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Periodic dynamics in the relativistic regime of an electromagnetic field induced by a time-dependent wire ☆

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Abstract

We consider the motion of a charged particle under the electromagnetic field generated by an electrically neutral infinite straight wire with a time-periodic oscillating (AC-DC) current. By using global continuation and topological degree, we identify a bi-parametric family of radially periodic motions. The proofs involve some delicate estimations of the induced electromagnetic field, which can be of independent interest.

MSC: 34B15; 58E05; 78A35; 83A05; 34C25

Keywords: Lorentz force equation; Electromagnetic field; Straight wire; Oscillating current; Radially periodic solution; Global continuation; Bessel functions

1. Introduction

According to classical Electrodynamics, the motion of a slowly accelerated charged particle $q(t)$ in an electromagnetic field is ruled by the classical Lorentz Force equation (LFE), which is one of the fundamental equations in Mathematical Physics and has its origin in the pioneering works of Poincaré [19] and Planck [18]. Due to its relevence in theory and applications, the LFE

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can be found in many classical textbooks on Physics and Electrodynamics, see for instance [12, Chapter 12], [11, Chapters 5 and 12] or [14, Chapter 3]. In this paper, we study the relativistic dynamics induced by the electromagnetic field created by an oscillating current in an infinitely long and infinitely thin straight wire. The wire is also assumed to be electrically neutral, i.e., at any time, every segment of it contains as many electrons as protons, so the charge density is null.

Without loss of generality, we normalize both the speed of light in vacuum and the charge-to-mass ratio to 1. The dynamical system under study is then the following equation

$$\frac{d}{dt}\left(\frac{\dot{q}(t)}{\sqrt{1-|\dot{q}(t)|^2}}\right) = E(t,q(t)) + \dot{q}(t) \times B(t,q(t)). \tag{1}$$

On the right-hand side, we have the common expression of the Lorentz force, with $E : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ and $B : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ being the electric and magnetic field respectively, while the left-hand side denotes the relativistic acceleration of the particle, which implies the characteristic speed limitation of Special Relativity. Introducing the diffeomorphism $\phi : B(0,1) \to \mathbb{R}^3$, $\phi(v) = \frac{v}{\sqrt{1-|v|^2}}$, the left-hand side is also known as a singular ϕ-Laplacian in the related (more mathematically oriented) literature (see for instance [5]).

Despite being a very classical equation, there has been a remarkable lack of qualitative and quantitative results about its dynamics until recently. One of the main reasons is that the proper mathematical tools for it were developed during the last quarter of the last century. Moreover, such techniques were conceived in an abstract mathematical framework, and their applications to (1) appear during the last 15 years. In 2008, the existence of solutions was proven in [5], by topological degree methods, for periodic, Dirichlet and Neumann problems associated to the general equation

$$\frac{d}{dt}\phi(\dot{q}(t)) = f(t,q,\dot{q}),$$

with f continuous, with d/dt denoting the total derivative with respect to the time variable. In particular, they provide an application to (1). More recently, a different approach is given in [3], by developing a critical point theory for (1) that can be applied to the periodic and Dirichlet problem for continuous electromagnetic fields. This work is based on Szulkin's variational method [20] for functionals having a regular part plus a lower semi-continuous term, applied to the Lagrangian of (1), namely

$$\mathcal{L}(t,q,\dot{q}) = 1 - \sqrt{1-|\dot{q}|^2} + \dot{q} \cdot \vec{A}(t,q) - \Phi(t,q),$$

where $\vec{A}$ and Φ compose an electromagnetic potential, i.e., the electromagnetic field given by

$$B(t,q) = \nabla \times \vec{A}(t,q), \qquad E(t,q) = -\partial_t \vec{A}(t,q) - \nabla\Phi(t,q), \tag{2}$$

solves Maxwell's equations uniquely for a specific distribution of charge and current. We are denoting by ∇ and $\nabla\times$ the gradient and the rotational with respect to q, while ∂_t is the time partial derivative. Notice that these derivatives are defined, in principle, in a distributional sense. In a second paper [4], the authors of [3] provide a Lusternik-Schnirelmann multiplicity theory in the same regime. However, the required conditions for the previously cited papers do not permit the presence of singularities in the fields.

Concerning singular electromagnetic fields, the existence of periodic solutions for a long range of potentials that admit isolated singularities is proven in [7], covering relevant physical cases like the Coulomb potential or the magnetic dipole. This was done by topological degree methods, so the approach to the singular problem using variational methods is still an open problem.

Focusing on the wire model, the case of a constant current is well-known. Due to the absence of dependence with respect to the time, the regime is magnetostatic, i.e., there is no electric field and the magnetic field $B(q)$ is obtained via Biot-Savart law, reducing it from Maxwell's equations in a rigorous way. The corresponding non-relativistic dynamics has been studied extensively in [2,9,10], together with other autonomous wire distributions with symmetries, like the circular loop, where the symmetries imply the existence of first integrals in the Newton-Lorentz equation, according to Noether's Theorem. In the case of an infinite wire, the conservation of the energy, linear and angular momenta imply that the system is totally integrable. In particular, the particles cannot reach the wire, every solution is radially periodic and there exists a unique radial equilibrium.

On the other hand, the Biot-Savart formula does not hold anymore for time-dependent currents and the electromagnetic field has to be obtained by solving Maxwell's equations

$$\begin{cases} \nabla \cdot B = 0, & \text{(Gauss's law for magnetism)} \\ \nabla \times E + \partial_t B = 0, & \text{(Maxwell-Faraday equation)} \\ \nabla \times B = \mu_0 \vec{J} + \partial_t E, & \text{(Ampère's law)} \\ \nabla \cdot E = \mu_0 \rho, & \text{(Gauss's law)} \end{cases} \tag{3}$$

where $\vec{J}$ and ρ denote the current and charge distribution respectively, and μ_0 is the vacuum permeability constant for unitary light speed. If the densities are compactly supported distributions, then the solutions are given by (2) via the introduction of retarded potentials. However, although the infinite wire does not have a compact support, a retarded potential still solves (3), obtained as the distributional limit of suitable solutions for approximated problems. We refer to the reader to [8] for the mathematical details. In the same paper it is studied the non-relativistic model for a T-periodic time-dependent perturbation, where the momenta are still conserved but no more the energy. Through this, the Newton-Lorentz equation is reduced to a two-dimensional hamiltonian system with one degree of freedom, which describes the radial dynamics of the particle. The authors show the existence of solutions such that the radial component is T-periodic and stable in the Liapunov sense, not just with respect of the radial initial conditions, but also in the phase plane. Both results are qualitative and hold only for small values of the perturbation. It can be seen as a complementary study to [13,15], where the case of an infinite wire with no current density and a time-dependent charge density is considered.

Just like in the Newtonian case, the cylindrical symmetry is still present in our model and the corresponding linear and angular relativistic momenta are conserved quantities. Therefore, we are able to reduce (1) to a planar hamiltonian system with one degree of freedom. We prove the existence of radially periodic solutions for an explicit interval of the perturbation parameter k (see Section 2 for details), which depends on the values of the current and the conserved momenta. While [8] used a perturbation argument, here the mathematical procedure is different, based on a global continuation by using the topological degree. Regarding the radial stability, we are sure that it holds for small values of k reasoning like in [8], where the third approximation method is used [17,21]. However, we have not been able to obtain a quantitative result, which remains as an open problem.

This paper is structured as follows. In Section 2 we construct rigorously the model from physical principles and the main results are presented. Section 3 collects some delicate estimations of the electromagnetic potential. In Section 4, the symmetries of the system are used to deduce the corresponding conserved quantities (linear and relativistic angular momentum) and reduce the problem to a Hamiltonian system with one degree of freedom. Finally, Section 5 contains the main proof for the existence of radially periodic solutions. It relies on a global continuation theorem for the Leray-Schauder degree that is presented in Appendix A and resembles the celebrated global continuation theorem by Capietto-Mawhin-Zanolin [6], adapted to our context.

2. Statements and main results

Let us fix the period $T > 0$. From now on, we denote by $C^n(\mathbb{R}/T\mathbb{Z})$ the Banach space of T-periodic functions of class C^n. We consider an infinitely long and infinitely thin straight wire carrying a current of the form $I_0 + kI(t)$, where $I_0 > 0$, $k \geq 0$ are constants and $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfies

$$\int_0^T I(t)dt = 0. \tag{4}$$

Notice that (4) implies the existence of a T-periodic primitive of $I(t)$, which we denote by $\mathcal{I}(t)$. Without loss of generality, let us assume the wire centered in the z-axis. Then, the corresponding current density is a vectorial distribution $\vec{J} = (0, 0, J)$ such that

$$J(f) = \int_{\mathbb{R}^2} [I_0 + kI(t)]\, f(t, 0, 0, z)\, dt\, dz, \qquad \text{for every } f \in \mathcal{D}(\mathbb{R}^4),$$

whit $\mathcal{D}(\mathbb{R}^4)$ denoting the space of test functions in $\mathbb{R}^4$. Furthermore, let us also consider that the wire is electrically neutral, i.e., at every time, every segment of it contains as many electrons as protons, and so the charge density ρ is null. In this situation, and assuming certain conditions of decaying at infinity, (3) admits as unique distributional solution to the electromagnetic field giving by (2), with $\Phi \equiv 0$ and $\vec{A}(t, q) = A(t, r)\mathbf{e}_z$, where

$$A(t, r) = -\frac{\mu_0}{2\pi}[a_0(r) + ka(t, r)], \tag{5}$$

and

$$a_0(r) = I_0 \ln r, \quad a(t, r) = \int_0^\infty \frac{I[t, r, \tau]}{\sqrt{\tau^2 + r^2}} d\tau.$$

Here r denotes the radial variable in the XY-plane, $\mathbf{e}_z$ is the positive unitary vector in the z-direction, and the bracket $[t, r, \tau] = \left(t - \sqrt{\tau^2 + r^2}\right)$ shows the delay effect of the potential.

More concretely, via the Lorenz gauge condition, this is done uncoupling (3) into two wave equations that are solving by the *retarded potentials* Φ and $\vec{A}$. We refer the reader to Proposition 1 in [8] for the mathematical details. By the same result, (5) is T-periodic and such that

$$\text{if } I(t) \in \mathcal{C}^n(\mathbb{R}/T\mathbb{Z}), \text{ then } a(t,r) \in \mathcal{C}^n\left([0,T] \times \mathbb{R}^+; \mathbb{R}\right). \tag{6}$$

Moreover, there exists a constant $C > 0$ for which

$$|a(t,r)| \leq C|\ln r|, \quad \text{when } r << 1.$$

Therefore, the blow up sign of (5) when r tends to 0 is controlled for all $t \in [0,T]$ by the logarithm term $a_0(r)$ if k is small enough. In this paper, we improve Proposition 1 in [8] by giving a quantitative version, being able to estimate explicitly not just C but also describing the asymptotic behaviour of $a(t,r)$ and its partial derivative $\partial_r a(t,r)$ when r tends to 0 or to $+\infty$. Concretely, for any $\hat{r} > 0$, let $K_{\hat{r}}$ be the quantity

$$K_{\hat{r}} := \frac{4I_0}{\max\left\{[2\left(1+\sqrt{2}\right)+\pi]\|I\|_\infty + \hat{r}\pi\|\dot{I}\|_\infty; \frac{4T^2}{3\pi}\|\ddot{I}\|_\infty\right\}}. \tag{7}$$

Theorem 1. *Let I_0 be positive and $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying (4). Then (5) is such that:*

i) For any $k \geq 0$, $\lim_{r\to+\infty} A(t,r) = -\infty$, uniformly in t.
ii) For any $k \in [0, K_{\hat{r}})$, $\lim_{r\to 0^+} A(t,r) = +\infty$, uniformly in t.
iii) For any $k \in [0, K_{\hat{r}})$,

$$0 > -\frac{I_0}{r}\left[1 - \frac{k}{K_{\hat{r}}}\right] > \frac{2\pi}{\mu_0}\partial_r A(t,r) > -\frac{I_0}{r}\left[1 + \frac{k}{K_{\hat{r}}}\right], \tag{8}$$

for all $(t,r) \in [0,T] \times [0,\hat{r}]$.

Moreover,

iv) If $k > 0$, then

$$\partial_r A(t,r) \sim r^{-1/2}\mathcal{G}(t,r), \quad \textit{when } r >> 1,$$

with $\mathcal{G}(t,r)$ being an oscillating function such that, given any $(t,r) \in [0,T] \times (0,+\infty)$, there exists an infinite number of points in [r, $+\infty$) *where $\mathcal{G}(t,r)$ changes its sign.*

Notice that the first three assertions are satisfied by the logarithm $a_0(r)$. Therefore, the holding of this picture is explicitly controlled with respect of the perturbation parameter k. Through this we are able to find invariant sets for the solutions in order to apply a fixed point argument. The proof of Theorem 1 is given in Section 3.

In our particular conditions, the dynamical system (1) is

$$\frac{d}{dt}\phi(\dot{q}(t)) = -\partial_t \vec{A}(t,q(t)) + \dot{q}(t) \times \left[\nabla_q \times \vec{A}(t,q(t))\right]. \tag{9}$$

Induced by the wire symmetries, let us consider the cylindrical coordinates $q = (r\cos\theta, r\sin\theta, z)$. We also denote by p_r, L and p_z the relativistic radial, angular and linear momenta respectively, defined as

$$p_r = \frac{\dot r}{\sqrt{1-\dot r^2 - r^2\dot\theta^2 - \dot z^2}}, \qquad L = \frac{r^2\dot\theta}{\sqrt{1-\dot r^2 - r^2\dot\theta^2 - \dot z^2}},$$
$$p_z = \frac{\dot z}{\sqrt{1-\dot r^2 - r^2\dot\theta^2 - \dot z^2}} + A(t,r). \tag{10}$$

Both L and p_z are first integrals of (9) for every $k \geq 0$. We rely on this fact to introduce the following definition, analogous to the one established in [8].

Definition 1. A solution $q(t) = (r(t), \theta(t), z(t))$ of (9) with angular momentum L and linear momentum p_z is called a (L, p_z)-solution. Moreover, it is *radially T-periodic* if $r(t+T) = r(t)$, for every $t \in \mathbb{R}$.

Now, we are ready to set up the main result about the existence of radially periodic solutions. Given T, $k > 0$, let us define

$$\mathbf{P}(k) = -\frac{\mu_0}{2\pi}\left[I_0 \ln T + k\frac{T^2}{3}\|\ddot I\|_\infty\right]. \tag{11}$$

Theorem 2. *Let I_0, L be positive, $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying (4) and $P_0 < \boldsymbol{P}(K_T)$. Take $\varepsilon > 0$ and $\bar k \in [0, K_{r_\varepsilon+T+\varepsilon}]$, where r_ε is the unique point satisfying the identity*

$$r^2\left[P_0 + \frac{\mu_0}{2\pi}\left(I_0 \ln r - K_{r+T+\varepsilon}\frac{T^2}{3}\|\ddot I\|_\infty\right)\right] = \frac{2\pi}{\mu_0}\frac{L^2}{I_0}\frac{C_1 + C_2(r+T+\varepsilon)}{C_2\varepsilon}, \tag{12}$$

with

$$C_1 = \|I\|_\infty\left[2\left(1+\sqrt{2}\right)+\pi\right], \text{ and } C_2 = \pi\ \|\dot I\|_\infty.$$

Then, there exists a positive constant $r_m < r_\varepsilon$, depending on $\bar k$ and $\boldsymbol{P}(\bar k)$, such that for all $(k, p_z) \in [0, \bar k] \times \left[P_0, \boldsymbol{P}(\bar k)\right]$, (9) admits at least one radially T-periodic (L, p_z)-solution with

$$r_m < r(t) < r_\varepsilon, \quad \text{for every } t \in \mathbb{R}.$$

Moreover,

$$\lim_{\varepsilon\to 0} r_\varepsilon = +\infty, \qquad \lim_{\varepsilon\to 0} K_{r_\varepsilon} = 0.$$

Remark 1. If compared with Theorem 1 in [8], the main improvement is the existence of radially T-periodic (L, p_z)-solutions for an explicit interval for the perturbation parameter k, which depends on L, T, I_0 and the norm of $I(t)$ in $W^{1,\infty}\left([0,T];\mathbb{R}^3\right)$. Moreover, these motions are located between the cylinders centered in the z-axis with radius r_m and r_ε, which also depend on the above constants.

3. Estimations for the potential

In this section, we prove Theorem 1 as a consequence of two previous results. The first one describes explicitly the asymptotic behaviour of (5) close to the wire and also very far from it. In particular, we have the following.

Proposition 1. *Let I_0 be positive and $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying (4). Then, for every $k \geq 0$:*

$$\lim_{r \to +\infty} A(t,r) = -\infty, \text{ uniformly in } t.$$

Moreover, let k_0 be the constant

$$k_0 = \frac{I_0}{\frac{T^2}{3\pi}\|\ddot{I}\|_\infty}.$$

Then, for any $k \in [0, k_0)$,

$$\lim_{r \to 0^+} A(t,r) = +\infty, \text{ uniformly in } t.$$

On the other hand, as the potential (5) is regular, we can proceed analogously for its derivative $\partial_r A(t,r)$.

Lemma 1. *Let I_0 be positive and $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying (4). Given any radius $\hat{r} > 0$, define the constant*

$$k_{\hat{r}} = \frac{4I_0}{[2\left(1+\sqrt{2}\right)+\pi]\|I\|_\infty + \hat{r}\pi\|\dot{I}\|_\infty}. \tag{13}$$

If $k \in [0, k_{\hat{r}})$, then

$$0 > -\frac{I_0}{r}\left[1 - \frac{k}{k_{\hat{r}}}\right] > \frac{2\pi}{\mu_0}\partial_r A(t,r) > -\frac{I_0}{r}\left[1 + \frac{k}{k_{\hat{r}}}\right],$$

for all $(t,r) \in [0,T] \times [0,\hat{r}]$. Moreover, if $k > 0$, then

$$\partial_r A(t,r) \sim r^{-1/2}\mathcal{G}(t,r), \text{ when } r >> 1,$$

where $\mathcal{G}(t,r)$ is a bounded and oscillating function such that, given any $(t,r) \in [0,T] \times (0,+\infty)$, there exists an infinite number of points in $[\mathrm{r}, +\infty)$ where $\mathcal{G}(t,r)$ changes its sign.

Once we prove both results, it is clear that Theorem 1 follows combining them.

3.1. Proof of Proposition 1

By Proposition 1 in [8], we know that

$$|a(t,r)| \leq \|I\|_\infty \ln\left(1+\sqrt{2}\right) + 2r^{-1}\|I\|_\infty,$$

for all $(t,r) \in [0,T] \times \mathbb{R}^+$. Consequently, for every $k \geq 0$,

$$-\frac{2\pi}{\mu_0} A(t,r) \geq I_0 \ln r - k|a(t,r)| \geq I_0 \ln r - k\|I\|_\infty \left(\ln\left(1+\sqrt{2}\right) + 2r^{-1}\right),$$

and the first assertion follows taking limits when $r \to +\infty$.

Complementary, recalling again Proposition 1 in [8], the Fourier series of $I(t)$ allows to write $a(t,r)$ as

$$\begin{aligned} a(t,r) = &-\frac{\pi}{2}\sum_{j\geq 1}\left[\alpha_j \cos\left(\lambda_j t\right) + \beta_j \sin\left(\lambda_j t\right)\right]\mathcal{Y}_0\left(\lambda_j r\right) \\ &+\frac{\pi}{2}\sum_{j\geq 1}\left[\alpha_j \sin\left(\lambda_j t\right) - \beta_j \cos\left(\lambda_j t\right)\right]\mathcal{J}_0\left(\lambda_j r\right), \end{aligned} \tag{14}$$

where $\mathcal{J}_0$, $\mathcal{Y}_0$ are the Bessel functions of first and second kind with zero index respectively, $\lambda_j = j2\pi T^{-1}$ and $\{\alpha_j, \beta_j\}_{j\geq 1}$ are the Fourier coefficients:

$$\alpha_j = \frac{2}{T}\int_0^T I(t)\cos(\lambda_j t)dt, \quad \beta_j = \frac{2}{T}\int_0^T I(t)\sin(\lambda_j t)dt.$$

Due to the regularity of I, integrating by parts twice we get the following:

$$\alpha_j = -\frac{T}{2\pi^2 j^2}\int_0^T \ddot{I}(t)\cos(\lambda_j t)dt, \quad \beta_j = -\frac{T}{2\pi^2 j^2}\int_0^T \ddot{I}(t)\sin(\lambda_j t)dt.$$

It is clear that $|\alpha_j|, |\beta_j| \leq \frac{T^2}{2\pi^2 j^2}\|\ddot{I}\|_\infty$, for all $j \geq 1$. However, let us be more precise:

$$\begin{aligned} \int_0^T |\sin(\lambda_j t)|dt &= \int_0^T \left|\sin\left(\tfrac{j2\pi}{T}t\right)\right|dt = j\int_0^{\frac{T}{j}} \left|\sin\left(\tfrac{j2\pi}{T}t\right)\right|dt \\ &= j\left[\int_0^{\frac{T}{2j}} \sin\left(\tfrac{j2\pi}{T}t\right)dt - \int_{\frac{T}{2j}}^{\frac{T}{j}} \sin\left(\tfrac{j2\pi}{T}t\right)dt\right] \\ &= \frac{T}{2\pi}\left[-\cos\left(\tfrac{j2\pi}{T}t\right)\Big|_0^{\frac{T}{2j}} + \cos\left(\tfrac{j2\pi}{T}t\right)\Big|_{\frac{T}{2j}}^{\frac{T}{j}}\right] = \frac{2T}{\pi}. \end{aligned}$$

Similarly, $\int_0^T |\cos(\lambda_j t)|dt = \frac{2T}{\pi}$ and

$$|\alpha_j|, |\beta_j| \le \frac{T^2}{\pi^3 j^2}\|\ddot{I}\|_\infty, \text{ for all } j \ge 1.$$

Therefore, as $\|\mathcal{J}_0\|_\infty = \mathcal{J}_0(0) = 1$ and $\sum_{j\ge 1} j^{-2} = \pi^2/6$, then

$$|a(t,r)| < \frac{T^2}{\pi^2}\|\ddot{I}\|_\infty \left[\sum_{j\ge 1}\frac{1}{j^2}|\mathcal{Y}_0\left(\lambda_j r\right)| + \frac{\pi^2}{6}\right], \quad \text{for all } (t,r) \in [0,T]\times\mathbb{R}^+.$$

Concerning $\mathcal{Y}_0(x)$, it is known that $|\mathcal{Y}_0(x)| < 1$, when $x > \sigma$, where σ[i] is the unique real root of the equation $\mathcal{Y}_0(x) + 1 = 0$. Thus, defining

$$\mathbf{r} = \sigma\frac{T}{2\pi}, \tag{15}$$

if $r \ge \mathbf{r}$, it is clear that $\lambda_j r > \sigma$, for every $j \in \mathbb{N}$. Furthermore,

$$|a(t,r)| < \frac{T^2}{3}\|\ddot{I}\|_\infty, \quad \text{for all } (t,r) \in [0,T]\times[\mathbf{r},\infty). \tag{16}$$

On the other hand, assume now $r \in (0,\mathbf{r})$ and let us define the number

$$j_r = \left\lfloor \frac{\mathbf{r}}{r} \right\rfloor \ge 1,$$

where $\lfloor\cdot\rfloor$ denotes the integer part function. Moreover, note that $\lambda_j r < \sigma$ when $j \le j_r$, and then $\left|\mathcal{Y}_0\left(\lambda_j r\right)\right| \ge 1$ if, and only if, $j \le j_r$. Therefore,

$$\begin{aligned}\sum_{j\ge 1}\frac{1}{j^2}|\mathcal{Y}_0\left(\lambda_j r\right)| &= \sum_{j\ge 1}^{j_r}\frac{1}{j^2}|\mathcal{Y}_0\left(\lambda_j r\right)| + \sum_{j>j_r}\frac{1}{j^2}|\mathcal{Y}_0\left(\lambda_j r\right)| \\ &\le \sum_{j\ge 1}^{j_r}\frac{1}{j^2}|\mathcal{Y}_0\left(\lambda_j r\right)| + \sum_{j>j_r}\frac{1}{j^2} < \sum_{j\ge 1}^{j_r}\frac{1}{j^2}|\mathcal{Y}_0\left(\lambda_j r\right)| + \sum_{j\ge 1}\frac{1}{j^2}.\end{aligned}$$

Now we recall the inequality (26), developed in Appendix B:

$$|\mathcal{Y}_0(x)| < \frac{2}{\pi}\left[\left|\ln\left(\frac{|x|}{2}\right)\right| + \gamma\right] + \frac{2}{\pi}\exp\left(\frac{x^2}{4}\right), \quad x \in \mathbb{R}.$$

In particular,

[i] $\sigma \approx 0,22583743107335437789$.

$$\left|\mathcal{Y}_0\left(\lambda_j r\right)\right| < \frac{2}{\pi}\left[\left|\ln\left(\frac{r\pi}{T}\right)\right| + \ln j + \gamma + \exp\left(\lambda_j^2 \frac{r^2}{4}\right)\right], \quad r > 0,$$

and

$$\begin{aligned}\frac{\pi}{2}\sum_{j\geq 1}^{j_r}\frac{1}{j^2}|\mathcal{Y}_0\left(\lambda_j r\right)| &< \sum_{j\geq 1}^{j_r}\frac{1}{j^2}\left[\left|\ln\left(\frac{r\pi}{T}\right)\right| + \ln j + \gamma + \exp\left(\frac{\sigma^2}{4}\right)\right] \\ &< \left[\left|\ln\left(\frac{r\pi}{T}\right)\right| + \gamma + \exp\left(\frac{\sigma^2}{4}\right)\right]\sum_{j\geq 1}\frac{1}{j^2} + \sum_{j\geq 1}\frac{\ln j}{j^2}.\end{aligned}$$

The second sum is also convergent and can be computed explicitly. More concretely,

$$\sum_{j\geq 1}\frac{\ln j}{j^2} = \frac{\pi^2}{6}\left(12\ln\mathcal{A} - \gamma - \ln(2\pi)\right),$$

where $\mathcal{A}$[ii] denotes the Glaisher–Kinkelin constant. Using this,

$$\sum_{j\geq 1}^{j_r}\frac{1}{j^2}|\mathcal{Y}_0\left(\lambda_j r\right)| < \frac{\pi}{3}\left[\left|\ln\left(\frac{r}{T}\right)\right| + \exp\left(\frac{\sigma^2}{4}\right) + 12\ln\mathcal{A} - \ln 2\right],$$

and then, for any $(t,r) \in [0,T] \times (0,\mathbf{r})$, we conclude that

$$|a(t,r)| < \frac{T^2}{3\pi}\|\ddot{I}\|_\infty\left(|\ln r| + \widetilde{C}\right), \tag{17}$$

with $\widetilde{C} = |\ln T| + \exp\left(\frac{\sigma^2}{4}\right) + 12\ln\mathcal{A} - \ln 2 + \pi$. Finally, by (17),

$$\begin{aligned}\frac{2\pi}{\mu_0}A(t,r) &= -I_0\ln r - ka(t,r) \geq -I_0\ln r - k|a(t,r)|, \\ &\geq -I_0\ln r - k\frac{T^2}{3\pi}\|\ddot{I}\|_\infty\left(|\ln r| + \widetilde{C}\right)\end{aligned}$$

when $(t,r) \in [0,T] \times (0,\mathbf{r})$, and the second assertion follows taking limit when $r \to 0$.

3.2. Proof of Lemma 1

Due to the smoothness of $a(t,r)$, by the Leibniz's rule we obtain $\partial_r a(t,r)$ deriving under the integral sign

$$\partial_r a(t,r) = -r\int_0^\infty \frac{1}{\tau^2 + r^2}\left[\dot{I}[t,r,\tau] + \frac{I[t,r,\tau]}{\sqrt{\tau^2 + r^2}}\right]d\tau.$$

[ii] $\mathcal{A} \equiv 1,2824271291006226$.

Fix $m > 0$ arbitrarily and let us consider the following decomposition:

$$\int_0^\infty \frac{\dot{I}[t,r,\tau]}{\tau^2+r^2}d\tau = \int_0^m \frac{\dot{I}[t,r,\tau]}{\tau^2+r^2}d\tau + \int_m^\infty \frac{\dot{I}[t,r,\tau]}{\tau^2+r^2}d\tau.$$

Integrating by parts, we get that

$$\begin{aligned}
\int_m^\infty \frac{\dot{I}[t,r,\tau]}{\tau^2+r^2}d\tau &= -\frac{I[t,r,\tau]}{\tau\sqrt{\tau^2+r^2}}\Bigg|_m^\infty + \int_m^\infty I[t,r,\tau]\frac{d}{d\tau}\frac{1}{\tau\sqrt{\tau^2+r^2}}d\tau \\
&= \frac{I[t,r,m]}{m\sqrt{m^2+r^2}} - \int_m^\infty \frac{I[t,r,\tau]}{\tau^2\left(\tau^2+r^2\right)}\left[\sqrt{\tau^2+r^2}+\frac{\tau^2}{\sqrt{\tau^2+r^2}}\right]d\tau \\
&= \frac{I[t,r,m]}{m\sqrt{m^2+r^2}} - \int_m^\infty \frac{I[t,r,\tau]}{\tau^2\sqrt{\tau^2+r^2}}d\tau - \int_m^\infty \frac{I[t,r,\tau]}{(\tau^2+r^2)^{3/2}}d\tau.
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
\partial_r a(t,r) = r\Bigg[&-\int_0^m \frac{\dot{I}[t,r,\tau]}{\tau^2+r^2}d\tau - \int_0^m \frac{I[t,r,\tau]}{(\tau^2+r^2)^{3/2}} \\
&- \frac{I[t,r,m]}{m\sqrt{m^2+r^2}} + \int_m^\infty \frac{I[t,r,\tau]}{\tau^2\sqrt{\tau^2+r^2}}d\tau d\tau\Bigg].
\end{aligned}$$

With respect of the first line, both terms can be bounded because

$$\left|\int_0^m \frac{f(\tau)}{\tau^2+r^2}d\tau\right| \le \|f\|_\infty \int_0^m \frac{1}{\tau^2+r^2}d\tau = \frac{\|f\|_\infty}{r}\arctan(m/r),$$

for any $f \in L^\infty(\mathbb{R})$. On the other hand,

$$\left|\int_m^\infty \frac{I[t,r,\tau]}{\tau^2\sqrt{\tau^2+r^2}}d\tau\right| < \|I\|_\infty \int_m^\infty \frac{1}{\tau^3}d\tau = \frac{\|I\|_\infty}{2m^2}.$$

Putting together all this, and choosing $m = r$, it follows that

$$|\partial_r a(t,r)| < \|I\|_\infty \frac{2\left(1+\sqrt{2}\right)+\pi}{4r} + \|\dot{I}\|_\infty \frac{\pi}{4}, \quad \text{for all } (t,r) \in [0,T]\times\mathbb{R}^+.$$

Moreover, fix $\hat{r} > 0$ arbitrarily. Then, by (13),

$$|\partial_r a(t,r)| < \frac{1}{r}\frac{I_0}{k_{\hat{r}}}, \quad \text{for all } (t,r) \in [0,T] \times [0,\hat{r}].$$

From this, the inequalities of Lemma 1 are obtained easily because

$$\partial_r A(t,r) = -\frac{I_0}{r} - k\partial_r a(t,r) \le -\frac{I_0}{r} + k\,|\partial_r a(t,r)| < -\frac{I_0}{r}\left[1 - \frac{k}{k_{\hat{r}}}\right]$$

and

$$\partial_r A(t,r) \ge -\frac{I_0}{r} - k\,|\partial_r a(t,r)| > -\frac{I_0}{r}\left[1 + \frac{k}{k_{\hat{r}}}\right].$$

To conclude, let us focus again in the Bessel functions. By 9.2.1 and 9.2.2 in [1], for $|x|$ large we have that

$$\mathcal{J}_0(x) = \sqrt{\frac{2}{\pi x}}\cos(x - \pi/4) + \Theta(|x|^{-3/2}),$$
$$\mathcal{Y}_0(x) = \sqrt{\frac{2}{\pi x}}\sin(x - \pi/4) + \Theta(|x|^{-3/2}),$$

where $\Theta(f(x))$ means that the remaining term is of the order of a certain function $f(x)$. Then, applying this in (14), we obtain that, for r large enough,

$$a(t,r) = r^{-1/2}\mathcal{F}(t,r) + \Theta(r^{-3/2}),$$

with

$$\mathcal{F}(t,r) = -\sqrt{\frac{\pi}{2}}\sum_{j\ge 1}\left[\alpha_j\cos\left(\lambda_j t\right) + \beta_j\sin\left(\lambda_j t\right)\right]\sin\left(\lambda_j r - \pi/4\right)\frac{1}{\lambda_j}$$
$$+\sqrt{\frac{\pi}{2}}\sum_{j\ge 1}\left[\alpha_j\sin\left(\lambda_j t\right) - \beta_j\cos\left(\lambda_j t\right)\right]\cos\left(\lambda_j r - \pi/4\right)\frac{1}{\lambda_j}.$$

Reasoning like we did above, it is not difficult to see that $\mathcal{F}(t,r)$ is well defined in $\mathbb{R} \times \mathbb{R}^+$ and that satisfies the properties of the statement. Finally, as $a(t,r)$ is regular, for r large enough we can write

$$\frac{2\pi}{\mu_0}\partial_r A(t,r) = -\frac{I_0}{r} + k\partial_r a(t,r) = \frac{k}{\sqrt{r}}\mathcal{G}(t,r) + \Theta(r^{-1}),$$

with $\mathcal{G}(t,r) = \partial_r\mathcal{F}(t,r)$, and the lemma is proven.

4. Hamiltonian formulation and magnetostatic regime

In general electrodynamics situations, the Lorentz Force equation (1) admits a Hamiltonian formulation, according to [3]. In our case, as Section 2 stated, the dynamical system is (9), that is hamiltonian for

$$H(t,p,q)=\sqrt{1+|p-\vec{A}(t,q)|^2}.$$

Similarly to the non-relativistic approach [8], let us consider the cylindrical change of variable $q=(r\cos\theta, r\sin\theta, z)$ and its associated basis $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$, where

$$\mathbf{e}_r=(\cos\theta,\sin\theta,0)\,,\ \mathbf{e}_\theta=(-\sin\theta,\cos\theta,0)\,.$$

In these coordinates, the time-derivative is $\dot{q}=\dot{r}\mathbf{e}_r+r\dot{\theta}\mathbf{e}_\theta+\dot{z}\mathbf{e}_z$ and, after some basic computations (see [22] for a similar procedure), we are able to reduce system (9) to

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\dot{r}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-\dot{z}^2}}\right)-\dfrac{r\dot{\theta}^2}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-\dot{z}^2}}=\dot{z}\partial_r A(t,r), \\ \dfrac{d}{dt}\left(\dfrac{r^2\dot{\theta}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-\dot{z}^2}}\right)=0, \\ \dfrac{d}{dt}\left(\dfrac{\dot{z}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-\dot{z}^2}}+A(t,r)\right)=0. \end{cases}$$

Remark 2. In this coordinates system, the characteristic Special Relativity effect for the limitation of the particle velocities is the vector $(\dot{r}, r\dot{\theta}, \dot{z})$ strictly contained in the interior of the unitary ball.

Observe that, as in [8], every symmetry implies a conservation law, being consistent according to Noether's Theorem, while there is a dynamical equation for the radial component $r(t)$. Consequently, this induces the definition (10) of the corresponding relativistic momenta (p_r, L, p_z) that, after some algebra, can be reversed obtaining:

$$\dot{r}=\frac{p_r}{\sqrt{1+(p_z-A)^2+p_r^2+L^2r^{-2}}},\quad r^2\dot{\theta}=\frac{L}{\sqrt{1+(p_z-A)^2+p_r^2+L^2r^{-2}}},$$
$$\dot{z}=\frac{p_z-A(t,r)}{\sqrt{1+(p_z-A)^2+p_r^2+L^2r^{-2}}}.$$

From these identities, it is clear that the motion of a charged particle is described by its radial component, depending implicitly on L, p_z and k. In particular, this allows to reduce (9) to the planar system

$$\begin{cases} \dot{r} = \dfrac{p_r}{\sqrt{1 + (p_z - A)^2 + p_r^2 + L^2 r^{-2}}}, \\ \\ \dot{p}_r = \dfrac{L^2 r^{-3} + (p_z - A)\partial_r A}{\sqrt{1 + (p_z - A)^2 + p_r^2 + L^2 r^{-2}}}, \end{cases} \tag{18}$$

that is Hamiltonian for the energy function

$$\mathcal{H}(t, r, p_r) = \sqrt{1 + (p_z - A)^2 + p_r^2 + L^2 r^{-2}}. \tag{19}$$

In summary, by the wire symmetries, the dynamics described in (9) and in (18) are equivalent. Therefore, we are reducing the Lorentz Force equation (1) to a planar Hamiltonian system of one degree of freedom for the radial component of the solutions, which is a similarity with the Newtonian approach.

Remark 3. Magnetostatic dynamics.

As (19) is integrable when $k = 0$, the energy of the system is conserved in that case. Physically, this induces a magnetostatic regime, because of the vanishing of the electric field, and the particles cannot collide with the wire. It is not difficult to see that, for any pair $(L, p_z) \in \mathbb{R}^+ \times \mathbb{R}$, (18) admits a unique equilibrium $(\bar{r}, 0)$. We call it (L, p_z)-equilibrium and is given by the identity

$$\bar{r}^2 \left[p_z + \frac{\mu_0}{2\pi} I_0 \ln \bar{r} \right] = \frac{2\pi}{\mu_0} \frac{L^2}{I_0}. \tag{20}$$

Furthermore, this point is the same that in the non-relativistic dynamics (see the definition of admissible triplet in [8]). By the sign of (20), observe that the (L, p_z)-equilibria are close to the wire only for p_z large. Also,

$$\bar{r} \in \left(\exp\left(-p_z \frac{2\pi}{\mu_0 I_0} \right), 1 \right), \quad \text{when } p_z > 0,$$

while $\bar{r} \geq 1$ in the complementary case. However, regardless of the sign of p_z, a particle with radially constant motion is always accelerated in the current direction because

$$\dot{z} = H_0^{-1} \left(p_z + \frac{\mu_0}{2\pi} I_0 \ln \bar{r} \right) > 0.$$

Again, this is also a similarity with the non-relativistic approach, where the unique difference is the presence of the Hamiltonian constant H_0. In fact, as $H_0 > 1$, $\dot{z}$ is smaller than in the Newtonian case, which is something expected.

5. Existence of radially periodic solutions

In this section we apply the topological degree arguments developed in Appendix A to the planar Hamiltonian system (18) in order to prove Theorem 2. Before of that, *suitable a priori bounds* are obtained using the asymptotic estimations of $A(t, r)$ developed in Section 3. As Theorem 2 was formulated in terms of equation (9), the equivalent version for (18) reads:

Theorem 3. *Let I_0, L be positive, $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying (4) and $P_0 < \boldsymbol{P}(K_T)$. Take $\varepsilon > 0$ and $\bar{k} \in [0, K_{r_\varepsilon+T+\varepsilon}]$, with r_ε defined in (12). Then, there exists a positive constant $r_m < r_\varepsilon$, depending on $\bar{k}$ and $\boldsymbol{P}(\bar{k})$, such that for all $(k, p_z) \in [0, \bar{k}] \times \left[P_0, \boldsymbol{P}(\bar{k})\right]$, (18) admits at least one T-periodic solution with*

$$r_m < r(t) < r_\varepsilon, \quad \text{for every } t \in \mathbb{R}.$$

Moreover,

$$\lim_{\varepsilon\to 0} r_\varepsilon = +\infty, \quad \lim_{\varepsilon\to 0} K_{r_\varepsilon} = 0.$$

From here to the end, we will refer to $(r(t), p_r(t))$ as a T-periodic solution of (18), where L, k and p_z will be specified depending of the case. By Remark 3, we recall that $|\dot{r}(t)| < 1$ for all $t \in \mathbb{R}$, thus the oscillation of $r(t)$ is bounded by the period, i.e.,

$$\max_{t\in[0,T]} r(t) - \min_{t\in[0,T]} r(t) < T.$$

On the other hand, by (18), the periodicity of the solution implies the existence of $t_0 \in [0, T]$ such that

$$r^3(t_0)\left[A(t_0, r(t_0)) - p_z\right]\partial_r A(t_0, r(t_0)) = L^2. \tag{21}$$

We begin with the lower bound.

Lemma 2. *Let I_0, L be positive, $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying (4). Then, for any $\bar{k} \in [0, K_T]$, there exists a constant $r_m > 0$, depending on $\bar{k}$ and $\boldsymbol{P}(\bar{k})$, such that*

$$r_m < \min\left\{r(t) : t \in [0, T],\ (k, p_z) \in [0, \bar{k}] \times \left(-\infty, \boldsymbol{P}(\bar{k})\right]\right\}.$$

Proof. Let us fix $(k, p_z) \in [0, \bar{k}] \times \left(-\infty, \mathbf{P}(\bar{k})\right]$. Recalling that $T > \mathbf{r}$, with $\mathbf{r}$ defined in (15), using (16) we get

$$A(t, T) > -\frac{\mu_0}{2\pi}\left[I_0 \ln T + k\frac{T^2}{3}\|\ddot{I}\|_\infty\right] \geq \mathbf{P}(\bar{k}), \quad \text{for all } t \in \mathbb{R}.$$

So, in particular, $A(t_0, T) - p_z > 0$. Furthermore, Theorem 1 state that $A(t, r)$ is strictly decreasing when $r \leq T$, and, because of the signs in (21), $r(t_0) > T$ necessarily.

Finally, define the set

$$\mathcal{Z}_m(\bar{k}) = \{r \in \mathbb{R}^+ : \mathbf{P}(\bar{k}) = A(t, r),\ \text{for some } (t, k) \in [0, T] \times [0, \bar{k}]\},$$

that is closed because A is continuous in (t, r, k). Therefore, it has a minimum $m(\bar{k}) := \min \mathcal{Z}_m(\bar{k}) > T$, and

$$\min_{t\in[0,T]} r(t) > \max_{t\in[0,T]} r(t) - T > m(\bar{k}) - T := r_m > 0. \quad \blacksquare$$

The above result gives an explicit set for k and p_z such that the particles with radially periodic motion do not collide with the wire. This cannot be doing for the upper bound, at least using (21), due to the oscillations of $A(t,r)$ for any $k > 0$ when r is large. More precisely, by iv) in Theorem 1,

$$r^3 \left[A(t,r) - p_z\right] \partial_r A(t,r) = -p_z r^{5/2} \mathcal{G}(t,r) + \Theta(r^2), \quad r >> 1.$$

Therefore, for any $L > 0$, there exists a sequence $\{(t_n, r_n)\}_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} r_n = +\infty$, and

$$r_n^3 \left[A(t_n,r_n) - p_z\right] \partial_r A(t_n,r_n) = L^2, \quad \text{for all } n \in \mathbb{N}.$$

However, under suitable conditions, the control for the decreasing of $A(t,r)$, stated in *iii)*-Theorem 1, allows to find explicit radius where there are no periodic solutions of (18). To this aim, let us recall the identity (12):

$$r^2 \left[P_0 + \frac{\mu_0}{2\pi} \left(I_0 \ln r - K_{r+T+\varepsilon} \frac{T^2}{3} \|\ddot{I}\|_\infty \right) \right] = \frac{2\pi}{\mu_0} \frac{L^2}{I_0} \frac{C_1 + C_2(r+T+\varepsilon)}{C_2 \varepsilon}.$$

Lemma 3. *Let I_0, L be positive, $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying (4) and $P_0 < \boldsymbol{P}(K_T)$. Then, for any $\varepsilon > 0$ there exists a unique r_ε satisfying (12). Moreover,*

i) $\lim\limits_{\varepsilon\to 0} r_\varepsilon = +\infty$.
ii) $r_\varepsilon \notin \left\{ r(t) : t \in [0,T],\ (k,p_z) \in [0, K_{r_\varepsilon+T+\varepsilon}] \times [P_0, \boldsymbol{P}(K_T)] \right\}$.

Proof. Fix $\varepsilon > 0$ arbitrarily and let us write (12) as $f_\varepsilon(r) = g_\varepsilon(r)$. Firstly, it is not difficult to see that $f_\varepsilon(r)$ has a unique critical point $\widetilde{r}$, which is a global minimum. Furthermore,

$$\lim_{r\to 0^+} f_\varepsilon(r) = 0^- \text{ and } \lim_{r\to+\infty} f_\varepsilon(r) = +\infty.$$

Then, as $g_\varepsilon(r)$ is an increasing linear function with $g_\varepsilon(0) > \frac{2\pi}{\mu_0}\frac{L^2}{I_0}$, the intersection point r_ε is unique.

Regarding the properties, *i)* is trivial because g_ε tends to $+\infty$ when $\varepsilon \to 0$. On the other hand, as K_r is decreasing in r, by (11) we have that

$$-\mathbf{P}(K_T) > \frac{\mu_0}{2\pi} \left(I_0 \ln T + K_{r+T} \frac{T^2}{3} \|\ddot{I}\|_\infty \right),$$

for any $r > 0$. As consequence,

$$P_0 + \frac{\mu_0}{2\pi} \left(I_0 \ln T - K_{r+T} \frac{T^2}{3} \|\ddot{I}\|_\infty \right) < P_0 - \mathbf{P}(K_T) < 0,$$

for any $r > 0$ and then $f_\varepsilon(T) < 0$, from where it follows that $r_\varepsilon > T$. In particular, $r_\varepsilon > \mathbf{r}$ and, by (16),

$$|a(t,r)| < \frac{T^2}{3}\|\ddot{I}\|_\infty, \text{ for all } (t,r) \in [0,T] \times [r_\varepsilon, +\infty). \tag{22}$$

Now fix $(k, p_z) \in (0, K_{r_\varepsilon+T+\varepsilon}] \times [P_0, \mathbf{P}(K_T)]$. Firstly, let us see that r_ε cannot satisfy (21). To this aim, defining the function

$$F(t,r) = r^3 \left[-p_z + A(t,r)\right] \partial_r A(t,r),$$

we can write (21) as $F(t_0, r(t_0)) = L^2$. By (22) and (12), we have that

$$\begin{aligned} p_z - A(t, r_\varepsilon) &= p_z + \frac{\mu_0}{2\pi}\left[I_0 \ln r_\varepsilon + k a(t, r_\varepsilon)\right] \\ &> P_0 + \frac{\mu_0}{2\pi}\left[I_0 \ln r_\varepsilon - K_{r_\varepsilon+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right] > 0, \end{aligned}$$

for every $t \in \mathbb{R}$. On the other hand, as $k \le K_{r_\varepsilon+T+\varepsilon}$, then $\partial_r A(t, r_\varepsilon)$ is strictly negative and we can write

$$F(t, r_\varepsilon) = r_\varepsilon^3 \left[p_z - A(t, r_\varepsilon)\right] |\partial_r A(t, r_\varepsilon)| .$$

Using (8),

$$\begin{aligned} F(t, r_\varepsilon) &> \frac{I_0\mu_0}{2\pi} r_\varepsilon^2 \left(P_0 + \frac{\mu_0}{2\pi}\left[I_0 \ln r_\varepsilon - K_{r_\varepsilon+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right]\right)\left[1 - \frac{K_{r_\varepsilon+T+\varepsilon}}{K_{r_\varepsilon+T}}\right] \\ &= \frac{I_0\mu_0}{2\pi} r_\varepsilon^2 \left(P_0 + \frac{\mu_0}{2\pi}\left[I_0 \ln r_\varepsilon - K_{r_\varepsilon+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right]\right)\frac{C_2\varepsilon}{C_1 + C_2(r_\varepsilon + T + \varepsilon)}, \end{aligned}$$

where in the last equality we have used the definition (7) of K_r. Then, from (12) it follows that $F(t, r_\varepsilon) > L^2$ for all $t \in [0, T]$ and, consequently,

$$r_\varepsilon \notin \{r(t_0) : r(t)\ T\text{-periodic with } (k, p_z) \in (0, K_{r_\varepsilon+T+\varepsilon}] \times [P_0, \mathbf{P}(K_T)]\}.$$

However, if there exists a $t_1 \ne t_0$ such that $r(t_1) = r_\varepsilon$, condition (21) must be satisfied at some point $r \in (r_\varepsilon, r_\varepsilon + T)$. Reasoning for any r in this set as we just did above for r_ε, we obtain that

$$\begin{aligned} F(t, r) &> \frac{I_0\mu_0}{2\pi} r^2 \left(P_0 + \frac{\mu_0}{2\pi}\left[I_0 \ln r - K_{r_\varepsilon+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right]\right)\left[1 - \frac{K_{r_\varepsilon+T+\varepsilon}}{K_{r_\varepsilon+T}}\right] \\ &> \frac{I_0\mu_0}{2\pi} r_\varepsilon^2 \left(P_0 + \frac{\mu_0}{2\pi}\left[I_0 \ln r_\varepsilon - K_{r_\varepsilon+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right]\right)\left[1 - \frac{K_{r_\varepsilon+T+\varepsilon}}{K_{r_\varepsilon+T}}\right], \end{aligned}$$

and therefore $F(t, r) > L^2$, for all $(t, r) \in [0, T] \times [r_\varepsilon, r_\varepsilon + T]$, which proves *ii)*. ■

Corollary 1. *Let I_0, L be positive, $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying (4) and $P_0 < \boldsymbol{P}(K_T)$. Then,*

$$r_\varepsilon > \bar{r}, \quad \textit{for any } (\varepsilon, p_z) \in \mathbb{R}^+ \times [P_0, \boldsymbol{P}(K_T)],$$

where $\bar{r}$ is the corresponding (L, p_z)-equilibrium defined in (20).

Proof. Fix $\varepsilon > 0$, $p_z \in [P_0, \mathbf{P}(K_T)]$ arbitrarily and let us write again (12) as $f_\varepsilon(r) = g_\varepsilon(r)$. By (20),

$$f_\varepsilon(\bar{r}) < \bar{r}^2\left[p_z + \frac{\mu_0}{2\pi} I_0 \ln \bar{r}\right] = \frac{2\pi}{\mu_0}\frac{L^2}{I_0} < g_\varepsilon(0) < g_\varepsilon(r_\varepsilon) = f_\varepsilon(r_\varepsilon).$$

Then, as $f_\varepsilon(r)$ is increasing for all $r \geq r_\varepsilon$, the result is proven. ■

Finally, we prove Theorem 3.

Proof. Let us obtain a bound for $p_r(t)$. To this aim, fix $\varepsilon > 0$ and take $(\bar{k}, p_z) \in [0, K_{r_\varepsilon+T+\varepsilon}] \times [P_0, \mathbf{P}(K_T)]$. Because of Lemmas 2 and 3, we assume the existence of a T-periodic solution such that $r(t) \in \{r \in \mathbb{R} : r_m < r < r_\varepsilon\}$. Then, by periodicity, there exists a $\bar{t} \in [0, T]$ such that $p_r(\bar{t}) = 0$ and, integrating $\dot{p}_r$ in $[\bar{t}, t]$, with t arbitrary, we obtain the a priori bound:

$$\begin{aligned}
|p_r(t)| &= \left|\int_{\bar{t}}^{t} \frac{L^2 r^{-3}(s) + (p_z - A(s, r(s)))\partial_r A(s, r(s))}{\sqrt{1 + (p_z - A(s, r(s)))^2 + p_r^2 + L^2 r^{-2}(s)}} ds\right| \\
&< \int_{\bar{t}}^{t} L^2 r^{-3}(s) ds + \int_{\bar{t}}^{t} \frac{|p_z - A(s, r(s))|\,|\partial_r A(s, r(s))|}{\sqrt{1 + (p_z - A(s, r(s)))^2 + p_r^2 + L^2 r^{-2}(s)}} ds \\
&< T\frac{L^2}{r_m^3} + \int_{\bar{t}}^{t} |\partial_r A(s, r(s))|\, ds < T\frac{L^2}{r_m^3} + C(\bar{k}) := P(\bar{k}) < \infty,
\end{aligned}$$

with $C(\bar{k}) = \max\{|\partial_r A(t, r)|;\ (t, r) \in [0, T] \times [r_m, r_\varepsilon]\}$. Due to this, we define the sets:

$$\begin{aligned}
\Omega_{\varepsilon,\bar{k}} &= \{x \in \mathbb{R}^2 : r_m < x_1 < r_\varepsilon,\ |x_2| < P(\bar{k})\}; \\
\Omega &= \{x \in X :\ \mathrm{Im}(x) \subset \Omega_{\varepsilon,\bar{k}}\}.
\end{aligned}$$

On the other hand, take $k = \lambda\bar{k}$, with $\lambda \in [0, 1]$, and let us consider the corresponding potential $A_\lambda(t, r)$ and hamiltonian function $\mathcal{H}_\lambda(t, r, p_r)$ giving by (5) and (19) respectively. By (6), $\mathcal{H}_\lambda(t, r, p_r)$ is T-periodic and regular in its domain for any $\lambda \in [0, 1]$. Furthermore, as $\mathcal{H}_0$ is autonomous, hypotheses i)-ii) of Appendix A are verified and the homotopic problem associated to (18) satisfies (23). Therefore, we conclude applying Corollary 2 in Appendix A to $f_0(r, p_r) = \mathcal{H}_0^2(r, p_r)$, for which it only remains to study its critical points in $\Omega_{\bar{k}}$:

$$\partial_r \mathcal{H}_0^2 = \left(p_z + \frac{I_0\mu_0}{2\pi}\ln r\right)\frac{I_0\mu_0}{\pi}\frac{1}{r} - 2\frac{L^2}{r^3};\ \partial_{p_r}\mathcal{H}_0^2 = 2p_r.$$

After basic computations, it follows that the equilibrium $(\bar{r}, 0)$, which belongs to the interior of $\Omega_{\bar{k}}$ by Corollary 1, is the unique critical point of $\mathcal{H}_0^2(r, p_r)$. Concerning the derivatives of second order, clearly

$$\partial^2_{r\,p_r}\mathcal{H}_0^2(r, p_r) = \partial^2_{p_r\,r}\mathcal{H}_0^2(r, p_r) = 0, \quad \text{and} \quad \partial^2_{p_r p_r}\mathcal{H}_0^2(r, p_r) = 2.$$

Moreover,

$$\partial^2_{rr}\mathcal{H}^2_0(r,p_r)=\frac{I_0^2\mu_0^2}{2\pi^2}\frac{1}{r^2}-\left(p_z+\frac{I_0\mu_0}{2\pi}\ln r\right)\frac{I_0\mu_0}{\pi}\frac{1}{r^2}+6\frac{L^2}{r^4}$$
$$=-\frac{\partial_r\mathcal{H}^2_0}{r}(r,p_r)+\frac{1}{r^2}\left(\frac{I_0^2\mu_0^2}{2\pi^2}+4\frac{L^2}{r^2}\right).$$

Finally, as $\partial_r\mathcal{H}^2_0(\bar r,0)=0$,

$$\left|\mathrm{Hess}\mathcal{H}^2_0(\bar r,0)\right|=\frac{2}{\bar r^2}\left(\frac{I_0^2\mu_0^2}{2\pi^2}+4\frac{L^2}{\bar r^2}\right)>0,$$

and the result is proven. ■

6. Conclusions

In this paper we study the relativistic dynamics induced by a periodically time dependent current along an infinitely long and infinitely thin straight wire. This situation is ruled by the Lorentz Force Equation (9), where the electromagnetic field is given by (2) and (5) and solves Maxwell (3) uniquely for the corresponding distributional current $\vec{J}$ as data. We extend the knowledge about (5), describing explicitly its asymptotic behaviour with respect of the perturbation parameter k, which has mathematical interest by itself.

Physically, there exist similarities with the Newtonian case studied in [8], but the mathematical procedure is different. The symmetries of the current imply that the corresponding linear and angular relativistic momenta are conserved quantities in our model. Therefore, (9) is reduced to the planar hamiltonian system with one degree of freedom (18), that describes the dynamic of the radial component. There we prove the existence of periodic solutions for an explicit interval of k, which depends on the values of the current and the conserved momenta. The proof is based on a topological degree argument, for which we are able to obtain *suitables a priori bounds* for the solutions. In other words, we find bounded invariant sets for the periodic dynamic, where we can compute the topological degree of (18) and prove the existence of radially periodic solutions of (9).

To conclude, let us briefly comment on some open questions arising from this research. First, it is rather straightforward to prove a qualitative (perturbative) result for the stability of the solutions by a similar reasoning like in [8]. However, we consider more interesting to find a quantitative result (an explicit range of k for stability), which remains open. On the other hand, it is natural to ask if there exist collisions with the wire when the regime is no longer magnetostatic, i.e., for $k>0$. Concerning this, observe that the second equation in (18) shows a repulsive force close to the wire for particles with not null angular momentum L. Nevertheless, we do not know if this fact is enough to avoid collisions. Finally, it would be interesting to approach the Newtonian dynamics making the light speed c tending to $+\infty$, and then to see if these periodic solutions survive in the limit. In our method, the obtained a-priori bounds go to infinity as $c\to+\infty$, so the question is far from trivial.

Data availability

No data was used for the research described in the article.

Appendix A. Global continuation of equilibria in autonomous Hamiltonian systems

Here we particularize Mawhin's coincidence degree theory for periodic Hamiltonian systems that are homotopic to an autonomous one. Concretely, consider a system of the form

$$S\dot{x}(t) = \nabla\mathcal{H}_\lambda(t, x(t)),\ x(0) = x(T),\ \lambda \in [0, 1], \tag{23}$$

where $S = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the standard symplectic matrix in $\mathbb{R}^{2N}$, $\mathcal{H}_\lambda : \mathbb{R} \times \mathcal{O} \to \mathbb{R}$ is T-periodic in t, $\mathcal{O} \subseteq \mathbb{R}^{2N}$ is an open set, and the gradient operator is computed in x. Furthermore, $\mathcal{H}_\lambda$ is a homotopy in λ verifying that:

i) $\mathcal{H}_1(t, x)$ is such that its gradient is a Carathéodory function.
ii) $\mathcal{H}_0(x)$ is autonomous and of class $\mathcal{C}^2$ in $\mathcal{O}$.

The construction of the topological degree for this kind of systems is a particular case of the general theory for nonlinear perturbations of Fredholm operators, see Section 2 in [16]. To this aim, let X be the Banach space of T-periodic functions in $\mathcal{C}\left(\mathbb{R}; \mathbb{R}^{2N}\right)$ with the uniform norm, and consider the metric space

$$X_{\mathcal{O}} = \{x \in X : \mathrm{Im}(x) \subset \mathcal{O}\}.$$

As usual, the subspace of constant functions in X is naturally identified with $\mathbb{R}^{2N}$. In addition, given any subset Ω of X, we denote by $\mathbf{cl}_{\mathbb{R}^{2N}}\Omega$ to the closure of Ω in $\mathbb{R}^{2N}$, and by $\partial_X\Omega$ to the boundary of the set in X. Moreover, let $Z_\Omega(\mathcal{H}_0)$ be the set of critical points of $\mathcal{H}_0$ in $\Omega \cap \mathbb{R}^{2N}$. Then, denoting by $\mathbf{sg}$ to the sign function and by $|\mathrm{Hess} f(x)|$ to the determinant of the Hessian matrix of $f(x)$, we present the next result, which we prove at the end of this appendix.

Theorem 4. *Let $\Omega \subset X_{\mathcal{O}}$ be open, bounded and such that:*

a) There is no $x \in \partial_X\Omega$ solving (23), for any λ.
b) All the critical points of $\mathcal{H}_0$ in $\mathbf{cl}_{\mathbb{R}^{2N}}\Omega$ are non-degenerate.
c) $\sum_{x \in Z_\Omega(\mathcal{H}_0)} \mathbf{sg}\,|Hess\mathcal{H}(x)| \neq 0$.

Then, (23) admits at least one T-periodic solution $x_\lambda(t)$, for any $\lambda \in [0, 1]$.

Remark 4. Observe that the last two assertions hold when the set $Z_\Omega(\mathcal{H}_0)$ has an odd number of points and they are all non-degenerate. Moreover, *b)* is equivalent to assume that 0 is a regular value for $\nabla\mathcal{H}_0$ in $\mathbf{cl}_{\mathbb{R}^{2N}}\Omega$, which is a necessary condition to define the Brouwer degree of a function of class $\mathcal{C}^1$ in $\Omega \cap \mathbb{R}^{2N}$.

On the other hand, if the homotopy has the form

$$\mathcal{H}_\lambda(, t, x) = f_\lambda(t, x)^\alpha, \text{ with } \alpha \in \mathbb{R} \setminus \{0\},$$

Theorem 4 is adapted as follows.

Corollary 2. *Let $\Omega \subset X_{\mathcal{O}}$ be open, bounded and such that there is no $x \in \partial_X \Omega$ solving (23) for any λ. Moreover, let $f_0(t,x)$ be a function without critical points at level 0 in $\mathbf{cl}_{\mathbb{R}^{2N}} \Omega$ and that verifies b and c in Theorem 4. Then, (23) admits at least one T-periodic solution $x_\lambda(t)$, for any $\lambda \in [0,1]$.*

Proof. The proof relies on the fact that the hypotheses about $f_0(x)$ are equivalent to assume that $\mathcal{H}_0(x) = f_0^\alpha(x)$ verifies b and c. To this aim, it is enough to compute the Hessian of $\mathcal{H}_0(x)$ and make some observations. Firstly, computing the partial derivatives of first order, we have that

$$\partial_{x_i} \mathcal{H}_0 = \partial_{x_i} f_0^\alpha = \alpha f_0^{\alpha-1} \partial_{x_i} f_0 = \alpha \mathcal{H}_0^{1-\alpha^{-1}} \partial_{x_i} f_0.$$

Then, if there is no $x \in \mathcal{Z}_\Omega(f_0)$ such that $f_0(x) = 0$, the set of critical points $\mathcal{Z}_\Omega(\cdot)$ coincides for $f_0(x)$ and $\mathcal{H}_0(x)$. On the other hand,

$$\partial^2_{x_j x_i} \mathcal{H}_0 = \alpha(\alpha-1)\mathcal{H}_0^{1-2\alpha^{-1}} \partial_{x_j} f_0 \partial_{x_i} f_0 + \alpha \mathcal{H}_0^{1-\alpha^{-1}} \partial^2_{x_j x_i} f_0,$$

and

$$\text{Hess}\mathcal{H}_0(x) = \alpha \mathcal{H}_0^{1-\alpha^{-1}}(x) \text{Hess} f_0(x), \quad \text{for any } x \in Z_\Omega(f_0).$$

Therefore,

$$|\text{Hess}\mathcal{H}_0(x)| = 0 \Leftrightarrow |\text{Hess} f_0(x)| = 0, \quad \text{for any } x \in Z_\Omega(f_0),$$

and Corollary 2 follows directly from this identity. ■

To conclude, we prove Theorem 4.

Proof. Firstly, some definitions are needed in order to rewrite (23) in an abstract form. To this aim, let L be the derivative operator, that is Fredholm and of index 0, and is defined in the subspace $D(L) = X \cap \mathcal{C}^1(\mathbb{R}; \mathbb{R}^{2N})$. We also introduce the mean value projector in X as Q, i.e.

$$Qx = \frac{1}{T} \int_0^T x(t)dt,$$

which verifies that $L + Q : D(L) \to X$ is a bijection. To calculate $(L+Q)^{-1}$, let $\mathcal{K}$ be the inverse of $L|_{D(L)\cap \text{Ker} Q}$, defined in the subspace of X of functions with null mean value. Then, it is standard to see that $(L+Q)^{-1} = Q + \mathcal{K}(I - Q)$, where I is the identity projector.

On the other hand, the Nemitsky operators $N_\lambda : X_{\mathcal{O}} \to X$ associated to (23) are the symplectic orthogonal gradients of $\mathcal{H}_\lambda(t,x)$, i.e.

$$N_\lambda x = -S\nabla \mathcal{H}_\lambda(t,x), \quad \forall x \in X_{\mathcal{O}}.$$

For any λ, the good properties of $\mathcal{H}_\lambda$ and the Arzelà-Ascoli Theorem imply the L-compactness of N_λ and then (23) can be reformulated as the fixed point problem:

$$x = \mathcal{F}_\lambda x := Q(I + N_\lambda)x + \mathcal{K}(I - Q)N_\lambda x, \quad x \in D(L) \cap X_{\mathcal{O}}.$$

Note that $\mathcal{F}_\lambda : X_{\mathcal{O}} \to X$ is completely continuous by the same arguments. Therefore, for any λ and any open and bounded subset Ω in $X_{\mathcal{O}}$ such that $0 \notin (I - \mathcal{F}_\lambda)\,[D(L) \cap \partial_X \Omega]$, the coincidence degree of $L - N_\lambda$ in $D(L) \cap \Omega$ is well defined as the Leray-Schauder degree of $I - \mathcal{F}_\lambda$:

$$d\,(L - N_\lambda, D(L) \cap \Omega) := d\,(I - \mathcal{F}_\lambda, D(L) \cap \Omega)\,. \tag{24}$$

Furthermore, if Ω is such that $0 \notin \cup_{\lambda \in [0,1]}(I - \mathcal{F}_\lambda)\,[D(L) \cap \partial_X \Omega]$, (24) is well defined for all λ. Then, as the degree is invariant by homotopy, we have that

$$d\,(L - N_\lambda, D(L) \cap \Omega) = d\,(L - N_0, D(L) \cap \Omega) = d_B\left(g, \mathbb{R}^{2N} \cap \Omega\right), \tag{25}$$

where the right term denotes the $2N$-dimensional Brouwer degree of $g(x) = -S\nabla\mathcal{H}_0(x)$. Moreover, the last equality follows from [6, Theorem 1].
Assume now that 0 is a regular value of g, i.e. $\mathcal{H}_0$ satisfies b). Then (25) is easily computed by classical properties of Brouwer degree:

$$d_B\left(g, \mathbb{R}^{2N} \cap \Omega\right) = \sum_{x \in \mathcal{Z}_\Omega(\mathcal{H}_0)} \mathbf{sg}|\nabla g(x)| = -\sum_{x \in \mathcal{Z}_\Omega(\mathcal{H}_0)} \mathbf{sg}|\mathrm{Hess}\mathcal{H}_0(x)|.$$

To conclude, if this sum is not null, $0 \in (L - N_\lambda)\,[D(L) \cap \Omega]$ for all λ and the theorem is proven. ■

Appendix B. Auxiliary functions

As some of the bounds obtained in Section 3 are computed using properties of the Bessel functions, we consider necessary to include this appendix just to comment some basic aspects of them. In particular, we need to justify the inequality

$$|\mathcal{Y}_0(z)| \leq \frac{2}{\pi}\left[\left|\ln\left(\frac{z}{2}\right)\right| + \gamma\right] + \frac{2}{\pi}\exp(z^2/4), \quad z \geq 0. \tag{26}$$

Before to define them, let us recall some brief notions about the Gamma function $\Gamma(x)$, which we use in our development.

Gamma function

Defined as the improper integral

$$\Gamma(x) = \int_0^{+\infty} s^{x-1}e^{-s}ds, \quad x > 0.$$

It is clear that Γ diverges in the origin, moreover

$$\Gamma(m) = (m-1)!, \quad m \in \mathbb{N}.$$

On the other hand, we denote by ψ_0 the Digamma function, defined as the logarithmic derivative of Γ, i.e. $\frac{\Gamma'(x)}{\Gamma(x)} = \psi_0(x)$. About this, we just remark the following property

$$\psi_0(x+1) = \psi_0(x) + \frac{1}{x}, \quad x > 0.$$

In particular, this implies:

$$\psi_0(m+1) = \psi_0(1) + \sum_{n=1}^{m} \frac{1}{n} = -\gamma + \sum_{n=1}^{m} \frac{1}{n}, \quad m \in \mathbb{N}, \tag{27}$$

where γ is commonly known as the Euler-Masheroni constant.

Bessel functions

Given $\nu \in \mathbb{R}$ and $z \geq 0$, the Bessel function of first kind, order ν and argument z, is defined as

$$\mathcal{J}_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m+\nu}}{m!\Gamma(m+\nu+1)}.$$

This expression is a linear combination of solutions for the Bessel equation

$$z^2 \frac{d^2 y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)y = 0,$$

which is also a solution by linearity. In particular, when $\nu = 0$ we get

$$\mathcal{J}_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m}}{(m!)^2}. \tag{28}$$

One of its basic properties is that $\mathcal{J}_\nu(z)$ is analytic for any $\nu \in \mathbb{R}$ and any $z \neq 0$. Therefore, the Bessel function of second kind and natural order $n \in \mathbb{N}$ is defined as the derivative

$$\mathcal{Y}_n(z) = \frac{1}{\pi}\left[\frac{\partial J_\nu}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}}{\partial \nu}\right]_{\nu=n}, \quad n \in \mathbb{N}.$$

It is not difficult to see that $\frac{\partial J_\nu}{\partial \nu}(z)|_{\nu=0} = -\frac{\partial J_{-\nu}}{\partial \nu}(z)|_{\nu=0}$. By this,

$$\mathcal{Y}_0(z) = \frac{2}{\pi}\frac{\partial J_\nu}{\partial \nu}(z)\Big|_{\nu=0} = \frac{2}{\pi}\left[\frac{\partial}{\partial \nu}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+\nu}}{m!\Gamma(m+\nu+1)}\right]_{\nu=0}$$

$$= \frac{2}{\pi}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+\nu}}{m!\Gamma(m+\nu+1)}\ln\left(\frac{z}{2}\right) - \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+\nu}}{m!\Gamma(m+\nu+1)}\psi_0(m+\nu+1)\right]_{\nu=0}.$$

Then, using (28) and (27), we obtain

$$\mathcal{Y}_0(z) = \frac{2}{\pi}\left[\ln\left(\frac{z}{2}\right)+\gamma\right]\mathcal{J}_0(z) - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m}}{(m!)^2}\left[1+\ldots+\frac{1}{m}\right].$$

Finally, as $|\mathcal{J}_0(z)| \le 1$ for all z, we get (26):

$$|\mathcal{Y}_0(z)| \le \frac{2}{\pi}\left[\left|\ln\left(\frac{z}{2}\right)\right|+\gamma\right] + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(z^2/4)^m}{m!}$$

$$= \frac{2}{\pi}\left[\left|\ln\left(\frac{z}{2}\right)\right|+\gamma\right] + \frac{2}{\pi}\exp(z^2/4).$$

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, vol. 55, US Government Printing Office, 1948.

[2] J. Aguirre, A. Luque, D. Peralta-Salas, Motion of charged particles in magnetic fields created by symmetric configurations of wires, Physica D 239 (2010) 654–674.

[3] D. Arcoya, C. Bereanu, P.J. Torres, Critical point theory for the Lorentz force equation, Arch. Ration. Mech. Anal. 232 (2019) 1685–1724.

[4] D. Arcoya, C. Bereanu, P.J. Torres, Lusternik–Schnirelmann theory for the action integral of the Lorentz force equation, Calc. Var. Partial Differ. Equ. 59 (2) (2020) 1–32.

[5] C. Bereanu, J. Mawhin, Boundary value problems for some nonlinear systems with singular ϕ-Laplacian, J. Fixed Point Theory Appl. 4 (2008) 57–75.

[6] A. Capietto, J. Mawhin, F. Zanolin, Continuation theorems for periodic perturbations of autonomous system, Trans. Am. Math. Soc. 329 (1) (1992) 41–72.

[7] M. Garzón, P.J. Torres, Periodic solutions for the Lorentz force equation with singular potentials, Nonlinear Anal., Real World Appl. 56 (2020) 103162.

[8] M. Garzón, S. Marò, Motions of a charged particle in the electromagnetic field induced by a non-stationary current, Phys. D, Nonlinear Phenom. 424 (2021) 132945.

[9] F.G. Gascón, D. Peralta-Salas, Motion of a charge in the magnetic field created by wires: impossibility of reaching the wires, Phys. Lett. A 333 (2004) 72–78.

[10] F.G. Gascón, D. Peralta-Salas, Some properties of the magnetic fields generated by symmetric configurations of wires, Physica D 206 (2005) 109–120.

[11] D.J. Griffiths, Introduction to Electrodynamics, Prentice Hall, New Jersey, 1962.

[12] J.D. Jackson, Classical Electrodynamics, third edition, John Wiley & Sons, 1999.

[13] C. King, A. Leśniewski, Periodic motion of atoms near a charged wire, Lett. Math. Phys. 39 (1997) 367–378.

[14] L.D. Landau, E.M. Lifschitz, The Classical Theory of Fields, Vol. 2, fourth edition, Butterworth-Heinemann, 1980.

[15] J. Lei, M. Zhang, Twist property of periodic motion of an atom near a charged wire, Lett. Math. Phys. 60 (2002) 9–17.

[16] J. Mawhin, Topological degree and boundary value problems for nonlinear differential equations, in: Topological Methods for Ordinary Differential Equations, Springer, Berlin, Heidelberg, 1993, pp. 74–142.
[17] R. Ortega, Periodic solutions of a Newtonian equation: stability by the third approximation, J. Differ. Equ. 128 (1996) 491–518.
[18] M. Planck, Das Prinzip der Relativität und die Grundgleichungen der Mechanik, Verh. Dtsch. Phys. Ges. 4 (1906) 136–141.
[19] H. Poincaré, Sur la dynamique de l'électron, Rend. Circ. Mat. Palermo 21 (1906) 129–176.
[20] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, Ann. Inst. Henri Poincaré, Anal. Non Linéaire (C) 3 (1986) 77–109.
[21] P.J. Torres, M. Zhang, Twist periodic solutions of repulsive singular equations, Nonlinear Anal. 56 (2004) 591–599.
[22] P.J. Torres, A.J. Ureña, M. Zamora, Periodic and quasi-periodic motions of a relativistic particle under a central force field, Bull. Lond. Math. Soc. 45 (2013) 140–152.