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# Prime ends dynamics on invariant Peano continua

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Dedicated to Professor Sanjurjo on the occasion of his 70th birthday

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ABSTRACT

The dynamics of a planar homeomorphism $h$ is simple on any non-separating Peano continuum $K$ that is invariant under $h$. This means that all limit sets on $K$ are either fixed points or periodic orbits. The map $h$ induces a homeomorphism $h^*$ on the space of prime ends associated to $K$. The goal of this paper is to show that in some cases the dynamics on prime ends can have a certain complexity. We construct a dissipative homeomorphism with attractor $K$ and $h_*$ a Denjoy map.



## 1. Introduction

Assume that $K \subset \mathbb{R}^2$ is a nonseparating Peano continuum. In addition, $K$ has an empty interior. The complement of $K$ on the Riemman sphere $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ is a simply connected domain, denoted by $\Omega = \mathbb{S}^2 \backslash K$. The space of Carathéodory prime ends, $\mathbb{P} = \mathbb{P}(\Omega)$, is homeomorphic to the unit circle. Intuitively speaking, each prime end in $\mathbb{P}$ describes one way of approaching the continuum $K$ from outside.

Let us now assume that $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a planar homeomorphism such that $K$ is invariant under $h$; that is,

$$h(K) = K.$$

The dynamics of $h$ on $K$ is rather simple. There exists an integer $N \geq 1$ such that all orbits lying in $K$ are convergent either to a fixed point or to a periodic orbit with minimal period $N$. Together with the

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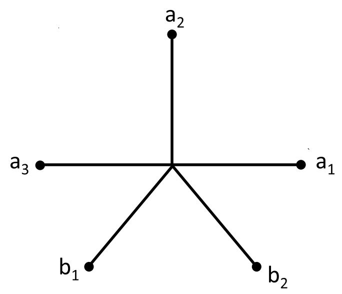


**Fig. 1.** 5-star.

dynamics on $K$, $h$ induces a homeomorphism on prime ends denoted by $h_* : \mathbb{P} \longrightarrow \mathbb{P}$. For simple continua, say finite graphs without loops, the dynamical properties of $h_*$ are similar to those of the restricted map $h : K \longrightarrow K$. However, this is not always the case for general continua. The goal of this paper will be to construct a continuum $K$ in the previous conditions and a planar and dissipative homeomorphism $h$ such that $K$ is the global attractor of $h$ and the induced map $h_*$ has a dynamics of Denjoy type. This shows that the dynamics induced on prime ends can be substantially more complex than the original dynamics on the continuum.

In some aspects our construction will be reminiscent of the Cantorian Sun introduced in several papers [5,7,11]. The continuum in those papers is constructed as a set of rays connecting the origin with all the points of a Cantor set placed at the unit circle. The dynamics on the set of rays is of Denjoy type. This continuum is not suitable for our paper since it is not locally connected. We will construct a simpler Sun, where the rays connect the origin to a countable set of points converging to the origin. The dynamics on the set of rays will be produced by an ergodic rotation but, to our surprise, the dynamics on prime ends will be of different type.

The rest of the paper is organized in two Sections. First we discuss the dynamical properties of a general homeomorphism on an invariant Peano continuum $K$. Later we construct the example mentioned above.

## 2. Dynamics on Peano continua

From now on $K \subset \mathbb{R}^2$ is a compact, connected and locally connected set. In addition, $K$ contains more than one point, has an empty interior in $\mathbb{R}^2$ and the complement $\mathbb{R}^2 \backslash K$ is connected.

The class of all planar homeomorphisms will be denoted by $\mathcal{H}(\mathbb{R}^2)$. Given $h \in \mathcal{H}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, the complete orbit $\{h^n(x)\}_{n\in\mathbb{Z}}$ is well defined. The corresponding $\omega$-limit set will be denoted by $L_\omega(x, h)$. For each integer $N \geq 1$, a periodic orbit with minimal period $N$ will be called an $N$-cycle.

**Proposition 2.1.** *Assume that $h \in \mathcal{H}(\mathbb{R}^2)$ is such that $h(K) = K$. Then, there exists an integer $N \geq 1$ such that for each $x \in K$, the limit set $L_\omega(x, h)$ is either a fixed point or an $N$-cycle*

**Remarks.** 1.- The set of periods of $K$ is either $\{1\}$ or $\{1, N\}$. The alternative $\{N\}$ is excluded by a result due to Cartwright-Littlewood and Bell ([2,1]). The homeomorphism $h$ always has at least one fixed point lying on $K$. This fixed point principle will be used several times.

2.- In the above Proposition the map is defined on the whole plane and this is essential. The conclusion does not hold for a general homeomorphism of $K$. As an example assume that $K$ is the 5-star shown in Fig. 1.

Define a homeomorphism $h : K \longrightarrow K$ such that $a_1 \mapsto a_2 \mapsto a_3 \mapsto a_1$ and $b_1 \mapsto b_2 \mapsto b_1$. Then a 2-cycle and a 3-cycle coexist and, according to Proposition 2.1, $h$ cannot be extended to a map in $\mathcal{H}(\mathbb{R}^2)$.

Homma's theorem [6] says that a homeomorphism of $K$ can be extended to a map in $\mathcal{H}(\mathbb{R}^2)$ if and only if the orientation of all $Y$-sets is simultaneously preserved or reversed. In the previous example the orientation of the $Y$-set with final points $\{a_1, a_2, a_3\}$ is preserved and the orientation of $\{a_1, b_1, b_2\}$ is reversed.

We will prove Proposition 2.1 as an application of the theory of prime ends. Following the exposition by Mather in [8] and due to the nice properties of our continuum, we observe that each prime end $\mathcal{P} \in \mathbb{P}$ can be described by an arc $\gamma$ ending at a point $\xi \in K$ and such that $\gamma \backslash \{\xi\} \subset \Omega$. The point $\xi$ is called the principal point of $\mathcal{P}$. All points in $K$ are principal points of some prime end. Two of these arcs produce the same prime end if they can be deformed homotopically in $\Omega$ (the common final point $\xi$ remains fixed through the process).

The disjoint union $\widehat{\Omega} = \Omega \cup \mathbb{P}$ becomes a surface with boundary $\mathbb{P}$. In fact the pairs $(\mathbb{D}, \mathbb{S}^1)$ and $(\widehat{\Omega}, \mathbb{P})$ are homeomorphic. Note that

$$\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\} \text{ and } \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Given $h \in \mathcal{H}(\mathbb{R}^2)$ with $h(K) = K$, the homeomorphism induced on prime ends is defined by the rule

$$h_* : \mathbb{P} \to \mathbb{P}, \quad h_*(\mathcal{P}_\gamma) = \mathcal{P}_{h(\gamma)},$$

where $\mathcal{P}_\gamma$ denotes the prime end defined by the arc $\gamma$.

The map $\widehat{h} : \widehat{\Omega} \longrightarrow \widehat{\Omega}$,

$$\widehat{h} = \begin{cases} h & on\ \Omega \\ h_* & on\ \mathbb{P} \end{cases}$$

is also a homeomorphism. In consequence, $h_*$ preserves the orientation in $\mathbb{P}$ if and only if $\widehat{h}$ has this property in $\widehat{\Omega}$. Since $\widehat{h}$ and $h$ coincide on the open set $\Omega$, the same can be said for $h$.

The map $\Pi : \mathbb{P} \to K$ sending each prime end into its principal point is continuous and onto. From the definition of $h_*$ we can deduce that the following diagram is commutative,

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{h_*} & \mathbb{P} \\ \Pi \downarrow & & \downarrow \Pi \\ K & \xrightarrow[h]{} & K \end{array}$$

In consequence, $\Pi$ defines a semi-conjugacy. This implies that

$$L_\omega(x, h) = \Pi(L_\omega(\mathcal{P}, h_*)) \tag{2.1}$$

for each $x \in K$ and $\mathcal{P} \in \mathbb{P}$ with $\Pi(\mathcal{P}) = x$. Moreover, we observe that when $\mathcal{P} \in \mathbb{P}$ is a periodic point of $h_*$ with period $N \geq 1$, then $x = \Pi(\mathcal{P})$ is a periodic point of $h$ with minimal period $N' \geq 1$, with $N'$ some divisor of $N$.

After these preliminaries we are ready for the proof.

**Proof of Proposition 2.1.** First of all we recall that if $L_\omega(x, h)$ is contained in $Fix(h)$, the set of fixed points of $h$, then $L_\omega(x, h)$ is either a singleton or a continuum (see [3] or Proposition 2 in Chapter 3 of [10]). In particular, if $L_\omega(x, h)$ is finite and $L_\omega(x, h) \subset Fix(h)$ then it is a singleton. From the identity

$$L_\omega(x,h) = \bigcup_{k=0}^{N-1} L_\omega(h^k(x), h^N)$$

we can also deduce that if $L_\omega(x,h)$ is finite and $L_\omega(x,h) \subset Fix(h^N)$ then $L_\omega(x,h)$ is a $N'$-cycle with $N'$ some divisor of $N$.

Assume now that $h \in \mathcal{H}(\mathbb{R}^2)$ is orientation reversing. Then, also $h_*$ is orientation reversing and the general theory of dynamics on $\mathbb{S}^1$ implies that for each $\mathcal{P} \in \mathbb{P}$, the limit set $L_\omega(\mathcal{P}, h_*)$ is either a 2-cycle or a fixed point. In view of (2.1) we find three possible configurations for $L_\omega(x,h)$, a 2-cycle, one fixed point or two fixed points. The third possibility is excluded by the above remark. Therefore, the conclusion of Proposition 2.1 holds with $N = 2$ or $N = 1$.

From now on we assume that $h$ is orientation preserving. In this case $h_*$ has a well defined rotation number, interpreted as an angle $\rho = \rho(h_*) \in \mathbb{T}$, with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We distinguish three cases:

**Case 1:** $\rho = \overline{0} = 2\pi\mathbb{Z}$.

Every limit set $L_\omega(\mathcal{P}, h_*)$ is a single fixed point and from (2.1), we deduce that the same can be said for $L_\omega(x,h)$.

**Case 2:** $\rho = 2\pi\frac{M}{N} + 2\pi\mathbb{Z}$ with $0 < M < N$ and $N, M$ integers relatively primes.

In this case, $L_\omega(\mathcal{P}, h_*)$ is an $N$-cycle for all $\mathcal{P} \in \mathbb{P}$. From (2.1) we deduce that for each $x \in K$, $L_\omega(x,h)$ is finite and contained in $Fix(h^N)$. In consequence $L_\omega(x,h)$ is an $N'$-cycle with $N'|N$. We will prove that either $N' = N$ or $N = 1$. Assume by contradiction that $1 < N' < N$. Thus, $h^{N'}$ has more than one fixed point lying in $K$. Moreover, $h^{N'} \in \mathcal{H}(\mathbb{R}^2)$, $h^{N'}(K) = K$ and $\rho(h^{N'}) = 2\pi\frac{MN'}{N} + 2\pi\mathbb{Z} \neq \overline{0}$.

This situation is not compatible with the following consequence of a result due to Cartwright and Littlewood (Corollary 2 in [2]): Assume that $f \in \mathcal{H}(\mathbb{R}^2)$ is orientation preserving with $\rho(f) \neq \overline{0}$ and $f(K) = K$, then $f$ has exactly one fixed point in $K$. The application of this result to $f = h^{N'}$ leads to the searched contradiction.

**Case 3:** $\rho = \omega + 2\pi\mathbb{Z}$, $\frac{\omega}{2\pi} \notin \mathbb{Q}$.

The general theory says that $h_*$ can be conjugate to an ergodic rotation or to a Denjoy map. The first step will be to discard the rotation.

By a contradiction argument assume that all orbits of $h_*$ are dense in $\mathbb{P}$. Then we select a point $y \in K$ with $h(y) = y$ and a prime end $\mathcal{Q} \in \mathbb{P}$ with $\Pi(\mathcal{Q}) = y$. Since $\Pi(h_*^n(\mathcal{Q})) = h^n(\Pi(\mathcal{Q})) = y$ for each $n \in \mathbb{Z}$, we deduce that $\Pi$ is constant on the dense set $\{h_*^n(\mathcal{Q}) : n \in \mathbb{Z}\}$. By continuity, we deduce that $\Pi(\mathcal{P}) = y$ for every $\mathcal{P} \in \mathbb{P}$. This is absurd since $K$ is not a singleton.

Once we know that $h_*$ is a Denjoy map, we find the Cantor set $C \subset \mathbb{P}$ such that

$$L_\omega(\mathcal{P}, h_*) = C \tag{2.2}$$

for every $\mathcal{P} \in \mathbb{P}$. Again take a fixed point $y \in K$ of $h$ and $\mathcal{Q} \in \mathbb{P}$ so that $\Pi(\mathcal{Q}) = y$. A similar reasoning implies that $\Pi(\widehat{\mathcal{Q}}) = y$ for every $\widehat{\mathcal{Q}} \in L_\omega(\mathcal{Q}, h_*)$. Therefore, $\Pi(C) = \{y\}$.

For an arbitrary $x \in K$ and $\mathcal{P} \in \mathbb{P}$ with $\Pi(\mathcal{P}) = x$, we combine (2.1) and (2.2) to obtain

$$L_\omega(x,h) = \Pi(L_\omega(\mathcal{P}, h_*)) = \Pi(C) = \{y\}.$$

Summing up, we can say that $h$ has a unique fixed point in $K$ attracting all orbits on this continuum. $\square$

In the next Section we construct an example corresponding to the third case of the previous proof.

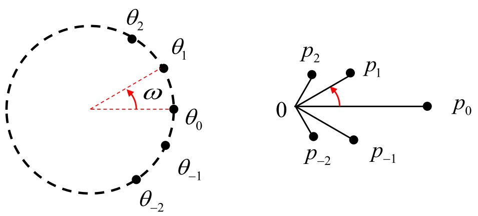


**Fig. 2.** Dynamics on $K$.

## 3. A curious attractor

A map $h \in \mathcal{H}(\mathbb{R}^2)$ is called dissipative if there exists a bounded set $B$ such that all compact sets are eventually attracted by $B$; that is, for each compact set $A \subset \mathbb{R}^2$ there exists an integer $n_*$ such that $h^n(A) \subset B$ if $n \geq n_*$. The attractor $\mathcal{A}_h$ is the largest compact and invariant set. It is well known that the attractor is a continuum and the results of the previous Section are applicable when $\mathcal{A}_h$ is locally connected and it has empty interior in $\mathbb{R}^2$.

The following result is a consequence of the previous definitions and it will be useful later.

**Lemma 3.1.** *Assume that $h \in \mathcal{H}(\mathbb{R}^2)$ is dissipative and $K \subset \mathbb{R}^2$ is a continuum with $h(K) = K$. In addition,*

$$h^n(x) \to K, \quad h^{-n}(x) \to \infty \quad as \quad n \to +\infty$$

*for each $x \in \mathbb{R}^2 \backslash K$. Then $\mathcal{A}_h = K$.*

The aim of this Section is to construct a dissipative homeomorphism $h$ whose attractor is a Peano continuum $K$ in the conditions of the paper such that $h_* : \mathbb{P} \to \mathbb{P}$ is a Denjoy map.

Assume that $\omega \in \mathbb{R}$ is not commensurable with $2\pi$; that is, $\frac{\omega}{2\pi} \notin \mathbb{Q}$, and let us fix an orbit of the rotation of angle $\omega$,

$$\mathcal{O}_\omega = \{\overline{\theta_n} : \theta_n = \theta_0 + n\omega, n \in \mathbb{Z}\}.$$

We consider the continuum

$$K = \bigcup_{n \in \mathbb{Z}} \sigma_n$$

where each $\sigma_n$ is the segment joining the origin to the point

$$p_n = \frac{1}{1 + |n|} e^{i\theta_n}.$$

See Fig. 2 for a pictorial description of $K$.

We observe that $K$ is locally connected because the lengths of the rays tend to 0. The space of prime ends $\mathbb{P}$ associated to the domain $\mathbb{S}^2 \backslash K$ has a curious structure, that can be better understood if we split the continuum in three parts. More precisely,

$$K = P_1 \cup P_2 \cup P_\infty$$

with $P_1 = \{p_n : n \in \mathbb{Z}\}$, $P_2 = \cup_{n \in \mathbb{Z}} \dot{\sigma}_n$, $P_\infty = \{0\}$. Here $\dot{\sigma}_n$ denotes the open segment $\sigma_n \backslash \{0, p_n\}$. For points in $P_1$, there is a unique prime end denoted by $\widehat{p_n} \in \mathbb{P}$ with $\Pi(\widehat{p_n}) = p_n$. To each point $x \in \dot{\sigma}_n$ there

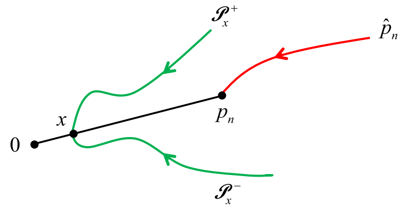


**Fig. 3.** Prime ends with impression outside the origin.

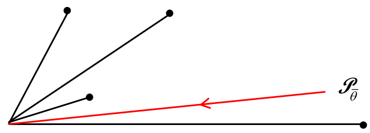


**Fig. 4.** A prime end with impression at the origin.

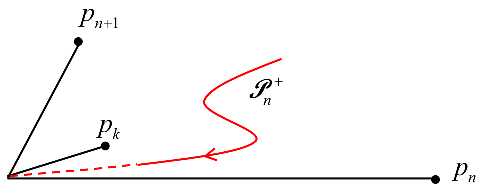


**Fig. 5.** Another prime end with impression at the origin.

correspond two prime ends $\mathcal{P}_x^{\pm} \in \mathbb{P}$ with $\Pi(\mathcal{P}_x^{\pm}) = x$. The paths defining these prime ends are described in Fig. 3.

All the remaining prime ends satisfy $\Pi(\mathcal{P}) = 0$. They can be of two kinds. For each $\overline{\theta} \in \mathbb{T}\backslash\mathcal{O}_\omega$, $\mathcal{P}_{\overline{\theta}}$ will be the prime end defined by the ray ending at the origin and having argument $\theta$, see Fig. 4.

Given $\overline{\theta}_n \in \mathcal{O}_\omega$ we construct two prime ends denoted by $\mathcal{P}_n^{\pm}$. They are defined by paths in $\mathbb{S}^2\backslash K$ ending at the origin and having the property (for $\mathcal{P}_n^+$): for any $k \in \mathbb{Z}$ such that $\overline{\theta_k}$ lies in the arc connecting $\overline{\theta}_n$ and $\overline{\theta}_{n+1}$, the path will eventually enter into the cone of points in the plane with arguments between $\overline{\theta}_n$ and $\overline{\theta}_k$, see Fig. 5.

Arcs are taken in the counter-clockwise sense. For $\mathcal{P}_n^-$, the arc $\overline{\theta_n\theta_{n+1}}$ is replaced by $\overline{\theta_{n-1}\theta_n}$.

It is convenient to be more precise on the construction of the arc defining $\mathcal{P}_n^+$. For each $m = 0, 1, 2, \ldots$ we consider $D_m$ the closed disk with center at the origin and radius $r_m = \frac{1}{m+1}$. The set $\partial D_m \backslash \bigcup_{|k|\leq m} \sigma_k$ has a finite number of connected components. For each $m \geq |n|$, exactly two of them will be open arcs in $\partial D_m$ having the point $r_m e^{i\theta_n}$ as one of the extreme points of the arc. These arcs will be denoted by $\Lambda_{m,n}^+$ and $\Lambda_{m,n}^-$, depending on whether $r_m e^{i\theta_n}$ is the initial or the final extreme. For a fixed $n \in \mathbb{Z}$ we pick up a sequence of points $\{z_{m,n}\}_{m\geq|n|}$ with $z_{m,n} \in \Lambda_{m,n}^+$. The arc defining $\mathcal{P}_n^+$ is constructed by a juxtaposition of arcs connecting these points. More precisely, $\gamma_{m,n}$ is an arc connecting $z_{m,n}$ and $z_{m+1,n}$ with the additional property

$$\dot{\gamma}_{m,n} = \gamma_{m,n}\backslash\{z_{m,n}, z_{m+1,n}\} \subset (int(D_m)\backslash D_{m+1}) \cap (\mathbb{R}^2\backslash K).$$

See Fig. 6.

The arc defining $\mathcal{P}_n^+$ is $\gamma = \bigcup_{m\geq|n|} \gamma_{m,n}$.

Experts in the theory of prime ends will notice that the choice of the points $z_{m,n}$ is irrelevant for the definition of the prime end. Indeed, $\mathcal{P}_n^+$ is defined by the sequence of cross-cuts $\{\Lambda_{m,n}^+\}_{m\geq|n|}$.

To achieve a complete description of the set $\mathbb{P}$ we must also prove that there are no additional prime ends. This is clear when the principal point is not the origin. Let us take an arbitrary $\mathcal{P} \in \mathbb{P}$ with $\Pi(\mathcal{P}) = 0$. We can find an associated parameterized arc $\gamma : [0, 1] \to \mathbb{R}^2$ with $\gamma(t) \in \mathbb{R}^2\backslash K$ if $t \in [0, 1[$ and $\gamma(1) = 0$. For $m$

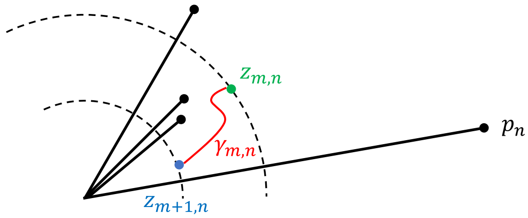


**Fig. 6.** Construction of $\mathcal{P}_n^+$.

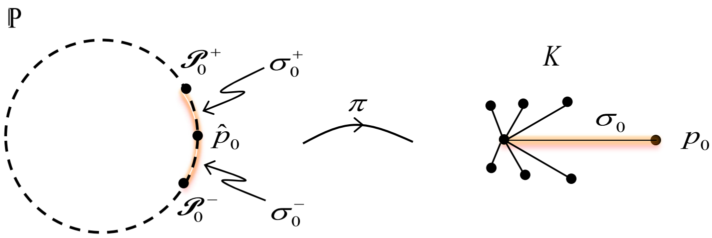


**Fig. 7.** The map Π.

large enough this arc will intersect the set $\partial D_m \backslash \bigcup_{|k|\leq m} \sigma_k$. In principle $\gamma(t)$ can touch several components but we are interested in the last one, denoted by $\Lambda_m(\mathcal{P})$. This means that $\Lambda_m(\mathcal{P})$ is the component of $\partial D_m \backslash \bigcup_{|k|\leq m} \sigma_k$ such that there exists $\tau \in ]0,1[$ satisfying $\gamma(\tau) \in \Lambda_m(\mathcal{P})$ and $|\gamma(t)| < r_m$ if $t \in ]\tau, 1]$.

The sequence of cross-cuts $\{\Lambda_m(\mathcal{P})\}$ provides an alternative way to define $\mathcal{P}$. Each arc $\Lambda_m(\mathcal{P})$ can be projected into the space of angles,

$$\Theta_m(\mathcal{P}) = \{\overline{\theta} \in \mathbb{T} : r_m e^{i\theta} \in \Lambda_m(\mathcal{P})\}.$$

The closure of each $\Theta_m(\mathcal{P})$ is a compact subset of $\mathbb{T}$. Moreover, $cl(\Theta_{m+1}(\mathcal{P})) \subset cl(\Theta_m(\mathcal{P}))$ and the diameter of these sets goes to zero as $m \to \infty$. In consequence this sequence defines a point $\overline{\varphi} \in \mathbb{T}$ as the limit,

$$\bigcap_m cl(\Theta_m(\mathcal{P})) = \{\overline{\varphi}\}.$$

Assume first that $\overline{\varphi} \notin \mathcal{O}_\omega$. Then $r_m e^{i\varphi} \in \Lambda_m(\mathcal{P})$ and the arc is equivalent to the ray of argument $\varphi$. This implies that $\mathcal{P} = \mathcal{P}_{\overline{\varphi}}$.

Otherwise $\overline{\varphi} = \overline{\theta}_n$ for some $n$. Then, for large $m$ the angle $\overline{\varphi}$ is one of the boundary points of $\Theta_m(\mathcal{P})$ for all $m$. As the sequence $\{\Theta_m(\mathcal{P})\}$ is decreasing, this boundary point cannot alternate. Either it is always the first ($\mathcal{P} = \mathcal{P}_n^+$) or the last ($\mathcal{P} = \mathcal{P}_n^-$).

From the previous discussion it is not hard to prove also that the prime ends $\{\mathcal{P}_{\overline{\theta}}\}_{\overline{\theta}\in\mathbb{T}\backslash\mathcal{O}_\omega}$ and $\{\mathcal{P}_n^\pm\}_{n\in\mathbb{Z}}$ are different.

Now we have a complete description of the set $\mathbb{P}$. To understand its topology we first observe that $\Pi^{-1}$ unfolds the segment $\sigma_n$ in two adjacent arcs $\sigma_n^+$ and $\sigma_n^-$. They are arcs connecting $\widehat{p_n}$ to $\mathcal{P}_n^+$ and $\mathcal{P}_n^-$, see Fig. 7.

The arcs $\sigma_n^\pm$ are closed and the corresponding open arcs will be denoted by $\dot{\sigma}_n^\pm$. The set

$$V = \bigcup_{n\in\mathbb{Z}} (\dot{\sigma}_n^+ \cup \dot{\sigma}_n^- \cup \{\widehat{p}_n\})$$

is a disjoint union of open arcs. Let us now prove that $V$ is dense in $\mathbb{P}$. First note that $\mathbb{P}\backslash V = \Pi^{-1}(0)$. Given $\mathcal{P} \in \mathbb{P}\backslash V$ we consider the sequence $\{\Theta_m(\mathcal{P})\}$ defined above. Let $\overline{\theta}_m = \overline{\theta_m}(\mathcal{P})$ be the final point of

$\Theta_m(\mathcal{P})$ and let $\sigma(m) \in \mathbb{Z}$ be the integer such that the point $r_m e^{i\theta_m}$ lies in the segment $[0, p_{\sigma(m)}]$. Then $\widehat{p}_{\sigma(m)}$ is in the closed neighborhood of $\mathcal{P}$ defined by the crosscut $\Lambda_m(\mathcal{P})$. In consequence $\widehat{p}_{\sigma(m)} \to \mathcal{P}$. Since $\widehat{p}_{\sigma(m)}$ belongs to $V$, we conclude that $\mathcal{P}$ is in the closure of $V$.

The complement $C = \mathbb{P} \backslash V$ is a Cantor set. Moreover, $C = A \cup I$, where

$$A = \{\mathcal{P}_n^{\pm} : n \in \mathbb{Z}\}$$

is the accessible set and

$$I = \{\mathcal{P}_{\overline{\theta}} : \overline{\theta} \in \mathbb{T} \backslash \mathcal{O}_\omega\}$$

is the unaccessible set.

The previous discussions can be summarized as follows: The map $\Pi$ defines the homeomorphisms between $\sigma_n^{\pm}$ and $\sigma_n$ while the Cantor set $C$ is collapsed to the origin.

Before we start the construction of the map $h$ it will be convenient to recall a preliminary result on the extension of maps on sets homeomorphic to $\mathbb{S}^1$. Assume that an orientation has been fixed on $X \cong \mathbb{S}^1$. The cyclic order on $X$ is defined as follows, given $x_1, x_2, x_3 \in X$, $x_1 \prec x_2 \prec x_3$ means that $x_2$ belongs to the open arc going from $x_1$ to $x_3$.

**Lemma 3.2.** *i) Assume that $D_1$ is a dense subset of $X$ and $\varphi : D_1 \to X$ preserves the cyclic order. Then $\varphi$ admits an extension $\widehat{\varphi} : X \longrightarrow X$ that also preserves the cyclic order.*
*ii) In addition, assume that $D_2 = \varphi(D_1)$ is also dense in $X$. Then the extension $\widehat{\varphi}$ is unique and it becomes a homeomorphism of $X$.*

This result is inspired by Lemma 3.3, page 140, in [4]. The proof is based on the same ideas and will be presented at the end of the paper.

We will apply this lemma with $X = \mathbb{P}$. The orientation will be chosen by the rule

$$\mathcal{P}_x^- \prec \widehat{p}_n \prec \mathcal{P}_x^+ \quad \text{if } x \in \dot{\sigma}_n.$$

To start the construction we take a sequence of homeomorphisms $H_n : cl(\sigma_n) \cong cl(\sigma_{n+1})$ with $H_n(0) = 0$. Then we define the map $H : V \cup A \to V \cup A$,

$$H(\mathcal{P}_x^{\pm}) = \mathcal{P}_{H_n(x)}^{\pm} \quad if\ x \in \dot{\sigma}_n, \quad H(\widehat{p}_n) = \widehat{p}_{n+1}, \qquad H(\mathcal{P}_n^{\pm}) = \mathcal{P}_{n+1}^{\pm}.$$

This map is bijective and continuous. Moreover,

$$\Pi(\mathcal{P}_1) = \Pi(\mathcal{P}_2) \Longleftrightarrow \Pi(H(\mathcal{P}_1)) = \Pi(H(\mathcal{P}_2)) \tag{3.1}$$

for any $\mathcal{P}_1, \mathcal{P}_2 \in V \cup A$.

Since $H_n$ is an order preserving map from $\sigma_n$ to $\sigma_{n+1}$, the cyclic order is preserved by $H$ on each arc $\dot{\sigma}_n^+ \cup \dot{\sigma}_n^- \cup \{\widehat{p}_n\}$. Indeed, $H$ preserves the cyclic order on the whole domain $V \cup A$. The reason for this is in the way we have chosen the sequence $\mathcal{O}_\omega$. Since the rays $\sigma_n$ have been labelled according to a rotation, the cyclic order is preserved in the space of rays. As a consequence, an ordering of the type

$$\sigma_n^+ \cup \sigma_n^- \prec \sigma_m^+ \cup \sigma_m^- \prec \sigma_{n+1}^+ \cup \sigma_{n+1}^-$$

implies

$$\sigma_{n+1}^{+} \cup \sigma_{n+1}^{-} \prec \sigma_{m+1}^{+} \cup \sigma_{m+1}^{-} \prec \sigma_{n+2}^{+} \cup \sigma_{n+2}^{-}.$$

We know that $V \cup A$ is dense in $\mathbb{P}$ so that Lemma 3.2 can be applied to deduce that $H$ can be extended to a homeomorphism $\widehat{H} : \mathbb{P} \longrightarrow \mathbb{P}$. Since $\Pi^{-1}(0) = C$ and $C$ is invariant under $\widehat{H}$, property (3.1) is also valid for $\widehat{H}$ and arbitrary prime ends $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{P}$.

Define the map $\Upsilon : \mathbb{P} \to \mathbb{T}$,

$$\Upsilon(\mathcal{P}) = \begin{cases} \overline{\theta}_n & if\ \mathcal{P} \in \sigma_n^{+} \cup \sigma_n^{-} \\ \overline{\theta} & if\ \mathcal{P} = \mathcal{P}_{\overline{\theta}}, \overline{\theta} \in \mathbb{T} \backslash \mathcal{O}_{\omega}. \end{cases}$$

This map is continuous and onto. It defines the semiconjugacy

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\widehat{H}} & \mathbb{P} \\ \Upsilon \downarrow & & \downarrow \Upsilon \\ \mathbb{T} & \xrightarrow[R_{\varpi}]{} & \mathbb{T} \end{array}$$

with $R_{\varpi}(\overline{\theta}) = \overline{\theta + \omega}$. In particular, the rotation number of $\widehat{H}$ is $\varpi$.

Our next step will be to extend $\widehat{H}$ to a homeomorphism of $\widehat{\Omega}$ such that all compact subsets of $\widehat{\Omega} \backslash \{\infty\}$ are attracted by $\mathbb{P}$. This can be made in many way and we describe one of them.

Since $\widehat{\Omega}$ is homeomorphic to

$$\mathbb{E} = \{z \in \mathbb{C} : |z| \geq 1\} \cup \{\infty\},$$

we take a homeomorphism $\Psi : \widehat{\Omega} \longrightarrow \mathbb{E}$. In particular, $\Psi(\mathbb{P}) = \mathbb{S}^1$. After this change of variables we make a radial and contractive extension. In polar coordinates $z = re^{i\varphi}$,

$$\widehat{\mathcal{H}} : \mathbb{E} \longrightarrow \mathbb{E}, \quad e^{i\varphi_1} = \Psi \widehat{H} \Psi^{-1}(e^{i\varphi}), \quad r_1 = \frac{1}{2} + \frac{1}{2} r.$$

For any orbit in $\mathbb{E}$, $r_{n+1} = \frac{1}{2} + \frac{1}{2} r_n$. This implies that $r_n \longrightarrow 1$ as $n \longrightarrow \infty$ and all orbits are attracted by $\mathbb{S}^1$. Also, if $r_0 > 1$, then $r_n \longrightarrow +\infty$ as $n \longrightarrow -\infty$. The map $\mathcal{H} = \Psi^{-1} \circ \widehat{\mathcal{H}} \circ \Psi$ has the searched dynamics.

Consider now the map

$$\mathcal{R} : \widehat{\Omega} \backslash \{\infty\} \longrightarrow \mathbb{R}^2, \ \ \mathcal{R} = id \quad \text{on } \Omega, \ \ \mathcal{R} = \Pi \quad \text{on } \ \mathbb{P}.$$

Since $K$ is locally connected this map is continuous and onto. We define $h$ in terms of the following commutative diagram

$$\begin{array}{ccc} (\mathbb{R}^2, K) & \xleftarrow[\mathcal{R}]{} & (\widehat{\Omega} \backslash \{\infty\}, \mathbb{P}) \\ h \downarrow & & \downarrow \mathcal{H} \\ (\mathbb{R}^2, K) & \xleftarrow[\mathcal{R}]{} & (\widehat{\Omega} \backslash \{\infty\}, \mathbb{P}) \end{array}$$

See Fig. 8 for an illustration of the behavior of $\mathcal{R}$.

The extension of property (3.1) to all prime ends implies that $h$ is well defined, continuous and one-to-one. The most delicate point is the continuity of $h$ at the points of the continuum. Assume that $\{x_n\}$ is a sequence in $\mathbb{S}^2 \backslash K$ converging to some $\xi \in K$. Since $h$ maps bounded sets into bounded set it is enough to

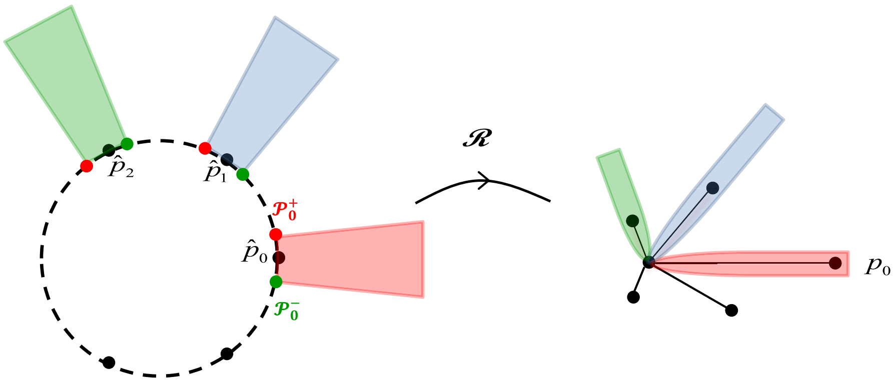


**Fig. 8.** The map $\mathcal{R}$.

prove that if the sequence $\{h(x_n)\}$ is also convergent then the limit must be $h(\xi)$. First we observe that the sequence $\{y_n\}$ with $y_n = \mathcal{R}^{-1}(x_n)$ is not necessarily convergent in $\widehat{\Omega}$. Anyway, the limit set of $\{y_n\}$ will be contained in $\mathbb{P}$ and, more precisely, in $\Pi^{-1}(\xi)$. Let us extract a sub-sequence $\{y_{\sigma(n)}\}$ converging to some $\mathcal{P} \in \mathbb{P}$ with $\Pi(\mathcal{P}) = \xi$. Then, $h(x_{\sigma(n)}) = h(\mathcal{R}(y_{\sigma(n)})) = \mathcal{R}(\mathcal{H}(y_{\sigma(n)})) \to \mathcal{R}(\mathcal{H}(\mathcal{P}))$. Since $\mathcal{H} = \widehat{H}$ and $\mathcal{R} = \Pi$ on $\mathbb{P}$, the limit of $\{h(x_{\sigma(n)})\}$ is $\Pi(\widehat{H}(\mathcal{P})) = h(\Pi(\mathcal{P})) = h(\xi)$. Therefore, the whole sequence $\{h(x_n)\}$ will converge to $h(\xi)$. At this moment we know that $h$ is a continuous embedding, but since it is obviously onto, the theorem of invariance of the domain implies that $h \in \mathcal{H}(\mathbb{R}^2)$. The restrictions of $h$ to $\mathbb{R}^2 \backslash K$ and $\widehat{\mathcal{H}}$ to $\{|z| > 1\}$ are conjugate. This implies that $h$ is orientation preserving. Also, $h$ is dissipative because the set

$$B = \mathcal{R}(\Psi^{-1}(\{z \in \mathbb{E} : 1 \leq |z| \leq 2\}))$$

attracts all compact subsets of $\mathbb{R}^2$. Given $x \in \mathbb{R}^2 \backslash K$ we know that if $n \to +\infty$,

$$dist(\mathcal{H}^n(x), \mathbb{P}) \to 0, \;\; \mathcal{H}^{-n}(x) \to \infty.$$

Since $\mathcal{R}$ is continuous, also

$$dist(h^n(x), K) \to 0, \;\; h^{-n}(x) \to \infty.$$

We can now invoke Lemma 3.1 to deduce that the global attractor is $K$.

It remains to prove that the induced map on prime ends coincides with the Denjoy map $\widehat{H}$. From the definition of $H$, we know that $h_* = H$ on $V$. The density of $V$ in $\mathbb{P}$ implies that $h_* = \widehat{H}$ everywhere.

To finish our discussion on the dynamics of the map $h$ we observe that the fixed point $x = 0$ attracts all orbits but it is unstable in the Lyapunov sense. The theory of unstable attractors is one of the topics of interest in the research of Professor Sanjurjo (see [9]).

**Appendix: Proof of Lemma 3.2.** First of all we observe that any map preserving the cyclic order is one-to-one. Given $x \in X$ we take two points $\xi, \eta \in D_1$ with $\xi \prec x \prec \eta$ and define the sets

$$\mathcal{L} = \{y \in D_1 : \xi \prec y \prec x\} \text{ and } \mathcal{R} = \{y \in D_1 : x \prec y \prec \eta\}.$$

The point $x$ is in the accumulation of both sets because $D_1$ is dense in $X$. We claim that the limits below exist,

$$\varphi(x^-) = \lim_{\substack{y \to x \\ y \in \mathcal{L}}} \varphi(x) \text{ and } \varphi(x^+) = \lim_{\substack{y \to x \\ y \in \mathcal{R}}} \varphi(x).$$

To prove the existence of $\varphi(x^-)$ we take any sequence $\{y_n\}$ satisfying $y_n \in D_1$, $y_n \to x$ and

$$\xi \prec y_0 \prec y_1 \prec \ldots \prec y_n \prec y_{n+1} \ldots \prec x.$$

Then,

$$\varphi(\xi) \prec \varphi(y_0) \prec \varphi(y_1) \prec \ldots \prec \varphi(y_n) \prec \varphi(y_{n+1}) \ldots \prec \varphi(\eta).$$

In consequence $\varphi(y_n)$ has a limit lying in the arc between $\varphi(\xi)$ and $\varphi(\eta)$. It is not hard to prove that this limit is independent of the chosen sequence $\{y_n\}$.

Once we have established the existence of $\varphi(x^-)$ and $\varphi(x^+)$, we observe that the role of the points $\xi$ and $\eta$ in the previous discussion can be played by arbitrary points in $D_1$. In consequence we obtain the property

$$\xi, \eta \in D_1,\ x \in X,\ \xi \prec x \prec \eta \Rightarrow \varphi(\xi) \prec \varphi(x^{\pm}) \prec \varphi(\eta).$$

Also, either $\varphi(x^-) = \varphi(x^+)$ or

$$\varphi(\xi) \prec \varphi(x^-) \prec \varphi(x^+) \prec \varphi(\eta). \tag{3.2}$$

For points $x$ lying in $D_1$ one of the following three cases must hold, $\varphi(x) = \varphi(x^-)$, $\varphi(x) = \varphi(x^+)$ or $\varphi(x^-) \prec \varphi(x) \prec \varphi(x^+)$. To prove **i)** it is enough to define

$$\widehat{\varphi}(x) = \begin{cases} \varphi(x) & if\ x \in D_1 \\ \varphi(x^-) & if\ x \in X \backslash D_1. \end{cases}$$

To prove **ii)** we observe that the continuity of $\widehat{\varphi}$ at $x \in X$ is equivalent to $\varphi(x^+) = \varphi(x^-)$. We claim that this is the case for all points $x \in X$. Otherwise, $\varphi(x^-) \neq \varphi(x^+)$ for some $x$ and condition (3.2) will hold. The arc $\gamma$ going from $\varphi(x^-)$ to $\varphi(x^+)$ is such that $\varphi(D_1) \cap (\gamma \backslash \{\varphi(x)\}) = \emptyset$. This is impossible if $D_2 = \varphi(D_1)$ is dense in $X$.

This indirect argument proves that $\widehat{\varphi}$ is continuous. Since it is also one-to-one and $X$ is homeomorphic to $\mathbb{S}^1$, we conclude that $\widehat{\varphi}$ is indeed a homeomorphism.

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