Desirable gambles based on pairwise comparisons[☆]

Serafín Moral

Dpto. Ciencias de la Computación e Inteligencia Artificial, CITIC-UGR (Centro de Investigación Tecnologías de Infor. y Comunicaciones),
 Universidad de Granada, Granada, 18071, Spain

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ABSTRACT

This paper proposes a model for imprecise probability information based on bounds on probability ratios, instead of bounds on events. This model is studied in the language of coherent sets of desirable gambles, which provides an elegant mathematical formulation and a more expressive power. The paper provides methods to check avoiding sure loss and coherence, and to compute the natural extension. The relationships with other formalisms such as imprecise multiplicative preferences, the constant odd ratio model, or comparative probability are analyzed.

1. Introduction

Imprecise probability [1] advocates for a logical view of probability in which the basic concepts are avoiding sure loss, coherence, and natural extension. There are several representations of imprecise probabilistic information, one of the most general ones is given by upper and lower previsions [2]. One of the problems associated with this representation is that, even in the finite case, checking these concepts implies solving complex linear programming problems [3]. Less general models are commonly used in which more efficient algorithms exist. Most of them, are based on assigning interval probabilities to events [4]. However, there are other alternatives. One of them is to provide intervals to probability ratios. This approach has been proposed in the field of robust probability [5,6], it has been called the constant odds-ratio model [1,7], but it has received little attention.

This paper proposes a general study of these models in a very general setting: it uses the theory of coherent sets of desirable gambles [8–10]. One important advantage is that it allows a simple and elegant description, including the infinite case. The approach is based on considering information that can be represented by gambles with only two non-zero values comparing the uncertainties associated with two elements, and for this reason, is called “pairwise desirability”. We provide methods for representing and computing with pairwise specifications, that in the finite case are based on matrices multiplication. We also express the problem of determining whether a generic gamble belongs to the natural extension as a generalized gain/loss max-flow problem [11]. As an example of pairwise desirability, we give a representation of equiprobable information. In the finite case, this model corresponds to the uniform distribution, but in the infinite case it is a very imprecise model (the associated credal set is very wide). For example, in the case of the positive integers, it contains the uniform distribution proposed by Walley [1, Section 1.9.5], but it contains many more distributions. Nonetheless, this notion of equiprobability has shown to be useful in some settings as the hierarchical inference based on model selection and averaging proposed in [12].

One important source of imprecise probability representations is the consideration of distortion models, obtained through a discounting of a precise probability by a degree (lack of confidence) [7,13]. There are several approaches to discounting. In this

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 E-mail address: smc@decsai.ugr.es.

paper, it is shown that the discounting based on reducing the payments associated with a gamble by a percentage is especially appropriate for pairwise comparisons as the result is always a very simple to compute pairwise comparison.

Finally, the paper also investigates the relationships of coherent sets of pairwise comparison gambles with other apparently unrelated formalisms such as the imprecise multiplicative preference relationships [14,15] which are the basis for the analytical hierarchy process. The results in the paper show that imprecise probabilities can provide a solid basis for the meaning and management of these preference relationships.

The paper is organized as follows: Section 2 provides the basic concepts and notation; Section 3 introduces the model and gives the main results for checking avoiding sure loss and computing natural extension; Section 4 studies a particular case of special interest: the representation of equiprobability; Section 5 is devoted to discounting and the constant-odd ration model; Section 6 investigates the relationships with multiplicative preference relationships; while Section 7 considers the similarities and differences with comparative probability [16,17,18]; finally Section 8 is devoted to the conclusions and future research.

2. Basic concepts and notation

Here we show the basic concepts for coherent set of desirable gambles and provide the basic notation. A more detailed exposition can be found in [8–10].

If X is a set, a gamble on X is a bounded mapping $f : X \rightarrow \mathbb{R}$. The set of all the gambles in X is denoted as $\mathcal{L}(X)$. Operations and orderings on gambles are defined elementwise. If f and g are gambles, then we say that $f > g$ if and only if $f(x) \geq g(x)$, $\forall x \in X$ and $f(x) > g(x)$ for, at least, one element $x \in X$. A real number a is identified with the constant gamble $f(x) = a$, $\forall x \in X$. A set of gambles, $D \subseteq \mathcal{L}(X)$, on X is said to be a coherent set of desirable gambles if and only if it satisfies the following conditions:

- D1. $0 \notin D$,
- D2. If $f > 0$ then $f \in D$,
- D3. if $f \in D$ and $c > 0$ then $cf \in D$,
- D4. if $f, g \in D$ then $f + g \in D$.

If D and D' are coherent sets of desirable gambles, then D is said to be less informative than D' if and only if $D \subseteq D'$.

A general set of gambles D is said to avoid sure loss if and only if there is a coherent set of gambles D' such that $D \subseteq D'$. In that case, there is a smallest coherent set (in the sense of inclusion, i.e. being less informative) containing D and this set is called the natural extension of D and denoted as \overline{D} .

It is well known that D does not avoid sure loss if and only if there are $f_i \in D, \alpha_i > 0, i = 1, \dots, m$ such that

$$0 \geq \alpha_1 f_1 + \dots + \alpha_m f_m.$$

If D avoids sure loss, then $g \in \overline{D}$ if and only if there are $f_i \in D, \alpha_i > 0, i = 1, \dots, m$ such that

$$g \geq \alpha_1 f_1 + \dots + \alpha_m f_m.$$

A linear prevision P on $\mathcal{L}(X)$ is a mapping $P : \mathcal{L}(X) \rightarrow \mathbb{R}$ satisfying,

- If f is a gamble, $P(f) \geq \inf f$,
- If f, g are gambles, $P(f + g) = P(f) + P(g)$,
- If f is a gamble and $c > 0$, $P(cf) = cP(f)$.

It is well known that a linear prevision is equivalent to a finitely additive probability [1]. If $A \subseteq X$, then $P(A)$ is equal to $P(I_A)$ where I_A is the indicator function of A . To simplify the notation, $P(\{x\})$ will be denoted by $P(x)$, for $x \in X$.

A set of desirable gambles, D , always defines a credal set, which is the convex set, $C(D)$, of linear previsions, P , satisfying $P(X) \geq 0$, $\forall X \in D$.

If we know the associated credal set $C(D)$, there are several coherent sets of desirable gambles which can define $C(D)$, but it is known that if $P(f) > 0$ is satisfied for all $P \in C(D)$, then $f \in D$.

The vacuous set of gambles, D_0 , represents the ignorance (least informative coherent set of desirable gambles) and it is given by $D_0 = \{f \in \mathcal{L}(X) \mid f > 0\}$. Its associated credal set, P_0 , contains all the additive probabilities, P , defined on X .

The definition of conditioning is very simple: if D is a coherent set of desirable gambles on X and $H \subseteq X, H \neq \emptyset$, then the conditioning of D to H is the coherent set of desirable gambles $D_H = \{g \in \mathcal{L}(H) : g.I_H \in D\}$, where I_H is the indicator function of H . There is an alternative definition in which gambles $g \in \mathcal{L}(X)$ are considered, but we have considered only gambles defined on the conditioning set H . If $x \in X \setminus H$, we take that $g.I_H(x) = 0$.

A coherent set of desirable gambles D always defines a coherent set of almost desirable gambles D^* , which is given by the set of gambles g such that $g + \epsilon \in D$ for all $\epsilon > 0$. The coherence properties of sets of almost desirable gambles are the same as for desirable gambles except that condition D1 is transformed into if $f < 0$ then $f \notin D^*$ and a new property is added: if $f + \epsilon \in D^*, \forall \epsilon > 0$, then $f \in D^*$.

3. Pairwise desirability

3.1. Representation

A gamble f is said to express a pairwise desirability if and only if it can be expressed as $f = aI_x + bI_y$, where $a, b \in \mathbb{R}$, $x, y \in X$, and I_x is the indicator function of $\{x\}$. A gamble like this solely makes sense when $a.b < 0$, as in another case it is trivial ($f > 0$) or undesirable under coherence ($f \leq 0$). Without loss of generality, we will assume that $a = 1, b < 0$, i.e. that the gamble is $f = I_x - bI_y$ with $b > 0$ (this can always be achieved by considering an equivalent gamble obtained through multiplying by a positive constant as a consequence of condition D3 for coherence).

For some fixed elements $x, y \in X$, the set $\{b \in \mathbb{R} \mid I_x - bI_y \text{ is desirable}\}$ should be an interval of the real numbers with lower value $-\infty$. The reason is that if $I_x - bI_y$ is desirable and $c \leq b$, then we have that $I_x - bI_y \leq I_x - cI_y$, and $I_x - cI_y$ should also be desirable. Also, as for $b \leq 0$, the gamble $I_x - bI_y > 0$ is always desirable, it should always include the interval $(-\infty, 0]$. We have the following options for the intervals:

- $(-\infty, a]$ where $a \geq 0$,
- $(-\infty, a)$ where $a > 0$,
- $(-\infty, +\infty)$.

To simplify the notation in the paper, we will represent the closed interval $(-\infty, a]$ by a , the open interval $(-\infty, a)$ by a^- , and the interval $(-\infty, +\infty)$ by $+\infty$. The set of possible representations of an interval will be denoted by $V = \{a : a \geq 0\} \cup \{a^- : a > 0\} \cup \{+\infty\}$. Given $v \in V$, the interval represented by v will be denoted as $i(v)$, in this way, $i(2) = (\infty, 2]$ and $i(3^-) = (\infty, 3)$.

An initial specification of pairwise desirable gambles is a mapping $t : X \times X \rightarrow V$, satisfying $t(x, x) = 1^-$, if $x \in X$. An initial specification t can be considered to be a representation of a set of pairwise desirable gambles, namely:

$$\mathcal{T} = \{I_x - bI_y : b \in \mathbb{R}, 0 < b \in i(t(x, y))\} \quad (1)$$

To simplify, we have decided not to include in \mathcal{T} the gambles $I_x - bI_y$ with $b \leq 0$, which are always trivially desirable.

In detail, \mathcal{T} is defined by the following rules:

- If $t(x, y) = 0$, no gamble $I_x - bI_y$, with $b > 0$ is initially desirable. This situation represents vacuous information about the desirability of gambles $I_x - bI_y$. Only positive gambles (with $b \leq 0$) would be desirable.
- If $t(x, y) = +\infty$, every gamble $I_x - bI_y$, with $b > 0$ is considered to be desirable. In this case, any probability, P , in the associated credal must satisfy $P(y) = 0$, as in other case making b large enough we could have $P(I_x - bI_y) < 0$.
- If $t(x, y) = a$, $a > 0$, then every gamble $I_x - bI_y$ with $b \in (0, a]$ is desirable. The two first rules represent extreme situations (none or all gambles are desirable). This situation represents an intermediate case: some are desirable and some are not. As we are considering general sets of desirable gambles, the set of desirable gambles $I_x - bI_y$ could include its boundary or not. This case represents the first situation, and the following case the second one.
- If $t(x, y) = a^-$, $a > 0$, then every gamble $I_x - bI_y$ with $b \in (0, a)$ is desirable.

We can define a total order on the set V given by $v \leq v'$ if and only if $i(v) \subseteq i(v')$. In this way, we have that if $a < b$ are real numbers, then $a^- < a < b^- < b$ as elements of V .

Taking this order into account, the set \mathcal{T} can also be expressed as:

$$\mathcal{T} = \{I_x - bI_y : b \in \mathbb{R}, 0 < b \leq t(x, y)\}$$

If $A \subseteq V$, then the supremum of A always exists. If we consider $A' = \{a \in \mathbb{R} : a \in A \text{ or } a^- \in A\}$, then if A is not upper bounded, its supremum is $+\infty$. If A' is upper bounded and its supremum is b , then the supremum of A is b if $b \in A$, and b^- if $b \notin A$.

The vacuous information can be represented by the pairwise specification $t_0(x, y) = 0$ if $x \neq y$ and $t_0(x, x) = 1^-$. Its associated set of desirable gambles \mathcal{T}_0 is equal to D_0 .

If X is a finite set with n elements, $X = \{x_1, \dots, x_n\}$, then t can be specified by an $n \times n$ matrix T in which the element T_{ij} is equal to $t(x_i, x_j)$, and $T_{ii} = 1^-$.

Example 1. Assume that $X = \{x_1, x_2, x_3\}$ and the initial specification given by the following matrix:

$$T = \begin{pmatrix} 1^- & 0.2 & 0.3 \\ 1 & 1^- & 0.5 \\ 2 & 1 & 1^- \end{pmatrix}$$

We are specifying that, for example, both $I_{x_1} - 0.3I_{x_3}$ and $I_{x_3} - I_{x_2}$ are desirable. This specification determines a set of gambles initially desirable. By the rules of coherence, we can deduce that other gambles are also desirable. In this case, we can deduce that $(I_{x_1} - 0.3I_{x_3}) + 0.3(I_{x_3} - I_{x_2}) = I_{x_1} - 0.3I_{x_2}$ is also desirable, but this gamble was not in the initial specification, as $t(x_1, x_2) = 0.2$ and only gambles $I_{x_1} - bI_{x_2}$ with $b \leq 0.2$ were initially desirable. An important question is whether we can deduce, by applying these rules, that a gamble $f \leq 0$ should be desirable, i.e. that the initial specification does not avoid sure loss. In the following, methods

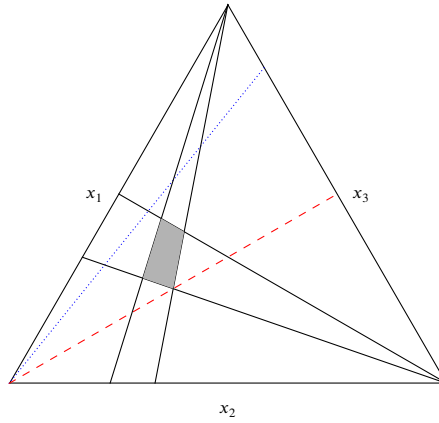


Fig. 1. The associated credal set.

to test avoiding sure loss and to compute the gambles that can be deduced from the initial specification (the natural extension) are provided.

From the point of view of the associated credal set, a pairwise desirable gamble can be interpreted as a bound on the probability ratios of x and y : if $I_x - bI_y$ is desirable, then if P is in the associated credal set, we have that $P(x) - bP(y) \geq 0$, i.e. that $\frac{P(x)}{P(y)} \geq b$, where it is considered $0/0 = 0$ and $a/0 = +\infty$, if $a > 0$, i.e. a pairwise desirable gamble provides a bound on the probability ratio of two elementary events.

If X has three elements, the set of probabilities defined on X can be represented on the probability triangle. A restriction such as $\frac{P(x_2)}{P(x_1)} = 1$ is represented by the line of points in which $P(x_1) = P(x_2)$, i.e. the points that are at the same distance of the triangle sides x_1 and x_2 : the dashed line in red in Fig. 1. The bound $\frac{P(x_2)}{P(x_1)} \geq 1$ delimits the region of the triangle above the dashed red line. Analogously, any bound $\frac{P(x_j)}{P(x_i)} \geq b$ determines a region of the triangle delimited by a line starting at a triangle vertex where the x_i and x_j sides intersect and ending on the opposite side. It is possible to notice in Fig. 1 the line corresponding to $\frac{P(x_1)}{P(x_2)} \geq 0.2$ (dotted blue color line).

In this way, the credal set associated with a set of pairwise gambles is delimited by lines going from a triangle vertex to the opposite side. In Fig. 1 the credal set associated with the specification of Example 1 is depicted in gray.

3.2. Avoiding sure loss and natural extension

If t is a specification and i is a natural number, then, t^i is recursively defined as follows:

$$t^1 = t, \quad t^{i+1} = \max\{t^i \circ t, t^i\}$$

where $t_1 \circ t_2(x, y) = \sup\{t_1(x, z) \cdot t_2(z, y) : z \in X\}$ and the product is the extension of the product of two real numbers, where $+\infty \cdot 0 = 0$, $+\infty \cdot b = +\infty$ ($b > 0$), $a^- \cdot 0 = 0$, and $a^- \cdot b^- = a^- \cdot b = a \cdot b^- = (a \cdot b)^-$, if $a, b > 0$.

Proposition 1. If t in an initial specification and \mathcal{T} is its associated set of gambles, then if $0 < b \leq t^i(x, y)$, we have that there are a number $m \leq i$ of gambles $I_{x_1} - b_1 I_{y_1}, \dots, I_{x_m} - b_m I_{y_m} \in \mathcal{T}$ and positive numbers $\alpha_1, \dots, \alpha_m$ such that

$$I_x - bI_y \geq \alpha_1(I_{x_1} - b_1 I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_m I_{y_m})$$

Proof. We will prove it by induction on i . For $i = 1$ the result is trivial.

Assume it is true for i and we will prove it for $i + 1$. If $0 < b \leq t^{i+1}(x, y)$ then either $b \leq t^i(x, y)$ and the result is obtained by induction or $b \leq t^i \circ t(x, y)$.

We have that $t^i \circ t(x, y) = \sup\{t^i(x, z) \cdot t(z, y) : z \in X\}$. If $t^i \circ t(x, y) \notin \{t^i(x, z) \cdot t(z, y) : z \in X\}$, we have that $t^i \circ t(x, y) = c^-$, where $c \in \mathbb{R}$ or $t^i \circ t(x, y) = +\infty$. As b is a real number, in both cases we can deduce that $b < t^i \circ t(x, y)$ and there is always a value $z \in X$ such that $b \leq t^i(x, z) \cdot t(z, y)$. Then there are b', b'' such that $b \leq b' \cdot b''$ and $b' \leq t^i(x, z)$, $b'' \leq t(z, y)$.

Now we can apply the induction hypothesis and there are gambles $I_{x_1} - b_1 I_{y_1}, \dots, I_{x_m} - b_m I_{y_m} \in \mathcal{T}$ and positive numbers $\alpha_1, \dots, \alpha_m$ with $m \leq i$ such that,

$$I_x - b' I_z \geq \alpha_1(I_{x_1} - b_1 I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_m I_{y_m})$$

Now, we have

$$I_x - bI_y \geq I_x - b'b''I_y = I_x - b'I_z + b'(I_z - b''I_y) \geq$$

$$\alpha_1(I_{x_1} - b_1I_{z_1}) + \dots + \alpha_m(I_{x_m} - b_mI_{z_m}) + b'(I_z - b''I_y)$$

and the result follows, as $I_z - b''I_y \in \mathcal{T}$ because $b'' \leq t(z, y)$. \square

Given $x, y \in X$, the sequence $\{t^i(x, y)\}_{i \in \mathbb{N}}$ is non decreasing and has a supremum. Let us call it $\bar{t}(x, y)$. It is clear that for $b \in \mathbb{R}$, we have that $b \leq \bar{t}(x, y)$ if and only if there is $i \geq 1$ such that $b \leq t^i(x, y)$. This is a consequence of the fact that if the supremum of $\{t^i(x, y)\}_{i \in \mathbb{N}}$ is not in the sequence, then it will be c^- or $+\infty$ and that if $b \in \mathbb{R}$ and $b \leq c^-$ (or $b \leq +\infty$), we necessarily have $b < c$ (or $b < +\infty$).

In the finite case, if T is a matrix specification, then \bar{T} will represent the matrix associated with \bar{t} . The function \bar{t} will be called the natural extension of t and the name will be justified by the results we prove in this section. First, we are going to prove a useful result.

Proposition 2. *If t is an initial pairwise specification on X and \bar{t} is its natural extension, then $\bar{t}(x, y) \cdot \bar{t}(y, z) \leq \bar{t}(x, z)$, $\forall x, y, z \in X$.*

Proof. We have that $t^{i+1}(x, z) \geq (t^i \circ t)(x, z) = \sup_{y \in X} (t^i(x, y) \cdot t(y, z))$.

So, for every $y \in X$, we have that $t^{i+1}(x, z) \geq (t^i(x, y) \cdot t(y, z))$

As $\bar{t}(x, z) \geq t^{i+1}(x, z)$ we have that:

$$\bar{t}(x, z) \geq (t^i(x, y) \cdot t(y, z)), \forall x, y, z \in X$$

As $\bar{t}(x, z)$ is an upper bound of values $\{t^i(x, y) \cdot t(y, z) : i \geq 1\}$, we have that

$$\bar{t}(x, z) \geq \sup_{i \geq 1} (t^i(x, y) \cdot t(y, z)) = \bar{t}(x, y) \cdot t(y, z), \forall x, y, z \in X$$

Now, by induction on i we are going to prove that $\bar{t}(x, z) \geq \bar{t}(x, y)t^i(y, z)$

For $i = 1$ the result is what we have proved, as $t^1 = t$.

Assume that it is true for i and we are going to prove it for $i + 1$. Assume that $r \in X$ is an arbitrary value, then by the hypothesis of induction $\bar{t}(x, r) \geq \bar{t}(x, y)t^i(y, r)$ and therefore:

$$\bar{t}(x, z) \geq \bar{t}(x, r)t(r, z) \geq \bar{t}(x, y)t^i(y, r)t(r, z)$$

As $r \in X$ is arbitrary, we obtain,

$$\bar{t}(x, z) \geq \sup_{r \in X} \bar{t}(x, y)t^i(y, r)t(r, z) = \bar{t}(x, y)(t^i \circ t)(y, z)$$

As we also have $\bar{t}(x, z) \geq \bar{t}(x, y)t(y, z)$ we find

$$\bar{t}(x, z) \geq \bar{t}(x, y) \max \{(t^i \circ t)(y, z), t(y, z)\} = \bar{t}(x, y)t^{i+1}(y, z)$$

and the result follows for $i + 1$.

Now, since we showed that $\bar{t}(x, z) \geq \bar{t}(x, y)t^i(y, z)$ for every $i \geq 1$, we conclude that

$$\bar{t}(x, z) \geq \sup_{i \geq 1} \bar{t}(x, y)t^i(y, z) = \bar{t}(x, y)\bar{t}(y, z). \quad \square$$

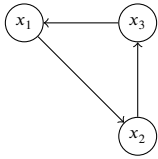
Example 2. Assuming the initial representation of Example 1, matrix T converges in one iteration to its limit:

$$\bar{T} = \begin{pmatrix} 1^- & \mathbf{0.3} & 0.3 \\ 1 & 1^- & 0.5 \\ 2 & 1 & 1^- \end{pmatrix}$$

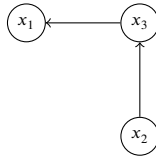
In boldface, we can see the only value which has been modified. This value, $\bar{T}(1, 2)$ has changed, as $T(1, 3) \cdot T(3, 2) = 0.3 \times 1 = 0.3$ which is greater than the original value 0.2. In Fig. 1 it is possible to see that the original restriction $\frac{P(x_1)}{P(x_2)} \geq 0.2$ (corresponding to the original value $T(1, 2)$) is redundant (dotted line in blue).

Lemma 1. *Any combination $\alpha(I_x - bI_y) + \beta(I_y - cI_z)$, where $0 < b \leq \bar{t}(x, y)$, $0 < c \leq \bar{t}(y, z)$, can be expressed as either one of the following options:*

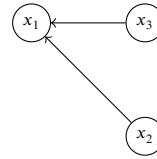
- $\alpha(I_x - bI_y) + \beta(I_y - cI_z) = \alpha'(I_x - bI_y) + \beta'(I_x - d'I_z)$, or
- $\alpha(I_x - bI_y) + \beta(I_y - cI_z) = \alpha'(I_x - d'I_z) + \beta'(I_y - cI_z)$,



(a) Graph from Expression (2).



(b) Graph from Example 4.



(c) Graph from Example 5.

Fig. 2. Graphs associated to expressions.

where $d' \leq \bar{t}(x, z)$.

Proof. We have that either $ab \geq \beta$ or $ab < \beta$. In the first case, we have:

$$\alpha(I_x - bI_y) + \beta(I_y - cI_z) = (\alpha - \frac{\beta}{b})(I_x - bI_y) + \frac{\beta}{b}(I_x - bI_y) + \beta(I_y - cI_z) = (\alpha - \frac{\beta}{b})(I_x - bI_y) + \frac{\beta}{b}(I_x - cbI_z),$$

and then, the first option holds with $\alpha' = (\alpha - \frac{\beta}{b})$, $\beta' = \frac{\beta}{b}$, and $d' = cb \leq \bar{t}(x, y)\bar{t}(y, z) \leq \bar{t}(x, z)$.

If $ab < \beta$, we have:

$$\alpha(I_x - bI_y) + \beta(I_y - cI_z) = \alpha(I_x - bI_y) + \alpha b(I_y - cI_z) + (\beta - \alpha b)(I_y - cI_z) = \alpha(I_x - bcI_z) + (\beta - \alpha b)(I_y - cI_z),$$

and the second option holds with $\alpha' = \alpha$, $\beta' = (\beta - \alpha b) > 0$, and $d' = bc \leq \bar{t}(x, y)\bar{t}(y, z) \leq \bar{t}(x, z)$. \square

Consider any expression,

$$\alpha_1(I_{x_1} - b_1I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_mI_{y_m})$$

where $\alpha_i > 0$. We can associate a graph G with it where there is a node for each x_1, \dots, x_m and each y_1, \dots, y_m , and where for each summand $(I_{x_j} - b_jI_{y_j})$, a directed link is added from x_j to y_j .

Example 3. Consider the expression:

$$0.5(I_{x_1} - 0.3I_{x_2}) + 0.2(I_{x_2} - 0.5I_{x_3}) + 2(I_{x_3} - 0.5I_{x_1}). \quad (2)$$

The associated graph can be seen in Fig. 2a.

We can prove the following lemmas.

Lemma 2. Assuming that $\bar{t}(x, x) < 1, \forall x \in X$, given an expression

$$\alpha_1(I_{x_1} - b_1I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_mI_{y_m}) \quad (3)$$

where $b_i \leq \bar{t}(x_i, y_i), \alpha_i > 0$, if the associated graph G has a cycle, then there is another expression

$$\alpha'_1(I_{x'_1} - b'_1I_{y'_1}) + \dots + \alpha'_m(I_{x'_\ell} - b'_\ell I_{y'_\ell})$$

where $b'_i \leq \bar{t}(x'_i, y'_i), \alpha'_i > 0, l \leq m$, that is strictly lower (according to the strict order relation, $<$, defined on gambles) than the original one, and with an associated graph G' that has no cycles. The expression may be such that $\ell = 0$ (there is no summand) and in this case, the expression is equal to 0.

Proof. We can prove it by induction. For $m = 1$ the only possibility of having a cycle is with an expression $\alpha(I_x - aI_x)$ with $a < 1$ and as $\alpha(I_x - aI_x) > 0$, the result follows with $\ell = 0$.

Assume that G has a cycle of length greater or equal to 2, z_1, \dots, z_h where $z_h = z_1$ and there is an arc from z_i to z_{i+1} . Now, for three consecutive nodes in the cycle z_i, z_{i+1}, z_{i+2} , in Expression (3) there will be two summands $\alpha_j(I_{x_j} - b_jI_{y_j})$ and $\alpha_k(I_{x_k} - b_kI_{y_k})$, where $x_j = z_i, y_j = z_{i+1}, x_k = z_{i+1}, y_k = z_{i+2}$. Consider their addition: $\alpha_j(I_{z_i} - b_jI_{z_{i+1}}) + \alpha_k(I_{z_{i+1}} - c_kI_{z_{i+2}})$. Now we can apply, Lemma 1 with $x = z_i, y = z_{i+1}, z = z_{i+2}$, and substitute this expression by one of the two possibilities provided in the lemma. In the associated graph after this substitution, there will be a direct arc from z_i to z_{i+2} . In this way, the length of the cycle is reduced, and we can transform Expression (3) in such a way that there is a cycle from a node z to itself. So, there will be a summand with the form $\beta(I_z - bI_z)$ with $b \leq \bar{t}(z, z) < 1$. As a consequence $\beta(I_z - bI_z) > 0$, and $\beta(I_z - bI_z)$ can be removed from Expression (3) obtaining another expression with fewer summands and that is strictly lower than the original one. Then we can apply the induction hypothesis and the result follows. \square

Example 4. We can apply this lemma to Expression (2) of Example 3. We obtain,

$$\begin{aligned}
 0.5(I_{x_1} - 0.3I_{x_2}) + 0.2(I_{x_2} - 0.5I_{x_3}) + 2(I_{x_3} - 0.5I_{x_1}) = \\
 0.5(I_{x_1} - 0.3I_{x_2}) + 0.15(I_{x_2} - 0.5I_{x_3}) + 0.05(I_{x_2} - 0.5I_{x_3}) + 2(I_{x_3} - 0.5I_{x_1}) = \\
 0.5(I_{x_1} - 0.15I_{x_3}) + 0.05(I_{x_2} - 0.5I_{x_3}) + 2(I_{x_3} - 0.5I_{x_1}) = \\
 0.5(I_{x_1} - 0.15I_{x_3}) + (I_{x_3} - 0.5I_{x_1}) + 0.05(I_{x_2} - 0.5I_{x_3}) + (I_{x_3} - 0.5I_{x_1}) = \\
 0.925I_{x_3} + 0.05(I_{x_2} - 0.5I_{x_3}) + (I_{x_3} - 0.5I_{x_1}) > 0.05(I_{x_2} - 0.5I_{x_3}) + (I_{x_3} - 0.5I_{x_1})
 \end{aligned}$$

We have that $0.5 \leq \bar{t}(x_2, x_3)$ and $0.5 \leq \bar{t}(x_3, x_1)$. The associated graph to the last expression has no cycles (see Fig. 2b).

Lemma 3. Assuming that $\bar{t}(x, x) < 1, \forall x \in X$, given an expression

$$\alpha_1(I_{x_1} - b_1I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_mI_{y_m}) \quad (4)$$

where $b_i \leq \bar{t}(x_i, y_i), \alpha_i > 0$, then there is another expression

$$\alpha'_1(I_{x'_1} - b'_1I_{y'_1}) + \dots + \alpha'_m(I_{x'_m} - b'_mI_{y'_m})$$

where $b'_i \leq \bar{t}(x_i, y_i), \alpha'_i > 0$ which is upper bounded by the original one and such that in the associated graph G' there are no paths of length 3, i.e. there are no three nodes $x \rightarrow y \rightarrow z$.

Proof. By Lemma 2, we can assume that the associated graph has no cycles.

The statement in the lemma is proved by induction in the length of the longest path in the associated graph. If this length is lower or equal than 2, then the result is trivial.

If the length is $I > 2$, then we can select a path z_1, z_2, \dots, z_I , we can consider the first 3 nodes and in the original expression, we are adding $\beta_1(I_{z_1} - c_1I_{z_2}) + \beta_2(I_{z_2} - c_2I_{z_3})$, where $c_1 \leq \bar{t}(z_1, z_2), c_2 \leq \bar{t}(z_2, z_3)$. Now, we can apply Lemma 1, and substitute these two summands by another two with the same sum, but in such a way that in the associated graph one link is created from z_1 to z_3 and one link is removed (either from z_1 to z_2 or from z_2 to z_3). The situation is that the length of this path is decreased by one and no other path has increased its length. If we repeat this for all the paths of length I , then the length of the longest path will be $I - 1$ and the result will follow by induction. \square

If the gamble f is equal to $\alpha_1(I_{x_1} - b_1I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_mI_{y_m})$ and in the graph associated with this expression there are no cycles, we have that $x_i \neq y_i$ for $i = 1, \dots, m$. If there are no paths of length 3 either, to compute $f(x)$ we are never adding a positive and a negative value for two pairwise gambles on the expression, $(I_{x_i} - b_iI_{y_i})$ and $(I_{x_j} - b_jI_{y_j})$. As a consequence, we will have that f will be positive on elements $\{x_1, \dots, x_m\}$, negative on points $\{y_1, \dots, y_m\}$, and null in the rest of the points. This fact will be very useful for reasoning with this type of expression.

Example 5. We start with the last expression of Example 4. We obtain,

$$\begin{aligned}
 0.05(I_{x_2} - 0.5I_{x_3}) + (I_{x_3} - 0.5I_{x_1}) = 0.05(I_{x_2} - 0.5I_{x_3}) + 0.025(I_{x_3} - 0.5I_{x_1}) + 0.975(I_{x_3} - 0.5I_{x_1}) = \\
 0.05(I_{x_2} - 0.25I_{x_1}) + 0.975(I_{x_3} - 0.5I_{x_1}),
 \end{aligned}$$

where $0.25 \leq \bar{t}(x_2, x_1)$ and $0.5 \leq \bar{t}(x_3, x_1)$. The graph associated with the last expression has no cycles, nor a path of length 3 (see Fig. 2c).

Proposition 3. If t is an initial specification, then its associated set of gambles \mathcal{T} avoids sure loss if and only if $\bar{t}(x, x) < 1, \forall x \in X$.

Proof. If $\bar{t}(x, x) \geq 1$ for some $x \in X$, then there will be a value j such that $t^j(x, x) \geq 1$ (we have to take into account that if $t^j(x, x) < 1$ for all $j \geq 0$, we would have $\bar{t}(x, x) = \sup_j t^j(x, x) \leq 1^-$). Now, by Proposition 1, taking $b = 1$ we have that,

$$0 = I_x - 1.I_x \geq \alpha_1(I_{x_1} - b_1I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_mI_{y_m})$$

where $(I_{x_i} - b_iI_{y_i}) \in \mathcal{T}$, and therefore, \mathcal{T} does not avoid sure loss.

On the other hand, if $\bar{t}(x, x) < 1$ for all $x \in X$, we will prove by contradiction that it is not possible that

$$0 \geq \alpha_1(I_{x_1} - b_1I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_mI_{y_m}) \quad (5)$$

Assume that there is an expression

$$0 \geq \alpha_1(I_{x_1} - b_1I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_mI_{y_m})$$

If the associated graph to the right-hand expression has a cycle, then we can apply Lemma 2 and obtain another expression in which the associated graph has no cycles and that is strictly lower than the original.

If the new expression is empty (the new expression is 0), we will have:

$$0 \geq \alpha_1(I_{x_1} - b_1 I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_m I_{y_m}) > 0,$$

which is contradictory.

So, we can assume that the Expression (5) is such that the graph associated with its right-hand part has no cycles. But then there will be a node x_i without incoming arcs. Then we will have that $\alpha_i(I_{x_i} - b_i I_{y_i})(x_i) > 0$ and $\alpha_j(I_{x_j} - b_j I_{y_j})(x_i) \geq 0$ if $j \neq i$, and above inequality will be impossible. \square

If X is finite, and we consider a weighted and directed graph with a node for each $x \in X$ and a weight $t(x, y)$ for a link from x to y , and define the value of a path as the product of the weights of its links, then $t^i(x, y)$ is the maximum weight of the paths going from x to y of length lower or equal to i : this is an immediate consequence of its recursive definition and it could be easily proved by induction. $\bar{t}(x, y)$ is the supremum of the values of the paths going from x to y . If $\bar{t}(x, x) < 1$, then all the cycles have a weight lower than 1, and only direct paths have to be considered, and we can conclude that $\bar{t}(x, y) = t^n(x, y)$, where $|X| = n$, i.e. t^i will converge after n iterations.

Now, we can prove the following result showing that if t avoids sure loss, then \bar{t} will represent the natural extension for pairwise comparisons.

Proposition 4. *If t is an initial specification such that $\bar{t}(x, x) < 1$ for all $x \in X$ and $\bar{\mathcal{T}}$ is the natural extension of \mathcal{T} , then a pairwise gamble $I_x - bI_y \in \bar{\mathcal{T}}$, if and only if $b \leq \bar{t}(x, y)$.*

Proof. The If part is a consequence of Proposition 1 and the fact that if $b \in \mathbb{R}$ and $b \leq \bar{t}(x, y)$ then we have that $b \leq t^i(x, y)$ for some $i \geq 0$.

Assume now that $I_x - bI_y \in \bar{\mathcal{T}}$, then there is an expression

$$I_x - bI_y \geq \alpha_1(I_{x_1} - b_1 I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_m I_{y_m}) \quad (6)$$

where $\alpha_i > 0$ and $b_i \leq t(x_i, y_i) \leq \bar{t}(x_i, y_i)$. We can apply Lemma 3 and assume that in the associated graph to the right-hand part of the above expression, there are no cycles and no paths of length 2.

We first show that we must have $x_i = x$ and that we can assume $y_i = y$ without loss of generality.

First, all the values x_1, \dots, x_m should be equal to x . If $x_i = x' \neq x$, then as this value does not appear in negative in the right part (there are no paths of length greater or equal than 2), then the value $(I_x - bI_y)(x') \leq 0$ and $(\alpha_1(I_{x_1} - b_1 I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_m I_{y_m}))(x') > 0$, which is in contradiction with inequality in (6).

Now, assume that a value y_i is different from y . We have $\alpha_i(I_{x_i} - b_i I_{y_i})(y_i) = -\alpha_i b_i$ and $\alpha_i(I_{x_i} - b_i I_{y_i})(x') \geq 0$, $\forall x' \neq y_i$, so removing this summand will increase the value of the right part of the inequality (6) only in point y_i by an amount of $\alpha_i b_i$.

As $y_i \neq x_j$ for all j (there are no paths of length greater or equal than 2), we obtain that

$$(\alpha_1(I_{x_1} - b_1 I_{y_1}) + \dots + \alpha_m(I_{x_m} - b_m I_{y_m}))(y_i) = - \sum_{y_j=y_i} \alpha_j b_j \leq -\alpha_i b_i.$$

As being $y_i \neq y$, we have that the left part $(I_x - bI_y)(y_i) \geq 0$, then we can remove summand $\alpha_i(I_{x_i} - b_i I_{y_i})$ from the right part and the inequality continues holding.

So we can assume the existence of an inequality,

$$I_x - bI_y \geq \alpha_1(I_x - b_1 I_y) + \dots + \alpha_m(I_x - b_m I_y)$$

where $b_i \leq \bar{t}(x, y)$, $\forall i$. Taking value of the two sides on x , we get that $1 \geq \sum_{i=1}^m \alpha_i$, and taking value on y we obtain:

$$-b \geq - \sum_{i=1}^m \alpha_i b_i$$

So,

$$b \leq \sum_{i=1}^m \alpha_i b_i \leq \frac{1}{\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i b_i$$

As we have a weighted average of values b_i with weights α_i and each one of them is lower or equal to $\bar{t}(x, y)$, the desired result $b \leq \bar{t}(x, y)$ is obtained. \square

Example 6. Let us consider the initial specification of Example 1 with the natural extension computed in Example 2. In Example 1, we proved that gamble $(I_{x_1} - 3I_{x_2})$ was desirable, though it was not initially desirable. Now, this fact is also a consequence of Proposition 1, as $3 \leq t^2(x_1, x_2) = 3$. This Proposition provides a sufficient condition for the desirability of a pairwise gamble.

Proposition 4 also provides a necessary condition for desirability and to prove it, Lemmas 2 and 3 play an important role. Let us illustrate it. Assume that a gamble $I_{x_1} - bI_{x_2}$ is in the natural extension of the initial representation of Example 1. Then, we know that

$$(I_{x_1} - bI_{x_2}) \geq \alpha_1(I_{x'_1} - b_1I_{y'_1}) + \cdots + \alpha_m(I_{x'_m} - b_mI_{y'_m})$$

where $\alpha_i > 0, (I_{x'_i} - b_iI_{y'_i}) \in \mathcal{T}$. To prove that in this situation we must have $b \leq \bar{t}(x_1, x_2)$ is not simple, because it is difficult to obtain conclusions about the gambles on the right side of the inequality. Lemmas 2 and 3 are useful because they simplify the type of expressions on the right side. Applying them, it is possible to show that there is another similar expression in which all the elements $x'_i = x_1$ and $y'_i = x_2$ (following the reasoning in the proof of Proposition 4) and where $b_i \leq \bar{t}(x_1, x_2) = 3$, i.e.

$$(I_{x_1} - bI_{x_2}) \geq \alpha_1(I_{x_1} - b_1I_{x_2}) + \cdots + \alpha_m(I_{x_1} - b_mI_{x_2})$$

Finally, it is simple to prove that in these conditions if all the $b_i \leq 3$, we must have $b \leq \bar{t}(x_1, x_2) = 3$. In conclusion, Lemmas 2 and 3 are useful tools to deduce the existence of simpler expressions from general linear combinations of pairwise gambles with positive coefficients.

3.3. Computing gambles in the natural extension

Proposition 4 characterizes pairwise gambles in the natural extension $\bar{\mathcal{T}}$. The problem we consider now is when a generic gamble g is in the natural extension. Assume a generic gamble g and define the sets:

$$P_g = \{x \in X : g(x) > 0\}, \quad N_g = \{x \in X : g(x) < 0\}.$$

We can assume that $N_g \neq \emptyset$, as otherwise the problem is trivial.

If a gamble g is in the natural extension, $\bar{\mathcal{T}}$, then we have that

$$g \geq \alpha_1(I_{x_1} - b_1I_{y_1}) + \cdots + \alpha_m(I_{x_m} - b_mI_{y_m}), \quad (7)$$

where $b_i \leq \bar{t}(x_i, y_i)$, for any $i = 1, \dots, m$. First, we are going to prove that in this case, there is a similar expression where $x_i \in P_g$ and $y_i \in N_g, \forall i$.

Proposition 5. If t is a pairwise specification that avoids sure loss and \bar{t} is its pairwise natural extension, then if $g \in \bar{\mathcal{T}}$, we have that there are finite sets $A \subseteq P_g, B \subseteq N_g$ such that

$$g \geq \sum_{(x,y) \in A \times B} \alpha_{x,y}(I_x - b_{x,y}I_y), \quad (8)$$

where $\alpha_{x,y} \geq 0, b_{x,y} \leq \bar{t}(x, y), \forall (x, y) \in A \times B$.

Proof. If this gamble is in the natural extension, by Lemma 3 we can assume the existence of a combination of pairwise gambles,

$$g \geq \alpha_1(I_{x_1} - b_1I_{y_1}) + \cdots + \alpha_m(I_{x_m} - b_mI_{y_m}), \quad (9)$$

where in the associated graph to the right part there are no cycles and no paths of length greater or equal to 2 and $b_i \leq \bar{t}(x_i, y_i), \forall i$. We must have $x_i \in P_g, \forall i$, as in other case, $g(x_i) \leq 0$, and $(\alpha_1(I_{x_1} - b_1I_{y_1}) + \cdots + \alpha_m(I_{x_m} - b_mI_{y_m}))(x_i) > 0$.

If we have $y_i \notin N_g$, then the summand $\alpha_i(I_{x_i} - b_iI_{y_i})$ can be removed and the inequality continues holding (we increase the right side only on point y_i to a value lower or equal than 0 and $g(y_i) \geq 0$).

So, we can assume that $x_i \in P_g$, and $y_i \in N_g, \forall i$. We can also join two summands $\alpha_i(I_{x_i} - b_iI_{y_i})$ and $\alpha_j(I_{x_j} - b_jI_{y_j})$ if $x_i = x_j$ and $y_i = y_j$, in $(\alpha_i + \alpha_j)(I_{x_i} - \max(b_i, b_j)I_{y_i})$, where $\max(b_i, b_j) \leq \bar{t}(x_i, y_j)$. So, considering $A = \{x_1, \dots, x_m\}, B = \{y_1, \dots, y_m\}$, the above expression can be rewritten as:

$$g \geq \sum_{(x,y) \in A \times B} \alpha_{x,y}(I_x - b_{x,y}I_y), \quad (10)$$

where $\alpha_{x,y} \geq 0, b_{x,y} \leq \bar{t}(x, y), \forall (x, y) \in A \times B$ (we can set $\alpha_{x,y} = b_{x,y} = 0$ if the pair (x, y) is not in the list $(x_1, y_1), \dots, (x_m, y_m)$). \square

If X is finite we can consider that sets A, B in the above expression are P_g and N_g , respectively, by considering that $\alpha_{x,y} = b_{x,y} = 0$ if $(x, y) \in P_g \times N_g \setminus A \times B$. As a consequence of this proposition, in this case, and when $\bar{t}(x, y) \neq \infty, \forall x, y \in X$, knowing whether $g \in \bar{\mathcal{T}}$ can be rephrased as a max flow problem with gain/loss factors [11]. A generalized max flow problem is determined by a directed

graph $G = (U, E)$, where there are two special nodes, S (source node) and V (sink node), and where each link ℓ has two associated values: a capacity $c(\ell)$ and a gain factor $\gamma(\ell)$. A flow F is a mapping from each link to a value such that for each link $F(\ell) \leq c(\ell)$ (the flow for a link should be lower or equal to its capacity). When a flow $F(\ell)$ is assigned to link ℓ connecting nodes R_1 and R_2 , it is said that the amount $F(\ell)$, is departing from R_1 and that the amount $\gamma(\ell) \cdot F(\ell)$ is arriving to node R_2 (due to the gain factor, $\gamma(\ell)$, of the link). It is assumed that for each intermediate node, the sum of the amounts departing from the outgoing links is equal to the sum of the amounts arriving from the incoming links. The net amount arriving to the sink is the difference between the sum of the arriving amounts minus the sum of the departing amounts.

In this case, the problem is as follows:

- There is a source node S and a sink node V .
- There is a node for each $x \in P_g$ and a node for each $y \in N_g$.
- There is a link from S to each node $x \in P_g$ with capacity $g(x)$ and gain factor of 1.
- There is a link from each node $y \in N_g$ and V with capacity $-g(x)$ and gain factor of 1.
- There is a link from each node $x \in P_g$ to each node $y \in N_g$ with unlimited capacity (it could be set to the value $\sum_{x \in P_g} g(x)$) and gain factor of $\bar{t}(x, y)$.

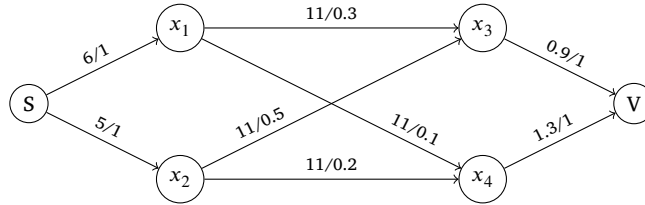
Problem (8) has a solution in $\alpha_{x,y}$ if and only if it is possible to send a flow such that the amount arriving at V is equal to $-\sum_{y \in N_g} g(y)$: the flow sent through the link from node $x \in P_g$ to $y \in N_g$ corresponds to the value $\alpha_{x,y}$. The flow arriving to y from x is $\alpha_{x,y} \bar{t}(x, y)$. The total amount arriving at t is equal to the sum of the capacities of all the links, so all the links should be at maximum capacity and for each $y \in N_g$ we have that $\sum_{x \in P_g} \alpha_{x,y} \bar{t}(x, y) = g(y)$, and Equation (8) is satisfied (with equality) in these points where $b_{x,y} = \bar{t}(x, y)$. We also have that the flow arriving to each point $x \in P_g$ is equal to $\sum_{y \in N_g} \alpha_{x,y}$, and it should be less or equal to the incoming flow to this node with is limited to $g(x)$. So the equation is also satisfied for values $x \in P_g$.

This is a well-known problem that can be solved in polynomial time with algorithms [11].

Example 7. Assume that $X = \{x_1, x_2, x_3, x_4\}$ and pairwise natural extension given by matrix:

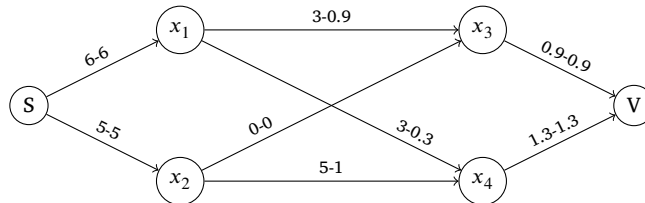
$$T = \begin{pmatrix} 1^- & 0.3 & 0.3 & 0.1 \\ 1 & 1^- & 0.5 & 0.2 \\ 2 & 1 & 1^- & 0.3 \\ 1 & 2 & 1 & 1^- \end{pmatrix}$$

The max flow problem associated with gamble $6I_{x_1} + 5I_{x_2} - 0.9I_{x_3} - 1.3I_{x_4}$ is represented in the following graph:



g is in the natural extension if and only if there is a flow in which an amount of $\sum_{y \in N_g} g(y)$ arrives at V . In that case the coefficient $\alpha_{x,y}$ is equal to the flow entering the link from x to y .

The flow depicted in the following figure (where $a - b$ means that a units enter the link and b units arrive at the link end) solves the problem:



It corresponds to the expression:

$$6I_{x_1} + 5I_{x_2} - 0.9I_{x_3} - 1.3I_{x_4} \geq 3(I_{x_1} - 0.3I_{x_3}) + 3(I_{x_1} - 0.1I_{x_4}) + 5(I_{x_2} - 0.2I_{x_4})$$

When there are values $t(x, y) = a^-$ or $t(x, y) = +\infty$, we can consider the approximate problem obtained by replacing each value $t(x, y) = a^-$ for another value $b < a^-$ but very close to it, and $t(x, y) = \infty$, by a very large value. If there is a solution to the approximate problem, then we are sure that there is a solution in the original case. If there is no solution to the new problem, we can expect that there is no solution to the original one, but there might be one. An exact solution could be obtained by solving the max flow problem with $t(x, y) = a - \epsilon$ whenever $t(x, y) = a^-$, where ϵ is an arbitrary very small value. For that, it would be necessary to adapt the existing algorithms to work with symbolic ϵ and ∞ values.

Another consequence of Proposition 5 is that to compute the conditioning of \bar{T} to a subset $H \subseteq X$, only the restriction of \bar{t} to $H \times H$ is necessary. To know whether $g \in \bar{T}_H$ for a $H \subseteq X$, we have to check whether there are $\alpha_{x,y} > 0, b_{x,y} < \bar{t}(x, y)$ such that $g \cdot I_H \geq \sum_{(x,y) \in P_g \cdot I_H \times N_g \cdot I_H} \alpha_{x,y} (I_z - b_{x,y} I_y)$. As $P_{g \cdot I_H} = P_g$ and $N_{g \cdot I_H} = N_g$, this is equivalent to $g \geq \sum_{(x,y) \in P_g \times N_g} \alpha_{x,y} (I_z - b_{x,y} I_y)$. And \bar{T}_H is the natural extension of the restriction of \bar{t} to $H \times H$ defined by the mapping \bar{t}_H defined on $H \times H$, by $\bar{t}_H(x, y) = \bar{t}(x, y)$. Observe that if we have an initial specification t the conditioning to H is, in general, not defined by the restriction t_H to $H \times H$, and we have that $(\bar{t}_H) \geq \bar{t}_H$.

4. Equiprobability

In [12] a concept of equiprobable information in the infinite case was introduced which is much weaker than the uniform distribution, but coincides with the uniform distribution in the finite case. Nonetheless, it was sufficient to serve as a basis to define a common framework integrating classical frequentist and Bayesian methods [12]. This generalization was a particular case of pairwise comparisons.

Equiprobability is represented by the initial representation $t_u(x, y) = 1^-$, for all $x, y \in X$, i.e. gamble $I_x - aI_y$, is desirable whenever $0 < a < 1$. It is immediate that $\bar{t}_u = t_u$. As a consequence, it avoids sure loss, and the natural extension of its associated set of desirable gambles, \bar{T}_u , is coherent. This definition is justified by the following result.

Proposition 6. P is in the credal set associated with t_u if and only if $P(x) = P(y), \forall x, y \in X$.

Proof. As $P(I_x - aI_y) = P(x) - aP(y)$, and this value is greater or equal than 0 for any $a < 1$ if and only if $P(x) \geq P(y)$, and as x, y are arbitrary we must have $P(x) = P(y)$, for all $x, y \in X$. \square

The credal set associated with the initial specification t_u is the same as the credal set associated with \bar{T}_u which is always coherent. One immediate consequence is that in the case that X is finite, the associated credal set contains only one probability, the uniform distribution P_u in X . In contrast with this situation, in the infinite case, there are many probabilities in the associated credal set.

Proposition 7. If X is infinite, then credal set associated with t_u is equal to the set of all the finitely additive probability measures in X such that $P(H) = 0$ for all finite $H \subseteq X$.

Proof. If P is in the associated credal set, we have that $P(x) = P(y)$ for all $x, y \in X$, and if X is infinite we must have that $P(x) = 0$ for all $x \in X$ (otherwise $P(X)$ would be infinite). If $P(x) = 0$ for all $x \in X$, then we will have $P(H) = 0$ for every finite set H . On the other hand, if $P(x) = 0$, for all $x \in X$, we have that $P(I_x - aI_y) = 1 - a \geq 0$ for all x, y and $a < 1$ and P is in the associated credal set. \square

This is a very large set of probabilities, more imprecise than the uniform distribution on the integers or in the real line [1, Section 2.9]. In particular, it contains all the probability measures associated with continuous densities (it is almost vacuous). If X is an interval of real numbers, one of these densities is the uniform density, but there are many more probabilities in the associated credal set.

Proposition 8. If X is finite, a general gamble g is desirable under t_u (i.e. it belongs to \bar{T}_u) if and only if $\sum_{x \in X} g(x) > 0$.

Proof. If a gamble g is desirable, then there are $\alpha_i > 0, b_i < 1, i = 1, \dots, m$, such that $g \geq \sum_{i=1}^m \alpha_i (I_{x_i} - b_i I_{y_i})$. As for each $I_{x_i} - b_i I_{y_i}$, we have $\sum_{x \in X} (I_{x_i} - b_i I_{y_i})(x) = 1 - b_i > 0$, we have that $\sum_{x \in X} g(x) > 0$.

On the other hand, if $\sum_{x \in X} g(x) > 0$, we have that $P_u(g) = \frac{\sum_{x \in X} g(x)}{|X|} > 0$ where $|X|$ is the number of elements of X . As P_u is the only probability in the credal set, we have that $g \in \bar{T}_u$. \square

As it was said, the conditioning of t_u to a subset $H \subseteq X$ is the uniform distribution on H . This is an immediate consequence of the fact that the conditioning of t_u is the pairwise specification associated with t_{uH} (t_u restricted to $H \times H$) which is the uniform specification on H . If H is finite, we obtain the uniform specification on a finite set, and therefore the following proposition is a consequence.

Proposition 9. If H is finite, the conditioning of $\bar{\mathcal{T}}_u$ to H will contain all the gambles g such that $\sum_{x \in H} g(x) > 0$, and the associated credal set is given by the uniform probability in H .

Proposition 10. If X is infinite, a gamble g is desirable under t_u (i.e. it belongs to $\bar{\mathcal{T}}_u$) if and only if N_g is finite and there is $H \subseteq P_g$ with H finite and $\sum_{x \in (H \cup N_g)} g(x) > 0$.

Proof. If g is desirable (i.e. $g \in \bar{\mathcal{T}}_u$), then we have

$$g \geq \alpha_1(I_{x_1} - a_1 I_{y_1}) + \dots + \alpha_m(I_{x_m} - a_m I_{y_m}),$$

where $a_i < 1$ for any i . Let us call $A = \{x_1, \dots, x_m\}$ and $B = \{y_1, \dots, y_m\}$. From the inequality, we can conclude that $N_g \subseteq B$ and therefore N_g is finite. Now, let H be the subset of $A \cup B$ where $g(x) > 0$. We have that

$$\sum_{x \in H \cup N_g} g(x) \geq \sum_{A \cup B} g(x) \geq \sum_{x \in A \cup B} \sum_{i=1}^m \alpha_i (I_{x_i} - a_i I_{y_i}) = \sum_{i=1}^m \alpha_i (1 - a_i) > 0.$$

The first inequality is true because g is lower or equal to 0 in the points belonging to $(A \cup B) \setminus (H \cup N_g)$.

On the other hand, if N_g is finite and there is a finite set H satisfying $\sum_{x \in (H \cup N_g)} g(x) > 0$, then the restriction of g to $N_g \cup H$, denoted by $g' = g|_{N_g \cup H}$, is a gamble that belongs to $\bar{\mathcal{T}}_u$ conditioned to the finite set $H \cup N_g$ by Proposition 9. Therefore $g' I_{N_g \cup H} \in \bar{\mathcal{T}}_u$. As $g \geq g' I_{N_g \cup H}$ (they are equal on $N_g \cup H$ and if $x \notin (N_g \cup H)$, we have $g(x) \geq 0 = g' I_{N_g \cup H}(x)$), we can conclude that $g \in \bar{\mathcal{T}}_u$. \square

5. Discounting and the constant odd ratio model

From an intuitive point of view, if we have a coherent set of desirable gambles, \mathcal{D} , discounting it by a factor $\delta \in [0, 1]$ is the process of building a new coherent set \mathcal{D}^δ which is less informative than \mathcal{D} ($\mathcal{D}^\delta \subseteq \mathcal{D}$) under the assumption that information represented by \mathcal{D} is unreliable with a degree $\delta \in [0, 1]$. This is a very generic idea and several concrete definitions are possible. An extensive study of these models in terms of credal sets can be found in [19]. Here we will consider a discounting model based on the constant odd ratio model [1,7]. The basic intuition of the constant odd ratio model is that after discounting by δ a gamble g is almost desirable if this gamble is almost desirable if all the gains are taxed by δ , i.e. if the gamble $g_\delta = (1 - \delta)g^+ - g^-$ is desirable, where $g^+ = \max\{g, 0\}$ and $g^- = \max\{-g, 0\}$. Here we consider an equivalent expression when $\delta < 1$ but that also works for desirable gambles when $\delta = 1$:

$$\mathcal{D}^\delta = \{g^+ - (1 - \delta)g^- : g \in \mathcal{D}\}. \quad (11)$$

That is, if in a gamble that was desirable under \mathcal{D} the payments are discounted by δ , then it becomes desirable after discounting. An important observation is that the above expressions should be applied to a coherent set of desirable gambles, as if they are applied to a generic set of gambles, the result can be less informative. The following example shows a case in which we have a set of gambles \mathcal{D} avoiding sure loss such that $(\mathcal{D}^\delta) \subsetneq (\bar{\mathcal{D}})^\delta$.

Example 8. Consider $X = \{x_1, x_2, x_3\}$ and \mathcal{D} the non-coherent set of gambles $\{I_{x_1} - I_{x_2}, I_{x_2} - I_{x_3}\}$ and $\delta = 0.5$. If we apply the discounting to \mathcal{D} , we have that $\mathcal{D}^\delta = \{I_{x_1} - 0.5I_{x_2}, I_{x_2} - 0.5I_{x_3}\}$. We have that $(I_{x_1} - I_{x_3}) \in \bar{\mathcal{D}}$, and the discounting of this gamble by $\delta = 0.5$ is $I_{x_1} - 0.5I_{x_3}$, however the most informative gamble comparing x_1 versus x_3 in $(\bar{\mathcal{D}})^\delta$ is $I_{x_1} - 0.25I_{x_3}$.

If t is a pairwise specification the discounting by $\delta \in [0, 1]$ is the pairwise specification:

$$t^\delta(x, y) = \begin{cases} (1 - \delta)t(x, y) & \text{if } x \neq y \\ 1^- & \text{otherwise} \end{cases} \quad (12)$$

If t is a pairwise specification avoiding sure loss and \bar{t} is its natural extension, then \bar{t}^δ will avoid sure loss and will be equal to its natural extension. This is an immediate consequence of the fact that $\bar{t}(x, y)\bar{t}(y, z) \leq \bar{t}(x, z)$, $\forall x, y, z \in X$, and if $x \neq y, y \neq z$, we have $\bar{t}^\delta(x, y)\bar{t}^\delta(y, z) = (1 - \delta)^2 \bar{t}(x, y)\bar{t}(y, z) \leq (1 - \delta)\bar{t}(x, z) \leq \bar{t}^\delta(x, z)$. In the case $x = y$ or $y = z$, the inequality $\bar{t}^\delta(x, y)\bar{t}^\delta(y, z) \leq \bar{t}^\delta(x, z)$ is always trivially satisfied. Therefore, $\bar{t}^\delta \circ \bar{t}^\delta = \bar{t}^\delta$, and \bar{t}^δ will avoid sure loss and will be equal to its natural extension.

It is immediate that if \mathcal{T} is the set of gambles associated with t , then \mathcal{T}^δ is the set of gambles associated with t^δ . From the above example, it is clear that if t is an initial specification and \bar{t} is its natural extension, then $(\bar{t}^\delta) \leq \bar{t}^\delta$, and we should compute the natural extension \bar{t} before discounting, as in another case, some information will be lost. The question now is the following: is it enough to discount \bar{t} or is it necessary to discount $\bar{\mathcal{T}}$? The following result shows that it is enough to discount \bar{t} and that the discounting the full set of gambles $\bar{\mathcal{T}}$ is the set of gambles associated with pairwise specification \bar{t}^δ . So, it is not necessary to discount the full set of desirable gambles $\bar{\mathcal{T}}$.

Proposition 11. If t is an initial pairwise specification avoiding sure loss, \bar{t} is its natural extension, and $\delta \in [0, 1]$, then the set of gambles associated with \bar{t}^δ is the discounting of \bar{T} by δ , which is denoted as \bar{T}^δ .

Proof. When $\delta = 1$ both sets are vacuous and the result is obtained.

Assuming, $\delta < 1$. According to Proposition 5, a gamble g is in \bar{T} if and only if there are finite sets $A \subseteq P_g, B \subseteq N_g$ and values $\alpha_{x,y} \geq 0, b_{x,y} \leq \bar{t}(x, y)$, where $(x, y) \in A \times B$, such that

$$g \geq \sum_{(x,y) \in A \times B} \alpha_{x,y} (I_x - b_{x,y} I_y)$$

So a gamble $g \in \bar{T}^\delta$ if and only if $g = f^+ - (1 - \delta)f^-$ where $f \in \bar{T}$ i.e.

$$f^+ - f^- \geq \sum_{(x,y) \in A \times B} \alpha_{x,y} (I_x - b_{x,y} I_y)$$

In the right member of the inequality we split the positive and the negative part obtaining the equivalent expression:

$$f^+ - f^- \geq \sum_{x \in A} \left(\sum_{y \in B} \alpha_{x,y} \right) I_x - \sum_{y \in B} \left(\sum_{x \in A} \alpha_{x,y} b_{x,y} \right) I_y.$$

As A and B are disjoint, if $\lambda < 1$ we can multiply the negative parts of both sides of the equality by the positive number $1 - \delta$ obtaining an equivalent expression:

$$f^+ - (1 - \delta)f^- \geq \sum_{x \in A} \left(\sum_{y \in B} \alpha_{x,y} \right) I_x - \sum_{y \in B} \left(\sum_{x \in A} (1 - \delta) \alpha_{x,y} b_{x,y} \right) I_y.$$

Rearranging again we obtain,

$$g = f^+ - (1 - \delta)f^- \geq \sum_{(x,y) \in A \times B} \alpha_{x,y} (I_x - (1 - \delta)b_{x,y} I_y).$$

If we call $c_{x,y} = (1 - \delta)b_{x,y}$, we obtain

$$g \geq \sum_{(x,y) \in A \times B} \alpha_{x,y} (I_x - c_{x,y} I_y),$$

where $c_{x,y} = (1 - \delta)b_{x,y} \leq (1 - \delta)\bar{t}(x, y) = \bar{t}^\delta(x, y)$, and this is equivalent to the fact that g is in the set of gambles associated with \bar{t}^δ . \square

The constant odd ratio model is a distortion model of a precise additive probability P_0 defined on a set X . It was introduced in [6], studied in [5] and in [1, Sect. 2.9] from a behavioral point of view. A detailed examination can be found in [7].

A precise probability P_0 can be represented by a pairwise comparison in the finite case.¹ A probability P_0 on a finite set X can be represented by the pairwise specification $\bar{t}(x, y) = \left(\frac{P_0(x)}{P_0(y)} \right)^-$, where it is considered that $(a/0)^- = \infty$, if $a > 0$ and $(0/0)^- = 0$. The discounting of this pairwise specification by δ is the pairwise specification $\bar{t}^\delta(x, y) = (1 - \delta) \left(\frac{P_0(x)}{P_0(y)} \right)^-$, and its associated credal set is the set of probabilities on X satisfying: $P(x) - (1 - \delta) \left(\frac{P_0(x)}{P_0(y)} \right) P(y) \geq 0, \forall x, y \in X$, i.e. $\frac{P(x)}{P(y)} \geq (1 - \delta) \left(\frac{P_0(x)}{P_0(y)} \right)$. In the finite case this is equivalent to $\frac{P(A)}{P(B)} \geq (1 - \delta) \left(\frac{P_0(A)}{P_0(B)} \right)$, for any $A, B \subseteq X$, and this is the credal set associated with the constant odd ratio model [1,7].

6. Multiplicative preference relationships

A multiplicative preference relationship [14] defined on a finite set X with n elements is a $n \times n$ matrix A with positive values satisfying $a_{i,j} \cdot a_{j,i} = 1$. The value $a_{i,j}$ represents the preference of x_i against x_j . In [15] these precise preference relationships were generalized to intervals. In this case, there is a lower matrix L and an upper matrix U , satisfying $L \leq U$, such that $u_{i,j} = 1/\ell_{j,i}$. Given this relationship, only one of the matrices is necessary and for the lower matrix, we have that: $\ell_{i,j} \cdot \ell_{j,i} = \ell_{i,j}/u_{i,j} \leq \ell_{i,j}/\ell_{i,j} = 1$.

There are clear formal connections between interval multiplicative preference relationships L and coherent sets of pairwise comparisons, with some minor differences:

- Interval multiplicative preference relationships are more related to the concept of almost desirability. If t is an initial pairwise comparison, we can consider its associated pairwise comparison in the limit, t^* given by $t^*(x_i, x_j) = a$, whenever $t(x_i, x_j) = a$ or

¹ In the infinite case, we can see in the example of equiprobability that very precise pairwise comparisons give rise to very imprecise credal sets.

$t(x_i, x_j) = a^-$. If T is the matrix determining t , then T^* , the matrix defining t^* is a matrix, T^* , equal to T but removing – from the exponents. The compatibility relation is now $t^*(x_i, x_j) \cdot t^*(x_j, x_i) \leq 1$. This pairwise comparison, t^* , defines a credal set, the set of probabilities P satisfying $P(I_{x_i}) - t^*(x_i, x_j)P(I_{x_j}) \geq 0$, i.e. $P(x_i)/P(x_j) \geq t^*(x_i, x_j)$.

- We allow 0 and ∞ values, which are not allowed in multiplicative preference relationships. Apart from that, it is immediate that L is a lower multiplicative relationship if and only if it is equal to the matrix T^* of a pairwise specification in the limit with positive values.
- Not allowing 0 values, a coherent pairwise comparison will be called maximal if $t(x_i, x_j) \cdot t(x_j, x_i) = 1^-$ for any $x_i, x_j \in X$. The associated pairwise comparison in the limit satisfies $t^*(x_i, x_j) \cdot t^*(x_j, x_i) = 1$. In this case, we have a precise multiplicative preference relationship.

Even, if formally pairwise specifications and imprecise multiplicative relationships are quite similar, they do not necessarily have the same meaning. However, of the different consistency conditions considered in [20], one of them, the Wang's et al. consistency [21], has a direct interpretation as a pairwise specification. According to this definition, an interval multiplicative preference is consistent if and only if the set

$$\{(p_1, \dots, p_m) : \ell_{i,j} \leq p_i/p_j, \sum_i p_i = 1, p_i > 0\},$$

is not empty, which is equivalent to avoiding sure loss in the associated credal set.

7. Comparative probability

If we have a finite set X it might be convenient that probabilistic information is assessed by comparing the probability of events: A is as probable as B ($A \geq B$), where $A, B \subseteq X$ [16]. A comparative probability ordering is a set of such assessments. Walley [1, Section 4.5] studies incomplete comparative probability orderings (the order relationship is not total) and its relation with imprecise probabilities: $A \geq B$ is equivalent to $I_A - I_B$ to be almost desirable, i.e. $P(A) \geq P(B)$. A review can be found in [17]. Miranda and Destercke [18] study comparative probability when the assessments are given comparing elementary events $x \geq y$, where $x, y \in X$ and they provide results about its associated credal set and the number of extreme probabilities. This study is extended to general comparative orderings in [22].

There are clear relationships between comparative probability and sets of pairwise desirable gambles. We summarize them in the following points:

- We consider desirability instead of almost desirability. In this sense, our model is more general. It is considered that A is preferred to B if $I_A - I_B$ is desirable. If $t(x, y) = 1$, we have that I_x is preferred to I_y and in the credal set $P(x) \geq P(y)$, but we can also have $t(x, y) = 1^-$, and in this case, I_x is preferred to $(1 - \epsilon)I_y$, for all $\epsilon > 0$, this also implies that $P(x) \geq P(y)$, but the situation is different. Traditional comparative probability cannot distinguish between these two situations. To have a more direct correspondence between the two models, we need to consider the associated comparison in the limit t^* obtained from the original specification.
- Comparative probability only considers comparisons between events, but without scaling them: i.e. relationships as $A \geq B$ but not $A \geq 0.3B$, our model allows that events are multiplied by positive numbers: $0.2x$ is preferred to $0.3y$, which is equivalent to $0.2I_x - 0.3I_y$ is desirable, i.e. $3/2 \leq t(x, y)$ for a given specification t .
- We only consider comparisons between elementary events, so comparative probability orderings are more general in this aspect. Our model is only a generalization of comparative probabilities on elementary events as considered in [18].

If we have a pairwise specification t avoiding sure loss and \bar{t} is its natural extension we can associate with it two comparative probability orderings defined on the elementary events:

- **Considering almost preference:** $x \geq y$ when $\bar{t}(x, y) \geq 1^-$. This defines a (pre)order relationship in X similar to the one considered in [18].
- **Considering preference:** $x > y$ when $\bar{t}(x, y) \geq 1$. This is a strict order relationship, but from the point of view of the credal set, it does not imply strict inequalities ($P(x) > P(y)$) as in strict comparative probability [23].

8. Conclusions and future work

In this paper, we have shown that the language of gambles is appropriate to represent desirability relationships involving pairs of elementary events, which are the counterparts of bounds on probability ratios when using the language of probability distributions and credal sets. The provided results constitute a solid basis for the application of these models in real cases. One advantage of this representation is that it does not change under conditioning, i.e. the conditional information of a pairwise specification is again a pairwise specification. In the future, we plan to study the combination of pieces of information including conditional and unconditional ones. We also plan to develop the details of the generalized max-flow algorithm associated with computing the natural extension for general gambles. In particular, as the graph problem does not have cycles, we will study if it is possible to take advantage of this fact to design more efficient algorithms. It will be also interesting to study the relationships with other particular

cases of imprecise probability models such as order-2 capacities, belief functions, or elementary probability intervals. Finally, another point of interest for future research will be to try to generalize the results about the extreme points of the associated credal set in the case of comparative probability [18] to the model presented in this paper.

CRedit authorship contribution statement

Serafín Moral: Writing – review & editing, Writing – original draft, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

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Data availability

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