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MASTER THESIS

Banach Algebras:

Local and 2-Local Isometries

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DECLARACIÓN

En cumplimiento de la normativa aprobada en Consejo de Gobierno de 4 de marzo de 2013, sobre Directrices de la Universidad de Granada para el desarrollo de la asignatura "Trabajo Fin de Máster" de sus títulos de máster (Art 8,4)

D.Dª ....

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Introduction

The final aim of this Master Thesis is to survey the theory of *Banach algebras* in order to study *local isometries* and *2-local isometries* on classical Banach spaces and algebras. The memory is separated into four different parts: whereas throughout the first and second chapters we shall fix notation and gather the main results and tools which are necessary to reach our goals, the remaining chapters will be exclusively based on the study of local and 2-local maps on certain Banach spaces.

Let X and Y be Banach spaces, and let us pick a class $S \subset L(X, Y)$, we shall say that a map T in $L(X, Y)$, the space of all linear maps from X to Y, is a local S-map if for each $x \in X$, there exists $T_x \in S$, depending on x, such that, $T(x) = T_x(x)$. Furthermore, we shall say that a (non-necessarily linear) mapping $\Delta : X \to Y$ is a 2-local S-Map if for every $x, y \in X$, there exists $T_{x,y} \in S$, depending on x and y, satisfying

$$T(x) = T_{x,y}(x) \text{ and } T(y) = T_{x,y}(y).$$

In order to give a short (and very easy) preview of what we are going to do along these pages we are going to state first a simplified version of a surprising and celebrated result which provides a necessary and sufficient condition for a linear functional on the Banach algebra $C(K)$, of all continuous complex-valued functions on a compact Hausdorff space K, to be multiplicative.

Theorem (Gleason-Kahane-Zelazko). *Let K be a compact Hausdorff space. If $T : A = C(K) \to \mathbb{C}$ is a continuous $\mathbb{C}$-linear functional, then the following statements are equivalent:*

i) *T is an homomorphism (i.e. T preserves products),*

ii) *$T(f) \in Im(f) = f(K), \ \forall \ f \in A.$* ■

On the one hand, the functional $\mathcal{I} : \mathcal{C}([0,1],\mathbb{R}) \to \mathbb{R}$ defined by

$$\mathcal{I}(f) := \int_0^1 f(t)dt, \ \ \forall \ f \in \mathcal{C}([0,1],\mathbb{R})$$

is a continuous $\mathbb{R}$-linear functional on the real Banach algebra $\mathcal{C}([0,1],\mathbb{R})$ of all real-valued continuous functions on $[0,1]$. By the Mean Value Theorem, for each continuous function f, we have

$$\mathcal{I}(f) = \int_0^1 f(t)dt = f(c)$$

for at least one $c \in]0,1[$. That is $\mathcal{I}(f)$ satisfies the hypothesis of the above result but clearly $\mathcal{I}$ is not multiplicative. So, the previous theorem only holds in the setting of complex Banach algebras.

On the other hand, the *Gleason-Kahane-Zelazko theorem* motivates the concept of *local homomorphism.* A *local homomorphism* T on a Banach algebra A is a linear functional $T : A \to \mathbb{C}$ such that for each $a \in A$ there exists a multiplicative linear functional $T_a \in A^*$ (which depends on f) such that $T_a(a) = T(a)$. When $A = C(K)$ as above, by applying $i) \Rightarrow ii)$ in the above theorem, we obtain that $T(f) = T_f(f) \in \text{Im}(f)$. Now, the implication $ii) \Rightarrow i)$ in the previous theorem shows that T is multiplicative.

We could also take into consideration those maps $T : C(K) \to \mathbb{C}$ (not necessarily linear) such that for each $f, g \in C(K)$, there exists a multiplicative linear functional $T_{f,g} \in C(K)^*$, depending on f and g, such that $T_{f,g}(f) = T(f)$ and $T_{f,g}g = Tg$. These maps will be called *2-local homomorphisms.*

Although the notions of local and 2-local homomorphism seems to be, a priori, *too* weak, the *Gleason-Kahane-Zelazko theorem* assures us that every local homomorphism is in fact an homomorphism. In this memory we are going to generalize these notions by replacing $C(K)$ with an arbitrary Banach Algebra A, and by considering a random class of maps $S \subseteq L(A)$ instead of the class of homomorphisms.

When notation is fixed, we shall study the following natural questions:

Is every local S-Map an element of the class S?

and, of course,

Is every 2-local S-Map an element of the class S?

So, the main core of this memory will consists on a detailed survey on the different solutions to these problems on some classical Banach spaces. We are particularly interested in the case in which S is the class of all the surjective isometries on those spaces.

The main target of the first chapter is introducing the main concepts which will be necessary for the comprehension of this memory. When notation is fixed, we will also remind some theorems which will be useful in the future. At first, we shall study the notion of *complex Banach algebra*. Then, we will deal about the natural concept which appears when we have a product, we mean the notion of *invertible elements*. Throughout the third section of the chapter we shall see how the concept of *spectrum of a matrix* can be generalized in the context of *complex Banach algebras*. Later, we shall estate some properties of homomorphisms on a complex Banach algebra. We will take into account that a C^**-algebra* is a very particular kind of complex Banach algebra which can be described, via *Gelfand-Naimark representation theorem* in a very concrete form. In addition, we shall introduce two important tools which are indispensable in order to reach our goal of study *local and 2-local isometries*,e the theory of extreme points and the theory of linear operators on Hilbert spaces. Finally we will introduce the Schatten clases, a family of Banach algebras.

The purpose of this memory is to study *local and 2-local surjective isometries* on classical Banach spaces and therefore, Knowing the form of the surjective linear isometries on that spaces will be essential. In the second chapter we shall start by recalling the precise definition of isometry and introducing a folklore result, the *Mazur-Ulam* theorem which relates surjective isometries and linearity. Throughout the first section we will study the surjective linear isometries on some Banach spaces as sequences spaces or the Schatten classes. Later, at the second section, we will introduce a weaker version of the notion of surjective linear *isometry*, the concept of partial isometry, which will be useful when we work with Hilbert spaces.

Finally, on the third chapter, we reach the main core of this memory. In this chapter we shall study *local surjective isometries* on certain Banach spaces. At first, we shall recall the abstract definition of *local map* and then, we show that all *local homomorphisms* on a complex Banach algebra are homomorphisms. At the last section, we will proof that every *local surjective linear isometry* on a Hilbert space is a true *surjective linear isometry* if and only if the Hilbert space has finite dimension. Finally, we will show that there exist several examples of classical Banach spaces (with infinite dimension) satisfying the desired property:

"Every local surjective isometry is a surjective isometry".

Those privileged spaces will be the spaces of continuous complex function defined on a compact Hausdorff space, and those sequence spaces and Schatten classes which are not isomorphic to a Hilbert space.

In the fourth chapter of this memory, we will show that it is possible to formulate a weaker version for the concept of *local map*. This new notion will be the concept of *2-local map*. We will start giving the precise definition of this concept and later, we will study the question

Is every *2-local S-map* an element of S?

for certain Banach spaces and certain classes of mapping just as we did at the previous chapter. As we will see, every *2-local homomorphism* on a complex Banach algebra is in fact a homomorphism. In addition, under the assumption of surjectivity, we will see that studying *2-local surjective isometries* to studying *local surjective isometries*. Finally, we will show that it is possible studying *2-local surjective isometries* although we do not suppose surjectivity and we will prove that in some sequence spaces every *2-local surjective isometry* is a true surjective linear isometry.

Chapter 1

Banach Algebras

This chapter is devoted to introduce the main concepts which are required along the memory. When notation is fixed, we shall also recall some theorems which will be useful in the future.

The first concept we are going to study is the notion of *complex Banach algebra*, which can be defined as a Banach space equipped with a product. Next, we will talk about the natural concept which appears when we have a product, we mean the notion of *invertible elements*. Throughout the third section of the chapter we shall see how the (possibly familiar) concept of *spectrum of a matrix* can be generalized in the context of *complex Banach algebras*. Later we shall estate some properties of multiplicative functionals on the dual space of a complex Banach algebra, which will be called *homomorphisms*, we will take into account that a *C^*-algebra* is a very particular kind of complex Banach algebra which can be described, via *Gelfand-Naimark representation theorem* in a very concrete form. In addition, we shall introduce in this chapter two important tools which are indispensable in order to reach our goal of study local and 2-local isometries, these tools are the theory of extreme points and a survey on the theory of linear operators on Hilbert spaces. Finally we will introduce the Schatten clases, which is a well known family of Banach algebras.

1.1 Complex Banach algebras

In this section we shall introduce the definition of complex Banach algebra. A complex Banach algebra is a particular case of complex Banach space where we can "associatively multiply" elements. Note that the possibility of being able to sum and multiply vectors will allow us, via Taylor series and completeness, define standard operations as exponential or logarithm mappings for certain elements in the space. After introducing Banach algebras, we shall provide several conccrete examples in order to show that many classical Banach spaces are in fact complex Banach algebras.

Definition 1.1 (*Complex Banach algebra*)**.** *Let $(A, ||\cdot||)$ be a complex Banach space and consider a bilinear map $\star : A \times A \to A$ defined by $\star(a, b) = a \star b$ such that:*

i) Associativity: $(a \star b) \star c = a \star (b \star c), \ \ \forall\ a, b, c \in A;$

ii) Continuity: $||a \star b|| \leq ||a|| \cdot ||b||, \ \ \forall\ a, b \in A.$

The triplet $(A, ||\cdot||, \star)$ is said to be a complex Banach algebra and the application $\star$ is usually called the product of A. When, in addition, the map $\star$ satisfies

iii) Commutativity: $a \star b = b \star a, \ \ \forall\ a, b \in A,$

we shall say that such a triplet is a commutative complex Banach algebra. If there exists an element $u \in A$ such that

iv) Existence of unit element: $u \star a = a \star u = u, \ \ \forall\ a \in A,$

the Banach algebra will be called unital.

Omit the name of the bilinear map $\star$ in the last definition and to apply the notation by juxtaposition is usual. So if we say that $(A, ||\cdot||)$ is a complex Banach algebra, we shall understand that there is a bilinear application $A \times A \to A$, $(a, b) \to ab$ satisfying $i)$ and $ii)$. Moreover, we shall abuse of the notation and say that A is a *complex Banach algebra*, so we shall have to understand that A is a Banach space for certain norm $||\cdot||$ and also, that $(A, ||\cdot||)$ is a complex Banach algebra in the previous sense. Is very easy to show the following result:

Proposition 1.2. *Let A be a unital complex Banach algebra. Then there exists a unique unit element.* ■

Along this memory we shall introduce several examples of complex Banach algebras in order to study some properties of the maps between them. Now we shall take into account that the concept recently exposed is not as abstract as we could think.

Example 1.3. *Let K be a compact Hausdorff space and consider*

$$\mathcal{C}(K) := \{f : K \to \mathbb{C} : f \text{ is continuous}\}$$

and the supremum norm $||\cdot||_\infty$. Then $(\mathcal{C}(K), ||\cdot||_\infty)$ is a commutative unital complex Banach algebra with the product $\mathcal{C}(K)\times\mathcal{C}(K) \to \mathcal{C}(K)$ given by pointwise multiplication, that is, $(fg)(t) = f(t)g(t)$.

Example 1.4. *The Banach space c_0 of complex sequences which are convergent to zero with the supremum norm is also a commutative complex Banach algebra if we consider the usual product of complex sequences. In this case, c_0 admits no unit.*

Example 1.5. *Let X be a complex Banach space and consider*

$$B(X) := \{T : X \to X \ : T \text{ is linear and bounded}\} \tag{1.1}$$

which is also a complex Banach space with the operator norm. Then, the product $B(X) \times B(X) \to B(X)$ given by composition, $TS = T \circ S$ makes $(B(X), ||\cdot||_\infty)$ into a non-commutative complex Banach algebra.

1.2 Invertible elements

The abstract notions of product and unit allow us to generalize the well-known concept of *invertible element*, which will be the main axis of this part of the memory. After giving a precise definition of this, we will state some interesting properties which will be useful in the future. Across this section A will denote a unital complex Banach algebra, with unit element $1 \in A$.

Definition 1.6 (*Invertible elements*)**.** *An element $a \in A$ is said to be*

i) Right invertible: if there exists an element $b \in A$ such that $ab = 1$. Such element $b \in A$ is said to be a right inverse of $a \in A$.

ii) Left invertible: if there exists an element $b \in A$ such that $ba = 1$. Such element $b \in A$ is said to be a left inverse of $a \in A$.

iii) Invertible: if there exists an element $b \in A$ such that $ab = ba = 1$. Such element $b \in A$ is said to be the inverse of $a \in A$.

The set consisting on all invertible elements in A will be denote by $Inv(A)$.

Remark 1.7. *It is very easy to see that if $a \in Inv(A)$ then there exists a unique inverse element of $a \in A$ which we shall denote a^{-1}. Also it is true that $a \in Inv(A)$ if and only if a is both left invertible and right invertible.*

The following result gives several properties of the inverse of an element. After a short deliberation, it is very easy to realise that they are generalizations of the statements which are proved in every basic course of complex variable.

Proposition 1.8. *The next statements are true:*

i) If $a \in A$ is such that $||a-1|| < 1$ then $a \in Inv(A)$ and $a^{-1} = \sum_{k=0}^{\infty}(1-a)^n$,

ii) Let be $a, b \in A$. If $||a-b|| < ||b^{-1}||^{-1}$ then $||a^{-1}|| \leq \dfrac{||b^{-1}||}{1-||1-b^{-1}a||}$,

iii) $Inv(A)$ is an open set in A,

iv) The map $Inv : Inv(A) \to Inv(A)$ defined by $Inv(a) = a^{-1}$, $\forall\, a \in A$ is a continuous mapping.

Proof. The proof can be found throughout in the first chapter of [BD73]. ■

1.3 Spectrum of an element in a complex Banach algebra

In this section we shall generalize the concept of spectrum of a matrix into the notion of spectrum of a certain element in a unital complex Banach algebra. We shall start by giving a rigorous definition and then we will show how that definition is a true generalization of the spectrum of a matrix with order $n \in \mathbb{N}$. It will be possible thanks to the fact that the vector space $\mathcal{M}_{n\times n}(\mathbb{C})$ can equipped with the norm given by the supremum norm that each matrix has when it is seen as the linear matrix which it represents (when a basis of $\mathbb{C}^n$ is fixed).

Let us fix a unital complex Banach algebra A for the rest of the section. For an element $a \in A$ we define the *spectrum of a in A* as the set

$$\sigma(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \notin \mathrm{Inv}(A)\}.$$

Also we define the *spectral radius of a in A* by

$$r(a) := \sup \{|\lambda| : \lambda \in \sigma(a)\}.$$

Note that when we set a compact Hausdorff space K for $f \in \mathcal{C}(K)$ we have $\sigma(f) = \{\lambda \in \mathbb{C} : \exists\, x \in K : f(x) - \lambda = 0\} = \mathrm{Im}(f)$. In addition, if $n \in \mathbb{N}$ and $M \in \mathcal{M}_{n\times n}(\mathbb{C})$ we have $\sigma(M) = \{\lambda \in \mathbb{C} : \mathrm{Ker}(M - \lambda Id) \neq 0\}$, that is $\sigma(M)$ is the set of all eigenvalues of M, as we expected.

Throughout the rest of this section we will state three classical results. The first assures that the spectrum of an element in a unital complex Banach algebra is always a non-empty compact set, and gives an upper bound for the modulus of the elements which forms the spectrum of that element.

Theorem 1.9 (Gelfand)**.** *Let A be a unital complex Banach algebra. Then, for each a in A, the spectrum $\sigma(a)$ is a non empty compact subset of $\mathbb{C}$ contained in the closed unit ball of $\mathbb{C}$ of center 0 and radius $||a||$.*

Proof. The proof can be seen at *Theorem 10.13* in [Rud91]. ∎

We shall say that a unital complex Banach algebra A is a *division algebra* if $A \setminus \{0\} = \mathrm{Inv}(A)$. The *Gelfan-Mazur theorem* is a milestone result in the theory of Banach algebras.

Theorem 1.10 (Gelfand-Mazur)**.** *If A is a division complex Banach algebra then A is isometrically isomorphic to the complex field.*

Proof. The proof can be seen in *Theorem 10.14* in [Rud91]. ∎

Finally, we recall the *Gelfand-Beurling formula*, an useful result which allows us to calculate the spectral radius of an element in a complex Banach algebra.

Theorem 1.11 (Gelfand-Beurling Formula)**.** *Let A be a unital complex Banach algebra. For every $a \in A$:*

$$r(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}.$$

Proof. The proof can be seen in *Theorem 10.13* in [Rud91]. ■

1.4 Homomorphisms

Now, it is time to consider the dual space of a complex Banach algebra i.e. the Banach space consisting of all linear continuous functionals whose domain is the former Banach algebra. In this space we can find a very particular kind of continuous linear functionals: the linear multiplicative functionals, which are usually called *homomorphisms*. After give the precise definition, we will show that homomorphisms are in fact elements of the dual space and finally we will state two results which relates homomorphisms on a complex Banach algebra and the spectrum of the elements in that complex Banach algebra.

Let A be a complex Banach algebra. We shall say that a functional $\phi : A \to \mathbb{C}$ is a *homomorphism* when $\phi(xy) = \phi(x)\phi(y), \ \forall\ x, y \in A$.

Proposition 1.12. *Let A be a complex Banach algebra and consider a multiplicative linear functional $\phi : A \to \mathbb{C}$. Then ϕ is continuous.*

Proof. Suppose $\phi : A \to \mathbb{C}$ is a linear multiplicative functional which is not continuous. Then there exists $x \in A$ such that $||x|| < 1$ and $\phi(x) = 1$. Define $y = \sum_{k=1}^{\infty} x^k$. Then $x + xy = y$ and

$$1 + \phi(y) = \phi(x) + \phi(x)\phi(y) = \phi(y)$$

which is not possible. ■

These theorems exhibit the relations between properties of different nature of a linear functional. The first one is the Gleason-Kahane-Zelazko theorem which has already been revisited in the introduction.

Theorem 1.13 (Gleason-Kahane-Zelazko)**.** *Let A be a complex Banach algebra. Let $f : A \to \mathbb{C}$ be a linear mapping. Then, the following statements are equivalent:*

i) f is a multiplicative linear functional;

ii) $f(x) \in \sigma(x), \ \forall\, x \in A$.

Proof. The proof can be seen in [KZ68] and [Zel68]. ■

We remark the fact that although a priori we do not assume continuity for the functional f, from both *i)* and *ii)* we can deduce that f is continuous. On the one hand, if f is an homomorphism, from Proposition 1.12 we obtain that f is also a continuous mapping. On the other hand if $f(x) \in \sigma(x), \ \forall\, x \in A$, we can apply the *Gelfand theorem* 1.9 to conclude that $|f(x)| \leq ||x||, \ \forall\, x \in A$ and hence f is a continuous map.

The following statement is a celebrated theorem which generalizes the previous result when no linearity is assumed. In this case only one of the implications holds true.

Theorem 1.14 (Kowalski–Slodkowski)**.** *Let A be a complex Banach algebra, and let $f : A \to \mathbb{C}$ be a mapping satisfying:*

i) $f(0) = 0$,

ii) $f(x) - f(y) \in \sigma(x - y), \ \forall\, x, y \in A$.

Then f is multiplicative and linear.

Proof. The proof can be seen in [KS80]. ■

We remark that *Gleason-Kahane-Zelazko theorem* 1.13 is a generalization of the first result stated in this memory. Following the ideas outlined in the Introduction we shall try to apply that theorem in order to prove that every local homomorphism is in fact a homomorphism. Now the reader can imagine the real purpose of the *Kowalsky-Slodkowski theorem* 1.14, we will apply it to prove that every 2-local homomorphism is a homomorphism.

1.5 C*-algebras

In this section we shall introduce the concept of C* -algebra. A C*-algebra will be a complex Banach algebra equipped with an algebra involution satisfying a geometric axiom. Later, we will apply the notion of involution to get the concepts of self-adjoint elements, positive elements, normal elements and unitary elements. Finally we state the *Russo-Dye theorem*, an important result which describes those linear maps on a unital C*-algebra preserving unitary elements.

Let B be complex Banach algebra. A map $* : B \to B$ defined by $a \to a^*$ is said to be an *algebra involution* if it satisfies:

i) $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*, \ \forall \ \lambda, \mu \in \mathbb{C}, \ , \ \forall \ a, b \in B,$

ii) $a^{**} = a, \ \forall \ a \in B,$

iii) $(ab)^* = b^* a^*, \ \forall \ a, b \in B.$

If $* : B \to B$ is an involution on a complex Banach algebra B and B satisfies the *Gelfand-Naimark axiom*

$$||a^* a|| = ||a||^2, \ \forall \ a \in B, \tag{GNA}$$

we shall say that B is a *C*-algebra*.

Let B be a C*-algebra. We shall say that an element $a \in B$ is a *self-adjoint element* if it satisfies $a = a^*$. A self-adjoint element $a \in B$ is said to be *positive* if $\sigma(a) \subset \mathbb{R}_0^+$, while the set consisting of all positive elements in B is usually denoted by B_0^+. When an element $a \in B$ satisfies $aa^* = a^* a$ we shall say that a is a *normal element*. A normal element $a \in B$ such that $a^* a = aa^* = 1$ is called *unitary*.

In order to state the *Russo-Dye theorem* we have to set some notation. Let B and C be two C*-algebras. A linear map $\phi : B \to C$ is said to a be a $*$-

map if $\phi(a^*) = \phi(a)^*$, $\forall\, a \in B$. A $*$-map $\phi : B \to C$ will be called a **-homomorphism* if $\phi(ab) = \phi(a)\phi(b)$, $\forall\, a, b \in B$, and a *Jordan *-homomorphism* if $\phi(a^2) = \phi(a)^2$, $\forall\, a \in B$. It is known that a *-linear map $\phi : B \to C$ is a Jordan *-homomorphism if and only if $\phi(a \circ b) = \phi(a) \circ \phi(b)$, for all $a, b \in B$, where $a \circ b := 1/2(ab + ba)$ (if and only if $\phi(a^2) = \phi(a)^2$, $\forall\, a \in B$ with $a^* = a$). A bijective *-homomorphism (respectively, Jordan *-homomorphism) is called a *-isomorphism (respectively, a Jordan *-isomorphism).

Theorem 1.15 (Russo and Dye)**.** *Every unitary preserving linear map on a unital C*-algebras is a Jordan *-homomorphism multiplied by a fixed unitary element.*

Proof. The proof can be found in Corollary 2 at [RD66]. ■

1.6 The Gelfand representation

In this section we shall see that certain Banach algebras can be represented as C*-subalgebras of the space of all continuous function defined on a compact Hausdorff space. These privileged complex Banach algebras will be commutative (unital and non-unital) C*-algebras and unital non commutative C*-algebras. We will survey some properties of positive elements as, for example, that it is possible to obtain a square root of every positive element in a C*-algebra. The proof of the results stated in this section are deeply developed throughout chapters 11 and 12 from [Rud91] and in chapter VIII from [Con90].

Given a commutative complex Banach algebra A, we start by defining the set $\Omega(A)$, as the set of all homomorphisms from A into $\mathbb{C}$ (excluding the case $\phi \equiv 0$). This set $\Omega(A)$ will be called *the character set of A*. First note that this set can be seen as a compact space.

Proposition 1.16. *Let A be a commutative unital complex Banach algebra. Then $(\Omega(A), \sigma^*(A^*, A))$ is a closed subset of $(A^*, \sigma^*(A^*, A))$, and hence, by the Banach-Alaoglu theorem, $(\Omega(A), \sigma^*(A^*, A))$ is a compact Hausdorff space.*

Proof. The proof of this result can be found at *theorem 11.9* from [Rud91]. ■

The next theorem is a classical result in the theory of complex Banach algebras. It allows us to define a homomorphism between a commutative unital complex Banach algebra and a space of continuous function defined on a compact space which, in addition, has a *good* behavior with respect to the spectrum of every element.

Theorem 1.17 (Gelfand representation)**.** *Let A be a commutative unital complex Banach algebra. Then, the Gelfand representation*

$$\begin{array}{rccc} G\colon & A & \longrightarrow & (\mathcal{C}(\Omega(A)), ||\cdot||_\infty) \\ & a & \longrightarrow & G(a) = \hat{a} \end{array}$$

where $G(a) := \hat{a} : \Omega(A) \to \mathbb{C}$ is defined by $G(a)(\phi) = \hat{a}(\phi) = \phi(a), \ \forall \ \phi \in \Omega(a)$ and satisfies:

i) G is a contractive homomorphism;

ii) For each $a \in A$ we have $\sigma(a) = Im(G(a)) = \sigma(\hat{a})$. In addition $||\hat{a}|| = \sup_{\phi\in\Omega(A)} \{\hat{a}(\phi)\} = r(a)$.

Proof. The proof of this result can be found at *theorem 11.9* from [Rud91]. ■

This result becomes very useful when we are dealing with C*-algebras. In such an environment the Gelfand representation acquires some additional properties. Depending on a C*-algebra is commutative and unital or if any of them fails we obtain different version of the Gelfand representation.

Theorem 1.18. *Let A be a commutative unital C^*-algebra. Then the Gelfand representation of A is an isometric $*$-homomorphism which preserves the spectrum of every element of A.*

Proof. This result is a consecuence of at *theorem 11.18* from [Rud91]. ■

In order to treat the case of a commutative non-unital C*-algebra D, for an element $a \in D$ we consider the set $\tilde{\Omega}(D) := \Omega(D) \cup \{0\}$. As before, this set has a very nice property. The proofs of the following results are developed at fifth section of Chapter VIII from[Con90].

Proposition 1.19. *Let D be a non-unitary commutative C^*-algebra. Then $(\tilde{\Omega}(D), \sigma^*(D^*, D)) \prec (B(D^*), \sigma^*(D^*, D))$ is a Hausdorff compact space.* ■

So we can state

Theorem 1.20. *Let D be a non-unital commutative C^*-algebra. Then the map $G : D \to \mathcal{C}_0(\tilde{\Omega}(D)) := \left\{ f \in \mathcal{C}(\tilde{\Omega}(D)) \ : \ f(0) = 0 \right\}$ defined by $G(a) := \hat{a} : \tilde{\Omega}(D) \to \mathbb{C}$, where as before $\hat{a}(\phi) = \phi(a), \ \ \forall \ \phi \in \tilde{\Omega}(D)$; is an isometric $*$-homomorphism such that*

$$\forall \ a \in D \ \sigma_D(a) \cup \{0\} = \sigma_{\mathcal{C}_0(\tilde{\Omega}(D))}(\hat{a})$$

■

We have a similar result for non-commutative C*-algebras

Theorem 1.21 (Gelfand-Naimark theorem)**.** *Let B be a general C^*-algebra. Then there exists a complex Hilbert space H and an isometric $*$-monomorphism (i.e. injective homomorphism) $\Theta : B \to B(H)$ i.e. we can see B as a closed self-adjoint subalgebra of $B(H)$.* ■

The Gelfand representation allows us, via the continuous functional calculus, to obtain a *square root* from a fixed positive element in a C*-algebra.

Proposition 1.22. *Let B be a C*-algebra. If $a \in B$ is a positive element then there exists a unique element $\sqrt{a} \in \langle 1, a\rangle$(i.e the smaller C*-subalgebra of B containing the elements 1 and a) such that $\sqrt{a}\sqrt{a} = a$. In addition $\sigma(\sqrt{a}) = \sqrt{\sigma(a)}$ and $\sqrt{a}$ is a positive element.* ■

Finally we state two theorem which will be useful in the future. The first result states that the set of positive elements in a C*-algebra is in fact a cone.

Theorem 1.23 (Fukamiya-Kelley-Vaught)**.** *Let B be a C*-algebras. Then the set B_0^+ consisting of all positive elements in B is a convex cone i.e. B_0^+ is a convex set satisfying $\lambda B_0^+ \subset B_0^+,\ \forall\ \lambda \in \mathbb{R}_0^+$.* ■

The last result in this section shows that we can write a positive element as a product of some (possibly other) element and its "conjugate".

Theorem 1.24 (Kaplansky)**.** *Let B be a unital C*-algebras. Then the positive cone B_0^+ satisfies:*

$$B_0^+ = \{bb^* \,:\, b \in B\} = \{b^*b \,:\, b \in B\}$$

■

1.7 Operators on Hilbert spaces

The goal of this section will be the *spectral resolution theorem* which will allow us *diagonalize* certain compact operators as matrices. Previously we will introduce the most important subclasses of operators on a Hilbert space and we will study some of their properties. The contents of this section can be found at chapter 12 in [Rud91]. Other interesting references are [KR83] and [KR86].

The first result of this section assures the existence of the adjoint operator of a continuous linear operator between two Hilbert spaces.

Theorem 1.25. *Let H and K be two Hilbert spaces and consider $T \in B(K, H)$. Then*

i) There exists a unique map $T^* : K \to H$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, $x \in H$, $y \in K$.

ii) $T^* \in L(K, H)$ y $T^{**} = T$;

iii) $||T|| = ||T^*|| = ||TT^*||^{\frac{1}{2}}$;

iv) If $T_1, T_2 \in B(H, K)$ and $\alpha \in \mathbb{K}$, then

$$(T_1 + T_2)^* = T_1^* + T_2^* \text{ y } (\alpha T)^* = \overline{\alpha} T^*$$

;

v) If Z is a Hilbert space and $S \in B(K, Z)$ then

$$(ST)^* = T^*S^*.$$

■

When $K = H$ the Banach space $B(H) := B(H, H)$ is an unital complex Banach algebra. The last theorem allow us to conclude that the map $T \to T^*$ is an involution which satisfies the Gelfand-Naimark Axiom and therefore $B(H)$ is a C*-algebras. We shall say that an operator $T : H \to H$ is a *self-adjoint operator* (resp. *positive operator*, *normal operator* or *unitary operator*) when T is a self-adjoint element(resp. positive element, normal element or unitary element) of $B(H)$. We shall also say that a conjugate linear operator $T : H \to H$ is an *anti-unitary operator* when it satisfies $< Tx, Ty >=< y, x >$, $\forall\ x, y \in H$.

The following results show some properties of these operators:

Proposition 1.26. *Let H be a Hilbert space and pick $T \in B(H)$. Then:*

i) If T *is a self-adjoint operator*

$$||T|| = \sup \{|\langle Tx, x\rangle| \ : x \in H, \ ||x|| = 1\},$$

ii) T *is a normal operator if and only if*

$$||Tx|| = ||T^{*}x|| \ , \ \ \forall \ x \in H,$$

iii) There exist unique self-adjoint operators A *and* B *such that* $T = A + iB$*. In addition* T *is a normal operator if and only if* $AB = BA$*,*

iv) $Im(T^{*})^{\perp} = Ker(T)$*.*

■

Proposition 1.27. *Let* H *be a Hilbert space and consider an anti-unitary operator* $T : H \to H$*. Then there exist a unique operator* $\hat{T} : H \to H$ *satisfying:*

$$\langle x, Ty\rangle = \langle y, \hat{T}x\rangle, \ \ \forall \ x, y \in H$$

Proof. Note that $< x, T\cdot >$ is a linear map and apply the Riesz Representation Theorem. ■

Proposition 1.28. *Let* H *be a Hilbert space and pick* $T \in B(H)$*. Then*

i) T *is a self-adjoint operator if and only if* $\langle Tx, x\rangle \in \mathbb{R} \ \forall x \in H$*,*

ii) T *is a positive operator if and only if* $\langle Tx, x\rangle \geq 0 \ \forall x \in H$*.*

■

Proposition 1.29. *Let* H *be a Hilbert space and consider two anti-unitary operators* $T_1, T_2 : H \to H$*. Then*

i) $T_1\hat{T}_1 = \hat{T}_1 T_1 = Id$*,* *ii) If $S \in B(H)$ then $(T_1 S^* T_2)^* = \hat{T}_2 S \hat{T}_1$.*

Proof. Pick $x \in H$ and apply T_1 is an "isometry" to compute:

$$||\hat{T}_1 T_1 x - x||^2 = \langle \hat{T}_1 T_1 x, \hat{T}_1 T x\rangle + \langle x, x\rangle - \langle \hat{T}_1 T_1 x, x\rangle - \langle x, \hat{T}_1 T_1 x\rangle = 0.$$

The other identity is similar. For the second statement we take $x, y \in H$ and operate again:

$$\langle T_1 S^* T_2 x, y\rangle = \overline{\langle y, T_1 S^* T_2 x\rangle} = \overline{\langle S^* T_2 x, \hat{T}_1 y\rangle} =$$
$$= \langle \hat{T}_1 y, S^* T_2 x\rangle = \langle S\hat{T}_1 y, T_2 x\rangle = \langle x, \hat{T}_2 S \hat{T}_1 y\rangle.$$

■

The next lemma is an interesting property relating projections and positive operators.

Lemma 1.30. *Let H be a Hilbert space and consider $R, S : H \to H$ two projections such that $(RS)^* = (RS)$ and $RS = SR$. Then RS is a positive operator.*

Proof. Pick $x \in H$. Then:

$$< RSx, x >=< R^2 Sx, x >=< RSx, R^* x >=< SRx, R^* x >=$$
$$=< S^2 Rx, R^* x >=< S^2 Rx, R^* x >=< SRx, S^* R^* x >=$$
$$=< SRx, (RS)^* x >= ||RSx||^2 \geq 0$$

and due to $x \in H$ was arbitrary we conclude the proof. ■

Let H be a Hilbert space. An operator $T \in B(H)$ is said to be a *finite rank operator* if $\dim(T(H)) < \infty$. The space consisting on all the finite rank operators on H will be denoted by $F(H)$. A linear map $T : H \to H$ such that $T(B_H)$ is a relatively compact subset is said to be a *compact operator on H*. Note that if T is a compact operator then $T \in B(H)$. We shall apply the notation $K(H)$ to denote the space consisting on all the compact operators on H. A remarkable property of compact operators is their useful characterization by sequences:

Proposition 1.31. *Let H be a Hilbert space an consider a linear map $T : H \to H$. Then T is a compact operator on H if and only if for every bounded sequence $\{x_n\}_{n\in\mathbb{N}} \prec H$ the sequence $\{Tx_n\}_{n\in\mathbb{N}}$ admits a convergent partial sequence.* ■

Compact operators and finite rank operators on a Hilbert space are closely related:

Theorem 1.32. *Let H be a Hilbert space. Then:*

i) *$K(H)$ is a closed ideal of $B(H)$ and $F(H) \subset K(H)$.*

ii) *If $T \in K(H)$ is a compact operator, then $\overline{T(H)}$ is separable. In addition, if $\{e_n\}$ is an orthonormal basis of $\overline{T(H)}$ then the orthogonal projections P_n, $n \in \mathbb{N}$ over the subspaces $\{e_1, ..., e_n\}$ satisfies $\{P_nT\} \to T$. Therefore $K(H) = \overline{F(H)}$.*

iii) *If $T \in B(H)$ then $T \in K(H)$ if and only if $T^* \in K(H)$.* ■

One of the most important result in compact operators theory is the *Spectral Resolution Theorem*, which allows us to "diagonalize"those operators. Let H be a Hilbert space and pick $T \in B(H)$. We shall say that $\lambda \in \mathbb{C}$ is an eigenvalue of T if $E_\lambda := \mathrm{Ker}(T - \lambda I) \neq \emptyset$. The set consisting on all the eigenvalues of T will be called *punctual spectrum of* T and denoted by $\sigma_p(T)$. The vector space E_λ receives the name of *eigenspace associated to* λ and their elements are called eigenvectors. When there exist an orthonormal base of H consisting on eigenvectors we shall say that T is diagonalizable. The punctual spectrum of normal compact operator has several well-known properties:

Proposition 1.33. *Let H be a Hilbert space, take a normal compact operator $T \in B(H)$, and for each $\lambda \in \mathbb{C}$ denote $E_\lambda :=$ Ker$(T - \lambda I)$. Then:*

(i) *There exists $\lambda \in \sigma_p(T)$ such that $|\lambda| = \|T\|$,*

(ii) *If* $\lambda \in \mathbb{C}^*$, *then* $\dim E_\lambda < \infty$,

(iii) *Take* $\lambda, \mu \in \mathbb{C}$ *such that* $\lambda \neq \mu$. *Then* $E_\lambda \subset E_\mu^\perp$, *i.e. if* P_λ *and* P_μ *denote the orthogonal projections over the eigenspaces* E_λ *and* E_μ *then*

$$P_\lambda P_\mu = P_\mu P_\lambda = 0.$$

(iv) *For each* $\varepsilon > 0$, *the set* $A_\varepsilon = \{\lambda \in \sigma_p(T) : |\lambda| \geq \varepsilon\}$ *has finite cardinal. Therefore,* $\sigma_p(T)$ *is a countable set whose unique potential accumulation point is zero.*

■

We state now the main result of this section:

Theorem 1.34 (Spectral resolution of a normal compact operator)**.** *Let* H *be a Hilbert space and take a normal compact operator* $T \in B(H)$. *Denote by* $P_\lambda : H \to E_\lambda$, $\lambda \in \sigma_p(T)$ *the orthogonal projection over* E_λ. *Then the family* $\{\lambda P_\lambda : \lambda \in \sigma_p(T)\}$ *is summable and*

$$T = \sum_{\lambda \in \sigma_p(T)} \lambda P_\lambda. \tag{$*$}$$

■

Note that if T is an operator in the form $(*)$, then T is a normal operator. In addition, due to $\overline{F(H)} \subset K(H)$ the operator T is also compact. Proposition 1.33 says that $\sigma_p(T)$ is at most countable and we can write T in the form

$$T = \sum_{n=1}^{\infty} \lambda_n P_{\lambda_n}$$

where $\lambda_n \in \sigma_p(T),\ \forall\, n \in \mathbb{N}$. Using the fact that $\dim(E_\lambda) < \infty$ we conclude that

Corollary 1.35 (Diagonalization of a normal compact operator). *Let H be a Hilbert space and consider a normal compact operator $T \in B(H)$. Then there exist an orthonormal countable basis $\{e_\gamma : \gamma \in \Gamma\}$ and a complex sequence $\{\alpha_\gamma\}_{\gamma\in\Gamma}$ such that the family $\{\alpha_\gamma e_\gamma \otimes e_\gamma : \gamma \in \Gamma\}$ is summable and*

$$T = \sum_{\gamma\in\Gamma} \alpha_\gamma e_\gamma \otimes e_\gamma.$$

Therefore, $T(x) = \sum_{\gamma\in\Gamma} \alpha_\gamma < x, e_\gamma > e_\gamma, \ \forall\, x \in H$ *and T is diagonalizable.* ■

The previous results can be summarized in an unique sentence:

"A linear operator on a Hilbert space is diagonalizable if and only if
it is a normal compact operator".

1.8 Extreme points

At this point we are going to introduce a key tool which shall be very useful when we study the isometries on a Banach space. First, we will see that the set of the extreme points for a fixed subset of a Banach space has a nice behaviour under surjective linear isometric transformations defined on the whole Banach space. Finally we will calculate the precise expression of the extreme points of the unit ball in concrete Banach spaces.

Let X be a complex Banach space. For $A \subset X$ we shall say that $x \in A$ is a *extreme point in* A if it satisfies that if $y, z \in A$ are points such that there exists $t \in]0,1[$ satisfying $x = ty + (1-t)z$ then $x = y = z$ holds. The set consisting on all the extreme points of A will be denoted by $\partial_e(A)$.

We start by developing some results which will be useful in the future.

Proposition 1.36. *Let X be a complex Banach space. Then $||x|| = 1, \ \forall\, x \in \partial_e(B_X)$.*

Proof. Since

$$0 = y + (-1)y, \ \ \forall \ y \in B_X,$$

then $0 \notin \partial_e B_X$. So let us take $x \in B_X \setminus \{0\}$. If $||x|| \neq 1$ we can write:

$$x = (1 - ||x||)0 + (||x||)\frac{x}{||x||}$$

and therefore $x \notin \partial_e(B_X)$. ■

Proposition 1.37. *Let X be a Banach space and consider a linear surjective map $T : X \to Y$ which preserves the norm (i.e $||Tx|| = ||x|| \ \forall \ x \in X$). Then $T(\partial_e(B_X)) = \partial_e(B_Y)$.*

Proof. Let $e \in \partial_e(B_X)$ and suppose $t \in]0, 1[$ such that $T(e) = (1 - t)y + (t)z$, with $y, z \in B_Y$. Then $e = (1 - t)T^{-1}(y) + (t)T^{-1}(z)$. Due to T is an isometry, $T^{-1}(y), T^{-1}(z) \in B_X$ and $T^{-1}(y) = T^{-1}(z) = e$ so we conclude that $y = z = T(e)$. Using the same argument with T^{-1} instead of T we can get the other inclusion. ■

Now, we are going to introduce a well known family of classical Banach spaces and then we will study the extreme point set of their unit ball in these spaces. For $1 \leq p \leq \infty$ we recall that

$$\ell_p := \left\{ x : \mathbb{N} \to \mathbb{C} : \ \sum_{n \in \mathbb{N}} |x(n)| < \infty \right\} \text{ if } p < \infty$$

$$\ell_\infty = \{x : \mathbb{N} \to \mathbb{C} : \ \exists \ M > 0 : \ |x(n)| < M, \ \ \forall \ n \in \mathbb{N}\}$$

are the usual sequences and it is known that that the following maps are norms over the spaces where they are defined:

$$|| \cdot ||_p : \ell_p \to \mathbb{R} \ ||x||_p := \left(\sum_{n \in \mathbb{N}} |x(n)| \right)^{\frac{1}{p}}, \ \ \forall \ x \in \ell_p$$

$$|| \cdot ||_\infty : \ell_\infty \to \mathbb{R} \ ||x||_\infty := \sup_{n \in \mathbb{N}} \{|x(n)| : \ n \in N\}, \ \ \forall \ x \in \ell_\infty(\Lambda)$$

Proposition 1.38. *If* $1 \leq p < \infty, p \neq 2$ *we have*

$$\partial_e(B_{\ell_p}) = \left\{\lambda e^k \,:\, \lambda \in \mathbb{T}, e^k(n) = \delta_{n,k}, \;\; \forall\, n \in \mathbb{N}, k \in \mathbb{N}\right\}$$

Proof. One of the inclusions are obvious. In order to prove the other one suppose that $x \in \partial_e B_{\ell_p}$ is such that there exist $n, m \in \mathbb{N}$ satisfying $n \neq m$ and $x_m x_n \neq 0$. Define $y, z \in \ell_p$ by setting $y(k) := x_n \delta_{n,k}$ and $z := x - y$. Then

$$||x||^p = |x_n|^p + ||z||^p = ||y||^p + ||z||^p$$

and

$$x = |x_n|^p \frac{y}{||y||^p} + ||z||^p \frac{z}{||z||^p}$$

Due to $1 = ||x||^p = ||z||^p + ||y||^p$ and $\frac{y}{||y||^p}, \frac{z}{||z||^p} \in \mathbb{S}_{\ell_p}$ we conclude that $\frac{y}{||y||^p} = \frac{z}{||z||^p}$ which is impossible. ■

Proposition 1.39. *The extreme points of* B_{ℓ_∞} *are*

$$\partial_e(B_{\ell_\infty}) = \{x \in \ell_\infty \,:\, |x_n| = 1, \;\; \forall\, n \in \mathbb{N}\}\,.$$

Proof. As before, one of the inclusions are clear. Suppose then $x \in B_{\ell_\infty}$ is such that there exists $m \in \mathbb{N}$ satisfying $|x_m| < 1$. If $|x_m| = 0$ we can define $x^+, x^- \in B_{\ell_\infty}$ by setting $x^+(k) := x_k + \delta_{k,m}$ and $x^-(k) := x_k - \delta_{k,m}, \;\; \forall\, k \in \mathbb{N}$. Thus $x = x^+ + x^-$ and cannot be an extreme point. Therefore, suppose that $0 < |x_m| < 1$ so that in this case $x = (x - x_m e^m) + x_m e^m$ which is in contradiction with the fact that $x \in \partial_e B_{\ell_\infty}$. ■

The last "exercise" in this section will be calculating the extreme points of the unit ball of the Banach space consisting on the continuous functions defined on a compact space. In order to do that, we will need a folklore result, the Urysohn's Lemma, whose statement says what follows:

Theorem 1.40 (Urysohn's Lemma). *Let K be a first countable compact Hausdorff space. Then, if $A, B \subset K$ are closed subsets of K satisfying $A \cap B \neq \emptyset$ there exists a continuous function $f : K \to [0,1]$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$.*

Proof. The proof can be consult at *Theorem 2.12* from [Rud87]. ∎

Proposition 1.41. *Let K be a first countable compact Hausdorff space. Then*

$$\partial_e B_{\mathcal{C}(K)} = \{f \in \mathcal{C}(K) : |f(x)| = 1, \ \forall\, x \in K\}.$$

Proof. Suppose first $f \in \mathcal{C}(K)$ such that $|f(x)| = 1, \ \forall\, x \in K$ and pick $g, h \in B_{\mathcal{C}(K)}$ satisfying

$$f(x) = tg(x) + (1-t)h(x), \ \forall\, x \in K$$

for some $t \in]0,1[$. In that case, due to $\partial_e B_{\mathbb{C}} = \mathbb{S}_{\mathbb{C}}$ and $|g(x)|, |h(x)| \leq 1, \ \forall\, x \in K$, we obtain $g(x) = h(x) = f(x), \ \forall\, x \in K$ and f is a extreme point. To prove the other inclusion let us pick $f \in \partial_e B_{\mathcal{C}(K)}$ and suppose there exists $x_0 \in K$ such that $|f(x_0)| < 1$. Due to f is continuous we can assure that for $\varepsilon = |f(x_0)|$ we can find $V = V^0 \subset K$ which contains x_0 and satisfying

$$|f(x) - f(x_0)| < \frac{1-\varepsilon}{2}, \ \forall\, x \in V.$$

Using Urysohn' s Lemma we can find $g \in \mathcal{C}(K)$ such that $g(x_0) = \frac{1-\varepsilon}{2}$, $g(x) = 0, \ \forall\, x \in K \setminus V$ and $||g||_{\mathcal{C}(K)} = \frac{1-\varepsilon}{2}$. In that case

$$||f \pm g||_{\mathcal{C}(V)} \leq ||f - f(x_0)||_{\mathcal{C}(V)} + ||f(x_0) \pm g||_{\mathcal{C}(V)} \leq \frac{1-\varepsilon}{2} + \varepsilon + \frac{1-\varepsilon}{2} = 1$$

and due to $g|_{K \setminus V} \equiv 0$ we conclude $||f \pm g||_{\mathcal{C}(K)} \leq 1$ and f cannot be an extreme point because

$$f = \frac{1}{2}(f+g) + \frac{1}{2}(f-g).$$

∎

1.9 Schatten classes and symmetric norm ideals

In this section we will introduce the family of Banach spaces known as *Schatten classes*. This classes will be a family of Banach spaces whose elements are compact operators defined on a Hilbert space. The interesting matter is that wi will change the topology and we will not consider the standard operator norm but we will define a new family of norms instead (one new norm for each class). After we state the precise definition, we will remark that in the family of all *Schatten classes* we can found some familiar spaces as, for example, a generalization of the space of matrix equipped with the trace norm. After that, we will recall the notion of *symmetric norm ideal*, which is a generalization of these classes. Then we will state some properties which shall be useful in order to work with these Banach spaces. As we shall prove, these spaces will have a large synergy with the concept of *local isometry*. The contents of this section are deeply developed at *Chapter 11* from [FJ07].

Let H be a Hilbert space and take a linear compact operator $T \in K(H)$. Let $p \in [1, \infty[$ and consider the eigenvalues of the positive compact (use the spectral resolution theorem) operator $|T| := \sqrt{T^*T}$ which are usually called *singular values of* T and denoted by $s_1(T) \geq s_2(T) \geq \cdots \geq s_n(T) \geq \ldots$. We define the *Schatten* p*-norm of* T by

$$||T||_{Sp} := \left(\sum_{n\in\mathbb{N}} s_n^p(T) \right)^{\frac{1}{p}}$$

provided the sum $\sum_{n\in\mathbb{N}} s_n^p(T)$ is finite. For a fixed $p \in [1, \infty[$ the subspace $C_p(H) \prec B(H)$ consisting on all the operators whose Schatten p-norm is finite is usually given the name of p*-th Schatten class*. It can be proved that $(C_p(H), ||\cdot||_{Sp})$ is in fact a complex Banach algebra. For the sake of simplicity we fix the notation $(C_\infty(H), ||\cdot||_{S\infty}) := (K(H), ||\cdot||_\infty)$.

The next Proposition is a collection of results which will be useful in order to work with Schatten classes.

Proposition 1.42. *Let H be a Hilbert space and pick $A \in C_p(H)$ where $p \in [1, \infty[$. Then*

i) If $U, V \in B(H)$ are unitary operators then $||UAV^*||_{S_p} = ||A||_{S_p}$,

ii) If $q \geq p$ then $||A||_{Sp} \geq ||A||_{Sq}$,

iii) If $p^* := \frac{1}{1-\frac{1}{p}}$ and $B \in C_{p^*}(H)$ then $|tr(B^*A)| \leq ||A||_{Sp}||B||_{Sp^*}$.

iv) $F(H)$ is a dense subset of $C_p(H)$ (for the norm $||\cdot||_{Sp}$).

■

Fixing the value of $p \in [1, \infty]$ we obtain some well-known norms: $||\cdot||_{S1}$ is called the trace norm and $||\cdot||_{S\infty}$ is the classical operator norm $||\cdot||_\infty$. The space $(C_2(H), ||\cdot||_{S2})$ is a Hilbert space and receives the name of *Hilbert-Schmidt class.*

Now we are going to generalize those classes into an unique concept. A *symmetric norm ideal* is an ideal $\mathcal{I}$ of $K(H)$ equipped with a norm ν which satisfies:

i) $(\mathcal{I}, \nu)$ is a Banach space,

ii) If $A \in \mathcal{I}$ is a rank one operator, then $\nu(A) = ||A||_\infty$,

iii) If $U, V \in B(H)$ are unitary operators then $\nu(UAV) = \nu(A)$ for every $A \in \mathcal{I}$,

iv) $F(H)$ are dense in $(\mathcal{I}, \nu)$.

We can easily see via Proposition 1.42 that the notion of symmetric norm ideal is a true generalization of all the Schatten classes. The following result is an interesting property which will be absolutely necessary in the future. It relates the norm of a symmetric norm ideal with the standard operator norm.

Proposition 1.43. *Let H be a Hilbert space and consider $(\mathcal{I}, \nu)$ a symmetric norm ideal. Then ν majorizes the operator norm.*

Proof. Take first a finite rank operator $A \in F(H)$. Now, consider scalar numbers $\lambda_1, ...\lambda_n \in \mathbb{C}$ and rank one operators $P_1, ..., P_n \in F(H)$ such that $A = \sum_{j=1}^{n} \lambda_j P_j$ and, by reordering indexes if it was necessary, we obtain $||A||_\infty = ||\lambda_1 P_1||_\infty$. Therefore

$$
\begin{aligned}
||A||_\infty = ||\lambda_1 P_1|| \leq \nu(\lambda_1 P_1) = \frac{1}{2}\nu(2\lambda_1 P_1) = \\
= \frac{1}{2}\nu(\lambda_1 P_1 + \sum_{j=2}^{m} \lambda_j P_j + \lambda_1 P_1 - \sum_{j=2}^{m} \lambda_j P_j) \leq \\
= \frac{1}{2}\left(\nu(\lambda_1 P_1 + \sum_{j=2}^{m} \lambda_j P_j) + \nu(\lambda_1 P_1 - \sum_{j=2}^{m} \lambda_j P_j)\right) \leq \frac{1}{2} 2\nu(A) = \nu(A)
\end{aligned}
$$

where we have used that ν is invariant by unitary transformations.

■

Chapter 2

Isometries

The purpose of this memory is to study *local and 2-local surjective isometries* on classical Banach spaces. Knowing the form of the surjective linear isometries on that spaces will be essential in order to do that. In this chapter we shall start by recalling the precise definition of isometry and introducing a folklore result, the *Mazur-Ulam* theorem which relates surjective isometries and linearity. Throughout the first section we will study the surjective linear isometries on some Banach spaces as sequences spaces or the Schatten classes. Later, at the second section, we will introduce a weaker version of the notion of surjective linear *isometry*, the concept of partial isometry, which will be useful when we work with Hilbert spaces.

Let X and Y be two Banach spaces. We shall say that a mapping $T : X \to Y$ is an *isometry* when $||Tx - Ty|| = ||x - y||, , \ \forall\, x, y \in X$. Note that no linearity is assumed on this definition, and the fact that if T is a linear mapping, then T is an isometry if and only if $||Tx|| = ||x||, , \ \forall\, x \in X$. We recall now Proposition 1.37 proved at the previous chapter in order to conclude that every surjective linear isometry *preserves* the set of the extreme points of the unit ball.

An interesting classical result relating isometries and linearity, whose proof can be found at *theorem 1.3.5* from [FJ02], is the celebrated Mazur-Ulam Theorem whose statement reads as follows

Theorem 2.1 (Mazur-Ulam). *If T is a surjective isometry from a normed space X onto a normed space Y such that $T(0) = 0$, then T is real linear.* ■

This result has an unquestionable strength. For example, we can easily conclude that every $\mathbb{C}$-homogeneous surjective isometry between two normed spaces is necessarily a $\mathbb{C}$-linear mapping.

2.1 Surjective isometries

In this section we are going to survey the precise form of surjective linear isometries on certain Banach spaces. As we shall see on the next chapter, finding exact descriptions for surjective linear isometries will be absolutely necessary in order to determine if a local or 2-local surjective isometry is a surjective linear isometry. Therefore, we will analyse surjective linear isometries on three families of Banach spaces: continuous functions on a compact Hausdorff space, sequence spaces and Schatten classes.

Continuous functions

One of the best known results describing isometries on spaces consisting on continuous functions on a compact set is the so-called Banach-Stone theorem, whose proof is based on the theory of extreme points which we studied at Chapter 1. Its statement reads as follows

Theorem 2.2 (Banach-Stone). *Let X and Y be two compact Hausdorff spaces, and consider $T : \mathcal{C}(X) \to \mathcal{C}(Y)$ a linear surjective isometry. Then there exists* *an homeomorphism $\phi : Y \to X$ and $g \in \mathcal{C}(Y, \mathbb{T})$ (i.e, g is a continuous function on Y such that $|g(y)| = 1, , \ \forall\ y \in Y$) satisfying*

$$(Tf)(y) = g(y)f(\phi(y)), , \ \forall\ y \in Y, \ , \ \forall\ f \in \mathcal{C}(X).$$

Proof. The proof of this result can be found at *theorem 2.1* [Con90]. ■

There also exists a version of the Banach-Stone theorem for continuous functions vanishing on a locally compact Hausdorff space vanishing at infinite.

Theorem 2.3 (Banach-Stone for $\mathcal{C}_0(X)$ spaces)**.** *Let X and Y be two locally compact Hausdorff spaces. If T is a linear isometry from $\mathcal{C}_0(X)$ onto $\mathcal{C}_0(Y)$, then X and Y are homeomorphic. Furthermore there exists an homeomorphism $\phi : Y \to X$ and $g \in \mathcal{C}(Y, \mathbb{T})$ satisfying*

$$(Tf)(y) = g(y)f(\phi(y)),\ \forall\ y \in Y,\ \ \forall\ f \in \mathcal{C}_0(X).$$

Proof. The proof of this result can be found at *theorem 6* from [GJ02]. ■

Sequence spaces

The *Banach-Stone theorem* is now in the folklore of the theory of surjective isometries between Banach spaces. Looking for a similar result on ℓ_p, $p \in [1, \infty] \setminus \{2\}$ (we introduce a useful tool:

Remark 2.4 (On the Stone–Čech compactification of $\mathbb{N}$)**.** *There exists a unique compact Hausdorff space $\beta\mathbb{N}$ satisfying:*

i) *$\mathbb{N}$ is a dense open subset of $\beta\mathbb{N}$,*

ii) *Each bounded continuous function on $\mathbb{N}$ extends to a unique continuous function on $\beta\mathbb{N}$,* *iii) If Y is a a compact Hausdorff space such that $\mathbb{N}$ is a dense open subset of Y then there exists a unique continuous map $\phi : \beta\mathbb{N} \to Y$ such that $\phi(n) = n, , \ \forall\, n \in \mathbb{N}$.*

Furthermore, if K is a compact space and $f : \mathbb{N} \to K$ is a continuous function there exists a unique continuous map $\beta f : \beta\mathbb{N} \to K$ such that $\beta f(n) = f(n), \ \forall\, n \in \mathbb{N}$. Another interesting property that we shall apply is that every homeomorphism between $\beta\mathbb{N}$ and itself sends natural numbers onto natural numbers. More information about the Stone–Čech compactification of natural numbers can be found at [Wal74]. ■

The following theorem is a surprising result which states the precise form of the surjective linear isometries on the Banach spaces $\{\ell_p : \ 1 \leq p \leq \infty, p \neq 2\} \cup \{c_0\}$. As we will see, such isometries are very similar to the ones which are given by the *Banach-Stone theorem* in the case of $C(K)$ spaces.

Theorem 2.5. *Let Y be a Banach space in the class $\{\ell_p : \ 1 \leq p \leq \infty, p \neq 2\} \cup \{c_0\}$, and let $T : Y \to Y$ a surjective linear isometry. Then there exist a sequence τ with terms of modulus 1 and a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that:*

$$T(f) = \tau \cdot f \circ \phi, \ \ \forall\, f \in Y$$

Proof. The case $Y = \ell_p$, $1 \leq p < \infty, p \neq 2$ is clear applying Propositions 1.37 and 1.38.

In addition, due to the fact that c_0 can be seen as the algebra of complex functions on $\mathbb{N}$ that vanish at infinity, we can apply the Banach-Stone theorem to conclude the desired result.

To prove the remaining case $Y = \ell_\infty$ we shall work with the Stone–Čech compactification of $\mathbb{N}$ (see Remark 2.4) and the fact that there exists an isometric isomorphism $\psi : \ell_\infty \to \mathcal{C}(\beta\mathbb{N})$ such that $\psi(f)(n) = f(n), \ \forall\, n \in \mathbb{N}, , \ \forall\, f \in \ell_\infty$.

If $T : \ell_\infty \to \ell_\infty$ is a surjective lineal isometry we can apply the Banach-Stone Theorem to the map $\hat{T} = \psi \circ T \circ \psi^{-1} : \mathcal{C}(\beta\mathbb{N}) \to \mathcal{C}(\beta\mathbb{N})$ and obtain $\hat{\tau} : \beta\mathbb{N} \to \mathbb{T}$ and a homeomorphism $\hat{\phi} : \beta\mathbb{N} \to \beta\mathbb{N}$ such that $\hat{T}(f) = \hat{\tau} \cdot \hat{f} \circ \hat{\phi}$. In that case,

$$T(f)(n) = \psi(T(f))(n) = \psi \circ T \circ \psi^{-1} \circ \psi(f)(n) = \hat{T}(\psi(f)) = \hat{\tau}(n) \cdot \psi(f) \circ \hat{\phi}(n)$$

for every $n \in \mathbb{N}$. Due to $\hat{\phi}$ is an homeomorphism, it can be proved that $\hat{\phi}(\mathbb{N}) = \mathbb{N}$ and $\phi := \hat{\phi}|_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ is a bijection. It only remains to take into account that $\tau := \hat{\tau}|_{\mathbb{N}}$ is a sequence whose terms have modulus one to conclude that

$$T(f)(n) = \tau(n) \cdot (f \circ \phi)(n), , \ \ \forall\, n \in \mathbb{N}.$$

■

Schatten classes and symmetric norm ideals

Schatten classes and *symmetric norm ideals* are the last examples of Banach spaces whose surjective linear isometries we are going to study. Whereas the *Arazy-Erdos theorem* shall be the result which classifies surjective linear isometries on Schatten classes, we will need the generalized version of this result (Sourour theorem) in order to study surjective linear isometries on a symmetric norm ideal. As we said on the last chapter, those spaces has a nice synergy with surjective linear isometries and hence we will able to precisely determine the form of all of them.

Theorem 2.6 (Arazy, Erdos). *Let H be a Hilbert space and consider a surjective linear isometry $\Phi : C_p(H) \to C_p(H)$, where $p \in [1, \infty[\backslash\{2\}$. Then one and only one of the next cases holds:*

i) There exist two unitary operators U, V on $B(H)$ such that

$$\Phi(A) = UAV, , \ \ \forall\, A \in C_p(H),$$

ii) There exist two anti-unitary operators U, V on $C_p(H)$ such that

$$\Phi(A) = UA^*V, , \quad \forall\ A \in B(H).$$

Proof. The proof can be found at [Erd94]. ■

In the next Chapter, we will see that, in order to study the local surjective isometries on Schatten classes, it is very useful being able to work with the concept of *symmetric norm ideal*, which generalizes all the Schatten classes. Therefore we need a similar result which generalizes *Arazy's theorem* in the case of *symmetric norm ideals*. This celebrated result was proved by Sourour [Sou81].

Theorem 2.7 (Sourour). *Let $(\mathcal{I}, \nu)$ be symmetric norm ideal which is not isomorphic to a Hilbert-Schmidt class of operators and consider a surjective linear isometry $\Phi : \mathcal{I} \to \mathcal{I}$. Then one and only one of the next cases holds:*

i) There exist two unitary operators U, V on $B(H)$ such that

$$\Phi(A) = UAV, \quad \forall\ A \in \mathcal{I}(H),$$

ii) There exist two anti-unitary operators U, V on $B(H)$ such that

$$\Phi(A) = UA^*V, \quad \forall\ A \in \mathcal{I}.$$

■

2.2 Partial isometries

In this section we will give a rigorous definition for the notion of *partial isometry*, which will be a weaker version of the concept *surjective local isometry* on a Hilbert space. After prove a characterization for an operator to be a *partial isometry* we will recall the *polar decomposition theorem*, which will allow us to write (in a unique way) a bounded linear operator on a Hilbert space as a composition of a partial isometry and a positive operator.

Let H be a Hilbert space. A linear operator $e : H \to H$ is said to be a *partial isometry* if there exist closed subspaces $H_1, H_2 \prec H$ such that $e\big|_{H_1} : H_1 \to H_2$ is a surjective linear isometry and $e\big|_{H_1^\perp} = 0$.

The following result states that the concept of partial isometry admits an algebraic treatment. As we will see, a bounded linear operator $e : H \to H$ on a Hilbert space H is a a partial isometry if and only if $ee^*e = e$. In addition, we will show that both the product of a partial isometry and its adjoint and the product of its adjoint and the original partial isometry are orthogonal projections onto certain subspaces of H.

Theorem 2.8. *Let H be a Hilbert space and consider a bounded linear operator $e : H \to H$. Then the following statements are equivalent:*

i) e is a partial isometry,

ii) There exist closed subspaces $H_1, H_2 \prec H$ such that $ee^* : H \to H$ is the orthogonal projection from H onto H_2 and $e^*e : H \to H$ is the orthogonal projection from H onto H_1,

iii) $ee^*e = e$.

Proof. *i)* $\Rightarrow$ *ii)* Take $H_1, H_2 \prec H$ closed subspaces such that $e\big|_{H_1} : H_1 \to H_2$ is a surjective linear isometry and $e\big|_{(H_1)^\perp} = 0$. Note first that $\mathrm{Ker}(e^*) = \mathrm{Im}(e)^\perp = H_2^\perp$ and $\mathrm{Im}(e^*) = \mathrm{Ker}(e)^\perp = H_1$. If $x \in H$ then

$$ee^*x = 0 \;\Leftrightarrow\; e^*x \in \mathrm{Ker}(e) = H_1^\perp \;\Leftrightarrow\; e^*x = 0 \;\Leftrightarrow\; x \in \mathrm{Ker}(e^*) = H_2^\perp.$$

Therefore $\mathrm{Ker}(ee^*) = H_2^\perp$ and $ee^* : H \to H$ is the orthogonal projection onto H_2. A similar argument is valid to prove that $e^*e : H \to H$ is the orthogonal projection onto H_1.

$ii) \Rightarrow i)$. Let's prove that $\mathrm{Ker}(e) = H_1^{\perp}$. Pick $x \in \mathrm{Ker}(e)$. Then $e^*ex = 0$ and $x \in \mathrm{Ker}(e^*e) = H_1^{\perp}$. If now we take $x \in H_1^{\perp}$ we obtain $e^*ex = 0$ and $ex \in \mathrm{Ker}(e^*) = \mathrm{Im}(e)^{\perp}$. Since $ex \in \mathrm{Im}(e)$ it must be $ex = 0$ and $x \in \mathrm{Ker}(e)$. With a similar reasoning we can notice that $\mathrm{Im}(e) = \mathrm{Ker}(e^*)^{\perp} = (H_2^{\perp})^{\perp} = H_2$ and $e : H_1 \to H_2$ is surjective. Finally, take $x \in H_1$ and apply that $e^*ex = x$ to conclude that

$$||ex||^2 =< ex, ex >=< x, e^*ex >=< x, x >= ||x||^2$$

and e is an isometry.

$i) \Rightarrow iii)$. Suppose that $H_1, H_2 \prec H$ are the closed subspaces such that $e\big|_{H_1} : H_1 \to H_2$ is a surjective linear isometry and $e\big|_{(H_1)^{\perp}} = 0$. Since $\mathrm{Im}(e) = H_2$ and ee^* is the orthogonal projection onto H_2, if $x \in H_1$, then $ee^*ex = ee^*(ex) = ex$. On the other hand, if $x \in H_1^{\perp}$ then $ex = 0$ and $ee^*ex = 0 = ex$. Since $H = H_1 \oplus H_1^{\perp}$ we obtain $ee^*ex = ex, , \ \forall\, x \in H$.

$iii) \Rightarrow ii)$. This case is clear. We only have to *multiply* by e^* the expression $ee^*e = e$. ■

The final theorem of this section will be a key ingredient for our later purposes. This result allows us to write a bounded linear operator $T : H \to H$ on a Hilbert space as the *product* of a partial isometry U and the absolute value of the operator T. Furthermore, such a decomposition will be unique. The name of the result has a clear meaning now due to this theorem is a kind of generalization of the decomposition of a complex number as the product of its modulus and an element in the torus $\mathbb{T}$.

Theorem 2.9 (Polar Decomposition)**.** *Let H be a Hilbert space and consider a bounded linear operator T in $B(H)$. Then there exists a unique partial isometry U in $B(H)$ such that $T = U|T|$. Furthermore, UU^* and U^*U are the orthogonal projections of H onto $\overline{T(H)}$ and $\overline{T^*(H)}$, respectively.*

Proof. The proof can be found at *theorem 7.5* from [Wei80]. ■

Chapter 3

Local isometries

Finally, we reach the main core of this memory. In this chapter we will study *local surjective isometries* on certain Banach spaces. At first, we shall introduce the abstract definition of *local map* and then, we show that all *local homomorphisms* on a complex Banach algebra are homomorphisms. At the last section, we will write the rigorous definition of *local surjective linear isometry* and then we will proof that every *local surjective linear isometry* on a Hilbert space is a true *surjective linear isometry* if and only if the Hilbert space has finite dimension. Finally, we will show that there exist several examples of classical Banach spaces (with infinite dimension) satisfying the desired property:

"Every local surjective isometry is a surjective isometry".

3.1 Local Maps

Let X and Y be two Banach spaces, and let us consider a subset $S \subset L(X,Y)$. We shall say that a linear mapping $T \in S$ is a *local S-map* if it satisfies the following property:

For each $x \in X$, there exists $T_x \in S$, depending on x, such that $T(x) = T_x(x)$.

We apply the notation T_x in order to emphasize the fact that it is an application that strongly depends on the element $x \in X$. Our main goal will be to find conditions on S in order to get that every S-local map becomes an element of S.

Throughout the following sections we will prove that being a *S-local map* is not always a weak property. The final purpose of this chapter will be show that depending on the Banach spaces X and Y and, of course, the class S we can proof that every *S-local map* is in fact an element of the class S.

3.2 Homomorphisms

In order to introduce a first example of such a class, we fix a complex Banach algebra A, and consider the $S \subset B(A, \mathbb{C})$ defined as the family:

$$S = \Omega(A) \cup \{0\} = \{\phi \in A^* : \phi(xy) = \phi(x)\phi(y), \ \forall\, x, y \in A\}$$

which is the class consisting on the homomorphisms of A. In the Introduction, a particular version of *Gleaso-Kahane-Zelazko theorem* allowed us to conclude that every *local homomorphism* on $\mathcal{C}(K)$ was in fact a *homomorphism*. In this section we will apply the generalized version of that same theorem (see 1.13) in order to conclude the same property shown in Introduction for *local homomorphisms* (i.e. *S-local map* for the class S defined as before) on a random Banach algebra.

Theorem 3.1. *Every local homomorphism on a Banach algebra is a homomorphism.*

Proof. We keep the same notation as before. Let T be a linear S-local map. Then, if $x \in A$ we can find $T_x \in S$ such that $T_x(x) = T(x)$. In that case, due to $T_x \in S$, we can apply $i) \Rightarrow ii)$ from Theorem 1.13 in order to obtain that $T(x) = T_x(x) \in \sigma(x)$. Since T is now on the hypothesis of $ii) \Rightarrow i)$ from the same theorem, we conclude that it is a multiplicative functional. ∎

We remark that the last theorem is not a mere exercise. Although its proof seems to be easy, the *Gleaso-Kahane-Zelazko theorem* is a very powerful result which allow us to give that short proof.

3.3 Surjective isometries

The second class of mappings which we are going to study will be the family of the *surjective isometries* in a Banach space X:

$$\mathrm{Isom}_S(X) := \{T \in B(X,X) : ||T(x)|| = ||x||, \ \forall\, x \in X \text{ and } T(X) = X\}$$

We remark that if T is a $\mathrm{Isom}_S(X)$-local map, then for each $x \in X$ we can find $T_x \in \mathrm{Isom}_S(X)$ such that $T(x) = T_x(x)$. In that case $||T(x)|| = ||T_x(x)|| = ||x||$, and thus T is an isometry. Thus, every local surjective isometry T is an injective map. Hence, our main goal in this section shall be checking when a $\mathrm{Isom}_S(X)$-local map on some concrete Banach space X is surjective.

Referring to a $\mathrm{Isom}_S(X)$-local map T by saying that T is a *local surjective isometry* on X is usual notation. A Banach space X such that every local surjective isometry is a surjective isometry is said to be *algebraically reflexive.*

Hilbert spaces

We first analyse *local surjective isometries* on a Hilbert space. As we shall see, Hilbert spaces have a particular behaviour for local isometries. In fact, we will show that a Hilbert space is algebraically reflexive if and only if it has finite dimension.

Theorem 3.2. *Let H be a Hilbert space. A bounded linear $Isom_S(H)$-local map is a surjective isometry if and only if H has a finite dimension.*

Proof. Suppose that H has infinite dimension. In addition, we suppose that $H = \ell_2(\mathbb{N})$, although it is similar when $H = \ell_2(\Lambda)$, with Λ a set of indices. We can consider the mapping $T : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ given by

$$T(x) : \mathbb{N} \to \mathbb{C},\, T(x)_n = x_{n+1},\ \ \forall\, n \in \mathbb{N} \setminus \{1\} \text{ and } T(x)_1 = 0,\ \ \forall\, x \in \ell_2(\mathbb{N}).$$

Obviously, T is well defined and it is not a surjective linear isometry because of $-e_1 \notin T(\ell_2(\mathbb{N}))$-. However, we can see that T is a $\mathrm{Isom}_S(\ell_2(\mathbb{N}))$-local map, because if $x \in \ell_2(\mathbb{N})$, we always can find a bounded linear surjective linear isometry T_x such that $T_x(x)_n = T(x)_n = x_{n-1},\ \forall\, n \in \mathbb{N}$ defining $T_x : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ as follows: if $x \neq 0$ we can apply Gram-Schmidt to obtain two orthonormal basis of $\ell_2(\mathbb{N})$ one of them, B_1; containing $\frac{x}{||x||}$ and the other one, B_2; containing $\frac{T(x)}{||T(x)||}$. We can consider as T_x one of the surjective linear isometries which maps $\frac{x}{||x||}$ into $\frac{T(x)}{||T(x)||}$ and the remaining elements of B_1 into the ones of B_2.

Reciprocally if H is a finite dimensional Hilbert space and T is a $\mathrm{Isom}_S(H)$-map then we can apply the dimension formula

$$\dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)) = \dim(H)$$

and the fact that T is an injective map to conclude that T is surjective. ∎

A perceptive reader should have taken into account the fact that when we studied surjective linear isometries in the previous chapter, we always *dodged* those spaces which were Hilbert spaces. A clear justification for this appears here: in order to study *local surjective isometries* on a Hilbert space we do not need such information.

Sequence spaces

Throughout this section we will approach the problem of show if the sequences spaces on the family

$$\{\ell_p : \ 1 \leq p \leq \infty, p \neq 2\} \cup \{c_0\}$$

are algebraically reflexive or not. Note that, thanks to the description provided in Theorem 2.5, we can now study local surjective linear isometries on these sequences spaces. As we suggested at the end of the last section, we intentionally omit the case ℓ_2, do to it is a Hilbert space and we know that it is not able to be algebraically reflexive.

Theorem 3.3. *Let be Y a Banach space on the family $\{\ell_p : \ 1 \leq p \leq \infty, p \neq 2\} \cup \{c_0\}$. Then every $Isom_S(Y)$-local map is a surjective linear isometry (i.e every Banach space on the family $\{\ell_p : \ 1 \leq p \leq \infty, p \neq 2\} \cup \{c_0\}$ is algebraically reflexive).*

Proof. Let Y be any of c_0, ℓ_p, for $p \in [1, \infty[\setminus \{2\}$. Consider $T : Y \to Y$ a $\mathrm{Isom}_S(Y)$-local map and, for $n \in \mathbb{N}$, denote $e^n : \mathbb{N} \to \mathbb{N}$ to the sequence given by $e^n(k) = \delta_{n,k}, \ \forall \ k \in \mathbb{N}$. Since T is a local surjective isometry for $n \in \mathbb{N}$ we can find a bijection $\phi_n : \mathbb{N} \to \mathbb{N}$ and $u_n : \mathbb{N} \to \mathbb{T}$ such that

$$T(e_n) = u_n e^n \circ \phi_n$$

which can be obviously rewritten in the form

$$T(e_n) = u_{k(n)} e^{k(n)}$$

for certain $k(n) \in \mathbb{N}$. Let's see that the mapping $k : \mathbb{N} \to \mathbb{N}$ given by $n \to k(n)$ is a bijection. First notice that k is well defined because for $n \in \mathbb{N}$ there is an unique (ϕ_n is a bijection) $k(n) := \phi_n^{-1}(n) \in \mathbb{N}$ such that $(e^n \circ \phi_n)(m) = \delta_{n,\phi_n(m)} = \delta_{k(n),m} = e^{k(n)}(m), \ \forall\, k \in \mathbb{N}$. Now, if $n_1 \neq n_2$ we have:

$$T(e_{n_1} + e_{n_2}) = T(e_{n_1}) + T(e_{n_2}) = u_{k(n_1)}e^{k(n_1)} + u_{k(n_2)}e^{k(n_2)}$$

and

$$T(e_{n_1} + e_{n_2}) = u_{n_1,n_2}(e^{n_1} + e^{n_2}) \circ \phi_{n_1,n_2}$$

If $k(n_1) = k(n_2)$ the first equation give us that $T(e_{n_1} + e_{n_2})$ has only a non zero term, whereas, because of ϕ_{n_1,n_2} is a bijection, the second one has two non zero terms. To prove surjectivity of k we choose a non zero terms sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset Y$ and denote $x := \sum_{n\in\mathbb{N}} \alpha_n e^n \in Y$. On the one hand we have:

$$T(x)(m) = \left(\sum_{n\in\mathbb{N}} \alpha_n T(e^n)\right)(m) = \left(\sum_{n\in\mathbb{N}} \alpha_n u_{k(n)} e^{k(n)}\right)(m) =$$
$$= \left(\sum_{n\in\mathbb{N}} \alpha_n u_{k(n)} e^{k(n)}\right)(m) = \sum_{n\in\mathbb{N}} \alpha_n u_{k(n)} \delta_{k(n),m}, \ \ \forall\, m \in \mathbb{N}$$

and, on the other hand:

$$T(x)(m) = T_x(x) = \left(\sum_{n\in\mathbb{N}} \alpha_n T_x(e^n)\right)(m) = \left(\sum_{n\in\mathbb{N}} \alpha_n u_n^x e^n \circ \phi_x\right)(m) =$$
$$= \left(\sum_{n\in\mathbb{N}} \alpha_n u_n^x \delta_{\phi_x(m),n}\right) = \{\phi_x \text{ is a bijection }\} = \alpha_{\phi_x(m)} u^x_{\phi_x(m)}, \ \ \forall\, m \in \mathbb{N}$$

Due to $0 \neq \alpha_{\phi_x(m)} u^x_{\phi_x(m)} = \sum_{n\in\mathbb{N}} \alpha_n u_{k(n)} \delta_{k(n),m}, \ \ \forall\, m \in \mathbb{N}$, we conclude that, if $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $k(n) = m$. Note that as consequence we have that $\mathrm{Im}(T)$ contains every finite sequence. Finally notice that, provided that T is an isometry, T is also surjective. If $\alpha := \sum_{n\in\mathbb{N}} \alpha_n e^n \in Y$ we can define $\beta := \sum_{n\in\mathbb{N}} \alpha_n \overline{u_n} e^{k^{-1}(n)} \in Y$ which satisfies $T(\beta) = \sum_{n\in\mathbb{N}} \alpha_n \overline{u_n} T(e^{k^{-1}(n)}) = \sum_{n\in\mathbb{N}} \alpha_n \overline{u_n} u_n e^n = \alpha$.

Next, we treat the remaining case of ℓ_∞. Using the same argument as before with the exception of choosing $\{\alpha_n\}_{n\in\mathbb{N}} \in c_0$ allow us to conclude that the range of T contains every cofinite sequence. Due to ℓ_∞ is a unital C*-algebras and T is an isometry, T maps unitary elements into unitary elements, we can apply 1.15 to obtain $\tau \in \ell_\infty$ and a C*-Homomorphism $J : \ell_\infty \to \ell_\infty$ such that $T(x) = \tau J(x), \ \forall \ x \in \ell_\infty$. It is clear that $\mathrm{Im}(J)$ contains every cofinite sequence. Using Theorem 3 of [MZ95], which asserts that every $*$-Homomorphism of ℓ_∞ whose range contains every finite sequence is automatically surjective, we obtain the surjectivity of J concluding the proof. ■

Continuous functions

Now we are going to study if the spaces $\mathcal{C}(K)$ are algebraically reflexive. Following the lines of the last subsection it is clear that we need (at least) knowing the precise form of the surjective linear isometries on the spaces of continuous functions.So, the *Banac-Stone theorem* 2.2 will be our most powerful tool in this environment.

Theorem 3.4. *Let K be a first countable compact Hausdorff space. Then every local surjective linear isometry $\Phi : \mathcal{C}(K) \to \mathcal{C}(K)$ is a surjective linear isometry.*

Proof. Let $\Phi : \mathcal{C}(K) \to \mathcal{C}(K)$ be a local surjective linear isometry. It is clear that Φ maps continuous functions with modulus one into continuous functions with modulus one. By the *Russo-Dye theorem* 1.15 we know that Φ is the product of a fixed continuous function with modulus one, which can be supposed to be identically 1, and a Jordan $*$-homomorphism. Due to in this case $\Phi(1) = 1$, we can assure that there exists $\phi \in \mathcal{C}(K, K)$ such that $\Phi(f) = f \circ \phi, \ \forall \ f \in \mathcal{C}(K)$ (see Theorem 3.4.3 in [KR83]). Since Φ is an isometry, by Urysohn 's lemma , it is also surjective. Lets prove that Φ is also injective. Suppose that there exist $x, y \in K$ such that $\phi(x) = \phi(y) = z$ and consider a monotone decreasing sequence $\{U_n\}_{n\in\mathbb{N}}$ of open sets in K such that $\cap_{n\in\mathbb{N}} U_n = \{z\}$. Using the Urysohn 's lemma again we can found, for each $n \in \mathbb{N}$, a continuous function $f_n : K \to [0,1]$ such that $f_n(z) = 1$ and $f_n|_{K\setminus U_n} \equiv 0$. Define now $f : K \to \mathbb{R}$ by $f = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} f_n$. It is clear that $f \in \mathcal{C}(K)$ and $f(t) = 0$ if and only if $t = z$. In addition, Φ is a local surjective linear isometry and there exists, by the *Banach-Stone theorem*, an homeomorphism $\psi : K \to K$ and a continuous function $g : K \to \mathbb{C}$ with modulus 1 such that $f \circ \phi = g \cdot f \circ \psi$. Then

$$\{x, y\} \subset (f \circ \phi)^{-1}(0) = (f \circ \psi)^{1}(0) = \psi^{-1}(0)$$

which is in contradiction with the injectivity of ψ. ■

Schatten classes and symmetric norm ideals

The last part of this chapter will treat about study *local surjective isometries* on the *Schatten classes*. In order to do that we will study the problem in the case of symmetric norm ideals. So, throughout this section, we shall work with a symmetric norm ideal $(\mathcal{I}, \nu)$ which is not isomorphic to a Hilbert-Schmidt class. Due to this case is a bit laborious we will study first some lemmas and propositions for the sake of simplicity.

Proposition 3.5. *Let $\Phi : \mathcal{I} \to \mathcal{I}$ be a local surjective linear isometry. Then Φ is an isometry and it maps partial isometries into partial isometries.*

Proof. We know that every local isometry is a isometry. In order to prove that Φ preserves partial isometries we shall apply Theorem 2.7. Taking a partial isometry $e : \mathcal{I} \to \mathcal{I}$ and applying that Φ is a local surjective linear isometry we obtain U_e, V_e unitary operators or anti-unitary operators such that

$$\begin{cases} \Phi e = U_e e V_e \\ \Phi e = U_e e^* V_e \end{cases}$$

and in both cases, applying the Proposition 1.29, we have

$$\begin{cases} \Phi e(\Phi e)^*\Phi e = U_e e V_e (V_e)^* e^* (U_e)^* U_e e V_e = U_e e e^* e V_e = \Phi e \\ \Phi e(\Phi e)^*\Phi e = U_e e^* V_e \hat{V}_e e \hat{U}_e U_e e^* V_e = U_e e^* e e^* V_e = \Phi e. \end{cases}$$

■

Lemma 3.6. *Let $\Phi : \mathcal{I} \to B(H)$ be a continuous linear mapping which preserves partial isometries in $\mathcal{I}$. Then Φ preserves orthogonality between partial isometries in the sense that if $U, V : H \to H$ are partial isometries in $\mathcal{I}$ such that $U^*V = UV^* = 0$ then $(\Phi U)^*\Phi V = \Phi U(\Phi V)^* = 0$.*

Proof. Pick $\alpha \in \mathbb{T}$ and note that $\alpha U + V$ is also a partial isometry in $\mathcal{I}$. Denote $P = \Phi U$ and $Q = \Phi V$. Using Φ preserves partial isometries:

$$(\alpha P + Q) = (\alpha P + Q)(\alpha P + Q)^*(\alpha P + Q) =$$
$$= \alpha P + PP^*Q + \alpha^2 PQ^*P + \alpha PQ^*Q + QP^*P + \overline{\alpha} QP^*Q + \alpha QQ^*P + \alpha Q =$$
$$= \alpha P + Q + (PP^*Q + \alpha^2 PQ^*P + QP^*P) + (\alpha PQ^*Q + \overline{\alpha} QP^*Q + \alpha QQ^*P)$$

multiplying by α we obtain:

$$\alpha^3 PQ^*P + \alpha^2(PQ^*Q + QQ^*P) + \alpha(PP^*Q + QP^*P) + QP^*Q = 0, \ \forall \alpha \in \mathbb{T}.$$

Due to this expression is valid for every $\alpha \in \mathbb{T}$, giving the values $\alpha = 1$, $\alpha = -1$, $\alpha = i$, $\alpha = -i$, we obtain a linear equations system whose unique solution is

$$PQ^*P = 0,\ PQ^*Q + QQ^*P = 0,\ PP^*Q + QP^*P = 0,\ QP^*Q = 0.$$

If we multiply the second equation by P^* from the left we obtain

$$P^*PQ^*Q + (Q^*P)^*(Q^*P) = 0.$$

Since P and Q are partial isometries, then P^*P and Q^*Q are projections and by the last equation the product P^*PQ^*Q is self-adjoint and it must be $P^*PQ^*Q =$

Q^*QP^*P. On the one hand we have by Lemma 1.30 that P^*PQ^*Q is a positive operator and, on the other hand, by the *Kaplansky theorem* 1.24 $-P^*PQ^*Q$ is a positive operator. Thus, $P^*PQ^*Q = 0$ and $Q^*P = 0$. A similar argument allows us to conclude that $QP^* = 0$. ■

Lemma 3.7. *Let H be a Hilbert space and consider $T : H \to H$ a finite rank operator. Then T can be written as a finite linear combination of pairwise orthogonal partial isometries of finite rank.*

Proof. Take $|T| = \sum_{j=1}^{n} \alpha_j P_j$ the spectral resolution of $|T| := \sqrt{T^*T}$ and denote by U the partial isometry obtained in the polar decomposition $T = U|T|$. Note that the operators $V_j := UP_j$ are pairwise orthogonal partial isometries. Since for $k \neq j \in \mathbb{N}$ we have

$$V_j V_k^* = UP_jP_k^*U^* = UP_jP_kU^* = 0$$
$$V_j^* V_k = P_j^*U^*UP_k = P_j^*P_k = 0$$

due to $P_j = P_j^*$ (use Proposition 1.26 and the fact that P_j is an orthogonal projection) and $U^*U : H \to H$ is an orthogonal projection onto $\mathrm{Im}(|T|) \supset \mathrm{Im}(P_k)$. To prove V_j is a partial isometry we compute:

$$V_k(V_k)^*V_k = UP_kP_k^*U^*UP_k = UP_kU^*UP_k = UP_kP_k = UP_k = V_k.$$

Finally it suffices to note that

$$T = U|T| = U\sum_{j=1}^{n}\alpha_j P_j = \sum_{j=1}^{n}\alpha_j UP_j = \sum_{j=1}^{n}\alpha_j V_j.$$

■

Lemma 3.8. *Let $\Phi : \mathcal{I} \to B(H)$ be a continuous linear mapping which preserves partial isometries on $\mathcal{I}$. Then Φ also preserves triple products in the sense that*

$$\Phi(AB^*A) = \Phi(A)\Phi(B)^*\Phi(A), \ \ \forall\ A, B \in \mathcal{I}.$$

Proof. Let's pick $A \in F(H)$ a finite rank operator. By Lemma 3.7 we can find partial isometries $V_1, V_2, ..., V_n$ and scalar numbers $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}$ such that

$$A = \sum_{j=1}^{n} \alpha_j V_j.$$

Now, applying that Φ preserves partial isometries, we compute

$$\Phi(AA^*A) = \Phi(\sum_{j=1}^{n} \alpha_j |\alpha_j|^2 V_j) = \sum_{j=1}^{n} \alpha_j |\alpha_j|^2 \Phi(V_j) =$$

$$= (\sum_{j=1}^{n} \alpha_j \Phi(V_j))(\sum_{j=1}^{n} \alpha_j \Phi(V_j))^*(\sum_{j=1}^{n} \alpha_j \Phi(V_j)) = \Phi(A)\Phi(A)^*\Phi(A).$$

Since Φ is continuous we obtain $\Phi(AA^*A) = \Phi(A)\Phi(A)^*\Phi(A), \ \forall\, A \in \mathcal{I}$. Take now $A, B \in \mathcal{I}$ and linearize that equation:

$$\Phi((A+B)(A^*+B^*)(A+B)) =$$

$$\Phi(AA^*A + AA^*B + AB^*A + AB^*B + BA^*A + BA^*B + BB^*A + BB^*B) =$$

$$= \Phi(AA^*A) + \Phi(AA^*B) + \Phi(AB^*A) + \Phi(AB^*B) +$$

$$+\Phi(BA^*A) + \Phi(BA^*B) + \Phi(BB^*A) + \Phi(BB^*B) =$$

$$= \Phi(A+B)\Phi(A+B)^*\Phi(A+B)$$

and it must be

$$\Phi(AA^*B + AB^*A + AB^*B + BA^*A + BA^*B + BB^*A) =$$

$$\Phi(A)\Phi(A^*)\Phi(B) + \Phi(A)\Phi(B)^*\Phi(A) + \Phi(A)\Phi(B)^*\Phi(B) +$$

$$+\Phi(B)\Phi(A)^*\Phi(A) + \Phi(B)\Phi(A)^*\Phi(B) + \Phi(B)\Phi(B)^*\Phi(A).$$

If we replace in this equation B by αB, $\alpha \in \mathbb{R}$ and then we choose some particular values of α we obtain

$$\Phi(AA^*B + AB^*A + BA^*A) =$$

$$= \Phi(A)\Phi(A^*)\Phi(B) + \Phi(A)\Phi(B)^*\Phi(A) + \Phi(B)\Phi(A)^*\Phi(A).$$

Replacing in the last equation B by iB:

$$i\Phi(AA^*B - AB^*A + BA^*A) =$$
$$= i\Phi(A)\Phi(A^*)\Phi(B) - i\Phi(A)\Phi(B)^*\Phi(A) + \Phi(B)\Phi(A)^*\Phi(A).$$

but if we multiply by i the same equation

$$i\Phi(AA^*B + AB^*A + BA^*A) =$$
$$= i\Phi(A)\Phi(A^*)\Phi(B) + i\Phi(A)\Phi(B)^*\Phi(A) + i\Phi(B)\Phi(A)^*\Phi(A).$$

thus

$$\Phi(AB^*A) = \Phi(A)\Phi(B)^*\Phi(A).$$

Since $A, B \in \mathcal{I}$ were arbitrary we obtain what we were looking for. ■

We will apply the notions of *weak- and strong- operator topologies*, which can be found at *section 5.1* from [KR83].

Lemma 3.9. *Let H be a Hilbert space. If $\{U_n : n \in \mathbb{N}\} \prec B(H)$ is a family of pairwise orthogonal partial isometries, then the series $\sum_{n\in\mathbb{N}} U_n$ converges strongly to an operator $T \in B(H)$. Moreover such operator T is also a partial isometry.*

Proof. Let $x \in H$ be a fixed element in H. Then

$$< U_n x, U_m x >=< x, U_n^* U_m x >= 0, \ \ \forall\, n, m \in \mathbb{N} : \, n \neq m.$$

and due to $U_n^* U_n$ is a we have

$$||U_n x||^2 =< U_n^* U_n x, x >= ||U_n^* U_n x||^2.$$

Since $\{U_n^* U_n : n \in \mathbb{N}\}$ is a family of orthogonal projections with zero being the unique point in the intersection of their rank it must be $\sum_{n\in\mathbb{N}} ||U_n x||^2 = \sum_{n\in\mathbb{N}} ||U_n^* U_n x||^2 < \infty$ and the series converges for a fixed $x \in H$. Moreover, since every finite partial sum of $\sum_n U_n$ is a partial isometry (induction taking into account that $U_n^* U_m = U_n U_m^* = 0$ when $n \neq m$) then the family of the partial sums is uniformly bounded and by the *Banach Steinhaus theorem* we obtain $T = \sum_{n=1}^{\infty} U_n \in B(H)$.

Let's prove that T is a partial isometry. Using the strong convergence of $\sum_n U_n$ we obtain that $\sum_n U_n^* U_n$ converges weakly to T^*T. Since the terms in $\sum_n U_n^* U_n$ are pairwise orthogonal projections this series converges strongly to T^*T. Using the multiplication is continuous on every bounded subset of $B(H)$ with respect to the strong operator topology, we conclude that $\sum_n U_n U_n^* U_n = TT^*T$ in the strong operator topology. But $U_n U_n^* U_n = U_n, \ \forall\, n \in \mathbb{N}$ and so $TT^*T = T$. ■

The proof of the following result consist on a series of technical considerations which can be read at [MZ99]:

Theorem 3.10. *Let H be a Hilbert space and consider a continuous triple homomorphism $\Phi \colon \mathcal{I} \to B(H)$. Then there exist a family of isometries $\{U_\alpha, U'_\alpha : \alpha \in A\}$ in $B(H)$ and a family of anti-isometries $\left\{V_\beta, V'_\beta : \beta \in B\right\} \prec B(H)$ such that the ranks of the operators in $\{U_\alpha, V_\beta : \alpha \in A, \beta \in B\}$ form a pairwise orthogonal family, the same occurs for the ranks of the operators in $\left\{U'_\alpha, V'_\beta : \alpha \in A, \beta \in B\right\}$ and:*

$$\Phi(A) = \sum_{\alpha \in A} U'_\alpha A U_\alpha^* + \sum_{\beta \in B} V'_\beta A^* V_\beta^*, \ \ \forall\, A \in \mathcal{I}.$$

Now we can prove the main result in this subsection:

Theorem 3.11. *Let $(\mathcal{I}, \nu)$ be a symmetric norm ideal in a Hilbert space H. Then every local surjective linear isometry on $\mathcal{I}$ is a surjective linear isometry unless $\mathcal{I}$ is isomorphic to the Hilbert-Schmidt class.*

Proof. Let $\Phi \colon \mathcal{I} \to \mathcal{I}$ be a local surjective linear isometry. Since the norm ν dominates the operator norm, this map is continuous when viewed as a mapping

$\Phi\colon \mathcal{I} \to B(H)$. The results proved before assure us that Φ is of the form

$$\Phi(A) = \sum_{\alpha \in A} U'_\alpha A U^*_\alpha + \sum_{\beta \in B} V'_\beta A^* V^*_\beta, \ \ \forall\ A \in \mathcal{I}$$

where U_α, U'_α are isometries , $\forall\ \alpha \in A$ and V_β, V'_β are anti-isometries , $\forall\ \beta \in B$ such that the rank of the operators U_α, V_β forms a pairwise orthogonal family and the same occurs for the ranks of the operators U'_α, V'_β.

Since Φ is locally surjective and we know the form of the surjective linear isometries of $\mathcal{I}$ (Theorem 2.7) we infer that Φ maps rank one operators into rank-one operators. Let us prove that the right side of the last equation reduces to only one term. Pick $\xi \in H \backslash \{0\}$ and define the rank one operator $\xi \otimes \xi\colon H \to H$ by $\xi \otimes \xi(x) =< x, \xi > \xi, \ \forall\ x \in H$. Then $\xi \otimes \xi$ is self-adjoint and

$$\Phi(\xi \otimes \xi)(h) = \sum_{\alpha \in A} U'_\alpha \xi \otimes \xi U^*_\alpha(h) + \sum_{\beta \in B} V'_\beta \xi \otimes \xi^* V^*_\beta(h) =$$

$$= \sum_{\alpha \in A} U'_\alpha(\xi) \otimes U_\alpha(\xi)(h) + \sum_{\beta \in B} V'_\beta(\xi) \otimes V_\beta(\xi)(h), \ \ \forall\ h \in H.$$

But, since $\Phi(\xi \otimes \xi)$ is a rank one operator and, for $\alpha_0 \in A$ and $\beta_0 \in B$, we have $\Phi(\xi \otimes \xi)(U_{\alpha_0}(\xi)) = ||U_{\alpha_0}(\xi)||^2 U'_\alpha(\xi)$ and $\Phi(\xi \otimes \xi)(V_{\beta_0}(\xi)) = ||V_{\beta_0}(\xi)||^2 V'_{\beta_0}(\xi)$ it must be either $\Phi(A) = U'AU^*, \ \forall\ A \in \mathcal{I}$ or $\Phi(A) = V'A^*V^*, \ \forall\ A \in \mathcal{I}$.

Suppose Φ is of the form

$$A \mapsto U'AU^*$$

where U', U are isometries. Since Φ is a local surjective isometry, $\Phi(A) = U_A A U_A^*$ has dense rank whenever A has and U' is surjective. In addition, if A is injective then $\Phi(A)$ is also injective and U^* is injective. Then U, U' are unitary operators (they are surjective linear isometries on H) and Φ is a surjective linear isometry.

■

As a particular case of this theorem we obtain that the Schatten classes are algebraically reflexive when it is not the Hilbert-Schmidt class.

Theorem 3.12. *If $p \neq 2$, then every local surjective linear isometry on $C_p(H)$ is a surjective linear isometry.*

Chapter 4

2-Local isometries

The previous chapter of this memory treated the notion of *local map*. We studied that certain classes $S \subset L(X,Y)$ of maps between concrete Banach spaces X and Y satisfy the property

Every local S-map is an element in S.

In this final chapter we will show that it is possible to formulate a weaker version for the concept of $S-$*local map*. This new notion will be the concept of *2-local* S*map*. We will start giving the precise definition of this concept and later, we will study the question

Is every *2-local* S*-map* an element of S?

for certain Banach spaces and certain classes of mapping just as we did at the previous chapter.

Let X and Y be two Banach spaces and let us consider a class of mappings $S \subset L(X,Y)$. We shall say that $\Delta : X \to Y$ is a *2-local* S*-map* if it satisfies

$$\text{For each } x,\ y \in X, \text{ there exists } T_{x,y} \in S,$$
$$\text{depending on } x \text{ and } y, \text{ such that } \Delta(x) = T_{x,y}(x) \text{ and } \Delta(y) = T_{x,y}(y).$$

We apply again the notation $T_{x,y}$ in order to highlight that it is an application that strongly depends on the elements $x, y \in X$. Note that in this definition we do not require that Δ was a linear mapping.

We should notice first that that the concept of *2-local S-map* is more general than the concept of *local S-map*, in the sense of every linear *2-local S-map* is in fact a *local S-map.* If $S \subset L(X, Y)$ is a class of functions, and $\Delta : X \to Y$ is a 2- local S-map, we can pick an element $x \in X$ we get a mapping $T_x := T_{x,0} \in S$ such that $T_x(x) = T_{x,0}(x) = \Delta(x)$.

In addition, every 2-local S-Map Δ is homogeneous. If $\lambda \in \mathbb{C}$ is a scalar number and we take $x \in X$ we obtain $T_{\lambda x,x} \in S$ satisfying $T_{\lambda x,x}x = \Delta(x)$ and $T_{\lambda x,x}(\lambda x) = \Delta(\lambda x)$ and it must be $\Delta\lambda(x) = T_{\lambda x,x}(\lambda x) = \lambda T_{\lambda x,x}(x) = \lambda\Delta(x)$.

4.1 Homomorphisms

Following the lines of the last chapter we start studying multiplicative functionals on a complex Banach algebra A. So we recall the the space

$$S = \{\phi \in A^* : \phi(xy) = \phi(x)\phi(y), \ \forall \ x, y \in A\}$$

As before, referring to usual to a *2-local S map* by saying that it is a *2-local homomorphism* is a usual nomenclature.

We show now the statement we all are expecting.

Theorem 4.1. *Every 2-local homomorphism on a Banach algebra A is a homomorphism.*

Proof. Let us pick Δ a 2-local homomorphism. Then, for $x, y \in A$ we can find $T_{x,y} \in S$ such that $T_{x,y}(x) = \Delta(x)$ and $T_{x,y}(y) = \Delta(y)$. Since $T_{x,y}$ is a multiplicative and linear functional, for each fixed $x, y \in A$, by applying Proposition 1.12

we infer that $T_{x,y}$ is also a continuous functional and we can apply the *Gleason-Kahane-Zelazko theorem* 1.13 to conclude that $T_{xy}(h) \in \sigma(h), \ \forall \, h \in A$. Thus $\Delta(x) - \Delta(y) = T_{x,y}(x) - T_{x,y}(y) = T_{x,y}(x-y) \in \sigma(x-y)$ and $\Delta(0) = T_{x,y}(0) = 0$ for every $x, y \in A$. If we apply now the *Kowalski-Slodowski theorem*1.14 we obtain that Δ is multiplicative just as we desired. ■

4.2 Surjective isometries

In this section, we shall study 2-local S-maps in the case of S being the set of the *surjective linear isometries* in a Banach space X.

$$\mathrm{Isom}_S(X) := \{T \in B(X,X) : ||T(x)|| = ||x||, \ \forall \, x \in X \text{ and } T(X) = X\}$$

$2 - \mathrm{Isom}_S(X)$-local maps are called 2-local surjective isometries on X.

It is very easy to prove that every 2-local isometry always is an isometry. Furthermore, by the *Mazur-Ulam theorem* it is clear that every 2-local isometry Δ which is also surjectivity is a linear mapping and, consequently, Δ is a local isometry. We remark this fact in a theorem.

Proposition 4.2. *Every surjective 2-local isometry is a local surjective isometry.*■

Sequence spaces

We have studied that under the assumption of surjectivity the problem of determining when a *2-local surjective isometry* is an surjective linear isometry on a certain Banach space is very similar to the problem of determining *local surjective linear isometries* on the same space. In this section, we will see that even if we drop the hypothesis of surjectivity we can obtain surprising results.

Theorem 4.3. *Let Y be a space in the set $\{\ell_p : \ 1 \leq p < \infty, p \neq 2\}$, and let $\Delta : Y \to Y$ be a 2-local surjective isometry. Then Δ is a surjective linear isometry.*

Proof. Suppose $\Delta \colon \ell_p \to \ell_p$ is a 2-local isometry on ℓ_p. Note that for $\lambda \in \mathbb{C}$ and $x \in \ell_p$

$$\Delta(\lambda x) = T_{\lambda x, x}(\lambda x) = \lambda T_{\lambda x, x}(x) = \lambda \Delta(x)$$

and hence Δ is homogeneous. By previous results we know that for every surjective linear isometry $S \colon \ell_p \to \ell_p$ there exist a bijection $\varphi \colon \mathbb{N} \to \mathbb{N}$ and a sequence $\tau \colon \mathbb{N} \to \mathbb{T}$ such that

$$S(f) = \tau \cdot f \circ \varphi, \ \ \forall \ f \in \ell_p$$

Suppose now that $x, y \in \ell_p$ with finite support, $x, y \in c_{00}$. We can find $T_{xy} \colon \ell_p \to \ell_p$ surjective linear isometry satisfying

$$\Delta(x) = T_{xy}(x), \ \Delta(y) = T_{xy}(y)$$

Thus $\Delta(x), \Delta(y) \in C_{00}$ and we can see them as elements in $\ell_2(\mathbb{N})$. Therefore

$$< \Delta(x), \Delta(y) >=< T_{xy}x, T_{xy}y >=< x, T_{xy}^* T_{xy} y >=< x, y > .$$

Using the bilinearity of the inner product

$$< \Delta(x+y) - \Delta(x) - \Delta(y), \Delta(x+y) - \Delta(x) - \Delta(y) >= 0$$

and

$$\Delta(x+y) = \Delta(x) + \Delta(y) \ \ \forall \ x, y \in \ell_p \cap c_{00}.$$

Since $\overline{c_{00}}^{\|\cdot\|_p} = \ell_p$ and Δ is continuous, we can assure that Δ is linear. Now we apply that a 2-local linear isometry is a local isometry. ■

Next we shall study the remaining space ℓ_∞. Previously, we have to set some notation. It is clear that, $e \in \ell_\infty$ is a partial isometry, if and only if $e_n(1 - |e_n|^2) = 0, \ \forall \ n \in \mathbb{N}$ and therefore $e_n \in \mathbb{T} \cup \{0\}, \ \forall \ n \in \mathbb{N}$.

Proposition 4.4. *Let T be surjective linear isometry on th Banach space X. Then $T(xy^*z) = Tx(Ty)^*Tz,\ \forall\ x, y, z \in l_\infty(\mathbb{N})$. As a consequence $T(|a|^2b) = |T(a)|^2T(b),\ \forall\ a, b \in l_\infty(\mathbb{N})$.*

Proof. If $T \in Isom_S(X)$ we can find a unitary sequence $\{u_n\}_{n\in\mathbb{N}}$ and a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that

$$Tf = u \cdot f \circ \sigma.$$

In that case,

$$T(xy^*z) = u(xy^*z) \circ \sigma = ux_\sigma \cdot (uy_\sigma)^*uz_\sigma = Tx(Ty)^*Tz,\ \forall\ x, y, z \in l_\infty(\mathbb{N}).$$

■

Finally we state the last theorem in this memory

Theorem 4.5. *If $\Delta\colon l_\infty(\mathbb{N}) \to l_\infty(\mathbb{N})$ is a 2-local surjective isometry then Δ is a surjective linear isometry, that is, $\Delta \in Isom_S(l_\infty(\mathbb{N}))$.*

Proof. Step 1. Δ preserves 'orthogonal' products . For $a, b \in l_\infty(\mathbb{N})$ such that $a \cdot b = 0$ we have $|a|^2b = a^*b = ab^*a = ba^*b = 0$. Pick $T_{a,b} \in Isom_S(l_\infty(\mathbb{N}))$ such that $T_{a,b}(a) = \Delta(a)$ and $T_{a,b}(b) = \Delta(b)$. Therefore:

$$\Delta(a)\Delta(b)^*\Delta(a) = T_{a,b}(a)T_{a,b}(b)^*T_{a,b}(a) = T_{a,b}(ab^*a) = 0,$$

and then $Ta \cdot Tb = 0$.

Step 2. If $\{e^n : n \in \mathbb{N}\}$ is the canonical basis of $l_\infty(\mathbb{N})$ (i.e $e^n_k = \delta_{nk},\ \forall\ k \in \mathbb{N}, ,\ \forall\ n \in \mathbb{N}$) then

i) $e^n \perp e^m$ and $\Delta(e^m) \perp \Delta(e^n),\ \forall\ n, m \in \mathbb{N} : m \neq n$.

ii) $\Delta(e^n)$ is a partial isometry.

i) is clear from *Step* 1. To prove *ii)* we check that the non-zero terms have norms with modulus one. It $m \in \mathbb{N}$ is fixed we can find $T_m \in Isom_S(l_\infty)$ such that

$$\Delta(e^m) = T_m(e^m) = u^m \cdot e^m \circ \sigma^m(m),$$

where $\{u_n^m\}_{n\in\mathbb{N}}$ is a sequence of elements in the complex sphere $\mathbb{T}$ $\sigma^m \colon \mathbb{N} \to \mathbb{N}$ a bijection. Now it is clear the $(\Delta(e^m))(n) \in \mathbb{T} \cup \{0\},\ \forall\, n \in \mathbb{N}$.

Step 3. There exist a map $\sigma \colon \mathbb{N} \to \mathbb{N}$ and $\{u_n\}_{n\in\mathbb{N}}$ a sequence of elements in the complex sphere $\mathbb{T}$ such that $\Delta(e^n) = u_{(n)} e^{\sigma(n)},\ \forall\, n \in \mathbb{N}$. We have just seen that $\Delta(e^n) = u^n \cdot e^n \circ \sigma^n,\ \forall\, n \in \mathbb{N}$. So, we can write this on the form

$$\Delta(e^n)(m) = u^n \cdot e^n \circ \sigma^n(m) = u_m^n \cdot \delta_{n,\sigma^n(m)}.$$

Since $\sigma^n \colon \mathbb{N} \to \mathbb{N}$ is a bijection these exist a unique $k_0 \in \mathbb{N}$ such that $\sigma^n(k_0) = n$. So we can define $\sigma \colon \mathbb{N} \to \mathbb{N}$ $\sigma(m) = (\sigma^m)^{-1}(m),\ \forall\, m \in \mathbb{N}$ and take $u_m := u^m_{(\sigma^m)^{-1}(m)},\ \forall\, m \in \mathbb{N}$. Then we have

$$\Delta(e^n)(m) = u_m^n \cdot \delta_{n,\sigma^n(m)} = u_m^n \cdot \delta_{n,\sigma(n)} = u_m^n \cdot e^{\sigma(n)}(m) = u_m e^{\sigma(n)}(m),$$

for all $m \in \mathbb{N}, n \in \mathbb{N}$.

Step 4. $\sigma \colon \mathbb{N} \to \mathbb{N}$ is injective. Suppose $m \neq n$. Then $e_n \perp e_m$ and $\Delta(e_n) \perp \Delta(e_m)$. In that case $u_n e^{\sigma(n)} \perp u_m e\sigma(m)$ and it must be $\sigma(n) \neq \sigma(m)$.

Proposition 4.6. *Let k be a natural number. If $T \in Isom_S(l_\infty(\mathbb{N}))$ and $s \in l_\infty(\mathbb{N})$ is a partial isometry with k non-zero terms, then $T(s)$ is a partial isometry with k non-zero terms.*

Proof. Take a sequence $\{u_n\}_{n\in\mathbb{N}} \preceq \mathbb{T}$ whose elements are in the complex sphere $\mathbb{T}$ and bijection $\sigma \colon \mathbb{N} \to \mathbb{N}$ such that $Tx = u \cdot x_\sigma,\ \forall\, x \in l_\infty(\mathbb{N})$. Suppose $\alpha_1, ..., \alpha_k \in \mathbb{C} \backslash \{0\}$ and $e_{j_1}, ..., e_{j_k} \in l_\infty(\mathbb{N})$ such that $e_{j_l}(m) = \delta_{j_l,m},\ \forall\, m \in \mathbb{N},\ \forall\, l \in [1,k]_\mathbb{N}$. If $x = \sum_{l=1}^k \alpha_l e_{j_l}$ then $Tx = \sum_{l=1}^k u_{j_l} \alpha_l \cdot e^{\sigma^{-1}(j_l)}$ which is a sequence with exactly k non-zero terms. $\blacksquare$

We continue with the proof of the theorem.

If $F \subseteq \mathbb{N}$ finite and $s = \sum_{j\in F} e_j$ then $\Delta(s) = \sum_{j\in F} \Delta e_j$. From *Step*3 and the fact that Δ is 2-local isometry we know that both s and Δs are partial isometries with exactly $\#F$ non-zero terms, on the other hand $\sum_{j\in F} \Delta e_j$ is also a *rank* $\#F$ partial isometry. In addition, if $j_0 \in F$

$$\Delta(e_{j_0})(\Delta s)^*\Delta(e_{j_0}) = \Delta(e_{j_0}),\ \ \forall\ j_0 \in F.$$

Then, due to $\Delta e_{j_0} = u_{j_0} e^{\sigma(j_0)}$ we have $(\Delta s)(\sigma(j_0)) = u_{j_0}$ and $\Delta e_{j_0} \succeq \Delta s,\ \ \forall\ j_0 \in F$. Using that Δs is a *rank* $\#F$ partial isometry it most be $\Delta s = \sum_{j\in F} \Delta e_j$.

If $x = \{x_n\}_{n\in\mathbb{N}} \in c_0$ and $s_m = \sum_{j=1}^{m} x_j e_j$ then $\lim_{\to\infty} \Delta s_m = \Delta x$. We only have to rewrite:

$$||\Delta x - \Delta s_m|| = ||T_{x,m}x - T_{x,m}s_m|| = ||x - s_m|| \to 0.$$

Let $x = \{x_n\}_{n\in\mathbb{N}} \in c_{00}$. Then $\Delta x = \sum_{k=1}^{\infty} x_k\lambda_k e_{\sigma(k)}$ for certain $\lambda = \{\lambda_n\}_{n\in\mathbb{N}} \preceq \mathbb{T}$. In particular $\Delta(c_0) \subseteq c_0$ and $\Delta(c_{00}) \subseteq c_{00}$. Set $s_m^n := \sum_{k=1}^{m} x_k e_k$, $p_m = \sum_{k=1}^{m} e_k$, $q_m = 1 - p_m,\ \ \forall\ m \in \mathbb{N}$. If we compute:

$$\Delta p_m(\Delta p_m)^*\Delta s_m(\Delta p_m)^*Tp_m = T_m(p_m p_m^* s_m p_m p_m^*) = T_m s_m = \Delta s_m$$

where we have used that

$$p_m p_m^*(k) = \left(\sum_{j=1}^{m} e_j(k)\right)\left(\sum_{j=1}^{m} e_j(k)\right) = \sum_{j=1}^{m} e_j(k) = p_m(k),\ \ \forall\ k \in \mathbb{N}$$

and

$$p_m s_m(k) = \left(\sum_{j=1}^{m} e_j(k)\right)\left(\sum_{j=1}^{m} x_j e_j(k)\right) = s_m(k),\ \ \forall\ k \in \mathbb{N}.$$

In that case

$$\Delta p_m(\Delta p_m)^* = \Delta\left(\sum_{j=1}^{m} e_j\right)\left(\Delta\sum_{j=1}^{m} e_j\right)^* = \left(\sum_{j=1}^{m} \Delta e_j\right)\left(\sum_{j=1}^{m} \Delta e_j\right)^* =$$
$$= \sum_{j=1}^{m} |\Delta e_j|^2 = \sum_{j=1}^{m} e^{\sigma(j)}$$

and

$$\Delta s_m = (\Delta p_m)(\Delta p_m)^* \Delta s_m = \left(\sum_{j=1}^{m} e^{\sigma(j)}\right) \Delta s_m.$$

However,

$$e_{\sigma(k)} T s_m = \Delta e_k (\Delta e_k)^* \Delta s_m = \Delta(e_k e_k^* s_m) = \Delta(x_k e_k) = x_k \Delta e_k.$$

Therefore

$$\Delta s_m = \Delta \left(\sum_{k=1}^{m} x_k e_k\right) = \Delta p_m (\Delta p_m)^* \Delta s_m = \left(\sum_{k=1}^{m} e_{\sigma(k)}\right) \Delta s_m = \sum_{k=1}^{m} x_k \Delta e_k.$$

In that case $\lim_{m\to\infty} \Delta \left(\sum_{k=1}^{m} x_k e_k\right) = \lim_{m\to\infty} \sum_{k=1}^{m} x_k \Delta e_k = \sum_{k=1}^{\infty} x_k \Delta e_k$ and the convergence of the series is given by the unconditional summability derived from the fact that $x \in c_0$.

$\sigma \colon \mathbb{N} \to \mathbb{N}$ is surjective. Take $x = \{\frac{1}{n}\}_{n\in\mathbb{N}} \in c_0$. From *Step*7 $\Delta x = u \cdot x \circ \sigma$ and $\Delta x(n) = \lambda_n \frac{1}{\sigma(n)}, \ \forall\, n \in \mathbb{N}$. Using Δ is a 2-local isometry we get $u^* = \{u^*\}_{n\in\mathbb{N}} \prec \mathbb{T}$ and a bijection $\sigma^x \colon \mathbb{N} \to \mathbb{N}$ such that $\Delta x = u^x \cdot x \circ \sigma^x$. So $\Delta x(u) = u_n^x \frac{1}{\sigma^*(n)} = \lambda_n \frac{1}{\sigma(n)}$ and $\sigma^x(n) = \sigma(n), \ \forall\, n \in \mathbb{N} \Rightarrow \sigma = \sigma^*$ and σ is surjective.

We recall for $z \in l_\infty(\mathbb{N})$ and $m \in \mathbb{N}$ the expressions

i) $\Delta p_m (\Delta p_m)^* \Delta z = \sum_{k=1}^{m} z_k \Delta e^k$

ii) $\Delta p_m (\Delta p_m)^* = \sum_{k=1}^{m} e^{\sigma(k)}$

If we see $l_\infty(\mathbb{N})$ as $l_1(\mathbb{N})^*$ and consider the weak $*$ topology it is very easy to notice that $\{\Delta e^k : k \in \mathbb{N}\}$ and $\{z_k \Delta e_k : k \in \mathbb{N}\}$ are weak $*$ sumable in $l_\infty(\mathbb{N})$ and $w * \lim_{m\to\infty} (\Delta p_m)(\Delta p_m)^* = 1$. Taking $*$-limit in $i)$ we conclude

$$\Delta z = w * \lim_{m\to\infty} \sum_{k=1}^{m} z_k \Delta e^k = w * \lim_{m\to\infty} \sum_{k=1}^{m} u^k z^k e^{\sigma(k)} = u \cdot z \circ \sigma.$$

■

At this stage, the reader should be convinced of the fact that relaxing the assumptions of surjectivity and linearity is not an easy work. This memory ends here, but the study of *local* and *2-local isometries* is not concluded, actually, it has just started. We could for example study *2-local maps* on Schatten classes or on other Banach spaces. Furthermore, we could choose other classes of functions (see [JP16] and [JP17]). The theory of *local and 2-local* maps covers a vast area in Functional Analysis and, nowadays, there still existing open problems in this respect. We treated here those spaces and classes of mappings which in our criteria seems to be more interesting or representative, or simply accesible to our background.

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