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Intra-facility equity in discrete and continuous *p*-facility location problems

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ABSTRACT

We consider facility location problems with a new form of equity criterion. Demand points have preference order on the sites where the plants can be located. The goal is to find the location of the facilities minimizing the envy felt by the demand points with respect to the rest of the demand points allocated to the same plant. After defining this new envy criterion and the general framework based on it, we provide formulations that model this approach in both the discrete and the continuous framework. The problems are illustrated with examples and the computational tests reported show the potential and limits of each formulation on several types of instances. Although this article is mainly focused on the introduction, modeling and formulation of this new concept of envy, some improvements for all the formulations presented are developed, obtaining in some cases better solution times.

1. Introduction

The notion of equity has played a very important role in decision theory and operations research over the years since it is one of the most common criteria to judge fairness. It is mainly due to the fact that satisfaction is envy-free, i.e., a solution of a decision process such that every involved decision-maker likes its own solution at least as much as the one of any other agent. In particular, the envy-freeness criterion has been used in many different problems related to fair resource allocation, fair queuing processes, fair auctions, and pricing problems, among others (see e.g. Brams and Fishburn, 2000; Chun, 2006; Dall'Aglio and Hill, 2003; Domínguez et al., 2022; Haake et al., 2002; Moulin, 2014; Ohseto, 2005; Pápai, 2003; Reijnierse and Potters, 1998; Shioura et al., 2006; Webb, 1999).

One of the most analyzed areas in operations research is facility location, whose goal is to find the most appropriate positions for a set of facilities providing services in order to satisfy the demand required by a set of users. Facility location problems are classified according to two main characteristics: the solution domain and the criteria used to evaluate the goodness of a solution. Concerning the solution domain, in case the placements of the services are to be chosen from a finite set of potential facilities, the problem is called a *Discrete Facility Location problem* (DFLP) while if the positions of the facilities are chosen from a continuous set, the problem is called a *Continuous Facility Location problem* (CFLP). Apart from the practical applications of these two frameworks, the main difference between these two settings stems from the tools that are applied to solve the problems. While in DFLP Integer Linear Programming is the most common tool used to solve, exactly, the problems, in CFLP one resorts to the use of convex analysis or global optimization tools due to the non-linear nature of the problems. The interested reader is referred to Laporte et al. (2019) for recent contributions in this area.

Concerning the evaluation criteria, the most usual measure to evaluate a given set of positions with respect to a set of customers is the sum of the overall allocation costs. This cost usually represents the access cost of customers to the facilities, and it is an accurate indicator of the efficiency of the logistic system. With such a measure one assumes that for the customers, the least costly the better, which is a natural assumption. This criterion is the one used in the well-known *p*-median (Hakimi, 1964) problem or the multifacility continuous location problem (Blanco et al., 2016). However, other measures have also been proposed in the literature.

In this paper, we introduce an equity criterion that can be incorporated to make decisions on different facility location problems. Measuring the goodness of the solutions to location problems by means of equity criteria is not new. However, still the body of literature analyzing facility location problems under the equity lens is scarce, being the maximum the most representative objective function in this area, giving rise to the well-known (discrete and continuous) *p*-center

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problems. In spite of that, some authors have already analyzed the use of criteria that promote equitable or fair solutions in some facility location problems.

Savas (1978) stated the insufficiency of efficiency and effectiveness measures in location models for public facilities. Halpern and Maimon (1981) considered a large number of tree networks in order to determine the agreement and disagreement of the solutions to location problems using the median, center, variance, and Lorentz measures. Mulligan (1991) designed a simple experiment consisting of locating a facility in an interval of the real straight line regarding three demand points. Apart from a comparative analysis of the optimal solutions for nine equality measures, he also provided the standardized travel distance curves for them. Erkut (1993) proposed a general framework for quantifying inequality and presented some axioms for the appropriateness of the inequality measures. He also showed that only two of his considered measures – the coefficient of variation and the Gini coefficient – hold both the scale-invariance property and the principle of transfers or Pigou-Dalton property. Berman and Kaplan (1990) addressed the equity question using taxes. Marín et al. (2010) considered, as a general measure of equity, the ordered median function applied to discrete p-facility location problems. Mesa et al. (2003) and Garfinkel et al. (2006) addressed some algorithmic aspects of equity measures on network location and routing. Bertsimas et al. (2012) proposed a fairness measure and incorporated it into resource allocation problems. More recently, Filippi et al. (2021) apply the conditional β-mean as a fairness measure for the bi-objective single-source capacitated facility location problem. Blanco and Gázquez (2023) introduced a new fairness measure for the maximal covering location problem combining the approach in Bertsimas et al. (2012) with ordered weighted averaging operators. A review of the existing literature on equity measurement in location theory and a discussion on how to select an appropriate measure of equity was given in the paper by Marsh and Schilling (1994). Also, the equality objective literature was reviewed in Eiselt and Laporte (1995) within a general discussion of objectives in location theory based on the physical concepts of pulling, pushing, and balancing forces.

The envy criterion has also been considered in location theory as an equity criterion. In this framework, envy is defined with respect to the revealed preference of each demand point for the sites of the potential serving facilities. It is assumed that the users (demand points) elicit their preferences for the sites where the plants can be located. A positive envy appears between two users when one of them is allocated to a plant strictly more preferred than the one the other is allocated to. The goal is to find the location of the facilities minimizing the total envy felt by the entire set of demand points. A limitation of this approach is that information is common knowledge. Thus, it is assumed that the decision-maker has previous complete knowledge of the preferences of all customers at the demand points or, alternatively, that all customers do not lie when they are asked about their preferences. Espejo et al. (2009) study a discrete facility location problem with an envy criterion. There, it is assumed that demand points feel envy with respect to all other better-located demand points. Instead, in this paper we propose a novel envy measure that does not require the decision-maker to have a complete knowledge of the preferences of all other users. Rather than that, only partial knowledge of the points that are allocated to the same facility is considered. Specifically, we assume that the users only feel envy with respect to the users allocated to the same facility. On the other hand, in Espejo et al. (2009) the envy is measured using only the preferences of the points, whereas in our case the dissatisfaction of the user is given by a function that is independent of the preference function.

Following the applications of the envy criterion to location theory, one can find an adaptation of the notion of envy to the system of ambulances location in Chanta et al. (2011) with the objective of minimizing the sum of envies among all users with respect to an ordered set of operating stations. Also in Chanta et al. (2014), the envy is redefined as differences in customers' satisfaction between users, where satisfaction is measured by the survival probability of each user.

In Table 1 we summarize some of the equity criteria that have been considered in location science, classified by the domain (discrete, continuous, network, or general), number of facilities to be located (multi or single), and main equity measures applied in the works, sorted by publication year. The most commonly used measures are the center (Current and Ratick, 1995; Hakimi, 1964; Jung et al., 2019; Liu and Salari, 2022; Xu et al., 2023), the absolute deviation and its variants (Berman and Kaplan, 1990; Kalcsics et al., 2015; McAllister, 2010; Morrill and Symons, 1977; Mulligan, 1991; Shehadeh and Snyder, 2023; Xu et al., 2023), Gini/envy (Chanta et al., 2011; Drezner and Drezner, 2011; Espejo et al., 2009; Romero et al., 2016; Shehadeh and Snyder, 2023; Wang and Zhang, 2021), and the range (Erkut, 1993; Kalcsics et al., 2015; Marín, 2011; Puerto et al., 2009; Shehadeh and Snyder, 2023; Xu et al., 2023). A detailed overview of all the equity measures that have been proposed in facility location can be found in the seminal papers (Barbati and Piccolo, 2016; Eiselt and Laporte, 1995; Marsh and Schilling, 1994). Nevertheless, as far as we know, an equitable version of a p-facility location problem, in both a discrete and a continuous domain, that avoids comparisons between customers allocated to different facilities has never been analyzed.

The need of measuring locally the envy naturally arises when a user feels envy of other users in their close neighborhood, or those for which one can share the information about its allocation. In several practical situations, as for instance the location of schools or health centers (as well as the allocation of users to these facilities) an ideal envy-free solution would be one in which all the users allocated to a same facility are equally satisfied. The determination of the position of the facilities and the allocation patterns incorporating this paradigm would avoid undesirable cases in which, to minimize (in average) the envy of all pairs of users, the satisfaction of a single user to its assigned facility is poor because compared to a user to which he/she will never know its allocation, the difference between their preferences is large. Additionally, in our approach, we force the users to be allocated to the most preferred open facility. Thus, the solutions obtained with our methodology assure a minimum envy among the users with the same preferred open facility. In Marsh and Schilling (1994), the authors classify equity criteria based on three different *dimensions*: reference distribution, scale and metric. Specifically, the reference distribution is the comparison value for each user. That perfect equity is considered as achieved when the effect on each user is equal to its associated level(s) in the reference distribution(s). The envy criterion is identified with the so-called *peer* reference distribution, that is, each user is compared with all other users. The equity measure that we propose could be seen then as a novel reference distribution which is not fully peer defined but determined by the set of users allocated to the same service. None of the previous works in the literature mentions this type of reference distribution in the list of equity measures and, in particular, its application to facility location problems.

The model addressed in this paper fits well with the location of services provided by public administrations where the planner (decision-maker) takes into account users' preferences but at the same time wishes to offer the same service quality to all users allocated to the same service center. This policy ensures that when users meet at the service center none of them have the incentive to complain on the basis of other users' quality of service. This may be applicable to outpatient consultations in public health systems where a central authority allocates population areas to outpatient centers or to the location of civic centers offering service to population areas. In the continuous setting, the intra-envy location applies to determine the position of High-Performance Computers (HPC) servers and job allocation to them such that the communication/energy costs of all jobs allocated to the same node have to be similar to ensure a fair comparison of the computational hardness of the jobs run in the same node (which are supposed to run under the same conditions) (Meng et al., 2015).

Table 1
Some works incorporating equity in facility location (D: discrete, C: continuous, N: network, M: multi-facility, S: single-facility).

Equity measure	Domain	#Facilities	Reference
Center	D	M	Hakimi (1964)
min variance	General	M	Morrill and Symons (1977)
Coulter index	D	M	Coulter (1980)
Weighted mean	D	M	Heiner et al. (1981)
Gravity	D	M	Segall (1989)
min abs dev	N	S	Berman and Kaplan (1990)
min abs dev, envy	C (line)	S	Mulligan (1991)
Range	D	M	Erkut (1993)
Center	D	M	Current and Ratick (1995)
max lex	C (line)	S	Kumar and Kleinberg (2006)
Range	N	M	Puerto et al. (2009)
Envy	D	M	Espejo et al. (2009)
min std	General	M	McAllister (2010)
sq envy among facilities	C	M	Drezner and Drezner (2011)
Range	D	M	Marín (2011)
Restricted distances	D	M	Batta et al. (2014)
Envy	D	M	Chanta et al. (2011)
Variance, Abs Dev, Range	N	2	Kalcsics et al. (2015)
Restricted satisfaction	D	M	Caramia and Mari (2016)
Envy	D	M	Romero et al. (2016)
Neighborhood radius center	D	M	Jung et al. (2019)
ℓ_p-norm aggregation	D	M	Bektaş and Letchford (2020)
Conditional β-mean	D	M	Filippi et al. (2021)
abs dev, range, var/std dev, envy, regret	D	M	Shehadeh and Snyder (2023)
Proportional share, envy	C (line)	M	Wang and Zhang (2021)
(α, λ)-fairness	C+D (Covering)	M	Blanco and Gázquez (2023)
Center	D	M	Liu and Salari (2022)
abs dev, range, center	D (Covering)	M	Xu et al. (2023)

In this work we introduce the notion of intra-envy and incorporate it to p-facility location problems, both in discrete and continuous domains. The intra-envy felt by user a to user b, allocated to the same facility, is defined as the difference between the satisfaction of user a and b in case b is more satisfied than a with its allocation, and zero otherwise. The overall intra-envy is the sum of all these pairwise intra envies. We develop different mathematical programming formulations for these problems. We start by constructing *natural* models by explicitly modeling the intra-envy. We also provide equivalent formulations based on the representation of the envy objective as an ordered median function. One of the main differences of our models with respect to classical p-median problems is that the closest assignment constraints (Espejo et al., 2012) are needed to assure that each user is allocated to its most desirable open facility, which would not be assured by the objective functions as in the standard location problems.

The main contributions of this paper are:

- We introduce a new equity measure, the intra-envy, and incorporate it to general continuous and discrete p-facility location problems.
- We further analyze continuous facility location problems under the intra-envy criterion and provide three mathematical optimization formulations for solving, exactly, the problems. The first formulation is based on explicitly representing the intra-envy with linear constraints. The other two formulations are based on rewriting the intra-envy problem as a ordered median problem.
- We study the use of intra-envy measures in p-facility discrete location problems and provide three different mathematical optimization formulations.
- We report the results of an extensive battery of computational experiments on continuous and discrete location instances, analyze the computational limitations of our approaches as well as the structural properties of the obtained solutions compared to the p-median and the standard minimum envy criteria.

The paper is organized as follows. The intra-envy location problem in a general framework is introduced in Section 2. At the end of the section we analyze the properties of this new equity measure. The specific study of the application of the intra-envy criterion to continuous and the discrete facility location problems, including formulations and improvements can be found in Sections 3 and 4. In Section 5 we analyze the results of a battery of computational experiments. We analyze both the computational performance of the proposed solution approaches and the structural comparison of the solutions obtained with the intra-envy, the standard envy, and the median criteria. Finally, some conclusions are drawn.

2. The general minimum p-intra-envy facility location problem

In this section we introduce the problem that we analyze and fix the notation for the remaining sections. We provide here a framework for the p-intra-envy facility location problem for general domains and cost functions.

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a finite set of demand points in $\mathbb{R}^d$ indexed in set $N = \{1, \ldots, n\}$. Abusing the notation, throughout this paper we refer to a demand point interchangeably by a_i or by the index i, for $i \in N$. Denote by $\mathcal{X}$ a (not necessarily finite) set of points also in $\mathbb{R}^d$ that represent the set of potential locations for a facility. In order to quantify the cost incurred by a demand point when it is allocated to a facility in $\mathcal{X}$, each demand point, a_i, is assumed to be endowed with a cost function $\Phi_i : \mathcal{X} \to \mathbb{R}_+$ that represents the cost of allocating the service demanded by the user i from each of the potential facilities in $\mathcal{X}$. These cost functions can be induced by distances or by general functions representing the dissatisfaction of the users for the different potential facilities.

We are also given $p \in \mathbb{Z}$ with $p \geq 1$, and we denote by $P = \{1, \ldots, p\}$ the index set of the facilities to be located. A p-Facility location problem consists of choosing p facilities from $\mathcal{X}$ minimizing certain cost function to the demand points. It is usual to assume, as we also do here, that users are allocated to the least costly (or equivalently, most preferred) open facility. We also assume that in case of ties, demand points are allocated to the facilities producing the less global intra-envy.

In this paper, we will measure the goodness of a selected set of facilities by the overall envy of the pairwise allocation costs between demand points allocated to the same facility. Specifically, if $\mathbf{X} = \{X_1, \ldots, X_p\} \subset \mathcal{X}$ is a given set of facilities and $j(\ell) :=$

$\arg\min_{j\in P} \Phi_\ell(X_j)$, i.e., the most preferred facility for the user, for each pair of demand points $i, k \in N$, we compute the intra-envy of i for j as:

$$\mathrm{IE}_{ik}(\mathbf{X}) = \begin{cases} \Phi_i(X_{j(i)}) - \Phi_k(X_{j(k)}) & \text{if } \Phi_k(X_{j(k)}) < \Phi_i(X_{j(i)}) \text{ and } j(i) = j(k), \\ 0 & \text{otherwise.} \end{cases}$$

That is, if the least costly open facility for i and k coincides and the allocation cost of k is smaller than the allocation cost for i, an intra-envy of i for k is incurred, equal to the difference between the allocation cost of i and the allocation cost of k.

The goal of the p-Intra-Envy Facility Location Problem (p-IEFLP, for short) is to choose p out of the facilities from $\mathcal{X}$ minimizing the overall sum of the intra-envy of all pairs of demand points allocated to the same facility. Formally, the problem can be formulated as follows:

$$\min_{\substack{\mathbf{X}\subseteq\mathcal{X}:\\ |\mathbf{X}|=p}} \sum_{i\in N}\sum_{k\in N} \mathrm{IE}_{ik}(\mathbf{X}). \tag{p-IEFLP}$$

Observe that, avoiding duplicates and zeros in the expression being minimized above, the objective function can be equivalently rewritten as:

$$\sum_{i\in N}\sum_{k\in N} \mathrm{IE}_{ik}(\mathbf{X}) = \sum_{i=1}^{n-1} \sum_{\substack{k=i+1:\\ j(k)=j(i)}}^{n} |\Phi_i(X_{j(i)}) - \Phi_k(X_{j(k)})|.$$

Using the usual allocation variables in facility location

$$x_{ij} = \begin{cases} 1 & \text{if plant } j \text{ is the least costly} \\ & \text{open facility for } i, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall i \in N, j \in P,$$

and the variables $X_1, \dots, X_p \in \mathcal{X}$ that represent the locations for the facilities, the problem can be stated as follows:

$$\min \sum_{j\in P}\sum_{i\in N}\sum_{\substack{k\in N:\\ i<k}} |\Phi_i(X_j) - \Phi_k(X_j)| x_{ij} x_{kj}, \tag{1}$$

$$\text{s.t.} \sum_{j\in P} x_{ij} = 1, \quad \forall i \in N, \tag{2}$$

$$\Phi_i(X_j)\cdot x_{ij} \leq \min_{\ell=1,\dots,p} \Phi_i(X_\ell), \quad \forall i \in N, j \in P, \tag{3}$$

$$X_1, \dots, X_p \in \mathcal{X}, \tag{4}$$

$$x_{ij} \in \{0,1\}, \quad \forall i \in N, j \in P.$$

The objective function (1) accounts for intra-envy between pairs of points allocated to the same facility. Constraints (2) assure that a single allocation is obtained for each demand point and (3) model the closest-assignment assumption (Espejo et al., 2012). The set of constraints (4) indicates the domain of the positions of the facilities on a set of potential facilities.

We analyze here the two main frameworks in facility location based on the nature of the set $\mathcal{X}$. On the one hand, we will study the *continuous* case in which $\mathcal{X} = \mathbb{R}^d$ and, then, the facilities are allowed to be located in the whole decision space. On the other hand, we will analyze the *discrete* case in which $\mathcal{X} = \{b_1, \dots, b_m\}$ is a given finite set of potential locations for the facilities.

Note that there are several differences when analyzing p-IEFLP under these two frameworks:

1. In the discrete case, the possible costs between the demand points and the potential facilities can be obtained in a pre-processing phase since the possibilities for the values of X_j are known and finite. Thus, one can compute a cost matrix $\Phi = (\Phi_i(b_j)) \in \mathbb{R}^{n\times m}$ that serves as input for the problem. In contrast, in the continuous case the costs can only be known when the coordinates of the facilities are computed and, then, their values have to be incorporated as decision variables to the problem. Furthermore, in a continuous problem, the shape of the cost functions $\Phi_1, \dots, \Phi_n$ has a significant impact in deriving a suitable mathematical programming formulation of the problem.
2. The closest-assignment constraints (3) must be treated differently for the continuous and the discrete case because of the knowledge of the cost values. For the discrete case, there are different approaches to incorporate linear constraints enforcing this requirement (see Espejo et al. (2012)). In the continuous case, this requirement must also be ensured using different, but simple, strategies.
3. The absolute values in the objective function measuring the intra-envies in case the costs are known, as in the discrete case, are constant values. In the continuous case these values are unknown and part of the decision problem.

In what follows, we illustrate the different situations when locating p facilities with different criteria, namely, the p-median, the p-envy (Espejo et al., 2009) and the p-intra-envy with the continuous and discrete frameworks.

Example 2.1. Consider the six demand points in the real line $\mathcal{A} = \{1, 2, 4, 6, 10, 14\}$. For the discrete problem, we assume that each demand point is also a potential facility. We aim to locate $p = 2$ facilities.

In the discrete setting, the dissatisfaction matrix (based on distances) can be prespecified as:

$$\Phi = \begin{pmatrix} 0 & 1 & 3 & 5 & 9 & 13 \\ 1 & 0 & 2 & 4 & 8 & 12 \\ 3 & 2 & 0 & 2 & 6 & 10 \\ 5 & 4 & 2 & 0 & 4 & 8 \\ 9 & 8 & 6 & 4 & 0 & 4 \\ 13 & 12 & 10 & 8 & 4 & 0 \end{pmatrix}.$$

In Fig. 1 we show the optimal solutions for the p-median, the p-envy, and the p-intra-envy problems (from top to bottom). The open facilities are highlighted with gray color circles and the arrows indicate the open facility where each of the users are allocated.

The optimal solution of the p-median problem on this instance, where the aim is to minimize the total distance between plants and their allocated users, is obtained locating plants at sites 2 and 14 and the optimal allocation pattern can be seen in the first picture of the figure. There, customer located at 1 is allocated to the facility in position 2 at a distance of 1 unit. Customer at 6, which is allocated to the same plant, and whose distance to the plant is 4, *feels* envy from 1 of $4 - 1 = 3$ units. However, customer at position 10 does not pay attention to customer at 1 since it is not allocated to its same plant. Using the notation introduced above, the intra-envy matrix induced for facilities located at positions 2 and 14 is

$$\mathrm{IE}(2, 14) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here, each row represents the envy of a customer with respect to each of the other demand points. The overall intra-envy for this solution is 17 units.

An optimal solution of the minimum envy problem previously studied in Espejo et al. (2009) for this instance is shown in the second picture of Fig. 1. There, instead of computing the envy only with respect to the demand points allocated to the same plant, the envy is calculated with respect to all the demand points, no matter if the demand points are allocated to the same facility or not. For this problem it is optimal to locate plants at 4 and 10, and the corresponding intra-envy matrix is in this case

$$\mathrm{IE}(2, 10) = \begin{pmatrix} 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}.$$

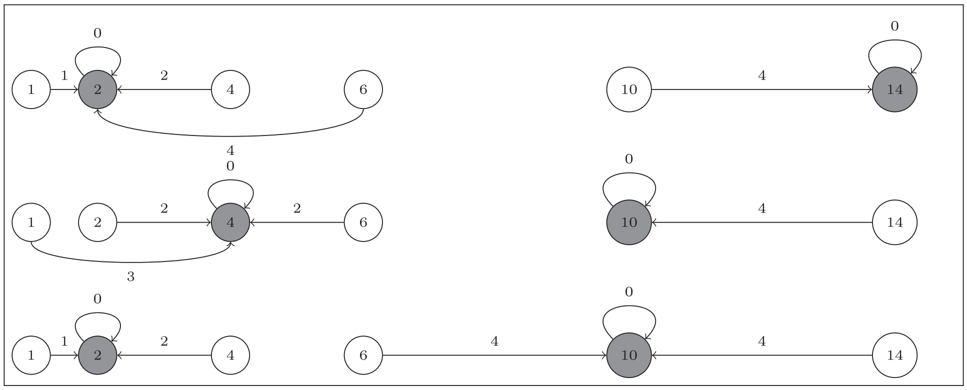

Fig. 1. Solution of the discrete problems of Example 2.1 (from top to bottom: *p*-median, *p*-envy and *p*-intra-envy problems).

The overall intra-envy is now 13 units.

In the third line of the figure we see the optimal solution to the discrete *p*-intra-envy location problem. The plants have been located at nodes 2 and 10. The envy matrix is

$$\mathrm{IE}(2,10)=\begin{pmatrix}0&1&0&0&0&0\\0&0&0&0&0&0\\1&2&0&0&0&0\\0&0&0&0&4&0\\0&0&0&0&0&0\\0&0&0&0&4&0\end{pmatrix}$$

being the total intra-envy equal to 12. Note that, for instance, user 6 which is at the same distance from 2 than from 10 feels a lower envy to its *neighbors* when it is allocated to 10 instead of being allocated to 2, implying an overall smaller intra-envy.

Example 2.2. In Fig. 2 we show the results for the 2-median problem, the 2-envy problem, and the 2-intra-envy problem on the plane (with cost function measured by the ℓ_1-norm) for the set of 6 demand points $\{(8,1),(1,13),(17,11),(18,15),(11,9),(19,7)\}$.

The solution of the 2-median problem on the plane is shown in the top picture. The facilities to be located are $X_1=(8,9)$ and $X_2=(18,11)$. The intra-envy matrix is in this case

$$\mathrm{IE}\left((8,9),(18,11)\right)=\begin{pmatrix}0&0&0&0&5&0\\3&0&0&0&8&0\\0&0&0&0&0&0\\0&0&3&0&0&0\\0&0&0&0&0&0\\0&0&4&1&0&0\end{pmatrix}$$

with overall intra-envy equal to 24.

In the second picture we show the result of solving a modified version of the minimum envy problem analyzed in Espejo et al. (2009) for the continuous case. The optimal facilities are located at positions $X_1=(8.5,15)$ and $X_2=(14.5,4)$. The intra-envy matrix in this case is

$$\mathrm{IE}\left((8.5,15),(14.5,4)\right)=\begin{pmatrix}0&0&0&0&1&2\\0&0&0&0&0&0\\0&0&0&0&1&2\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix}$$

with an overall intra-envy of 7. In this solution, all the demand points wish to be positioned at the same distance with respect to their closest facilities. Specifically, in all pairwise comparisons between the distances, a maximum difference of 2 units is observed.

Finally, in the bottom picture of Fig. 2 we show the solution of the 2-intra-envy problem in the continuous case. The two facilities are at $X_1=(4,6.5)$, and $X_2=(19,12)$, being the intra-envy matrix

$$\mathrm{IE}\left((4,6.5),(19,12)\right)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&2&2&0&0\end{pmatrix}$$

with an overall intra-envy of 4 units. The envy is computed only comparing pairs of demand points allocated to the same facility. The ideal solution is that in which all the points allocated to a facility are at the same distance to it. In the optimal solution, points numbered as 1, 2 and 5 are allocated to the same facility at the same distance, and then the envy among those points is zero.

2.1. Equity properties

There are some desirable properties for a measure to be considered *equitable* (see, for example, Barbati and Piccolo (2016), Marsh and Schilling (1994) and Mulligan (1991), among some others). In Barbati and Piccolo (2016), the authors classify these properties into two types: axiomatic binary properties and axiomatic computable properties, the first being those that can be either verified or not by an equality measure whereas the second are those that can be verified with different degrees of intensity. The axioms of the first type are: the transfer principle, the scale invariance principle, the normalization property and the impartiality property. The second type of axioms are: analytical traceability property, Pareto optimality solution and adequacy. The authors propose also three so-called *optimization properties* (that can be also classified into binary or computable): transformation invariance, asymptotic property, and the monotonic property.

In what follows we detail each of these properties and analyze the verification of them by the overall intra-envy measure:

- *Transfer principle*: This property is also known as the Pigou-Dalton property and a measure verifies it if it prioritizes satisfying users allocated to its most preferred facilities to those allocated to less preferred facilities. Since the intra-envy is a Gini-like measure, this property is verified (see e.g., Levy et al. (2006)).
- *Scale invariance principle and normalization*: This property states that the measure is not affected by the units for which the dissatisfaction or cost is measured. Clearly, this property is verified by the intra-envy.
- *Impartiality property*: This property states that equity only depends on social factors and data and not from other aspects like race, color, age or political. As already mentioned in Barbati and Piccolo (2016), in the location context this property is automatically satisfied because users are not distinguished according to these aspects.

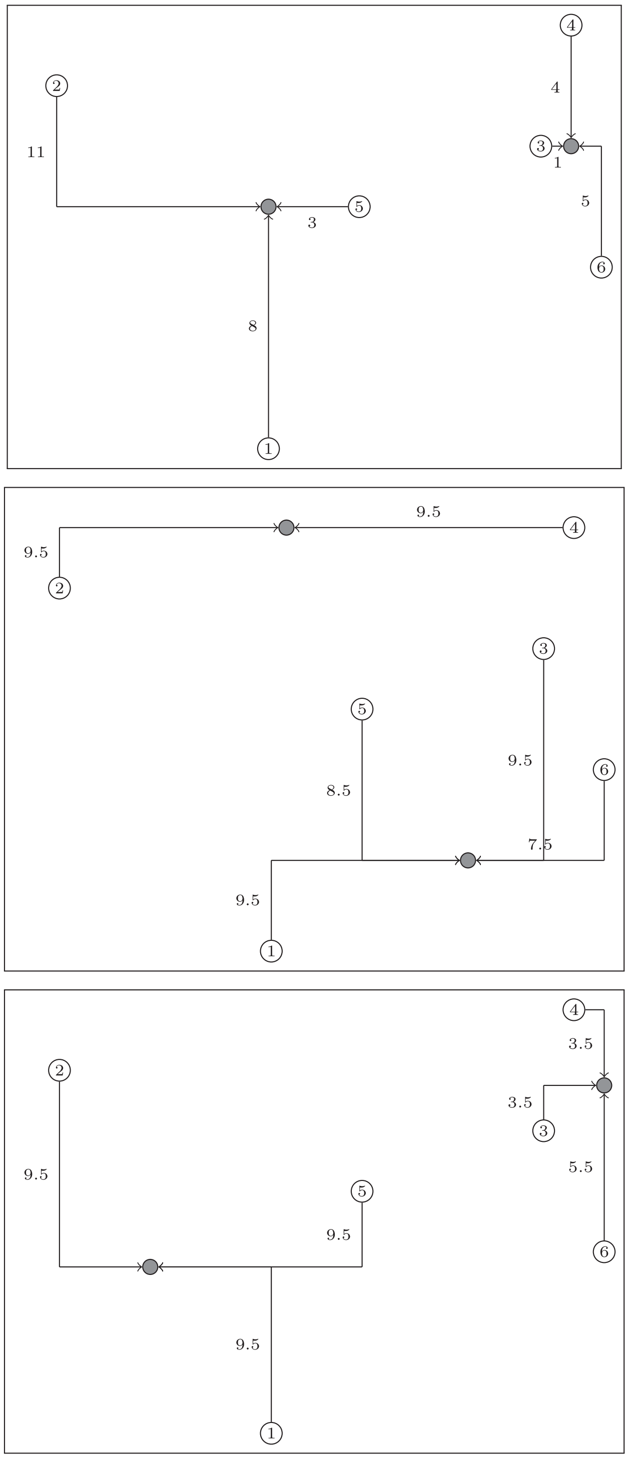

Fig. 2. Solution of the continuous problems of Example 2.2 (from top to bottom: p-median, p-envy and p-intra-envy problems).

- *Analytical traceability property*: This axiom refers to the computational tractability of the measure. The intra-envy of a given set of p facilities can be easily computed.
- *Pareto optimality*: This property implies that as the solution improves, none of the individuals or groups being affected will be worse off. This property is not satisfied by the intra-envy, but neither by the Gini index, the global envy or the absolute deviation.
- *Appropriateness*: This property assures that a measure can be easily understood and adequately defined in its decision making context. It is verified by the intra-envy.
- *Transformation invariance*: This axiom was proposed in Drezner et al. (2009) and is verified by a measure if changing with the same transformation the position of all the users and the position of the facilities, we obtain the same value for the measure considered. Thus, it is satisfied by the intra-envy.
- *Asymptotic and Monotonic property*: These properties analyze the trend of the different positions of the facilities and the demand points. The measure of this property is based on simulation and would require a further study. Based on the results in Barbati and Piccolo (2016), the performance of the intra-envy should be similar to the one described as AD in the mentioned paper.

3. The continuous minimum p-intra-envy facility location problem

In this section we will study the p-IEFLP assuming that the p facilities to be located are allowed to be positioned in the whole space. In this case, we assume that the preference function of user i with respect to a facility located at $X \in \mathbb{R}^d$ is given by the ℓ_1-norm based distance in $\mathbb{R}^d$, i.e.,

$$\Phi_i(X) = \|a_i - X\|_1 = \sum_{\ell=1}^{d} |a_{i\ell} - X_l|.$$

We provide three alternative mathematical programming formulations for the problem. The first formulation is based on representing envy as pairwise differences of distances. In contrast, in the second and third formulations, we exploit the structure of the intra-envy as an ordered median function of the distances. In both formulations, in order to adequately represent the distance in terms of the variables defining the positions of the facilities (which are part of the decision), we use the following sets of decision variables:

$X_{j\ell}$: ℓth coordinate of the jth facility, $\forall j \in P, \ell = 1, \dots, d$.

$\phi_{ij} = \Phi_i(X_j) = \|a_i - X_j\|_1, \quad \forall i \in N, j \in P.$

We also use the allocation variables x_{ij} described in the previous section.

First formulation for the continuous p-IEFLP

In the first formulation, in addition, we use the following variables to model the intra-envy between two demand points i and k for the set of p facilities $\mathbf{X} = \{X_1, \dots, X_p\}$:

$\theta_{ik} = \mathrm{IE}_{ik}(\mathbf{X}), \quad \forall i, k \in N (k > i).$

With the above sets of variables, the continuous p-IEFLP problem can be formulated as the following mathematical programming problem that we denote as (M_1^C):

$$\min \sum_{i \in N} \sum_{\substack{k \in N: \\ k > i}} \theta_{ik} \tag{5}$$

$$\text{s.t.} \sum_{j=1}^{p} x_{ij} = 1, \quad \forall i \in N, \tag{6}$$

$$\theta_{ik} \geq \phi_{ij} - \phi_{kj} - U(2 - x_{ij} - x_{kj}), \quad \forall i < k \in N, j \in P, \tag{7}$$

$$\theta_{ik} \geq \phi_{kj} - \phi_{ij} - U(2 - x_{ij} - x_{kj}), \quad \forall i < k \in N, j \in P, \tag{8}$$

$$\phi_{ij} \leq \phi_{i\ell} + U(1 - x_{ij}), \quad \forall i \in N, j \neq \ell \in P, \tag{9}$$

$$\phi_{ij} = \Phi_i(X_j), \quad \forall i \in N, j \in P, \tag{10}$$

$$X_1, \dots, X_p \in \mathbb{R}^d, \tag{11}$$

$$\theta_{ik} \geq 0, \quad \forall i, k \in N, k > i, \tag{12}$$

$$x_{ij} \in \{0,1\}, \quad \forall i \in N, j \in P \tag{13}$$

where U is a big enough constant. The reader may note that different $U's$ could be estimated for the different families of constraints, namely (7)–(8) and (9), although we keep it simpler for the sake of presentation.

In this model the objective function (5) accounts for the intra-envy felt by every customer with respect to all other customers. Constraints (7) and (8) allow to adequately represent the envy between users i and k, i.e., either $\mathrm{IE}_{ik}(X) = \Phi_i(X_j) - \Phi_k(X_j)$ or $\mathrm{IE}_{ki}(X) = \Phi_k(X_j) - \Phi_i(X_j)$ in case i and k are allocated to facility j. Constraints (9) assure the closest-assignment assumption. Constraints (10) are the representation of the ℓ_1 distances between demand points and the facilities. For $i \in N$ and $j \in P$, the constraint can be linearly modeled as follows:

$$w_{ij\ell} \le a_{i\ell} - X_{j\ell} + U(1-\xi_{ij\ell}), \quad \forall \ell = 1,\dots,d \tag{14}$$

$$w_{ij\ell} \ge a_{i\ell} - X_{j\ell}, \quad \forall \ell = 1,\dots,d, \tag{15}$$

$$w_{ij\ell} \le -a_{i\ell} + X_{j\ell} + U\xi_{ij\ell}, \quad \forall \ell = 1,\dots,d, \tag{16}$$

$$w_{ij\ell} \ge -a_{i\ell} + X_{j\ell}, \quad \forall \ell = 1,\dots,d, \tag{17}$$

$$\phi_{ij} = \sum_{\ell=1}^{d} w_{ij\ell}, \tag{18}$$

$$w_{ij\ell} \ge 0, \quad \forall \ell = 1,\dots,d,$$

$$\xi_{ij\ell} \in \{0,1\}, \quad \forall \ell = 1,\dots,d$$

where two sets of additional auxiliary variables are used: $w_{ij\ell} = |a_{i\ell} - X_{j\ell}|$, and $\xi_{ij\ell}$ that takes values 1 in case $w_{ij\ell} = a_{i\ell} - X_{j\ell} \ge 0$ and zero otherwise. Constraints (15) and (17) ensure that $w_{ij\ell} \ge \max\{a_{i\ell} - Xj\ell, -a_{i\ell} + X_{j\ell}\}$ while constraints (14) and (16) assure that $w_{ij\ell} \le \max\{a_{i\ell} - X_{j\ell}, -a_{i\ell} + X_{j\ell}\}$, defining adequately the absolute value $|a_{i\ell} - X_{j\ell}|$. Constraints (18) define the ℓ_1 distance by means of the sum of the absolute values of the differences at all the coordinates between the demand point and the facility. Note that with this representation of the distance, $\|X_j - a_i\|_1$ can be expressed by the sum of the adequate w-variables (18).

A k-sum based formulation for the continuous p-IEFLP

The second formulation that we propose is based on the following observation.

Lemma 3.1. *Let $\phi_{ij} = \Phi_i(X_j)$ if a_i is allocated to X_j and zero otherwise, $\phi_{(k)j}$ the kth largest distance in the sequence of distances from all the demand points to X_j, and k_j the number of points allocated to the jth facility. The intra-envy function can be written as:*

$$\sum_{i\in N}\sum_{k\in N} \mathrm{IE}_{ik}(X_1,\dots,X_p) = \sum_{j\in P}\sum_{k\in N}(k_j - 2k + 1)\phi_{(k)j}.$$

Proof. Let $X_1,\dots,X_p \in \mathbb{R}^d$ be the chosen facilities, and $\phi_{\cdot j} = (\phi_{1j},\dots,\phi_{nj})$ the allocation costs of all the demand points to X_j (assuming that the allocation cost of a demand point non allocated to X_j is zero). Sorting ϕ_j in non increasing order results in the vector $(\phi_{(1)j}, \phi_{(2)j}, \dots, \phi_{(k_j)j}, 0, \dots, 0)$ with $\phi_{(1)j} \ge \phi_{(2)j} \ge \cdots \ge \phi_{(k_j)j} \ge 0$.

With this notation, the intra-envy of the demand point sorted in the kth position for facility $j \in P$ can be computed as:

$$\sum_{i=k+1}^{k_j}(\phi_{(k)j} - \phi_{(i)j}) = (k_j - k)\phi_{(k)j} - \sum_{i=k+1}^{k_j}\phi_{(i)j}$$
$$= (k_j - k)\phi_{(k)j} - \phi_{(k+1)j} - \cdots - \phi_{(k_j)j}.$$

Adding up the above expression for all k we obtain that the intra-envy for the jth facility can be written as:

$$\begin{aligned}\sum_{k=1}^{k_j-1}\sum_{i=k+1}^{k_j}(\phi_{(k)j} - \phi_{(i)j}) &= \sum_{k=1}^{k_j-1}\left((k_j - k)\phi_{(k)j} - \phi^j_{(k+1)} - \cdots - \phi_{(k_j)j}\right)\\ &= (k_j - 1)\phi^j_{(1)} + (k_j - 3)\phi^j_{(2)} + \cdots + (1-k_j)\phi_{(k_j)j}\\ &= \sum_{k=1}^{k_j}(k_j - 2k + 1)\phi_{(k)j}\\ &= \sum_{k\in N}(k_j - 2k + 1)\phi_{(k)j}. \quad \square\end{aligned}$$

The above result identifies the objective function of p-IEFLP as an ordered median function of the allocation costs given by Φ (see Mesa et al. (2003)). This type of functions has been widely studied in location science (see e.g., Marín et al. (2009)) and several representations are possible to embed these sortings in a mathematical programming formulation. One of the most effective representations is through the so-called k-sums which is based on expressing the ordered median function as a weighted sum of k-sums $S_k(\phi_{\cdot j}) = \sum_{\ell=1}^{k}\phi_{(\ell)j}$.

Lemma 3.2. *Let $j \in P$, $X_j \in \mathbb{R}^d$ and $\phi_{\cdot j} = (\phi_{1j},\dots,\phi_{nj})$ the allocation costs of all the demand points to X_j. Then:*

$$\sum_{k\in N}(k_j - 2k + 1)\phi_{(k)j} = 2\sum_{k\in N: k<k_j}\sum_{\ell=1}^{k}\phi_{(\ell)j} - (2n + 1 - k_j)\sum_{\ell=1}^{n}\phi_{(\ell)j}.$$

Proof. Observe that defining:

$$\Delta_k^j = \begin{cases} 2 & \text{if } k < k_j,\\ 1 - k_j & \text{if } k = k_j,\\ 0 & \text{otherwise,}\end{cases}$$

we obtain

$$\begin{aligned}\sum_{k\in N}(k_j - 2k + 1)\phi_{(k)j} &= \sum_{k\in N}\Delta_k^j S_k(\phi_{\cdot j}) = \sum_{k\in N}\Delta_k^j\sum_{\ell=1}^{k}\phi_{(\ell)j}\\ &= 2\sum_{k\in N:k<k_j} S_k(\phi^j) + (1-k_j)S_{k_j}(\phi^j)\\ &= 2\sum_{k\in N} S_k(\phi^j) - 2(n - k_j + 1)\\ &\quad\times \sum_{\ell=1}^{n}\phi_{\ell j} + (1-k_j)\sum_{\ell=1}^{n}\phi_{\ell j}\\ &= 2\sum_{k\in N} S_k(\phi_{\cdot j}) - (2n + 1 - k_j)\sum_{\ell=1}^{n}\phi_{\ell j}\end{aligned}$$

since $\phi_{(k)j} = 0$ for all $k > k_j$ and then $S_k(\phi_{\cdot j}) = S_{k_j}(\phi^j) = \sum_{l=1}^{n}\phi_{lj}$. In the last equation the expression $\sum_{\ell=1}^{n}\phi_{\ell j}$ is added up and removed to aggregate in the first addend all the k-sums. $\square$

With the above results, the objective function of p-IEFLP can be rewritten as:

$$\sum_{i\in N}\sum_{k\in N}\mathrm{IE}_{ik}(\mathbf{X}) = 2\sum_{j\in P}\sum_{k\in N}\sum_{\ell=1}^{k}\phi_{(\ell)j} - \sum_{j\in P}(2n + 1 - k_j)\sum_{\ell=1}^{n}\phi_{\ell j}.$$

Different representations of k-sums are possible when they are incorporated to optimization problems (see e.g., Blanco et al. (2014), Marín et al. (2020), Ogryczak and Tamir (2003) and Puerto et al. (2017)). Specifically, we use the one proposed in Blanco et al. (2014) to derive the following mathematical formulation for the continuous p-EIFLP where, additionally to the above-mentioned variables, we use the set of auxiliary variables $\alpha_{ij} = \sum_{\ell\in N}\phi_{\ell j}x_{ij}$ in order to represent the expression $k_j\sum_{\ell=1}^{n}\phi_{\ell j} = \left(\sum_{i\in N}x_{ij}\right)\sum_{\ell\in N}\phi_{\ell j} = \sum_{i,\ell\in N}x_{ij}\phi_{\ell j}$ in the objective function. The following formulation for the problem was denoted by $\mathrm{M}_2{}^C$.

$$\min\ 2\sum_{j\in P}\sum_{k\in N}\left(\sum_{\ell\in N}u_{k\ell j}+\sum_{i\in N}v_{kij}\right)-(2n+1)\sum_{j\in P}\sum_{i\in N}\phi_{ij}+\sum_{j\in P}\sum_{i\in N}\alpha_{ij} \tag{19}$$

$$\text{s.t. } \sum_{j\in P}x_{ij}=1\quad \forall i\in N, \tag{20}$$

$$w_{ij\ell}\le a_{i\ell}-X_{j\ell}+U(1-\xi_{ij\ell}),\quad \forall i\in N, j\in P, \ell=1,\dots,d \tag{21}$$

$$w_{ij\ell}\ge a_{i\ell}-X_{j\ell},\quad \forall i\in N, j\in P, \ell=1,\dots,d, \tag{22}$$

$$w_{ij\ell}\le -a_{i\ell}+X_{j\ell}+U\xi_{ij\ell},\quad \forall i\in N, j\in P, \ell=1,\dots,d \tag{23}$$

$$w_{ij\ell}\ge -a_{i\ell}+X_{j\ell},\quad \forall i\in N, j\in P, \ell=1,\dots,d, \tag{24}$$

$$\phi_{ij}\le \sum_{\ell=1}^{d}w_{ij\ell}+U(1-x_{ij}),\quad \forall i\in N, j\in P, \tag{25}$$

$$\phi_{ij}\ge \sum_{\ell=1}^{d}w_{ij\ell}-U(1-x_{ij}),\quad \forall i\in N, j\in P, \tag{26}$$

$$\phi_{ij}\le Ux_{ij},\quad \forall i\in N, j\in P, \tag{27}$$

$$\alpha_{ij}\ge \sum_{\ell\in N}\phi_{\ell j}-U(1-x_{ij}),\quad \forall i,\ell\in N, j\in P, \tag{28}$$

$$u_{k\ell j}+v_{kij}\ge \phi_{ij},\quad \forall i,k\in N, j\in P, \tag{29}$$

$$u_{kij}, v_{kij}\ge 0,\quad \forall k,i\in N, j\in P, \tag{30}$$

$$\phi_{ij}\ge 0, x_{ij}\in\{0,1\},\quad \forall i\in N, j\in P, \tag{31}$$

$$\alpha_{ij}\ge 0,\quad \forall i\in N, j\in P. \tag{32}$$

The objective function represents the overall p-intra-envy as detailed in the above comments. Constraints (20) are the single-allocation constraints. Constraints (21)–(24) ensure the correct definition of the absolute values $w_{ij\ell}=|a_{i\ell}-X_{j\ell}|$ required to derive the ℓ_1 distances between demand points and the facilities. Constraints (25) and (26) assure that, in case the demand point a_i is allocated to facility j, then, the cost of allocating such a point to that facility, ϕ_{ij}, is defined as the ℓ_1-norm based distance between a_i and X_j. Otherwise, by constraints (27) the cost ϕ_{ij} is fixed to zero. Constraints (28) (and the minimization of the α-variables) ensure the correct definition of the α-variables. Finally, constraints (29) allow computing adequately the k-sums.

Remark 3.1. Apart from the k-sum representation applied above based on Blanco et al. (2014), other representations are possible. Specifically, in Ogryczak and Tamir (2003) the authors provide an alternative formulation that has been widely used in the literature. There, it is proved that

$$S_k(\phi_{\cdot j}) = \begin{array}{ll} \min & kt_{kj}+\sum_{i\in N}w_{kij}\\ \text{s.t.} & t_{kj}+w_{kij}\ge\phi_{ij},\quad \forall i\in N\\ & t_{kj}\ge 0,\\ & w_{kij}\ge 0,\quad \forall i\in N.\end{array}$$

This representation can be embedded in the following formulation that we call $\mathrm{M}_3{}^C$:

$$\min\ 2\sum_{j\in P}\sum_{k\in N}\Big(kt_{kj}+\sum_{i\in N}w_{kij}\Big)-(2n+1)\sum_{j\in P}\sum_{i\in N}\phi_{ij}+\sum_{j\in P}\sum_{i\in N}\alpha_{ij} \tag{33}$$

$$\text{s.t. (20)–(28)} \tag{34}$$

$$t_{kj}+w_{kij}\ge\phi_{ij},\quad \forall i,k\in N, j\in P \tag{35}$$

$$\phi_{ij}\ge 0, x_{ij}\in\{0,1\},\quad \forall i\in N, j\in P, \tag{36}$$

$$t_{kj}\ge 0,\quad \forall k\in N, j\in P, \tag{37}$$

$$w_{kij}\ge 0,\quad \forall i\in N, k\in N, j\in P, \tag{38}$$

$$\alpha_{ij}\ge 0,\quad \forall i\in N, j\in P. \tag{39}$$

4. The discrete minimum p-intra-envy facility location problem

In this section we analyze the case in which the set of potential positions for the facilities is a finite set, i.e., $\mathcal{X}=\{b_1,\dots,b_m\}\subseteq\mathbb{R}^d$. We denote by $M=\{1,\dots,m\}$ the index set for the potential facilities. We assume that $1\le p\le m-1$ plants have to be located.

In this situation, the distances/costs between the users and all the potential sites can be computed in a preprocessing phase. We denote by $C=(\phi_{ij})_{n\times m}$ a costs matrix, where $\phi_{ij}=\Phi_i(b_j)$ is the measure of the dissatisfaction user i will feel if he is allocated to site j.

In this case, the problem is reduced to choosing p facilities out of the m potential facilities minimizing the overall envy of the demand points. Thus, apart from the x-variables indicating the allocation of users to plants and the θ-variables used to model the envy between demand points, already defined in the previous sections, we use the following set of variables:

$$y_j=\begin{cases}1 & \text{if a plant is located at site } j,\\ 0 & \text{otherwise,}\end{cases}\qquad \forall j\in M.$$

With the above notation, the discrete minimum p-intra-envy facility location problem can be formulated as the following mixed integer linear programming problem that we denote by M_1^D:

$$\min \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\theta_{ik} \tag{40}$$

$$\text{s.t. } \sum_{j\in M}y_j=p, \tag{41}$$

$$\sum_{j\in M}x_{ij}=1,\quad \forall i\in N, \tag{42}$$

$$x_{ij}\le y_j,\quad \forall i\in N,\ j\in M, \tag{43}$$

$$y_j+\sum_{\substack{\ell\in M:\\ \phi_{ij}<\phi_{i\ell}}}x_{i\ell}\le 1,\quad \forall i\in N,\ j\in M, \tag{44}$$

$$\theta_{ik}\ge|\phi_{ij}-\phi_{kj}|(x_{ij}+x_{kj}-1),\quad \forall i<k\in N,\ j\in M, \tag{45}$$

$$y_i\in\{0,1\},\quad \forall j\in M, \tag{46}$$

$$x_{ij}\in\{0,1\},\quad \forall i\in N,\ j\in M. \tag{47}$$

Constraints (41)–(43) are the classical p-median constraints that assure that p services are open, each customer is allocated to a single plant and demand points are allowed to be allocated only to open plants. Constraints (44) are the closest assignment constraints. They avoid allocating a customer to plants which are less desired than others that are open and have been chosen among several alternative closest assignment constraints developed in the literature based on the analysis made in Espejo et al. (2009) and the experience of the authors. Constraints (45) ensure the adequate definition of the envy variables θ. Note that θ_{ik} takes value $|\phi_{ij}-\phi_{ik}|$ in case i and k are allocated to a common plant j, i.e, whenever $x_{ij}\cdot x_{kj}=1$ (term which is linearized in the constraint as $x_{ij}+x_{kj}-1$). These constraints can be strengthened by the following ones:

$$\theta_{ik}\ge|\phi_{ij}-\phi_{kj}|(x_{ij}+x_{kj}-y_j),\qquad \forall i<k\in N,\ j\in M, \tag{48}$$

since y_j will take value 1 whenever $x_{ij}x_{kj}$ will take value 1.

The above formulation is based on the classical variables and constraints of discrete location problems. A reduced formulation can be derived using only the y-variables, with the following formulation, named M_2^D:

$$\min \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\theta_{ik} \tag{49}$$

$$\text{s.t. } \sum_{j\in M}y_j=p, \tag{50}$$

$$\theta_{ik}\ge|\phi_{ij}-\phi_{kj}|\Big(y_j-\sum_{\substack{\ell\in M:\\ \phi_{i\ell}<\phi_{ij}\text{ or}\\ \phi_{k\ell}<\phi_{kj}}}y_\ell\Big),\quad \forall i<k\in N,\ j\in M, \tag{51}$$

$$\theta_{ik} \geq 0, \quad \forall i < k \in N, \tag{52}$$

$$y_j \in \{0,1\}, \forall j \in M. \tag{53}$$

Constraints (51) make θ_{ik} to take value $|\phi_{ij} - \phi_{kj}|$ when (i) plant j is opened, and (ii) no other plant ℓ is opened if it is closer to i or k than plant j. This means that the closest plant (or one of the closest plants in case of tie) to both, i and k, is j, and therefore i and k will be both allocated to j and the envy of this allocation will be (at least, in case of tie) $|\phi_{ij} - \phi_{kj}|$.

The following set of valid inequalities can be used to tighten the formulation:

$$\theta_{ik} \geq \sum_{j\in J} |\phi_{ij} - \phi_{kj}|(y_j - \sum_{\substack{\ell\in M:\\ \phi_{i\ell}<\phi_{ij} \text{ or}\\ \phi_{k\ell}<\phi_{kj}}} y_\ell), \quad \forall J \subseteq M.$$

These inequalities were incorporated to the above formulation sequentially in the branch-and-bound tree by separating them with the following strategy. Let $\bar{y} \in [0,1]^m$ and $\bar{\theta} \in \mathbb{R}_+^{n\times n}$ be a feasible relaxed solution. For each $i,k \in N$, let $J_{ik} = \{j \in M : \bar{y}_j > \sum_{\substack{\ell\in M:\\ \phi_{i\ell}<\phi_{ij} \text{ or}\\ \phi_{k\ell}<\phi_{kj}}} \bar{y}_\ell\}$ and $\rho_{ik} = \sum_{j\in J_{ik}} |\phi_{ij} - \phi_{kj}|(\bar{y}_j - \sum_{\substack{\ell\in M:\\ \phi_{i\ell}<\phi_{ij} \text{ or}\\ \phi_{k\ell}<\phi_{kj}}} \bar{y}_\ell)$. If $\bar{\rho}_{ik} > \bar{\theta}_{ik}$, then, incorporate the cut:

$$\theta_{ik} \geq \sum_{j\in J_{ik}} |\phi_{ij} - \phi_{kj}|(y_j - \sum_{\substack{\ell\in M:\\ \phi_{i\ell}<\phi_{ij} \text{ or}\\ \phi_{k\ell}<\phi_{kj}}} y_\ell).$$

4.1. k-sums based formulation

With the same ideas applied to reformulate the continuous problem into an ordered median problem, we obtain:

$$\sum_{i\in N}\sum_{k\in N} \mathrm{IE}_{ik}(\mathbf{X}) = 2\sum_{j\in P}\sum_{k\in N}\sum_{\ell=1}^{k} \phi_{(\ell)j} - \sum_{j\in P}(2n+1-k_j)\sum_{\ell=1}^{n}\phi_{\ell j}x_{\ell j}$$

where k_j is the number of demand points allocated to plant j. Thus the following formulation that we call M_3^D is valid for the problem:

$$\min \; 2\sum_{j\in P}\sum_{k\in N}\left(\sum_{\ell\in N} u_{k\ell j} + \sum_{i\in N} v_{kij}\right) - (2n+1)\sum_{j\in P}\sum_{i\in N}\phi_{ij}x_{ij} + \sum_{j\in P}\sum_{i\in N}\phi_{ij}\alpha_{ij} \tag{54}$$

$$\text{s.t.} \sum_{j\in M} y_j = p, \tag{55}$$

$$\sum_{j\in M} x_{ij} = 1, \quad \forall i \in N, \tag{56}$$

$$x_{ij} \leq y_j, \quad \forall i \in N, \; j \in M, \tag{57}$$

$$y_j + \sum_{\substack{\ell\in M:\\ \phi_{ij}<\phi_{i\ell}}} x_{i\ell} \leq 1, \quad \forall i \in N, \; j \in M, \tag{58}$$

$$\alpha_{ij} \geq \sum_{\ell\in N} x_{\ell j} - (n-p)(1-x_{ij}), \quad \forall i,\ell \in N, j \in P, \tag{59}$$

$$u_{k\ell j} + v_{kij} \geq \phi_{ij}x_{ij}, \quad \forall i,\ell,k \in N(\ell \leq k), j \in P, \tag{60}$$

$$u_{kij}, v_{kij} \geq 0, \quad \forall k,i \in N, j \in P, \tag{61}$$

$$x_{ij} \in \{0,1\}, \quad \forall i \in N, j \in P, \tag{62}$$

$$\alpha_{ij} \geq 0, \quad \forall i,\ell \in N, j \in P. \tag{63}$$

5. Computational study

In this section we provide the results of our computational experience in order to evaluate the performance of the proposed approaches. All the formulations were coded in Python 3.7 in an iMac with 3.3 GHz with an Intel Core i7 with 4 cores and 16 GB 1867 MHz DDR3 RAM. We used Gurobi 9.1.2 as optimization solver. A time limit of 2 h was fixed for all the instances.

In order to produce a set of test instances, we use a similar strategy to the one in Espejo et al. (2009). We generate two different types of instances for each combination of parameters, n, p and d. For the instances of type `random`, the demand points are uniformly generated in $[0,100]^d$. In the instances of type `blob` the points were generated as isotropic Gaussian blobs in $[0,100]^d$ with $\lceil \frac{n}{3} \rceil$ cluster centers and standard deviation 1. Whereas uniform random instances are the most frequent when generating random instances to test location science problems, the blob instances more adequately represent the behavior of users which are usually geographically clustered, simulating a higher concentration of users around certain points of interest. The generated instances are available at https://github.com/vblancoOR/intraenvy.

For the discrete problem, the set of potential sites for the facilities are assumed to be the whole set of demand points and the cost matrix Φ is pre-computed using the ℓ_1-norm.

We tested the formulations on a testbed of five instances for each combination of type (`random` and `blob`), $d \in \{2,3\}$, $n \in \{10,20,30,40,50\}$. Thus, we have generated 100 random instances. We solved the different models for $p \in \{2,3,5,7,10,15,20,25,30,35,40\}$ with $p \leq \frac{3n}{4}$.

For each of these instances, we run the following formulations for the minimum p-intra-envy facility location problem:

Discrete		Continuous	
$\mathtt{M1}^D$:	(40)–(47)	$\mathtt{M1}^C$:	(5)–(13)
$\mathtt{M2}^D$:	(49)–(53)	$\mathtt{M2}^C$:	(19)–(32)
$\mathtt{M3}^D$:	(54)–(63)	$\mathtt{M3}^C$:	(33)–(39)

Combining the different approaches, the two different domains, and the generated instances we solved 4568 problems. The complete results of our experiments are available in the github repository mentioned above. As expected, higher intra-envies occur in case p is small compared to n (more users are to be allocated to the same facility). Thus, in the analysis drawn in the following sections we focus on the results for those problems where $p \leq 7$ (1000 problems for continuous instances and 1723 for discrete instances).

5.1. p-median, p-envy and p-intraEnvy measures

The first experiment that we run is devoted to determining the convenience of the *intra-envy* model in our instances. For each instance, we solve three different facility location models for each of the two different solution domains (discrete and continuous) that differ in their optimization criterion, namely, p-median, envy, and intra-envy. In the p-median model, the goal is to minimize the overall sum of the distances from each user to its closest facility, i.e.:

$$\min_{\substack{\mathbf{X}\subseteq\mathcal{X}:\\ |\mathbf{X}|=p}} \sum_{i\in N} \min_{j=1,\ldots,p} \Phi_i(X_{j(i)}). \tag{p-Median}$$

The (global) envy model aims to minimize the overall envy felt by the users, no matter to which facility are allocated, that is, defining the envy between two users $i,k \in N$ for a given set of facilities $\mathbf{X} \subset \mathcal{X}$ as:

$$\mathrm{Envy}_{ik}(\mathbf{X}) = \begin{cases} \Phi_i(X_{j(i)}) - \Phi_k(X_{j(k)}) & \text{if } \Phi_k(X_{j(i)}) < \Phi_i(X_{j(k)}),\\ 0 & \text{otherwise.}\end{cases}$$

The envy problem consists of:

$$\min_{\substack{\mathbf{X}\subseteq\mathcal{X}:\\ |\mathbf{X}|=p}} \sum_{i\in N}\sum_{k\in N} \mathrm{Envy}_{ik}(\mathbf{X}). \tag{p-Envy}$$

For each of these solutions, we evaluate the three different objective functions and analyze the results which are shown in Figs. 3 to 8. In these figures, the results for the p-median problem are highlighted in green color, those for the envy problem in orange, and the results of the intra-envy problem in blue color.

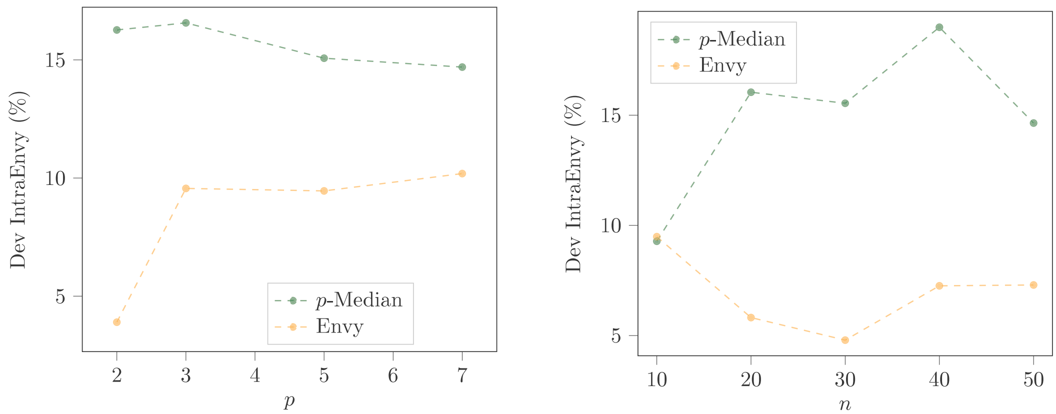

Fig. 3. Average deviations of p-median and envy solutions in the intra-envy measure with respect to the best intra-envy solution by p (left) and n (right) parameters for the discrete instances.

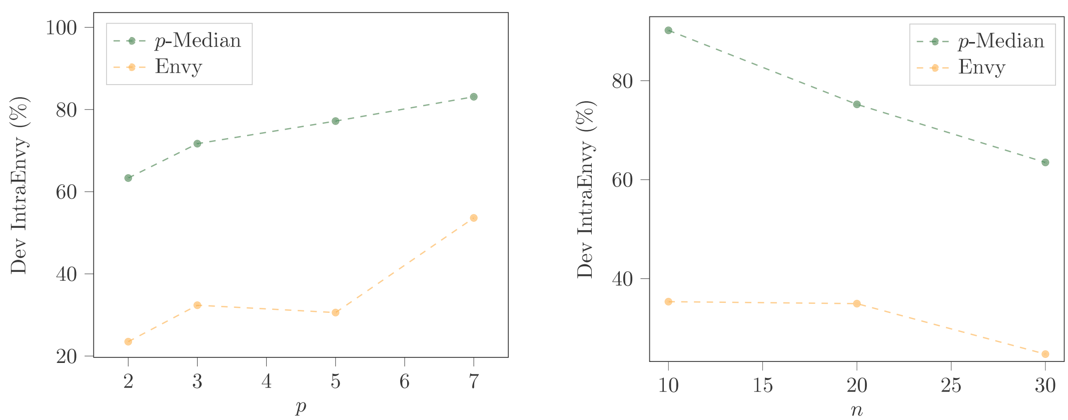

Fig. 4. Average deviations of p-median and envy solutions in the intra-envy measure with respect to the best intra-envy solution by p (left) and n (right) parameters for the continuous instances.

In Figs. 3 and 4 we report the average deviations (by the values of n and p), in terms of intra-envy, on the solutions of the models that do not consider such a criterion in their objectives, namely p-median and envy, for both the discrete and the continuous instances. The first observation that can be drawn is that the solutions of the Intra-Envy model do not coincide with those obtained with the other models. Specifically, for the discrete instances one can find instances for which the p-median problem obtains solutions deviated from the optimal intra-envy in more than 20%. For the continuous instances, the difference is even more impressive with deviations close to 90% (caused by very small intra-envy values). For the discrete instances, this deviation decreases with the values of p, whereas for the continuous instances, the situation is the opposite: The larger the p the larger the deviation. It is caused by the fact that the continuous instances allow more flexibility and an overall intra-envy close to zero as p increases. Observe that a zero intra-envy solution is one in which all the users allocated to a facility assume the same allocation costs. If the closest-assignment constraints were not present, it could be obtained by *clustering* users in the same ℓ_1-norm orbit around a point (the center). Since we also force the users to be allocated to their closest facility, a close to zero-envy solution can be also obtained by co-locating all the facilities at the same position and deciding the clusters of users by similar distances to the single center. This situation is not possible in the discrete instances, but in the continuous instances, as far as p increases, the facilities tend to co-locate. Nevertheless, this is not the situation of the median objective, where co-location is not a valid strategy.

The high deviation of the p-median problem with respect to the intra-envy measure can be also observed when averaging by the value of n. There, whereas the envy model seems to obtain stable solutions with respect to the intra-envy, the p-median problem is on average deviated from the intra-envy in more than 10% for the discrete instances and more than 70% for the continuous instances. We observed that the deviation for the random instances is 10% smaller than the obtained for the blobs instances.

Figs. 5 and 6 show the results of measuring the median objective (overall allocation costs) on the envy-based solutions with respect to the p-median solutions for the two types of instances. This deviation is known as the price of fairness according to Bertsimas et al. (2011). As expected, solutions with small global envy result in solutions with higher overall median-based allocation costs. Nevertheless, the deviations in these costs of the obtained solutions for the discrete instances are not that large, being the overall extra cost of the intra-envy model less than 9% with respect to the best p-median solution. The performance of the continuous instances is again different with respect to efficiency. On the one hand, the average deviations are larger than 15% in all cases. Nevertheless, the behaviors of the envy and the intra-envy solutions are similar, being the envy model slightly more efficient (in average 10% for the continuous instances and 2% for the discrete ones) than the intra-envy model.

In Figs. 7 and 8 we evaluate the p-median and the intra-envy model in terms of the global envy objective. In both types of instances, although the p-median model seems to deviate more from the envy model for small values of p, the deviations for larger p are neglectable for the discrete instances, being the overall global envy for this model similar to those of the envy model. Nevertheless, the intra-envy model results in solutions that differ from those of the envy model, being the deviation consistently close to its average (around 3.5%).

Finally, in Figs. 9 and 10 we analyze the results obtained for the different datasets that we tested in our experiments, namely, `random`

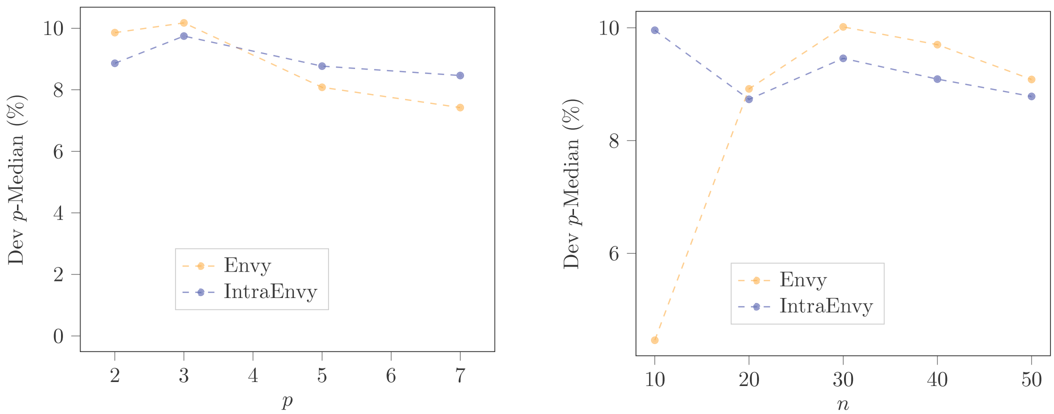

Fig. 5. Average deviations of envy and intra-envy solutions in the median measure with respect to the best median solution by p (left) and n (right) parameters for the discrete instances.

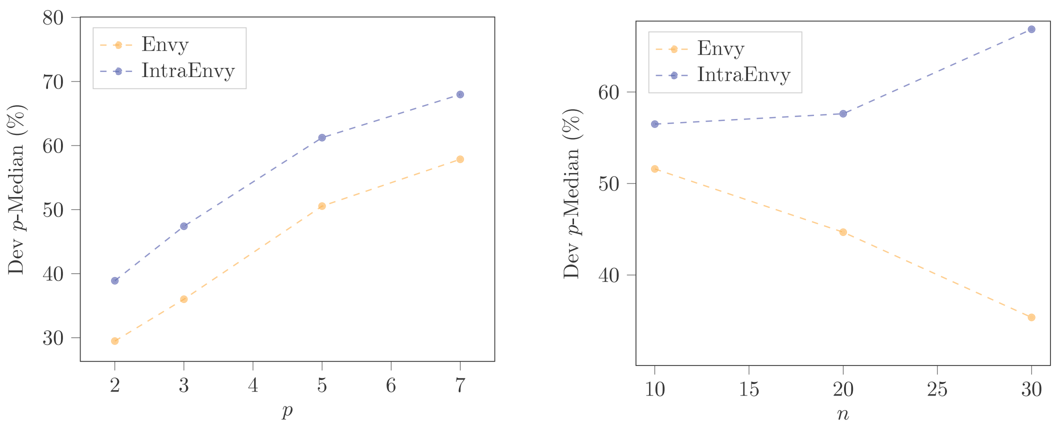

Fig. 6. Average deviations of envy and intra-envy solutions in the median measure with respect to the best median solution by p (left) and n (right) parameters for the continuous instances.

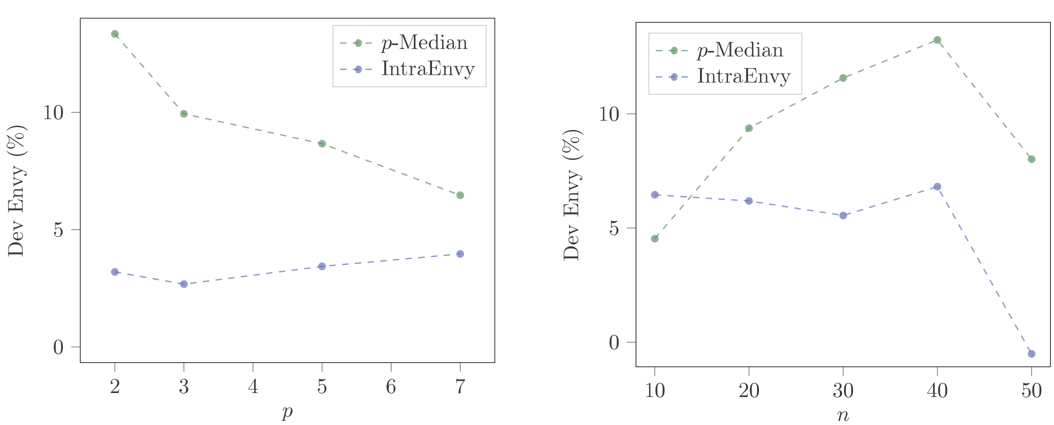

Fig. 7. Average deviations of p-median and intra-envy solutions in the global-envy measure with respect to the best envy solution by p (left) and n (right) parameters for the discrete instances.

and `blob`. We represent in those figures the average values of the optimal p-median, envy and intra-envy solutions aggregated by n and p, respectively. As expected, under similar random generation strategies (except the clustering), we obtain better solutions for the `blobs` instances. Recall that in these instances the demand points are already clustered.

Summarizing, the solutions obtained with the intra-envy model clearly differ from those obtained with the classical p-median and the envy model proposed in Espejo et al. (2009). Although determining facilities that exhibit minimum intra-envy has a direct impact on the transportation costs of the solution, in the discrete instances these extra costs are small enough to assume them in case one desires to avoid envies among the demand points allocated to the same facility, whereas in the continuous instances, they are similar to those obtained with the envy model.

5.2. Computational performance

In this section we analyze the computational performance of the different intra-envy formulations that we propose here. Specifically, we

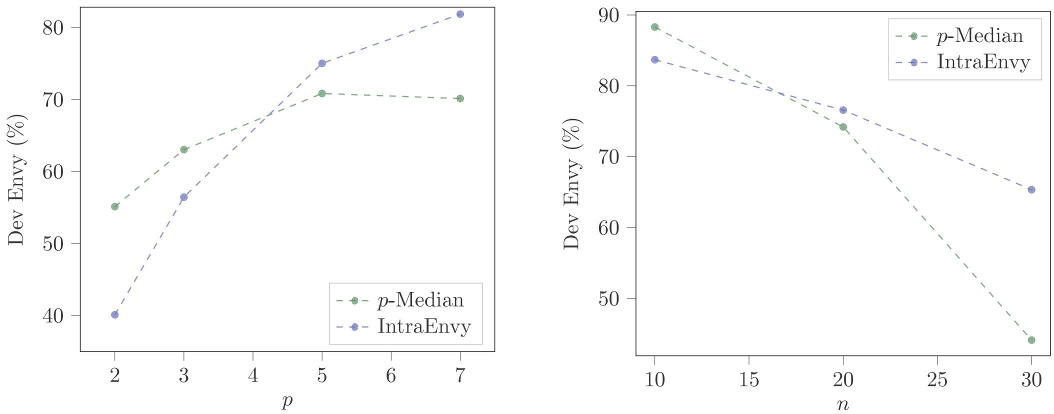

Fig. 8. Average deviations of p-median and intra-envy solutions in the global-envy measure with respect to the best envy solution by p (left) and n (right) parameters for the continuous instances.

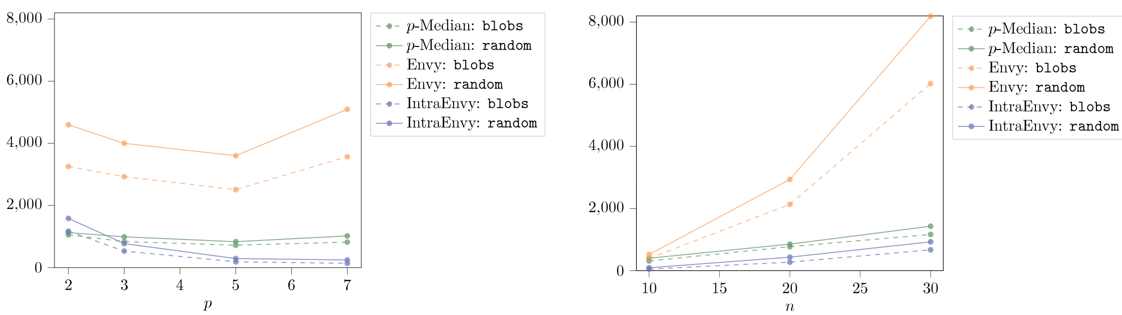

Fig. 9. Optimal median, envy and intra-envy values for the two types of continuous instances.

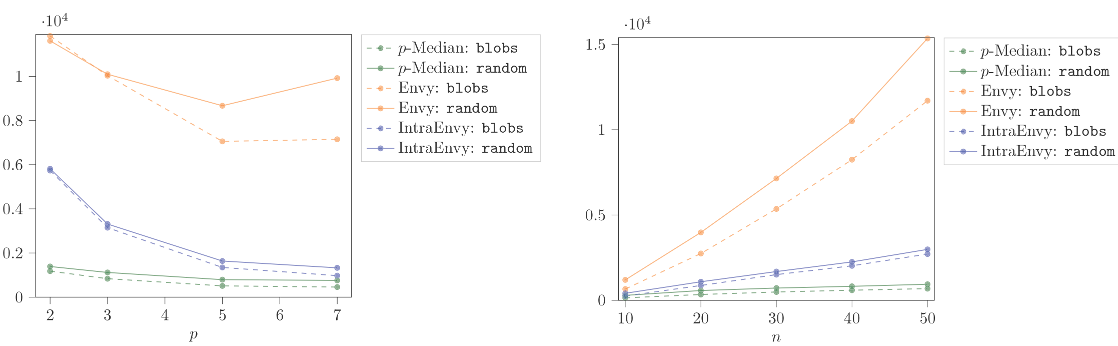

Fig. 10. Optimal median, envy and intra-envy values for the two types of discrete instances.

study the computational difficulty of solving each of the formulations by means of the consumed CPU time and the MIPGap, for those instances that we were not able to solve optimally within the time limit. We summarize here the obtained results, whereas the detailed results are available for the interested reader in https://github.com/vblancoOR/intraenvy.

In Table 2 we show the average CPU time, in seconds, required by the different formulations for solving the discrete location instances with n up to 30. We average our results by n (number of users), p (number of facilities to be located), type of instance (`random` or `blobs`) and integer programming formulation for the p-intra-envy problem (M1, M2, and M3). In case that any of the averaged instances are not optimally solved before the time limit, we write TL. The results of solving 300 different instances are summarized.

One can observe from these results that model M3, the one based on the ordered median reformulation of the problem, is not competitive with the other formulations. Specifically, even for instances with $n = 20$, formulation M2 required much more CPU time to solve the instances than the rest of the formulations, for both the random and the blobs instances. Note that the ordered median representation of the problem

Table 2
CPU times for discrete instances with $n \in \{10, 20, 30\}$.

n	p	BLB			RND		
		M1^D	M2^D	M3^D	M1^D	M2^D	M3^D
10	2	0.01	0.03	0.56	0.02	0.03	0.69
	3	0.01	0.03	0.61	0.03	0.03	0.63
	5	0.01	0.03	0.61	0.01	0.03	0.70
20	2	1.89	0.32	307.16	1.82	0.40	737.49
	3	1.05	0.42	76.28	1.42	0.53	807.96
	5	0.14	0.51	1050.24	0.54	1.00	6585.21
	7	0.06	1.01	4247.95	0.19	1.22	6525.11
30	2	402.80	2.90	TL	412.41	103.39	TL
	3	392.30	5.36	TL	131.35	9.74	TL
	5	4.38	125.07	TL	9.44	635.34	TL
	7	1.34	164.79	TL	4.34	1173.37	TL

Table 3
Computational results for discrete instances with $n \in \{40, 50\}$.

n	p	CPU time		%Unsolved		%MIPGap		%RootGap	
		$M1^D$	$M2^D$	$M1^D$	$M2^D$	$M1^D$	$M2^D$	$M1^D$	$M2^D$
40	2	1558.92	182.05	0%	0%	0%	0%	54%	100%
	3	2024.13	380.26	0%	0%	0%	0%	46%	100%
	5	1008.44	3944.59	0%	25%	0%	10%	30%	100%
	7	421.95	6235.06	0%	60%	0%	33%	21%	100%
50	2	6756.18	558.82	80%	0%	49%	0%	65%	100%
	3	7090.85	2933.55	95%	0%	48%	0%	57%	100%
	5	6619.14	6994.48	70%	90%	21%	70%	40%	100%
	7	4435.05	TL	40%	100%	6%	89%	30%	100%

requires $3n^2p$ auxiliary variables (u- and v-variables in (54)–(63)) apart from the $np+n$ decision variables (x and y) whereas M1 (resp. M2) uses n^2 auxiliary variables (θ) and the $np + p$ (resp. n) decision variables. We do not observe a significant difference in the behavior of the three formulations neither in the two dimensions ($d = 2, 3$) nor the type of instance (BLB or RND).

All the instances of these sizes were solved up to optimality with formulations M1 and M2. Nevertheless, M3 could not optimally solve any of the instances with $n = 30$ and 50% of the instances with $n = 20$.

Comparing M1 and M2 for these instances, it seems that M2 requires less CPU time for solving the instances with small values of p ($p \in \{2, 3\}$) whereas M1 consumes less CPU time than M2 for instances with larger values of p ($p \in \{5, 7\}$).

In Table 3 we show the results obtained with formulations M1 and M2 for the larger instances. We report there, for both formulations, the consumed CPU time in seconds, the percentage of instances not optimally solved within the time limit, the percent MIPGap, and the percent deviation of the best obtained solutions with respect to the relaxed solution after exploring the root node of the branch and bound tree.

The same behavior of M1 and M2 was observed for these instances, that is, M2 outperformed M1 for small values of p whereas M1 obtained better results for larger values of p. This performance is also observed in the number of unsolved instances and the MIPGap.

The information about the rootgap provides details about the weakness of the relaxed polyhedron induced by M2 with respect to M1 which makes it more difficult to solve larger instances. In Table 4 we report the results obtained for the continuous instances, organized similarly to those of the largest discrete instances for the three formulations we propose. We observe that the first formulation, M1, seems to have a better performance than the others in both consumed CPU time and number of optimally solved instances. Concretely, M1 solved 47% of the instances whereas M2 and M3 only solved 19% and 15% of them, respectively. Moreover, the MIPGap for the unsolved instances within the time limit was smaller in the case of M1. As expected, the MIPGap and the root relaxation gaps were very large since the nonconvex terms that appear in the formulation to represent the closest distance between users and facilities were reformulated using *big M* constraints. Regarding CPU times, the *easiest* instances seem to be those with large values of p, even if they use the largest number of variables for a fixed value of n. These times were consistently paired with smaller root gap relaxations.

Comparing the discrete and the continuous instances, the latter exhibit a higher computational difficulty: Only a few instances with 30 users could be solved up to optimality, whereas in the discrete case this size extended to 50 users.

Finally, in Fig. 11 we show the computational performance profile of each of the models for the two different frameworks that we analyze here (discrete: left, continuous: right). There, we show in the x-axis the CPU time (log scale), and in the y-axis the percentage of instances that were optimally solved by each of the models that we propose. In the plots one can observe that most of the discrete instances (∼60%) for small values of p that we analyze were optimally solved by M1 or M2 in less than 500 s. In contrast, M3 only solve ∼20% of them in the same time. The situation for the continuous instances is similar, although, as already mentioned, the problem is even more challenging than the discrete version: only ∼30% of the instances were solved to optimality in less than 500 s and M1 is the method that solves the more instances within this time.

6. Conclusions

We introduce in this paper the p-intra-envy facility location problem in order to determine the optimal position of p services by minimizing the envy felt by the users allocated to the same facility. This problem allows to find local fair solutions of p-facility location taking into account the realistic assumption that users are not usually compared with all rest of users but with those that make use of the same facility. Furthermore, we provide a general framework for the problem which is valid for the two most common solution domains in facility location, discrete and continuous.

We derive different MILP formulations for the discrete and continuous versions of the problem, assuming that the distance measure for the continuous problems is the ℓ_1-norm. The results of an extensive battery of computational experiments are reported. Apart from comparing computationally the different formulations, we analyze the solutions evaluating the proposed intra-envy measure, the global envy, and the median functions.

Future research on the topic includes the study of valid inequalities for the different models that we propose. For larger instances, it would be helpful to design heuristic approaches that assure good quality solutions in smaller computing times.

CRediT authorship contribution statement

Víctor Blanco: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **Alfredo Marín:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **Justo Puerto:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Data availability

The data used in our experiments is publicly available in Github (link in the manuscript).

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Table 4
Computational results for continuous instances.

n	p	CPU time			%Unsolved			%MIPGap			%RootGap		
		M1^C	M2^C	M3^C	M1^C	M2^C	M3^C	M1^C	M2^C	M3^C	M1^C	M2^C	M3^C
10	2	20.81	68.76	108.92	0%	0%	0%	0%	0%	0%	100%	100%	100%
	3	22.72	286.76	398.21	0%	0%	0%	0%	0%	0%	50%	50%	50%
	5	5.41	9.11	17.60	0%	0%	0%	0%	0%	0%	0%	0%	0%
20	2	2017.43	TL	TL	15%	100%	100%	8%	100%	100%	100%	100%	100%
	3	6681.26	TL	TL	85%	100%	100%	77%	100%	100%	100%	100%	100%
	5	TL	TL	TL	100%	100%	100%	100%	100%	100%	100%	100%	100%
	7	5085.74	6557.49	6359.81	60%	80%	85%	60%	80%	84%	60%	80%	80%
30	2	7151.42	TL	TL	95%	100%	100%	75%	100%	100%	95%	90%	95%
	3	TL	TL	TL	100%	100%	100%	100%	100%	100%	100%	100%	95%
	5	TL	TL	TL	100%	100%	100%	100%	100%	100%	100%	100%	100%
	7	TL	TL	TL	100%	100%	100%	100%	100%	100%	100%	70%	95%

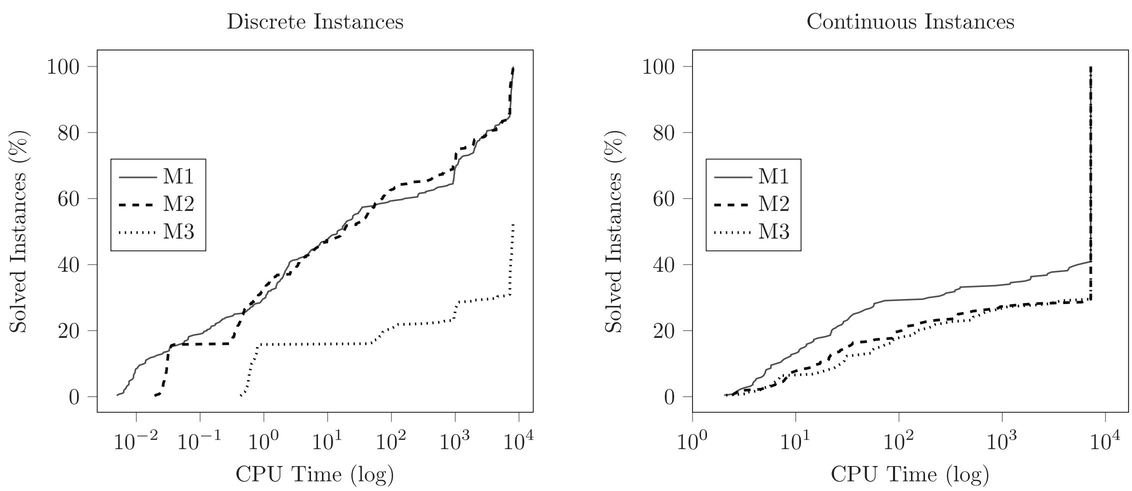

Fig. 11. Performance profiles for the intra-envy models (left: discrete, right: continuous).

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