Towards Adaptive Maps

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Abstract

A "standard" map provides a simplified representation of the real world but is not able to adapt to the needs of different users having different preferences. On the contrary, Adaptive Maps are aimed at representing the world as seen through the eyes of the user depicting the suitability/difficulty of the paths between points of interest according to the user's preferences. Adaptive maps consider, simultaneously, multiple attributes like travel time, distance, slopes, etc.

In this paper a model for Adaptive Maps is presented. A generation and a visualization methods are proposed. To illustrate the results, five different Adaptive Maps are generated and visualized considering different attributes and preferences.

1 Introduction

A map is a depiction that emphasizes relationships among elements of some space. The maps are categorized according to two different criteria as:

1. General cartography or thematic cartography. General cartography maps provide a large amount of general information for widespread public use (e.g. general purpose maps) while thematic cartography maps focus on specific geographic themes and are thought for an specific audience (e.g. population density maps).

2. Topographic or topological. Topographic maps represent the shape and features of the surface of the Earth (e.g. relief maps) while topological maps are a diagram that shows simplified information avoiding unnecessary details (e.g. transit maps). In topographic maps the relation between depicted elements is based on the euclidean distance while in topological maps this measure is not the euclidean distance.

Our interest lies on general cartography and topological maps. In contrast to general purpose maps that help to understand features of the depicted area (e.g. the location of sites of interest), the idea is to adapt the visualization of a map to the features that affect the final user.

The concept of Adaptive Maps was firstly explored at [16]. The concept is not defined but illustrated with some theoretical examples. To understand the context in which the Adaptive Maps appears, imagine a pedestrian that wants to select a route on a map between two points but has difficulties on walking steeped slopes. Then, a relief map and a street map are both required to select which route is most affordable. However, a topological map depicting distance and elevation features at the same time could help the pedestrian to directly choose the route based on the distances shown in such supposed topological map. Now, on the same map, if a driver is interested on routes without traffic the features have changed and the required topological map will not be the same: it has to be adapted to the new user's preferences. For that purpose, it is required to construct a topological map for each one of the users due to their different preferences.

Multiple topological maps are constructed to illustrate certain information like in public transit maps [3, 11]. Those maps have the simplification of the network as main objective with the purpose of helping to visualize specific features. Another example of topological map are the area cartograms [13]. In those representations geographic variables are visualized as spatial objects whose size is proportional to certain variable strength. In contrast to area cartograms, the information in the Adaptive Maps is related to routes (instead of areas) and to the user's preferences.

To the best of our knowledge, the definition of the Adaptive Map, its generation and visualization are proposed here for the first time. A similar concept that was no further developed was that of "subjective space" [12]. The subjective space was defined as a geographical space such as perceived from an specific subject, like a mental map. In order to generate it, the subject has to previously know the space. In contradistinction, the Adaptive Maps' idea is to show how the space is according to the subject's perception without previously knowing the space itself.

The objective of this paper is to define the concept of "Adaptive Map" and propose a methodology to achieve its construction and visualization. The purpose of those maps is to help different users to see a "personalized" map reflecting their needs.

The paper is organized as follows. Section 2 introduces the model and parameters of the Adaptive Maps. The construction method is explained in Section 3. It consists on two steps. The first step is the generation of the Measurement Matrix as explained in Section 3.1 and the second step is the visualization process, detailed in Section 3.2. The application of the proposed method is shown in Section 4 where details on the generation and visualization of Adaptive Maps with one and multiple attributes are given. Finally, the conclusions are detailed in Section 5.

2 Adaptive Maps: Model

In this section both a definition and a model of Adaptive Maps are provided.

An Adaptive Map is a modification of a general purpose map in terms of the user's preferences. An Adaptive Map departs from a general map, a set of *points of interest* (POI), (locations on a general map, like restaurants, museums, parks, etc. that are of interest for the user) and information of multiple attributes values, e.g. distance among points, slopes among points, quality of the road among points, traffic, etc. The relevance of each attribute is associated with the user preferences. Then, an Adaptive Map represents the difficulty (for a specific user) to travel from each POI to any other showing the world as seen through the eyes of the user. A model formulation for the Adaptive Map is presented below.

The model departs from a directed multilayer graph (a graph with multiple links between nodes) [6] $G = \{N, A, D\}$ where $N = \{1, ..., n\}$ is a finite set of nodes and D is a set of dimensions (layers) $d_1, d_2, \dots, d_{|D|}$, where every d_k with $k = \{1, 2, \dots, |D|\}$ stands for a kind of measure that can be represented by numerical values (e.g., elevation, latitude, longitude or distance), or nominal characteristics (like the edge's street type, e.g., path, ban or pedestrian). The use of layers lead to each pair of nodes having multiple arcs: one for each measure from the considered layers $d_1, d_2, \dots, d_{|D|}$. That way, the arcs in A, the links between nodes, consist on triplets (n_i, n_j, d_k) where $n_i, n_j \in N$ and $d_k \in D$. As it is assumed that the graph is complete, the number of arcs that connect two nodes is equal to the number of layers $|D|$. In other words, between each pair of nodes there are exactly $|D|$ number of arcs. The set of POIs is denoted as $P \subseteq N$.

The values of the elements from A will be noted as $a_{ij}^k \in [0, 1]$ (without loss of generality, it is assumed that they are normalized), $i \neq j$ with $i, j \in N$ and $k = \{1, 2, \dots, |D|\}$. For example, suppose that $|D| = 2$, being $d_1 = \textit{distance}$, $d_2 = \textit{slope}$ and given a pair of nodes, 2 and 6, there are two arcs with different values $a_{2,6}^1 = 0.3$ and $a_{2,6}^2 = 0.5$ meaning the distance and the slope between the nodes.

In this proposal, it is assumed 1) that the nominal characteristics can be translated into numerical values and 2) lower values in the attributes means more suitable arc. That means, the user considers a more suitable path between POIs if those attributes have lower values.

Summarizing, the proposed model requires:

G a directed multigraph $G = \{N, A\}$,

k, d index and set of layers that represent the measures of interest,

i, N index and set of nodes of the graph G,

i, j, k, A indexes and set of arcs. The arc's value from node i to node j on the layer d_k is denoted as a_{ij}^k,

and

k, W index and set of weights that represent the importance of the measure from layer d^k,

m, P index and set of POI, $P \subseteq N$.

Figure 1 shows an example of the adaptive map model where $|N| = 5$ nodes, $|P| = 3$ POIs and $|D| = 2$ measures.

3 Adaptive Maps: Construction

The construction of the Adaptive Map implies first, the combination (according with the user preferences) of the information from several layers, leading to a *Subjective Matrix* containing the relations among the POI, and secondly, the visualization of the map according to the Subjective Matrix and the set P (the POI). Both procedures are explained next.

3.1 Generation of a Subjective Matrix

The Subjective Matrix is an user dependent matrix that represents the difficulty for the user to travel between every pair of POIs.

Let's suppose that the user is only interested on the distance feature. Then, just a single layer associated with the Euclidean distance should be considered. In this case, to obtain the Subjective Matrix, it is necessary

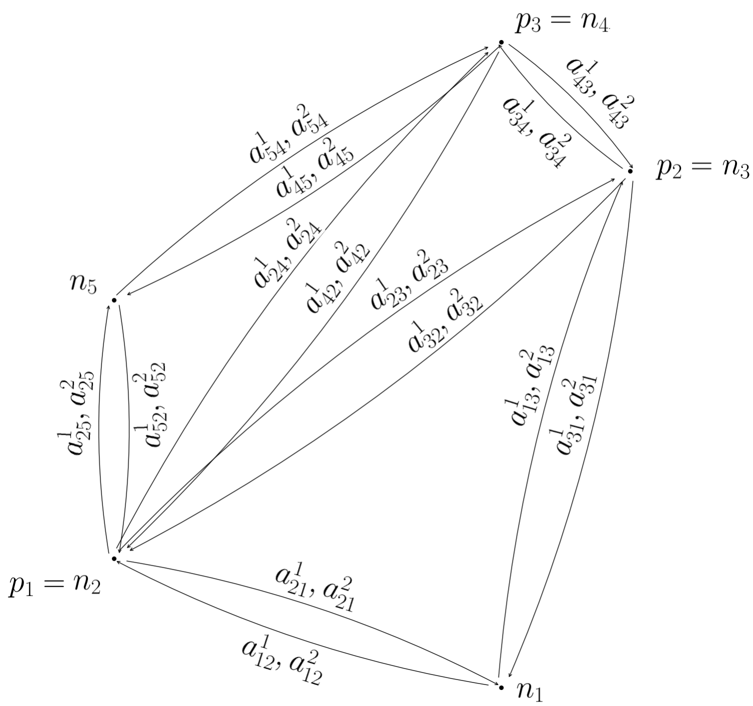


Figure 1: Elements of the model with $|N| = 5$, $|P| = 3$, a subset of N and $|D| = 2$. In this example, there are 2 layers so each pair of nodes has two arcs.

to run a shortest path algorithm for every pair of points $s, t \in P$ to represent the travelling effort for the user.

When other type of layers are considered and the user preferences are taken into account, then the shortest path algorithm becomes useless. In turn, a Personalized Route Problem (PRP) [14] should be solved between every pair of POIs in $P \subseteq N$, obtaining the personalized measurement of the path between them. That means that the PRP is solved for every pair of starting and ending points $s, t \in P$ with $s \neq t$.

The PRP model is:

$$\min \quad \sum_{i,j \in N(i \neq j)} \sum_{k=1}^{|D|} x_{ij} w_k a_{ij}^k, \tag{1}$$

s.t.

$$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & \text{if } i = s, \\ 0 & \text{if } i \neq s, t, \\ -1 & \text{if } i = t, \end{cases} \tag{2}$$

$$\sum_{k=1}^{|D|} w_k = 1 \quad and \quad 0 \leq w_k \leq 1 \quad \forall k, \tag{3}$$

$$x_{ij} \in \{0, 1\} \quad \forall a_{i,j}^k \in A. \tag{4}$$

The objective function 1 minimizes the value of the path (according to a weighted measure). Constraint 2 ensures that the arcs of the path are connected by a node to the next arc and that the path starts on s node and ends on t node. Constraint 3 is a normalization condition for the weights. Finally, in 4 the binary decision variables are defined: $x_{ij} = 1$ if the arc a_{ij}^k is part of the solution path, and 0 otherwise.

The solutions for the solved PRPs are organized as a Subjective Matrix called M, shown in Table 1, where the value $C(p_i, p_j)$ represents some notion of difficulty that the user may perceive (according to his/her preferences) to traverse the path joining p_i with p_j, where $p_i, p_j \in P$, $i, j = \{1, \ldots, m\}$.

M	p_1	$\ldots$	p_i	$\ldots$	p_m
p_1	0	$\ldots$	$C(p_1, p_i)$	$\ldots$	$C(p_1, p_m)$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
p_i	$C(p_i, p_1)$	$\ldots$	0	$\ldots$	$C(p_i, p_m)$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
p_m	$C(p_m, p_1)$	$\ldots$	$C(p_m, p_i)$	$\ldots$	0

Table 1: Subjective Matrix M. The value $C(p_i, p_j)$ stands for the measure of the path joining p_i with p_j.

3.2 Visualization

As has been previously stated, a map is a depiction that emphasizes relationships among elements of some space. At this point, the elements (the POI) and their relations (in the Subjective Matrix) are already defined. Now, the visualization of the Adaptive Map implies two steps: the relocation of the POI and the modification of the original map's image.

3.2.1 Relocation of POI

At this stage, the original POIs should be relocated according to the values in the Subjective Matrix M. This is achieved by finding the set of new locations $Q = \{q_1, q_2, \ldots, q_m\}$, $q_i \in \Re^2$ for the points $p_1, p_2, \ldots, p_m \in P$ in such a way that the distances among q_i, q_j reflects as much as possible $C(p_i, p_j) \in M$.

As the path attributes may come from a non-metric space (thus their combination is also a non-metric space), determining Q is a complex problem and, to the best of our knowledge, can not be solved exactly.

In any case, under the hypothesis that an approximated solution may fit our needs, this problem can be formulated as a Distance Geometry Problem [9] as well as a Multidimensional Scaling (MDS) Problem [1]. Due to the availability of software implementations, the solution is obtained solving a classical Multidimensional Scaling (classical MDS) problem.

MDS problem departs from a matrix known as dissimilarity matrix M' (in our case, the Subjective Matrix normalized) and calculates the location of the POI such that the distances between the points are approximately equal to the values on the matrix. The solution from a classical MDS problem is a coordinate matrix that minimizes a loss function called *strain* given by:

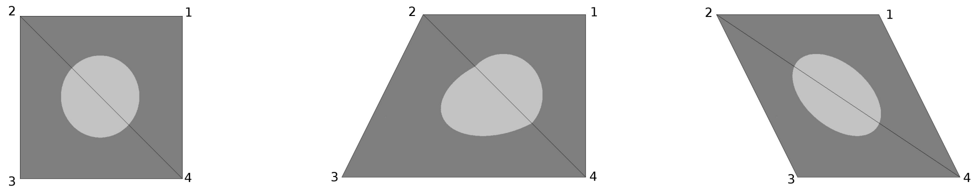
$$Strain(q_1, \ldots, q_m) = \left(\frac{\sum_{i,j} (b_{i,j} - \langle q_i, q_j \rangle)^2}{\sum_{i,j} b_{i,j}^2} \right)^{1/2} \tag{5}$$

where $b_{i,j}$ are coefficients from the matrix $B = -\frac{1}{2} J M'^2 J$, $J = I_m - \frac{1}{N}\mathbb{O}$ is a centering matrix and $\mathbb{O}$ a N-by-N matrix of 1.

The output of MDS is the set Q of the corresponding final locations on the Adaptive Map for the initial POI $p_1, \ldots, p_m \in P$. Note that those locations are invariant under reflection, translation and rotation.

3.2.2 Image modification

The final step of the visualization consists on modifying the original image of the map to locate the POI P at the new locations indicated by $q_i \in Q$, thus obtaining the image of the Adaptive Map.



(a) (b) (c)

Figure 2: Modification of an image obtained by mapping the vertices into different positions. In (a), the original image with 4 points in the set P' and a triangulation $T_{p'}$. In (b) and (c) the image's modifications considering two different sets Q of the vertices' final locations.

Firstly, for each point of interest $p_i \in P$, the coordinates $p'_i[0,1]^2 \in P'$ on the original map image are identified. The coordinates P' depend on the considered original image.

Secondly, a correspondence is established between the points in P' and Q by mapping each vertex p'_i to q_i and applying a piecewise linear homeomorphism (a bijective continuous function between topological spaces) [4, 8]. At the end, the original image is "distorted" using the q_i locations as anchor points. A way to do this is to obtain a triangulation $T_{p'}$ of P' (a subdivision into triangles) of the image. Then, the image of each triangle from $T_{p'}$ is drawn as a transformed triangle according to the vertices $q_i \in Q$, assuming that the homeomorphism is linear on each triangle. Figure 2 illustrates the idea.

It may happen that a triangle superposes a second triangle, in that case, if one of the two triangles is faced down, it will not be drawn (as it is usual in Computer Graphics when the normal vector of a polygon is opposite).

Even though multiple triangulation methods are suitable for this step, and after talking with Computer Graphics experts, the Delaunay Triangulation [7, 17] is applied because its properties help to achieve a proper Adaptive Map image. The Delaunay Triangulation $DT_{p'}$ of a set of points P' in 2-dimensional space is a triangulation such that no point in P' is inside the circumcircle of any triangle (a circle which passes through all the vertices of the triangle) in $DT_{p'}$. The Delaunay Triangulation's property of maximizing the minimum angle is of interest for the visualization of Adaptive Maps. It means that the smallest angle in the Delaunay triangulation is at least as large as the smallest angle in any other triangulation of the points P'. In simpler words, the Delaunay Triangulation avoids narrow triangles and, as a consequence, obtains better results in terms of image quality.

	Latitude	*Longitude*
p_1	37.206687	-3.606632
p_2	37.202209	-3.618311
p_3	37.190825	-3.622208
p_4	37.187247	-3.591872
p_5	37.178639	-3.612509
p_6	37.173052	-3.583150
p_7	37.163225	-3.569172
p_8	37.159979	-3.591418
p_9	37.149748	-3.608442
p_{10}	37.165262	-3.606178

Table 2: Example A: Geographical location of the POI.

4 Examples

In this section different examples are shown to illustrate the usefulness of our proposal. Example A consists on visualizing three Adaptive Maps, each one using a single attribute. Example B shows the process of generation and visualization of Adaptive Maps with multiple attributes.

4.1 Example A: One attribute examples

We consider the 10 POI shown in Table 2 which are located in the city of Granada (Spain).

We take into account three layers ($|D| = 3$):

1. Distance on foot. The true distance when walking from one point to other point.
2. Distance on bicycle. The true distance when cycling from one point to other point. Due to traffic restrictions on roads for bicycles, this distance is usually longer than the distance on foot.
3. Time on bicycle. The time that takes when cycling from one point to other point.

To generate 3 different Adaptive Maps, 3 different weights distributions are set:

- $W_1 = \{1, 0, 0\}$,

- $W_2 = \{0, 1, 0\}$ and
- $W_3 = \{0, 0, 1\}$.

For this example with one single attribute, the Subjective Matrix M is calculated using the GrapHopper's Matrix API [2]. This API gives the measurements according to one attribute between two geographical points. The Subjective Matrices M_k and the Q locations are shown in Tables 4, 5 and 6 (Appendix A).

The visualization of every *Adaptive Map k* is achieved as explained in Section 3.2. First, the classical MDS problem is solved using the *cmdscale* function on *R* [10] to determine the points' location $q_i \in Q$ on the *Adaptive Map k*. Secondly, the Delaunay Triangulation $DT_{p'}$ is calculated using the *Delaunay* function on *Python* [5] and each triangle is drawn by mapping the vertices p'_i into q_i.

The final Adaptive Maps are shown in Fig. 3 and Fig. 4. The position of the POI vary from one Adaptive Map to other.

Figure 3 (a) shows part of a typical tourist map while Fig. 3 (b) shows the adaptive map when considering walking distance. We can observe that in the later, p_1, p_3 and p_5, p_9 look closer than in the original map.

Figure 4 (a) shows the adaptive map in terms of distance on bicycle while (b) shows the time on bicycle. In the former, the distance between p_3, p_6 looks larger than when we considered distance. In turn, the points p_8, p_9 look close in the distance map but far apart in the time map. A similar situation occurs for p_8, p_{10}.

In general the point p_9 is farther on all the Adaptive Maps. Actually, the point p_9 has more elevation than any other point of interest and the paths leading to that point consist on zigzag roads. That makes the point p_9 harder to reach under all the considered attributes.

4.2 Example B: Multiple attributes examples

In this example two Adaptive Maps are generated and visualized departing from a map located on Granada (Spain) with 5 POI. Point locations are shown in Table 3. In this case, the Subjective Matrices M_1 and M_2 are calculated as explained in Section 3.1: a PRP associated to each $C(p_i, p_j)$ value is solved using the software PRoA (an acronym for Personalized Route Assistant), an Android application available on Google Play [15].

Four layers are considered, $|D| = 4$:

1. Distance on foot. The real distance when walking from one point to other point.
2. Upward slopes. The upward inclination of the street.

	Latitude	*Longitude*
p_1	37.169053	-3.597
p_2	37.167762	-3.596
p_3	37.167129	-3.590
p_4	37.168013	-3.589
p_5	37.169709	-3.593

Table 3: Example B: Geographical location of the POI.

3. Downward slopes. The downward inclination of the street.

4. Motor vehicles zones. This attribute means that the user wants to reduce zones with motor vehicles, preferring pedestrian zones, path, parks, etc.

Two different Adaptive Maps are generated by setting 2 different weights distributions:

- $W_1 = \{0.1, 0.45, 0.45, 0\}$ and
- $W_2 = \{0.3, 0, 0, 0.7\}$.

The weights distribution W_1 for the *Adaptive Map 1* has the following meaning: it is equally important to minimize attributes 2 and 3 (the user does not want slopes) and it is somehow important to reduce the distance. Passing through motor vehicles zones is irrelevant.

The weights distribution W_2 for the *Adaptive Map 2* says that the user does not care about slopes, it is very important to reduce the motor vehicles zones and the distance on foot should be also taken into account.

For each weight distribution W_k, the Subjective Matrix M_k is calculated by solving a PRP for every pair of points $p_i, p_j \in P$, with $i \neq j$ (see Section 3.1). The Subjective Matrices M_k and the Q locations are shown in tables 7 and 8.

Once the matrix M_k is calculated, the visualization of the *Adaptive Map k* is achieved as explained in Section 3.2. First, the classical MDS problem is solved to determine the set Q, the points' location on the *Adaptive Map k* (using the *cmdscale* function on *R* [10]). Secondly, every triangle from the Delaunay Triangulation $DT_{p'}$ (calculated using the *Delaunay* function on *Python* [5]) is drawn by mapping each vertex p'_i into q_i. In this example $DT_{p'}$ consists on 3 triangles.

The final Adaptive Maps are shown in Fig. 5. On top, the original map appears. In b) the *Adaptive Map 1* (considering upward, downward slopes and distance) brings points p_3 and p_4 closer, because they have similar elevation and the path between them barely has slopes. In c) the *Adaptive Map 2* (trying to avoid areas with motor vehicles and minimizing distance) the points p_2 and p_5 are closer because the path between them consists on pedestrian zones (that area corresponds to a park).

5 Conclusions

In this paper, the concept of Adaptive Maps has been presented and its model has been described. From an operational point of view, a method to generate and visualize these Adaptive Maps was also proposed. Examples using single and multiple attributes showed the benefits of our proposal.

Despite the quality of the final Adaptive Map images, there is still room for improvement at several stages. One is the calculation of the new locations of the points of interest. Right now, Multidimensional Wcaling is used but other techniques like Distance Geometry [9] should be explored. Another aspect is visualization. The proposed triangulation and mapping methods result in good map images in practice but further comparison will be a matter of interest.

Acknowledgments

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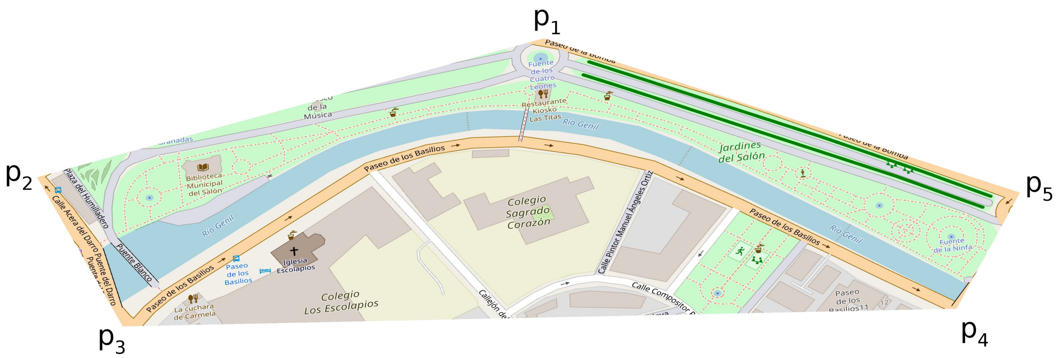
(a) (b)

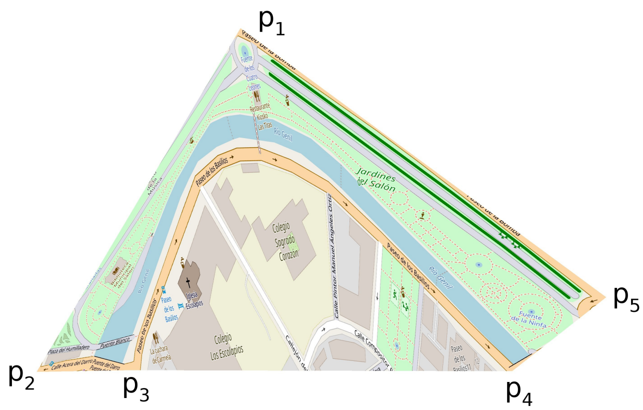
Figure 3: Adaptive Maps visualization. In a) the general purpose map and b) *Adaptive Map 1*, distance on foot

(a) (b)

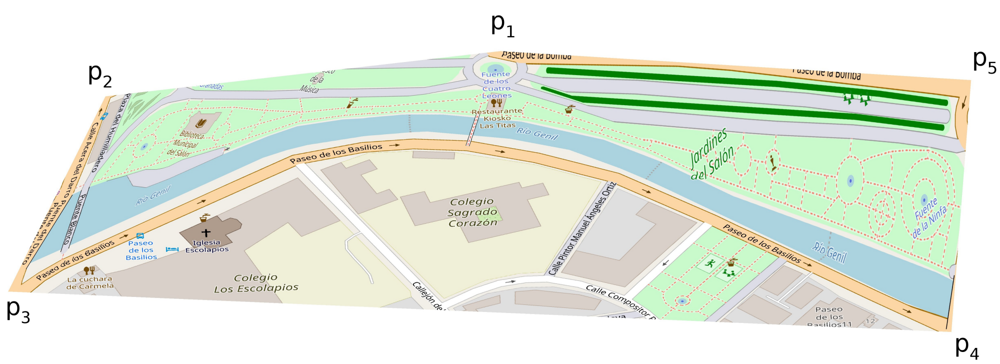
Figure 4: Adaptive Maps visualization. In a) *Adaptive Map 2*, distance on bicycle and b) *Adaptive Map 3*, time on bicycle.



(a)



(b)



(c)

Figure 5: Adaptive Maps visualization. In a) the general purpose map, b) *Adaptive Map 1*, reducing upward, downward slopes and distance, and c) *Adaptive Map 2*, reducing motor vehicles and distance.

Appendix A

M_1	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}
p_1	0	1553	2783	3614	4240	5674	6972	6064	7408	5387
p_2	1553	0	1674	3867	2975	5936	7234	6326	6807	4693
p_3	2783	1674	0	3427	2118	5098	6396	5409	5950	3837
p_4	3614	3867	3427	0	2471	2571	4609	3745	5222	3292
p_5	4240	2975	2118	2471	0	3600	4684	3172	3840	1727
p_6	5674	5936	5098	2571	3600	0	2569	3076	5083	3416
p_7	6972	7234	6396	4609	4684	2569	0	2692	4966	3903
p_8	6064	6326	5409	3745	3172	3076	2692	0	2415	1698
p_9	7408	6807	5950	5222	3840	5083	4966	2415	0	2303
p_{10}	5387	4693	3837	3292	1727	3416	3903	1698	2303	0

(a)

	x	y
q_1	0.124	0.488
q_2	−0.075	0.498
q_3	−0.086	0.351
q_4	−0.117	0.060
q_5	−0.161	−0.150
q_6	−0.362	−0.375
q_7	−0.033	−0.336
q_8	0.269	−0.414
q_9	0.286	−0.204
q_{10}	0.155	0.081

(b)

Table 4: Example A: a) Subjective Matrix M_1, b) locations of points Q.

M_2	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}
p_1	0	1791	3561	4106	4228	6889	7403	7260	8082	5822
p_2	2136	0	2212	5009	3498	7750	8264	6915	8075	5423
p_3	2915	2240	0	4637	2190	6474	6988	5562	6605	4308
p_4	3968	4346	3496	0	2915	4978	5492	5349	6172	3912
p_5	4074	3223	2313	3239	0	4780	5294	3572	4394	2080
p_6	6833	6546	5747	2681	4731	0	2543	3428	5198	4188
p_7	7614	7327	6529	5555	5362	2801	0	3347	5117	4527
p_8	6491	6282	5349	4432	3207	4319	2980	0	2393	1849
p_9	8044	7193	6227	6317	4118	6922	5569	2732	0	2458
p_{10}	5655	4803	3870	4055	1728	4510	4457	2079	2672	0

(a)

	x	y
q_1	0.064	0.506
q_2	−0.034	0.468
q_3	−0.051	0.320
q_4	−0.139	0.084
q_5	−0.138	−0.135
q_6	−0.409	−0.335
q_7	−0.097	−0.308
q_8	0.222	−0.408
q_9	0.387	−0.274
q_{10}	0.207	0.109

(b)

Table 5: Example A: a) Subjective Matrix M_2, b) locations of points Q.

M_3	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}
p_1	0	392	738	1423	899	2137	1949	1694	1838	1320
p_2	624	0	471	1652	743	2327	2139	1538	1710	1151
p_3	831	526	0	1637	503	2127	1939	1325	1420	938
p_4	967	969	786	0	640	1656	1468	1213	1357	839
p_5	1086	765	559	1320	0	1675	1488	856	999	468
p_6	1732	1605	1402	1434	1091	0	605	826	1202	949
p_7	2081	1953	1750	2023	1321	1290	0	748	1124	997
p_8	1621	1474	1277	1563	750	1585	875	0	490	401
p_9	1924	1603	1351	1986	880	2113	1413	598	0	535
p_{10}	1491	1170	973	1538	447	1636	1195	514	590	0

(a)

	x	y
q_1	0.093	0.431
q_2	−0.016	0.388
q_3	−0.042	0.284
q_4	−0.084	0.087
q_5	−0.108	−0.101
q_6	−0.385	−0.178
q_7	−0.117	−0.228
q_8	0.060	−0.465
q_9	0.373	−0.345
q_{10}	0.243	0.146

(b)

Table 6: Example A: a) Subjective Matrix M_3, b) locations of points Q.

Appendix B

M_1	p_1	p_2	p_3	p_4	p_5
p_1	0.000	0.111	0.486	0.581	0.411
p_2	0.108	0.000	0.392	0.462	0.359
p_3	0.480	0.390	0.000	0.071	0.428
p_4	0.604	0.514	0.125	0.000	0.512
p_5	0.406	0.371	0.442	0.435	0.000

(a)

	x	y
q_1	−0.271	−0.087
q_2	−0.174	−0.085
q_3	0.217	−0.086
q_4	0.314	−0.010
q_5	−0.062	0.266

(b)

Table 7: Example B: a) Subjective Matrix M_1, b) locations of points Q.

M_2	p_1	p_2	p_3	p_4	p_5
p_1	0.000	0.416	1.004	1.022	0.529
p_2	0.405	0.000	1.125	1.129	0.634
p_3	1.013	1.104	0.000	0.374	0.636
p_4	1.037	1.141	0.389	0.000	0.661
p_5	0.533	0.636	0.624	0.553	0.000

(a)

	x	y
q_1	−0.457	0.083
q_2	−0.568	−0.140
q_3	0.512	−0.186
q_4	0.534	0.102
q_5	−0.005	0.135

(b)

Table 8: Example B: a) Subjective Matrix M_2, b) locations of points Q.