $\hat{\boldsymbol{\theta}}$. The key of the proposed prognostics formulation is that $p(x_n|x_{n-1}, \hat{\boldsymbol{\theta}})$ is given by the Markov chains damage model introduced in Section 2 parameterised by $\hat{\boldsymbol{\theta}}$, as shown further below. Note that the conditioning on $\hat{\boldsymbol{\theta}}$ in $p(x_n|x_{n-1}, \hat{\boldsymbol{\theta}})$ automatically conveys the conditioning $p(x_n|y_{1:n-1}, \hat{\boldsymbol{\theta}})$ and $p(x_n|y_{1:n}, \hat{\boldsymbol{\theta}})$ in Equations (9) and (10). However, this conditioning is dropped for the sake of simpler notation.

By continually iterating between Equations (9) and (10), a sequential state estimation of the damage state can be obtained every time a new measurement is available. However, this recursive scheme is only a theoretical solution, given that the integrals in Equations (9) and (10) are analytically intractable in most of the cases. This typically leads to the adoption of specific stochastic simulation methods, such as *particle filters* [31], which provide a sample approximation of the posterior PDF in Equation (9) based on "particles" (weighted samples), while avoiding the calculation of the aforementioned integrals. These sampling methods are powerful, but they are subjected to well-known computational and algorithmic complexities [21, 32]. However, we note here an exception to such analytical intractability arising when the state space is discrete and consists of a finite number of states, as stated by hypothesis b) in Section 2. Under this modelling assumption, the recursive filtering equations (9) and (10) can be shown to be obtained as closed-form expressions, given by:

$$p(x_n|y_{1:n-1}) = \sum_{i=0}^{s} \omega^i_{n|n-1} \delta(x_n - \bar{x}^i_n) \,, \tag{11a}$$

$$p(x_n|y_{1:n}) = \sum_{i=0}^{s} \omega^i_{n|n} \delta(x_n - \bar{x}^i_n) \,, \tag{11b}$$

where $\omega^i_{n|n-1}$ and $\omega^i_{n|n}$ are the prediction and updating probabilities of the i-th damage state $\bar{x}^i_n$, respectively, satisfying $\sum_{j=0}^{s} \omega^j_{n|n-1} = \sum_{j=0}^{s} \omega^j_{n|n} = 1$. By substituting Equations (11a) and (11b) into (10) and (9) respectively, and after some algebraic manipulation explained in Appendix A, the prediction and updating probabilities are obtained as:

$$\omega^i_{n|n-1} = \sum_{j=0}^{s} \omega^j_{n-1|n-1} p(\bar{x}^i_n|\bar{x}^j_{n-1}) \,, \tag{12a}$$

$$\omega^i_{n|n} = \frac{\omega^i_{n|n-1} p(y_n|\bar{x}^i_n)}{\sum_{j=0}^{s} \omega^j_{n|n-1} p(y_n|\bar{x}^j_n)} \,, \tag{12b}$$

where $p(\bar{x}_n^i|\bar{x}_{n-1}^j)$ in Equation (12a) is the $(j+1, i+1)$ element of the transition probability matrix $\mathbf{P}_{n-1}$ of the Markov chains damage model defined in Section 2, and $p(y_n|\bar{x}_n^i)$ in Equation (12b) is the measurement equation particularised for damage state $\bar{x}_n^i$. This measurement equation can be conservatively defined by assuming that the discrepancy between the damage state $\bar{x}_n^i$ and the measurement y_n is a zero-mean Gaussian variable, i.e., $y_n = \bar{x}_n^i + v_n$, with $v_n \sim \mathcal{N}(0, \sigma_n)$, which is supported by the Principle of Maximum Information Entropy (PMIE) [33]; therefore:

$$p(y_n|\bar{x}_n^i) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(- \frac{(y_n - \bar{x}_n^i)^2}{2{\sigma_n}^2} \right), \tag{13}$$

with σ_n being the standard deviation of the measurement error v_n.

From this standpoint, it can be observed that the prediction and updating probabilities in Equations (12a) and (12b) can be analytically obtained in one single computational step. This is more easily revealed by rewriting those equations as follows:

$$\mathbf{p}_{n|n-1} = \mathbf{p}_{n-1|n-1} \cdot \mathbf{P}_{n-1}\,, \tag{14a}$$

$$\mathbf{p}_{n|n} = \frac{\mathbf{p}_{n|n-1} \cdot diag\left(p(y_n|\bar{x}_n^0), \ldots, p(y_n|\bar{x}_n^s)\right)}{\mathbf{p}_{n|n-1} \cdot \left(p(y_n|\bar{x}_n^0), \ldots, p(y_n|\bar{x}_n^s)\right)^T}\,, \tag{14b}$$

where $\mathbf{p}_{n|n-1}$ and $\mathbf{p}_{n|n}$ are the prediction and updating matrices given by $\mathbf{p}_{n|n-1} = (\omega_{n|n-1}^0, \omega_{n|n-1}^1 \ldots, \omega_{n|n-1}^s)$ and $\mathbf{p}_{n|n} = (\omega_{n|n}^0, \omega_{n|n}^1, \ldots, \omega_{n|n}^s)$, respectively. Note that Equations (14a) and (14b) are simple matrix multiplications, so that by continuously iterating between them, an analytical prediction and updating of the damage state is sequentially obtained as long as new measurements are collected.

3.2. Sequential reliability estimation and prognostics

If a particular damage state $\xi \in \mathcal{X}$ is identified as a damage threshold, interpreted here as a functional limit beyond which the system is no longer performing its intended functionality, a prediction of the time-dependent reliability at future time $n+\ell$ can be obtained as:

$$\mathcal{R}_{n+\ell|n} \triangleq P(x_{n+\ell} \leqslant \xi|y_{1:n}) = \int_{\mathcal{X}} p(x_{n+\ell}|y_{1:n}) \mathbb{I}_{[x_{n+\ell} \leqslant \xi]} dx_{n+\ell}\,, \tag{15}$$

where $p(x_{n+\ell}|y_{1:n})$ is a probabilistic prediction of the future damage state ℓ-steps forward in time given data up to current time n, and $\mathbb{I}_{[x_{n+\ell} \leqslant \xi]}$ makes unity when $x_{n+\ell} \leqslant \xi$, and 0 otherwise. By Total Probability Theorem, the ℓ-step ahead predictive PDF can be obtained as:

$$\begin{aligned} p(x_{n+\ell}|y_{1:n}) &= \int_{\mathcal{X}} p(x_{n+\ell}|x_{n:n+\ell-1}, y_{1:n}) p(x_{n:n+\ell-1}|y_{1:n}) dx_{n:n+\ell-1} \\ &= \int_{\mathcal{X}} \left[\prod_{t=n+1}^{n+\ell} p(x_t|x_{t-1}) \right] p(x_n|y_{1:n}) dx_{n:n+\ell-1}\,, \end{aligned} \tag{16}$$

where $p(x_n|y_{1:n})$ is the updating PDF of the damage state at time n, and $p(x_t|x_{t-1})$ is the state transition equation representing the future evolution of the degraded system for times $t > n$. Replacing $p(x_n|y_{1:n})$ in Equation (16) by its discrete-state expression (Eq. (11b)), and by noting that $p(x_t|x_{t-1})$ is given by the probability transition matrices of the Markov chains damage model, a discrete state-space expression of the predictive PDF can be shown to be obtained as:

$$p(x_{n+\ell}|y_{1:n}) = \sum_{i=0}^{s} \omega^i_{n+\ell|n} \delta(x_{n+\ell} - \bar{x}^i_{n+\ell}) \,, \tag{17}$$

where $\omega^i_{n+\ell|n}$ is the predictive probability of the i-th damage state at future time $n+\ell$, given data up to current time n. After some algebraic manipulation analogous to that carried out in Equation (A.2), the predictive probabilities can be shown to be obtained analytically as

$$\mathbf{p}_{n+\ell|n} = \mathbf{p}_{n|n} \cdot \prod_{j=n}^{n+\ell-1} \mathbf{P}_j \,, \tag{18}$$

where $\mathbf{p}_{n+\ell|n} = (\omega^0_{n+\ell|n}, \omega^1_{n+\ell|n}, \ldots, \omega^s_{n+\ell|n})$. Therefore, by substituting Equation (17) in Equation (15), the time-dependent reliability can be straightforwardly obtained as:

$$\mathcal{R}_{n+\ell|n} = \sum_{i=0}^{s} \omega^i_{n+\ell|n} \mathbb{I}_{[\bar{x}^i_{n+\ell} \leqslant \xi]} \,, \tag{19}$$

which basically consists on adding the probabilities of the states below the threshold. Finally, it is noted that the computation of the time-dependent reliability as given by Equation (19) implicitly conveys a probability-based prediction of the remaining useful life (RUL) of the degraded system [34], as:

$$P(RUL_n \leqslant \ell|y_{1:n}) = 1 - \mathcal{R}_{n+\ell|n} \,. \tag{20}$$

Therefore, sequential estimations of both the time-dependent reliability and the RUL of the system can be straightforwardly obtained using the proposed prognostics approach every time a new measurement is collected. The reader is referred to Figure 1 for a schematic overview of the proposed methodology.

3.3. Minimum required set of data for prognostics

A typical criticism of stochastic degradation models such as the proposed in Section 2 is that they are based on data, and therefore, their predictions will be hardly extensible to other datasets different from those used for model calibration. However, Chiachío *et al.* [20] showed that there exist a case specific amount of calibration data beyond which the model parameters do not gain significant information. This observation is extended here to investigate the minimum required amount of data for prognostics purposes. This is carried out by computing the *prospective evidence* [23] of the stochastic model, which provides the overall probability of the model to predict a set of data different to the one used for model calibration. To this end, let consider a degradation model denoted by $\mathcal{M}$ represented by model parameters $\boldsymbol{\theta}$, which are calibrated using a dataset $\mathcal{D}_k$, consisting of k experimental sequences of degradation, i.e., $\mathcal{D}_k = \{y_{1:N}^{(1)}, \dots, y_{1:N}^{(k)}\}$, such that $\mathcal{D}_{k-1} \subset \mathcal{D}_k$. Considering an unobserved set of data $\mathcal{D}^*$, the prospective evidence of model $\mathcal{M}$ calibrated for dataset $\mathcal{D}_k$ is given by the conditional probability:

$$p(\mathcal{D}^*|\mathcal{D}_k, \mathcal{M}) = \int_{\boldsymbol{\Theta}} p(\mathcal{D}^*|\boldsymbol{\theta}, \mathcal{D}_k, \mathcal{M}) p(\boldsymbol{\theta}|\mathcal{D}_k, \mathcal{M}) d\boldsymbol{\theta}\,, \tag{21}$$

where $p(\boldsymbol{\theta}|\mathcal{D}_k, \mathcal{M})$ is the *posterior* PDF of the model parameters according to the calibration subset $\mathcal{D}_k$, and $p(\mathcal{D}^*|\boldsymbol{\theta}, \mathcal{D}_k, \mathcal{M})$ is the *prospective likelihood function*, which provides a probabilistic measure of how well the model specified by $\boldsymbol{\theta}$ calibrated using $\mathcal{D}_k$ predicts the unobserved dataset $\mathcal{D}^*$. Both the posterior PDF and the prospective likelihood can be obtained using the Bayesian inference methodology explained in [20], so it is not repeated here.

The evaluation of the integral in Equation (21) is not trivial except for some particular cases where Laplace's method of asymptotic approximation can be used [35]. One straightforward way to approximate the prospective evidence that is adopted in this paper is by considering the probability integral in Equation (21) as a mathematical expectation of the likelihood with respect to the posterior $p(\boldsymbol{\theta}|\mathcal{D}_k, \mathcal{M})$. This leads to the direct Monte Carlo method, as follows:

$$p(\mathcal{D}^*|\mathcal{D}_k, \mathcal{M}) \approx \frac{1}{T_s} \sum_{m=1}^{T_s} p(\mathcal{D}^*|\boldsymbol{\theta}^{(m)}, \mathcal{D}_k, \mathcal{M})\,, \tag{22}$$

where the $\boldsymbol{\theta}^{(m)}$ are T_s samples drawn from the posterior $p(\boldsymbol{\theta}|\mathcal{D}_k, \mathcal{M})$. By evaluating Equation (22) for a nested sequence of calibration datasets $\mathcal{D}_1 \subset \dots \mathcal{D}_k \dots \subset \mathcal{D}_K$, a sequence of K values for the prospective evidence are obtained, so that the minimum required set of calibration data for prognostics is determined as the minimum k such that $p(\mathcal{D}^*|\mathcal{D}_{k+1}, \mathcal{M}) \approx p(\mathcal{D}^*|\mathcal{D}_k, \mathcal{M})$. Further details are provided in the following section in the context of a case study.

4. Application of methodology to fatigue test data

The proposed prognostics methodology is illustrated here using a case study about stochastic fatigue damage evolution in metallic materials. In particular, the Virkler's *et al.* fatigue dataset [22] is adopted, consisting of 68 crack growth histories from 68 nominally identical center-cracked aluminium panels subjected to identical loading conditions. The dataset is represented in Figure 2 for illustration purposes. During the test, the number of loading cycles required for the tip of the crack to reach a predetermined increment of 0.2 [mm] was recorded from an initial crack length of $a_0 = 9$ [mm] to a final length of $a_f = 49.8$ [mm]. More details regarding the experimental set-up and the measurement procedure are reported in [22]. For this case study, the damage state at a particular load cycle n is obtained for the j-th data sequence as a normalised crack length denoted by $y_n^{(j)}$, as follows:

$$y_n^{(j)} = \begin{cases} \dfrac{a_n^{(j)} - a_0}{a_f - a_0}, & \text{if} \quad a_n^{(j)} \leqslant a_f \\ 1, & \text{if} \quad a_n^{(j)} > a_f \end{cases} \tag{23}$$

where $a_n^{(j)}$ is the crack length of the j-th specimen measured after n fatigue loads, and a_0, a_f are the initial and final crack lengths, respectively, whose values are specified above.

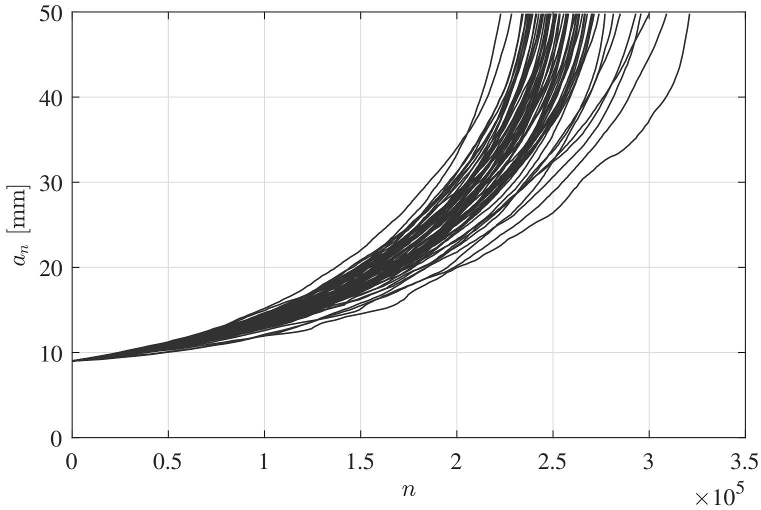


Figure 2: Experimental sequences of fatigue crack length from Virkler's *et al.* dataset [22].

Damage data, as given by Equation (23), are split into two disjoint subsets for model calibration and prognostics purposes; namely, *calibration subset* $\mathcal{D}$, and *prediction subset* $\mathcal{D}^*$, respectively. As explained in Section 3.3, the size of the required calibration dataset is determined by sequentially computing the prospective evidence $p(\mathcal{D}^*|\mathcal{D}_k, \mathcal{M})$ of the model using a calibration dataset $\mathcal{D}_k = \{y_{1:N}^{(1)}, \ldots, y_{1:N}^{(k)}\}$ with an increasing number k of damage sequences. To this end, 12 out of the 68 damage sequences are randomly chosen to constitute the prediction dataset $\mathcal{D}^*$, while the 56 remaining sequences are randomly distributed into a nested sequence of 56 calibration datasets $\{\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{56}\}$ such that $\mathcal{D}_{k-1} \subset \mathcal{D}_k$, with $k = 2, \ldots, 56$. Therefore, the minimum required set of calibration data for prognostics is chosen as the minimum k such that $p(\mathcal{D}^*|\mathcal{D}_{k+1}, \mathcal{M}) \approx p(\mathcal{D}^*|\mathcal{D}_k, \mathcal{M})$. As evident from the results shown in Figure 3, the Markov chains damage model reaches an asymptotic prognostics performance after having being calibrated with the subset $\mathcal{D}_{14}$. In other words, the model has extracted enough information from data just by calibrating it using the $14/56 = 25\%$ of the available calibration dataset. The model settings adopted to obtain these results are specified in the next section.

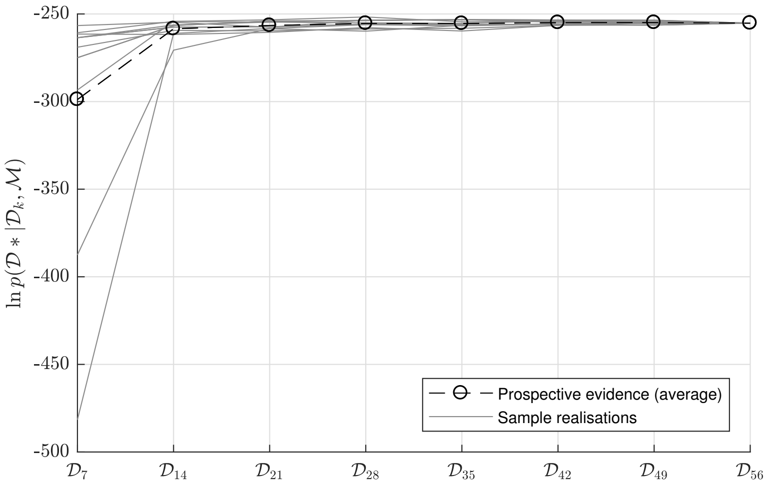


Figure 3: Prospective log-evidence of the Markov chains damage model calibrated using an increasing amount of data. The subset $\mathcal{D}_k$ includes the data in subset $\mathcal{D}_{k-1}$, for $k = 2, \ldots, 56$. The sample realisations (in gray colour) correspond to different combinations of the calibration dataset into subsets.

4.1. Model calibration for prognostics

The minimum required amount of calibration data $\mathcal{D}_{14}$, as explained above, are employed to infer the optimal values of parameters $\hat{\boldsymbol{\theta}}$ for the Markov chains damage model according to the methodology explained in Section 2.2. A model parameterisation given by $\boldsymbol{\theta} = \{p, \theta_1, \theta_1', \theta_2, \theta_2'\}$ with a Markov chain assembly of $s = 40$ states is adopted here, which was revealed as an optimal model configuration after some initial trials. In addition, a modified time unit is adopted to speed up the model simulations as normal practice using this class of models [7], whereby 1 *duty cycle* in the modified time scale equals to 1000 load cycles in the natural scale. With these settings, the optimal model parameters $\hat{\boldsymbol{\theta}}$ for the calibration dataset are found by solving the optimisation problem in Equation (8), resulting $\hat{\boldsymbol{\theta}} = \{0.661, 0.094, 0.436, 0.310, 0.716\}$. For this optimisation problem, a general purpose GA from the Optimisation Toolbox of Matlab ® is used with the configuration parameters specified in Table 1. In order to investigate the correctness of the parameter estimation, the estimated values $\hat{\boldsymbol{\theta}}$ are compared with those obtained using a full Bayesian inference method, as introduced in [20]. This is shown in Figure 5, where it is observed that the estimated values of model parameters lay within the region of high probability of the *posterior* PDF $p(\boldsymbol{\theta}|\mathcal{D}_{14})$. To numerically solve the Bayesian inference, the Metropolis-Hastings algorithm [36, 37] is applied with a multivariate Gaussian for the proposal PDF with identical standard deviation in each dimension. The standard deviation is appropriately selected through initial test runs so that the acceptance rate is within the suggested range $[0.2, 0.4]$ [38].

Moreover, as a proof of accuracy of the parameter estimation, the CDFs of the damage states $F(x_n|\hat{\boldsymbol{\theta}})$ predicted by the model parameterised by $\hat{\boldsymbol{\theta}}$ are compared in Figure 4 against their experimental counterpart $F(y_n)$, for a representative subset of times (load cycles) covering the whole process. As evident from the

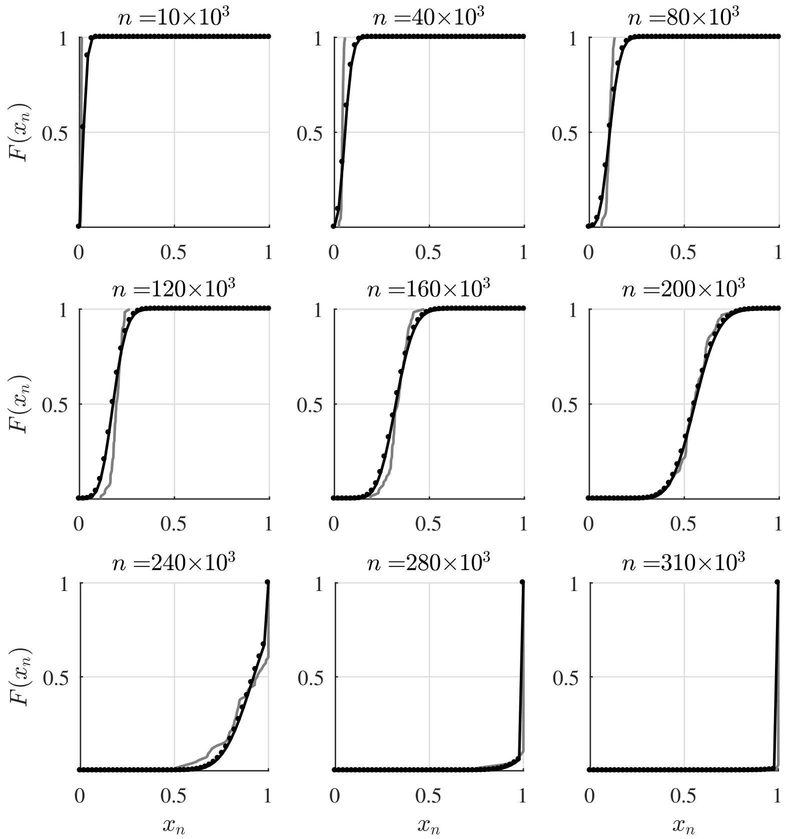


Figure 4: Cumulative distribution functions (CDFs) of the damage state x_n at several times n along the process as given by the Markov chains model (dotted black line), in comparison with the corresponding experimental CDFs obtained from data (solid gray line).

results, the agreement between the modelled and empirical damage states is remarkable at the considered times, which reveals the suitability of the proposed parameter estimation method.

4.2. Damage prognostics and reliability

The Markov chains model given by model parameters $\hat{\boldsymbol{\theta}}$ is embedded within the filtering framework described in Section 3.1 to obtain sequential estimations and predictions about the damage state of the structure. To this end, the set of particle weights $\{\omega^i_{n|n-1}\}_{i=0}^{s}$ representing the predicted state of damage at a particular time $n-1$ are sequentially updated using Equation (12b) on the basis of a new available measurement at time n. This measurement is taken from the prediction dataset $\mathcal{D}^*$ (not used for model calibration), so as to ensure that the model is working within a proper prognostics framework. The updated particles are then used as input to Equation (12a) to make predictions about the future states of degradation in absence of new data. This process is repeated every time a new measurement is collected, as depicted in Figure 1.

Parameter	Value
Population size	50
Generations	80
Prob. of crossover	0.80
Prob. of mutation	0.010
Prob. of selection	0.70
Selection method	RWS [39]
Crossover method	Two points [39]

Table 1: Configuration parameters for the GA-based optimisation problem in Eq. (8).

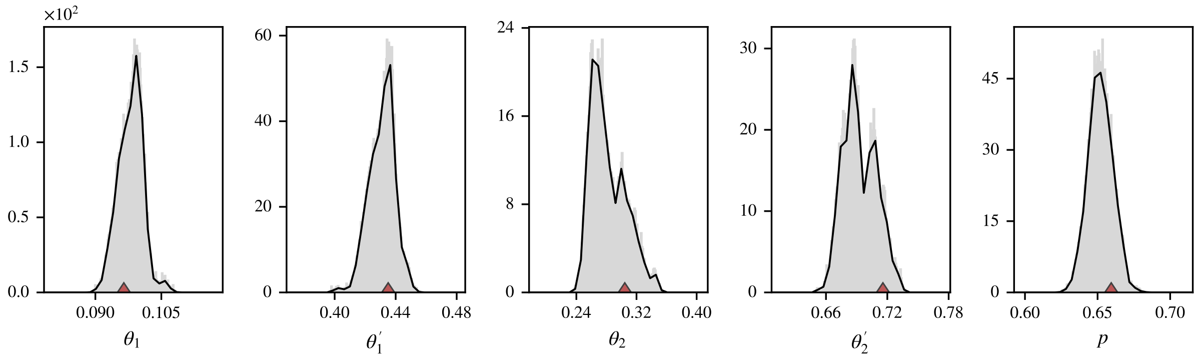
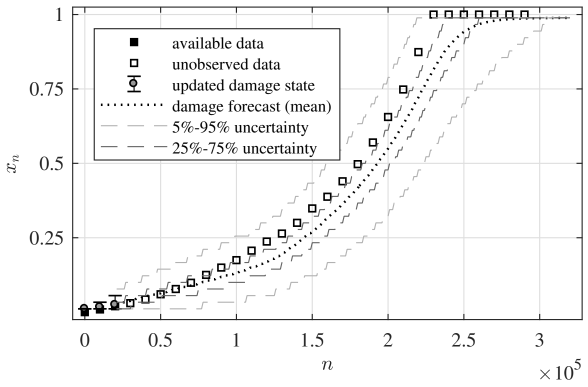


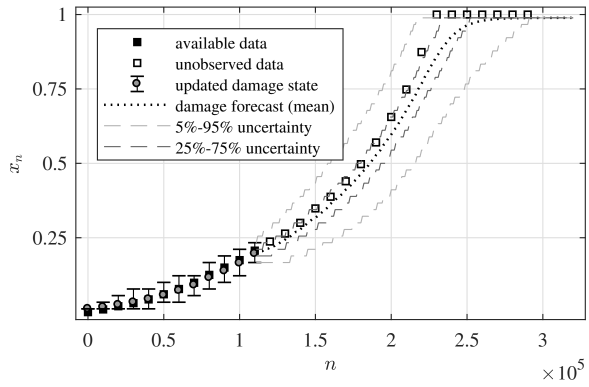
Figure 5: Posterior PDFs of model parameters obtained using Bayesian inference. The marker (red triangle) indicates the values for $\hat{\boldsymbol{\theta}}$ obtained using the parameter estimation method in Section 2.2.

Results for sequential state estimation and future state predictions are shown in Figures 6a to 6c for a sample damage sequence taken from the prediction dataset. Observe that the damage prognostics model produces increasingly accurate predictions of the upcoming damage states with quantified uncertainty. This prediction uncertainty takes into account both the inherent uncertainty of the Markov chains model along with the measurement error. For this case study, the degradation process is assumed to start from the non-damage state, therefore $\mathbf{p}_{0|0} = (1, 0, \ldots, 0)$ in Equation (14a). In regards to the measurement uncertainty, the standard deviation of the measurement error parameter is set to σ_n=0.025, (recall Eq. (13)), taking it as known.

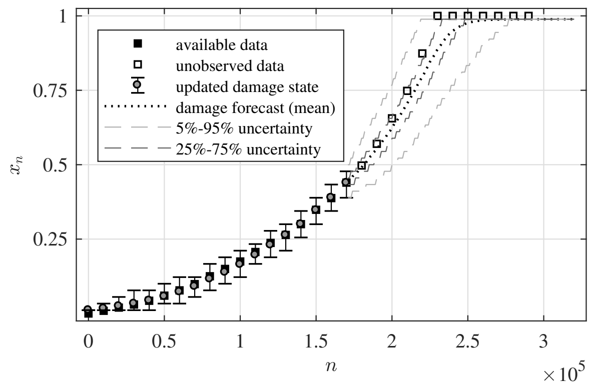
From the prediction of damage $\ell \in \mathbb{N}$ steps forward in time, an estimation of the future reliability associated to the degradation process is subsequently obtained using the time-dependent reliability methodology explained in Section 3.2. To this end, the predicted reliability $\mathcal{R}_{n+\ell|n}$ is estimated for each $\ell > 1$ as the probability of the damage states laying below a predefined damage threshold ξ, as defined in Equation (15). The results are shown in Figures 7a to 7c for selected cycles $n = \{20, 110, 170\} \times 10^3$. For this example, the thresh-



(a) Time of prediction: $n = 20,000$ cycles



(b) Time of prediction: $n = 110,000$ cycles



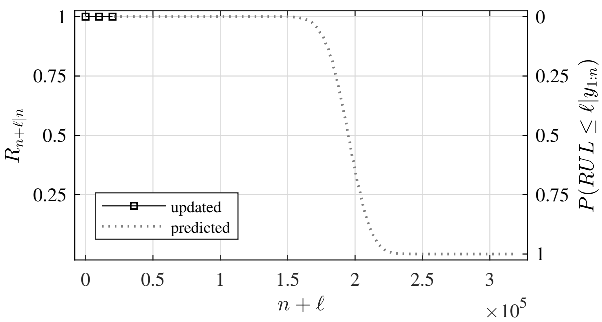
(c) Time of prediction: $n = 170,000$ cycles

Figure 6: Sequential estimation and future prediction of damage at several prediction times n along the process. The uncertainty of the predicted damage states is represented using dashed gray lines for the 25%-75% probability band (darker gray lines) and the 5%-95% probability band.

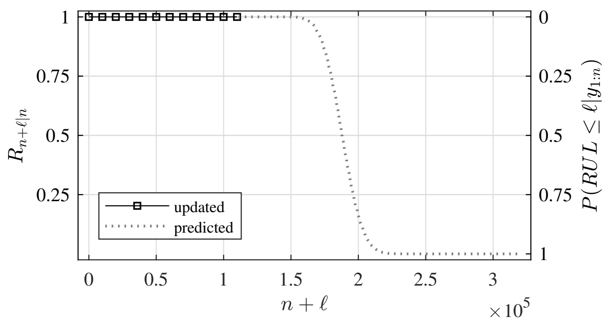
old is set to $\xi = 0.5$, therefore $\mathcal{R}_{n+\ell|n}$ has the meaning of the probability of the predicted damage state at time $n + \ell$ to take values below the 50% of the absorbing (failure) damage state. In addition, according to Equation (20), the computation of the time-dependent reliability $\mathcal{R}_{n+\ell|n}$ implicitly conveys a probabilistic prediction of the remaining time until the degradation reaches the established damage threshold. Therefore, the time-dependent reliability curves shown in Figures 7a to 7c can also be read as the complementary of the probability $P(RUL_n \leqslant \ell|y_{1:n})$, as depicted in the right hand vertical axes of those figures. From them, probability-based estimations about the RUL of the degraded structure, such as the median and the 25%-75% percentiles, can be straightforwardly obtained every time a new measurement is collected.

5. Discussion

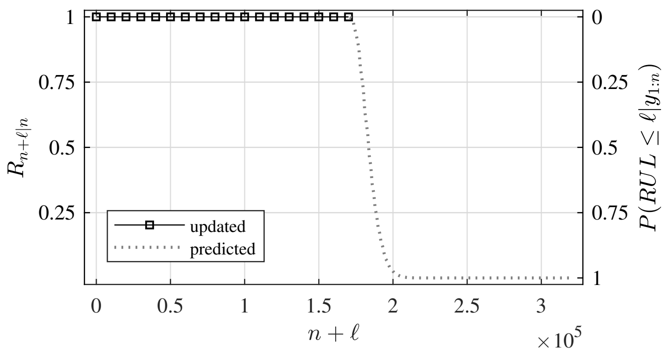
In this paper, an efficient computational and modelling framework has been presented for condition-based prognostics of complex degradation processes for which a physics-based degradation model is seldom



(a) Time of prediction: $n = 20,000$ cycles



(b) Time of prediction: $n = 110,000$ cycles



(c) Time of prediction: $n = 170,000$ cycles

Figure 7: Time-dependent reliability updating (square markers) and future reliability prediction (dotted line) at several prediction times n along the process. In the right-hand vertical axis, updated probability distribution of the remaining useful life (RUL) of the degraded structure.

known in practice. The methodology has been exemplified using the case study presented in the previous section based on Virkler's fatigue damage dataset. As a first interpretation of the results in Figure 6, the proposed methodology is able to accurately anticipate the evolution of the future stages of degradation with quantified uncertainty after an initial learning stage using limited data. This is reflected in Figures 6a to 6c, where the predicted evolution of damage (dotted black line) is shown to be in agreement with the upcoming damage data (empty squares) after assimilating a limited amount of condition monitoring data. This early correctness of the prognostics results can be interpreted as a consequence of having used *enough* amount of data for model calibration, which is supported by the method presented in Section 3.3. In terms of the prediction uncertainty, it is also observed in Figure 6 that the spread of the future damage predictions (represented by the 5%-95% and 25%-75% probability bands) tends to decrease as the model is sequentially updated with new condition data, as long as it is available.

The referred dependence on historical data for prognostics may arguably denote a limitation of the proposed prognostics methodology; notwithstanding, we emphasise here that the aim of such a methodology is to tackle *complex* degradation processes, for which condition monitoring data constitute the available source of physical information. In fact, this dependence on condition monitoring data suggests the type of engineering applications where the proposed methodology can be useful. For example, we can point out as potential applications structures or infrastructures subjected to periodic degradation-repair cycles, such as wind turbines, railway track assets, and aeronautic structures, among others, where condition monitoring data from past maintenance cycles can be used to make predictions about the current maintenance cycle according to the proposed prognostics framework. In this context, the methodology for the minimum required amount of calibration data presented in Section 3.3 constitutes an useful piece of information, since it enables a principled estimation about the amount of data, and, therefore, the minimum number of assets (e.g., the number of wind turbines within a wind farm) that need to be monitored to get an accurate predictive model for prognostics.

Finally, in the context of the extensibility of the proposed methodology to a real-life asset management scenario, it can be noted as a potential benefit the possibility to incorporate repair and corrective maintenance actions into the prognostics model. These could be effectively implemented by allowing non-zero "backward" transition probabilities in the elementary transition probability matrix of the Markov chains damage model; e.g., $1 < p(\bar{x}^i_{n+1}|\bar{x}^j_n) \leqslant 0$ with $j > i$, meaning that once the j-th damage state is reached, there is a probability different to zero to get the system to a lesser state of damage i. These *repair probabilities* can be inferred from maintenance data or properly defined by the modeller according to a particular maintenance policy. From this standpoint, infrastructure asset management activities such as optimal operation and maintenance planning can be carried out based on condition-based prognostics results, which constitutes an immediate next step of this research.

6. Concluding remarks

A condition-based prognostics methodology has been developed in this paper for complex degradation processes for which condition monitoring data constitute the only available source of physical information. The prognostics method relies on a stochastic damage model based on Markov chains calibrated using historical condition monitoring data, which is subsequently embedded within a filtering algorithm for sequential state estimation and remaining useful life (RUL) prediction. Given the discrete and low dimensional state space assumptions prescribed by the Markov chains damage model, the filtering *prediction* and *updating* equations and the subsequent RUL estimations were obtained as closed-form expressions, which makes this method specially suitable for applications within a real-time structural health monitoring context. The proposed methodology is general but in this paper it was exemplified using highly scattered fatigue damage data taken from the literature. The results demonstrated the efficiency and the accuracy of the proposed method for sequentially predicting the future stages of degradation and the remaining useful life of the degraded structures after a short model adaptation period using limited data. More research effort is needed to exploit the full potential of the proposed prognostics methodology in the context of predictive asset management modelling.

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Appendix A. Particle filter based on Markov chains

We use Equations (11a) and (11b) for the prediction and updating equations written as functions of the state probabilities:

$$p(x_n|y_{1:n-1}) = \sum_{i=0}^{s} \omega_{n|n-1}^{i} \delta(x_n - \bar{x}_n^i)\,, \tag{A.1a}$$

$$p(x_n|y_n) = \sum_{i=0}^{s} \omega_{n|n}^{i} \delta(x_n - \bar{x}_n^i)\,. \tag{A.1b}$$

From the Total Probability Theorem:

$$\begin{aligned} p(x_n|y_{1:n-1}) &= \int_{\mathcal{X}} p(x_n|x_{n-1}) p(x_{n-1}|y_{1:n-1}) dx_{n-1} \\ &= \int_{\mathcal{X}} \sum_{i=0}^{s} p(\bar{x}_n^i|x_{n-1}) \delta(x_n - \bar{x}_n^i) \sum_{l=0}^{s} \omega_{n-1|n-1}^{l} \delta(x_{n-1} - \bar{x}_{n-1}^l) dx_{n-1} \\ &= \sum_{i=0}^{s} \sum_{l=0}^{s} p(\bar{x}_n^i|\bar{x}_{n-1}^l) \cdot \omega_{n-1|n-1}^{l} \delta(x_n - \bar{x}_n^i) \\ &= \sum_{i=0}^{s} \sum_{l=0}^{s} p_{n-1}^{l,i} \cdot \omega_{n-1|n-1}^{l} \delta(x_n - \bar{x}_n^i) \\ &\triangleq \mathbf{p}_{n-1|n-1} \cdot \mathbf{P}_{n-1}\,. \end{aligned} \tag{A.2}$$

Given Equations (A.1a) and (A.2), it follows that

$$\omega_{n|n-1}^{i} = \sum_{l=0}^{s} p_{n-1}^{l,i} \omega_{n-1|n-1}^{l}\,. \tag{A.3}$$

From Bayes' Theorem:

$$\begin{aligned} p(x_n|y_{1:n}) &= \frac{p(x_n|y_{1:n-1}) p(y_n|x_n)}{\int_{\mathcal{X}} p(x_n|y_{1:n-1}) p(y_n|x_n) dx_n} \\ &= \frac{\sum_{i=0}^{s} \omega_{n|n-1}^{i} p(y_n|x_n) \delta(x_n - \bar{x}_n^i)}{\sum_{i=0}^{s} \omega_{n|n-1}^{i} p(y_n|\bar{x}_n^i)}\,. \end{aligned} \tag{A.4}$$

Then given Equations (A.1b) and (A.4) we have:

$$\omega^i_{n|n} = \frac{\omega^i_{n|n-1} p(y_n|\bar{x}^i_n)}{\sum_{l=0}^{s} \omega^l_{n|n-1} p(y_n|\bar{x}^l_n)}\,, \tag{A.5}$$

where $\omega^i_{n|n-1}$ was obtained at the preceding cycle $n-1$ using Equation (A.3), and

$$p(y_n|\bar{x}^i_n) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(- \frac{(y_n - \bar{x}^i_n)^2}{2{\sigma_n}^2} \right), \tag{A.6}$$

as explained in Section 3.1.

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