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# Networked distributed fusion estimation under uncertain outputs with random transmission delays, packet losses and multi-packet processing

R. Caballero-Águila[a,*], A. Hermoso-Carazo[b], J. Linares-Pérez[b]

[a] Departamento de Estadística e I. O., Universidad de Jaén, Paraje Las Lagunillas s/n, Jaén, 23071 Spain
[b] Departamento de Estadística e I.O., Universidad de Granada, Campus Fuentenueva s/n, Granada 18071, Spain

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## Abstract

This paper investigates the distributed fusion estimation problem for networked systems whose mul- tisensor measured outputs involve uncertainties modelled by random parameter matrices. Each sensor transmits its measured outputs to a local processor over different communication channels and random failures –one-step delays and packet dropouts–are assumed to occur during the transmission. White sequences of Bernoulli random variables with different probabilities are introduced to describe the ob- servations that are used to update the estimators at each sampling time. Due to the transmission failures, each local processor may receive either one or two data packets, or even nothing and, when the current measurement does not arrive on time, its predictor is used in the design of the estimators to compensate the lack of updated information. By using an innovation approach, local least-squares linear estimators (filter and fixed-point smoother) are obtained at the individual local processors, without requiring the signal evolution model. From these local estimators, distributed fusion filtering and smoothing estimators weighted by matrices are obtained in a unified way, by applying the least-squares criterion. A simula- tion study is presented to examine the performance of the estimators and the influence that both sensor uncertainties and transmission failures have on the estimation accuracy.

## 1. Introduction

Data fusion techniques are crucial to address the signal estimation problem in multi-sensor network systems (see e.g. [1], [2], [3]). Actually, by combining the information (sensor outputs or local estimators) from multiple sensors, more meaningful and precise signal estimators, with better performance than each local estimator, are obtained. Depending on the way of processing raw data, the centralized and distributed fusion architectures are the most common information fusion techniques. In the centralized fusion method, the raw data from multiple sensors are directly sent to a fusion center, where the signal estimation is performed. Under the distributed fusion method, first local signal estimators are obtained and, afterwards, these local estimators are sent to a fusion center to be combined by using some optimal or suboptimal fusion criterion. Although the centralized fusion scheme is theoretically optimal, the processing of all raw data at a single fusion center can be either ineffective or unfeasible due to communication overload, especially when the number of sensors is large. In contrast, despite not being optimal, the distributed architecture reduces the computational burden, it is more suitable for fault detection and isolation, and more robust in the presence of random transmission failures, due to its parallel structure. For this reason, distributed fusion estimation has received significant attention over the last few decades and it has been successfully applied in many interesting fields. The review papers [4] and [5] can be examined to get an overall view on the evolution of the distributed fusion estimation problem for multi-sensor network systems.

The classical estimation algorithms assume that the model parameters are precisely known and the information is transmitted over perfect connections. However, these assumptions does not typically hold in a multi-sensor network environment, where random perturbations usually affect both the measurement devices and the communication resources.

On the one hand, some random phenomena (e.g. multiplicative noise uncertainties, missing and fading measurements or sensor gain degradation) usually lead to packet errors in the measured outputs and may potentially deteriorate the performance of the estimators. Hence, the traditional estimation problems must face new challenges to deal with these random phenomena (see e.g. [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], and references therein). A unified framework to model the random disturbances in the output measurements is provided by the use of random measurement matrices.

For this reason, in the last years, the estimation problem in systems with random parameter matrices has become an interesting research topic among the scientific community (see e.g. [16], [17], [18], [19], [20], [21], [22], [23], [24], and references therein).

On the other hand, the communication resources may be affected by random phenomena due to different causes (e.g. imperfect communication channels, network congestion or random failures in the transmission mechanism). These uncertainties yield random failures when the sensors transmit their measurements through the network. The main transmission uncertainties are random delays and packet dropouts, which can clearly deteriorate the performance of the estimators. For this reason, a wide variety of new fusion estimation algorithms have been designed to incorporate the effects of these transmission random failures (see e.g. [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], and references therein).

In network systems subject to transmission losses, a hot issue that arises in the estimation problem is how to compensate the packet dropouts. The most common compensation procedures are the zero-input and the hold-input mechanisms, where either nothing or the latest successfully transmitted measurement is used in the estimation, respectively, if the current data is lost. Recently, a more general compensation framework, including the zero-input and hold-input procedures as special cases, is proposed in [38]. A different compensation mechanism to deal with losses, developed in [39], has been recently considered in some investigations (see e.g. [40], [41], and references therein). In this new approach, the observation predictor (i.e., the estimator of the lost measurement based on the information received previously) is used as compensator; this compensation technique improves significantly the estimator performance since, in case of loss, all the previous measurements successfully received are considered, instead of using only the last one.

In relation with the transmission packet dropouts, in order to avoid losses as far as possible, [25] and [26] propose transmitting each packet several times. However, since multiple transmissions may cause network congestion problems that must be also avoided, [27], [28] and, more recently, [35] suggest transmitting the packet just once at every sampling time. The aforementioned papers also assume that each packet is either received on time, one-step delayed or lost; hence, only one packet or no packet is used to update the estimator at each moment. However, in practice, due to one or multiple random delays, multiple packets may arrive at each sampling time, so the estimation performance can be improved by processing all the received packets, instead of only one (see e.g. [42], [43], [44], [45], and references therein). More specifically, in [42] and [43] two packets may arrive at each sampling time, in which case both are used to update the estimation; hence, the algorithms in these papers have better performance than those where one packet is received at most. In [42] the last available measurement is used as compensation when no packet is received, whereas in [43] the observation predictor is used for compensation as long as the current measurement does not arrive on time owing to delays or losses. Compared with [42], the computational cost is reduced in [43] and more accurate estimators are obtained. In [44], the results in [42] are extended to multi-sensor systems with multiplicative noise uncertainties in the state and measurement matrices, and the distributed fusion filtering problem is addressed under these assumptions. Finally, more general results including multi-step delays can be found in [45].

Motivated by the above discussion, this paper considers the distributed fusion estimation problem in multi-sensor network systems with measurements perturbed by random parameter matrices subject to random transmission failures. More precisely, the key ideas of the proposed observation model and distributed fusion estimators are summarized as follows:

1. The current observation model considers random parameter matrices in the measured outputs, which allows us to deal with a wide variety of real situations, where the measured outputs present uncertainties that cannot be described only by additive disturbances; thus, a unified framework is provided to manage different simultaneous network-induced phenomena.

2. Besides the uncertainties in the measured outputs, random one-step delays and packet dropouts with different rates at each sensor are considered during transmission. As in [43], in order to avoid congestion problems in the network, at each sampling time the packet is transmitted just once, so either one packet, two packets or no packet will reach the local processor.

3. Also, as in [43], to compensate the non-punctual arrival of a packet, such packet is replaced by its estimator based on the information received previously, thus providing better estimations than the algorithms in which the hold-input compensation mechanism is used.

4. The distributed fusion estimation problem is addressed using covariance information, without requiring full knowledge of the state-space model generating the signal process, thus providing a general frame to deal

with different kinds of signals. Actually, the proposed algorithms are also applicable to the conventional formulation using the state-space model, even in the presence of state-dependent multiplicative noise.

5. The innovation approach is used to obtain local estimation algorithms, which are recursive and computationally simple; moreover, the proposed algorithms (filtering and fixed-point smoothing) do not use the state augmentation technique, thus reducing the computational cost in comparison with most existing estimation algorithms dealing with random delays and packet dropouts.

6. Distributed fusion estimators weighted by matrices are obtained from the local estimators by applying the least-squares criterion, for which the cross-correlation matrices between any two local estimators need to be previously calculated.

The covariance-based distributed fusion problem for networked systems whose sensor measured outputs are perturbed by random parameter matrices and suffer random transmission delays and packet dropouts is addressed. It is assumed that, at each instant of time, the local processor may receive either one packet, two packets, or nothing; when the current measurement does not arrive on time, its predictor is used as compensator. The major contributions and novelties of this paper are highlighted as follows:

($i$) Unlike previous authors' papers concerning random measurement matrices and random transmission delays and losses, where only one packet is processed to update the estimator at each moment, this is the first time that covariance-based estimation algorithms are obtained under the assumption that either one packet, two packets, or nothing may arrive to the local processors at each sampling time. As a consequence, it is expected that the current estimators outperform those in [40] since they use more information for estimation update (see Figure 7).

($ii$) As compared with some authors' preceding papers with packet dropouts (e.g. [20]), where the last measurement successfully received is used to compensate the data packets from the different sensors that do not reach the local processors, in the current work, inspired by [43], the observation predictor is used as compensation when the current measurement does not arrive on time but, in contrast, multi-sensor information is considered in the current paper.

(*iii*) In contrast to [42], [43] and [44], where the state augmentation technique is used, great efforts are devoted to obtain local estimators for each sensor without making use of augmented systems, thus reducing the computational cost.

The remaining sections of the paper are organized as follows. Section 2 presents the assumptions on the signal process, as well as the models of the multi-sensor measured outputs with random parameter matrices and the measurements received by the local processors with random delays and packet losses. Section 3 provides the main results: the derivation of the local least-squares linear filter and fixed-point smoother is presented in Section 3.1, and the distributed fusion estimators are obtained in Section 3.2 as a matrix-weighted linear combination of the local estimators. A simulation study is presented in Section 4 to show the performance of the proposed estimators, and some concluding remarks are drawn in Section 5.

**Notation:** The notation throughout the paper is standard. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the $n$-dimensional Euclidean space and the set of all $m \times n$ real matrices, respectively. For a matrix $A$, $A^T$ and $A^{-1}$ denote its transpose and inverse, respectively. $I_{n\times n}$ and $0_{n\times n}$ denote the $n \times n$ identity matrix and zero matrix, respectively. If the dimensions of a vector or a matrix are not explicitly stated, they are assumed to be compatible with algebraic operations. For any function $G_{k,s}$, depending on the time instants $k$ and $s$, we will write $G_k = G_{k,k}$ for simplicity; analogously, $F^{(i)} = F^{(ii)}$ will be written for any function $F^{(ij)}$, depending on the sensors $i$ and $j$. Finally, $\delta_{k,s}$ denotes the Kronecker delta function.

## 2. Problem formulation and observation model

This paper addresses the least-squares (LS) linear estimation problem of a discrete-time random signal from multi-sensor noisy measurements, perturbed by random parameter matrices, using the distributed fusion method. It is assumed that each sensor transmits its measured outputs to a local processor over imperfect communication channels and random failures can occur during transmission. Specifically, in order to avoid congestion problems in the network, it is assumed that, at each time instant, a packet at the sensor side is transmitted just once and the transmissions are subject to one-step random delays and packet dropouts. The following situations can arise: on the one hand, when a sensor data packet is one-step delayed, it will be processed at the next sampling time together with the current packet, provided that this one arrives on time; on the other, if a data packet is lost, it will never arrive at the local processor; as a consequence, at each sampling time, either one data packet, two data packets or nothing can be received.

The design of the distributed fusion algorithms will be carried out under a covariance-based approach; that is, the evolution model generating the signal process is not required and only the mean and covariance functions of the signal are necessary for the estimation. More precisely, we assume that the signal mean function is zero and its covariance function is factorizable, according to the following assumption:

***(A1)*** *The $n_x$-dimensional signal process $\{x_k\}_{k\geq 1}$ has zero mean and its autocovariance function is expressed in a separable form, $E\big[x_k x_s^T\big] = A_k B_s^T,\ s \leq k$, where $A_k, B_s \in \mathbb{R}^{n_x \times n}$ are known matrices.*

***Remark 1:*** Some important practical applications, where the signal autocovariance function can be obtained and factorized according to *(A1)*, are described in the following items:

– If the system matrix $\Phi$ in the state-space model of a stationary signal is available, the signal autocovariance function is $E[x_k x_s^T] = \Phi^{k-s} E[x_s x_s^T]$, for $s \leq k$, and assumption *(A1)* is clearly satisfied taking $A_k = \Phi^k$ and $B_s = E[x_s x_s^T](\Phi^{-s})^T$.

– Similarly, when the signal obeys a linear evolution model, $x_k = \Phi_{k-1} x_{k-1} + w_{k-1}$, its covariance function can be expressed as $E[x_k x_s^T] = \Phi_{k,s} E[x_s x_s^T]$, for $s \leq k$, where $\Phi_{k,s} = \Phi_{k-1} \cdots \Phi_s$, so taking $A_k = \Phi_{k,0}$ and $B_s = E[x_s x_s^T](\Phi_{s,0}^{-1})^T$, it is clear that assumption *(A1)* is satisfied.

– Furthermore, *(A1)* even covers situations where the system matrix in the state-space model is singular, and the above factorization, $E[x_k x_s^T] = \Phi_{k,0}\Phi_{s,0}^{-1} E[x_s x_s^T],\ s \leq k$, is not feasible, although a different factorization must be used in such cases (see e.g. [20]).

– Processes with finite-dimensional, possibly time-variant, state-space models have semi-separable covariance functions, $E[x_k x_s^T] = \sum_{l=1}^{r} a_k^l b_s^{lT},\ s \leq k$ (see [46]), and this structure is a particular case of that assumed, just taking $A_k = \big(a_k^1, a_k^2, \ldots, a_k^r\big)$ and $B_s = \big(b_s^1, b_s^2, \ldots, b_s^r\big)$.

– Also, uncertain systems with state-dependent multiplicative noise, as those considered in [7], meet this assumption, as it will be shown in Section 4.

Hence, the structural assumption *(A1)* on the signal autocovariance function covers both stationary and non-stationary signals, providing a unified context to deal with a large number of different practical situations and it is not necessary to obtain specific algorithms for each of them. Finally, note also that, although a state-space model can be generated from covariances, when only this kind of information is available, it is preferable to address the estimation problem directly using covariances, thus obviating the need of previous identification of the state-space model.

*2.1. Multi-sensor measured outputs with random parameter matrices*

Let us assume that the signal measurements are provided by $m$ sensors according to the following model:

$$z_k^{(i)} = H_k^{(i)} x_k + v_k^{(i)}, \qquad k \geq 1, \qquad i = 1, \ldots, m, \tag{1}$$

where $z_k^{(i)} \in \mathbb{R}^{n_z}$ is the measured output of the $i$-th sensor at time $k$, $H_k^{(i)} \in \mathbb{R}^{n_z \times n_x}$, and $v_k^{(i)} \in \mathbb{R}^{n_z}$ is the measurement noise vector.

It is well-known that, in a wide variety of real situations, the measured outputs $z_k^{(i)}$ can be subject not only to the additive noises $v_k^{(i)}$, but also to other stochastic disturbances from multiple sources, such as missing or fading measurements caused by the degradation or aging of measuring instruments, or the presence of multiplicative noise, due to different reasons, such as interferences or intermittent failures, among others. Each of these situations would require the derivation of a new and different estimation algorithm, since the conditions necessary to implement the conventional ones are not met. To overcome this issue, a global framework to model these random phenomena and an estimation algorithm suitable to address all the aforementioned cases are provided by assuming that the measurement matrices $H_k^{(i)}$ are random parameter matrices. Specifically, the following assumption on the measurement matrices in (1) is considered:

***(A2)*** $\{H_k^{(i)}\}_{k\geq 1}$, $i = 1, \ldots, m$, *are independent sequences of independent random parameter matrices with known means,* $E\big[H_k^{(i)}\big] = \overline{H}_k^{(i)}$. *Moreover, by denoting* $h_{pq}^{(i)}(k)$ *the* $(p,q)$*-th entry of* $H_k^{(i)}$*, the expectations* $E\big[h_{pq}^{(i)}(k) h_{p'q'}^{(j)}(k)\big]$ *are also assumed to be known, for* $p, p' = 1, \ldots, n_z$ *and* $q, q' = 1, \ldots, n_x$.

Regarding the sensor measurement noises of the different sensors, they will be assumed to be correlated at the same time; specifically, the following assumption is required:

***(A3)*** *The measurement noises* $\{v_k^{(i)}\}_{k\geq 1}$, $i = 1, \ldots, m$, *are second-order zero-mean white processes with* $E\big[v_k^{(i)} v_s^{(j)T}\big] = R_k^{(ij)} \delta_{k,s}, \;\; i, j = 1, \ldots, m.$

### *2.2. Measurements received by the local processors*

As it has been previously indicated, at each sensor side $i = 1, \ldots, m$, every data packet, $z_k^{(i)}$, is assumed to be transmitted only once and this transmission is subject to random one-step delays and losses. As a consequence, at the sampling time $k$, the local processor can either receive only the current output $z_k^{(i)}$, only the previous output $z_{k-1}^{(i)}$, both $z_k^{(i)}$ and $z_{k-1}^{(i)}$, or nothing. As in [40], when the current measurement, $z_k^{(i)}$, is not received on time, its predictor $\widehat{z}_{k/k-1}^{(i)}$ will be used for compensation. More precisely, the following model for $y_k^{(i)}$, the measurement received from the $i$-th sensor, $i = 1, \ldots, m$, is considered:

$$y_k^{(i)} = \begin{pmatrix} (1-\gamma_k^{(i)}) z_k^{(i)} + \gamma_k^{(i)} \widehat{z}_{k/k-1}^{(i)} \\ \psi_k^{(i)} z_{k-1}^{(i)} \end{pmatrix}, \quad k \geq 2; \qquad y_1^{(i)} = \begin{pmatrix} (1-\gamma_1^{(i)}) z_1^{(i)} \\ 0 \end{pmatrix}, \tag{2}$$

where $\{\gamma_k^{(i)}\}_{k\geq 1}$ and $\{\psi_k^{(i)}\}_{k\geq 2}$ denote sequences of random variables verifying the following assumption:

***(A4)*** $\Big\{\big(\gamma_k^{(i)}, \psi_{k+1}^{(i)}\big)^T\Big\}_{k\geq 1}$, $i = 1, \ldots, m$, *are independent sequences of independent random vectors, such that:*

- $\{\gamma_k^{(i)}\}_{k\geq 1}$, $i = 1, \ldots, m$, *are sequences of Bernoulli random variables with known probabilities,* $P\big(\gamma_k^{(i)} = 1\big) = \overline{\gamma}_k^{(i)}$.
- $\{\psi_k^{(i)}\}_{k\geq 2}$, $i = 1, \ldots, m$, *are sequences of Bernoulli random variables such that the conditional probabilities* $P\big(\psi_k^{(i)} = 1/\gamma_{k-1}^{(i)} = 1\big)$ *are known. From now on, we will denote* $\overline{\psi}_k^{(i)} \equiv P\big(\psi_k^{(i)} = 1\big) = P\big(\psi_k^{(i)} = 1/\gamma_{k-1}^{(i)} = 1\big)\overline{\gamma}_{k-1}^{(i)}, \quad k \geq 2.$

Finally, the following independence hypothesis is also assumed.

***(A5)*** *For $i = 1, \dots, m$, the signal, $\{x_k\}_{k\geq1}$, and the processes $\{H_k^{(i)}\}_{k\geq1}$, $\{v_k^{(i)}\}_{k\geq1}$ and $\left\{\left(\gamma_k^{(i)}, \psi_{k+1}^{(i)}\right)^T\right\}_{k\geq1}$ are mutually independent.*

***Remark 2.*** From (2), it is clear that, for each $i = 1, \dots, m$, $\gamma_k^{(i)} = 0$ means that the output at the current sampling time, $z_k^{(i)}$, arrives on time to be processed for the estimation, while $\gamma_k^{(i)} = 1$ means that this output is either delayed or dropped out, in which case its predictor, $\widehat{z}_{k/k-1}^{(i)}$, is used as compensator. Also, for each $i = 1, \dots, m$, $\psi_k^{(i)} = 1$ means that $z_{k-1}^{(i)}$ is processed at the sampling time $k$ (because it was one-step delayed) and $\psi_k^{(i)} = 0$ means that $z_{k-1}^{(i)}$ is not processed at the sampling time $k$ (because it was either received at time $k-1$ or dropped out). Since $\gamma_{k-1}^{(i)} = 0$ implies $\psi_k^{(i)} = 0$, it is clear that the value of $\psi_k^{(i)}$ is conditioned by that of $\gamma_{k-1}^{(i)}$.

Summing up, from (2) we have that $\gamma_k^{(i)} = 0$ and $\psi_k^{(i)} = 1$ means that $y_k^{(i)} = \left( z_k^{(i)T} \;,\; z_{k-1}^{(i)T} \right)^T$; that is, the outputs of the $i$th-sensor at the instants $k$ and $k-1$ are both received at time $k$. If $\gamma_k^{(i)} = 0$ and $\psi_k^{(i)} = 0$, we have $y_k^{(i)} = \left( z_k^{(i)T} \;,\; 0 \right)^T$. When $\gamma_k^{(i)} = 1$, the packet $z_k^{(i)}$ is not available at time $k$ and its predictor, $\widehat{z}_{k/k-1}^{(i)}$, is then used for estimation; in this case, as above, if $\psi_k^{(i)} = 1$, we have $y_k^{(i)} = \left( \widehat{z}_{k/k-1}^{(i)T} \;,\; z_{k-1}^{(i)T} \right)^T$, while $y_k^{(i)} = \left( \widehat{z}_{k/k-1}^{(i)T} \;,\; 0 \right)^T$ when $\psi_k^{(i)} = 0$.

In order to simplify the mathematical derivations, for $i = 1, \dots, m$, the observation model (2) is rewritten as follows:

$$\begin{aligned} y_k^{(i)} &= \big(1-\gamma_k^{(i)}\big)C_0 z_k^{(i)} + \psi_k^{(i)} C_1 z_{k-1}^{(i)} + \gamma_k^{(i)} C_0 \widehat{z}_{k/k-1}^{(i)}, \quad k \geq 2; \\ y_1^{(i)} &= \big(1-\gamma_1^{(i)}\big)C_0 z_1^{(i)}, \end{aligned} \tag{3}$$

where $C_0 = \left( I_{n_z\times n_z} \;,\; 0_{n_z\times n_z} \right)^T$ and $C_1 = \left( 0_{n_z\times n_z} \;,\; I_{n_z\times n_z} \right)^T$.

***Remark 3:*** The fusion estimation algorithms require expressions of some correlation matrices which are presented below:

*(a)* For $i, j = 1, \dots, m$, the model assumptions guarantee the independence between $\{x_k, x_s\}$ and $\{H_k^{(i)}, H_s^{(j)}\}$; hence, using *(A1)*, we have:

$$E\big[H_k^{(i)} x_k x_s^T H_s^{(j)T}\big] = E\big[H_k^{(i)} E[x_k x_s^T] H_s^{(j)T}\big] = E\big[H_k^{(i)} A_k B_s^T H_s^{(j)T}\big], \;\; s \leq k,$$

for arbitrary $i, j = 1, \ldots, m$, where $E\big[H_k^{(i)} A_k B_s^T H_s^{(j)T}\big] = \overline{H}_k^{(i)} A_k B_s^T \overline{H}_s^{(j)T}$ for $j \neq i$ or $s \neq k$, and the entries of $E\big[H_k^{(i)} A_k B_k^T H_k^{(i)T}\big]$ are computed by:

$$\Big(E\big[H_k^{(i)} A_k B_k^T H_k^{(i)T}\big]\Big)_{pq} = \sum_{a=1}^{n_x} \sum_{b=1}^{n_x} E[h_{pa}^{(i)}(k) h_{qb}^{(i)}(k)] (A_k B_k^T)_{ab},$$

for $p, q = 1, \ldots, n_z$.

*(b)* From (1), the output processes in the different sensors $\{z_k^{(i)}\}_{k \geq 1}$, $i = 1, \ldots, m$, have zero mean, and the matrices $\Sigma_{k,s}^{z^{(ij)}} \equiv E\big[z_k^{(i)} z_s^{(j)T}\big]$, are obtained by:

$$\Sigma_{k,s}^{z^{(ij)}} = E\big[H_k^{(i)} A_k B_s^T H_s^{(j)T}\big] + R_k^{(ij)} \delta_{k,s}, \quad s \leq k, \quad i, j = 1, \ldots, m.$$

*(c)* Taking into account assumption *(A4)* and denoting

$$\xi_k^{(i)} = \big(1 - \gamma_k^{(i)}\big) C_0 z_k^{(i)} + \psi_k^{(i)} C_1 z_{k-1}^{(i)}, \ \ k \geq 2; \quad \xi_1^{(i)} = y_1^{(i)} = \big(1 - \gamma_1^{(i)}\big) C_0 z_1^{(i)}, \tag{4}$$

we obtain that, for $i, j = 1, \ldots, m$, the matrices $\Sigma_k^{\xi^{(ij)}} \equiv E\big[\xi_k^{(i)} \xi_k^{(j)T}\big]$ are given by:

$$\begin{aligned} \Sigma_k^{\xi^{(ij)}} &= E\big[(1 - \gamma_k^{(i)})(1 - \gamma_k^{(j)})\big] C_0 \Sigma_k^{z^{(ij)}} C_0^T + E\big[\psi_k^{(i)} \psi_k^{(j)}\big] C_1 \Sigma_{k-1}^{z^{(ij)}} C_1^T \\ &\quad + \big(1 - \overline{\gamma}_k^{(i)}\big) \overline{\psi}_k^{(j)} C_0 \Sigma_{k,k-1}^{z^{(ij)}} C_1^T + \overline{\psi}_k^{(i)} \big(1 - \overline{\gamma}_k^{(j)}\big) C_1 \Sigma_{k,k-1}^{z^{(ji)T}} C_0^T, \quad k \geq 2; \\ \Sigma_1^{\xi^{(ij)}} &= E\big[(1 - \gamma_1^{(i)})(1 - \gamma_1^{(j)})\big] C_0 \Sigma_1^{z^{(ij)}} C_0^T, \end{aligned} \tag{5}$$

where, from the properties of the Bernoulli distribution, it is clear that

$$E\big[(1 - \gamma_k^{(i)})(1 - \gamma_k^{(j)})\big] = \begin{cases} 1 - \overline{\gamma}_k^{(i)}, & i = j, \\ (1 - \overline{\gamma}_k^{(i)})(1 - \overline{\gamma}_k^{(j)}), & i \neq j, \end{cases}$$

and

$$E\big[\psi_k^{(i)} \psi_k^{(j)}\big] = \begin{cases} \overline{\psi}_k^{(i)}, & i = j, \\ \overline{\psi}_k^{(i)} \overline{\psi}_k^{(j)}, & i \neq j. \end{cases}$$

## 3. Distributed fusion estimation problem

In the distributed fusion method, each single sensor sends its measured outputs, through an unreliable network, to a local processor where local estimators are computed using only the measurements received from the sensor itself; after that, all the local estimators are transmitted, over perfect connections, to a fusion center where the distributed estimators are generated (see Figure 1). In this section, the LS linear estimation problem from the observations with random delays and packet dropouts defined by (1) and (3) is addressed under the distributed fusion method. First (Section 3.1), using an innovation approach, local LS linear estimators (including filtering and fixed-point smoothing estimators) are obtained by recursive algorithms. Second (Section 3.2), the proposed distributed estimators are designed as the LS matrix-weighted linear combinations of the local linear estimators.

### *3.1. Derivation of the local LS linear estimators*

This section is concerned with the problem of obtaining, at each local processor $i$, for $i = 1, \ldots, m$, recursive algorithms for the local LS linear filter and fixed-point smoothers, by using an innovation approach. These algorithms provide also the estimation error covariance matrices, which are used to measure the accuracy of the local estimators when the LS optimality criterion is used.

*Innovation technique* (for details, see [46]). For each $i = 1, \ldots, m$, the innovation at time $k$ is defined as $\mu_k^{(i)} = y_k^{(i)} - \widehat{y}_{k/k-1}^{(i)}$, where $\widehat{y}_{k/k-1}^{(i)}$ is the LS linear estimator of $y_k^{(i)}$ based on $y_s^{(i)}$, $s \leq k-1$, and $\widehat{y}_{1/0}^{(i)} = E[y_1^{(i)}] = 0$. So, each set of innovations, $\{\mu_1^{(i)}, \ldots, \mu_L^{(i)}\}$, is obtained by linear transformations of the corresponding observations, $\{y_1^{(i)}, \ldots, y_L^{(i)}\}$, and the LS linear estimator of any random vector $w_k$ based on the observations $y_1^{(i)}, \ldots, y_L^{(i)}$, which will be denoted as $\widehat{w}_{k/L}^{(i)}$, can be calculated by taking the orthogonal projection of $w_k$ in the linear space generated by the innovations $\mu_1^{(i)}, \ldots, \mu_L^{(i)}$. Namely, denoting $\Pi_l^{(i)} = E\big[\mu_l^{(i)}\mu_l^{(i)T}\big]$ and $\mathcal{W}_{k,l}^{(i)} = E\big[w_k^{(i)}\mu_l^{(i)T}\big]$, the following general expression is obtained:

$$\widehat{w}_{k/L}^{(i)} = \sum_{l=1}^{L} \mathcal{W}_{k,l}^{(i)} \Pi_l^{(i)-1} \mu_l^{(i)}, \quad i = 1, \ldots, m. \tag{6}$$

*One-stage observation predictor.* Since $\psi_k^{(i)}$ and $H_{k-1}^{(i)}$ are correlated with the innovation $\mu_{k-1}^{(i)}$, to simplify the derivation of the observation predictor, $\widehat{y}_{k/k-1}^{(i)}$, the observations (3) are rewritten as follows:

$$y_k^{(i)} = (1-\gamma_k^{(i)})C_0 z_k^{(i)} + \gamma_k^{(i)} C_0 \widehat{z}_{k/k-1}^{(i)} + \overline{\psi}_k^{(i)} C_1 \overline{H}_{k-1}^{(i)} x_{k-1} + V_{k-1}^{(i)}, \quad k \geq 2, \tag{7}$$

where $V_k^{(i)} = \psi_{k+1}^{(i)} C_1 z_k^{(i)} - \overline{\psi}_{k+1}^{(i)} C_1 \overline{H}_k^{(i)} x_k, \quad k \geq 1, \quad i = 1, \dots, m.$

Taking into account that, for $i = 1, \dots, m$, $\psi_{k+1}^{(i)}$ and $H_k^{(i)}$ are independent of $\mu_1^{(i)}, \dots, \mu_{k-1}^{(i)}$, from the general expression (6), it is easy to see that $\widehat{V}_{k/k}^{(i)} = \mathcal{V}_k^{(i)} \Pi_k^{(i)-1} \mu_k^{(i)}, \quad k \geq 1$, where $\mathcal{V}_k^{(i)} \equiv E\big[V_k^{(i)} \mu_k^{(i)T}\big] = E\big[V_k^{(i)} y_k^{(i)T}\big]$. Hence, according to the projection theory, $\widehat{y}_{k/k-1}^{(i)}, \ i = 1, \dots, m$, satisfy:

$$\widehat{y}_{k/k-1}^{(i)} = C_0 \overline{H}_k^{(i)} \widehat{x}_{k/k-1}^{(i)} + \overline{\psi}_k^{(i)} C_1 \overline{H}_{k-1}^{(i)} \widehat{x}_{k-1/k-1}^{(i)} + \mathcal{V}_{k-1}^{(i)} \Pi_{k-1}^{(i)-1} \mu_{k-1}^{(i)}, \quad k \geq 2. \tag{8}$$

The general expression (6) for the LS linear estimators as a linear combination of the innovations, along with (8) for the one-stage observation predictor, are the starting points to derive the recursive filtering and fixed-point smoothing algorithms.

Note that the determination of $\widehat{y}_{k/k-1}^{(i)}$ requires that of the linear predictor, $\widehat{x}_{k/k-1}^{(i)}$, and filter, $\widehat{x}_{k-1/k-1}^{(i)}$, which are simultaneously obtained in the next section.

### *3.1.1. Local filtering algorithm*

For notational simplicity, for $i = 1, \dots, m$, we will use the following matricial operators and they will be applied to the matrices $D_k = A_k, B_k$ that define the signal covariance function (see *(A1)*):

$$\begin{aligned} \overline{\mathcal{H}}_{D_k}^{(i)} &\equiv (1-\overline{\gamma}_k^{(i)}) C_0 \overline{H}_k^{(i)} D_k + \overline{\psi}_k^{(i)} C_1 \overline{H}_{k-1}^{(i)} D_{k-1}, \quad k \geq 2; \\ \overline{\mathcal{H}}_{D_1}^{(i)} &\equiv (1-\overline{\gamma}_1^{(i)}) C_0 \overline{H}_1^{(i)} D_1. \end{aligned} \tag{9}$$

*Expression for the local predictors and filter of the signal.* For each $i = 1, \dots, m$, we use (6) to express $\widehat{x}_{k/L}^{(i)} = \sum_{l=1}^{L} \mathcal{X}_{k,l}^{(i)} \Pi_l^{(i)-1} \mu_l^{(i)}, \ L \leq k$, and we proceed to calculate the coefficients

$$\mathcal{X}_{k,l}^{(i)} = E\big[x_k \mu_l^{(i)T}\big] = E\big[x_k y_l^{(i)T}\big] - E\big[x_k \widehat{y}_{l/l-1}^{(i)T}\big], \ 1 \leq l \leq k.$$

The independence hypotheses and the separable form of the signal covariance *(A1)* lead to

$$
\begin{aligned}
&E\big[x_k y_l^{(i)T}\big] = A_k \overline{\mathcal{H}}_{B_l}^{(i)T} + \overline{\gamma}_l^{(i)} E[x_k \widehat{x}_{l/l-1}^{(i)T}] \overline{H}_l^{(i)T} C_0^T, \quad 2 \le l \le k;\\
&E\big[x_k y_1^{(i)T}\big] = A_k \overline{\mathcal{H}}_{B_1}^{(i)T},
\end{aligned}
$$

with $\overline{\mathcal{H}}_{B_l}^{(i)}$ given in (9). Now, using (8) for $\widehat{y}_{l/l-1}^{(i)}$, together with (6) for $\widehat{x}_{l/l-1}^{(i)}$ and $\widehat{x}_{l-1/l-1}^{(i)}$, the following expression for the filter coefficients is obtained:

$$
\begin{aligned}
\mathcal{X}_{k,l}^{(i)} &= A_k \overline{\mathcal{H}}_{B_l}^{(i)T} - \sum_{j=1}^{l-1} \mathcal{X}_{k,j}^{(i)} \Pi_j^{(i)-1} \Big( (1-\overline{\gamma}_l^{(i)}) C_0 \overline{H}_l^{(i)} \mathcal{X}_{l,j}^{(i)} + \overline{\psi}_l^{(i)} C_1 \overline{H}_{l-1}^{(i)} \mathcal{X}_{l-1,j}^{(i)} \Big)^T \\
&\quad - \mathcal{X}_{k,l-1}^{(i)} \Pi_{l-1}^{(i)-1} \mathcal{V}_{l-1}^{(i)T}, \qquad 2 \le l \le k;\\
\mathcal{X}_{k,1}^{(i)} &= A_k \overline{\mathcal{H}}_{B_1}^{(i)T},
\end{aligned}
$$

which guarantees that $\mathcal{X}_{k,l}^{(i)} = A_k J_l^{(i)}, \ 1 \le l \le k$, with $J_l^{(i)}$ given by

$$
J_l^{(i)} = \overline{\mathcal{H}}_{B_l}^{(i)T} - \sum_{j=1}^{l-1} J_j^{(i)} \Pi_j^{(i)-1} J_j^{(i)T} \overline{\mathcal{H}}_{A_l}^{(i)T} - J_{l-1}^{(i)} \Pi_{l-1}^{(i)-1} \mathcal{V}_{l-1}^{(i)T}, \ \ l \ge 2; \quad J_1^{(i)} = \overline{\mathcal{H}}_{B_1}^{(i)T}.
$$

Then, by defining

$$
O_k^{(i)} \equiv \sum_{l=1}^{k} J_l^{(i)} \Pi_l^{(i)-1} \mu_l^{(i)}, \quad k \ge 1,
$$

it is clear from (6) that the signal predictors and filter are given by

$$
\widehat{x}_{k/L}^{(i)} = A_k O_L^{(i)}, \quad L \le k; \quad k \ge 1, \tag{10}
$$

and, from (8), the following expression for the observation predictor, $\widehat{y}_{k/k-1}^{(i)}$, is obtained,

$$
\widehat{y}_{k/k-1}^{(i)} = \big(\overline{\mathcal{H}}_{A_k}^{(i)} + \overline{\gamma}_k^{(i)} C_0 \overline{H}_k^{(i)} A_k\big) O_{k-1}^{(i)} + \mathcal{V}_{k-1}^{(i)} \Pi_{k-1}^{(i)-1} \mu_{k-1}^{(i)}, \quad k \ge 2. \tag{11}
$$

Moreover, by defining

$$
r_k^{(i)} \equiv E\big[O_k^{(i)} O_k^{(i)T}\big] = \sum_{l=1}^{k} J_l^{(i)} \Pi_l^{(i)-1} J_l^{(i)T}, \quad k \ge 1,
$$

the following expression for the coefficients $\mathcal{V}_k^{(i)} = E\big[V_k^{(i)} y_k^{(i)T}\big]$ is easily obtained:

$$\mathcal{V}_k^{(i)} = -\overline{\psi}_{k+1}^{(i)}(1-\overline{\gamma}_k^{(i)})C_1\overline{H}_k^{(i)}A_k\big(B_k - A_k r_{k-1}^{(i)}\big)^T\overline{H}_k^{(i)T}C_0^T, \quad k \geq 1. \tag{12}$$

Finally, since, from the Orthogonal Projection Lemma (OPL), the estimation error is uncorrelated with all the observations or, equivalently, uncorrelated with the corresponding innovations, $E\big[y_k^{(i)}O_{k-1}^{(i)T}\big] = E\big[\widehat{y}_{k/k-1}^{(i)}O_{k-1}^{(i)T}\big]$ and $E\big[y_k^{(i)}\mu_{k-1}^{(i)T}\big] = E\big[\widehat{y}_{k/k-1}^{(i)}\mu_{k-1}^{(i)T}\big]$; then, using (11) for $\widehat{y}_{k/k-1}^{(i)}$ and since $J_k^{(i)} = E[O_k^{(i)}\mu_k^{(i)T}]$, the following expressions for the matrices $\mathcal{O}_{k,k-1}^{(i)} \equiv E\big[y_k^{(i)}O_{k-1}^{(i)T}\big]$ and $\mathcal{Y}_{k,k-1}^{(i)} \equiv E\big[y_k^{(i)}\mu_{k-1}^{(i)T}\big]$, are also easily obtained:

$$\begin{aligned}\mathcal{O}_{k,k-1}^{(i)} &= \big(\overline{\mathcal{H}}_{A_k}^{(i)} + \overline{\gamma}_k^{(i)}C_0\overline{H}_k^{(i)}A_k\big)r_{k-1}^{(i)} + \mathcal{V}_{k-1}^{(i)}\Pi_{k-1}^{(i)-1}J_{k-1}^{(i)T}, \quad k \geq 2.\\ \mathcal{Y}_{k,k-1}^{(i)} &= \big(\overline{\mathcal{H}}_{A_k}^{(i)} + \overline{\gamma}_k^{(i)}C_0\overline{H}_k^{(i)}A_k\big)J_{k-1}^{(i)} + \mathcal{V}_{k-1}^{(i)}, \quad k \geq 2.\end{aligned} \tag{13}$$

Based on the above results, the following local filtering algorithm is deduced.

**Theorem 1.** *Under assumptions (A1)-(A5), for each $i = 1, \ldots, m$, the local LS linear filter, $\widehat{x}_{k/k}^{(i)}$, and the corresponding error covariance matrix, $\Sigma_{k/k}^{(i)} \equiv E\big[(x_k - \widehat{x}_{k/k}^{(i)})(x_k - \widehat{x}_{k/k}^{(i)})^T\big]$, are given by*

$$\widehat{x}_{k/k}^{(i)} = A_k O_k^{(i)}, \quad k \geq 1, \tag{14}$$

$$\Sigma_{k/k}^{(i)} = A_k\big(B_k - A_k r_k^{(i)}\big)^T, \quad k \geq 1,$$

*where the vectors $O_k^{(i)}$ and the matrices $r_k^{(i)} = E[O_k^{(i)}O_k^{(i)T}]$ are recursively obtained from*

$$O_k^{(i)} = O_{k-1}^{(i)} + J_k^{(i)}\Pi_k^{(i)-1}\mu_k^{(i)}, \quad k \geq 1; \quad O_0^{(i)} = 0, \tag{15}$$

$$r_k^{(i)} = r_{k-1}^{(i)} + J_k^{(i)}\Pi_k^{(i)-1}J_k^{(i)T}, \quad k \geq 1; \quad r_0^{(i)} = 0,$$

*and the matrices $J_k^{(i)} = E[O_k^{(i)}\mu_k^{(i)T}]$ satisfy*

$$J_k^{(i)} = \overline{\mathcal{H}}_{B_k}^{(i)T} - r_{k-1}^{(i)}\overline{\mathcal{H}}_{A_k}^{(i)T} - J_{k-1}^{(i)}\Pi_{k-1}^{(i)-1}\mathcal{V}_{k-1}^{(i)T}, \quad k \geq 2; \quad J_1^{(i)} = \overline{\mathcal{H}}_{B_1}^{(i)T},$$

*with $\overline{\mathcal{H}}_{A_k}^{(i)}$ and $\overline{\mathcal{H}}_{B_k}^{(i)}$ given in (9).*

*The innovations, $\mu_k^{(i)}$, are given by*

$$\mu_k^{(i)} = y_k^{(i)} - \big(\overline{\mathcal{H}}_{A_k}^{(i)} + \overline{\gamma}_k^{(i)} C_0 \overline{H}_k^{(i)} A_k\big) O_{k-1}^{(i)} - \mathcal{V}_{k-1}^{(i)} \Pi_{k-1}^{(i)-1} \mu_{k-1}^{(i)}, \quad k \geq 2; \quad \mu_1^{(i)} = y_1^{(i)},$$

*with $\mathcal{V}_k^{(i)}$ given in (12) and, finally, their covariances, $\Pi_k^{(i)} = E\big[\mu_k^{(i)} \mu_k^{(i)T}\big]$, are obtained by*

$$\Pi_k^{(i)} = \Sigma_k^{\xi^{(i)}} + \overline{\gamma}_k^{(i)} \big(\mathcal{O}_{k,k-1}^{(i)} - C_0 \overline{H}_k^{(i)} A_k r_{k-1}^{(i)}\big) A_k^T \overline{H}_k^{(i)T} C_0^T - \overline{\mathcal{H}}_{A_k}^{(i)} \mathcal{O}_{k,k-1}^{(i)T} - \mathcal{V}_{k-1}^{(i)} \Pi_{k-1}^{(i)-1} \mathcal{Y}_{k,k-1}^{(i)T}, \quad k \geq 2; \quad \Pi_1^{(i)} = \Sigma_1^{\xi^{(i)}},$$

*where $\Sigma_k^{\xi^{(i)}}$, $\mathcal{O}_{k,k-1}^{(i)}$ and $\mathcal{Y}_{k,k-1}^{(i)}$ are given in (5) and (13), respectively.*

*3.1.2. Local fixed-point smoothing algorithm*

A recursive algorithm for the local LS linear smoothers, $\widehat{x}_{k/k+h}^{(i)}$, at the fixed point $k$, for any $h \geq 1$, is presented in the following theorem.

**Theorem 2.** *Under assumptions (A1)-(A5), for each $i = 1, \ldots, m$, the local LS linear fixed-point smoothers, $\widehat{x}_{k/k+h}^{(i)}$, are calculated by*

$$\widehat{x}_{k/k+h}^{(i)} = \widehat{x}_{k/k+h-1}^{(i)} + \mathcal{X}_{k,k+h}^{(i)} \Pi_{k+h}^{(i)-1} \mu_{k+h}^{(i)}, \quad k \geq 1, \quad h \geq 1, \tag{16}$$

*with initial condition given by the local filter, $\widehat{x}_{k/k}^{(i)}$, and $\mathcal{X}_{k,k+h}^{(i)} = E\big[x_k \mu_{k+h}^{(i)T}\big]$ verifying*

$$\begin{aligned} \mathcal{X}_{k,k+h}^{(i)} &= \big(B_k - E_{k,k+h-1}^{(i)}\big) \overline{\mathcal{H}}_{A_{k+h}}^{(i)T} - \mathcal{X}_{k,k+h-1}^{(i)} \Pi_{k+h-1}^{(i)-1} \mathcal{V}_{k+h-1}^{(i)T}, \quad h \geq 1; \\ \mathcal{X}_{k,k}^{(i)} &= A_k J_k^{(i)}. \end{aligned} \tag{17}$$

*The matrices $E_{k,k+h}^{(i)} = E[x_k O_{k+h}^{(i)T}]$ satisfy the following recursive formula*

$$E_{k,k+h}^{(i)} = E_{k,k+h-1}^{(i)} + \mathcal{X}_{k,k+h}^{(i)} \Pi_{k+h}^{(i)-1} J_{k+h}^{(i)T}, \quad h \geq 1; \quad E_{k,k}^{(i)} = A_k r_k^{(i)}. \tag{18}$$

*The fixed-point smoothing error covariance matrices, $\Sigma_{k/k+h}^{(i)} \equiv E\big[(x_k - \widehat{x}_{k/k+h}^{(i)})(x_k - \widehat{x}_{k/k+h}^{(i)})^T\big]$, are obtained by*

$$\Sigma_{k/k+h}^{(i)} = \Sigma_{k/k+h-1}^{(i)} - \mathcal{X}_{k,k+h}^{(i)} \Pi_{k+h}^{(i)-1} \mathcal{X}_{k,k+h}^{(i)T}, \quad k \geq 1, \quad h \geq 1,$$

*with initial condition given by the filtering error covariance matrix $\Sigma_{k/k}^{(i)}$.*

*The filter $\widehat{x}_{k/k}^{(i)}$, the innovations $\mu_{k+h}^{(i)}$ and their covariance matrices, $\Sigma_{k/k}^{(i)}$ and $\Pi_{k+h}^{(i)}$, and the matrices $J_{k+h}^{(i)}$, are obtained from Theorem 1.*

**Proof.** See Appendix A. □

***Remark 4:*** As in the Kalman filter, since the signal has dimension $n_x$, from theorems 1 and 2, it is easily deduced that the computational cost of the proposed local estimators has the order of magnitude $O(n_x^3)$. The computational cost of the estimators in [42], [43] and [44] is clearly higher since augmented signal vectors with greater dimension than $n_x$ are considered. Actually, as it is indicated in [43] and [44], the computational cost of the estimators in [42] has the order of magnitude $O\big((n_x+5n_z)^3\big)$, which is higher than that of the estimators in [43] and [44] with the magnitude $O\big((n_x+3n_z)^3\big)$ and $O\big((n_x+n_z)^3\big)$, respectively. Hence, in comparison to the estimators in [42], [43] and [44], the proposed estimators provide a significant reduction of the computational cost.

*3.2. Derivation of the distributed LS fusion linear estimators*

As already mentioned, once the local LS linear estimators have been obtained, our next objective is to derive distributed fusion estimators $\widehat{x}^{(D)}_{k/k+h}$, $k\geq 1$, $h\geq 0$, as matrix-weighted linear combinations of the corresponding local estimators, $\widehat{x}^{(i)}_{k/k+h}$, $i=1,\ldots,m$, in which the weight matrices are computed by minimizing the mean squared estimation error.

For this purpose, we consider the following stacked vectors, constituted by the local estimators: $\widehat{X}_{k/k+h}=\Big(\widehat{x}^{(1)T}_{k/k+h},\ldots,\widehat{x}^{(m)T}_{k/k+h}\Big)^T$. By applying the LS criterion (see e.g. [20]), it is easy to prove that the proposed distributed estimators satisfy:

$$= E\big[x_k\widehat{X}^T_{k/k+h}\big]\Big(E\big[\widehat{X}_{k/k+h}\widehat{X}^T_{k/k+h}\big]\Big)^{-1}\widehat{X}_{k/k+h},\quad k\geq 1,\quad h\geq 0. \tag{19}$$

Since $E\big[\widehat{X}_{k/k+h}\widehat{X}^T_{k/k+h}\big]=\Big(E\big[\widehat{x}^{(i)}_{k/k+h}\widehat{x}^{(j)T}_{k/k+h}\big]\Big)_{i,j=1,\ldots,m}$ and, from the OPL, $E\big[x_k\widehat{X}^T_{k/k+h}\big]=\Big(E\big[\widehat{x}^{(1)}_{k/k+h}\widehat{x}^{(1)T}_{k/k+h}\big],\ldots,E\big[\widehat{x}^{(m)}_{k/k+h}\widehat{x}^{(m)T}_{k/k+h}\big]\Big)$, the derivation of the distributed estimators in (19) only requires to know the cross-covariance matrices between the local ones $E\big[\widehat{x}^{(i)}_{k/k+h}\widehat{x}^{(j)T}_{k/k+h}\big]$, $i,j=1,\ldots,m$, $h\geq 0$, which will be obtained in a recursive way by starting from the cross-covariance between the local filters.

From expressions (14) and (15), the cross-covariance between any two local filtering estimators, $\widehat{x}^{(i)}_{k/k}$ and $\widehat{x}^{(j)}_{k/k}$ will be obtained from a recursive formula for $r^{(ij)}_k\equiv E\big[O^{(i)}_kO^{(j)}_k\big]$, requiring also the cross-covariance matrices

$\Pi_k^{(ij)} \equiv E\big[\mu_k^{(i)}\mu_k^{(j)}\big]$. Also, from expression (16) for the local smoothing estimators, the cross-covariance between any two local smoothers $\widehat{x}_{k/k+h}^{(i)}$ and $\widehat{x}_{k/k+h}^{(j)}$, $h>0$, requires the expectations $\Phi_{k,k+h}^{(ij)} \equiv E\big[\widehat{x}_{k/k+h-1}^{(i)}\mu_{k+h}^{(j)T}\big]$ which, in turn, will be obtained from $\Lambda_{k,k+h-1}^{(ij)} \equiv E\big[\widehat{x}_{k/k+h-1}^{(i)}O_{k+h-1}^{(j)T}\big]$.

*3.2.1. Preliminary results*

In this section, we present some lemmas that provide the aforementioned expectations, which are necessary to calculate the cross-covariance matrices between the local estimators; the assumptions and notation in these lemmas are the same as those of the previous sections.

**Lemma 1.** *For $i,j=1,\dots,m$, the matrices $r_k^{(ij)} = E\big[O_k^{(i)}O_k^{(j)T}\big]$ are obtained by*

$$r_k^{(ij)} = r_{k-1}^{(ij)} + J_{k-1,k}^{(ij)}\Pi_k^{(j)-1}J_k^{(j)T} + J_k^{(i)}\Pi_k^{(i)-1}J_k^{(ji)T}, \quad k\geq 1; \qquad r_0^{(ij)}=0, \quad (20)$$

*where the matrices $J_k^{(ij)} = E[O_k^{(i)}\mu_k^{(j)T}]$ are given by*

$$J_k^{(ij)} = J_{k-1,k}^{(ij)} + J_k^{(i)}\Pi_k^{(i)-1}\Pi_k^{(ij)}, \quad k\geq 1, \quad (21)$$

*and $J_{k-1,k}^{(ij)} = E[O_{k-1}^{(i)}\mu_k^{(j)T}]$ are calculated by*

$$\begin{aligned} J_{k-1,k}^{(ij)} &= \big(r_{k-1}^{(i)} - r_{k-1}^{(ij)}\big)\overline{\mathcal{H}}_{A_k}^{(j)T} + J_{k-1}^{(i)}\Pi_{k-1}^{(i)-1}\mathcal{V}_{k-1}^{(ji)T} \\ &\quad - J_{k-1}^{(ij)}\Pi_{k-1}^{(j)-1}\mathcal{V}_{k-1}^{(j)T}, \quad k\geq 2; \qquad J_{0,1}^{(ij)}=0, \quad i\neq j. \\ J_{k-1,k}^{(i)} &= 0, \quad k\geq 1, \end{aligned} \quad (22)$$

*with $\mathcal{V}_k^{(ij)} = E\big[V_k^{(i)}y_k^{(j)T}\big]$ given by*

$$\mathcal{V}_k^{(ij)} = \overline{\psi}_{k+1}^{(i)}(1-\overline{\gamma}_k^{(j)})C_1R_k^{(ij)}C_0^T, \quad k\geq 1, \quad i\neq j. \quad (23)$$

**Proof.** See Appendix B. □

**Lemma 2.** *For $i,j=1,\dots,m$, $i\neq j$, the innovation cross-covariance matrices, $\Pi_k^{(ij)} = E\big[\mu_k^{(i)}\mu_k^{(j)}\big]$, satisfy*

$$\begin{aligned} \Pi_k^{(ij)} &= \Sigma_k^{\xi^{(ij)}} + \overline{\gamma}_k^{(j)}\Big(\overline{\mathcal{H}}_{A_k}^{(i)}r_{k-1}^{(j)} + \mathcal{V}_{k-1}^{(ij)}\Pi_{k-1}^{(j)-1}J_{k-1}^{(j)T}\Big)A_k^T\overline{H}_k^{(j)T}C_0^T \\ &\quad - \overline{\mathcal{H}}_{A_k}^{(i)}\Big(\mathcal{O}_{k,k-1}^{(j)T} + J_{k-1,k}^{(ij)}\Big) - \mathcal{V}_{k-1}^{(ij)}\Pi_{k-1}^{(j)-1}\mathcal{Y}_{k,k-1}^{(j)T} \\ &\quad - \mathcal{V}_{k-1}^{(i)}\Pi_{L-1}^{(i)-1}\Pi_{k-1,k}^{(ij)}, \quad k\geq 2; \qquad \Pi_1^{(ij)} = \Sigma_1^{\xi^{(ij)}}, \end{aligned} \quad (24)$$

*where* $\Pi^{(ij)}_{k-1,k} = E\big[\mu^{(i)}_{k-1}\mu^{(j)}_{k}\big]$ *is obtained by*

$$\Pi^{(ij)}_{k-1,k} = \big(J^{(i)}_{k-1} - J^{(ji)}_{k-1}\big)^T \overline{\mathcal{H}}^{(j)T}_{A_k} + \mathcal{V}^{(ji)T}_{k-1} - \Pi^{(ij)}_{k-1}\Pi^{(j)-1}_{k-1}\mathcal{V}^{(j)T}_{k-1}, \quad k \geq 2. \tag{25}$$

**Proof.** See Appendix C. □

**Lemma 3.** *For* $i,j = 1,\ldots,m$, *the expectations* $\Phi^{(ij)}_{k,L} = E\big[\widehat{x}^{(i)}_{k/L-1}\mu^{(j)T}_{L}\big]$ *satisfy*

$$\begin{aligned}
\Phi^{(ij)}_{k,L} &= \Big(\Lambda^{(i)}_{k,L-1} - \Lambda^{(ij)}_{k,L-1}\Big)\,\overline{\mathcal{H}}^{(j)T}_{A_L} + \mathcal{X}^{(i)}_{k,L-1}\Pi^{(i)-1}_{L-1}\mathcal{V}^{(ji)T}_{L-1} \\
&\quad - \Big(\mathcal{X}^{(i)}_{k,L-1}\Pi^{(i)-1}_{L-1}\Pi^{(ij)}_{L-1} + \Phi^{(ij)}_{k,L-1}\Big)\,\Pi^{(j)-1}_{L-1}\mathcal{V}^{(j)T}_{L-1}, \quad L > k \geq 1, \quad i \neq j; \\
\Phi^{(ij)}_{k} &= A_k J^{(ij)}_{k-1,k}, \quad k \geq 1, \quad i \neq j; \\
\Phi^{(i)}_{k,L} &= 0, \quad L \geq k \geq 1.
\end{aligned} \tag{26}$$

**Proof.** See Appendix D. □

**Lemma 4.** *For* $i,j = 1,\ldots,m$, *the expectations* $\Lambda^{(ij)}_{k,L} = E\big[\widehat{x}^{(i)}_{k/L}O^{(j)T}_{L}\big]$ *satisfy*

$$\begin{aligned}
\Lambda^{(ij)}_{k,L} &= \Lambda^{(ij)}_{k,L-1} + \Phi^{(ij)}_{k,L}\Pi^{(j)-1}_{L}J^{(j)T}_{L} + \mathcal{X}^{(i)}_{k,L}\Pi^{(i)-1}_{L}J^{(ji)T}_{L-1,L} \\
&\quad + \mathcal{X}^{(i)}_{k,L}\Pi^{(i)-1}_{L}\Pi^{(ij)}_{L}\Pi^{(j)-1}_{L}J^{(j)T}_{L}, \quad L > k \geq 1; \quad \Lambda^{(ij)}_{k} = A_k r^{(ij)}_{k}, \quad k \geq 1.
\end{aligned}$$

**Proof.** See Appendix E. □

*3.2.2. Distributed filtering and fixed-point smoothing estimators*

Using (14) and (16), a recursive expression for the cross-covariance matrices between any two local estimators, which depends on the matrices calculated in the previous lemmas, is immediately obtained and the distributed estimators, $\widehat{x}^{(D)}_{k/k+h}$, are calculated from (19). Also, from assumption *(A1)* and expression (19), it is easy to derive a formula for the error covariance matrices, $\Sigma^{(D)}_{k/k+h} \equiv E\big[(x_k - \widehat{x}^{(D)}_{k/k+h})(x_k - \widehat{x}^{(D)}_{k/k+h})^T\big]$. These results are presented in the following theorem.

**Theorem 3.** *Let* $\widehat{X}_{k/k+h} = \Big(\widehat{x}^{(1)T}_{k/k+h},\ldots,\widehat{x}^{(m)T}_{k/k+h}\Big)^T$ *be the vector constituted by the local estimators calculated from the algorithms in theorems 1 and 2; then, the distributed filtering and smoothing estimators are given by*

$$\widehat{x}^{(D)}_{k/k+h} = \boldsymbol{\Xi}_{k/k+h}\left(\boldsymbol{K}_{k/k+h}\right)^{-1}\widehat{X}_{k/k+h}, \quad k \geq 1, \quad h \geq 0,$$

*with* $\boldsymbol{K}_{k/k+h} = \left(K^{(ij)}_{k/k+h}\right)_{i,j=1,\dots,m}$ *and* $\boldsymbol{\Xi}_{k/k+h} = \left(K^{(1)}_{k/k+h}, \dots, K^{(m)}_{k/k+h}\right)$, *where* $K^{(ij)}_{k/k+h} = E\big[\widehat{x}^{(i)}_{k/k+h}\widehat{x}^{(j)T}_{k/k+h}\big]$, $i,j = 1,\dots,m$, *are obtained by*

$$\begin{aligned}K^{(ij)}_{k/k+h} &= K^{(ij)}_{k/k+h-1} + \Phi^{(ij)}_{k,k+h}\Pi^{(j)-1}_{k+h}\mathcal{X}^{(j)T}_{k,k+h} + \mathcal{X}^{(i)}_{k,k+h}\Pi^{(i)-1}_{k+h}\Phi^{(ji)T}_{k,k+h}\\ &\quad+\mathcal{X}^{(i)}_{k,k+h}\Pi^{(i)-1}_{k+h}\Pi^{(ij)}_{k+h}\Pi^{(j)-1}_{k+h}\mathcal{X}^{(j)T}_{k,k+h}, \quad k\geq 1, \quad h>0;\\ K^{(ij)}_{k/k} &= A_k r^{(ij)}_k A^T_k,\end{aligned}$$

*and the matrices* $r^{(ij)}_k$, $\Pi^{(ij)}_{k+h}$ *and* $\Phi^{(ij)}_{k,k+h}$ *are obtained in lemmas 1, 2 and 3, respectively.*

*The error covariance matrices of the distributed estimators are computed by*

$$\Sigma^{(D)}_{k/k+h} = A_k B^T_k - \boldsymbol{\Xi}_{k/k+h}\boldsymbol{K}^{-1}_{k/k+h}\boldsymbol{\Xi}^T_{k/k+h}, \quad k\geq 1, \quad h\geq 0.$$

## 4. Simulation study

In this section, a numerical simulation example is presented with a dual purpose: on the one hand, for illustrating some of the different sensor uncertainties covered by the current measurement model (1) with random measurement matrices and, on the other, for analyzing the performance of the proposed distributed filtering and fixed-point smoothing algorithms and how the estimation accuracy is influenced by the sensor uncertainties and the random transmission delays and packet losses.

*Signal process.* Consider the same signal process as that in [40]; specifically, a discrete-time scalar signal generated by the following model with state-dependent multiplicative noise:

$$x_{k+1} = \big(0.9 + 0.01\varepsilon_k\big)x_k + w_k, \quad k\geq 0,$$

where $x_0$ is a standard Gaussian variable and $\{w_k\}_{k\geq 0}$, $\{\varepsilon_k\}_{k\geq 0}$ are zero-mean Gaussian white processes with unit variance. Assuming that $x_0$, $\{w_k\}_{k\geq 0}$ and $\{\varepsilon_k\}_{k\geq 0}$ are mutually independent, the signal covariance function is given by $E[x_k x_s] = 0.9^{k-s}D_s$, $s\leq k$, where $D_s = E[x^2_s]$ is obtained by:

$$D_s = (0.9^2 + 0.01^2)D_{s-1} + 1, \; s\geq 1; \quad D_0 = 1;$$

hence, assumption *(A1)* is satisfied taking, for example, $A_k = 0.9^k$ y $B_s = 0.9^{-s}D_s$.

*Sensor measured outputs.* Consider four sensors which provide scalar measurements of the signal according to model (1), where the additive noises are defined as $v_k^{(i)} = c_i \eta_k,\ i = 1, 2, 3, 4$, with $c_1 = 1$, $c_2 = 0.5$, $c_3 = c_4 = 0.75$, and $\{\eta_k\}_{k\geq 1}$ is a zero-mean Gaussian white process with variance 0.5. Clearly, these noises are correlated, with $R_k^{(ij)} = 0.5 c_i c_j,\ \ k \geq 1;\ \ \ i, j = 1, 2, 3, 4$. The random measurement matrices are defined by $H_k^{(i)} = \theta_k^{(i)} C_k^{(i)}$, for $i = 1, 2, 3$, where $C_k^{(1)} = 0.82$, $C_k^{(2)} = 0.75$, $C_k^{(3)} = 0.74$, and $H_k^{(4)} = \theta_k^{(4)}\big(0.75 + 0.95\varphi_k\big)$, with $\{\varphi_k\}_{k\geq 1}$ a zero-mean Gaussian white process with unit variance and $\{\theta_k^{(i)}\}_{k\geq 1},\ i = 1, 2, 3, 4$, white processes with the following time-invariant probability distributions:

- $\{\theta_k^{(1)}\}_{k\geq 1}$ are uniformly distributed over $[0.2, 0.7]$.
- $P\big(\theta_k^{(2)} = 0\big) = 0.3,\ \ P\big(\theta_k^{(2)} = 0.5\big) = 0.3,\ \ P\big(\theta_k^{(2)} = 1\big) = 0.4,\ \ k \geq 1.$
- For $i = 3, 4$, $\{\theta_k^{(i)}\}_{k\geq 1}$ are Bernoulli random variables with $P\big(\theta_k^{(i)} = 1\big) = \overline{\theta}^{(i)},\ \ k \geq 1.$

Note that the random measurement parameters $H_k^{(i)},\ i = 1, 2, 3, 4$, model different sensor uncertainties; namely, *continuous and discrete gain degradation* in sensors 1 and 2, respectively, *missing measurements* in sensor 3, and both *missing measurements and multiplicative noise* in sensor 4.

*Observations with random packet dropouts.* Now, according to the theoretical study, we assume that the available measurements used for the estimation, $y_k^{(i)}$, $i = 1, 2, 3, 4$, are modeled as in (2):

$$y_k^{(i)} = \begin{pmatrix} (1-\gamma_k^{(i)}) z_k^{(i)} + \gamma_k^{(i)} \widehat{z}_{k/k-1}^{(i)} \\ \psi_k^{(i)} z_{k-1}^{(i)} \end{pmatrix}, \quad k \geq 2; \qquad y_1^{(i)} = \begin{pmatrix} (1-\gamma_1^{(i)}) z_1^{(i)} \\ 0 \end{pmatrix},$$

where, for $i = 1, 2, 3, 4$, $\{\gamma_k^{(i)}\}_{k\geq 1}$ and $\{\psi_k^{(i)}\}_{k\geq 2}$ are sequences of independent Bernoulli random variables whose distributions are determined by the following probabilities:

- $\overline{\gamma}^{(i)} \equiv P\big(\gamma_k^{(i)} = 1\big)$, which, for all $k$, is the probability that the measurement $z_k^{(i)}$ is delayed or lost and, hence, it is not received at time $k$.

- $\overline{\psi}_{\gamma}^{(i)} \equiv P\big(\psi_k^{(i)} = 1/\gamma_{k-1}^{(i)} = 1\big)$, which, for all $k$, is the probability that the measurement $z_{k-1}^{(i)}$ is received at the current time $k$, knowing that it was not received at the previous one, $k-1$.

- $\overline{\psi}^{(i)} \equiv P\big(\psi_k^{(i)} = 1\big) = \overline{\psi}_{\gamma}^{(i)}\overline{\gamma}^{(i)}$, which, for all $k$, is the probability that the measurement $z_{k-1}^{(i)}$ is received and processed at the current time $k$.

Finally, in order to apply the proposed algorithms, it is assumed that all the processes involved in the observation equations satisfy the independence hypotheses imposed on the theoretical model.

A MATLAB program has been designed to obtain the local and distributed fusion estimators, as well as the corresponding error variance matrices, and fifty iterations have been run to show the feasibility and effectiveness of the proposed filtering an fixed-point smoothing algorithms. The estimation accuracy has been examined by analyzing the error variances for different probabilities of the Bernoulli random variables which model the uncertainties of the third and fourth sensors, $\overline{\theta}^{(i)}$, $i = 3, 4$, and several values of the probabilities $\overline{\gamma}^{(i)}$, and the conditional probabilities $\overline{\psi}_{\gamma}^{(i)}$, $i = 1, 2, 3, 4$, have also been considered.

*Performance of the distributed fusion filtering and smoothing estimators.* Let us assume that $\overline{\theta}^{(i)} = 0.5$, $i = 3, 4$, $\overline{\gamma}^{(i)} = 0.1i$, $i = 1, 2, 3, 4$, and consider the same conditional probability for the four sensors, $\overline{\psi}_{\gamma}^{(i)} = 0.5$, $i = 1, 2, 3, 4$. In Figure 2, the error variances of the local filtering estimators and both the distributed filtering and smoothing error variances are displayed. Figure 2 shows, on the one hand, that the error variances of the distributed fusion estimators are smaller than those of every local filter (that is, as expected, the distributed fusion filtering estimators outperform all the local ones), and, on the other, that the error variances corresponding to the distributed fusion smoothers are quite less than those of the distributed filters. From this figure it is also deduced that the accuracy of the smoothers at each fixed-point, $k$, is better as the number of available observations, $k + h$, increases, although this improvement is practically imperceptible for $h > 3$. Similar results are obtained for other values of the probabilities $\overline{\theta}^{(i)}$, $\gamma^{(i)}$ and $\overline{\psi}_{\gamma}^{(i)}$.

*Influence of the missing measurement phenomenon in sensors 3 and 4.* Considering again $\overline{\gamma}^{(i)} = 0.1i$ and $\overline{\psi}_{\gamma}^{(i)} = 0.5$, $i = 1, 2, 3, 4$, and, in order to show the effect of the missing probabilities, $1-\overline{\theta}^{(i)}$, in sensors $i=3,4$, the distributed filtering error variances are displayed in Figure 3 for different values of these probabilities. Specifically, in Figure 3 (a), it is assumed that $\overline{\theta}^{(3)} = \overline{\theta}^{(4)}$, with value range from 0.5 to 0.9, and, in Figure 3 (b), $\overline{\theta}^{(3)}$ is varied from 0.5 to 0.9 and $\overline{\theta}^{(4)} = 0.5$. From these figures, it is observed that the performance of the distributed fusion filter is indeed influenced by the probabilities $\overline{\theta}^{(i)}$, $i=3,4$, and, as expected, it is confirmed that the distributed filtering error variances become smaller as $1-\overline{\theta}^{(i)}$ decreases, which means that the performance of the distributed fusion filters improves as the probability of missing measurements decreases. Analogous results are obtained for the distributed fusion smoothers and considering other values of the probabilities.

*Influence of the probabilities $\overline{\gamma}^{(i)}$.* For $\theta^{(i)} = 0.5$, $i=3,4$, $\overline{\psi}_{\gamma}^{(i)} = 0.5$, $i=1,2,3,4$, different values for the probabilities of measurements not arriving on time, $\overline{\gamma}^{(i)}$, $i=1,2,3,4$, have been considered to analyze the influence of the random delays and packet dropouts on the performance of the distributed estimators. Figure 4 shows the distributed filtering error variances considering the same probabilities in the four sensors, $\overline{\gamma}^{(i)} = \overline{\gamma}$, $i=1,2,3,4$; specifically, the distributed filter performance is analyzed when $\overline{\gamma}$ is varied from 0.1 to 0.9. From this figure it is concluded that, as $\overline{\gamma}$ decreases, the distributed filtering error variances become smaller, which means that, as expected, the smaller the probabilities of transmission failures are, the better estimations are obtained. Analogous results are observed for the distributed fusion smoothers.

Both, the filtering and smoothing error variances, have also been compared assuming different values of the probabilities $\overline{\gamma}^{(i)}$ for each sensor, obtaining similar results to the previous ones in all the considered situations; consequently, analogous conclusions are deduced from these comparisons, some of which are shown in Figure 5. Taking into account that the behavior of the error variances is analogous in all the iterations, only the results at a specific iteration ($k=50$) are displayed in Figure 5.

*Influence of the conditional probabilities.* Considering, as in Figure 4, $\overline{\theta}^{(i)} = 0.5$, $i=3,4$, and $\overline{\gamma}^{(i)} = \overline{\gamma}$, $i=1,2,3,4$, we analyze the influence of the conditional probabilities $\overline{\psi}_{\gamma}^{(i)}$, $i=1,2,3,4$, on the performance of the proposed distributed filtering estimators. Specifically, assuming the same conditional probabilities in the four sensors, $\overline{\psi}_\gamma^{(i)} = \overline{\psi}_\gamma$, $i = 1, 2, 3, 4$, Figure 6 displays the distributed filtering error variances at $k = 50$ versus $\overline{\psi}_\gamma$, for $\overline{\gamma}$ varying from 0.1 to 0.9. This figure shows that, for each value of $\overline{\gamma}$, the error variances decrease when the conditional probability increases. This result was expected since, fixed an arbitrary value of $\overline{\gamma}$, the increasing of $\overline{\psi}_\gamma$ entails that of $\overline{\psi}$, the probability of processing at the current time the delayed measurement in the previous time. Also, we observe that the decreasing of the error variances is more evident for higher values of $\overline{\gamma}$, which was also expected since $\overline{\psi} = \overline{\psi}_\gamma \overline{\gamma}$ and, hence, $\overline{\gamma}$ specifies the increasing rate of $\overline{\psi}$ with respect to $\overline{\psi}_\gamma$.

*Comparison of filtering error variances.* Finally, the error variances of the proposed local and distributed filtering estimators, which use one or two data packets at each time instant, are compared with those of the estimators obtained processing a single packet. Two cases are considered in this comparison: *1)* filtering estimators in [40], that use the measurement predictor as compensation in case of loss, i.e. to compensate the non-punctual arrival of a packet, such packet is replaced by its estimator based on the information previously received, and *2)* filtering estimators in [20] which use the hold-input compensation mechanism, i.e. estimators that use the last measurement received when a packet is lost. More specifically, assuming as in Figure 2 that $\overline{\theta}^{(i)} = 0.5$, $i = 3, 4$ and $\overline{\gamma}^{(i)} = 0.1i$, $i = 1, 2, 3, 4$, Figure 7 displays the error variances of the local and distributed filtering estimators for the three cases we are considering. For a better visualization of the results, only the local estimators of sensors 3 and 4 are displayed in this figure, but analogous conclusions are obtained for the other two sensors. From this figure it is observed that, for both local and distributed estimation, the error variances of the proposed filters are less than those of the filters in [40] and, consequently, the proposed algorithms provide better estimations than those in [40]. This conclusion was expected since, although both of them use the measurement predictor as compensator, under the observation model in the current paper the estimator can additionally use the previous measurement. Figure 7 also shows that both algorithms, the proposed one and that in [40], provide better estimators than the algorithms in [20]. This fact was also expected since, as indicated above, the filters in [20] only use the last measurement received when a packet is lost, while the proposed estimators and those in [40] use all the previous measurements successfully received when the current one does not arrive on time.

## 5. Conclusion

The main conclusions of this paper are summarized as follows:

- A solution to the distributed fusion estimation problem is provided for a class of multi-sensor network systems with random parameter matrices.
- The measured outputs of each sensor are transmitted to a local processor and random failures (one-step delays and packet dropouts) can occur during such local transmissions.
- As in [43], every data packet is assumed to be transmitted just once but, due to the random delays and packet losses, the estimator may receive either one packet, two packets, or nothing. When the current measurement does not arrive on time, its predictor is used as compensator in the design of the estimators.
- By an innovation approach, recursive algorithms for the local filtering and fixed-point smoothing estimators have been designed without requiring full knowledge of the signal evolution model, but only the first and second order moments of the processes involved in the observation model.
- In contrast to most existing estimation algorithms dealing with random delays and packet dropouts, the proposed ones do not require any state vector augmentation technique, thus being computationally more simple.
- A numerical example has been presented to show how uncertain systems with state-dependent multiplicative noise can be covered by the proposed model and how the estimation accuracy is influenced by the transmission failure probabilities.

*Further research topics:*

- An immediate future research could be the extension of the current work to the case of systems with correlation in both the measurement matrices and the additive observation noises.

- It would be also interesting the use of the current model to deal with the losses in the case of connected sensor networks, where each sensor node exchanges information with some other nodes in the network according to a given topology.

- Another important challenge to be addressed in future research is considering the case of imperfect communication also between the local estimators and the fusion node, to cover more general and realistic situations.

- Finally, concerning the packet dropouts modelling, more general situations where the transmission packet losses are bounded and driven by a finite-state Markov process could be also considered.

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## Appendix A. Proof of Theorem 2

**Proof.** Using the general expression (6), the local estimators are written as $\widehat{x}^{(i)}_{k/k+h} = \sum_{l=1}^{k+h} \mathcal{X}^{(i)}_{k,l} \Pi^{(i)-1}_l \mu^{(i)}_l, \ h \geq 1$; hence, it is clear that the local smoothers are recursively obtained by (16) from the filter, $\widehat{x}^{(i)}_{k/k}$. The recursive relation (17) for $\mathcal{X}^{(i)}_{k,k+h} = E\big[x_k y^{(i)T}_{k+h}\big] - E\big[x_k \widehat{y}^{(i)T}_{k+h/k+h-1}\big], \ h \geq 1$, is derived as follows:

- On the one hand, the independence assumptions, together with *(A1)* and (9), lead us to

$$E\big[x_k y^{(i)T}_{k+h}\big] = B_k \overline{\mathcal{H}}^{(i)T}_{A_{k+h}} + \overline{\gamma}^{(i)}_{k+h} E\big[x_k O^{(i)T}_{k+h-1}\big] A^T_{k+h} \overline{H}^{(i)T}_{k+h} C^T_0, \quad h \geq 1.$$

- On the other, using expression (11) for $\widehat{y}^{(i)}_{k+h/k+h-1}$, is clear that

$$E\big[x_k \widehat{y}^{(i)T}_{k+h/k+h-1}\big] = E\big[x_k O^{(i)T}_{k+h-1}\big] \big(\overline{\mathcal{H}}^{(i)}_{A_{k+h}} + \overline{\gamma}^{(i)}_{k+h} C_0 \overline{H}^{(i)}_{k+h} A_{k+h}\big)^T \\ + \mathcal{X}^{(i)}_{k,k+h-1} \Pi^{(i)-1}_{k+h-1} \mathcal{V}^{(i)T}_{k+h-1}, \quad h \geq 1.$$

Therefore, denoting $E^{(i)}_{k,k+h} = E\big[x_k O^{(i)T}_{k+h}\big]$, expression (17) holds and, using (15) for $O^{(i)}_{k+h}$, the recursive expression (18) for the matrices $E^{(i)}_{k,k+h}$ is also clear.

Finally, using (16) for the smoothers $\widehat{x}^{(i)}_{k/k+h}$, the recursive formula for the fixed-point smoothing error covariance matrices, $\Sigma^{(i)}_{k/k+h}$, is immediately deduced. □

## Appendix B. Proof of Lemma 1

**Proof.** Using (15) for $O^{(i)}_k$ and taking into account that $J^{(ij)}_{s,k} = E[O^{(i)}_s \mu^{(j)T}_k]$, for $s = k-1, k$, we get (20) for $r^{(ij)}_k$. Expression (21) for $J^{(ij)}_k = E[O^{(i)}_k \mu^{(j)T}_k]$ is directly obtained using again (15).

To derive (22) for $J^{(ij)}_{k-1,k} = E[O^{(i)}_{k-1}\mu^{(j)T}_k] = E[O^{(i)}_{k-1} y^{(j)T}_k] - E[O^{(i)}_{k-1}\widehat{y}^{(j)T}_{k/k-1}]$ we proceed in the following way:

- $E[O^{(i)}_{k-1} y^{(j)T}_k]$ is obtained by using (7) for $y^{(j)}_k$ and calculating the resulting expectations as follows:

  - First, by applying the OPL, $E\big[O^{(i)}_{k-1} z^{(j)T}_k\big] = E\big[O^{(i)}_{k-1}\widehat{x}^{(i)T}_{k/k-1}\big]\overline{H}^{(j)T}_k$ and $E\big[O^{(i)}_{k-1} x^T_{k-1}\big] = E\big[O^{(i)}_{k-1}\widehat{x}^{(i)T}_{k-1/k-1}\big]$.
  - Next, using (10) for the local predictor and filter of the signal, together with the definition of $r^{(ij)}_{k-1}$, we otain $E\big[O^{(i)}_{k-1}\widehat{x}^{(i)T}_{s/k-1}\big] = r^{(i)}_{k-1} A^T_s$, for $s = k-1, k$, and $E\big[O^{(i)}_{k-1}\widehat{z}^{(j)T}_{k/k-1}\big] = r^{(ij)}_{k-1} A^T_k \overline{H}^{(j)T}_k$.
  - Finally, using (15) for $O^{(i)}_{k-1}$ and the uncorrelation between $O^{(i)}_{k-2}$ and $V^{(j)}_{k-1}$, we get $E\big[O^{(i)}_{k-1} V^{(j)T}_{k-1}\big] = J^{(i)}_{k-1}\Pi^{(i)-1}_{k-1}\mathcal{V}^{(ji)T}_{k-1}$, with $\mathcal{V}^{(ji)}_k = E\big[V^{(j)}_k \mu^{(i)T}_k\big] = E\big[V^{(j)}_k y^{(i)T}_k\big]$.

  Then, from (9) for $\overline{\mathcal{H}}^{(j)}_{A_k}$, we conclude that

$$E[O^{(i)}_{k-1} y^{(j)T}_k] = r^{(i)}_{k-1}\overline{\mathcal{H}}^{(j)T}_{A_k} + \overline{\gamma}^{(j)}_k r^{(ij)}_{k-1} A^T_k \overline{H}^{(j)T}_k C^T_0 + J^{(i)}_{k-1}\Pi^{(i)-1}_{k-1}\mathcal{V}^{(ji)T}_{k-1}. \quad (27)$$

- To obtain $E[O^{(i)}_{k-1}\widehat{y}^{(j)T}_{k/k-1}]$ we use (11) for $\widehat{y}^{(j)}_{k/k-1}$, and the definitions of $r^{(ij)}_k$ and $J^{(ij)}_k$ lead us to

$$E[O^{(i)}_{k-1}\widehat{y}^{(j)T}_{k/k-1}] = r^{(ij)}_{k-1}\left(\overline{\mathcal{H}}^{(j)}_{A_k} + \overline{\gamma}^{(j)}_k C_0 \overline{H}^{(j)}_k A_k\right)^T - J^{(ij)}_{k-1}\Pi^{(j)-1}_{k-1}\mathcal{V}^{(j)T}_{k-1}.$$

From both expectations, we easily conclude that expression (22) holds for $J_{k-1,k}^{(ij)}$. Finally, expression (23) for $\mathcal{V}_k^{(ji)}$ is easily obtained from the definition of $V_k^{(j)}$. □

### Appendix C. Proof of Lemma 2

**Proof.** To obtain expression (24) for $\Pi_k^{(ij)},\ i \neq j$, first we write

$$\Pi_k^{(ij)} = E\big[y_k^{(i)} y_k^{(j)T}\big] - E\big[y_k^{(i)} \widehat{y}_{k/k-1}^{(j)T}\big] - E\big[\widehat{y}_{k/k-1}^{(i)} \mu_k^{(j)T}\big].$$

- From (3) and (4), we get $y_k^{(i)} = \xi_k^{(i)} + \gamma_k^{(i)} C_0 \widehat{z}_{k/k-1}^{(i)}$, and using that $\widehat{z}_{k/k-1}^{(i)} = \overline{H}_k^{(i)} A_k O_{k-1}^{(i)}$, together with the definition of $r_k^{(ij)}$, we obtain

$$\begin{aligned} E\big[y_k^{(i)} y_k^{(j)T}\big] = \Sigma_k^{\xi^{(ij)}} &+ \overline{\gamma}_k^{(i)} C_0 \overline{H}_k^{(i)} A_k E[O_{k-1}^{(i)} y_k^{(j)T}] \\ &+ \overline{\gamma}_k^{(j)} E[y_k^{(i)} O_{k-1}^{(j)T}] A_k^T \overline{H}_k^{(j)T} C_0^T \\ &- \overline{\gamma}_k^{(i)} \overline{\gamma}_k^{(j)} C_0 \overline{H}_k^{(i)} A_k r_{k-1}^{(ij)} A_k^T \overline{H}_k^{(j)T} C_0^T. \end{aligned}$$

- Using (7) for $y_k^{(i)}$, an analogous procedure to that used to derive (27), leads us to

$$\begin{aligned} E\big[y_k^{(i)} \widehat{y}_{k/k-1}^{(j)T}\big] = &\ \overline{\mathcal{H}}_{A_k}^{(i)} \mathcal{O}_{k,k-1}^{(j)T} - \overline{\gamma}_k^{(i)} C_0 \overline{H}_k^{(i)} A_k \left( E[O_{k-1}^{(i)} y_k^{(j)T}] - J_{k-1,k}^{(ij)} \right) \\ &- \mathcal{V}_{k-1}^{(ij)} \Pi_{k-1}^{(j)-1} \mathcal{Y}_{k,k-1}^{(j)T}, \end{aligned}$$

  where we have used that $E\big[O_{k-1}^{(i)} \widehat{y}_{k/k-1}^{(j)T}\big] = E\big[O_{k-1}^{(i)} y_k^{(j)T}\big] - E\big[O_{k-1}^{(i)} \mu_k^{(j)T}\big]$ and the definition of $J_{k-1,k}^{(ij)}$.

- Finally, from (11) for $\widehat{y}_{k/k-1}^{(i)}$ and the definitions of $J_{k-1,k}^{(ij)}$ and $\Pi_{k-1,k}^{(ij)}$, we obtain that

$$E[\widehat{y}_{k/k-1}^{(i)} \mu_k^{(j)}] = \left( \overline{\mathcal{H}}_{A_k}^{(i)} + \overline{\gamma}_k^{(i)} C_0 \overline{H}_k^{(i)} A_k \right) J_{k-1,k}^{(ij)} + \mathcal{V}_{k-1}^{(i)} \Pi_{k-1}^{(i)-1} \Pi_{k-1,k}^{(ij)}.$$

From the above items, using (27) for $E[O_{k-1}^{(i)} y_k^{(j)T}]$ and after some manipulations, expression (24) for $\Pi_k^{(ij)}$ is obtained.

Next, we prove expression (25) for $\Pi_{k-1,k}^{(ij)} = E\big[\mu_{k-1}^{(i)} y_k^{(j)T}\big] - E\big[\mu_{k-1}^{(i)} \widehat{y}_{k/k-1}^{(j)T}\big]$. Taking into account (7) for $y_k^{(j)}$ and using again a similar procedure to that used to derive (27), we obtain

$$E\big[\mu_{k-1}^{(i)} y_k^{(j)T}\big] = J_{k-1}^{(i)T} \overline{\mathcal{H}}_{A_k}^{(j)T} + \overline{\gamma}_k^{(j)} J_{k-1}^{(ji)T} \overline{H}_k^{(j)T} C_0^T + \mathcal{V}_{k-1}^{(ji)T};$$

using now expression (11) for $\widehat{y}_{k/k-1}^{(j)}$ and the definition of $J_k^{(ij)}$, we have

$$E\big[\mu_{k-1}^{(i)} \widehat{y}_{k/k-1}^{(j)T}\big] = J_{k-1}^{(ji)T} \big(\overline{\mathcal{H}}_{A_k}^{(j)} + \overline{\gamma}_k^{(j)} C_0 \overline{H}_k^{(j)} A_k\big)^T + \Pi_{k-1}^{(ij)} \Pi_{k-1}^{(j)-1} \mathcal{V}_{k-1}^{(j)T}.$$

The above relations lead us to expression (25) for $\Pi_{k-1,k}^{(ij)}$, and Lemma 2 is proven. □

## Appendix D. Proof of Lemma 3

**Proof.** Taking into account expression (11) for $\widehat{y}_{k/k-1}^{(j)}$ and the definition of $\Lambda_{k,L}^{(ij)}$, we have that

$$\begin{aligned}\Phi_{k,L}^{(ij)} = E\big[\widehat{x}_{k/L-1}^{(i)} y_L^{(j)T}\big] - \Lambda_{k,L-1}^{(ij)} \big(\overline{\mathcal{H}}_{A_L}^{(j)} + \overline{\gamma}_L^{(j)} C_0 \overline{H}_L^{(j)} A_L\big)^T \\ - E\big[\widehat{x}_{k/L-1}^{(i)} \mu_{L-1}^{(j)T}\big] \Pi_{L-1}^{(j)-1} \mathcal{V}_{L-1}^{(j)T}.\end{aligned}$$

- On the one hand, using again (7) for $y_k^{(j)}$ and an analogous reasoning to that used to derive (27), we obtain

$$\begin{aligned}E\big[\widehat{x}_{k/L-1}^{(i)} y_L^{(j)T}\big] = \Lambda_{k,L-1}^{(i)} \overline{\mathcal{H}}_{A_L}^{(j)T} + \Lambda_{k,L-1}^{(ij)} \left(\overline{\gamma}_L^{(j)} C_0 \overline{H}_L^{(j)} A_{L-1}\right)^T \\ + \mathcal{X}_{k,L-1}^{(i)} \Pi_{L-1}^{(i)-1} \mathcal{V}_{L-1}^{(ji)T}.\end{aligned}$$

- On the other, from (6), it is clear that $\widehat{x}_{k/L-1}^{(i)} = \widehat{x}_{k/L-2}^{(i)} + \mathcal{X}_{k,L-1}^{(i)} \Pi_{L-1}^{(i)-1} \mu_{L-1}^{(i)}$, and, hence, $E\big[\widehat{x}_{k/L-1}^{(i)} \mu_{L-1}^{(j)T}\big] = \Phi_{k,L-1}^{(ij)} + \mathcal{X}_{k,L-1}^{(i)} \Pi_{L-1}^{(i)-1} \Pi_{L-1}^{(ij)}$.

Substituting the above expectations, expression (26) for $\Phi_{k,L}^{(ij)}$, $L > k \geq 1$, $i \neq j$, is immediately derived. From (10) and the definition of $J_{k-1,k}^{(ij)}$, the initial condition, $\Phi_k^{(ij)} = A_k J_{k-1,k}^{(ij)}$, is clear, and, from the OPL, we have that $\Phi_{k,L}^{(i)} = 0$, $L \geq k \geq 1$, so Lemma 3 is proven. □

## Appendix E. Proof of Lemma 4

**Proof.** From (16) and (15), $\widehat{x}^{(i)}_{k/L} = \widehat{x}^{(i)}_{k/L-1} + \mathcal{X}^{(i)}_{k,L}\Pi^{(i)-1}_{L}\mu^{(i)}_{L}$, and $O^{(j)}_{L} = O^{(j)}_{L-1} + J^{(j)}_{L}\Pi^{(j)-1}_{L}\mu^{(j)}_{L}$, respectively, and the expression of $\Lambda^{(ij)}_{k,L}$, $L > k$, is immediately deduced. The initial condition, $\Lambda^{(ij)}_{k} = A_k r^{(ij)}_{k}$, is also directly obtained from (14) and the definition of $r^{(ij)}_{k}$, so Lemma 4 is proven. □

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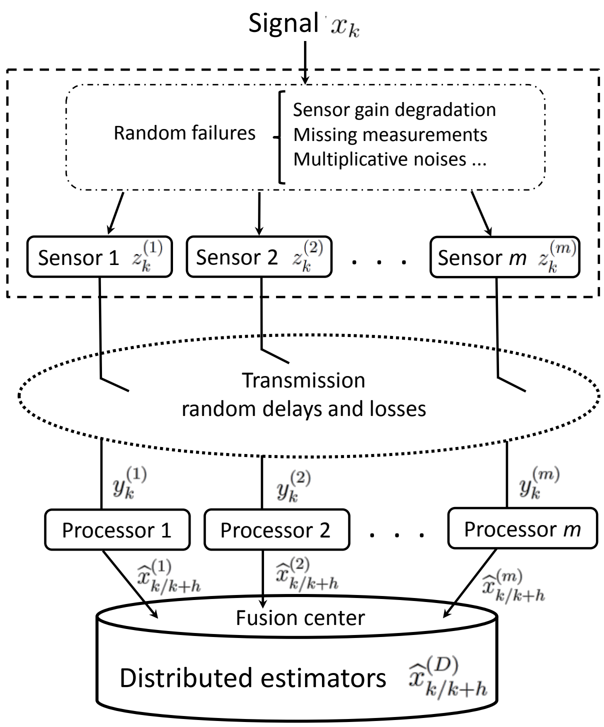


Figure 1: Conceptual diagram of distributed fusion filtering estimation.

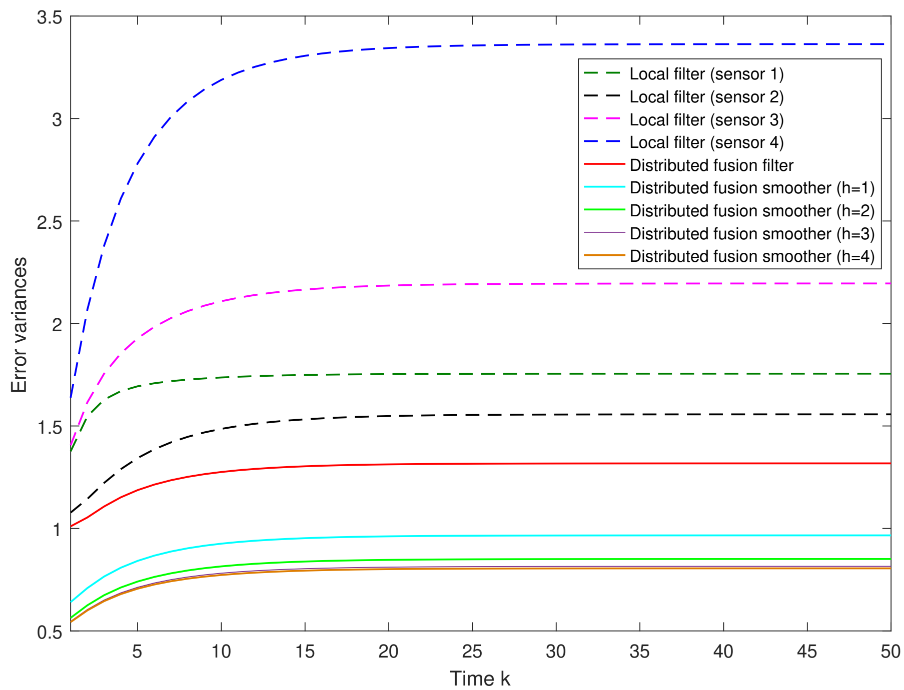


Figure 2: Error variance comparison of the local filters and distributed fusion filter and smoothers.

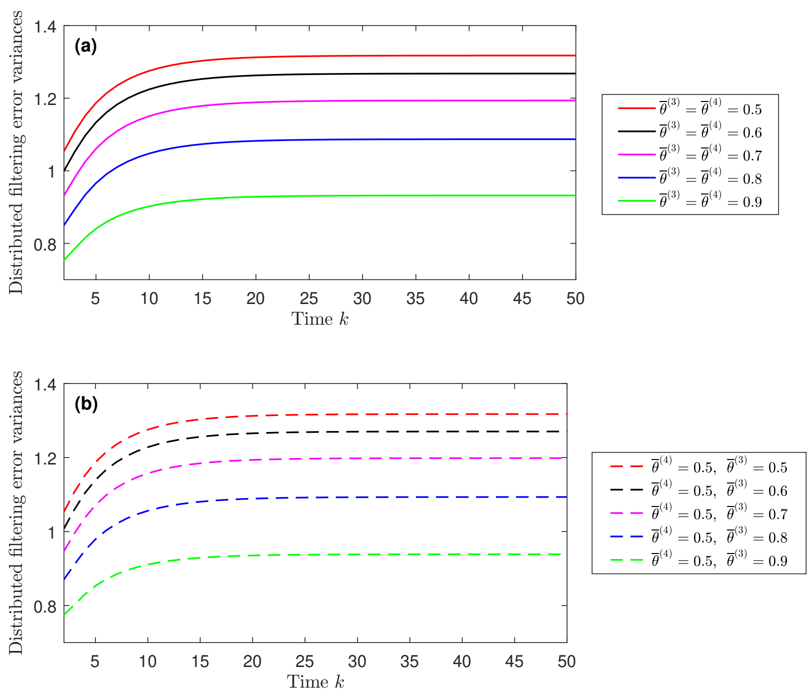


Figure 3: Distributed filtering error variances for different values of $\overline{\theta}^{(3)}$ and $\overline{\theta}^{(4)}$.

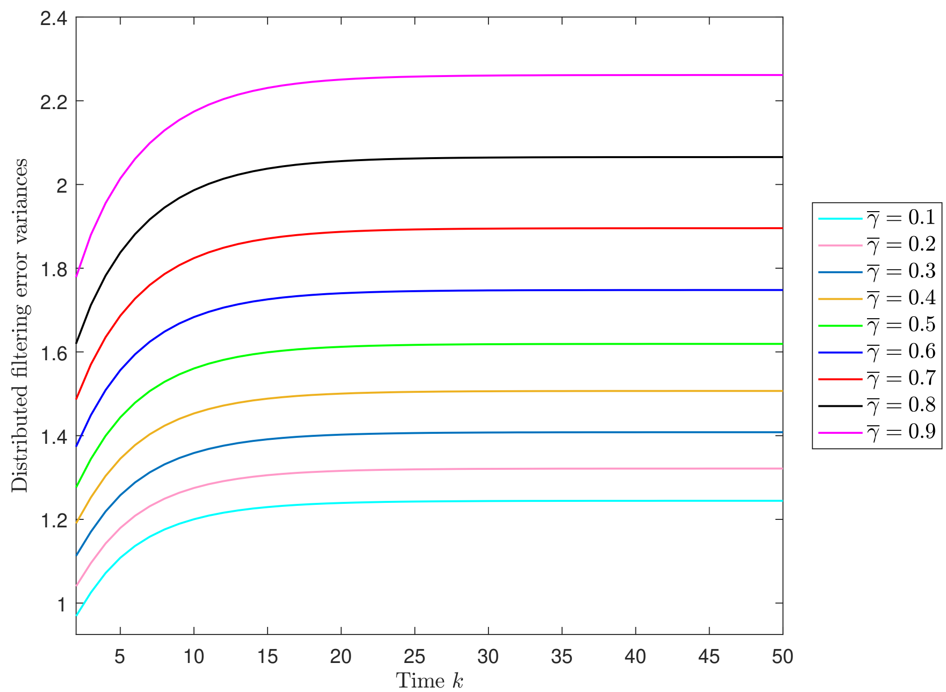


Figure 4: Distributed filtering error variances when $\overline{\theta}^{(i)} = 0.5$, $i = 3, 4$, $\overline{\psi}_{\gamma}^{(i)} = 0.5$ and $\gamma^{(i)} = \gamma$, for $i = 1, 2, 3, 4$, with $\overline{\gamma}$ varying from 0.1 to 0.9.

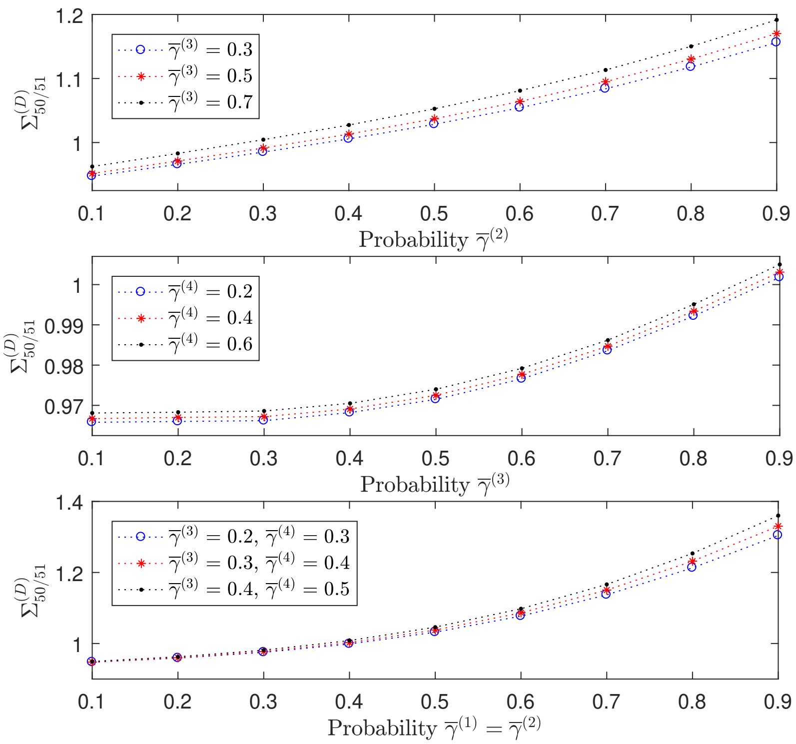


Figure 5: Distributed smoothing error variances $\Sigma^{(D)}_{50/51}$ when $\overline{\theta}^{(i)} = 0.5$, $i = 3, 4$, $\overline{\psi}^{(i)}_{\gamma} = 0.5$, $i = 1, 2, 3, 4$, for different values of the probabilities $\overline{\gamma}^{(i)}$: (a) $\overline{\gamma}^{(1)} = \overline{\gamma}^{(4)} = 0.1$, $\overline{\gamma}^{(3)} = 0.3, 0.5, 0.7$, versus $\overline{\gamma}^{(2)}$; (b) $\overline{\gamma}^{(1)} = 0.1$, $\overline{\gamma}^{(2)} = 0.2$, $\overline{\gamma}^{(4)} = 0.4, 0.6, 0.8$, versus $\overline{\gamma}^{(3)}$; (c) different values of $\overline{\gamma}^{(3)}$ and $\overline{\gamma}^{(4)}$ versus $\overline{\gamma}^{(1)} = \overline{\gamma}^{(2)}$.

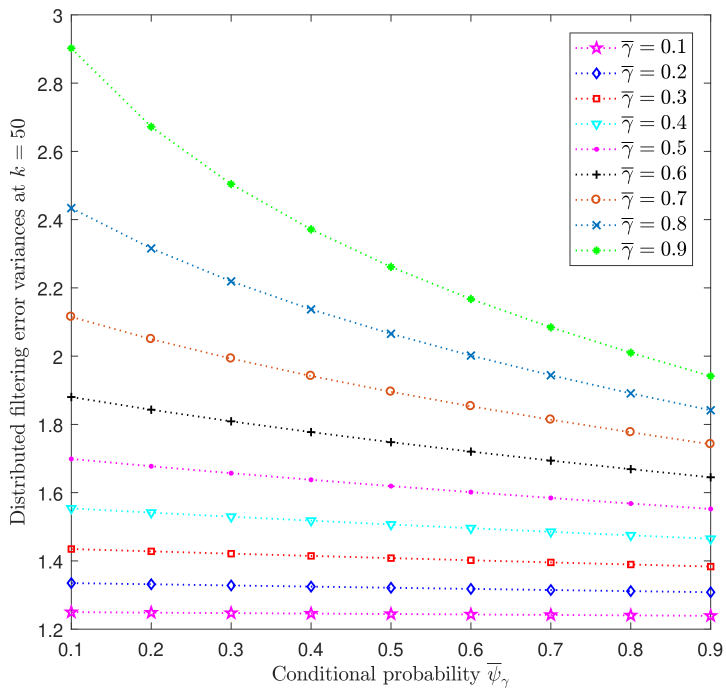


Figure 6: Distributed filtering error variances at $k = 50$, versus $\overline{\psi}_\gamma$, for $\overline{\gamma}$ varying from 0.1 to 0.9., when $\overline{\theta}^{(i)} = 0.5$, $i = 3, 4$.

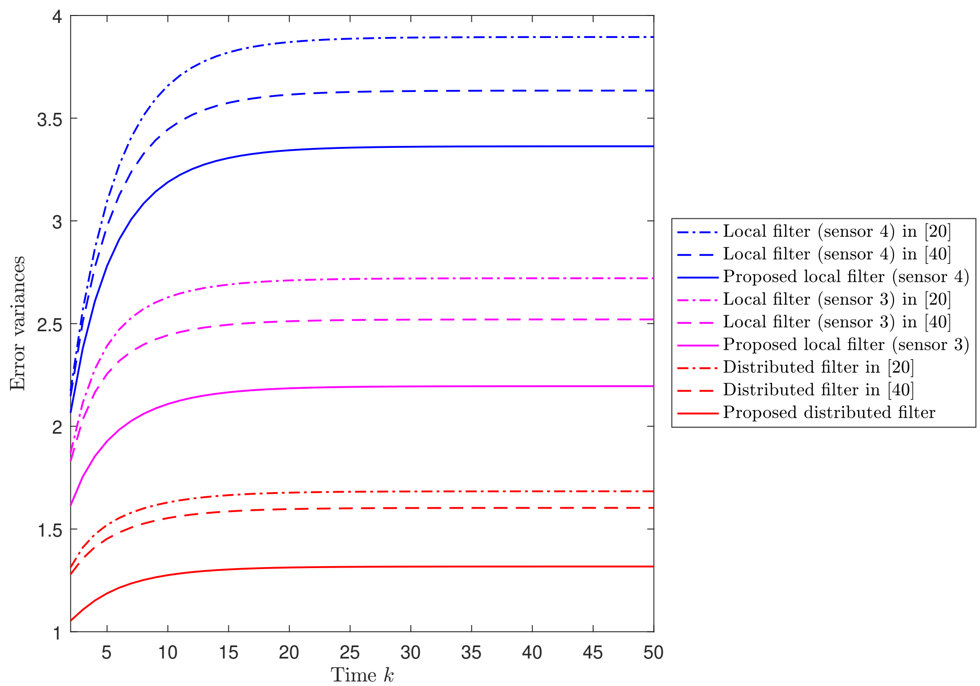


Figure 7: Comparison of filtering error variances when $\overline{\theta}^{(i)} = 0.5$, $i = 3, 4$ and $\overline{\gamma}^{(i)} = 0.1i$, $i = 1, 2, 3, 4$.