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The Spin group and its applications

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Resumen

Este trabajo se centra en las propiedades y aplicaciones del álgebra de Clifford $\mathbb{G}_{p,m}$, asociada al espacio cuadrático no degenerado $\mathbb{R}^{p,m}$, y de su subgrupo más importante, el grupo espín (ortócrono) $\mathrm{Spin}^+_{p,m}$. En particular, demostramos que este grupo es el recubridor doble del grupo de rotaciones (generalizado), $\mathrm{SO}^+_{p,m}$. De hecho, en los casos euclídeo y loretziano, los grupos $\mathrm{Spin}(n)$ y $\mathrm{Spin}^+(1,n)$ son simplemente conexos, es decir, son los recubridores universales del grupo de rotaciones $\mathrm{SO}(n)$ y del grupo de Lorentz (restringido) $\mathrm{SO}^+(1,n)$. Particularizando al caso $n = 3$, discutimos algunas aplicaciones del álgebra de Clifford $\mathbb{G}_3$ y de sus subgrupos $\mathrm{Spin}(3)$ y $\$\mathrm{pin}^+(1,3) \cong \mathrm{Spin}^+(1,3)$ a la relatividad especial, el electromagnetismo y la mecánica cuántica.

El trabajo está estructurado en tres capítulos. En el capítulo 1, partiendo de la identidad $v^2 = vv = q(v)$ para todo $v \in \mathbb{R}^{p,m,z}$ (donde q es una forma cuadrática de signatura arbitraria (p, m, z)), se construye el álgebra $\mathbb{G}_{p,m,z}$ y su producto fundamental, el producto geométrico. Éste álgebra no solo contiene una copia de los números reales y del espacio vectorial $\mathbb{R}^n$, sino que también posee una estructura graduada que permite representar los diferentes subespacios lineales de $\mathbb{R}^n$. A partir del producto geométrico se definen dos nuevos productos: el producto exterior, que se usa para construir áreas, volúmenes e hipervolúmenes orientados; y el producto escalar, que permite, en el caso no degenerado, medir el tamaño de estos hipervolúmenes. Además, las involuciones del álgebra (análogas a la conjugación de números complejos) permiten escribir expresiones algebraicas de forma sencilla y compacta.

En el capítulo 2, se definen, dentro del álgebra de Clifford no degenerada $\mathbb{G}_{p,m}$, los grupos $\mathrm{Pin}_{p,m}$, $\mathrm{Spin}_{p,m}$ y $\mathrm{Spin}^+_{p,m}$, que permiten realizar transformaciones ortogonales de vectores en el espacio cuadrático $\mathbb{R}^{p,m}$ mediante un homomorfismo de grupos sobreyectivo $\rho : (\mathrm{S})\mathrm{Pin}^{(+)}_{p,m} \to (\mathrm{S})\mathrm{O}^{(+)}_{p,m}$. La sobreyectividad de esta aplicación se deduce del teorema de Cartan-Dieudonné, que conseguimos demostrar de forma sencilla gracias a las operaciones y propiedades de las álgebras de Clifford. Por otro lado, dotando a $\mathbb{G}_{p,m}$ de una topología natural, esta álgebra adquiere, además, una estructura de variedad real analítica, y se observa que los grupos $\mathrm{Pin}_{p,m}$, $\mathrm{Spin}_{p,m}$ y $\mathrm{Spin}^+_{p,m}$ están contenidos en el grupo $\mathbb{G}^\times_{p,m}$ de elementos invertibles del álgebra, que resulta ser un grupo de Lie. Esta estructura de grupo de Lie nos permite demostrar que $\mathrm{Spin}^+_{p,m}$ es arcoconexo y que el homomorfismo ρ es también un recubridor doble. Como consecuencia, obtenemos que el grupo de rotaciones (generalizado) $\mathrm{SO}^+_{p,m}$ es conexo.

Como mencionábamos arriba, en los casos euclídeo y lorentziano, este recubridor doble resulta ser también un recubridor universal. La demostración de este resultado no se completa hasta el capítulo 3, donde probamos que el grupo fundamental de $\mathrm{SO}(3)$ es $\mathbb{Z}_2$. Para cerrar el capítulo, se muestra que el álgebra $\mathbb{G}_{p,m}$ contiene una copia isométrica de $\mathbb{R}^{m+1,p}$ y otra isomorfa de $\mathrm{Spin}^+_{m+1,p}$ (que denotaremos por $\$\mathrm{pin}^+_{m+1,p}$), lo que nos permite usar el álgebra de Clifford euclídea, $\mathbb{G}_n$, para modelar también el espacio lorentziano $\mathbb{R}^{1,n}$ y sus transformaciones de Lorentz.
Finalmente, en el capítulo 3 se estudia la estructura y aplicaciones del álgebra de Clifford $\mathbb{G}_3$ asociada al espacio euclídeo $\mathbb{R}^3$. La forma particularmente simple que toma el grupo espín en dimensiones bajas ($n \leq 4$) nos permite demostrar que $\mathrm{Spin}(3)$ es simplemente conexo, lo cual implica, por ser $\rho : \mathrm{Spin}(3) \to \mathrm{SO}(3)$ un recubridor doble, que el grupo fundamental de $\mathrm{SO}(3)$ es $\mathbb{Z}_2$, como anticipábamos arriba. Este resultado topológico-algebraico se ilustra físicamente mediante el truco del cinturón de Dirac. Para ello, seguimos la explicación dada en [22]. Además, se establece un isomorfismo entre $\mathrm{Spin}(3)$ y los cuaterniones unitarios, deduciéndose la fórmula de representación de rotaciones usando cuaterniones.

En cuanto a las aplicaciones en física, utilizamos el espacio de paravectores $\mathbb{P}_{1,3} = \mathbb{G}_3^0 \oplus \mathbb{G}_3^1$ y el grupo $\$\mathrm{pin}^+(1,3) \subseteq \mathbb{G}_3^\times$ para modelar el espacio-tiempo de Minkowski y las transformaciones de Lorentz, respectivamente. Además, el elemento $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, también llamado *el pseudoescalar*, dota al álgebra $\mathbb{G}_3$ de una estructura compleja natural, que es utilizada para definir la dualidad de Hodge. Usando esta operación, recuperamos el producto vectorial usual de $\mathbb{R}^3$ como el dual del producto exterior, lo que nos permite reescribir de forma compacta las cuatro ecuaciones de Maxwell como una sola ecuación en $\mathbb{G}_3$.

Para terminar el capítulo, nos centramos en las aplicaciones en mecánica cuántica. En primer lugar, se muestra que las matrices de Pauli inducen un isomorfismo entre $\mathbb{G}_3$ y $\mathbb{C}^{2\times 2}$, que restringido a los grupos $\mathrm{Spin}(3)$ y $\$\mathrm{pin}^+(1,3)$ da lugar a los isomorfismos

$$\mathrm{Spin}(3) \cong \mathrm{SU}(2), \quad \$\mathrm{pin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C}).$$

El isomorfismo dado por las matrices de Pauli también nos permite modelar los espinores (de Pauli y de Weyl) como elementos del ideal mínimo por la izquierda $\mathbb{G}_3 P_3$, donde $P_3 = \frac{1}{2}(1 - \mathbf{e}_3)$ es un idempotente primitivo. Por último, partiendo de la representación de las transformaciones de Lorentz usando el grupo $\$\mathrm{pin}^+(1,3)$, deducimos la ecuación de Dirac, revelando una estrecha conexión entre los espinores de Dirac (y de Weyl) y ciertos elementos de dicho grupo espín.

Contents

Introduction 5

1 Clifford (geometric) algebras 8

1.1 Heuristics . . . 8

1.2 Formal definition of the Clifford Algebra . . . 9

1.3 The wedge and scalar products . . . 12

1.4 Two more involutions . . . 19

2 The Spin group 22

2.1 Cartan-Dieudonné Theorem . . . 22

2.2 The Pin and Spin groups . . . 26

2.3 Topological and Lie group properties . . . 31

2.4 The paravector space . . . 38

3 Applications 41

3.1 The (geometric) algebra of physical space . . . 41

3.2 3D rotations and quaternions . . . 43

3.2.1 Relation to quaternions . . . 44

3.3 The topology of $\mathrm{Spin}(3)$ and $\mathrm{SO}(3)$ . . . 45

3.3.1 Dirac's belt trick . . . 46

3.4 Paravector model of spacetime . . . 48

3.5 Maxwell's Equation(s) . . . 51

3.6 Pauli matrices and spinors . . . 52

3.7 \$pin$^+(1,3)$ and the Dirac equation . . . 55

Conclusions 58

References 59

Appendices 62

A Quadratic spaces 62

B Covering spaces 65

Introduction

Throughout history, several number systems have been developed to address increasingly complex mathematical problems, obtaining the following chain of successive extensions

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Among these number systems, the ones which are most important to physics are those which allow a geometric interpretation of the numbers and of the operations between them.

The most familiar example the real numbers, $\mathbb{R}$, which can be thought of as oriented segments on a line. In this view, a real number $a \in \mathbb{R}$ encodes both a magnitude $|a| \geq 0$ and, if it is non-zero, also a direction along a one-dimensional space (a and $-a$ represent opposite directions). Moreover, the operations between real numbers have a clear geometric interpretation, namely, addition represents translations and multiplication represents dilations.

When we extend the real numbers by introducing an imaginary unit i satisfying $i^2 = -1$, we obtain the complex numbers $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$. A complex number $z = x + iy$ can be used to represent the vector (x, y) in the plane $\mathbb{R}^2$, and again, the operations between complex numbers can be geometrically interpreted. Addition and multiplication by a real number still represent translations and dilations, respectively. Moreover, multiplication by a unit complex number, which are of the form $w = \cos\theta + i\sin\theta$, represents a rotation of the plane around the origin by an angle of θ. In this sense, complex numbers can be viewed as two-dimensional geometric numbers, which allow a clear correspondence between algebraic manipulations and geometric transformations in the plane.

Motivated by how the complex numbers are able to model planar geometry and transformations efficiently, the Irish mathematician and physicist W.R. Hamilton tried to generalize the complex numbers to a number system which could model 3-dimensional geometry. The result of his investigation was the invention of the quaternions

$$\mathbb{H} = \{w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} : w, x, y, z \in \mathbb{R}\},$$

which are obtained by extending the real numbers with three imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfying the rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Quaternions form a four-dimensional number system that is no longer commutative but retains associativity. The imaginary or pure part $\mathbf{q} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ of the quaternion $q = w + \mathbf{q}$ can be used to represent vectors in $\mathbb{R}^3$, and addition of pure quaternions and multiplication of a pure quaternion by a real number again represent translations (vector addition) and dilations in $\mathbb{R}^3$, respectively. Moreover, the rotation of the vector $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ by an angle of θ around the axis given by the unit vector $\hat{\mathbf{a}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, is achieved via the formula $q\mathbf{v}q^*$, where $q = \cos\frac{\theta}{2} + \hat{\mathbf{a}}\sin\frac{\theta}{2}$ is a unit quaternion and $q^* = \cos\frac{\theta}{2} - \hat{\mathbf{a}}\sin\frac{\theta}{2}$ is its quaternion conjugate (we will prove this formula in chapter 3).

In order to generalize these "geometric number systems" to arbitrary dimensions we first have to formalize what we mean by "geometric number". On the one hand, a number can be thought of as something that can be added, subtracted and multiplied.

On the other hand, vectors are the basic mathematical objects used to model geometry. Therefore, the mathematical structure which formalizes the idea of a "geometric number system" is that of an (associative and unital) *algebra*, which is a vector space equipped with a product that is bilinear, associative and has a multiplicative identity. The three sets we have seen above ($\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$) are examples of *real algebras*, where the scalars of the vector space are the real numbers.

In this work, we will generalize these geometric numbers to n dimensions, starting from the real vector space $\mathbb{R}^n$ and defining a suitable "geometric" product between vectors. This will lead to the definition of a *Clifford or geometric algebra*, after the 19th century English mathematician W.K. Clifford, who first defined them. Within this algebra there will be a special group, the *Spin group*, which generalizes the unit quaternions and allows us to perform rotations in spaces of any dimension.

The work is divided into three chapters, and the objectives of the TFG proposal are as follows:

1. Study of previous material on Lie groups and the rotation group $\mathrm{SO}(n)$, proving that it is connected.
2. Construction of $\mathrm{Spin}(n)$ as a subgroup of the group of units of the Clifford algebra $\mathbb{G}_n$, showing that it is the universal cover of $\mathrm{SO}(n)$.
3. Identification of the group $\mathrm{Spin}(3)$ with $\mathrm{SU}(2)$ and the unit quaternions.
4. Explanation of *Dirac's belt trick*.
5. Description of some applications of $\mathrm{Spin}(n)$ in particle physics.

In chapter 1, we introduce the Clifford algebra $\mathbb{G}_{p,m,z}$ associated to the quadratic space $\mathbb{R}^{p,m,z} = (\mathbb{R}^n, q)$, where q is a quadratic form of arbitrary signature (p, m, z). Just as complex numbers are constructed from the identity $i^2 = -1$, defining

$$v^2 = vv = q(v) \in \mathbb{R}$$

for all $v \in \mathbb{R}^{p,m,z} = (\mathbb{R}^n, q)$, leads to the construction of the algebra $\mathbb{G}_{p,m,z}$ and its fundamental product, the *geometric product*. We follow [26] for the heuristics of the construction and [1] for the formal definition. Not only does the algebra $\mathbb{G}_{p,m,z}$ contain a copy of the real numbers and the vector space $\mathbb{R}^n$, but it also has a graded structure which lets us represent the different linear subspaces of $\mathbb{R}^n$. From the geometric product, we define two more products: the wedge or exterior product, which is used to construct oriented areas (*bivectors*), volumes (*trivectors*) and hyper-volumes (*k-vectors*) starting from the vectors in $\mathbb{R}^n$; and the dot or scalar product, which is used to measure the size of these hyper-volumes (when they are non-degenerate). The involutions of the algebra, just like complex conjugation in $\mathbb{C}$, enable expressing certain algebraic manipulations in a clean and compact way. In developing the different products and involutions of the algebra, we mainly follow [20].

In chapter 2 we define, within the non-degenerate Clifford algebra $\mathbb{G}_{p,m}$, the groups $\mathrm{Pin}_{p,m}$, $\mathrm{Spin}_{p,m}$ and $\mathrm{Spin}^+_{p,m}$, which are employed to perform orthogonal transformations of vectors in the quadratic space $\mathbb{R}^{p,m}$ via a surjective group homomorphism $\rho : (\mathrm{S})\mathrm{Pin}^{(+)}_{p,m} \to (\mathrm{S})\mathrm{O}^{(+)}_{p,m}$. The surjectivity of the above map relies on the Cartan-Dieudonné theorem (Theorem 2.9), which is proven in a simple way thanks to the algebraic power of Clifford algebras. The main references for this part of the chapter are

[18] and [20]. On the other hand, when a topology is introduced in $\mathbb{G}_{p,m}$ (in fact, we give it a real analytic manifold structure) we shall see that all these groups lie within the Lie group $\mathbb{G}_{p,m}^{\times}$ of invertible elements of the algebra. The Lie group structure of $\mathbb{G}_{p,m}^{\times}$ is used to show that the group $\mathrm{Spin}_{p,m}^{+}$ is path-connected and that the group homomorphism ρ is, in fact, a double covering map. As a consequence, we conclude the connectedness of the group $\mathrm{SO}_{p,m}^{+}$. The main references followed to proof these results are [20], [11] and [14]. Secondary references also consulted are [17] and [10].

Furthermore, in the Euclidean and Lorentzian cases the above double covering turns out to be a universal covering map. The proof of this result is not completed until chapter 3, when we show that the fundamental group of the rotation group $\mathrm{SO}(3)$ is $\mathbb{Z}_2$. In addition, we use two important facts about Lie groups (Lemma 2.52 and Theorem 2.49) which are not proven here, but can be found in [7] and [29], respectively. Hence, in this chapter we fulfill objectives 1 and 2 of the TFG proposal.

To finish the chapter, following [25] and [19] we show in section 2.4 that the algebra $\mathbb{G}_{p,m}$ contains both an isometric copy of $\mathbb{R}^{m+1,p}$ and an isomorphic image of $\mathrm{Spin}_{m+1,p}^{+}$ (denoted by $\$\mathrm{pin}_{m+1,p}^{+}$), allowing the use of the Euclidean algebra $\mathbb{G}_n$ to model also the Lorentzian space $\mathbb{R}^{1,n}$ and its Lorentz transformations.

In the final chapter (chapter 3), we focus on the structure and applications of the Clifford algebra $\mathbb{G}_3$ associated with physical space $\mathbb{R}^3$. Theorem 2.30 is used to show that $\mathrm{Spin}(3)$ is simply connected, which implies, by means of the double cover $\rho : \mathrm{Spin}(3) \to \mathrm{SO}(3)$, that fundamental group of the rotation group $\mathrm{SO}(3)$ is $\mathbb{Z}_2$, as we anticipated above. This algebraic-topological result is illustrated physically using Dirac's belt trick. The explanation of this trick follows closely the one given in [22] and the images employed are also taken from that video (except Figure 5, which is taken from [21]). Moreover, an isomorphism between $\mathrm{Spin}(3)$ and the unit quaternions is found, and the quaternion rotation formula is deduced.

As for the application in physics, following [3] and [19], the paravector space $\mathbb{P}_{1,3} = \mathbb{G}_3^0 \oplus \mathbb{G}_3^1$ and the group $\$\mathrm{pin}^{+}(1,3) \subseteq \mathbb{G}_3^{\times}$ are used to model Minkowski spacetime and Lorentz transformations of four-vectors, respectively. Moreover, we show how the unit trivector $\mathrm{i} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ (also called the *pseudoscalar*) provides the algebra $\mathbb{G}_3$ with a natural complex structure, and is used to define the operation of *Hodge duality* from which we recover the usual cross product in $\mathbb{R}^3$ from the wedge product. By (formally) replacing the cross products with wedge products, we are able to compactly rewrite the four Maxwell's equations as a single equation in $\mathbb{G}_3$.

Finally, delving into the realm of quantum mechanics, we show, following [19], that the Pauli matrices give an isomorphism between $\mathbb{G}_3$ and $\mathbb{C}^{2\times 2}$, which, when restricted to the groups $\mathrm{Spin}(3)$ and $\$\mathrm{pin}^{+}(1,3)$ results in the isomorphisms

$$\mathrm{Spin}(3) \cong \mathrm{SU}(2) \quad \text{and} \quad \$\mathrm{pin}^{+}(1,3) \cong \mathrm{SL}(2,\mathbb{C}).$$

The Pauli matrices isomorphism also allows us to model two-component column spinors in $\mathbb{C}^2$ as members of the minimal left ideal $\mathbb{G}_3 P_3$, where $P_3 = \frac{1}{2}(1 - \mathbf{e}_3)$ is a minimal idempotent. Furthermore, following [5] and [6], we derive the Dirac equation starting solely from the representation of Lorentz transformations using the Spin group $\$\mathrm{pin}^{+}(1,3)$, revealing a deep connection between Dirac (and Weyl) spinors and elements of $\$\mathrm{pin}^{+}(1,3)$. Therefore, in this chapter we fulfill objectives 3, 4 and 5 of the TFG proposal.

1 Clifford (geometric) algebras

In this chapter we introduce *Clifford algebras*, also known as *geometric algebras*, which can be thought of as an extension of the real numbers obtained when a "geometric" product is defined between vectors of $\mathbb{R}^n$. This algebra has a rich structure with several operations that emerge from the geometric product and which will allow us to produce elements that naturally represent not only vectors (oriented line segments) but also oriented areas, volumes, and higher-dimensional subspaces of $\mathbb{R}^n$. The main references used for this chapter are [1], [20] and [26].

1.1 Heuristics

The first step to construct a Clifford algebra is to equip $\mathbb{R}^n$ with the quadratic form

$$q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+m} x_j^2$$

of arbitrary signature (p, m, z), upgrading it to the quadratic space[1] $\mathbb{R}^{p,m,z} := (\mathbb{R}^n, q)$. The quadratic form allows us to define the square of any vector $v \in \mathbb{R}^{p,m,z}$ as follows

$$v^2 = vv := q(v). \tag{1.1}$$

From this simple axiom, we will construct the Clifford algebra associated to $\mathbb{R}^{p,m,z}$ only by requiring the product to be associative and distributive.

In the following we will denote the metric associated to q by g, i.e., $q(x) = g(x, x)$, and the canonical basis vectors of $\mathbb{R}^n$ by $e_1, \ldots, e_n$, which are also an orthonormal (Sylvester) basis of $\mathbb{R}^{p,m,z}$. If we replace $v = e_i + e_j$ with $i \neq j$ in 1.1 and use $g(e_i, e_j) = 0$, we obtain

$$q(e_i) + q(e_j) = q(e_i + e_j, e_i + e_j) = (e_i + e_j)^2 = e_i^2 + e_i e_j + e_j e_i + e_j^2$$

and since $e_i^2 = q(e_i) =: \varepsilon_i$ we get

$$e_i e_j = -e_j e_i \quad \forall i \neq j. \tag{1.2}$$

Now, consider a formal product of the basis vectors $e_{\alpha_1} e_{\alpha_2} ... e_{\alpha_k}$ with $1 \leq \alpha_i \leq n$, $k \geq 0$ where $k = 0$ corresponds to the empty product, which equals 1 by definition. Using associativity and 1.2 we can rearrange the factors according to increasing indices and, since $e_i^2 = \varepsilon_i$ is a scalar, each e_i will appear at most to the first power. For example,

$$e_3 e_5 e_1 e_5 e_4 e_3 e_5 = e_3(-e_1 e_5) e_5 (-e_3 e_4) e_5 = (e_1 e_3) e_5^2 (-e_3 e_4 e_5) = -\varepsilon_5 e_1 e_3^2 e_4 e_5 = -\varepsilon_3 \varepsilon_5 e_1 e_4 e_5.$$

Therefore, every product of basis vectors is proportional to one containing at most n factors and where the factors are arranged according to increasing indices: $e_{\beta_1} e_{\beta_2} ... e_{\beta_k} \propto e_{\alpha_1} e_{\alpha_2} ... e_{\alpha_l}$ where $1 \leq \alpha_1 < \alpha_2 < ... < \alpha_l \leq n$ and $0 \leq l \leq n$. We will denote a set of indices $\{\alpha_1, \alpha_2, ..., \alpha_l\}$ such that $1 \leq \alpha_1 < \alpha_2 < ... < \alpha_l \leq n$ with $0 \leq l \leq n$ by a capital letter A and the product $e_{\alpha_1} e_{\alpha_2} ... e_{\alpha_l}$ by e_A. If $l = 0$ then $A = \emptyset$ and $e_\emptyset = 1$.

Since every product is proportional to one of the form e_A, it will suffice to know how to multiply two of such elements in order to multiply any element of the algebra.

[1] See section A in the Appendix for a review of quadratic spaces.

Given two sets $A, B \subseteq \{1, ..., n\}$, consider the corresponding elements $e_A = e_{\alpha_1}e_{\alpha_2}...e_{\alpha_l}$ and $e_B = e_{\beta_1}e_{\beta_2}...e_{\beta_r}$. Their product would have the form $e_Ae_B = e_{\alpha_1}e_{\alpha_2}...e_{\alpha_l}e_{\beta_1}e_{\beta_2}...e_{\beta_r}$. To simplify this product we first rearrange the factors in increasing order.

We start by looking at the element e_{β_1}. If $\alpha_l > \beta_1$ we swap e_{α_l} and e_{β_1} using 1.2, acquiring a minus sign. Next, if $\alpha_{l-1} > \beta_1$ we swap $e_{\alpha_{l-1}}$ and e_{β_1} acquiring another minus sign. We repeat this until $\alpha_i = \beta_1$ or there are no more e_{α_i} at the left of e_{β_1}. We continue by looking at e_{β_2} and repeating the same procedure: we swap e_{α_i} and e_{β_2} using 1.2 and acquiring a -1 until $\alpha_i = \beta_2$ or there are no more e_{α_i} at the left of e_{β_2}. Once we have done this process for all $\beta_j \in B$, the product e_Ae_B will be ordered with increasing (but not necessarily strictly increasing) indices. The total number of negative signs acquired during this process is equal to the number of swaps. Since we had to swap every time $\alpha_i > \beta_j$ with $\alpha_i \in A$ and $\beta_j \in B$ we conclude that the overall sign acquired is

$$\prod_{\substack{\alpha\in A\\ \beta\in B}} s(\alpha,\beta), \quad \text{where } s(\alpha,\beta) = +1 \text{ if } \alpha \leq \beta, \text{ and } -1 \text{ if } \alpha > \beta. \tag{1.3}$$

The product e_Ae_B can be further simplified using that $e_i^2 = q(e_i) = \varepsilon_i$ is a scalar. If an element e_i is a factor of both e_A and e_B, i.e., if $i \in A \cap B$, it means that, after rearranging the factors, it will appear in e_Ae_B as $e_i^2 = \varepsilon_i \in \{1, -1, 0\}$. Therefore, e_Ae_B will be proportional to the product $e_{\gamma_1}...e_{\gamma_s}$ where γ_i is either in A or in B, but not in both. This means that γ_i is in the *symmetric difference* of A and B,

$$A \triangle B := (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A) \subseteq \{1, ..., n\}.$$

We would therefore have

$$e_Ae_B = c(A,B)e_{A\triangle B}, \quad \text{where} \quad c(A,B) := \prod_{\substack{\alpha\in A\\ \beta\in B}} s(\alpha,\beta) \prod_{i\in A\cap B} \varepsilon_i \in \mathbb{R}.$$

The first product in $c(A,B)$ takes care of the sign changes due to rearrangement of the factors according to increasing indices and the second factor accounts for the squares of the repeated e_i.

1.2 Formal definition of the Clifford Algebra

Motivated by the above discussion we make the following definitions.

Definition 1.1. *We define the set of Clifford or geometric numbers as the 2^n-dimensional real vector space*

$$\mathbb{G}(n) := \operatorname{span}_{\mathbb{R}}(\{e_A\}_{A\subseteq\Omega_n})$$

where $\Omega_n := \{1, ..., n\}$ and e_A is a different symbol for each subset $A \subseteq \Omega_n$. In other words, $\mathbb{G}(n)$ is the free $\mathbb{R}$-vector space over $\{e_A\}_{A\subseteq\Omega_n}$.

If for every $k \in \{0, ..., n\}$ we define the $\binom{n}{k}$-dimensional subspace

$$\mathbb{G}^k(n) := \operatorname{span}_{\mathbb{R}}(\{e_A\}_{A\subseteq\Omega_n:|A|=k}),$$

where $|A|$ denotes the cardinal of $A \subseteq \Omega_n$, we can write

$$\mathbb{G}(n) = \bigoplus_{k=0}^{n} \mathbb{G}^k(n).$$

Definition 1.2. *Let* $(\varepsilon_1, ..., \varepsilon_n) = (\overbrace{1, ..., 1}^{p \text{ times}}, \overbrace{-1, ..., -1}^{m \text{ times}}, \overbrace{0, ..., 0}^{z \text{ times}})$ *be the signature of* $\mathbb{R}^{p,m,z}$. *We define the Clifford or geometric product between basis elements of* $\mathbb{G}(n)$ *by*

$$e_A e_B := c(A, B) e_{A \triangle B} \quad \forall A, B \subseteq \Omega_n, \tag{1.4}$$

where

$$c(A, B) := \prod_{\substack{\alpha \in A \\ \beta \in B}} s(\alpha, \beta) \prod_{i \in A \cap B} \varepsilon_i \in \mathbb{R},$$

and extend it univocally by bilinearity to all of $\mathbb{G}(n)$.

Remark 1.3. *We give it the name geometric product because its definition relies on the quadratic form or, equivalently, on the metric defined on* $\mathbb{R}^n$, *which allows us to talk about fundamental geometric notions such as length or orthogonality.*

Proposition 1.4. $\mathbb{G}(n)$ *with the geometric product becomes an associative* $\mathbb{R}$*-algebra with identity element* $e_\varnothing$.

Proof. The symmetric difference satisfies the following properties:

(i) Associativity: $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

(ii) Distributivity of $\cap$: $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$.

With the above definition, we have

$$(e_A e_B) e_C = \prod_{\substack{a \in A \\ b \in B}} s(a, b) \prod_{i \in A \cap B} \varepsilon_i \prod_{\substack{d \in A \triangle B \\ c \in C}} s(d, c) \prod_{j \in (A \triangle B) \cap C} \varepsilon_j \, e_{A \triangle B \triangle C}.$$

Let us rewrite this expression in a more symmetric way. First, consider the third factor. If we let d range not over $A \triangle B = (A \setminus B) \cup (B \setminus A)$ but first over all A and then over all B, then any $d \in A \cap B$ will appear twice, and will contribute a factor of $s(d, c)^2$. However, since $s(d, c)^2 = 1$ no change is introduced:

$$\prod_{\substack{a \in A \\ c \in C}} s(a, c) \prod_{\substack{b \in B \\ c \in C}} s(b, c) = \prod_{\substack{d \in A \triangle B \\ c \in C}} s(d, c) \prod_{\substack{d \in A \cap B \\ c \in C}} s(d, c)^2 = \prod_{\substack{d \in A \triangle B \\ c \in C}} s(d, c).$$

On the other hand, we have

$$\begin{aligned} (A \triangle B) \cap C = (A \cap C) \triangle (B \cap C) &= [(A \cap C) \cup (B \cap C)] \setminus (A \cap B \cap C), \\ A \cap B \cap C \subseteq A \cap B \implies (A \cap B) \cup [(A \triangle B) \cap C] &= (A \cap B) \cup (A \cap C) \cup (B \cap C), \\ (A \cap B) \cap (A \triangle B) = \varnothing &\implies (A \cap B) \cap (A \triangle B) \cap C = \varnothing \end{aligned}$$

which imply

$$\prod_{i \in A \cap B} \varepsilon_i \prod_{j \in (A \triangle B) \cap C} \varepsilon_i = \prod_{i \in (A \cap B) \cup [(A \triangle B) \cap C]} \varepsilon_i = \prod_{i \in (A \cap B) \cup (A \cap C) \cup (B \cap C)} \varepsilon_i.$$

Thus, the product $(e_A e_B) e_C$ can be rewritten as

$$\prod_{\substack{a \in A \\ b \in B}} s(a, b) \prod_{\substack{a \in A \\ c \in C}} s(a, c) \prod_{\substack{b \in B \\ c \in C}} s(b, c) \prod_{i \in (A \cap B) \cup (A \cap C) \cup (B \cap C)} \varepsilon_i \, e_{A \triangle B \triangle C},$$

from which associativity, $(e_A e_B)e_C = e_A(e_B e_C)$, clearly follows.

In addition, since empty products are 1 by definition, we have that

$$e_\emptyset e_A = e_{\emptyset\triangle A} = e_A = e_{A\triangle\emptyset} = e_A e_\emptyset \quad \forall A \subseteq \Omega_n,$$

so $e_\emptyset$ is the identity element of the algebra. □

Notation 1.5. *The algebra $G(n)$ with the geometric product (1.4) will be denoted by $\mathbb{G}_{p,m,z}$ and is called the Clifford or geometric algebra associated to $\mathbb{R}^{p,m,z}$, after the 19th century English mathematician W.K. Clifford, who first defined it. Consequently, the subspace $\mathbb{G}^k(n)$ will be denoted by $\mathbb{G}^k_{p,m,z}$.*

Proposition 1.6. *If we (canonically) identify $\mathbb{R} \equiv \mathbb{G}^0_{p,m,z}$ and $\mathbb{R}^n \equiv \mathbb{G}^1_{p,m,z}$ via*

$$\iota : \mathbb{R} \times \mathbb{R}^{p,m,z} \to \mathbb{G}^0_{p,m,z} \times \mathbb{G}^1_{p,m,z}$$
$$(\alpha, v^i e_i) \mapsto (\alpha e_\emptyset, v^i e_{\{i\}})$$

then, the basis elements of $\mathbb{G}_{p,m,z}$ can be written as

$$1, \quad e_A = e_{\alpha_1} e_{\alpha_2} ... e_{\alpha_k},$$

where $k \in \Omega_n$ and $A = \{1 \leq \alpha_1 < \alpha_2 < ... < \alpha_k \leq n\}$. In particular, we have that $\mathbb{G}_{p,m,z}$ is generated (as an algebra) by the basis vectors $e_1, ..., e_n$ of $\mathbb{R}^n \equiv \mathbb{G}^1_{p,m,z}$, which satisfy the relations

$$e_i^2 = q(e_i) \quad \text{and} \quad e_i e_j + e_j e_i = 0 \quad \text{for } i \neq j. \tag{1.5}$$

Proof. Proceeding by induction, assume that $e_{A_k} = e_{\alpha_1} e_{\alpha_2} ... e_{\alpha_k}$ for all $A_k = \{\alpha_1 < \alpha_2 < ... < \alpha_k\} \subseteq \Omega_n$ with $1 \leq k \leq n-1$. Then, using Definition 1.2 we have that any set of the form $A_{k+1} := A_k \sqcup \{\alpha_{k+1}\}$, with $n \geq \alpha_{k+1} > \alpha_i$ for all $\alpha_i \in A_k$, can be written as

$$e_{\alpha_1} e_{\alpha_2} ... e_{\alpha_k} e_{\alpha_{k+1}} = e_{A_k} e_{\alpha_{k+1}} = c(A_k, \{\alpha_{k+1}\}) e_{A_k \triangle \{\alpha_{k+1}\}} = e_{A_{k+1}}.$$

Since the base case $e_{\{i\}} \equiv e_i$ is trivial, we obtain the first part of the Proposition. As a consequence, we have that the smallest subalgebra containing the basis vectors is the whole algebra $\mathbb{G}_{p,m,z}$.

If $i = j$, then $e_i e_i = e_{\{i\}} e_{\{i\}} = \varepsilon_i e_{\{i\}\triangle\{i\}} = \varepsilon_i e_\emptyset = \varepsilon_i = q(e_i)$.

For the last part, let us analyze the two cases. If $i \neq j$, we can assume without loss of generality that $i > j$ then

$$e_i e_j = e_{\{i\}} e_{\{j\}} = (-1) e_{\{i\}\triangle\{j\}} = -e_{\{i,j\}} = -e_{\{j,i\}} = -e_j e_i.$$

□

Notation 1.7. *The elements of $\mathbb{G}_{p,m,z}$) are called Clifford numbers. They are also called multivectors. On the other hand, an element of $\mathbb{G}^k_{p,m,z}$ is called a k-vector or a grade-k Clifford number (also referred to as a grade-k multivector). Instead of saying 0-vectors, 1-vectors, 2-vectors, and so on, we will often employ the terminology: scalars, vectors, bivectors, etc.*

With the above identifications we have that

$$\alpha e_A = \alpha(e_A e_\emptyset) = e_A(\alpha e_\emptyset) = e_A \alpha \quad \forall A \subseteq \Omega_n,$$

so scalars commute with every element of the algebra, and in particular with vectors in $\mathbb{R}^n \equiv \mathbb{G}^1_{p,m,z}$. On the other hand, since $g(e_i, e_j) = 0$ for all $i \neq j$, we can condense the relations 1.5 as

$$\{e_i, e_j\} = 2g(e_i, e_j),$$

where we have introduced the *anticommutator* of e_i and e_j, defined as $\{a, b\} := ab + ba$ for all $a, b \in \mathbb{G}_{p,m,z}$. One easily checks that the anticommutator is bilinear. Thus, given two vectors $a, b \in \mathbb{R}^n$ we can write $a = \sum_i a^i e_i$ and $b = \sum_j b^j e_j$ and we have that

$$ab + ba = \{a, b\} = \sum_{i,j} a^i b^j \{e_i, e_j\} = 2\sum_{i,j} a^i b^j g(e_i, e_j) = 2g(a, b) \tag{1.6}$$

From this, we recover the basic identity (1.1) we stipulated at the beginning of this section

$$v^2 = g(v, v) = q(v) \quad \forall v \in \mathbb{R}^n. \tag{1.7}$$

The following theorem shows that the above property indeed characterizes Clifford algebras.

Theorem 1.8 (Universal property). *Let $\mathcal{A}$ be an associative $\mathbb{R}$-algebra with identity $1_\mathcal{A}$. If $f : \mathbb{R}^{p,m,z} \to \mathcal{A}$ is a linear map such that*

$$f(v)^2 = q(v)1_\mathcal{A} \quad \forall v \in \mathbb{R}^n, \tag{1.8}$$

then f extends uniquely to an $\mathbb{R}$-algebra homomorphism $\hat{f} : \mathbb{G}_{p,m,z} \to \mathcal{A}$.

Proof. We follow the proof in [20, Thm. 2.8]. Given $A = \{\alpha_1 < ... < \alpha_k\} \subseteq \Omega_n$ with $k \in \Omega_n$, we know that $e_A = e_{\alpha_1}...e_{\alpha_k}$ and we set $a_A := a_{\alpha_1}...a_{\alpha_k}$ where $a_{\alpha_i} = f(e_{\alpha_i})$. If $k = 0$ we have that $e_\emptyset \equiv 1 \in \mathbb{R}$ and we set $a_\emptyset := 1_\mathcal{A}$. Equation (1.8) implies, substituting $v = e_i + e_j$, that

$$a_i a_j + a_j a_i = 0 \text{ if } i \neq j, \quad a_i^2 = q(e_i)1_\mathcal{A}. \tag{1.9}$$

If we want to extend f to an algebra homomorphism we must define $\hat{f} : \mathbb{G}_{p,m,z} \to \mathcal{A}$ by $\hat{f}(e_A) := a_A$ for all $A \subseteq \Omega_n$, and extend linearly to $\mathbb{G}_{p,m,z}$. Let us check that $\hat{f}$ is indeed an algebra homomorphism, in which case it is uniquely determined by construction. On the one hand, $\hat{f}(e_A e_B) = \hat{f}(c(A, B)e_{A\triangle B}) = c(A, B)a_{A\triangle B}$. On the other hand, repeating the same procedure as in section 1.1, we show that the relations (1.9) imply $\hat{f}(e_A)\hat{f}(e_B) = a_A a_B = c(A, B)a_{A\triangle B}$. Finally, from the linearity of $\hat{f}$ we conclude that $\hat{f}(x + y) = \hat{f}(x) + \hat{f}(y)$ and $\hat{f}(xy) = \hat{f}(x)\hat{f}(y)$ for all $x, y \in \mathbb{G}_{p,m,z}$. □

1.3 The wedge and scalar products

The geometric product of two vectors $a, b \in \mathbb{R}^n$ can be written as

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) = a * b + a \wedge b \tag{1.10}$$

where we have defined

$$a * b := \frac{1}{2}(ab + ba) = g(a, b) \in \mathbb{R}, \quad a \wedge b := \frac{1}{2}(ab - ba) = -b \wedge a. \tag{1.11}$$

as the symmetric and antisymmetric parts of the geometric product, respectively.

As a consequence, we have that $a \perp b$ (i.e., $g(a,b) = 0$) if and only if $ab = a \wedge b = -b \wedge a = -ba$. Hence, if $g = 0$ is the completely degenerate or trivial metric, then $ab = a \wedge b$ for all $a, b \in \mathbb{R}^n$, so the Clifford and $\wedge$ products of vectors coincide. Motivated by this fact, we will define a new product between Clifford numbers (which will in fact extend $\wedge$, see Remark 1.17) by taking $g = 0$ or, equivalently, $(p,m,z) = (0,0,n)$ in Definition 1.2, so

$$c(A,B) := \prod_{\substack{\alpha \in A \\ \beta \in B}} s(\alpha,\beta) \prod_{i \in A \cap B} \varepsilon_i = \begin{cases} 0 \text{ if } A \cap B \neq \emptyset, \\ \prod_{\substack{\alpha \in A \\ \beta \in B}} s(\alpha,\beta) \text{ otherwise.} \end{cases}$$

Definition 1.9. *Introducing the notation $(P) = 0$ if proposition P is false and $(P) = 1$ if P is true, we define the wedge or exterior product between basis Clifford numbers as*

$$e_A \wedge e_B := (A \cap B = \emptyset) e_A e_B = (A \cap B = \emptyset) \prod_{\substack{\alpha \in A \\ \beta \in B}} s(\alpha,\beta) e_{A \cup B} \quad \forall A, B \subseteq \Omega_n \tag{1.12}$$

and extend it (univocally) by bilinearity to all of $\mathbb{G}(n)$.

Proposition 1.10. *The wedge product satisfies the following properties:*

(i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *(associativity),*

(ii) $1 \wedge a = a \wedge 1 = a$ *(identity element),*

for all $a, b, c \in \mathbb{G}_{p,m,z}$. Moreover, when it acts on the basis vectors $e_1, \ldots, e_n$ one has that:

(iii) $e_i \wedge e_j = -e_j \wedge e_i$ *for all $i, j \in \Omega_n$. In particular, $e_i \wedge e_i = 0$.*

(iv) $e_A = e_{\alpha_1} \wedge e_{\alpha_2} \wedge \ldots \wedge e_{\alpha_k}$, *where $A = \{1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k \leq n\}$ with $0 \leq k \leq n$.*

Proof. All the properties are obtained particularizing Propositions 1.4 and 1.6 to the $(p,m,z) = (0,0,n)$ case, since $a \wedge b = ab$ for all $a, b \in \mathbb{G}_{0,0,n}$. □

Proposition 1.11. *If $v, w \in \mathbb{R}^n$, then $v \wedge w = -w \wedge v$. Therefore, if $u_1, \ldots, u_k \in \mathbb{R}^n$ the product $u_1 \wedge \ldots \wedge u_k$ is antisymmetric under the exchange of any two adjacent vectors.*

Proof. If we write $v = v^i e_i$, from property (iii) of Proposition 1.10 we deduce that

$$v \wedge v = \sum_{i,j} v^i v^j e_i \wedge e_j = \sum_{i<j} v^i v^j e_i \wedge e_j + \sum_{j<i} v^i v^j e_i \wedge e_j = \sum_{i<j} v^i v^j e_i \wedge e_j - \sum_{j<i} v^j v^i e_j \wedge e_i = 0.$$

From this, it follows that $0 = (v+w) \wedge (v+w) = v \wedge w + w \wedge v$. □

Remark 1.12. *Properties (i) and (ii) of Proposition 1.10 show that $\mathbb{G}(n)$ equipped with the wedge product forms an associative and unital algebra. In fact, we show below that $(\mathbb{G}(n), \wedge)$ is isomorphic (as algebras) to $\Lambda(\mathbb{R}^n)$, the Grassmann or exterior algebra associated to $\mathbb{R}^n$. This justifies the notation for the product defined in Definition 1.9. Moreover, since by construction $(\mathbb{G}(n), \wedge) \cong \mathbb{G}_{0,0,n}$, we also obtain the isomorphism of algebras $\Lambda(\mathbb{R}^n) \cong \mathbb{G}_{0,0,n}$. The structure of the algebra $(\mathbb{G}(n), \wedge)$ is therefore independent of the signature (p,m,z).*

Corollary 1.13. *The algebras $(\mathbb{G}(n), \wedge)$ and $\Lambda(\mathbb{R}^n)$ isomorphic.*

Proof. Since $\dim \Lambda(\mathbb{R}^n) = 2^n = \dim \mathbb{G}(n)$ and we have by Proposition 1.11 that $v \wedge v = 0$ for all $v \in \mathbb{G}^1(n)$. Therefore, by the universal property of $\Lambda(\mathbb{R}^n)$, the injection $\mathbb{R}^n \to \mathbb{G}(n)$, $e_i \mapsto e_{\{i\}} \in \mathbb{G}^1(n)$ extends to an isomorphism of algebras $\Lambda(\mathbb{R}^n) \to (\mathbb{G}_{p,m,z}, \wedge)$. □

Corollary 1.14. *From Proposition 1.6 and property (iv) of Proposition 1.10 we have that*

$$e_{\alpha_1} e_{\alpha_2} ... e_{\alpha_k} = e_A = e_{\alpha_1} \wedge e_{\alpha_2} \wedge ... \wedge e_{\alpha_k},$$

where $0 \leq k \leq n$ and $A = \{1 \leq \alpha_1 < \alpha_2 < ... < \alpha_k \leq n\}$. Moreover, since both sides are totally antisymmetric in their k indices (by (1.5)), we actually have the equality

$$e_{i_1} ... e_{i_k} = e_{i_1} \wedge ... \wedge e_{i_k},$$

where the indices $(i_1, ..., i_k)$ are all distinct but not necessarily in increasing order.

Notation 1.15. *Given a sequence of distinct indices $(i_1, ..., i_k)$ with $1 \leq i_j \leq n$ we will denote the product $e_{i_1} ... e_{i_k} = e_{i_1} \wedge ... \wedge e_{i_k}$ by $e_{i_1 i_2 ... i_k}$ where convenient.*

Proposition 1.16. *Let $a_1, ..., a_k \in \mathbb{R}^n$. Then*

$$a_1 \wedge .... \wedge a_k = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) a_{\sigma(1)} ... a_{\sigma(k)}.$$

Proof. Define $f, h : (\mathbb{R}^n)^k := \mathbb{R}^n \times \overset{\text{k times}}{\cdots} \times \mathbb{R}^n \to \mathbb{G}_{p,m,z}$ by

$$f(a_1, ..., a_k) := a_1 \wedge .... \wedge a_k, \quad h(a_1, ..., a_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) a_{\sigma(1)} ... a_{\sigma(k)}.$$

Since both f and h are multilinear and totally antisymmetric, we have that $\phi \in \{f, h\}$ satisfies

(i) $\phi(a_1^{i_1} e_i, ..., a_k^{i_k} e_k) = a_1^{i_1} ... a_k^{i_k} \phi(e_{i_1}, ..., e_{i_k})$ (using Einstein's summation convention),

(ii) $|\{i_1, .., i_k\}| < k \implies \phi(e_{i_1}, ..., e_{i_k}) = 0,$

(iii) If $(i_1, .., i_k) = \sigma(\alpha_1, ..., \alpha_k)$ where $\sigma \in S_k$, then $\phi(e_{i_1}, ..., e_{i_k}) = \operatorname{sgn}(\sigma) \phi(e_{\alpha_1}, ..., e_{\alpha_k})$.

Therefore, it suffices to show that f and h coincide when acting on $(e_{\alpha_1}, ..., e_{\alpha_k})$ with $1 \leq \alpha_1 < \alpha_2 < ... < \alpha_k \leq n$. This is easy since using Corollary 1.14 we have that

$$e_{\alpha_1} \wedge ... \wedge e_{\alpha_k} = e_{\alpha_1} ... e_{\alpha_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) e_{\alpha_{\sigma(1)}} ... e_{\alpha_{\sigma(k)}}.$$

□

Remark 1.17. *The above proposition shows that the product $\wedge$ defined in Definition 1.9 indeed extends the one defined in equation (1.11), which was denoted in boldface by $\boldsymbol{\wedge}$. Therefore, in the following we will no longer use the symbol $\boldsymbol{\wedge}$ to denote the wedge product of vectors.*

In addition to the wedge product, in (1.11) we also defined the scalar product of two vectors. We can extend it to arbitrary multivectors by noting that $e_A e_B$ is a scalar if $A \triangle B = \emptyset$, i.e., $A = B$.

Definition 1.18. *We define the dot or scalar product between basis Clifford numbers as*

$$e_A \cdot e_B := (A = B)e_A e_B = (A = B)e_A^2 = (A = B)(-1)^{|A|(|A|-1)/2} \prod_{i \in A} q(e_i) \in \mathbb{R} \qquad (1.13)$$

for all $A, B \subseteq \Omega_n$, and extend it (univocally) by bilinearity to all of $\mathbb{G}_{p,m,z}$.

Using the above definition we see that $e_i \cdot e_j = \delta_{ij} e_i^2 = g(e_i, e_j)$, which implies, using linearity, that $a \cdot b = g(a, b)$ for all $a, b \in \mathbb{G}^1_{p,m,z} \equiv \mathbb{R}^{p,m,z}$. Thus, it indeed extends the scalar product in equation (1.11), which was denoted by $*$. Therefore, we will no longer use the symbol $*$ to denote the scalar product of vectors.

Apart from products, other important operations in the algebra $\mathbb{G}_{p,m,z}$ are *involutions*, i.e., maps of the form $j : \mathbb{G}_{p,m,z} \to \mathbb{G}_{p,m,z}$ with $j^2 = j \circ j = Id$.

Definition 1.19. *We define the reversion on basis Clifford numbers as*

$$e_A^\dagger = e^\dagger_{\alpha_1\alpha_2\ldots\alpha_k} := e_{\alpha_k\alpha_{k-1}\ldots\alpha_1} = (-1)^{k(k-1)/2} e_{\alpha_1\alpha_2\ldots\alpha_k} = (-1)^{|A|(|A|-1)/2} e_A,$$

for all $A = \{1 \leq \alpha_1 < \ldots < \alpha_k \leq n\} \subseteq \Omega_n$ and extend it (univocally) by linearity to all of $\mathbb{G}_{p,m,z}$.

In the third equality we have used that

$$(k-1) + (k-2) + \ldots + 2 + 1 = \frac{1}{2}k(k-1), \qquad (1.14)$$

which is the total number of adjacent transpositions needed to go from $(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$ to $(\alpha_1, \alpha_2, \ldots, \alpha_k)$.

Proposition 1.20. *The reversion satisfies the following properties.*

(i) It is an antiautomorphism: $(ab)^\dagger = b^\dagger a^\dagger$ for all $a, b \in \mathbb{G}_{p,m,z}$.

(ii) It is an involution: $(a^\dagger)^\dagger = a$ for all $a \in \mathbb{G}_{p,m,z}$.

(iii) $(x^{-1})^\dagger = (x^\dagger)^{-1}$ for all $x \in \mathbb{G}^\times_{p,m} := \{a \in \mathbb{G}_{p,m,z} : a \text{ is invertible }\}$.

Proof. (i) Given $a, b \in \mathbb{G}_{p,m,z}$ we can write $a = \sum_A a^A e_A$ and $b = \sum_B b^B e_B$, where the sums go over all $A \subseteq \Omega_n$. The product ab becomes

$$ab = \sum_{A,B} a_A b^B e_A e_B = \sum_{A,B} a_A b^B c(A, B) e_{A \triangle B}.$$

If we set $k = |A|$, $l = |B|$ and $m = |A \cap B|$, from elementary set theory we know that $|A \triangle B| = k + l - 2m$. Then, using linearity we can write

$$(ab)^\dagger = \sum_{A,B} a_A b^B c(A, B)(e_{A \triangle B})^\dagger = \sum_{A,B} a_A b^B c(A, B)(-1)^{(k+l-2m)(k+l-2m-1)/2} e_{A \triangle B}.$$

On the other hand,

$$b^\dagger a^\dagger = b^B a^A e_B^\dagger e_A^\dagger = b^B a^A (-1)^{k(k-1)/2 + l(l-1)/2} c(B, A) e_{B \triangle A}.$$

From Definition 1.3 we deduce that $c(B, A) = (-1)^{kl+m} c(A, B)$ and since we also have that

$$k(k-1)/2 + l(l-1)/2 + kl + m \equiv \frac{1}{2}(k + l - 2m)(k + l - 2m - 1) (\text{mod}\, 2)$$

and $B \triangle A = A \triangle B$, we obtain the desired result $(ab)^\dagger = b^\dagger a^\dagger$.

(ii) We know from Equation (1.14) that $k(k-1)$ is even, so $(e_A^\dagger)^\dagger = (-1)^{k(k-1)} e_A = e_A$ for all $A \subseteq \Omega_n$, and the result follows by linearity.

(iii) $xx^{-1} = 1$ implies that $1 = (xx^{-1})^\dagger = (x^{-1})^\dagger x^\dagger$. Thus, $(x^{-1})^\dagger = (x^\dagger)^{-1}$.

$\square$

Corollary 1.21. *$(v_1 \wedge ... \wedge v_k)^\dagger = v_k \wedge ... \wedge v_1$ for all $v \in \mathbb{R}^n$.*

Proof. Using Proposition 1.16 and property (i) above, we have

$$(v_1 \wedge .... \wedge v_k)^\dagger = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)(v_{\sigma(1)}...v_{\sigma(k)})^\dagger = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) v_{\sigma(k)}...v_{\sigma(1)} = v_k \wedge .... \wedge v_1.$$

$\square$

The next proposition allows us to combine the three operations that we have introduced in this section.

Proposition 1.22. *If $u_1, ..., u_k$ and $v_1, ..., v_k$ are vectors then*

$$(u_1 \wedge ... \wedge u_k) \cdot (v_1 \wedge ... \wedge v_k)^\dagger = \begin{vmatrix} u_1 \cdot v_1 & \dots & u_1 \cdot v_k \\ \vdots & \ddots & \vdots \\ u_k \cdot v_1 & \dots & u_k \cdot v_k \end{vmatrix}.$$

Proof. Define $f, g : (\mathbb{R}^n)^k \times (\mathbb{R}^n)^k \to \mathbb{R}$ by

$$f((u_1, ..., u_k), (v_1, ..., v_k)) = (u_1 \wedge ... \wedge u_k) \cdot (v_k \wedge ... \wedge v_1),$$
$$g((u_1, ..., u_k), (v_1, ..., v_k)) = \begin{vmatrix} u_1 \cdot v_1 & \dots & u_1 \cdot v_k \\ \vdots & \ddots & \vdots \\ u_k \cdot v_1 & \dots & u_k \cdot v_k \end{vmatrix}.$$

Write $u_j = \sum_i u_j^i e_i$ and $v_j = \sum_i v_j^i e_i$ for $j = 1, ..., k$. Since both f and g are linear and totally antisymmetric in the u_i's and the v_i's we have that $h \in \{f, g\}$ satisfies:

(i) $h((u_1^{i_1} e_{i_1}, ..., u_k^{i_k} e_{i_k}), (v_1^{j_1} e_{j_1}, ..., v_k^{j_k} e_{j_k})) = u_1^{i_1} v_1^{j_1} ... u_k^{i_k} v_k^{j_k} h((e_{i_1}, ..., e_{i_k}), (e_{j_1}, ..., e_{j_k}))$ (using Einstein's summation convention),

(ii) $|\{i_1, .., i_k\}|$ or $|\{j_1, .., j_k\}| < k \implies h((e_{i_1}, ..., e_{i_k}), (e_{j_1}, ..., e_{j_k})) = 0$.

(iii) If $(i_1, .., i_k) = \sigma(\alpha_1, ..., \alpha_k)$ and $(j_1, .., j_k) = \tau(\beta_1, ..., \beta_k)$, where $\sigma, \tau \in S_k$, then $h((e_{i_1}, ..., e_{i_k}), (e_{j_1}, ..., e_{j_k})) = \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) h((e_{\alpha_1}, ..., e_{\alpha_k}), (e_{\beta_1}, ..., e_{\beta_k}))$.

Therefore it suffices to show that f and g coincide when acting on $((e_{\alpha_1}, ..., e_{\alpha_k}), (e_{\beta_1}, ..., e_{\beta_k}))$ where $1 \leq \gamma_1 < ... < \gamma_k \leq n$ with $\gamma \in \{\alpha, \beta\}$. If $A = \{\alpha_1, ..., \alpha_k\}$ and $B = \{\beta_1, ..., \beta_k\}$, the result follows from

$$(e_{\alpha_1}...e_{\alpha_k}) \cdot (e_{\beta_k}...e_{\beta_1}) = (A = B) e_{\alpha_k}^2 ... e_{\alpha_1}^2 = (A = B) \det(e_{\alpha_p} \cdot e_{\alpha_q})_{1 \leq p,q \leq k} = \det(e_{\alpha_p} \cdot e_{\beta_q})_{1 \leq p,q \leq k}.$$

Indeed, if $A \neq B$, then there exists $\alpha_i \notin B$, so $e_{\alpha_i} \neq e_{\beta_j}$ for all $\beta_j \in B$ and $e_{\alpha_i} \cdot e_{\beta_j} = 0$ for all $1 \leq j \leq k$. Hence, $\det(e_{\alpha_p} \cdot e_{\beta_q})_{1 \leq p,q \leq k} = 0$. $\square$

We note from Proposition 1.22 that, in the Euclidean case, the volume of the parallelepiped formed by the linearly independent (l.i.) vectors $u_1, ..., u_k$ is given by

$$\sqrt{(u_1 \wedge ... \wedge u_k) \cdot (u_1 \wedge ... \wedge u_k)^\dagger},$$

just as the length of the vector v is given by $\sqrt{v \cdot v} = \sqrt{v \cdot v^\dagger}$. Nevertheless, we shall see that the product $u_1 \wedge ... \wedge u_k$ still represents a k-dimensional subspace within $\mathbb{R}^n$ regardless of the signature of the metric g. We follow [20].

Definition 1.23. *Elements of the form $u_1 \wedge u_2 \wedge ... \wedge u_k$ with $u_1, ..., u_k \in \mathbb{R}^n$ are called k-blades or simple k-vectors.*

Proposition 1.24. *If $u_1, ..., u_k \in \mathbb{R}^n$, then $u_1 \wedge ... \wedge u_k \in \mathbb{G}^k_{p,m,z}$. In fact, if $b_1, ..., b_n$ is any basis of $\mathbb{R}^n$ and we define $A := (u^i_j)^{1 \le i \le n}_{1 \le j \le k}$ where $u_j = \sum_i u^i_j b_i$, we have that*

$$u_1 \wedge ... \wedge u_k = \sum_{K \subseteq \Omega_n : |K| = k} \det(A^K) b_K, \tag{1.15}$$

where, if $K = \{1 \le i_1 < ... < i_k \le n\}$, $b_K = b_{i_1} \wedge ... \wedge b_{i_k}$ and A^K denotes the $k \times k$ matrix minor obtained from A by deleting the rows not in K.

Proof. Since by Remark 1.12 we know that the structure of $(\mathbb{G}_{p,m,z}, \wedge)$ is independent of the signature (p, m, z), we can assume that the metric g is non-degenerate, i.e., $z = 0$. In that case, we can always find vectors $a_1, ..., a_n$ such that $b_i \cdot a_j = \delta_{ij}$ (see $\flat$ and $\sharp$ isomoprhisms in [9, p. 104]).

Let us show, that $c^J = \det A^J$ is a solution of the equation

$$u_1 \wedge ... \wedge u_k = \sum_{K \subseteq \Omega_n : |K| = k} c^K b_K \tag{1.16}$$

with $c^K \in \mathbb{R}$. Indeed, call $B := u_1 \wedge ... \wedge u_k$ and assume that equation (1.16) holds. Then taking $J \subseteq \Omega_n$ with $|J| = k$, we would have that

$$(B \cdot a^\dagger_J) = \sum_{K \subseteq \Omega_n : |K| = k} c^K (b_K \cdot a^\dagger_J) \tag{1.17}$$

On the other hand, using Proposition 1.22 we have that $(B \cdot a^\dagger_J) = \det A^J$ and $(b_K \cdot a^\dagger_J) = \delta^J_K$. From Equation (1.17) we conclude that

$$\det A^J = c^J,$$

as we wanted to show. □

Theorem 1.25. *The vectors $u_1, ..., u_k$ are l.i. if and only if $u_1 \wedge ... \wedge u_k \neq 0$.*

Proof. Without loss of generality assume that $c_1 u_1 + ... + c_k u_k = 0$ with $c_1 \neq 0$. Then

$$u_1 \wedge u_2 \wedge ... \wedge u_k = (-c_1^{-1}(c_2 u_2 + ... + c_k u_k)) \wedge u_2 \wedge ... \wedge u_m = 0$$

since $u_i \wedge u_i = 0$.

To prove the converse, assume that $B := u_1 \wedge u_2 \wedge ... \wedge u_k = 0$. Write $u_j = \sum_i u_j^i e_i$ and consider the matrix $A = (u_j^i)_{1\le j\le k}^{1\le i\le n}$. Using Proposition 1.24 we can expand B as a linear combination of the basis k-vectors, e_K,

$$0 = B = \sum_{K\subseteq\Omega_n:|K|=k} \det(A^K)e_K \tag{1.18}$$

If the rank of A is r, we deduce from 1.18 that $r < k$, since all the $k \times k$ minors are zero. If $r = 0$ we automatically get that the u_j are linearly dependent. If $r > 0$, we can assume without loss of generality that $d := \det(u_j^i)_{1\le i,j\le r} \neq 0$. In that case, we can pick $i \in \Omega_n$ and $j \in \Omega_k$ and consider the $(r+1)\times(r+1)$ matrix

$$B_{i,j} := \begin{bmatrix} u_1^1 & \dots & u_r^1 & u_j^1 \\ \vdots & \ddots & \vdots & \vdots \\ u_1^r & \dots & u_r^r & u_j^r \\ u_1^i & \dots & u_r^i & u_j^i \end{bmatrix}.$$

We have that $\det B_{i,j} = 0$ since the rank of A is r. Expanding $\det B_{i,j}$ along the bottom row for fixed i, we obtain

$$u_1^i c_1 + \cdots + u_r^i c_r + u_j^i d = 0, \tag{1.19}$$

where the c_l are independent of the choice of i (but dependent on j). Hence, we have that equation (1.19) holds for all $1 \le i \le n$, and we can write

$$c_1u_1 + \cdots + c_ru_r + du_j = 0,$$

so the vectors $u_1, \dots, u_k$ are l.i., as we wanted to show. □

Corollary 1.26. *At the beginning of the section we saw that if $a, b \in \mathbb{G}^1_{p,m,z}$ are vectors, then $a \perp b$ if and only if $ab = a \wedge b$. As a consequence of the above theorem, we also have that $a||b$ (i.e. $a = \lambda b$ for some $\lambda \in \mathbb{R}$) if and only if $ab = a \cdot b$.*

Corollary 1.27. *If $B = u_1 \wedge u_2 \wedge ... \wedge u_k \neq 0$ is a non-zero k-blade and $v \in \mathbb{R}^n$ then*

$$v \wedge B = 0 \iff v \in \operatorname{span}_{\mathbb{R}}\{u_1, ..., u_k\}.$$

Hence, to every non-zero k-blade $B = u_1 \wedge u_2 \wedge ... \wedge u_k \neq 0$ there corresponds a unique k-dimensional subspace

$$B^\wedge := \{v \in \mathbb{R}^n : v \wedge B = 0\} = \operatorname{span}_{\mathbb{R}}\{u_1, ..., u_k\}.$$

Conversely, if $U \subseteq \mathbb{R}^n$ is a k-dimensional subspace of $\mathbb{R}^n$ we can take a basis $u_1, u_2, ..., u_k$ of U and obtain a non-zero k-blade $B := u_1 \wedge u_2 \wedge ... \wedge u_k$ such that $U = B^\wedge$.

Remark 1.28. *Proposition 1.24 states that every k-blade is a k-vector. However, the converse is not true in general, e.g., if $p + m + z = n \ge 4$ we have that $e_1 \wedge e_2 + e_3 \wedge e_4 \in \mathbb{G}^2_{p,m,z}$ is a 2-vector but not a 2-blade, as we prove below. This is why k-blades are also called simple k-vectors.*

Proof. Assume $e_1 \wedge e_2 + e_3 \wedge e_4 = u_1 \wedge u_2$, with $u_1, u_2 \in \mathbb{R}^n$ where $n \ge 4$. Then we would have that $(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4) = 0$ since

$$(u_1 \wedge u_2) \wedge (u_1 \wedge u_2) = -(u_1 \wedge u_1) \wedge (u_2 \wedge u_2) = 0,$$

but

$$(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4) = 2e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0,$$

since e_1, e_2, e_3, e_4 are l.i. in $\mathbb{R}^n$ ($n \geq 4$). □

To finish the discussion on wedge products, let $B = u_1 \wedge ... \wedge u_k \neq 0$ be a non-zero blade and $b_1, ..., b_k$ any basis of $B^\wedge$, so we can write $u_j = \sum_i u_j^i b_i$ and $A = (u_j^i)_{1 \leq i,j \leq k}$. Extend $b_1, ..., b_k$ to a basis $b_1, ..., b_k, b_{k+1}, ..., b_n$ of $\mathbb{R}^n$. Using Proposition 1.24 we conclude that

$$B = \det(A) b_1 \wedge ... \wedge b_k.$$

Therefore, each non-zero k-blade $B = u_1 \wedge ... \wedge u_k \neq 0$ contains three pieces of information:

(i) the k-dimensional subspace $B^\wedge$,

and, once a basis $b_1, ..., b_k$ of $B^\wedge$ is chosen,

(ii) a magnitude relative to $b_1 \wedge ... \wedge b_k$, given by $|\det A|$,

(iii) an orientation relative to $b_1 \wedge ... \wedge b_k$, given by $\operatorname{sgn}(\det A)$.

Hence, we see that k-blades represent oriented k-dimensional subspaces with magnitude, generalizing the notion of vectors as oriented line segments.

1.4 Two more involutions

In addition to the reversion, we can obtain another involution of the algebra $\mathbb{G}_{p,m,z}$ by sending each basis vector to its opposite, $e_i \mapsto \tilde{e}_i := -e_i$. It extends to $\mathbb{G}_{p,m,z}$ as follows.

Definition 1.29. *The grade involution is defined on basis Clifford numbers by*

$$e_A \mapsto \tilde{e}_A := (-1)^{|A|} e_A \quad \forall A \subseteq \Omega_n$$

and extended by linearity to all of $\mathbb{G}_{p,m,z}$.

Remark 1.30. *Given* $u \in \mathbb{G}_{p,m,z}$*, we will use the notations* $\tilde{u} = (u)\tilde{}$ *for its grade involution.*

Proposition 1.31. *The grade involution satisfies the following properties:*

(i) It is an automorphism: $\widetilde{ab} = \tilde{a}\tilde{b}\ \forall a, b \in \mathbb{G}_{p,m,z}$.

(ii) It is an involution: $\tilde{\tilde{a}} = a\ \forall a \in \mathbb{G}_{p,m,z}$.

(iii) $(x^{-1})\tilde{} = (\tilde{x})^{-1}$ *for all* $x \in \mathbb{G}^\times_{p,m}$. *Thus, we can write* $\tilde{x}^{-1}$ *without ambiguity.*

Proof. (i) Given $a, b \in \mathbb{G}_{p,m,z}$ we can write $a = \sum_A a^A e_A$, $b = \sum_B b^B e_B$. And the product ab becomes

$$ab = \sum_{A,B} a^A b^B e_A e_B = \sum_{A,B} a^A b^B c(A, B) e_{A \triangle B}.$$

Using linearity we can write

$$\widetilde{ab} = \sum_{A,B} a^A b^B c(A, B) \tilde{e}_{A \triangle B} = \sum_{A,B} a^A b^B c(A, B)(-1)^{|A \triangle B|} e_{A \triangle B}. \tag{1.20}$$

Since $|A \triangle B| = |A| + |B| - 2|A \cap B|$, one has that $(-1)^{|A \triangle B|} = (-1)^{|A|+|B|}$. From this, Equation 1.20 becomes

$$\begin{aligned}\sum_{A,B} a^A b^B (-1)^{|A|+|B|} c(A,B) e_{A\triangle B} &= \sum_{A,B} a^A b^B (-1)^{|A|+|B|} e_A e_B \\ &= \sum_A a^A (-1)^{|A|} e_A \sum_B b^B (-1)^{|B|} e_B = \sum_A a^A \tilde{e}_A \sum_B b^B \tilde{e}_B = \tilde{a}\tilde{b}.\end{aligned}$$

(ii) It follows from $(\tilde{e}_A)\tilde{} = (-1)^{|A|}\tilde{e}_A = (-1)^{2|A|} e_A = e_A$ for all $A \subseteq \Omega_n$.

(iii) $xx^{-1} = 1$ implies that $1 = (xx^{-1})\tilde{} = \tilde{x}(x^{-1})\tilde{}$. Thus, $(x^{-1})\tilde{} = (\tilde{x})^{-1}$.

□

The grade involution splits the algebra into two linear subspaces

$$\begin{aligned}\mathbb{G}^+_{p,m,z} &:= \{a \in \mathbb{G}_{p,m,z} : \tilde{a} = a\} = \bigoplus_{\text{k even}} \mathbb{G}^k_{p,m,z}, \\ \mathbb{G}^-_{p,m,z} &:= \{a \in \mathbb{G}_{p,m,z} : \tilde{a} = -a\} = \bigoplus_{\text{k odd}} \mathbb{G}^k_{p,m,z},\end{aligned}$$

so that $\mathbb{G}_{p,m,z} = \mathbb{G}^+_{p,m,z} \oplus \mathbb{G}^-_{p,m,z}$. The elements of $\mathbb{G}^+_{p,m,z}$ and $\mathbb{G}^-_{p,m,z}$ are said to have *even* and *odd grade*, respectively. Moreover, from the binomial theorem we obtain that $\dim \mathbb{G}^+_{p,m,z} = \dim \mathbb{G}^-_{p,m,z} = \frac{1}{2}\dim \mathbb{G}_{p,m,z} = 2^{p+m+z-1}$, since

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = \sum_{\text{k even}} \binom{n}{k} - \sum_{\text{k odd}} \binom{n}{k}.$$

Proposition 1.32. *The subspace $\mathbb{G}^+_{p,m,z}$ is a subalgebra of $\mathbb{G}_{p,m,z}$ called the even subalgebra.* $\dim \mathbb{G}^+_{p,m,z} = 2^{n-1}$.

Proof. Since $(ab)\tilde{} = \tilde{a}\tilde{b}$, we have that

$$\begin{aligned}\mathbb{G}^+_{p,m,z}\mathbb{G}^+_{p,m,z} &\subseteq \mathbb{G}^+_{p,m,z}, \quad \mathbb{G}^-_{p,m,z}\mathbb{G}^-_{p,m,z} \subseteq \mathbb{G}^+_{p,m,z}, \\ \mathbb{G}^+_{p,m,z}\mathbb{G}^-_{p,m,z} &\subseteq \mathbb{G}^-_{p,m,z}, \quad \mathbb{G}^-_{p,m,z}\mathbb{G}^+_{p,m,z} \subseteq \mathbb{G}^-_{p,m,z}.\end{aligned}$$

From the first inclusion and the fact that $1 \equiv e_\emptyset \in \mathbb{G}^+_{p,m,z}$, we conclude that $\mathbb{G}^+_{p,m,z} \subseteq \mathbb{G}_{p,m,z}$ is a subalgebra of $\mathbb{G}_{p,m,z}$. □

Notation 1.33. *When $z = 0$ (resp., $m = z = 0$), we will write $\mathbb{R}^{p,m}$ and $\mathbb{G}_{p,m}$ (resp., $\mathbb{R}^n$ and $\mathbb{G}_n$) instead of $\mathbb{R}^{p,m,0}$ and $\mathbb{G}_{p,m,0}$ (resp., $\mathbb{R}^{n,0,0}$ and $\mathbb{G}_{n,0,0}$). In the $m = z = 0$ case, we may also write $\mathbb{R}^{n,0}$ when we wish to clearly distinguish the quadratic space $\mathbb{R}^{n,0,0}$ from the vector space $\mathbb{R}^n$.*

The following proposition gives a useful isomorphism between the even subalgebra associated to the quadratic space $\mathbb{R}^{m+1,p}$ and the Clifford algebra associated with $\mathbb{R}^{p,m}$.

Proposition 1.34. $\mathbb{G}^+_{m+1,p} \cong \mathbb{G}_{p,m}$.

Proof. Let $(e'_1, \dots, e'_p, f'_1, \dots, f'_m)$ and $(e_1, \dots, e_{m+1}, f_1, \dots, f_p)$ be the canonical o.n. bases of $\mathbb{R}^{p,m}$ and $\mathbb{R}^{m+1,p}$, respectively, so that $e_i^2 = 1 = e_i'^2$ and $f_j^2 = -1 = f_j'^2$. Define $\phi : \mathbb{R}^{p,m} \to \mathbb{G}^+_{m+1,p}$ by setting

$$\phi(e'_i) := f_i e_{m+1}\ , i = 1, \dots, p, \quad \phi(f'_i) := e_i e_{m+1}\ , i = 1, \dots, m, \tag{1.21}$$

and extending by linearity to $\mathbb{R}^{p,m}$. We have that $\phi(e'_i)\phi(e'_j) = f_i e_{m+1} f_j e_{m+1} = -f_i f_j = f_j f_i = f_j e_{m+1}^2 f_i = -f_j e_{m+1} f_i e_{m+1} = -\phi(e'_j)\phi(e'_i)$ for $i \neq j$, and one can verify analogously that

$$\phi(f'_i)\phi(f'_j) = -\phi(f'_j)\phi(f'_i), \quad \phi(e'_i)\phi(f'_j) = -\phi(f'_j)\phi(e'_i), \text{ for } i \neq j,$$
$$\phi(e'_i)^2 = -f_i^2 = 1 = e_i'^2, \quad \phi(f'_j)^2 = -e_j^2 = -1 = f_j'^2.$$

From these relations (proceeding as in (1.6) to obtain (1.7)), it follows that $\phi(v)^2 = q(v)$ $\forall v \in \mathbb{R}^{p,m}$, so Theorem 1.8 tells us that we can extend ϕ uniquely to an algebra homomorphism $\hat{\phi} : \mathcal{G}_{m+1,p} \to \mathcal{G}^+_{p,m}$. Using (1.21) and $\hat{\phi}(e'_i f'_j) = f_i e_{m+1} e_j e_{m+1} = e_j f_i$ for $j < m+1$, we obtain that $\{e_i e_j, e_i f_j : i \neq j\} \subseteq \hat{\phi}(\mathcal{G}_{p,m})$, and since the bivectors $\{e_i e_j, e_i f_j : i \neq j\}$ generate $\mathcal{G}^+_{p,m}$ (as an algebra) we conclude the surjectivity of $\hat{\phi}$. Injectivity follows from the fact that $\dim \mathcal{G}^+_{m+1,p} = 2^{p+m} = \dim \mathcal{G}_{p,m}$. □

Finally, combining the grade involution and the reversion we obtain one more involution of $\mathcal{G}_{p,m,z}$.

Definition 1.35. *Given $a \in \mathcal{G}_{p,m,z}$ we define its Clifford conjugate as $\bar{a} := (\tilde{a})^\dagger$.*

Proposition 1.36. *Clifford conjugation satisfies the following properties:*

(i) $\bar{a} = (a^\dagger)^\sim$ *for all $a \in \mathcal{G}_{p,m}$. Thus we can write $\bar{a} = \tilde{a}^\dagger$ without ambiguity.*

(ii) It is an antiautomorphism: $\overline{(ab)} = \bar{b}\bar{a}$ *for all $a, b \in \mathcal{G}_{p,m}$.*

(iii) It is an involution: $\bar{\bar{a}} = a$ *for all $a \in \mathcal{G}_{p,m}$.*

(iv) $\bar{x}^{-1} = \overline{x^{-1}}$ *for all $x \in \mathcal{G}^\times_{p,m}$.*

Proof. We proceed as with the other involutions.

(i) It follows by linearity from $(\tilde{e}_A)^\dagger = (-1)^{|A|} e_A^\dagger = (-1)^{|A|+|A|(|A|-1)/2} e_A = (e_A^\dagger)^\sim$ for all $A \subseteq \Omega_n$.

(ii) $\overline{(ab)} = (\tilde{a}\tilde{b})^\dagger = (\tilde{b})^\dagger (\tilde{a})^\dagger = \bar{b}\bar{a}$.

(iii) Using (i) we get that $\bar{\bar{a}} = (((\tilde{a})^\dagger)^\dagger)^\sim = a$.

(iv) From $1 = xx^{-1}$ it follows that $1 = \overline{(x^{-1})}\bar{x}$, so $\overline{x^{-1}} = \bar{x}^{-1}$.

□

2 The Spin group

In this chapter, we introduce several subgroups of the non-degenerate Clifford algebra $\mathbb{G}_{p,m}$ which will allow us to perform orthogonal transformations of the quadratic space $\mathbb{R}^{p,m} \equiv \mathbb{G}^1_{p,m}$. The most important of these subgroups is the *(orthochronous) Spin group*, denoted by $\mathrm{Spin}^+_{p,m}$, which in the Euclidean and Lorentzian cases turns out to be the universal double cover of the rotation group SO_n and the restricted Lorentz group $\mathrm{SO}^+_{1,n}$, respectively. We finish the section introducing the *paravector space*, which allows us to use the Euclidean algebra $\mathbb{G}_n$ to handle both the Euclidean space $\mathbb{R}^n$ and the Lorentzian space $\mathbb{R}^{1,n}$. The main references used for this chapter are [18], [20], [11], [14] and [25]. As secondary references, we have also consulted [10] and [17].

We begin by discussing how orthogonal transformations of $\mathbb{R}^{p,m}$ are represented within the Clifford algebra $\mathbb{G}_{p,m}$.

2.1 Cartan-Dieudonné Theorem

The generalization of reflections and rotations to arbitrary non-degenerate quadratic spaces[2] is obtained through the notion of an *isometry* or *orthogonal transformation*, which is a linear map $f : (V, g) \to (V, g)$ such that $g(a, b) = g(f(a), f(b))$ for all $a, b \in V$. The last condition is equivalent to $q(f(a)) = q(a)$ for all $a \in V$. Since g is non-degenerate, one sees that f is automatically injective and, since V is finite dimensional, it is also a linear automorphism. Thus, we make the following definitions

$$\begin{aligned}
\mathrm{End}(V) &:= \{f : V \to V : f \text{ is linear}\},\\
\mathrm{Gl}(V) &:= \{f \in \mathrm{End}(V) : \ f \text{ is bijective}\},\\
\mathrm{O}(V, g) &:= \{f \in \mathrm{End}(V) : q \circ f = q\} = \{f \in \mathrm{Gl}(V) : q \circ f = q\}.
\end{aligned}$$

It is easy to check that they are actually groups. We are interested in the case $(V, g) = \mathbb{R}^{p,m}$, so we will use the notation $\mathrm{GL}_n(\mathbb{R}) := \mathrm{Gl}(\mathbb{R}^n)$ and $\mathrm{O}_{p,m} := \mathrm{O}(\mathbb{R}^{p,m})$.

Let us now show how we can represent orthogonal transformations within the Clifford algebra $\mathbb{G}_{p,m}$. Recall that when W is a non-degenerate subspace of $V := \mathbb{R}^{p,m}$ we can write $V = W \oplus W^\perp$.[3] Thus, for every $v \in \mathbb{R}^{p,m} \equiv \mathbb{G}^1_{p,m}$ with $v^2 \neq 0$ we can set $W = \mathrm{span}_{\mathbb{R}}(v)$ and write every $a \in \mathbb{R}^{p,m}$ uniquely as $a = a_\parallel + a_\perp$ with $a_\parallel \in W$ and $a_\perp \in W^\perp = \{v\}^\perp$. Clifford algebras allow us to find the previous decomposition easily by noting that

$$a = avv^{-1} = (a \cdot v + a \wedge v)v^{-1} = (a \cdot v)v^{-1} + (a \wedge v)v^{-1}. \tag{2.1}$$

It is clear that $(a \cdot v)v^{-1} = \frac{a \cdot v}{v^2} v \in \mathrm{span}_{\mathbb{R}}(v)$. To show that $(a \wedge v)v^{-1} \in \{v\}^\perp$ we need the following proposition.

Proposition 2.1. *The linear map $P_v : \mathbb{G}^1_{p,m} \to \mathbb{G}^1_{p,m}$, $P_v(a) = (a \cdot v)v^{-1}$ satisfies*

(i) $P_v(a) = a$ *for all* $a \parallel v$.

(ii) $P_v(a) = 0$ *for all* $a \perp v$.

[2] See section A in the Appendix for a review of quadratic spaces.
[3] See Definition A.7 and Proposition A.9 in the Appendix.

(iii) $P_v^2 = P_v \circ P_v = P_v$.

Proof. (i) Follows from linearity and the fact that $P_v(v) = (v \cdot v)v^{-1} = v^2 v^{-1} = v$.

(ii) $P_v(a) = (a \cdot v)v^{-1} = 0$.

(iii) Since $q(v) = v^2 \neq 0$, we can write $a = a_\parallel + a_\perp$ as above. Then, using (i) and (ii),

$$P_v(P_v(a)) = P_v(a_\parallel) = a_\parallel = P_v(a).$$

□

Corollary 2.2. *$(a \wedge v)v^{-1} \in \{v\}^\perp$ for all $a \in \mathbb{G}^1_{p,m}$. Thus, $a_\parallel = (a \cdot v)v^{-1}$ and $a_\perp = (a \wedge v)v^{-1}$ for all $a \in \mathbb{G}^1_{p,m}$.*

Proof. It suffices to show that $P_v((a \wedge v)v^{-1}) = 0$. Since $P_v(a) = (a \cdot v)v^{-1}$, using (2.1) we obtain

$$P_v(a) = P_v((a \cdot v)v^{-1} + (a \wedge v)v^{-1}) = P_v(P_v(a)) + P_v((a \wedge v)v^{-1}),$$

and since $P_v^2 = P_v$, we conclude that $P_v((a \wedge v)v^{-1}) = 0$. □

Proposition 2.3. *Given $v \in (\mathbb{R}^{p,m})^\times := \{v \in \mathbb{R}^{p,m} : q(v) \neq 0\}$, then the reflection through the hyperplane $\{v\}^\perp$, namely $\rho_v : \mathbb{R}^{p,m} \to \mathbb{R}^{p,m}$, $a = a_\parallel + a_\perp \mapsto \rho_v(a) := a - 2g(a,v)\frac{v}{q(v)} = -a_\parallel + a_\perp$, can be written as $\rho_v(a) = va\tilde{v}^{-1}$. In addition, we have that $\rho_v \in \mathrm{O}_{p,m}$.*

Proof. Using that $\tilde{v}^{-1} = \tilde{v}/v^2 = -v^{-1}$, we have that

$$\begin{aligned} \rho_v(a) &= -\,a_\parallel + a_\perp = -(a \cdot v)v^{-1} + (a \wedge v)v^{-1} = -(a \cdot v - a \wedge v)v^{-1} \\ &= -\,(v \cdot a + v \wedge a)v^{-1} = -vav^{-1} = \tilde{v}av^{-1} = va\tilde{v}^{-1}. \end{aligned}$$

For the second part, we have that for any $a \in \mathbb{R}^{p,m} \equiv \mathbb{G}^1_{p,m}$

$$q(\rho_v(a)) = \rho_v(a)^2 = (-1)^2 vav^{-1}vav^{-1} = va^2v^{-1} = a^2vv^{-1} = a^2 = q(a).$$

□

Moreover, since $\mathrm{O}_{p,m}$ is a group, we also have that if $v_1, ..., v_k \in (\mathbb{R}^{p,m})^\times$, then $\rho_{v_1} \circ \rho_{v_2} \circ \cdots \circ \rho_{v_k} \in \mathrm{O}_{p,m}$, and

$$\rho_{v_1} \circ \rho_{v_2} \circ \cdots \circ \rho_{v_k}(a) = v_1v_2 \ldots v_k a\tilde{v}_k^{-1} \ldots \tilde{v}_2^{-1}\tilde{v}_1^{-1} = (v_1v_2 \ldots v_k)a(\widetilde{v_1v_2 \ldots v_k})^{-1} \quad \forall a \in \mathbb{R}^{p,m}.$$

Motivated by this result, we define the following group.

Definition 2.4. $\mathrm{P}_{p,m} := \{v_1v_2 \ldots v_k : k \geq 0,\, v_i \in (\mathbb{R}^{p,m})^\times \equiv (\mathbb{G}^1_{p,m})^\times\} \subseteq \mathbb{G}^\times_{p,m}$*, where $S^\times$ denotes the set of invertible elements of $S \subseteq \mathbb{G}_{p,m}$. An element $x \in \mathrm{P}_{p,m}$ is called a versor, so we refer to $\mathrm{P}_{p,m}$ as the group of versors.*

Lemma 2.5. *If $x \in \mathbb{G}_{p,m}$ satisfies $xv = v\tilde{x}$ for all $v \in \mathbb{R}^{p,m}$, then $x \in \mathbb{R}$.*

Proof. We follow the proof in [18, Prop. 2.4.]. Write $x = x_+ + x_-$ where $x_\pm \in \mathbb{G}^\pm_{p,m}$. Since $\mathbb{G}_{p,m} = \mathbb{G}^+_{p,m} \oplus \mathbb{G}^-_{p,m}$ and $\tilde{x}_\pm = \pm x_\pm$, we have that the condition $xv = v\tilde{x}$ for all $v \in \mathbb{R}^{p,m}$ is equivalent to

$$x_\pm v = \pm v x_\pm \iff v x_\pm = \pm x_\pm v \tag{2.2}$$

for all $v \in \mathbb{R}^{p,m}$. Since the vectors $e_{\alpha_1} e_{\alpha_2} ... e_{\alpha_k}$ with $1 \leq \alpha_1 < \alpha_2 < ... < \alpha_k \leq n$ and $0 \leq k \leq n$ form a basis of $\mathbb{G}_{p,m}$, we can write $x_\pm = a^\pm_0 + e_1 a^\pm_1$ where the $a^\pm_i$ do not contain e_1 as a term. Applying the grade involution yields $\tilde{x}_\pm = (a^\pm_0)\tilde{} - e_1 (a^\pm_1)\tilde{} = \pm x_\pm = \pm(a^\pm_0 + e_1 a^\pm_1)$, so $(a^\pm_0)\tilde{} = \pm a^\pm_0$ and $(a^\pm_1)\tilde{} = \mp a^\pm_1$, i.e., $a^\pm_0 \in \mathbb{G}^\pm_{p,m}$ and $a^\pm_1 \in \mathbb{G}^\mp_{p,m}$. Hence, since $e_j e_1 = -e_1 e_j$ for all $j \neq 1$, we have that

$$a^\pm_0 e_1 = \pm e_1 a^\pm_0, \quad a^\pm_1 e_1 = \mp e_1 a^\pm_1.$$

Substituting $v = e_1$ in (2.2) we obtain

$$e_1 a^\pm_0 + e_1^2 a^\pm_1 = \pm(a^\pm_0 e_1 + e_1 a^\pm_1 e_1) = \pm(\pm e_1 a^\pm_0 \mp e_1^2 a^\pm_1) = e_1 a^\pm_o - e_1^2 a^\pm_1,$$

so $a^\pm_1 = 0$. This implies that $x_\pm = a^\pm_o$, which do not contain e_1 as a term, so we can write $x_\pm = b^\pm_0 + e_2 b^\pm_1$ where the $b^\pm_i$ do not contain e_1 nor e_2 as terms. Proceeding as above, one shows by induction that $x_\pm$ do not contain any of the $e_1, \ldots, e_n$ as terms. Hence, $x_\pm \in \mathbb{G}^0_{p,m} \equiv \mathbb{R}$ and $x = x_+ + x_- \in \mathbb{R}$. □

Proposition 2.6. *The map $\rho : \mathrm{P}_{p,m} \to \mathrm{O}_{p,m}$ given by $x \mapsto \rho(x) := \rho_x$, where*

$$\rho_x(a) := x a \tilde{x}^{-1} \quad \forall a \in \mathbb{R}^{p,m},$$

defines a group homomorphism with kernel $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$.

Proof. Given $x, y \in \mathrm{P}_{p,m}$ we have that

$$\rho_{xy}(a) = (xy)a(\widetilde{xy})^{-1} = xya(\tilde{x}\tilde{y})^{-1} = xya\tilde{y}^{-1}\tilde{x}^{-1} = (\rho_x \circ \rho_y)(a)$$

for all $a \in \mathbb{R}^{p,m}$, so ρ is a group homomorphism.

In order to prove that $\ker \rho = \mathbb{R}^\times$ we first show that $\mathbb{R}^\times \subseteq \mathrm{P}_{p,m}$. Since $p + m = n \neq 0$, we can take $v \in \mathbb{G}^1_{p,m}$ such that $|v^2| = 1$, and we have that $tv \in (\mathbb{G}^1_{p,m})^\times$ for all $t \in \mathbb{R}^\times$. Therefore, $t = \pm t v^2 = \pm (tv) v \in \mathrm{P}_{p,m}$, depending on the sign of $v^2 = \pm 1$. Next, we show that $\mathbb{R}^\times \subseteq \ker \rho$. Take $t \in \mathbb{R}^\times$, then

$$\rho_t(v) = tv\tilde{t}^{-1} = tt^{-1}v = v$$

for all $v \in \mathbb{R}^n$. To prove the other inclusion assume that $x \in \ker \rho \subseteq \mathrm{P}_{p,m}$, i.e.,

$$xv\tilde{x}^{-1} = v \iff xv = v\tilde{x}$$

for all $v \in \mathbb{R}^{p,m}$. Lemma 2.5 tells us that $x \in \mathbb{R}$. Moreover, since $x \in \mathrm{P}_{p,m} \subseteq \mathbb{G}^\times_{p,m}$, we conclude that $x \in \mathbb{R}^\times$. □

The following important theorem tells us that $\rho : \mathrm{P}_{p,m} \to \mathrm{O}_{p,m}$ is actually surjective.

Theorem 2.7 (Cartan-Dieudonné). *Let g be a non-degenerate metric on a n-dimensional vector space V. Then every element $f \in \mathrm{O}(V, g)$ can be written as a product of k reflections*

$$f = \rho_{v_1} \circ \cdots \circ \rho_{v_k},$$

where $k \leq n$ and $v_i \in V$ such that $q(v_i) = g(v_i, v_i) \neq 0$.

A complete proof can be found in [1, p. 129]. Here, we restrict ourselves to proving a weaker version of the theorem (which is sufficient for our purposes) where the bound $k \leq n$ is relaxed to $k \leq 2n$. The proof takes advantage of the geometric algebra $\mathbb{G}_{p,m}$ and is based on the following lemma (we follow [20, pp. 66-67]).

Lemma 2.8. *If $x, y \in (\mathbb{R}^{p,m})^\times$ and $x^2 = y^2$ then we have that either*

(i) $x - y \in (\mathbb{R}^{p,m})^\times$*, in which case* $\rho_{x-y}(x) = y$*, or*

(ii) $x + y \in (\mathbb{R}^{p,m})^\times$*, in which case* $\rho_{x+y}(x) = -y$*.*

Proof. Reasoning by contradiction, assume that

$$(x-y)^2 = x^2 - xy - yx + y^2 = 0, \quad (x+y)^2 = x^2 + xy + yx + y^2 = 0,$$

which implies (summing both equations) that $x^2 + y^2 = 0$. This is possible only if $x^2 = y^2 = 0$, since by hypothesis $x^2 = y^2$. However, $x, y \in (\mathbb{R}^{p,m})^\times$, so $x^2 \neq 0 \neq y^2$.

For the "*in which case*" part of the lemma assume that $x - y \in (\mathbb{R}^{p,m})^\times$, then

$$(x-y)x(\widetilde{x-y})^{-1} = -(x^2 - yx)(x-y)^{-1} = -(y^2 - yx)(x-y)^{-1} = -y(y-x)(x-y)^{-1} = y,$$

where we have used that $x^2 = y^2$ by hypothesis. The result for $x + y$ is obtained by replacing $y \leftrightarrow -y$ in the previous calculation. □

Theorem 2.9 (Cartan-Dieudonné, weaker version)**.** *Every isometry $f \in \mathrm{O}_{p,m}$, with $n = p + m$, can be written as a product of k reflections*

$$f = \rho_{v_1} \circ \cdots \circ \rho_{v_k},$$

where $v_i \in (\mathbb{R}^{p,m})^\times$ and $k \leq 2n$. Moreover, if $p = 0$ or $m = 0$, then we recover the optimal bound $k \leq n$.

Proof. Let $\{o_1, \ldots, o_n\} =: \{o_{0,1}, \ldots, o_{0,n}\}$ be any orthonormal basis of $\mathbb{R}^{p,m}$ and $f \in \mathrm{O}_{p,m}$. If we set $f_i := f(o_i)$ for all $1 \leq i \leq n$, we have that $\{f_1, \ldots, f_n\}$ is another orthonormal basis, since $f_i^2 = f(o_i)^2 = o_i^2 = \pm 1$ and $f_i \cdot f_j = g(f(o_i), f(o_j)) = g(o_i, o_j) = 0$ for all $1 \leq i \neq j \leq n$. In particular, $o_1^2 = f_1^2 \neq 0$, so applying Lemma 2.8 we have three possibilities:

(i) $o_1 = f_1$ (which implies that $o_1 + f_1 \in (\mathbb{R}^{p,m})^\times$).

(ii) $o_1 - f_1 \in (\mathbb{R}^{p,m})^\times$. Then, taking $v_1 = o_1 - f_1$ we have that $\rho_{v_1}(o_1) = f_1$.

(iii) $o_1 \neq f_1$, but $o_1 + f_1 \in (\mathbb{R}^{p,m})^\times$. Then, taking $v_1 = o_1 + f_1$ and $v_2 = f_1$ we have that

$$\rho_{v_2} \circ \rho_{v_1}(o_1) = \rho_{v_2}(-f_1) = -f_1(-f_1)f_1^{-1} = f_1,$$

Since $\rho_{v_2} \circ \rho_{v_1} = \rho_{v_2 v_1}$, the above result is equivalent to saying that there exists a versor $\theta_1 \in \{1, o_1 - f_1, f_1(o_1 + f_1)\}$ such that $\rho_{\theta_1}(o_1) = f_1$. Furthermore, since $\rho_{\theta_1} \in \mathrm{O}_{p,m}$, if we call $o_{1,i} := \rho_{\theta_1}(o_i)$ for all $1 \leq i \leq n$, we obtain that the set

$$\rho_{\theta_1}(\{o_1, o_2, \ldots, o_n\}) = \{o_{1,1}, o_{1,2}, \ldots, o_{1,n}\}$$

is again orthonormal with $f_i^2 = o_i^2 = o_{1,i}^2 \neq 0$ for all i and $o_{1,1} = f_1$. In particular, $f_2^2 = o_2^2 = o_{1,2}^2 \neq 0$.

Proceeding by induction, take $k \in \{1, \dots, n-1\}$ and assume that there exists an orthonormal set $\{o_{k,1}, \dots, o_{k,n}\}$ with $f_i^2 = o_{k,i}^2 \neq 0$ for all i and $o_{k,j} = f_j$ for all $j \leq k$. In particular, $f_{k+1}^2 = o_{k,k+1}^2 \neq 0$ so, by Lemma 2.8, there exists a versor $\theta_{k+1} \in \{1, o_{k,k+1} - f_{k+1}, f_{k+1}(o_{k,k+1} + f_{k+1})\}$ such that $\rho_{\theta_{k+1}}(o_{k,k+1}) = f_{k+1}$ and the set $\rho_{\theta_{k+1}}(\{o_{k,i}\}) := \{o_{k+1,i}\}$ is orthonormal with $f_i^2 = o_{k,i}^2 = o_{k+1,i}^2 \neq 0$. Moreover, since $f_j \cdot f_{k+1} = 0$ and $f_j \cdot o_{k,k+1} = o_{k,j} \cdot o_{k,k+1} = 0$ for all $j \leq k$, we also have that

$$o_{k+1,j} = \rho_{\theta_{k+1}}(o_{k,j}) = \rho_{\theta_{k+1}}(f_j) = \theta_{k+1} f_j \tilde{\theta}_{k+1}^{-1} = f_j \theta_{k+1} \theta_{k+1}^{-1} = f_j$$

for all $j \leq k$, so actually $o_{k+1,j} = f_j$ for all $j \leq k+1$.

Thus, $o_{n,j} = f_j = f(o_j)$ for all $j \leq n$, where $o_{n,j} = \rho_{\theta_n}(o_{n-1,j}) = \rho_{\theta_n} \circ \cdots \circ \rho_{\theta_2} \circ \rho_{\theta_1}(o_j)$ for all $j \leq n$, so $f = \rho_{\theta_n} \circ \cdots \circ \rho_{\theta_2} \circ \rho_{\theta_1}$. In addition, since each θ_i is of the form $\theta_i = \prod_{j=1}^{l_i} v_{i,j}$ with $v_{i,j} \in (\mathbb{R}^{p,m})^\times$ and $0 \leq l_i \leq 2$, we obtain that

$$f = \rho_{\theta_n} \circ \cdots \circ \rho_{\theta_1} = \rho_{v_1} \circ \cdots \circ \rho_{v_k},$$

where $v_i \in (\mathbb{R}^{p,m})^\times$ and $k \leq 2n$.

On the other hand, if $p = 0$ or $m = 0$, i.e., if q is definite, then $q(v) = v^2 = 0$ if and only if $v = 0$. Therefore, we have that either $o_i = f_i$ or $(o_i - f_i)^2 \neq 0$ and it suffices to consider only the first two posibilities (i) and (ii) above. Thus, in this case we can take $\theta_i \in \{1, o_i - f_i\}$, so that $0 \leq l_i \leq 1$ and $k \leq n$. □

2.2 The Pin and Spin groups

We know that every $x \in \mathrm{P}_{p,m}$ verifies that $x \in \mathbb{G}_{p,m}^\times$, that is, x is invertible, and $\rho_x(v) = xv\tilde{x}^{-1} \in \mathbb{R}^{p,m}$ for all $v \in \mathbb{R}^{p,m}$, i.e. $x\mathbb{R}^{p,m}\tilde{x}^{-1} \subseteq \mathbb{R}^{p,m}$.

Definition 2.10. *The Clifford-Lipschitz group, or C-L group for short, is defined as*

$$\tilde{P}_{p,m} := \{x \in \mathbb{G}_{p,m}^\times : x\mathbb{R}^{p,m}\tilde{x}^{-1} \subseteq \mathbb{R}^{p,m}\}.$$

That is, the C-L group is the biggest group within $\mathbb{G}_{p,m}$ that leaves $\mathbb{R}^{p,m}$ invariant under the action of ρ.

Thus, we have the inclusion $\mathrm{P}_{p,m} \subseteq \tilde{P}_{p,m}$ and, in fact, we are going to show that the other inclusion also holds.

Definition 2.11. *We introduce the norm mapping $N : \mathbb{G}_{p,m} \to \mathbb{G}_{p,m}$ defined by*

$$N(x) := \bar{x}x$$

for all $x \in \mathbb{G}_{p,m}$.

Proposition 2.12. *The restriction of N to $\tilde{P}_{p,m} \subseteq \mathbb{G}_{p,m}$ yields a group homomorphism $N : \tilde{P}_{p,m} \to \mathbb{R}^\times$.*

Proof. Let $x \in \tilde{P}_{p,m}$. Then $xv\tilde{x}^{-1} \in \mathbb{R}^{p,m}$ for all $v \in \mathbb{R}^{p,m}$, so

$$xv\tilde{x}^{-1} = (xv\tilde{x}^{-1})^\dagger = \bar{x}^{-1}vx^\dagger \iff \bar{x}xv = vx^\dagger\tilde{x} = v\widetilde{\bar{x}x}$$

for all $v \in \mathbb{R}^{p,m}$. By Lemma 2.5 we have that $N(x) = \bar{x}x \in \mathbb{R}$. Moreover, since $x \in \mathbb{G}^{\times}_{p,m}$, it follows that $N(x) \in \mathbb{R}^{\times}$ with $N(x)^{-1} = x^{-1}\bar{x}^{-1} = x^{-1}\overline{(x^{-1})}$.

We now observe that if $x, y \in \tilde{P}_{p,m}$, then $N(xy) = \overline{xy}xy = \bar{y}\bar{x}xy = N(x)\bar{y}y = N(x)N(y)$, so N is indeed a group homomorphism. $\square$

Corollary 2.13. *The map $\rho : \tilde{P}_{p,m} \to \mathrm{O}_{p,m}$, $x \mapsto \rho_x$, where $\rho_x(v) := xv\tilde{x}^{-1}$ for all $v \in \mathbb{R}^{p,m}$, is a well-defined group homomorphism with kernel $\mathbb{R}^{\times}$.*

Proof. Take $x \in \tilde{P}_{p,m}$ and $v \in \mathbb{R}^{p,m}$, we have that

$$N(\rho_x(v)) = N(xv\tilde{x}^{-1}) = N(x)N(v)N(\tilde{x})^{-1} = N(x)N(x)^{-1}N(v) = N(v),$$

where we have used that $N(\tilde{x}) = x^{\dagger}\tilde{x} = \widetilde{\bar{x}x} = \widetilde{N(x)} = N(x)$, since $N(x) \in \mathbb{R}^{\times}$. Therefore, $\rho_x(v)^2 = -N(\rho_x(v)) = -N(v) = v^2$ and $\rho_x \in \mathrm{O}_{p,m}$, and the map is well-defined. That it is a group homomorphism with $\ker \rho = \mathbb{R}^{\times}$ is proven as in Prop. 2.6, since $\mathrm{P}_{p,m} \subseteq \tilde{\mathrm{P}}_{p,m}$. $\square$

Theorem 2.14. *The group of versors (Def. 2.4) and the C-L group (Def. 2.10) are the same:*

$$\mathrm{P}_{p,m} = \tilde{P}_{p,m}.$$

Proof. It suffices to show that $\tilde{P}_{p,m} \subseteq \mathrm{P}_{p,m}$. Take $x \in \tilde{P}_{p,m}$. By the above corollary we have that $\rho_x \in \mathrm{O}_{p,m}$. On the other hand, by Theorem 2.9 we know that $\rho : \mathrm{P}_{p,m} \to \mathrm{O}_{p,m}$ is surjective, so there exists $y \in \mathrm{P}_{p,m}$ such that $\rho_x = \rho_y$. Hence, $\rho_{xy^{-1}} = \mathbb{1}$, so $xy^{-1} = \lambda \in \mathbb{R}^{\times}$ and $x = \lambda y \in \mathrm{P}_{p,m}$. $\square$

The above theorem tells us that $\mathrm{P}_{p,m}$ is actually the biggest possible group within $\mathbb{G}_{p,m}$ that leaves $\mathbb{R}^{p,m}$ invariant under the action of ρ. However, it is more convenient to work with smaller subgroups of $\mathrm{P}_{p,m}$.

Definition 2.15. $\mathrm{Pin}_{p,m} := \{v_1 v_2 \dots v_k \in \mathrm{P}_{p,m} : k \geq 0,\ |v_i| = \sqrt{|q(v_i)|} = 1\}$.

Proposition 2.16. *The restriction $\rho : \mathrm{Pin}_{p,m} \to \mathrm{O}_{p,m}$ defines a surjective group homomorphism.*

Proof. We know from Proposition 2.6 that $\rho : \mathrm{P}_{p,m} \to \mathrm{O}_{p,m}$ is a group homomorphism, so its restriction to the subgroup $\mathrm{Pin}_{p,m} \leq \mathrm{P}_{p,m}$ is also one. On the other hand, for every $v \in (\mathbb{R}^{p,m})^{\times}$ we have that

$$\rho_{tv}(a) = (tv)a(\widetilde{tv})^{-1} = tt^{-1}va\tilde{v}^{-1} = va\tilde{v}^{-1} = \rho_v(a)$$

for all $t \in \mathbb{R} \setminus \{0\}$ and $a \in \mathbb{R}^n$. Therefore, for every $v \in (\mathbb{R}^{p,m})^{\times}$ we have that $\hat{v} := \frac{v}{\sqrt{|q(v)|}}$ is a unit vector, i.e., $\hat{v}^2 = q(\hat{v}) = \pm 1$, and $\rho_{\hat{v}} = \rho_v$. Surjectivity now follows from Theorem 2.9. $\square$

Just as the even subalgebra $\mathbb{G}^{+}_{p,m}$ plays an important role within $\mathbb{G}_{p,m}$, the subgroup of $\mathrm{P}_{p,m}$ formed by even versors will also be relevant in our discussion of orthogonal transformations.

Definition 2.17. $\mathrm{SP}_{p,m} := \{v_1 \dots v_k \in \mathrm{P}_{p,m} :\ k \text{ is even}\} = \mathrm{P}_{p,m} \cap \mathbb{G}^{+}_{p,m}$.

The second equality follows from the fact that $x \in \mathbb{G}^{+}_{p,m}$ if and only if $\tilde{x} = x$, and $\tilde{x} = \tilde{v}_1 \dots \tilde{v}_j = (-1)^j v_1 \dots v_j = x$ if and only if j is even.

Proposition 2.18. $\mathrm{SO}_{p,m} := \{f \in \mathrm{O}_{p,m} : \det f = 1\} = \rho(\mathrm{SP}_{p,m})$.

Proof. By the Theorem 2.14, we have that every $f \in \mathrm{SO}_{p,m}$ is of the form ρ_x with $x \in \mathrm{P}_{p,m}$. On the other hand, for any $v \in (\mathbb{R}^{p,m})^\times$ we know that ρ_v is the reflection through the hyperplane $\{v\}^\perp$, so $\det(\rho_v) = -1$. Thus, taking $x = v_1 \dots v_j \in \mathrm{P}_{p,m}$ we have that $\det(\rho_x) = \det(\rho_{v_1} \circ \dots \circ \rho_{v_j}) = \det(\rho_{v_1}) \dots \det(\rho_{v_j}) = (-1)^j$, so $\rho_x \in \mathrm{SO}_{p,m}$ if and only if $x \in \mathrm{SP}_{p,m}$. □

Since $\mathrm{SO}_{p,m}$ is the (generalized or indefinite) rotation group, the above proposition tells us that any rotation can be written as the product of an even number of reflections. The corresponding even subgroup of $\mathrm{Pin}_{p,m}$ is

Definition 2.19. $\mathrm{Spin}_{p,m} := \{v_1 \dots v_{2k} \in \mathrm{Pin}_{p,m} : k \in \mathbb{N}\} = \mathrm{Pin}_{p,m} \cap \mathbb{G}^+_{p,m}$.

Proposition 2.20. $\mathrm{SO}_{p,m} = \rho(\mathrm{Spin}_{p,m})$.[4]

Proof. The proof is basically the same as in Prop. 2.18. By Proposition 2.16 we have that every $f \in \mathrm{SO}_{p,m}$ is of the form ρ_x with $x \in \mathrm{Pin}_{p,m}$. On the other hand, for any $v \in (\mathbb{R}^{p,m})^\times$ we know that ρ_v is the reflection through the hyperplane $\{v\}^\perp$, so $\det(\rho_v) = -1$. Thus, taking $x = v_1 \dots v_j \in \mathrm{Pin}_{p,m}$ we have that $\det(\rho_x) = \det(\rho_{v_1} \circ \dots \circ \rho_{v_j}) = \det(\rho_{v_1}) \dots \det(\rho_{v_j}) = (-1)^j$, so $\rho_x \in \mathrm{SO}_{p,m}$ if and only if $x \in \mathrm{Spin}_{p,m}$. □

Definition 2.21. *An isometry $f \in \mathrm{O}_{p,m}$ is called proper if $\det f = 1$, i.e. if $f \in \mathrm{SO}_{p,m}$, and improper otherwise. The group $\mathrm{SO}_{p,m}$ of proper isometries is also called the special orthogonal group.*

Next, we prove that Pin coincides with the group of unit versors

$$\widetilde{\mathrm{Pin}}_{p,m} := \{x \in \mathrm{P}_{p,m} : N(x) = \pm 1\}$$

.

Lemma 2.22. *$\rho : \widetilde{\mathrm{Pin}}_{p,m} \to \mathrm{O}_{p,m}$ is a group homomorphism with kernel $\{\pm 1\}$.*

Proof. Since $\rho : \widetilde{\mathrm{Pin}}_{p,m} \to \mathrm{O}_{p,m}$ is the restriction of $\rho : \widetilde{\mathrm{P}}_{p,m} \to \mathrm{O}_{p,m}$ to $\widetilde{\mathrm{Pin}}_{p,m} \subseteq \widetilde{\mathrm{P}}_{p,m}$, it is a group homomorpism with $\ker \rho|_{\widetilde{\mathrm{Pin}}_{p,m}} = \ker \rho \cap \widetilde{\mathrm{Pin}}_{p,m} = \mathbb{R}^\times \cap \widetilde{\mathrm{Pin}}_{p,m} = \{\pm 1\}$. □

Proposition 2.23. $\mathrm{Pin}_{p,m} = \widetilde{\mathrm{Pin}}_{p,m}$.

Proof. For very $x = v_1 \dots v_k \in \mathrm{Pin}_{p,m}$ we have that $N(x) = N(v_1) \dots N(v_k) = (-1)^k v_1^2 \dots v_k^2 = \pm 1$, so $\mathrm{Pin}_{p,m} \subseteq \widetilde{\mathrm{Pin}}_{p,m}$. Take $x \in \widetilde{\mathrm{Pin}}_{p,m}$. By the above lemma we have that $\rho_x \in \mathrm{O}_{p,m}$. On the other hand we know that $\rho : \mathrm{Pin}_{p,m} \to \mathrm{O}_{p,m}$ is surjective, so there is a $y \in \mathrm{Pin}_{p,m}$ such that $\rho_x = \rho_y$. Hence $\rho_{xy^{-1}} = \mathbb{1}$, which implies that $xy^{-1} = \lambda \in \{\pm 1\}$ and $x = \lambda y \in \mathrm{Pin}_{p,m}$. □

Corollary 2.24. $\mathrm{Spin}_{p,m} = \{x \in \mathrm{SP}_{p,m} : N(x) = \pm 1\}$.

Motivated by the above results, we make the following definitions.

[4] This statement is an example of Serre's joke: Pin(n) is to O(n) as Spin(n) is to SO(n), stated in the classic reference [2, p 3., line 17].

Definition 2.25. *We define the orthochronous Pin and Spin groups as*

$$\mathrm{Pin}^{+}_{p,m} := \{x \in \mathrm{Pin}_{p,m} : N(x) = 1\}, \quad \mathrm{Spin}^{+}_{p,m} := \{x \in \mathrm{Spin}_{p,m} : N(x) = 1\}.$$

Their images will be denoted by $\mathrm{O}^{+}_{p,m} := \rho(\mathrm{Pin}^{+}_{p,m}) \subseteq \mathrm{O}_{p,m}$ *and* $\mathrm{SO}^{+}_{p,m} := \rho(\mathrm{Spin}^{+}_{p,m}) \subseteq \mathrm{SO}_{p,m}$. *Elements of* $\mathrm{Spin}^{+}_{p,m}$ *are called rotors and* $\mathrm{Spin}^{+}_{p,m}$ *is also called the rotor group.*

The name *orthochronous* comes from terminology in the Lorentzian case, where a vector v such that $v^2 > 0$ is said to be *timelike*. The condition $N(x) = 1$ implies that $x \in \mathrm{Pin}_{p,m}$ contains an even number of vector factors v_i such that $N(v_i) = -v_i^2 = -1$, which implies that $\rho_x \in \mathrm{O}^{+}_{p,m}$ involves an even number of timelike reflections. Therefore, ρ_x preserves the orientation of the subspace spanned by the timelike basis vectors $\mathrm{span}_{\mathbb{R}}(e_1, \ldots, e_p) \subseteq \mathbb{R}^{p,m}$ and is said to be *orthochronous*. Hence, the group $\mathrm{SO}^{+}_{p,m} = \mathrm{SO}_{p,m} \cap \mathrm{O}^{+}_{p,m}$ consists of proper and orthochronous isometries.

Theorem 2.26. *The map* $\rho : (\mathrm{S})\mathrm{Pin}^{(+)}_{p,m} \to (\mathrm{S})\mathrm{O}^{(+)}_{p,m}$, $x \mapsto \rho(x) := \rho_x$, *where* $\rho_x(a) = xa\tilde{x}^{-1}$ *for all* $a \in \mathbb{R}^{p,m}$, *defines a surjective group homomorphism with kernel* $\{+1, -1\}$.

Proof. We already know that $\rho : (\mathrm{S})\mathrm{Pin}^{(+)}_{p,m} \to (\mathrm{S})\mathrm{O}^{(+)}_{p,m}$ is a surjective map, and since it is the restriction of $\rho : \mathrm{P}_{p,m} \to \mathrm{O}_{p,m}$ to the subgroup $(\mathrm{S})\mathrm{Pin}^{(+)}_{p,m}$, we get, from Proposition 2.6, that it is a group homomorphism with $\ker \rho|_{(\mathrm{S})\mathrm{Pin}^{(+)}_{p,m}} = \mathbb{R}^{\times} \cap (\mathrm{S})\mathrm{Pin}^{(+)}_{p,m} \supseteq \{+1, -1\}$. Therefore, if $x = v_1 \ldots v_k \in \ker \rho|_{(\mathrm{S})\mathrm{Pin}^{(+)}_{p,m}}$, then

$$x^2 = xx^{\dagger} = (v_1 \ldots v_k)(v_k \ldots v_1) = v_1^2 \ldots v_k^2 = \pm 1,$$

where we have used that $x^{\dagger} = x$ for all $x \in \mathbb{R}$. Hence, $x \in \{+1, -1\}$ and $\ker \rho|_{(\mathrm{S})\mathrm{Pin}^{(+)}_{p,m}} = \{+1, -1\} \cong \mathbb{Z}_2$. □

Corollary 2.27. *We have the following group isomorphisms:*

$$\begin{aligned} \mathrm{Pin}_{p,m}/\mathbb{Z}_2 &\cong \mathrm{O}_{p,m}, \quad & \mathrm{Spin}_{p,m}/\mathbb{Z}_2 &\cong \mathrm{SO}_{p,m}, \\ \mathrm{Pin}^{+}_{p,m}/\mathbb{Z}_2 &\cong \mathrm{O}^{+}_{p,m}, \quad & \mathrm{Spin}^{+}_{p,m}/\mathbb{Z}_2 &\cong \mathrm{SO}^{+}_{p,m}. \end{aligned}$$

Proof. The isomorphisms follow from the above theorem and the first isomorphism theorem. □

The following proposition tells us that the group(s) $(\mathrm{S})\mathrm{Pin}^{(+)}_{p,m}$ can be obtained from the group $\mathrm{Spin}^{+}_{p,m}$, so it is enough to study the structure of the latter to deduce the properties of the former.

Proposition 2.28. *If* $m = 0$ *(but* $n \geq 1$*) we have that*

$$\mathrm{Spin}_n := \mathrm{Spin}_{n,0} = \mathrm{Spin}^{+}_{n,0}, \quad \mathrm{Pin}_n := \mathrm{Pin}_{n,0} = \mathrm{Spin}_n \bigsqcup e\,\mathrm{Spin}_n,$$

where e *is any vector satisfying* $e^2 = 1$. *In addition, if* $p, m \geq 1$, *then*

$$\begin{aligned} \mathrm{Spin}_{p,m} &= \mathrm{Spin}^{+}_{p,m} \bigsqcup e_+e_-\mathrm{Spin}^{+}_{p,m}, \\ \mathrm{Pin}^{+}_{p,m} &= \mathrm{Spin}^{+}_{p,m} \bigsqcup e_-\mathrm{Spin}^{+}_{p,m}, \\ \mathrm{Pin}_{p,m} &= \mathrm{Spin}^{+}_{p,m} \bigsqcup e_+e_-\mathrm{Spin}^{+}_{p,m} \bigsqcup e_-\mathrm{Spin}^{+}_{p,m} \bigsqcup e_+\mathrm{Spin}^{+}_{p,m}, \end{aligned}$$

where e_+ *and* e_- *are any vectors satisfying* $e_{\pm}^2 = \pm 1$.

Proof. If $m = 0$, then $\bar{x}x = v_1^2 \dots v_{2k}^2 = (1)^{2k} = 1$ for all $x = v_1 \dots v_{2k} \in \mathrm{Spin}_n$ and we obtain the first equality. Now, let $x = v_1 \dots v_j \in \mathrm{Pin}_n$ with j odd. Since $n \neq 0$, we can take $e \in \mathbb{G}_n^1 \equiv \mathbb{R}^n$ such that $e^2 = 1$. Then, x can be written as $x = ey$ with $y = ex \in \mathrm{Spin}_n$.

For the $p \neq 0 \neq m$ case, we can take $e_+, e_- \in \mathbb{R}^{p,m}$ such that $e_\pm^2 = \pm 1$. If $x = v_1 \dots v_j \in \mathrm{Pin}_{p,m}$ we have four possibilities:

(i) j is even and $\bar{x}x = 1$, i.e., $x \in \mathrm{Spin}_{p,m}^+$.

(ii) j is even and $\bar{x}x = -1$. Calling $e_{+-} := e_+ e_-$, we have that $\overline{e_{+-}} e_{+-} = e_+^2 e_-^2 = -1$ and $e_{+-}^{-1} = e_{+-}$. Thus, $y = e_{+-}x \in \mathrm{Spin}_{p,m}^+$, since $\bar{y}y = \bar{x}\bar{e}_{+-}e_{+-}x = -\bar{x}x = 1$, and $x = e_{+-}y$.

(iii) j is odd and $\bar{x}x = 1$. Then, $y = -e_- x \in \mathrm{Spin}_{p,m}^+$, since $\bar{y}y = \bar{x}\bar{e}_- e_- x = \bar{x}x = 1$, and $x = e_- y$.

(iv) j is odd and $\bar{x}x = -1$. Then, $y = e_+ x \in \mathrm{Spin}_{p,m}^+$, since $\bar{y}y = \bar{x}\bar{e}_+ e_+ x = -\bar{x}x = 1$, and $x = e_+ y$.

□

Corollary 2.29. *For $n \geq 1$, we have that*

$$\mathrm{SO}_n = \mathrm{SO}_n^+ , \quad \mathrm{O}_n = \mathrm{SO}_n \bigsqcup \rho_e \mathrm{SO}_n ,$$

where ρ_e is any reflection with $e^2 = 1$. Moreover, if $p, m \geq 1$, then

$$\begin{aligned} \mathrm{SO}_{p,m} &= \mathrm{SO}_{p,m}^+ \bigsqcup \rho_{e_+}\rho_{e_-} \mathrm{SO}_{p,m}^+ , \\ \mathrm{O}_{p,m}^+ &= \mathrm{SO}_{p,m}^+ \bigsqcup \rho_{e_-} \mathrm{SO}_{p,m}^+ , \\ \mathrm{O}_{p,m} &= \mathrm{SO}_{p,m}^+ \bigsqcup \rho_{e_+}\rho_{e_-} \mathrm{SO}_{p,m}^+ \bigsqcup \rho_{e_-} \mathrm{SO}_{p,m}^+ \bigsqcup \rho_{e_+} \mathrm{SO}_{p,m}^+ , \end{aligned}$$

where ρ_{e_+} and ρ_{e_-} are timelike and spacelike reflections ($e_\pm^2 = \pm 1$), respectively.

Proof. It suffices to apply the map ρ to the sets in the above proposition. □

We finish this section with a useful characterization of the Spin group in low dimensions.

Theorem 2.30. *If $n = p + m \leq 4$, then*

$$\mathrm{Spin}_{p,m}^+ = \{R \in \mathbb{G}_{p,m}^+ : N(R) = \bar{R}R = 1\} = \{R \in \mathbb{G}_{p,m} : \bar{R}R = 1 = R^\dagger R\}.$$

Proof. Call $A = \{R \in \mathbb{G}_{p,m}^+ : N(R) = \bar{R}R = 1\}$. Since $\mathrm{Spin}_{p,m}^+ = \{x \in \Gamma_{p,m} \cap \mathbb{G}_{p,m}^+ : N(x) = 1\}$ we have that $\mathrm{Spin}_{p,m}^+ \subseteq A$. For the other inclusion, take $R \in A$ and $v \in \mathbb{R}^{p,m}$. It suffices to show that $\tilde{R}vR^{-1} = RvR^\dagger \in \mathbb{R}^{p,m}$, where we have used that $R^{-1} = \bar{R}$ and $R \in \mathbb{G}_{p,m}^+$, so $\tilde{R} = R$ and $\bar{R} = R^\dagger$. Note that for $n \leq 4$, vectors are the only elements in $\mathbb{G}_{p,m}$ satisfying $\tilde{x} = -x$ and $x^\dagger = x$. Let us check that $RvR^\dagger$ satisfies these relations. Indeed, $(RvR^\dagger)^\sim = \tilde{R}\tilde{v}\tilde{R}^\dagger = -RvR^\dagger$ and $(RvR^\dagger)^\dagger = (R^\dagger)^\dagger v^\dagger R^\dagger = RvR^\dagger$, so $\tilde{R}vR^{-1} = RvR^\dagger \in \mathbb{R}^{p,m}$ and $R \in \mathrm{Spin}_{p,m}^+$.

The second equality follows from the fact that $R \in \mathbb{G}_{p,m}^+$ if and only if $\tilde{R} = R$, or, equivalently, $R^\dagger = \bar{R}$. □

2.3 Topological and Lie group properties

We have seen above that the surjective homomorphism $\rho : (\mathrm{S})\mathrm{Pin}_{p,m}^{(+)} \to (\mathrm{S})\mathrm{O}_{p,m}^{(+)}$ lets us represent orthogonal transformations within the geometric algebra $\mathbb{G}_{p,m}$. In this section we will see that ρ has also nice topological properties, namely, it turns out to be a (double) covering map[5] which is also a universal covering when restricted to the Euclidean and Lorentzian Spin groups Spin_n and $\mathrm{Spin}_{1,n}^{+}$.

First of all, let us equip $\mathbb{G}_{p,m}$ with a natural topology. Recall that $\mathbb{G}_{p,m}$ is a real vector space of dimension $N = 2^{p+m}$, so we can rename its basis vectors as $\{b_1, b_2, \ldots, b_N\} := \{e_A : A \subseteq \Omega_n\}$. Therefore, we have a vector space isomorphism $\Phi : \mathbb{G}_{p,m} \to \mathbb{R}^N$ via $x = \sum_i x^i b_i \mapsto \Phi(x) := (x^1, x^2, \ldots, x^N)$. If $\|\cdot\|_{\mathbb{R}^N}$ is a norm in $\mathbb{R}^N$, we define a norm $\|\cdot\|$ in $\mathbb{G}_{p,m}$ by $\|x\| := \|\Phi(x)\|_{\mathbb{R}^N}$. With this norm, $\mathbb{G}_{p,m}$ becomes a topological space homeomorphic to $\mathbb{R}^N$.

The homeomorphism Φ allows us, in fact, to view $\mathbb{G}_{p,m}$ as an (analytic) N-dimensional manifold, since $(\mathbb{G}_{p,m}, \Phi)$ is a global chart. With respect to this analytic manifold structure, multiplication in $\mathbb{G}_{p,m}$ is also analytic. Indeed, $(a, x) \mapsto ax$ is bilinear and, therefore, analytic. The following proposition shows that the analytic structure in $\mathbb{G}_{p,m}$ is inherited by the group of invertible elements $\mathbb{G}_{p,m}^{\times} \subseteq \mathbb{G}_{p,m}$, becoming a *Lie group*.

Definition 2.31. *A Lie group G is a group which is endowed with a smooth manifold structure*[6] *such that the group operations*

$$G \times G \to G, (g, h) \mapsto gh \quad \text{and} \quad G \to G, g \mapsto g^{-1}$$

are also smooth.

Proposition 2.32. *$\mathbb{G}_{p,m}^{\times}$ is a Lie group.*

Proof. We want to define an analytic structure on $\mathbb{G}_{p,m}^{\times}$ and show that the group operations of multiplication and inversion are also analytic with respect to this structure.

First, consider the map $l : \mathbb{G}_{p,m} \to \mathrm{End}(\mathbb{G}_{p,m})$, $a \mapsto l(a) := l_a$, where $l_a(x) := ax$ for all $x \in \mathbb{G}_{p,m}$, which is continuous since it is linear. Moreover, it is injective since $l_a(x) = 0$ for all $x \in \mathbb{G}_{p,m}$ implies, in particular that $l_a(1) = a = 0$. Therefore, l^{-1} exists and is linear (in particular, continuous), which implies that $l : \mathbb{G}_{p,m} \to l(\mathbb{G}_{p,m})$ is an homeomorphism.

Fixing the above basis $\{b_1, b_2, \ldots, b_N\}$ of $\mathbb{G}_{p,m}$, we can identify $\mathbb{G}_{p,m} \equiv \mathbb{R}^N$ and $\mathrm{End}(\mathbb{G}_{p,m}) \equiv M_N(\mathbb{R})$, endowing them with a manifold structure (the identifications serve as global charts).[7] In addition, we see that $a \in \mathbb{G}_{p,m}^{\times} \iff l_a \in \mathrm{GL}_N(\mathbb{R})$, and $(l_a)^{-1} = l_{a^{-1}}$. The sufficient condition is clear. For the necessary condition, take $b := l_a^{-1}(1)$. Then $ab = l_a(l_a^{-1}(1)) = 1$ and $b = a^{-1}$.

Therefore, $l(\mathbb{G}_{p,m}^{\times}) = l(\mathbb{G}_{p,m}) \cap \mathrm{GL}_N(\mathbb{R})$ and, since $\det : M_N(\mathbb{R}) \to \mathbb{R}$ is continuous, we have that $\mathrm{GL}_N(\mathbb{R}) = \det^{-1}(\mathbb{R}^{\times})$ is open in $M_N(\mathbb{R})$, so $l(\mathbb{G}_{p,m}^{\times})$ is open in $l(\mathbb{G}_{p,m})$ and $\mathbb{G}_{p,m}^{\times} = l^{-1}(l(\mathbb{G}_{p,m}^{\times}))$ is open in $\mathbb{G}_{p,m}$. Since above we have seen that $\mathbb{G}_{p,m}$ is an analytic manifold with analytic multiplication, we conclude that $\mathbb{G}_{p,m}^{\times}$ inherits an analytic

[5]See section B in the Appendix for a review of covering spaces.

[6]It turns out that even if the Lie group is endowed with just a C^1 differentiable structure, this structure contains a unique analytic structure (see discussion in [29, p. 110]).

[7]If we had chosen a different basis, the differentiable structure would be the same, since the change of charts is linear.

manifold structure with respect to which, the restriction $\mathbb{G}^{\times}_{p,m} \times \mathbb{G}^{\times}_{p,m} \mapsto \mathbb{G}^{\times}_{p,m}$, $(a, x) \mapsto ax$ is analytic.

Finally, since inversion in $\mathrm{GL}_N(\mathbb{R})$ is analytic ($M^{-1} = \det(M)^{-1}\mathrm{adj}(M)$ is a rational function of the coordinates) and l^{-1} is linear, we also have that $a \mapsto a^{-1} = l^{-1}(l(a^{-1})) = l^{-1}(l(a)^{-1})$ is analytic. Therefore, $\mathbb{G}^{\times}_{p,m}$ is an (analytic) Lie group. □

Proposition 2.33. *The exponential map* $\exp : \mathbb{G}_{p,m} \to \mathbb{G}_{p,m}$,

$$A \mapsto e^A := \exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

is a well-defined analytic map.

Proof. The injection $l : \mathbb{G}_{p,m} \to \mathrm{End}(\mathbb{G}_{p,m})$ from the above proposition allows us to induce an operator norm, $\|\cdot\|_o$ in $\mathbb{G}_{p,m}$, defined as

$$||A||_o := \sup\{\frac{||Ax||}{||x||} : x \in \mathbb{G}_{p,m} \setminus \{0\}\}.$$

Since $\mathbb{G}_{p,m}$ is finite-dimensional, all norms in $\mathbb{G}_{p,m}$ are equivalent and the associated topology is the same for all of them. Therefore, since $\|A^n\|_o \leq \|A\|_o^n$, we have that exp converges absolutely for all A and, applying Weierstrass M-test, uniformly on compact sets. Thus, it is well defined and analytic on $\mathbb{G}_{p,m}$, since each function $A \mapsto \frac{A^k}{k!}$ is analytic. □

Proposition 2.34. *Let $A, B \in \mathbb{G}_{p,m}$. If $[A, B] := AB - BA = 0$, then $e^{A+B} = e^A e^B$.*

Proof. Following [14], since both e^A and e^B converge absolutely, we can write

$$e^A e^B = \sum_{k=0}^{\infty} \frac{A^k}{k!} \sum_{l=0}^{\infty} \frac{B^l}{l!} = \sum_{k=0,l=0}^{\infty} \frac{A^k B^l}{k!l!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k+l=n} \frac{n!}{k!l!} A^k B^l = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}.$$

If $AB = BA$, the binomial theorem

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$$

applies and we obtain the desired result. □

Corollary 2.35. *For each $A \in \mathbb{G}_{p,m}$, $e^A \in \mathbb{G}^{\times}_{p,m}$ with $(e^A)^{-1} = e^{-A}$.*

Proposition 2.36. *Given $A \in \mathbb{G}_{p,m}$, the map $\mathbb{R} \to \mathbb{G}^{\times}_{p,m}$, $t \mapsto e^{tA}$ is analytic.*

Proof. Again, $\|(tA)^n\|_o \leq |t|^n \|A\|_o^n = |t|^n$, so applying Weierstrass M-test once more, we conclude the analyticity of the map above, since each term in the series is a polynomial function of t. □

Lemma 2.37. *Given $B \in \mathbb{G}^2_{p,m}$, the map $\mathrm{ad}_B : \mathbb{R}^{p,m} \equiv \mathbb{G}^1_{p,m} \to \mathbb{G}_{p,m}$ defined by*

$$\mathrm{ad}_B(v) := [B, v] := Bv - vB$$

for all $v \in \mathbb{R}^{p,m}$, verifies that $\mathrm{ad}_B(\mathbb{R}^{p,m}) \subseteq \mathbb{R}^{p,m}$.

Proof. Expand $B = \sum_{i<j} B^{ij} e_i e_j$ and $v = \sum_k v^k e_k$. Since ad_B is linear, we have that $[B, v] = \sum_{i<j} \sum_k B^{ij} v^k [e_i e_j, e_k]$. Therefore, it suffices to consider the following two cases:

(i) $B = e_i e_j$, $v = e_k$ with i, j, k all different. Then $e_i e_j e_k - e_k e_i e_j = e_i e_j e_k - e_i e_j e_k = 0$.

(ii) $B = e_i e_j$, $v = e_i$. Then $e_i e_j e_i - e_i e_i e_j = -2e_i^2 e_j = -2q(e_i)e_j \in \mathbb{R}^{p,m}$. The case $v = e_j$ follows from this one, since $e_i e_j = -e_j e_i$ for all $i \neq j$.

□

Proposition 2.38. *For every bivector $B \in \mathbb{G}^2_{p,m}$, we have that $\pm e^{tB} \in \mathrm{Spin}^+_{p,m}$ for all $t \in \mathbb{R}$.*

Proof. Since the involutions are continuous (they are linear), we have that $(e^B)\tilde{} = e^{\tilde{B}} = e^B$ and $(e^B)^\dagger = e^{B^\dagger} = e^{-B}$. It follows that $e^{tB} \in \mathbb{G}^+_{p,m}$ and $N(e^{tB}) = \overline{e^{tB}} e^{tB} = (e^{tB})^\dagger e^{tB} = 1$ for all $t \in \mathbb{R}$. Therefore, it suffices to show that $e^{tB} v e^{-tB} \in \mathbb{R}^{p,m}$ for all $t \in \mathbb{R}$. In order to do that, we define the function $f : \mathbb{R} \to \mathbb{G}_{p,m}$, $f(t) := e^{tB} v e^{-tB}$ for all $t \in \mathbb{R}$. Since the map $t \mapsto e^{tB}$ is analytic and also multiplication in $\mathbb{G}_{p,m}$ and inversion in $\mathbb{G}_{p,m}$, we have that f is analytic. If we differentiate with respect to t we obtain that

$$f'(t) = Be^{tB} v e^{-tB} - e^{tB} v B e^{-tB} = e^{tB}[B, v]e^{-tB} = e^{tB} \mathrm{ad}_B(v) e^{-tB},$$

where we have used that $xe^x = e^x x$ for all $x \in \mathbb{G}_{p,m}$. From here, one easily shows by induction that $f^{(n)}(t) = e^{tB} \mathrm{ad}^n_B(v) e^{-tB}$, where $\mathrm{ad}^n_B = \mathrm{ad}_B \circ \cdots \circ \mathrm{ad}_B$ (n times). From the above lemma, we deduce that $f^{(n)}(0) = \mathrm{ad}^n(v) \in \mathbb{R}^n$ and the series

$$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f^n(0)$$

converges in $\mathbb{R}^{p,m}$ for all $t \in \mathbb{R}$, since $\mathbb{R}^{p,m}$ is a linear subspace of $\mathbb{G}_{p,m}$ and, thus, closed with respect to $\| \cdot \|$. □

Lemma 2.39. *If $a, b \in \mathbb{G}^1_{p,m}$ and $R := ab = a \cdot b + a \wedge b$ verifies $RR^\dagger = 1$, then $R = \epsilon e^{\lambda a \wedge b}$ for some $\epsilon \in \{+1, -1\}$ and $\lambda \in \mathbb{R}$.*

Proof. $R = ab = a \cdot b + a \wedge b$ and $R^\dagger = a \cdot b - a \wedge b$, so $RR^\dagger = (a \cdot b)^2 - (a \wedge b)^2$ and the condition $RR^\dagger = 1$ reads $(a \cdot b)^2 = 1 + (a \wedge b)^2$. We can distinguish 3 cases:

(i) $(a \wedge b)^2 > 0$. In that case $(a \cdot b)^2 > 1$, so either $a \cdot b > 1$ or $a \cdot b < 1$. Hence, there exists a unique $\phi > 0$ such that $a \cdot b = \pm \cosh \phi$. Then $(a \wedge b)^2 = (a \cdot b)^2 - 1 = \cosh^2 \phi - 1 = \sinh^2 \phi$. Since $\phi > 0$ we have that $\sinh \phi > 0$, so we can write $a \wedge b = \sinh \phi \mathbf{j}$ with $\mathbf{j} := a \wedge b / \sinh \phi$ and $\mathbf{j}^2 = 1$. Therefore, $R = a \cdot b + a \wedge b = \pm \cosh \phi + \sinh \phi \mathbf{j} = \pm e^{\pm \phi \mathbf{j}} = \pm e^{\pm \phi a \wedge b / \sinh \phi}$.

(ii) $(a \wedge b)^2 = 0$. In that case $(a \cdot b)^2 = 1$ so $(a \cdot b) = \pm 1$ and $R = \pm 1 + a \wedge b = \pm(1 \pm a \wedge b) = \pm e^{\pm a \wedge b}$.

(iii) $(a \wedge b)^2 < 0$. In that case $(a \cdot b)^2 < 1$, so $-1 < a \cdot b < 1$ and there exists a unique $\theta \in]0, \pi[$ such that $a \cdot b = \cos \theta$. Then $(a \wedge b)^2 = (a \cdot b)^2 - 1 = \cos^2 \theta - 1 = \sin^2 \theta$. Since $\theta \in]0, \pi[$ we have that $\sin \theta > 0$, so we can write $a \wedge b = \sin \theta \mathbf{i}$ with $\mathbf{i} := a \wedge b / \sin \theta$ and $\mathbf{i}^2 = 1$. Therefore, $R = a \cdot b + a \wedge b = \cos \theta + \sin \theta \mathbf{i} = e^{\theta \mathbf{i}} = e^{\theta a \wedge b / \sin \theta}$.

□

We are now ready to prove the main results of this section.

Theorem 2.40. $\mathrm{Spin}^+_{p,m}$ *is path-connected to* 1 *(hence connected) when* $p \geq 2$ *or* $m \geq 2$.

Proof. We follow [20, Thm. 6.15].Take $x = v_1 \dots v_{2k} \in \mathrm{Spin}^+_{p,m}$. Since $x\bar{x} = v_1^2 \dots v_{2k}^2 = 1$ we have that there is an even number of v_i such that $v_i^2 = 1$ and an even number of v_j such that $v_j^2 = -1$. In addition, if $a, b \in (\mathbb{R}^{p,m})^\times$ we can write $ab = aba^{-1}a = b'a$, where $b' = -\rho_a(b)$ satisfies $b'^2 = q(b)^2 = b^2$. Therefore, we can rearrange the factors in x so that

$$x = v_1 \dots v_{2k} = a_1^+ b_1^+ \dots a_s^+ b_s^+ a_1^- b_1^- \dots a_t^- b_t^- = R_1^+ \dots R_s^+ R_1^- \dots R_t^-$$

where $R_i^\pm = a_i^\pm b_i^\pm$ and $(a_i^\pm)^2 = (b_i^\pm)^2 = \pm 1$. Since $R_i^\pm (R_i^\pm)^\dagger = (a_i^\pm)^2 (b_i^\pm)^2 = (\pm 1)^2 = 1$, using Lemma 2.39 we can write $R_i^\pm = \epsilon^\pm e^{\lambda_i^\pm a^\pm \wedge b^\pm}$ with $\epsilon^\pm \in \{+1, -1\}$ and $\lambda_i^\pm \in \mathbb{R}$. From Proposition 2.38 we know that $\epsilon^\pm e^{ta^\pm \wedge b^\pm} \in \mathrm{Spin}^+_{p,m}$ for all $t \in \mathbb{R}$, so each $R_i^\pm$ is path-connected to either $+1$ or -1 within $\mathrm{Spin}^+_{p,m}$. In addition, since $p \geq 2$ or $m \geq 2$ we can find two orthogonal vectors u_1, u_2 such that $u_1^2 = u_2^2 = \pm 1$, so that $(u_1 \wedge u_2)^2 = (u_1 u_2)^2 = -1$ and $e^{tu_1 u_2} = \cos t + u_1 u_2 \sin t \in \mathrm{Spin}^+_{p,m}$ for all $t \in \mathbb{R}$, so $+1$ and -1 are also path-connected within $\mathrm{Spin}^+_{p,m}$. Therefore, $x = R_1^+ \dots R_s^+ R_1^- \dots R_t^-$ is path-connected to 1 within $\mathrm{Spin}^+_{p,m}$. □

Theorem 2.41. *The map* $\rho : (\mathrm{S})\mathrm{Pin}^{(+)}_{p,m} \to (\mathrm{S})\mathrm{O}^{(+)}_{p,m} \cong (\mathrm{S})\mathrm{Pin}^{(+)}_{p,m}/\mathbb{Z}_2$ *is a double covering,*[8] *which is non-trivial if* $p \geq 2$ *or* $m \geq 2$.

Proof. To prove this result we will use Proposition B.7 and Lemma B.8. Call $X = (\mathrm{S})\mathrm{Pin}^{(+)}_{p,m}$ and consider the natural action $\mathbb{Z}_2 \times X \to X$, $1 \cdot x = x$ and $-1 \cdot x = -x$ for all $x \in X$. Since $\mathbb{Z}_2 \cong \{+1, -1\}$ is finite and $X \subseteq \mathcal{G}_{p,m} \cong \mathbb{R}^{2^{p+m}}$ is Hausdorff, it suffices to check that the homeomorphism $l_{-1} = -\mathbb{1}_X$ has no fixed points. Assume that there exists $x \in X$ such that $l_{-1}(x) = -x = x$, then $x = 0$, but $0 \notin X$ since $0 \notin \mathcal{G}^\times_{p,m}$. Therefore, $\rho : X \to X/\mathbb{Z}_2$ is a covering map. In addition, $p^{-1}([x]) = [x] = \{x, -x\}$ for all $x \in X$, so ρ is indeed a double covering.

Assume now that $p \geq 2$ or $m \geq 2$ and that the covering is trivial, i.e., $X = \bigsqcup_\alpha V_\alpha$ where the restriction $p : V_\alpha \to X/\mathbb{Z}_2$ is an homeomorphism for each α. In that case, we have that $1 \in V_1$ and $-1 \in V_{-1}$ with $V_1 \cap V_{-1} = \emptyset$, since otherwise $p(1) = p(-1) = [1] \in X/\mathbb{Z}_2$ and one of the restrictions $p|_{V_1}$ or $p|_{V_{-1}}$ would not be bijective. Hence, 1 and -1 belong to different connected components. However, both $1, -1 \in \mathrm{Spin}^+_{p,m} \subseteq (\mathrm{S})\mathrm{Pin}^{(+)}_{p,m}$ and, by the above theorem, we know that $\mathrm{Spin}^+_{p,m}$ is path-connected. We have, therefore, arrived at a contradition, so the covering must be non-trivial. □

Corollary 2.42. $\mathrm{SO}^+_{p,m}$ *is (path-)connected when* $p \geq 2$ *or* $m \geq 2$. *In particular,* $\mathrm{SO}_n = \mathrm{SO}^+_{n,0}$ *is (path-)connected for* $n \geq 2$.

Proof. We know from Theorem 2.40 that $\mathrm{Spin}^+_{p,m}$ is path-connected when $p \geq 2$ or $m \geq 2$ and from the above theorem that $\rho : \mathrm{Spin}^+_{p,m} \to \mathrm{SO}^+_{p,m}$ is continuous (and surjective). Hence, $\mathrm{SO}^+_{p,m}$ is also path-connected. □

[8] See section B in the Appendix for a review of covering spaces.

Recall that a topological space X is said to be *simply connected* if it is path-connected and its fundamental group (based on any point), denoted by $\pi_1(X)$, is the trivial (one-element) group.

Corollary 2.43. *When $p \geq 2$ or $m \geq 2$, $\mathrm{SO}^+_{p,m}$ cannot be simply connected and $\mathrm{Spin}^+_{p,m}$ is simply connected if and only if $\pi_1(\mathrm{SO}^+_{p,m}) \cong \mathbb{Z}_2$.*

Proof. Since $\rho : \mathrm{Spin}^+_{p,m} \to \mathrm{SO}^+_{p,m}$ is a double covering, we have that $|\rho^{-1}(x_0)| = 2$ for all $x_0 \in \mathrm{SO}^+_{p,m}$ and we also know that $\mathrm{Spin}^+_{p,m}$ is path-connected when $p \geq 2$ or $m \geq 2$. The result follows from part (ii) in Theorem B.13. □

Let us focus now on the cases of interest in physics, namely, the Euclidean and Lorentzian cases. We will later show (Theorem 3.8 and Corollary 3.10) that Spin_3 and $\mathrm{Spin}^+_{1,3}$ are simply connected, so they are the universal covers of SO_3 and $\mathrm{SO}^+_{1,3}$, respectively, and $\pi_1(\mathrm{SO}_3) \cong \mathbb{Z}_2 \cong \pi_1(\mathrm{SO}^+_{1,3})$. In fact, we have that $\pi_1(\mathrm{SO}_n) \cong \mathbb{Z}_2 \cong \pi_1(\mathrm{SO}^+_{1,n})$ for all $n \geq 3$, so Spin_n and $\mathrm{Spin}^+_{1,n}$ are, respectively, the universal covers of SO_n and $\mathrm{SO}^+_{1,n}$ for all $n \geq 3$. However, proving the general result would require us to develop notions from the theory of Lie groups which go beyond the scope of this work. Therefore, we give here a proof that relies on three results (Lemma 2.52 and Theorems 2.44 and 2.49) that will not be proven.

Theorem 2.44. *Let G be a Lie group, and let H be a closed subgroup of G. Then H has a unique manifold structure which makes H into a Lie subgroup of G. In particular, H is a Lie group.*

Proof. See [29, Thm. 3.42, p. 110]. □

Proposition 2.45. $\mathrm{SO}^+_{1,n} = \{\Lambda \in \mathrm{SO}_{1,n} : \Lambda^0_0 > 0\} = \{\Lambda \in \mathrm{SO}_{1,n} : \Lambda^0_0 \geq 1\}$.

Proof. If $\Lambda \in \mathrm{O}_{1,n}$, then

$$1 = q(e_0) = q(\Lambda e_0) = (\Lambda^0_0)^2 - \sum_i (\Lambda^i_0)^2 \implies |\Lambda^0_0| \geq 1,$$

where $e_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{1,n}$. Therefore, $\Lambda^0_0 > 0$ if and only if $\Lambda^0_0 \geq 1$ and

$$\mathrm{SO}_{1,n} = U^+ \sqcup U^- \quad \text{with} \quad U^+ = \mathrm{SO}_{1,n} \cap \{\Lambda^0_0 > 0\}, \quad U^- = \mathrm{SO}_{1,n} \cap \{\Lambda^0_0 < 0\}, \tag{2.3}$$

which are open and non-empty sets ($\mathbb{1} \in \mathrm{U}^+$ and $\rho_{e_0}\rho_{e_1} \in U^-$). Moreover, from Corollaries 2.29 and 2.42 we have that

$$\mathrm{SO}_{1,n} = V^+ \sqcup V^- \quad \text{with} \quad V^+ = \mathrm{SO}^+_{1,n}, \quad V^- = \rho_{e_0}\rho_{e_1}\mathrm{SO}^+_{1,n}, \tag{2.4}$$

which are connected sets. From (2.3), we can write $V^+ = (V^+ \cap U^+) \sqcup (V^+ \cap U^-)$. Then, from the connectivity of V^+ and the fact that $\mathbb{1} \in V^+ \cap U^+ \neq \emptyset$, we conclude that $V^+ \cap U^- = \emptyset$. Therefore, $V^+ \subseteq U^+$.

For the other inclusion, using (2.4) and the previous inclusion we can write

$$V^- = \mathrm{SO}_{1,n} \setminus V^+ = (U^+ \setminus V^+) \sqcup U^- = (U^+ \cap V^-) \sqcup U^-.$$

From the connectivity of V^- and the fact that $U^- \neq \emptyset$, we conclude that $U^+ \setminus V^+ = U^+ \cap V^- = \emptyset$. Therefore, $V^+ = U^+$, as we wanted to show. □

Corollary 2.46. *SO_n and $\mathrm{SO}^+_{1,n-1}$ are Lie groups.*

Proof. We have that $\mathrm{SO}_n = f^{-1}(\{0\} \times \{1\})$ and $\mathrm{SO}^+_{1,n-1} = h^{-1}(\{0\} \times \{1\} \times [1,\infty))$, where

$$f : \mathrm{GL}_n(\mathbb{R}) \mapsto \mathbb{R}^{n\times n} \times \mathbb{R}, \quad R \mapsto (R^t R - \mathbb{1}, \det R)$$
$$h : \mathrm{GL}_n(\mathbb{R}) \mapsto \mathbb{R}^{n\times n} \times \mathbb{R} \times \mathbb{R}, \quad \Lambda \mapsto (\Lambda^t \eta \Lambda - \eta, \det \Lambda, \Lambda^0_0)$$

are continuous functions. Since $\mathrm{GL}_n(\mathbb{R})$ is a Lie group (in fact, it is the prototypical example of Lie group), the result follows from Theorem 2.44. □

Definition 2.47. *Let M be a (smooth) manifold and G a Lie group acting (smoothly)*[9] *on M. We say that the action is transitive if for any $x_1, x_2 \in M$, there exists $g \in G$ such that $g \cdot x_1 = x_2$. The isotropy group at x is the subgroup $G_x := \{g \in G : g \cdot x = x\}$.*

Remark 2.48. *Since the action $G \times M \to M$ is smooth, the map $l_x : G \to M$, $g \mapsto gx$ is also smooth. Hence, $G_x = l_x^{-1}(x)$ is a closed subgroup of G.*

Theorem 2.49. *If $G \times M \to M$ is a transitive action, then G/G_x is diffeomorphic to M for all $x \in M$.*

Proof. See [29, Thm. 3.62, p. 123]. □

Definition 2.50. *The (upper) hyperbolic space in $\mathbb{R}^{1,n}$ is defined as*

$$H^n = \{x = (x^0, x^1, \ldots, x^n) \in \mathbb{R}^{1,n} : q(x) = 1,\ x^0 \geq 1\}.$$

Theorem 2.51. *We have the following diffeomorphisms*

$$\mathrm{SO}_{n+1}/\widetilde{\mathrm{SO}}_n \cong S^n, \quad \mathrm{SO}^+_{1,n}/\widetilde{\mathrm{SO}}_n \cong H^n,$$

where $\widetilde{\mathrm{SO}}_n = \left\{\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right) : A \in \mathrm{SO}_n\right\} \cong \mathrm{SO}_n$ *and S^n is the n-sphere.*

Proof. Calling, $G_0(n) = \mathrm{SO}_{n+1}$, $M_0(n) = S^n$ and $G_1(n) = \mathrm{SO}^+_{1,n}$, $M_1(n) = H^n$, it suffices to show, using Theorem 2.49, that the map $\alpha_\epsilon : G_\epsilon(n) \times M_\epsilon(n) \to M_\epsilon(n)$, $(A, p) \mapsto Ap$ defines a smooth and transitive action whose isotropy group at $e_0 = (1, 0, \ldots, 0)^t$ is $\widetilde{SO}_n$.

That α_ϵ is well-defined follows from the fact that

$$q_\epsilon(\alpha_\epsilon(x)) = q_\epsilon(x) = 1 \quad \forall x \in M_\epsilon,$$

where $q_{0/1}$ is the Euclidean/Lorentzian quadratic form. Thus, $\alpha_\epsilon(x) \in M_\epsilon$ for all $x \in M_\epsilon$. On the other hand, the fact that matrix multiplication is associative and smooth implies that it indeed defines an smooth action.

To prove transitivity, it suffices to show that for any $x_{\epsilon,0} \in M_\epsilon(n)$, there exists $L_\epsilon \in G_\epsilon(n)$ such that $L_\epsilon e_0 = x_{\epsilon,0}$, where $e_0 = (1, 0, \ldots, 0) \in M_\epsilon(n)$. Since $q_\epsilon(x_{\epsilon,0}) = 1$, we can extend to an o.n. basis $\{x_{\epsilon,0}, x_{\epsilon,1}, \ldots, x_{\epsilon,n}\}$ of $V_\epsilon(n)$, where $V_0 = \mathbb{R}^n$ and $V_1 = \mathbb{R}^{1,n}$, which implies that

$$q_\epsilon(x_{\epsilon,i}) = q_\epsilon(e_i) \tag{2.5}$$

[9] See B.5 for the definition of (continuous) group action. We say that the action is *smooth* if the topological space X is a smooth manifold and the homeomorphism l_g is also smooth for all $g \in G$.

for all $i = 0, 1, \dots, n$. Next, let $L_\epsilon : V_\epsilon \to V_\epsilon$ be the unique linear map such that $L_\epsilon(e_i) := x_{\epsilon,i}$ for all $i = 0, 1, \dots, n$. Equation (2.5) implies that $L_0 \in G_0(n) = \mathrm{SO}_n$ and $L_1 \in \mathrm{SO}_{1,n}$, since, except changing $x_{\epsilon,n} \leftrightarrow -x_{\epsilon,n}$, we may assume that $\det L_\epsilon = 1$. Finally, since $x_{1,0} \in H^n$ we have that

$$0 < 1 \leq (x_{1,0})^0 = (L_1)^0_\mu (e_0)^\mu = (L_1)^0_0.$$

Hence, $L_1 \in \mathrm{SO}^+_{1,n} = G_1(n)$ and we are done.

Let us now calculate the isotropy group at e_0. Suppose that $L \in G_\epsilon$ satisfies $Le_0 = e_0$, then

$$L = \left(\begin{array}{c|c} 1 & B \\ \hline 0 & A \end{array}\right).$$

The condition $L^t g_\epsilon L = g_\epsilon$ implies

$$\begin{pmatrix} 1 & 0 \\ B^t & A^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pm\mathbb{1}_n \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & B \\ B^t & B^t B \pm A^t A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pm\mathbb{1}_n \end{pmatrix}.$$

Therefore $B = 0$ and $A^t A = \mathbb{1}_n$. Finally, since $1 = \det L = \det A$, we obtain the desired result. □

Lemma 2.52. *Let G be a connected Lie group and $H \leq G$ a closed subgroup of G. If G/H is simply connected, then $\pi_1(G) \cong \pi_1(H)/N$, where N is a normal subgroup of $\pi_1(H)$.*

Proof. See [7, Prop. 5, p. 59]. □

Since $\mathbb{R}^n$ is homeomorphic to H^n via

$$x = (x^1, \dots, x^n) \mapsto (x^0, x^1, \dots, x^n),$$

where $x^0 = \sqrt{1 + (x^1)^2 + \cdots + (x^n)^2} \geq 1$ and the fact that S^n is simply connected for all $n \geq 2$ is known from the Topolgy II course, we obtain the desired result.

Theorem 2.53. $\pi_1(\mathrm{SO}^+_{1,n}) \cong \pi_1(\mathrm{SO}_n) \cong \mathbb{Z}_2$ *for all* $n \geq 3$.

Proof. By Lemma 2.52 and Theorem 2.51, we have that $\pi_1(G_\epsilon(n)) \cong \pi_1(\mathrm{SO}_n)/N_n$, with N_n a normal subgroup of $\mathrm{SO}_n = G_0(n-1)$. As we have mentioned above, we will show below (Theorem 3.8) that $\pi_1(\mathrm{SO}_3) \cong \mathbb{Z}_2$. Therefore, by induction we have that $\pi_1(G_\epsilon(n))$ is either trivial or isomorphic to $\mathbb{Z}_2$ for all $n \geq 3$. However, from Corollary 2.43 we know that $\pi_1(G_\epsilon(n))$ cannot be trivial for $n \geq 2$. Hence $\pi_1(G_\epsilon(n)) \cong \mathbb{Z}_2$ for all $n \geq 3$. □

Corollary 2.54. Spin_n *and* $\mathrm{Spin}^+_{1,n}$ *are simply connected for* $n \geq 3$. *Hence, they are the universal covers of* SO_n *and* $\mathrm{SO}^+_{1,n}$*, respectively.*

Proof. Follows from the above theorem and Corollary 2.43. □

2.4 The paravector space

To finish the chapter, we introduce the *paravector space*, an important quadratic space which lives within the algebra $\mathbb{G}_{p,m}$ in addition to $\mathbb{R}^{p,m} \equiv \mathbb{G}^1_{p,m}$. It will allow us to perform orthogonal transformations of the space $\mathbb{R}^{m+1,p}$ without leaving $\mathbb{G}_{p,m}$ via an isomorphic image of $\mathrm{Spin}^+_{m+1,p} \subseteq \mathbb{G}_{m+1,p}$, which we will denote by $\$\mathrm{pin}^+_{m+1,p} \subseteq \mathbb{G}_{p,m}$. In particular, it allows us to treat the proper transformations of the Lorentzian space $\mathbb{R}^{1,n}$ within the Euclidean geometric algebra $\mathbb{G}_n$, which will be useful for the applications of our formalism to physics in the next chapter. The main references are [25] and [19].

Definition 2.55. *The paravector space is defined as*

$$\mathbb{P}_{m+1,p} := \mathbb{G}^0_{p,m} \oplus \mathbb{G}^1_{p,m}.$$

Notation 2.56. *An element $x = t + \mathbf{x} \in \mathbb{P}_{m+1,p}$, where $t \in \mathbb{G}^0_{p,m} \equiv \mathbb{R}$ and $\mathbf{x} \in \mathbb{G}^1_{p,m} \equiv \mathbb{R}^{p,m}$, is called a paravector.*

We can equip $\mathbb{P}_{m+1,p}$ with the quadratic form

$$x \mapsto N(x) = \bar{x}x = (t - \mathbf{x})(t + \mathbf{x}) = t^2 - \mathbf{x}^2 = t^2 - q(\mathbf{x}) \in \mathbb{R},$$

so $\mathbb{P}_{m+1,p}$ becomes a quadratic space isometric with $\mathbb{R}^{m+1,p} = \mathbb{R} \oplus \mathbb{R}^{m,p}$. From above, we deduce that $N(x) = x\bar{x} = N(\bar{x})$ for all $x \in \mathbb{P}_{m+1,p}$. Every $x \in \mathbb{P}_{m+1,p}$ such that $N(x) \neq 0$ is invertible with inverse $x^{-1} = \bar{x}/N(x)$, so $\mathbb{P}^\times_{m+1,p} = \{x \in \mathbb{P}_{m+1,p} : N(x) \neq 0\}$. The metric associated to the quadratic form N is obtained via the polarization formula

$$N(x,y) = \frac{1}{2}(N(x+y) - N(x) - N(y)) = \frac{1}{2}((\bar{x}+\bar{y})(x+y) - \bar{x}x - \bar{y}y) = \frac{1}{2}(\bar{x}y + \bar{y}x).$$

We have an analogous result to Proposition 2.3 for paravectors.

Proposition 2.57. *Let $p \in \mathbb{P}^\times_{m+1,p}$. If $\rho'_p : \mathbb{P}_{m+1,p} \equiv \mathbb{R}^{m+1,p} \to \mathbb{P}_{m+1,p}$ denotes the reflection through $\{p\}^\perp$, then*

$$\rho'_p(x) = -p\bar{x}\tilde{p}^{-1} \quad \forall x \in \mathbb{P}_{m+1,p}.$$

Proof. $\rho'_p(x) = x - 2pN(p,x)/N(p) = \frac{\bar{p}px - p(\bar{p}x + \bar{x}p)}{\bar{p}p} = -p\bar{x}p/N(p) = -p\bar{x}\tilde{p}^{-1}$, where we have used that $N(p) = \bar{p}p = p\bar{p}$ and $\bar{p} = \tilde{p}$ for all $p \in \mathbb{P}_{m+1,p}$. □

Composing two reflections we get

$$\begin{aligned}\rho'_{p_1}(\rho'_{p_2}(x)) = \rho'_{p_1}(p_2\bar{x}\tilde{p}_2^{-1}) = p_1\overline{p_2\bar{x}\tilde{p}_2^{-1}}\tilde{p}_1^{-1} = p_1 p_2^{-1} x \tilde{p}_2 \tilde{p}_1^{-1} = \\ p_1\overline{p_2}xp_2^{-1}\overline{p_1}^{-1} = (p_1\overline{p_2})x(\overline{p_1}p_2)^{-1} = (p_1\overline{p_2})x(\widetilde{p_1\overline{p_2}})^{-1}\end{aligned}$$

and, by induction,

$$\rho'_{p_1} \circ \rho'_{p_2} \circ \cdots \circ \rho'_{p_k}(x) = -(p_1\overline{p_2}\ldots p_k)\bar{x}((p_1\overline{p_2}\ldots p_k)^{\sim})^{-1} = \rho'_{p_1\overline{p_2}\ldots p_k}(x)$$

if k is odd and

$$\rho'_{p_1} \circ \rho'_{p_2} \circ \cdots \circ \rho'_{p_k}(x) = (p_1\overline{p_2}\ldots\overline{p_k})x((p_1\overline{p_2}\ldots\overline{p_k})^{\sim})^{-1} = \rho_{p_1\overline{p_2}\ldots\overline{p_k}}(x)$$

if k is even, where $\rho : \mathrm{P}_{p,m} \to \mathrm{O}_{p,m}$ is the action from the previous sections.

Therefore, if we define the *paraversor group* as

$$\$\mathrm{P}_{m+1,p} := \{p_1\overline{q_1}\dots p_k\overline{q_k} : k \geq 0,\ p_i, q_i \in \mathbb{P}^\times_{m+1,p}\}$$

we have that $\rho_g \in \mathrm{SO}_{m+1,p}$ for all $g \in \$\mathrm{P}_{m+1,p}$, in fact, $\rho(\$\mathrm{P}_{m+1,p}) = \mathrm{SO}_{m+1,p}$ by the Cartan-Dieudonné theorem (Theorem 2.9).

Proposition 2.58. $\$\mathrm{P}_{m+1,p} = \{p_1 \dots p_k : k \geq 0,\ p_i \in \mathbb{P}^\times_{m+1,p}\}$.

Proof. Call $A := \{p_1 \dots p_k : p_i \in \mathbb{P}^\times_{m+1,p}\}$. Since $\overline{q} \in \mathbb{P}^\times_{m+1,p} \iff q \in \mathbb{P}^\times_{m+1,p}$ we have that $\$\mathrm{P}_{m+1,p} \subseteq A$. Conversely, take $p_1 \dots p_k \in A$, since $e_0 := 1 \in \mathbb{P}^\times_{m+1,p}$ and $\overline{e_0} = e_0 = 1$ we have that $p_1 \dots p_k = p_1\overline{e_0} \dots p_k\overline{e_0} \in \$\mathrm{P}_{m+1,p}$. □

The following results allow us to relate the spaces $\mathbb{P}_{m+1,p}$ and $\mathbb{G}^1_{m+1,p}$, which are both isometric to $\mathbb{R}^{m+1,p}$, and the groups $\$\mathrm{P}_{m+1,p}$ and $\mathrm{SP}_{m+1,p}$, which represent (generalized) rotations in those quadratic spaces, respectively.

Lemma 2.59. *The algebra isomorphism $\phi : \mathbb{G}_{p,m} \to \mathbb{G}^+_{m+1,p}$ of Prop. 1.34 preserves Clifford conjugation, i.e., $\overline{\phi(x)} = \phi(\bar{x})$.*

Proof. Using the same notation as in Prop. 1.34, it suffices to check it for the generators $\{e'_1, \dots, e'_p, f'_1, \dots, f'_m\}$ of $\mathbb{G}_{p,m}$.

(i) $\overline{\phi(e'_i)} = \overline{f_i e_{m+1}} = e_{m+1} f_i = -f_i e_{m+1} = \phi(-e'_i) = \phi(\overline{e'_i})$.

(ii) $\overline{\phi(f'_j)} = \overline{e_j e_{m+1}} = e_{m+1} e_j = -e_j e_{m+1} = \phi(-f'_j) = \phi(\overline{f'_i})$.

□

Theorem 2.60. *Let $\phi : \mathbb{G}_{p,m} \to \mathbb{G}^+_{m+1,p}$ be the algebra isomorphism of Prop. 1.34. Then*

(i) The map $u : \mathbb{P}_{m+1,p} \to \mathbb{G}^1_{m+1,p} \equiv \mathbb{R}^{m+1,p}$, $u(x) := \phi(x)e_{m+1}$ is a bijective isometry with inverse $u^{-1}(y) = \phi^{-1}(ye_{m+1})$. In the following, we will treat u as an identification, so $\mathbb{P}_{m+1,p} \equiv \mathbb{G}_{m+1,p} \equiv \mathbb{R}^{m+1,p}$.

(ii) The restriction of ϕ to the group $\$\mathrm{P}_{m+1,p}$ gives a group isomorphism $\phi : \$\mathrm{P}_{m+1,p} \to \mathrm{SP}_{m+1,p}$. Moreover, for any $g \in \$\mathrm{P}_{m+1,p}$, the following diagram commutes:

$$\begin{tikzcd}
\mathbb{P}_{m+1,p} \arrow[r, "\rho_g"] \arrow[d, "u"'] & \mathbb{P}_{m+1,p} \arrow[d, "u"] \\
\mathbb{G}^1_{m+1,p} \arrow[r, "\rho_{\phi(g)}"'] & \mathbb{G}^1_{m+1,p}
\end{tikzcd}$$

Proof. We use the same notation as in Prop. 1.34.

(i) Take $x = t + \sum_{i=1}^p x^i e'_i + \sum_{j=1}^m x^j f'_j \in \mathbb{P}_{m+1,p}$, then

$$\phi(x) = t + \sum_{i=1}^p x^i f_i e_{m+1} + \sum_{j=1}^m x^j e_j e_{m+1} = (te_{m+1} + \sum_{i=1}^p x^i f_i + \sum_{j=1}^m x^j e_j)e_{m+1},$$

so $u(x) := \phi(x)e_{m+1} = te_{m+1} + \sum_{j=1}^{m} x^j e_j + \sum_{i=1}^{p} x^i f_i \in \mathbb{G}^1_{m+1,p}$. In addition, using the above lemma,

$$u(x)^2 = -\overline{u(x)}u(x) = -\overline{e_{m+1}}\,\overline{\phi(x)}\phi(x)e_{m+1} = \bar{x}xe_{m+1}\phi(1)e_{m+1} = \overline{x}x = N(x).$$

(ii) Take $g = p_1 \dots p_k \in \$\mathrm{P}_{m+1,p}$, then $\phi(g) = \phi(p_1)\dots\phi(p_k) = u(p_1)e_{m+1}\dots u(p_k)e_{m+1} \in \mathrm{SP}_{m+1,p}$, with $e_{m+1}^2 = 1 \neq 0$ and $u(p_i)^2 = N(p_i) \neq 0$. Since ϕ is an algebra homomorphism and $\$\mathrm{P}_{m+1,p}$ and SP are subgroups of $\mathbb{G}_{p,m}$ and $\mathbb{G}_{m+1,p}$, respectively, we have that $\phi : \$\mathrm{P}_{m+1,p} \to \mathrm{SP}_{m+1,p}$ is a group homomorphism. Conversely, take $h = u_1v_1\dots u_kv_k \in \mathrm{SP}_{m+1,p}$. By part (i) we have that $u_i = u(p_i) = \phi(p_i)e_{m+1}$ and $v_i = u(q_i) = \phi(q_i)e_{m+1}$ with $p_i, q_i \in (\mathbb{P}_{m+1,p})^\times$, so

$$h = \phi(p_1)e_{m+1}\phi(q_1)e_{m+1}\dots\phi(p_k)e_{m+1}\phi(q_k)e_{m+1}.$$

Moreover, if $q_i = t + \sum_{i=1}^{p} x^i e_i' + \sum_{j=1}^{m} x^j f_j' \in \mathbb{P}_{m+1,p}$, then

$$e_{m+1}\phi(q_i)e_{m+1} = e_{m+1}(te_{m+1} + \sum_{j=1}^{m} x^j e_j + \sum_{i=1}^{p} x^i f_i) = t - \sum_{i=1}^{p} x^i f_i e_{m+1} - \sum_{j=1}^{m} x^j e_j e_{m+1} = \phi(\overline{q_i}),$$

and $h = \phi(p_i)\phi(\overline{q_i})\dots\phi(p_k)\phi(\overline{q_k})$. Therefore, $\phi^{-1}(h) = p_1\overline{q_1}\dots p_k\overline{q_k} \in \$\mathrm{P}_{m+1,p}$. For the diagram part, we have that for any $g \in \$\mathrm{P}_{m+1,p}$

$$\begin{aligned}\rho_{\phi(g)}(u(p)) &= \phi(g)u(p)\phi(g)^{-1} = \phi(g)\phi(p)e_{m+1}\phi(g)^{-1} = \phi(gp)(\phi(g)e_{m+1})^{-1}\\ &= \phi(gp(e_{m+1}\phi(\tilde{g}))^{-1} = \phi(gp)\phi(\tilde{g})^{-1}e_{m+1} = \phi(gp\tilde{g}^{-1})e_{m+1} = u(gp\tilde{g}^{-1}) = u(\rho_g(p)),\end{aligned}$$

where we have used that $\phi(g) \in \mathrm{SP}_{m+1,p} \subseteq \mathbb{G}^+_{m+1,p}$, so $\tilde{\phi}(g) = \phi(g)$.

□

In addition, since Clifford conjugation is preserved under ϕ we also obtain the following result.

Corollary 2.61. *Restricting the isomorphism $\phi : \$\mathrm{P}_{m+1,p} \to \mathrm{SP}_{m+1,p}$ we obtain*

$$\begin{aligned}\$\mathrm{pin}_{m+1,p} &:= \{s \in \$\mathrm{P}_{m+1,p} : N(s) = \pm 1\} \cong \phi(\$\mathrm{pin}_{m+1,p}) = \mathrm{Spin}_{m+1,p},\\ \$\mathrm{pin}^+_{m+1,p} &:= \{s \in \$\mathrm{P}_{m+1,p} : N(s) = 1\} \cong \phi(\$\mathrm{pin}^+_{m+1,p}) = \mathrm{Spin}^+_{m+1,p}.\end{aligned}$$

The above isomorphisms allow us to represent proper orthogonal transformations in $\mathbb{R}^{m+1,p}$ using the group $\$\mathrm{pin}_{m+1,p}$.

Corollary 2.62. *The map $\rho : \$\mathrm{pin}^{(+)}_{m+1,p} \to \mathrm{SO}^{(+)}_{m+1,p}$ is a surjective group homomorphism with kernel $\{+1, -1\}$.*

Proof. Since $\phi : \$\mathrm{pin}^{(+)}_{m+1,p} \to \mathrm{Spin}^{(+)}_{m+1,p}$ is a group homomorphism and, treating $u : \mathbb{P}_{m+1,p} \to \mathbb{G}^1_{m+1,p}$ as an identification, we can write $\rho = \rho \circ \phi$ (abuse of notation from the diagram above), the result follows from Theorem 2.26. □

3 Applications

In this final chapter we discuss the applications in physics of the formalism we have developed so far. In particular, we focus on the Clifford algebra $\mathbb{G}_3$ associated with physical space $\mathbb{R}^3$. In addition, thanks to the paravector space, we will be able to model spacetime and electromagnetism within the algebra $\mathbb{G}_3$. Finally, taking advantage of the complex structure provided by the pseudoscalar in $\mathbb{G}_3$, we will also discuss some applications of Clifford algebras to quantum theory (see [6] for more applications). The main references for this chapter are [3], [5], [13] and [19].

3.1 The (geometric) algebra of physical space

Physical space is modelled by the Euclidean 3-dimensional space $\mathbb{R}^3$, which is equipped with the quadratic form

$$q(\mathbf{x}) := x_1^2 + x_2^2 + x_3^2, \tag{3.1}$$

for all $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. Its associated geometric algebra, $\mathbb{G}_3$, is generated by the canonical basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ under the relations

$$\mathbf{e}_i^2 = \mathbf{e}_i \cdot \mathbf{e}_i = q(\mathbf{e}_i) = 1, \quad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i, \tag{3.2}$$

for all $i \neq j$. A basis of the full $8 = 2^3$ dimensional (real) algebra is then

scalar	vectors	bivectors	trivector
1	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	$\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3$	$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$

We see that while scalars and vectors have positive square, the basis bivectors and the basis trivector behave as imaginary units, since for $i \neq j$

$$\begin{aligned}
(\mathbf{e}_i\mathbf{e}_j)^2 &= \mathbf{e}_i\mathbf{e}_j\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_i\mathbf{e}_j^2\mathbf{e}_i = -\mathbf{e}_i^2 = -1, \\
(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^2 &= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1^2\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_3 = (\mathbf{e}_2\mathbf{e}_3)^2 = -1.
\end{aligned}$$

In addition, the basis trivector shares an important property with scalars, namely, that it commutes with all the elements of the algebra.

Proposition 3.1. *If $Z(\mathbb{G}_3) := \{x \in \mathbb{G}_3 : xy = yx \ \forall y \in \mathbb{G}_3\}$ denotes the center of the algebra $\mathbb{G}_3$, then $Z(\mathbb{G}_3) = \mathbb{G}_3^0 \oplus \mathbb{G}_3^3$. Moreover, we have the isomorphism of algebras $Z(\mathbb{G}_3) \cong \mathbb{C}$.*

Proof. We know that $\mathbb{G}_3^0 \equiv \mathbb{R} \subseteq Z(\mathbb{G}_3)$. Next, we show that $\mathbb{G}_3^3 \subseteq Z(\mathbb{G}_3)$. It suffices to check that $\mathbf{e}_{123}$ commutes with the generators. Indeed, we have that

$$\begin{aligned}
\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) &= \mathbf{e}_1^2\mathbf{e}_2\mathbf{e}_3 = (\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)\mathbf{e}_1, \\
\mathbf{e}_2(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) &= -\mathbf{e}_1\mathbf{e}_2^2\mathbf{e}_3 = (\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)\mathbf{e}_2, \\
\mathbf{e}_3(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) &= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3^2 = (\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)\mathbf{e}_3.
\end{aligned}$$

On the other hand, if $0 \neq \mathbf{v} \in \mathbb{G}_3^1$ is any non-zero vector, we know that $\mathbf{v}\mathbf{w} = -\mathbf{w}\mathbf{v}$ for any $\mathbf{w} \in \mathbf{v}^\perp$, so $\mathbf{v} \notin Z(\mathbb{G}_3)$. Moreover, let $B = c_{12}\mathbf{e}_{12} + c_{13}\mathbf{e}_{13} + c_{23}\mathbf{e}_{23} \in \mathbb{G}_3^2$ be any bivector. The condition $B \in Z(\mathbb{G}_3)$ implies, using (i) and (ii) in the proof of Lemma 2.37, that

$$[B, \mathbf{e}_1] = -2c_{12}\mathbf{e}_2 - 2c_{13}\mathbf{e}_3 = 0 \implies c_{12} = c_{13} = 0$$
$$[B, \mathbf{e}_2] = c_{23}[\mathbf{e}_{23}, \mathbf{e}_2] = -2c_{23}\mathbf{e}_{23} = 0 \implies B = 0.$$

This proves the first part of the proposition. For the other part, note that since $\mathbf{e}_{123}^2 = -1$, the desired isomorphism of algebras is obtained via $Z(\mathbb{G}_3) \to \mathbb{C}$, $a + b\mathbf{e}_{123} \mapsto a + bi$. □

Therefore, the basis trivector behaves like a unit imaginary scalar, so we will denote it by $\mathrm{i} := \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ and call it the *(unit) pseudoscalar*. A general trivector will then be referred to as a *pseudoscalar*. The main difference between scalars and pseudoscalars is that the latter change sign under parity. Indeed, if we send $\mathbf{e}_i \mapsto -\mathbf{e}_i$, then $\mathrm{i} \mapsto (-\mathbf{e}_1)(-\mathbf{e}_2)(-\mathbf{e}_3) = -\mathrm{i}$.

Remark 3.2. *Notice the slightly different typography between* i *and* i, *so that we can distinguish the pseudoscalar from the imaginary unit of the complex numbers when convenient.*

The pseudoscalar also allows us to define the concept of *(Hodge) duality*.

Definition 3.3. *We define the (Hodge) dual of the Clifford number* $c \in \mathbb{G}_3$ *as*

$$\star c := \mathrm{i}c^\dagger = c^\dagger \mathrm{i},$$

where $c^\dagger$ *is the reverse of* c *(recall Def. 1.19).*

Proposition 3.4. *The map* $\mathbb{G}_3^k \to \mathbb{G}_3^{3-k}$, $c \mapsto \star c$, $k = 0, 1$, *defines a vector space isomorphism.* $\mathbb{G}_3^k$ *and* $\mathbb{G}_3^{3-k}$ *are said to be dual subspaces.*

Proof. The case $k = 0$ ($\star\mathbb{G}_3^0 = \mathbb{G}_3^3$) is clear. For the other isomorphism take $\mathbf{a} \in \mathbb{G}_3^1$, then (using that $\star$ is linear)

$$\star\mathbf{a} = a_1 \star \mathbf{e}_1 + a_2 \star \mathbf{e}_2 + a_3 \star \mathbf{e}_3 = a_1\mathbf{e}_{23} - a_2\mathbf{e}_{13} + a_3\mathbf{e}_{12} \in \mathbb{G}_3^2, \tag{3.3}$$

where we have written $\mathbf{e}_{ij} = \mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$ for $i \neq j$. That it defines an isomorphism follows from the fact that Hodge duality is involutory:

$$\star(\star c) = \star(c^\dagger \mathrm{i}) = \mathrm{i}(c^\dagger \mathrm{i})^\dagger = \mathrm{i}\mathrm{i}^\dagger c = -\mathrm{i}^2 c = c \quad \forall c \in \mathbb{G}_3,$$

so $\star$ is its own inverse. □

Duality also allows us to recover the cross product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ from its wedge product $\mathbf{a} \wedge \mathbf{b}$.

Proposition 3.5. *We have that* $\mathbf{a} \times \mathbf{b} = \star(\mathbf{a} \wedge \mathbf{b}) = -\mathrm{i}(\mathbf{a} \wedge \mathbf{b})$ *for all* $\mathbf{a}, \mathbf{b} \in \mathbb{G}_3^1 \equiv \mathbb{R}^3$.

Proof. Indeed, if $\mathbf{a} = \sum_i a^i\mathbf{e}_i$ and $\mathbf{b} = \sum_i b^i\mathbf{e}_i$, using $\mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i$ we obtain

$$\mathbf{a} \wedge \mathbf{b} = (a^2b^3 - a^3b^2)\mathbf{e}_2 \wedge \mathbf{e}_3 + (a^1b^3 - a^3b^1)\mathbf{e}_1 \wedge \mathbf{e}_3 + (a^1b^2 - a^2b^1)\mathbf{e}_1 \wedge \mathbf{e}_2,$$

so looking at ((3.3)), we conclude that $\mathbf{a} \wedge \mathbf{b} = \star(\mathbf{a} \times \mathbf{b}) = \mathrm{i}(\mathbf{a} \times \mathbf{b})$, which is equivalent to the desired result. □

The main advantage of using the wedge product over the cross product is that the latter is not associative, i.e., in general $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Moreover, the wedge product can be defined in any dimension, while the cross product is only well defined in three dimensions, since it requires to have an essentially unique normal vector to a plane. We will later see that formulas which involve the cross product (like Maxwell's equations) greatly simplify by expressing them in terms of the wedge product.

Finally, let us use duality to write the basis bivectors in terms of the basis vectors and the pseudoscalar. Looking at (3.3), we have that

$$\mathbf{e}_{12} = \star\mathbf{e}_3 = \mathrm{i}\mathbf{e}_3, \quad \mathbf{e}_{13} = -\star\mathbf{e}_2 = -\mathrm{i}\mathbf{e}_2, \quad \mathbf{e}_{23} = \star\mathbf{e}_1 = \mathrm{i}\mathbf{e}_1.$$

Therefore, identifying $Z(\mathbb{G}_3) \equiv \mathbb{C}$ via $a + \mathrm{i}b \simeq a + ib$ (the isomorphism of Prop. 3.1), we can regard $\mathbb{G}_3$ as a complex algebra spanned by $\{\mathbf{e}_0 := 1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and a general element $p \in \mathbb{G}_3$ can be written as

$$p = \alpha + \mathbf{v} + \mathrm{i}\mathbf{w} + \mathrm{i}\beta \simeq (\alpha + i\beta) + (\mathbf{v} + i\mathbf{w}),$$

where $\alpha, \beta \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in \mathbb{G}_3^1 \equiv \mathbb{R}^3$. The involutions of the algebra then act as follows:

$$\begin{aligned} \tilde{p} &= \alpha - \mathbf{v} + \mathrm{i}\mathbf{w} - \mathrm{i}\beta, \\ p^\dagger &= \alpha + \mathbf{v} - \mathrm{i}\mathbf{w} - \mathrm{i}\beta, \\ \bar{p} &= \alpha - \mathbf{v} - \mathrm{i}\mathbf{w} + \mathrm{i}\beta, \end{aligned}$$

and we can use them to project onto the different parts of a clifford number $p \in \mathbb{G}_3$

$$\begin{aligned} (p)_{even} &:= \frac{1}{2}(p + \tilde{p}) = \alpha + \mathrm{i}\mathbf{w}, \quad (p)_{odd} := \frac{1}{2}(p - \tilde{p}) = \mathbf{v} + \mathrm{i}\beta, \\ (p)_{\Re} &:= \frac{1}{2}(p + p^\dagger) = \alpha + \mathbf{v}, \quad (p)_{\Im} := \frac{1}{2}(p - p^\dagger) = \mathrm{i}(\beta + \mathbf{w}), \\ (p)_S &:= \frac{1}{2}(p + \bar{p}) = \alpha + \mathrm{i}\beta, \quad (p)_V := \frac{1}{2}(p - \bar{p}) = \mathbf{v} + \mathrm{i}\mathbf{w}. \end{aligned}$$

3.2 3D rotations and quaternions

Since $n = 3 \leq 4$, recall from Proposition 2.28 and Theorem 2.30 that we can write (except for a slight change of notation)

$$\mathrm{Spin}(3) = \mathrm{Spin}^+(3,0) = \{R \in \mathbb{G}_3^+ : \bar{R}R = 1\} = \{R \in \mathbb{G}_3^+ : \bar{R}R = 1 = R^\dagger R\}, \tag{3.4}$$

where in the last equation we have used that $R \in \mathbb{G}_3^+$ if and only if $R^\dagger = \bar{R}$. A rotation in $\rho_R \in \mathrm{SO}(3)$ is then represented by $R \in \mathrm{Spin}(3)$ via

$$\rho_R(\mathbf{x}) = R\mathbf{x}\tilde{R}^{-1} = R\mathbf{x}\bar{R} = R\mathbf{x}R^\dagger \quad \forall \mathbf{x} \in \mathbb{G}_3^1 \equiv \mathbb{R}^3,$$

where we have used that $\tilde{R} = R$, $R^{-1} = \bar{R} = R^\dagger$ for all $R \in \mathrm{Spin}(3) \subseteq \mathbb{G}_3^+$.

Let us now express a rotor $R \in \mathbb{G}_3^+$ in a form that will reveal the geometric content of the rotation ρ_R. If we write $R = \alpha - i\mathbf{w} \in \mathbb{G}_3^+$, the condition $\bar{R}R = 1$ then becomes

$$(\alpha + \mathrm{i}\mathbf{w})(\alpha - \mathrm{i}\mathbf{w}) = \alpha^2 - \mathrm{i}^2\mathbf{w}^2 = \alpha^2 + w^2 = 1, \tag{3.5}$$

where $w = |\mathbf{w}| = \sqrt{q(\mathbf{w})}$. Note that we can always write $\mathbf{w} = w\hat{\mathbf{w}}$ with $\hat{\mathbf{w}}^2 = 1$. Condition (3.5) implies that $-1 \leq \alpha \leq 1$, so there exists a unique $\theta \in [0, 2\pi]$ such that $\alpha = \cos\frac{\theta}{2}$. Moreover, since $w = |\mathbf{w}| \geq 0$, condition (3.5) then implies that $w = \sin\frac{\theta}{2}$. Therefore, using that $(\mathrm{i}\hat{\mathbf{w}})^2 = -1$, we can write

$$R = \cos\frac{\theta}{2} - \mathrm{i}\hat{\mathbf{w}}\sin\frac{\theta}{2} = e^{-\mathrm{i}\hat{\mathbf{w}}\frac{\theta}{2}}, \tag{3.6}$$

which is the *canonical form* of the rotor $R \in \mathrm{Spin}(3)$. To see how the rotation ρ_R acts on a general vector $\mathbf{x} \in \mathbb{R}^3$ we decompose $\mathbf{x} = \mathbf{x}_\parallel + \mathbf{x}_\perp$, where $\mathbf{x}_\parallel$ and $\mathbf{x}_\perp$ are the components of $\mathbf{x}$ parallel and orthogonal to $\hat{\mathbf{w}}$. Now, since $\hat{\mathbf{w}}\mathbf{x}_\parallel = \mathbf{x}_\parallel\hat{\mathbf{w}}$ and $\hat{\mathbf{w}}\mathbf{x}_\perp = -\mathbf{x}_\perp\hat{\mathbf{w}}$, then

$$\rho_R(\mathbf{x}) = R\mathbf{x}R^\dagger = R\mathbf{x}_\parallel R^\dagger + R\mathbf{x}_\perp R^\dagger = RR^\dagger\mathbf{x}_\parallel + R^2\mathbf{x}_\perp = \mathbf{x}_\parallel + R^2\mathbf{x}_\perp,$$

and the second term is

$$R^2\mathbf{x}_\perp = (\cos\theta - \mathrm{i}\hat{\mathbf{w}}\sin\theta)\mathbf{x}_\perp = \cos\theta\,\mathbf{x}_\perp + \sin\theta\,\hat{\mathbf{w}} \times \mathbf{x}_\perp, \tag{3.7}$$

where we have used that $\hat{\mathbf{w}}\mathbf{x}_\perp = \hat{\mathbf{w}} \wedge \mathbf{x}_\perp$ and $\hat{\mathbf{w}} \times \mathbf{x}_\perp = \star(\hat{\mathbf{w}} \wedge \mathbf{x}_\perp) = -\mathrm{i}\hat{\mathbf{w}}\mathbf{x}_\perp$. Therefore, ρ_R is the (right-handed) rotation about the axis $\mathrm{span}_\mathbb{R}(\hat{\mathbf{w}})$ by an angle θ. In particular, if we take $\hat{\mathbf{w}} = \mathbf{e}_1$, then the matrix representing the rotation ρ_R in the canonical basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is

$$\rho_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(3).$$

Hence, we see that expressing R in its canonical form (3.6) reveals the geometric content of the rotation ρ_R, namely, the axis and angle of rotation.

3.2.1 Relation to quaternions

Recall that the algebra of the quaternions, $\mathbb{H}$, is a four dimensional real vector space with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, where the three symbols $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Remark 3.6. *Note that* $\mathbf{i}$ *is in boldface to distinguish it from the pseudoscalar* i.

It turns out that quaternions can be viewed as the even subalgebra of $\mathbb{G}_3$,

$$\mathbb{G}_3^+ = \mathrm{span}_\mathbb{R}\{1, \mathrm{i}\mathbf{e}_1, \mathrm{i}\mathbf{e}_2, \mathrm{i}\mathbf{e}_3\}.$$

Theorem 3.7. *An isomorphism of real algebras* $\kappa : \mathbb{G}_3^+ \to \mathbb{H}$ *is obtained via*

$$-\mathrm{i}\mathbf{e}_1 \mapsto \mathbf{i}, \quad -\mathrm{i}\mathbf{e}_2 \mapsto \mathbf{j}, \quad -\mathrm{i}\mathbf{e}_3 \mapsto \mathbf{k}. \tag{3.8}$$

and extending as an algebra homomorphism. In addition, the restriction of κ *to the group* $\mathrm{Spin}(3) \subseteq \mathbb{G}_3$ *defines a group isomorphism*

$$\mathrm{Spin}(3) \cong \mathbb{H}_1,$$

where $\mathbb{H}_1 := \{q \in \mathbb{H} : q^*q = 1\}$ *denotes the group of unit quaternions*

Proof. To show that κ defines an algebra isomorphism follows from the fact that the bivectors $-\mathrm{i}\mathbf{e}_1, -\mathrm{i}\mathbf{e}_2, -\mathrm{i}\mathbf{e}_3$ also satisfy

$$(-\mathrm{i}\mathbf{e}_1)^2 = (-\mathrm{i}\mathbf{e}_2)^2 = (-\mathrm{i}\mathbf{e}_3)^2 = -1, \quad (-\mathrm{i}\mathbf{e}_1)(-\mathrm{i}\mathbf{e}_2)(-\mathrm{i}\mathbf{e}_3) = -\mathrm{i}^4 = -1.$$

For the second part, note that if we take $R = \alpha - \mathrm{i}(w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3) \in \mathbb{G}_3^+$, then $\bar{R} = \alpha + \mathrm{i}(w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3)$, so $\kappa(\bar{R}) = \kappa(R)^*$, where $*$ denotes quaternion conjugation. Hence, we also obtain that

$$\kappa(\mathrm{Spin}(3)) = \{\kappa(R) \in \kappa(\mathbb{G}_3^+) : \kappa(\bar{R})\kappa(R) = 1\} = \{q \in \mathbb{H} : q^*q = 1\} = \mathbb{H}_1.$$

$\square$

The isomorphism (3.8) also allows us to recover the traditional quaternion rotation formula. Start with a vector $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \in \mathbb{G}_3^1$. Its associated pure quaternion will be denoted by $\underline{\mathbf{x}} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathrm{Im}(\mathbb{H})$, and we have that $\underline{\mathbf{x}} = \kappa(-\mathrm{i}\mathbf{x})$, where multiplication by $-\mathrm{i}$ stablishes a vector space isomorphism $\mathbb{G}_3^1 \to -\mathrm{i}\mathbb{G}_3^1 = \mathbb{G}_3^2$. From the discussion above, we know that if we rotate the vector $\mathbf{x}$ by an angle θ about the axis given by the unit vector $\hat{\mathbf{a}}$, we obtain the rotated vector

$$\mathbf{y} = R\mathbf{x}R^\dagger = R\mathbf{x}\bar{R}, \text{ where } R = \cos\frac{\theta}{2} - \mathrm{i}\hat{\mathbf{a}}\sin\frac{\theta}{2} \in \mathrm{Spin}(3).$$

Then, using that $\mathrm{i} \in Z(\mathbb{G}_3)$ and that κ is an isomorphism we have the correspondence

$$\mathbf{y} = R\mathbf{x}\bar{R} \iff -i\mathbf{y} = R(-i\mathbf{x})\bar{R} \iff \kappa(-i\mathbf{y}) = \kappa(R)\kappa(-i\mathbf{x})\kappa(\bar{R}) \iff \underline{\mathbf{y}} = q\underline{\mathbf{x}}q^*,$$

where $q = \kappa(R) = \cos\frac{\theta}{2} + \underline{\hat{\mathbf{a}}}\sin\frac{\theta}{2} \in \mathbb{H}_1$, is the unit quaternion associated with the rotation around the axis $\underline{\hat{\mathbf{a}}} = \kappa(-i\hat{\mathbf{a}}) \in \mathrm{Im}(\mathbb{H})$.

3.3 The topology of Spin(3) and SO(3)

Recall that $\mathbb{G}_3$ becomes a topological space by demanding that the map $\tau : \mathbb{G}_3 \to \mathbb{R}^8$, $p = \alpha + \mathbf{v} + \mathrm{i}\mathbf{w} + \mathrm{i}\beta \mapsto \tau(p) = (\alpha, \mathbf{v}, \mathbf{w}, \beta)$, where $\mathbf{v} = (v_1, v_2, v_3), \mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$, be an homeomorphism. Then, taking into account (3.4) and (3.5), we obtain the homeomorphism

$$\mathrm{Spin}(3) \cong \{(a, \mathbf{0}, \mathbf{w}, 0) \in \mathbb{R}^8 : \alpha^2 + w_1^2 + w_2^2 + w_3^2 = 1\} \cong \mathbb{S}^3, \tag{3.9}$$

where $\mathbb{S}^3 = \{(a, b, c, d) \in \mathbb{R}^4 : a^2 + b^2 + c^2 + d^2 = 1\}$ is the 3-sphere. Since all the spheres $\mathbb{S}^n$ are simply connected for $n \geq 2$, we obtain the result we promised in Theorem 2.53.

Theorem 3.8. *The group* Spin(3) *is simply connected. This in turn implies, using the double covering map* $\rho : \mathrm{Spin}(3) \to \mathrm{SO}(3)$ *of Theorem 2.41 and Corollary 2.43, that*

$$\mathrm{SO}(3) \cong \mathbb{S}^3/\{1, -1\} \cong \mathbb{RP}^3, \quad \pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2, \tag{3.10}$$

where the $\cong$ *on the left is an homeomorphism between topological spaces, while the one on the right is an isomorphism of groups.*

The way of interpreting the homeomoprhism on the left of (3.10) is the following: we saw above that any rotation $\rho_{\theta,\hat{\mathbf{a}}} \in \mathrm{SO}(3)$ is determined by a unit vector $\hat{\mathbf{a}} \in \mathbb{R}^3$, which defines the axis of rotation, and a right-handed angle $\theta \in [0, 2\pi]$. However, if $\alpha \in [0, \pi]$, we have that $\rho_{\pi+\alpha,\hat{\mathbf{a}}} = \rho_{\pi-\alpha,-\hat{\mathbf{a}}}$, since the rotation by an angle $\pi + \alpha$ with $\alpha \in [0, \pi]$ coincides with the rotation by an angle $\alpha - \pi \in [-\pi, 0]$. Therefore, it suffices to consider angles in the range $[0, \pi]$. Then, we see that every point $p \in B(0,1) := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \leq 1\}$ corresponds to a rotation $\rho_{\theta,\hat{\mathbf{a}}} \in \mathrm{SO}(3)$ by taking $\theta := \pi\|p\| \in [0, \pi]$ and $\hat{\mathbf{a}} := \frac{p}{\|p\|} \in \mathbb{S}^2$. Moreover, this correspondence is unique except for the points on the boundary $\partial B(0,1) = \mathbb{S}^2$. In that case, the point $p = \hat{\mathbf{a}}$ and its antipode $-p = -\hat{\mathbf{a}}$ represent the same rotation, since $\rho_{\pi,\hat{\mathbf{a}}} = \rho_{\pi,-\hat{\mathbf{a}}}$. Hence, antipodal points of the boundary $\mathbb{S}^2$ of $B(0,1)$ must be identified, giving us the real projective space $\mathbb{RP}^3$, whose points correspond uniquely to rotations in $\mathrm{SO}(3)$. Moreover, this correspondence is continuous, i.e., points close in $\mathbb{RP}^3$ represent rotations that differ little from each other.

On the other hand, the isomorphism of groups on the right of (3.10) tells us that there are two classes of closed loops in $\mathrm{SO}(3) \cong \mathbb{RP}^3$: those that can be continuously contracted to a point, and those that cannot, with any two non-contractible loops being homotopic (i.e., continuously deformable) to one another. To find the generator of the group $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ it suffices to find a non-trivial loop, i.e., one that is not contractible. We will take the identity rotation $\mathbb{1} \in \mathrm{SO}(3)$ as our base point. Then, consider the path $\gamma : [0, 2\pi] \to \mathrm{SO}(3)$ given by

$$t \mapsto R(t) := e^{-\mathrm{i}\mathbf{e}_1 t/2} \in \mathrm{Spin}(3) \mapsto \gamma(t) := \rho_{R(t)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}.$$

Since $\gamma(0) = \mathbb{1} = \gamma(2\pi)$, we see that it defines a closed loop in $\mathrm{SO}(3)$. However, $R : [0, 2\pi] \to \mathrm{Spin}(3)$, $t \mapsto R(t) = e^{-\mathrm{i}\mathbf{e}_1 t/2}$, is the unique lifting of the loop γ under ρ which satisfies $R(0) = 1$,[10] and we see that $R(2\pi) = -1 \neq R(0)$. Therefore, γ cannot be (continuously) contracted to $\gamma(0) = \mathbb{1} = \gamma(2\pi)$, because that would mean that the lift R can also be contracted to a point *while keeping the endpoints fixed*, which is a contradiction given that the endpoints, $R(0)$ and $R(2\pi)$, are different. Hence, we can write $\pi_1(\mathrm{SO}(3)) = \{[\mathbb{1}], [\gamma]\}$ and we have that $[\mathbb{1}] = [\gamma] * [\gamma] = [\gamma * \gamma]$, i.e. the 4π twist $\gamma * \gamma$ is contractible (nulhomotopic) to the identity rotation $\mathbb{1}$.

Rigorously speaking, this result is a consequence of the homotopy lifting theorem (Theorem B.11), which implies (except for a reparameterization $[0, 1] \leftrightarrow [0, 2\pi]$) that if γ were homotopic to the constant loop $\varepsilon_{\mathbb{1}} \equiv \mathbb{1}$, then the liftings $R = \tilde{\gamma}$ and $\varepsilon_1 = \tilde{\varepsilon}_{\mathbb{1}} \equiv 1$ would have to end at the same points and be homotopic. However $R(2\pi) = -1 \neq 1 = \varepsilon_1(2\pi)$.

3.3.1 Dirac's belt trick

A physical illustration of the above topological result (Theorem 3.8) can be demonstrated using a belt and is often attributed to Paul Dirac, which is why it is known as *Dirac's belt trick*. The trick proceeds as follows:

(i) Start with a flat/untwisted belt.

[10] see Prop. B.10 in the Appendix.

(ii) Perform a 4π twist of the belt.

(iii) The goal is to untwist the belt, but one is not allowed to rotate either end of the belt. However, one is allowed to move the ends of the belt freely in space, as long as one does not turn them while doing so.

The correspondence between configurations of the belt and rotations in space is established by placing a coordinate frame (i.e., a set of axes) on top of the belt. If we let the the frame slide along the belt, it will rotate according to the twist on the belt and trace a path in the space of rotations, represented as the quotient $\mathbb{RP}^3$ of $B(0,1)$, see Figures 1 to 4. An illustration of Dirac's belt trick is shown in Figure 5. [11]

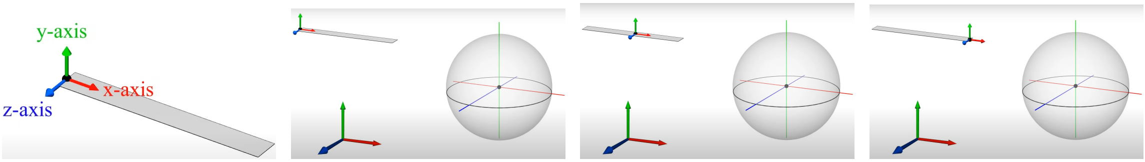

Figure 1: First image: a belt with a set of axes on top. Three last images: as the frame slides along the belt it performs the identity rotation, i.e., it does not rotate at all.

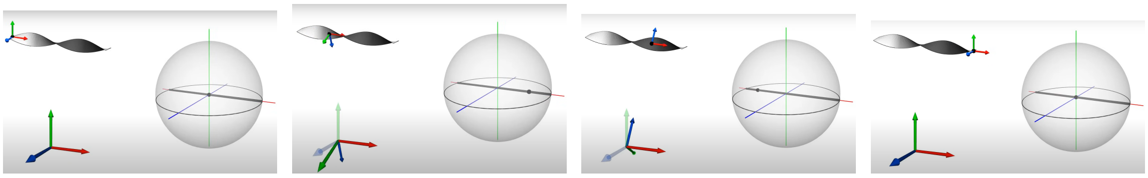

Figure 2: The 2π twist corresponds to the path that performs a 2π rotation. Notice how from the second to the third image, the point that traces the path jumped from one side to the other due to the identification of antipodal points in the boundary.

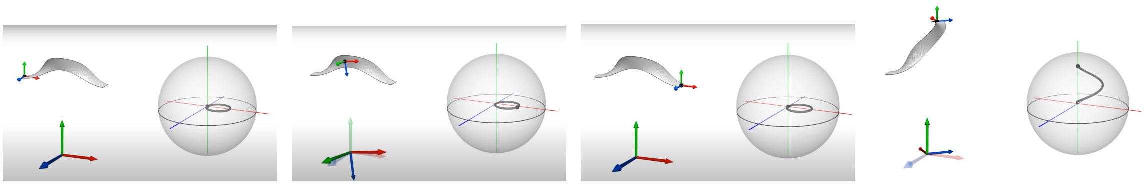

Figure 3: Examples of a closed (first three images) and an open (last image) path in the space of rotations. The rule of not being allowed to rotate the ends of the belt while trying to untwist it corresponds to requiring that the path in the space of rotations be a closed loop (at the second end, the frame must return to its initial state due to the total twisting of the belt being an even multiple of π).

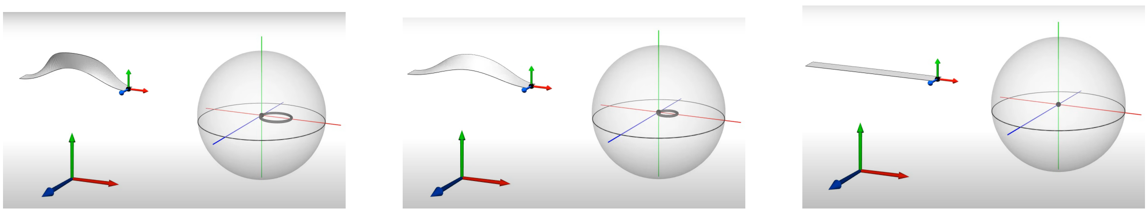

Figure 4: Flattening/untwisting the belt corresponds to contracting the loop to a point.

[11] All the images are taken from [22], except for Figure 5, which is taken from [21].

Figure 5: Dirac's belt trick (from top-left to bottom-right): a belt with a 4π twist in it can be made flat without rotating the ends of the belt. This shows that the loop in SO(3) associated with a 4π rotation is contractible to a point.

3.4 Paravector model of spacetime

From section 2.4 we know that the paravector space $\mathbb{P}_{1,3} = \mathbb{G}_3^0 \oplus \mathbb{G}_3^1$ is equipped with the quadratic form

$$N(x) = \bar{x}x = t^2 - \mathbf{x}^2 = t^2 - q(\mathbf{x}),$$

for all $x = t + \mathbf{x} \in \mathbb{P}_{1,3}$. The associated metric is obtained via the polarization formula

$$g_N(x, x') = \frac{1}{2}(\bar{x}x' + \bar{x}'x) = (\bar{x}x')_S = tt' - \frac{1}{2}(\mathbf{x}\mathbf{x}' + \mathbf{x}'\mathbf{x}) = tt' - \mathbf{x} \cdot \mathbf{x}'.$$

Therefore, we can identify Minkowski spacetime $\mathbb{R}^{1,3}$ with the paravector space $\mathbb{P}_{1,3}$ by sending

$$(x^0, x^1, x^2, x^3)^t \mapsto x = x^\mu \mathbf{e}_\mu \in \mathbb{P}_{1,3},$$

where we have defined $\mathbf{e}_0 := 1$ and used Einstein's summation convention. With this identification, using Theorem 2.30, and Corollaries 2.62 and 2.61, we have that a (restricted) Lorentz transformation $\Lambda \in \mathrm{SO}^+(1,3)$ is represented by a rotor $L \in \$\mathrm{pin}^+(1,3)$, where

$$\$\mathrm{pin}^+(1,3) = \{s \in \mathbb{G}_3 : \bar{s}s = 1\}, \tag{3.11}$$

via the spin covering $\rho : \$\mathrm{pin}^+(1,3) \to \mathrm{SO}^+(1,3)$

$$\Lambda(x) = \rho_L(x) = Lx\tilde{L}^{-1} = LxL^\dagger,$$

where we have used that $L^{-1} = \bar{L}$ for all $L \in \$\mathrm{pin}^+(1,3)$. From now on, we will refer to the rotor $L \in \$\mathrm{pin}^+(1,3)$ also as a Lorentz transformation, since Λ can be obtained from L. There are two basic types of Lorentz transformations: rotations and boosts; from which we can construct a general one.

A Lorentz transformation $R \in \$\mathrm{pin}^+(1,3)$ is a rotation if and only if $R^\dagger = \bar{R} = R^{-1}$. If we write $R = \alpha - \mathbf{v} - i\mathbf{w} + i\beta \in \mathbb{G}^3$, the previous condition is equivalent to

$$\alpha - \mathbf{v} + i\mathbf{w} - i\beta = \alpha + \mathbf{v} + i\mathbf{w} + i\beta \implies R = \alpha - i\mathbf{w} \in \mathbb{G}_3^+.$$

Therefore, R is a rotation if and only if $R \in \mathrm{Spin}(3) = \{s \in \mathbb{G}_3^+ : \bar{s}s = 1\} \subseteq \mathrm{Spin}^+(1,3)$, i.e., R is a usual rotation of 3D euclidean space. We saw above that the rotor $R = \alpha - i\mathbf{w}$

can be written in the form

$$R = \cos\frac{\theta}{2} - \mathrm{i}\hat{\mathbf{w}}\sin\frac{\theta}{2} = e^{-\mathrm{i}\hat{\mathbf{w}}\frac{\theta}{2}},$$

where $\hat{\mathbf{w}}$ is a unit vector defining an axis and θ is the right-handed angle of rotation around that axis. Now, if $x = t + \mathbf{x}$ is a paravector, then, decomposing as above $\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}$, where $\mathbf{x}_{\parallel}$ and $\mathbf{x}_{\perp}$ are the components of $\mathbf{x}$ parallel and orthogonal to $\hat{\mathbf{w}}$, we obtain that

$$x' = RxR^{\dagger} = R(t + \mathbf{x}_{\parallel})R^{\dagger} + R\mathbf{x}_{\perp}R^{\dagger} = (t + \mathbf{x}_{\parallel}) + R^2\mathbf{x}_{\perp},$$

where the second term is of the form (3.7). Therefore, a rotation around the axis $\hat{\mathbf{w}}$ mixes the perpendicular directions $\mathbf{x}_{\perp}$ and $\hat{\mathbf{w}} \times \mathbf{x}_{\perp}$, and leaves the time axis $\mathbf{e}_0$ and the parallel direction $\mathbf{x}_{\parallel}$ invariant.

The other basic type of Lorentz transformation is a boost, which is defined as a rotor $B \in \$\mathrm{pin}^+(1,3)$ such that $B^{\dagger} = B$ and $(B)_S > 0$. If we write $B = \alpha - \mathbf{v} - \mathrm{i}\mathbf{w} + \mathrm{i}\beta$, the first condition is equivalent to

$$\alpha - \mathbf{v} + \mathrm{i}\mathbf{w} - \mathrm{i}\beta = \alpha - \mathbf{v} - \mathrm{i}\mathbf{w} + \mathrm{i}\beta \implies B = \alpha - \mathbf{v} \in \mathbb{P}_{1,3}.$$

The condition $\bar{B}B = 1$ now implies that $\alpha^2 - v^2 = 1$, where $v = |\mathbf{v}|$. Since $(B)_S = \alpha > 0$, we have that $\alpha \geq 1$ and there exists a unique $\phi \geq 0$ such that $\alpha = \cosh\frac{\phi}{2}$. Since $v = |\mathbf{v}| \geq 0$, we have that $v = \sinh\frac{\phi}{2}$ and, writing $\mathbf{v} = v\hat{\mathbf{v}}$ with $\hat{\mathbf{v}}^2 = 1$, we obtain that

$$B = \cosh\frac{\phi}{2} - \hat{\mathbf{v}}\sinh\frac{\phi}{2} = e^{-\hat{\mathbf{v}}\frac{\phi}{2}}. \tag{3.12}$$

The boost B transforms a general paravector $x = t + \mathbf{x}$ as follows

$$x' = B\mathbf{x}B^{\dagger} = B(t + \mathbf{x}_{\parallel})B + B\mathbf{x}_{\perp}B = B^2(t + \mathbf{x}_{\parallel}) + \mathbf{x}_{\perp}, \tag{3.13}$$

and the second term is

$$\begin{aligned} B^2(t + \mathbf{x}_{\parallel}) &= (\cosh\phi - \hat{\mathbf{v}}\sinh\phi)(t + \mathbf{x}_{\parallel}) \\ &= (\cosh\phi\, t - \sinh\phi\, \hat{\mathbf{v}} \cdot \mathbf{x}_{\parallel}) + (\cosh\phi\, \mathbf{x}_{\parallel} - \sinh\phi\, \hat{\mathbf{v}}t), \end{aligned} \tag{3.14}$$

where $\mathbf{x}_{\parallel}$ and $\mathbf{x}_{\perp}$ are the parallel and orthogonal components of $\mathbf{x}$ with respect to $\hat{\mathbf{v}}$ and we have used that $\hat{\mathbf{v}}\mathbf{x}_{\parallel} = \hat{\mathbf{v}} \cdot \mathbf{x}_{\parallel}$. Hence, as opposed to a rotation, we see that the boost B is a hyperbolic rotation that mixes the time axis $\mathbf{e}_0$ and the parallel direction $\mathbf{x}_{\parallel}$, but leaves the perpendicular component $\mathbf{x}_{\perp}$ invariant.

Theorem 3.9. *Every $L \in \$\mathrm{pin}^+(1,3)$ can be uniquely decomposed as $L = BR = RB'$, where B, B' are boosts and R is a rotation.*

Proof. Call $A := LL^{\dagger}$. Then

$$\bar{A}A = \overline{(LL^{\dagger})}(LL^{\dagger}) = \bar{L}^{\dagger}\bar{L}LL^{\dagger} = (L\bar{L})^{\dagger} = 1$$

and $A^{\dagger} = (L^{\dagger})^{\dagger}L^{\dagger} = LL^{\dagger} = A$. Moreover, if we write $L = \alpha + \mathbf{v} + i\mathbf{w} + \mathrm{i}\beta$, then $L^{\dagger} = \alpha + \mathbf{v} - i\mathbf{w} - \mathrm{i}\beta$ and

$$(LL^{\dagger})_S = ((\alpha + \mathbf{v})^2 - \mathrm{i}^2(\beta + \mathbf{w})^2)_S = \alpha^2 + \mathbf{v}^2 + \mathbf{w}^2 + \beta^2 > 0,$$

since otherwise $L = 0$, but L must satisfy $\bar{L}L = 1$. Hence, A is a boost and, after reparametrizing $\phi \mapsto 2\phi$, we can write $A = e^{-\hat{\mathbf{v}}\phi}$ as in (3.12). If we define $B := A^{1/2} = e^{-\hat{\mathbf{v}}\phi/2}$ as the (positive) square root of A, we have that B is also a boost. Define now, $R := \bar{B}L$, so that $\bar{R}R = \bar{L}B\bar{B}L = 1$ and

$$R^\dagger R = L^\dagger \bar{B}^\dagger \bar{B} L = L^\dagger \bar{B}^2 L = L^\dagger \overline{LL^\dagger} L = L^\dagger \bar{L}^\dagger \bar{L} L = (\bar{L}L)^\dagger = 1.$$

Hence, R is a rotation and $L = BR$. For the other decomposition, simply define $B' := R^\dagger B R$, which satisfies

$$\begin{aligned}
\bar{B}'B' &= \bar{R}\bar{B}\bar{R}^\dagger R^\dagger B R = \bar{R}\bar{B}BR = \bar{R}R = 1, \\
(B')^\dagger &= R^\dagger B^\dagger (R^\dagger)^\dagger = R^\dagger B R = B',
\end{aligned}$$

so it is a boost and $L = BR = RR^\dagger BR = RB'$. For uniqueness assume that $L = B_1 R_1 = BR$, which implies $(B_1)^2 = LL^\dagger = B^2$. Since $B_1 = e^{-\hat{\mathbf{v}}_1\phi_1/2}$ and $B = e^{-\hat{\mathbf{v}}\phi/2}$ with $\phi_1, \phi \geq 0$, $B_1^2 = B^2$ implies $B_1 = B$ and $R_1 = \bar{B}_1 L = \bar{B}L = R$. The uniqueness of the other decomposition is proven analogously. □

Corollary 3.10. *Denoting the set of boosts by $\mathfrak{B}$, we have the following homeomorphisms:*

$$\$\mathrm{pin}^+(1,3) \cong \mathfrak{B} \times \mathrm{Spin}(3) \cong \mathbb{R}^3 \times \mathbb{S}^3.$$

Therefore, $\$\mathrm{pin}^+(1,3)$ is simply connected and, thus, the universal cover the of restricted Lorentz group $\mathrm{SO}^+(1,3)$.

Proof. The map $\$\mathrm{pin}^+(1,3) \to \mathfrak{B} \times \mathrm{Spin}(3)$, $L = BR \mapsto (B, R)$ is bijective due to the uniqueness of the decomposition. Moreover, since Clifford multiplication, reversion, Clifford conjugation and the exponential are continuous maps, we conclude that

$$L \mapsto LL^\dagger = e^{-\hat{\mathbf{v}}\phi} \in \mathfrak{B} \mapsto B = (LL^\dagger)^{1/2} = e^{-\hat{\mathbf{v}}\frac{\phi}{2}} \in \mathfrak{B},$$

$L \mapsto R = \bar{B}L$ and the inverse $(B, R) \mapsto L = BR$ are also continuous, obtaining the first homeomorphism. For the second one, note that

$$\mathfrak{B} = \{B \in \$\mathrm{pin}^+(1,3) : B^\dagger = B, (B)_S > 0\} = \{\alpha - \mathbf{v} \in \mathbb{P}_{1,3} : \alpha^2 - |\mathbf{v}|^2 = 1, \alpha > 0\}$$

is homeomorphic to $\mathbb{R}^3$ via $\mathbf{v} \in \mathbb{R}^3 \mapsto \sqrt{1 + |\mathbf{v}|^2} - \mathbf{v} \in \mathfrak{B}$. The result now follows from (3.9). □

The name *boost* for a hyperbolic rotation comes from the fact that it relates the physical quantities seen in a frame S, which is taken to be at rest, with those seen in a "boosted" frame S' that moves at a constant velocity $\mathbf{v}$ with respect to S. Indeed, if we use natural units ($c = 1$), then the spacetime position of a particle at rest in S is given by the paravector $x = \tau + \mathbf{0} = \tau$, while its spacetime position as viewed from S' is $x' = t' + \mathbf{x}'$, with $\mathbf{x}' = -\mathbf{v}t'$. Both quantities will be related by a Lorentz transformation $L \in \$\mathrm{pin}^+(1,3)$, so that

$$t' + \mathbf{x}' = L(t + \mathbf{0})L^\dagger = tLL^\dagger \implies LL^\dagger = \frac{t'}{t}(1 - \mathbf{v}).$$

On the other hand, using Theorem 3.9 we can write $L = BR$ with B a boost and R a rotation. Therefore,

$$LL^\dagger = B^2 = \frac{t'}{\tau}(1 - \mathbf{v}),$$

and using $\bar{B}B = 1$, we obtain

$$1 = \bar{B}^2 B^2 = \left(\frac{t'}{\tau}\right)^2 (1 - \mathbf{v}^2) \implies \frac{t'}{\tau} = (1 - v^2)^{-\frac{1}{2}} =: \gamma = \gamma(v),$$

where $v = |\mathbf{v}|$. Then, $B^2 = \gamma - \gamma v \hat{\mathbf{v}}$ and $1 = \gamma^2 - (\gamma v)^2$. The speed limit $v \leq c = 1$ implies that $\gamma \geq 1$, so we can write $\gamma = \cosh \phi$ and $\gamma v = \sinh \phi$ with $\phi \geq 0$. Hence, taking the square root of B^2, we can write

$$B = \cosh \frac{\phi}{2} - \hat{\mathbf{v}} \sinh \frac{\phi}{2} = e^{-\hat{\mathbf{v}}\frac{\phi}{2}} \quad \text{with} \quad v = \tanh \phi,$$

where the hyperbolic angle $\phi \geq 0$ is called the *rapidity*. In addition, the transformations (3.13) and (3.14) can now be written as

$$t' = \gamma(t - \mathbf{v} \cdot \mathbf{x}_{\parallel}), \quad \mathbf{x}'_{\parallel} = \gamma(\mathbf{x}_{\parallel} - \mathbf{v}t), \quad \mathbf{x}'_{\perp} = \mathbf{x}_{\perp}.$$

In particular, for a boost of velocity $\mathbf{v} = v\mathbf{e}_1$, the matrix ρ_B can be written as

$$\rho_B = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}^+(1,3).$$

3.5 Maxwell's Equation(s)

We have seen so far how the geometric algebra $\mathbb{G}_3$ models Lorentz transformations in spacetime. Here, we briefly show that $\mathbb{G}_3$ can also be used to model electrodynamics in a natural and compact way. In particular, we rewrite the four Maxwell's equations as single equation in $\mathbb{G}_3$. For more applications of $\mathbb{G}_3$ to electrodynamics see [4].

Recall that if $\mathbf{E}$, $\mathbf{B}$, ρ and $\mathbf{j}$ denote, respectively, the electric and magnetic fields and the charge and current densities, then the four Maxwell's equations are

$$\begin{aligned} \nabla \cdot \mathbf{E} = \rho, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}. \end{aligned}$$

We can view all the elements formally within the algebra $\mathbb{G}_3$ by performing the substitution $\mathbf{x} = (x^1, x^2, x^3) \mapsto \mathbf{x} = x^k \mathbf{e}_k$, so, for example, $\nabla = \sum_k \mathbf{e}_k \partial_k$ with $\partial_k = \frac{\partial}{\partial x^k}$. We say "formally", because the operator $\nabla = \sum_k \mathbf{e}_k \partial_k$ is not actually an element of $\mathbb{G}_3$. We can also formally replace the cross products by their dual wedge products using Proposition 3.5 and rearrange the terms

$$\begin{aligned} \nabla \cdot \mathbf{E} = \rho, \quad \partial_t \mathbf{B} - \mathrm{i} \nabla \wedge \mathbf{E} = 0, \\ \nabla \cdot \mathbf{B} = 0, \quad \partial_t \mathbf{E} + \mathrm{i} \nabla \wedge \mathbf{B} = -\mathbf{j}. \end{aligned}$$

Next, we multiply the homogeneous equations by the pseudoscalar i and sum all four equations to obtain

$$(\nabla \cdot \mathbf{E}) + (\mathrm{i} \nabla \cdot \mathbf{B}) + (\mathrm{i} \partial_t \mathbf{B} + \nabla \wedge \mathbf{E}) + (\partial_t \mathbf{E} + \mathrm{i} \nabla \wedge \mathbf{B}) = \rho - \mathbf{j}.$$

Finally, we can bring the pseudoscalar inside the differential operators and (again, formally) use the fundamental identity of the geometric product

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

to obtain the result

$$\partial_t(\mathbf{E} + \mathrm{i}\mathbf{B}) + \nabla \cdot (\mathbf{E} + \mathrm{i}\mathbf{B}) + \nabla \wedge (\mathbf{E} + \mathrm{i}\mathbf{B}) = \partial_t(\mathbf{E} + \mathrm{i}\mathbf{B}) + \nabla(\mathbf{E} + \mathrm{i}\mathbf{B}) = (\partial_t + \nabla)(\mathbf{E} + \mathrm{i}\mathbf{B}) = \rho - \mathbf{j}.$$

We can write this equation in a more suggestive way by defining the *gradient paravector* $\partial := \partial_t - \nabla$,[12] the *Faraday parabivector* $\mathbf{F} := \mathbf{E} + \mathrm{i}\mathbf{B} \in \mathbb{G}_3^1 \oplus \mathbb{G}_3^2$, the *current paravector* $j := \rho + \mathbf{j}$ and applying (formally) the grade involution. The result is the single *Maxwell's equation*[13]

$$\tilde{\partial}\mathbf{F} = \tilde{j}.$$

3.6 Pauli matrices and spinors

We now turn our attention to the applications of the algebra $\mathbb{G}_3$ to quantum mechanics. Following [19], recall that the (free) Schrödinger equation is obtained from the non-relativistic energy of a particle with mass m, momentum $\mathbf{p}$ and potential energy $U = U(\mathbf{x})$ by replacing E and $\mathbf{p}$ with their quantum mechanical operators $E \equiv i\hbar\frac{\partial}{\partial t}$ and $\mathbf{p} \equiv -i\hbar\nabla$, respectively,

$$E = \frac{\mathbf{p}^2}{2m} + U \implies i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi. \tag{3.15}$$

The wave function ψ maps each spacetime point $(t, \mathbf{x})$ to a complex number $\psi(t, \mathbf{x})$, whose modulus square $|\psi(t, \mathbf{x})|^2$ gives the probability density of finding the particle at position $\mathbf{x}$ at a time t.

If we want to describe an electron in an electromagnetic field, the potential energy becomes $U = -eV$ and we need to substitute $\mathbf{p} = -i\hbar\nabla$ with the generalized momentum $\vec{\pi} := \mathbf{p} - e\mathbf{A}$, where e is the charge of the electron, V is the electric potential and $\mathbf{A}$ the magnetic vector potential. Thus, (3.15) becomes

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\vec{\pi}^2\psi - eV\psi. \tag{3.16}$$

The Stern-Gerlach experiment (1922), showed that a beam of silver atoms passing through an inhomogeneous magnetic field was split in two, suggesting that the electron had an intrinsic *spin* angular momentum, which could take two values: spin *up* or spin *down*. In 1927, Pauli accounted for the spin of the electron by introducing the *Pauli matrices*

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.17}$$

[12]The reason for the minus sign comes from using the inverse metric tensor $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ to raise the indices in $\partial = \mathbf{e}_\mu\partial^\mu = \eta^{\mu\nu}\mathbf{e}_\mu\partial_\nu$ (using Einstein's summation convention).

[13]From this equation, we can also recover the tensorial (covariant) form of Maxwell's equations, see [4, section 3.1., p. 90].

which satisfy the commutation and anticommutation relations

$$[\sigma_i, \sigma_j] := \sigma_i\sigma_j - \sigma_j\sigma_i = 2i\varepsilon_{ijk}\sigma_k \tag{3.18}$$
$$\{\sigma_i, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}_2, \tag{3.19}$$

where ε_{ijk} is the 3-dimensional Levi-Civita symbol. The above relations imply that if $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are two vectors, then

$$(\vec{\sigma}\cdot\mathbf{a})(\vec{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}\,\mathbb{1} + i\vec{\sigma}\cdot(\mathbf{a}\times\mathbf{b}), \tag{3.20}$$

where $\vec{\sigma}\cdot\mathbf{a} := \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3$ is called tha *Pauli vector* associated to $\mathbf{a} \in \mathbb{R}^3$.

In addition, unlike a usual vector, the operator $\vec{\pi} = -i\hbar\nabla - e\mathbf{A}$ has a non-zero cross-product with itself:

$$(\vec{\pi}\times\vec{\pi})\psi = \vec{\pi}\times(-i\hbar\nabla\psi - e\mathbf{A}\psi) = ie\hbar(\nabla\times(\mathbf{A}\psi) + \mathbf{A}\times\nabla\psi) =$$
$$ie\hbar((\nabla\times\mathbf{A})\psi - \mathbf{A}\times\nabla\psi + \mathbf{A}\times\nabla\psi) = ie\hbar\mathbf{B}\psi.$$

Pauli substituted $\vec{\pi}^2 = \vec{\pi}\cdot\vec{\pi}$ by $(\vec{\sigma}\cdot\vec{\pi})^2 = \vec{\pi}^2\mathbb{1} - e\hbar\,\vec{\sigma}\cdot\mathbf{B}$ in (3.15) obtaining the *Schrödinger-Pauli equation*:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}[\vec{\pi}^2 - e\hbar\,\vec{\sigma}\cdot\mathbf{B}]\psi - eV\psi, \tag{3.21}$$

where the term $\frac{e\hbar}{2m}\vec{\sigma}\cdot\mathbf{B}$ describes the coupling of the electron's spin to the magnetic field. Since $\vec{\sigma}\cdot\mathbf{B} \in \mathbb{C}^{2\times2}$, the wave function ψ which we operate upon must have now to components

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \in \mathbb{C}_2. \tag{3.22}$$

The above column vector is called a *Pauli spinor*.

The surprising fact is that the relations (3.19) are precisely the relations (3.2) satisfied by the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and (3.20) has the same form as the geometric product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{G}_3^1$

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b} = \mathbf{a}\cdot\mathbf{b} + \mathrm{i}(\mathbf{a}\times\mathbf{b}). \tag{3.23}$$

Therefore, the Pauli matrices are simply a matrix representation of the generators of the Clifford algebra $\mathbb{G}_3$, and we could describe Pauli's achievement by saying that he upgraded the quadratic space $\mathbb{R}^3$ (equipped with the dot product) to the algebra $\mathbb{G}_3$ (equipped with the geometric product (3.23)).

One important property about the Pauli matrices (3.17) is that any matrix in $\mathbb{C}^{2\times2}$ can be written as a linear combination of the form

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \frac{a+d}{2}\mathbb{1} + \frac{b+c}{2}\sigma_1 + i\frac{b-c}{2}\sigma_2 + \frac{a-d}{2}\sigma_3. \tag{3.24}$$

Therefore, the Pauli matrices generate $\mathbb{C}^{2\times2}$ as a *complex* algebra. On the other hand, we know that if we identify $Z(\mathbb{G}_3) \equiv \mathbb{C}$ via $a + \mathrm{i}b \simeq a + ib$, the algebra $\mathbb{G}_3$ can also be regarded as a complex algebra spanned by $\{\mathbf{e}_0 := 1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Theorem 3.11. *We obtain an isomorphism of complex algebras $\mathbb{G}_3 \cong \mathbb{C}^{2\times 2}$ mapping*[14]

$$1 \mapsto \underline{1} := \mathbb{1}, \quad \mathbf{e}_1 \mapsto \underline{\mathbf{e}_1} := \sigma_1, \quad \mathbf{e}_2 \mapsto \underline{\mathbf{e}_2} := \sigma_2, \quad \mathbf{e}_3 \mapsto \underline{\mathbf{e}_3} := \sigma_3, \tag{3.25}$$

and extending by (complex) linearity. Therefore, the isomorphism $\mathbb{G}_3 \to \mathbb{C}^{2\times 2}$ is given explicitly by

$$s = s_0 + s_1\mathbf{e}_1 + s_2\mathbf{e}_2 + s_3\mathbf{e}_3 \mapsto \underline{s} = \begin{pmatrix} s_0 + s_3 & s_1 - is_2 \\ s_1 + is_2 & s_0 - s_3 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and the inverse is given by (3.24) replacing $\mathbb{1} \leftrightarrow 1, \sigma_k \leftrightarrow \mathbf{e}_k$.

Proof. Since both $\mathbb{G}_3$ and $\mathbb{C}^{2\times 2}$ are four-dimensional when considered as complex algebras, it suffices to show that the map is an algebra homomorphism, which follows from the fact that the anticommutation relations (3.19) are precisely the relations (3.2) satisfied by the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. □

As a consequence, we have the following identifications

$$s \simeq \underline{s} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \tilde{s} \simeq \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix} = \mathrm{cof}(\underline{s})^*,$$
$$s^\dagger \simeq \begin{pmatrix} a^* & c^* \\ c^* & d^* \end{pmatrix} = (\underline{s}^t)^* = (\underline{s})^\dagger, \quad \bar{s} \simeq \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \mathrm{cof}(\underline{s})^t = \mathrm{adj}(\underline{s}),$$

where z^*, $\mathrm{cof}(A)$ and $\mathrm{adj}(A)$ denote the complex conjugate of $z \in \mathbb{C}$ and the cofactor and adjoint matrices of $A \in \mathbb{C}^{2\times 2}$, respectively. Note that Hermitian conjugation in $\mathbb{C}^{2\times 2}$ coincides with reversion in $\mathbb{G}_3$, so there is no problem in using the same symbol for both operations.

Theorem 3.12. *Restricting the above algebra isomorphism to the groups* $\mathrm{Spin}(3)$ *and* $\$\mathrm{pin}^+(1,3)$ *we obtain the following group isomorphisms*

$$\mathrm{Spin}(3) \cong \mathrm{SU}(2) := \{U \in \mathbb{C}^{2\times 2} : U^\dagger = U^{-1}, \det U = 1\},$$
$$\$\mathrm{pin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C}) := \{L \in \mathbb{C}^{2\times 2} : \det L = 1\}.$$

Proof. From the above identifications we have that $\underline{\bar{s}}\underline{s} = \mathrm{adj}(\underline{s})\underline{s} = \det(\underline{s})\mathbb{1}$ for all $s \in \mathbb{G}_3$. Therefore, if we recall equations (3.4) and (3.11), we obtain the desired result

$$\underline{\mathrm{Spin}(3)} = \{\underline{s} \in \mathbb{C}^{2\times 2} : \det(\underline{s})\mathbb{1} = \mathbb{1} = \underline{s}^\dagger\underline{s}\}, \quad \underline{\$\mathrm{pin}^+(1,3)} = \{\underline{s} \in \mathbb{C}^{2\times 2} : \det(\underline{s})\mathbb{1} = \mathbb{1}\}.$$

□

Now, in the Clifford algebra $\mathbb{G}_3$ the vector $\mathbf{B} \in \mathbb{R}^3 \equiv \mathbb{G}_3^1$ is rotated as $\mathbf{B} \mapsto \mathbf{B}' = R\mathbf{B}R^\dagger$ with $R \in \mathrm{SU}(2)$, so the corresponding vector $\vec{\sigma} \cdot \mathbf{B} \in \mathbb{C}^{2\times 2}$ will rotate as $\vec{\sigma} \cdot \mathbf{B} \mapsto \vec{\sigma} \cdot \mathbf{B}' = U(\vec{\sigma} \cdot \mathbf{B})U^\dagger$, where $U = \underline{R} \in \mathrm{SU}(2)$. If we want (3.21) to be covariant under rotations, we have to require that the Pauli spinor $\psi \in \mathbb{C}^2$ transforms as

$$\psi \mapsto \psi' = U\psi. \tag{3.26}$$

In that case, (3.21) is fulfilled if and only if

$$i\hbar\frac{\partial \psi'}{\partial t} = \frac{1}{2m}[\vec{\pi}^2 - e\hbar\,\vec{\sigma} \cdot \mathbf{B}']\psi' - eV\psi'.$$

[14] Do not confuse this underbar with the one used for quaternions at the end of section 3.2.1.

Therefore, just as a vector can be defined as a list of real numbers $x \in \mathbb{R}^3$ such that it transforms as

$$x \mapsto x' = Rx, \text{ with } R \in \mathrm{SO}(3),$$

we can also define a Pauli spinor as a list of complex numbers $\psi \in \mathbb{C}^2$ that transforms as

$$\psi \mapsto \psi' = U\psi, \text{ with } U \in \mathrm{SU}(2),$$

being $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$ the universal double cover of $\mathrm{SO}(3)$. Analogously, a spacetime vector is an element $x \in \mathbb{R}^{1,3}$ which transforms as

$$x \mapsto x' = \Lambda x, \text{ with } \Lambda \in \mathrm{SO}^+(1,3),$$

and a *Weyl spinor* is defined as a column $\psi \in \mathbb{C}^2$ which transforms as

$$\psi \mapsto \psi' = L\psi, \text{ with } L \in \mathrm{SL}(2,\mathbb{C}),$$

being $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}^+(1,3)$ the universal double cover of $\mathrm{SO}^+(1,3)$.

We can also represent a spinor $\psi \in \mathbb{C}^2$ within the algebra $\mathbb{G}_3$ by considering the following injection

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2 \mapsto \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}. \tag{3.27}$$

Matrices of the type (3.27) form a *left ideal* $S \subseteq \mathbb{C}^{2\times 2}$, i.e.,

$$M\psi \in S, \quad \psi + \psi' \in S$$

for all $M \in \mathbb{C}^{2\times 2}$ and $\psi, \psi' \in S$. Moreover, this left ideal is *minimal*, since it contains no other left ideal apart from itself and the $\{0\}$ ideal. Finally, noting that

$$S = \mathbb{C}^{2\times 2}\underline{P_3}, \quad \text{where} \quad \underline{P_3} = \frac{1}{2}(\mathbb{1} + \sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and taking into account the isomorphism $\mathbb{G}_3 \cong \mathbb{C}^{2\times 2}$ given by (3.25), we can view spinors as members of the minimal left ideal $\mathbb{G}_3 P_3$, with $P_3 = \frac{1}{2}(1 + \mathbf{e}_3)$.

3.7 $\$\mathrm{pin}^+(1,3)$ and the Dirac equation

We finish this chapter by showing how the Dirac equation emerges naturally from the treatment of Lorentz transformations using the group $\$\mathrm{pin}^+(1,3)$. Consider a free particle of mass m moving with (constant) four-momentum $p = mu$ in the lab frame. We can relate the particle's rest frame four-momentum m with the one observed in lab frame using a rotor $\Lambda \in \$\mathrm{pin}^+(1,3)$ via

$$p = \Lambda m \Lambda^\dagger = m\Lambda\Lambda^\dagger = mu. \tag{3.28}$$

Following W.E. Baylis ([5]), we will refer to such a rotor as an *eigenspinor*. The *eigen* part refers to the fact that it relates the rest frame of the particle (its *own* frame) to the lab frame. The *spinor* part refers to the following property of rotors: if $\Lambda \in \$\mathrm{pin}^+(1,3)$ is a (spacetime) rotor, it acts on a paravector $x \in \mathbb{P}_{1,3}$ (spacetime vector) as $x \mapsto \Lambda x \Lambda^\dagger$. However, if we apply another Lorentz transformation $L \in \$\mathrm{pin}^+(1,3)$, then the final paravector will be $L\Lambda x \Lambda^\dagger L^\dagger = (L\Lambda)x(L\Lambda)^\dagger$, so the rotor Λ changes as $\Lambda \mapsto L\Lambda$ under a Lorentz transformation $L \in \$\mathrm{pin}^+(1,3)$, whereas a spacetime vector changes as $x \mapsto \rho_L(x) = LxL^\dagger$ with $\rho_L \in \mathrm{SO}^+(1,3)$. Therefore, noting that $\$\mathrm{pin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})$, we see that Λ transforms as a Weyl spinor.

The eigenspinor may have a proper-time dependence $\Lambda = \Lambda(\tau)$ even though the four-velocity u is constant for a free particle. The restrictions $\bar{\Lambda}(\tau)\Lambda(\tau) = 1$ and $\Lambda(\tau)\Lambda^\dagger(\tau) = u$ imply that the *rest frame spacetime rotation rate* $\Omega_r := 2\bar{\Lambda}\dot{\Lambda}$, where $\dot{\Lambda} = \frac{d}{d\tau}\Lambda$, is an imaginary vector, i.e., $\Omega_r \in \mathbb{G}_3^2 = \star\mathbb{G}_3^1 = \mathrm{i}\mathbb{G}_3^1$. Indeed, using $\dot{\Lambda} = \frac{1}{2}\Lambda\Omega_r$, we have that

$$
\begin{aligned}
0 &= \frac{d}{d\tau}(\bar{\Lambda}(\tau)\Lambda(\tau)) = \dot{\bar{\Lambda}}\Lambda + \bar{\Lambda}\dot{\Lambda} = \frac{1}{2}(\bar{\Omega}_r + \Omega_r) = (\Omega_r)_S \\
0 &= \frac{d}{d\tau}(\Lambda(\tau)\Lambda(\tau)^\dagger) = \dot{\Lambda}\Lambda^\dagger + \Lambda\dot{\Lambda}^\dagger = \frac{1}{2}\Lambda(\Omega_r + \Omega_r^\dagger)\Lambda^\dagger \implies (\Omega_r)_\Re = 0.
\end{aligned}
$$

Hence, we can write $\Omega_r = -\mathrm{i}\mathbf{e}\omega_r$ for certain $\mathbf{e} \in \mathbb{G}_3^1$ and $\omega_r > 0$. Since the particle is free, we will assume that Ω_r is constant. Thus, we can integrate the equation $\dot{\Lambda} = \frac{1}{2}\Lambda\Omega_r$ to obtain

$$\Lambda(\tau) = \Lambda(0)e^{-\mathrm{i}\mathbf{e}\omega_r\tau/2}.$$

Moreover, since (3.28) is invariant under any rotation of the rest frame $\Lambda \mapsto \Lambda R_r$ with $R_r \in \mathrm{Spin}(3)$, we can choose R_r such that $R_r\mathbf{e}_3R_r^\dagger = \mathbf{e}$, so $\mathbf{e}R_r = R_r\mathbf{e}_3$ and

$$\Lambda \mapsto \Psi_p := \Lambda R_r = \Lambda(0)e^{-\mathrm{i}\mathbf{e}\omega_r\tau/2}R_r = u_p e^{-\mathrm{i}\mathbf{e}_3\omega_r\tau/2},$$

where $u_p := \Lambda(0)R_r \in \$\mathrm{pin}^+(1,3)$ is a constant rotor. Therefore, we can write (3.28) as

$$p = m\Psi_p\Psi_p^\dagger \iff p\widetilde{\Psi}_p = m\Psi_p \iff \tilde{p}\Psi_p = m\widetilde{\Psi}_p \tag{3.29}$$

where $\widetilde{\Psi}_p = \overline{\Psi}_p^\dagger$. On the other hand, since the proper time τ is Lorentz-invariant and $u = 1$, $x = \tau$ in the rest frame, then

$$\tau = u_\mu x^\mu = (\bar{u}x)_S = (\frac{\bar{p}}{m}x)_S = p_\mu x^\mu/m.$$

Using the above equation, we can express the eigenspinor Ψ_p as a function of the spacetime position x

$$\Psi_p(x) = u_p e^{-\mathrm{i}\mathbf{e}_3\frac{\omega_r}{2m}(\bar{p}x)_S},$$

and Ψ_p satisfies

$$\partial_\mu\Psi_p = -\mathrm{i}\frac{\omega_r}{2m}p_\mu\Psi_p\mathbf{e}_3 \iff p_\mu\Psi_p = \mathrm{i}\frac{2m}{\omega_r}\partial_\mu\Psi_p\mathbf{e}_3.$$

Therefore, if we assume that the frequency ω_r is such that $\hbar = \frac{2m}{\omega_r}$, then (3.29) can be written as

$$\tilde{p}\Psi_p = \mathrm{i}\hbar\tilde{\partial}\Psi_p\mathbf{e}_3 = m\widetilde{\Psi}_p, \quad \Psi_p = u_p e^{-\frac{\mathrm{i}}{\hbar}(\bar{p}x)_S\mathbf{e}_3},\ u_p \in \$\mathrm{pin}^+(1,3). \tag{3.30}$$

We note that except for the factor of $\mathbf{e}_3$, the left equation in (3.31) has the form of the free Dirac equation, and the eigenspinor Ψ_p has the form of a plane wave of momentum p. Therefore, we will refer to

$$\mathrm{i}\hbar\tilde{\partial}\Psi\mathbf{e}_3 = m\tilde{\Psi},\ \text{where } \Psi = \Psi(x) \in \mathbb{G}_3 \tag{3.31}$$

as the *Dirac-Baylis equation*,[15] and we have that the eigenspinor

$$\Psi_p = u_p e^{-\frac{i}{\hbar}\langle \bar{p}x\rangle_S \mathbf{e}_3} \in \$\mathrm{pin}^+(1,3), \tag{3.32}$$

is a solution for any four-momentum p such that $\bar{p}p = m^2$.

We will obtain the usual form of the Dirac equation by projecting (3.31) onto the minimal left ideal $\mathbb{G}_3 P_3$. To show that, we use that

$$1 = P_3 + \tilde{P}_3, \text{ where } P_3 = \frac{1}{2}(1 + \mathbf{e}_3).$$

Since $P_3\tilde{P}_3 = \frac{1}{2}(1 - \mathbf{e}_3^2) = 0$, we have that $\mathbb{G}_3 = \mathbb{G}_3 P_3 \oplus \mathbb{G}_3 \tilde{P}_3$ and the ideals $\mathbb{G}_3 P_3$ and $\mathbb{G}_3 \tilde{P}_3$ are (linearly) independent of each other. Hence, (3.31) is fulfilled if and only if the two equations

$$i\hbar\tilde{\partial}\Psi P_3 = m\widetilde{\Psi} P_3, \quad -i\hbar\tilde{\partial}\Psi \tilde{P}_3 = m\widetilde{\Psi}\tilde{P}_3$$

are fulfilled, where we have used that $\mathbf{e}_3 P_3 = P_3$ and $\mathbf{e}_3\tilde{P}_3 = -\tilde{P}_3$. The second equation can be sent to the left ideal $\mathbb{G}_3 P_3$ by applying the grade involution, so that

$$i\hbar\tilde{\partial}\Psi P_3 = m\widetilde{\Psi} P_3, \quad i\hbar\partial\widetilde{\Psi} P_3 = m\Psi P_3.$$

This two equations can be combined into the matrix equation

$$\begin{pmatrix} 0 & i\hbar\tilde{\partial} \\ i\hbar\partial & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\Psi} P_3 \\ \Psi P_3 \end{pmatrix} = m \begin{pmatrix} \widetilde{\Psi} P_3 \\ \Psi P_3 \end{pmatrix}.$$

If we write

$$\begin{pmatrix} 0 & i\hbar\tilde{\partial} \\ i\hbar\partial & 0 \end{pmatrix} = i\hbar\gamma^\mu\partial_\mu, \text{ with } \gamma^0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^k := \begin{pmatrix} 0 & \mathbf{e}_k \\ -\mathbf{e}_k & 0 \end{pmatrix} \in (\mathbb{G}_3)^{2\times 2}, \text{ and } \psi = \begin{pmatrix} \widetilde{\Psi} P_3 \\ \Psi P_3 \end{pmatrix}$$

we obtain that (3.31) is equivalent to

$$i\hbar\gamma^\mu\partial_\mu\psi = m\psi,$$

which, except by taking the matrix representation $\mathbb{G}_3 \cong \mathbb{C}^{2\times 2}$ of Theorem 3.11, is the Dirac equation in its usual form. Moreover, we see that $\psi_L = \tilde{\Psi} P_3$ and $\psi_R = \Psi P_3$ correspond to left and right chiral Weyl spinors, respectively. In particular, if $\Psi_p \in \$\mathrm{pin}^+(1,3)$ is the eigenspinor (3.32), then

$$\psi_p = \begin{pmatrix} \tilde{\Psi}_p P_3 \\ \Psi_p P_3 \end{pmatrix} = \begin{pmatrix} u_p^L \\ u_p^R \end{pmatrix} e^{-\frac{i}{\hbar}p_\mu x^\mu}, \tag{3.33}$$

where $u_p^L = \tilde{u}_p P_3$ and $u_p^R = u_p P_3$. Taking their matrix representations

$$u_p \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}), \quad u_p^L \simeq \begin{pmatrix} d^* & 0 \\ -b^* & 0 \end{pmatrix} \simeq \begin{pmatrix} d^* \\ -b^* \end{pmatrix} \in \mathbb{C}^2, \quad u_p^R \simeq \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \simeq \begin{pmatrix} a \\ c \end{pmatrix} \in \mathbb{C}^2$$

we see that (3.33) represents a positive energy momentum eigenstate of the Dirac equation (see [15, Section 2.4.2]).

We have thus derived the Dirac equation starting from the Lorentz transformation (3.28), revealing a close relationship between eigenspinors $\Lambda \in \$\mathrm{pin}^+(1,3)$, which transform the dynamical quantities from the rest frame of a classical particle to the lab frame, and the usual Dirac bispinors $\psi \in \mathbb{C}^4$.

[15] The reason for this name is that I first found this equation in [5].

Conclusions

We have seen that, starting from the quadratic space $\mathbb{R}^{p,m,z} = (\mathbb{R}^n, q)$, the simple axiom

$$v^2 = v \cdot v = q(v)$$

is enough to construct the full Clifford algebra $\mathbb{G}_{p,m,z}$ and its geometric product. The elements of this algebra may well be called "geometric numbers", since they extend the real numbers by incorporating vectors (oriented line segments) and their products (k-vectors), which represent oriented hypervolumes in $\mathbb{R}^{p,m,z}$. Using the geometric product and the involutions in the non-degenerate Clifford algebra $\mathbb{G}_{p,m}$, we have constructed the groups $\mathrm{Pin}_{p,m}$, $\mathrm{Spin}_{p,m}$ and $\mathrm{Spin}^+_{p,m}$, which allow us to perform orthogonal transformations in $\mathbb{R}^{p,m}$ via the group homomorphism $\rho : (\mathrm{S})\mathrm{Pin}^{(+)}_{p,m} \to (\mathrm{S})\mathrm{O}^{(+)}_{p,m}$.

The Clifford algebra formalism helped us address the objectives in our TFG proposal efficiently, and even generalize the Euclidean case results to arbitrary non-degenerate signatures. For example, the proofs of the Cartan-Dieudonné theorem and the path-connectedness of $\mathrm{Spin}^+_{p,m}$ given here (based on [20]), are made entirely within $\mathbb{G}_{p,m}$ and take advantage of its properties. These results imply, in turn, the surjectivity of the above homomorphism, ρ, and the connectedness of the group $\mathrm{SO}^+_{p,m}$, of which the rotation group $\mathrm{SO}(n)$ is a particular case.

Moreover, in dimension 3 the situation is particularly nice. The Clifford algebra $\mathbb{G}_3$ lets us model both physical space $\mathbb{R}^3$ and Minkowski spacetime $\mathbb{R}^{1,3}$, as well as rotations and boosts using the groups $\mathrm{Spin}(3)$ and $\$\mathrm{pin}^+(1,3) \cong \mathrm{Spin}^+(1,3)$. The simple form of these groups allows us to show that they are simply connected and, thus, the universal covers of the rotation group $\mathrm{SO}(3)$ and the (restricted) Lorentz group $\mathrm{SO}^+(1,3)$. However, in order to generalize this result to arbitrary dimensions we had to go beyond the Clifford algebra formalism, and use three important facts about Lie groups taken from [7] and [29], which are not proven here.

Apart from its applications to the geometry of $\mathbb{R}^3$ and special relativity, we have also rewritten the four Maxwell's equations as a single equation in $\mathbb{G}_3$. This result indicates that working with the geometric product, instead of the usual dot and cross products (which can be recovered from the former by splitting it into symmetric and antisymmetric parts), might simplify other physical equations.

Finally, we have also seen some applications of the algebra $\mathbb{G}_3$ to quantum mechanics. The Pauli matrices give us an isomorphism of algebras $\mathbb{G}_3 \cong \mathbb{C}^{2\times 2}$, which allows us to model column spinors $\psi \in \mathbb{C}^2$ as members of the minimal left ideal $\mathbb{G}_3 P_3$, where $P_3 = \frac{1}{2}(1 + \mathbf{e}_3)$. In addition, the Pauli matrices isomorphism restricts to the group isomorphisms

$$\mathrm{Spin}(3) \cong \mathrm{SU}(2) \quad \text{and} \quad \$\mathrm{pin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C}).$$

In addition, we have derived the Dirac equation starting from the representation of Lorentz transformations using the Spin group $\$\mathrm{pin}^+(1,3)$, obtaining a correspondence between momentum eigenstates $\psi_p = u_p e^{-\frac{i}{\hbar} p_\mu x^\mu} \in \mathbb{C}^4$ of the Dirac equation and certain elements of the group $\$\mathrm{pin}^+(1,3)$.

The above results let us conclude that Clifford algebras provide, especially through the use of their Spin group(s), a natural language to model physics and geometry, often simplifying equations and highlighting their geometric content.

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Appendices

A Quadratic spaces

In this section we follow [16]. Unless otherwise stated, V will denote a real n-dimensional vector space.

Definition A.1. *A metric on a vector space V is a 2-covariant symmetric tensor, that is, a bilinear map $g : V \times V \to \mathbb{R}$ that satisfies $g(u,v) = g(v,u)$ $\forall u,v \in V$.*

Definition A.2. *A map $q : V \to \mathbb{R}$ is called a quadratic form on V if there exists a metric g on V such that $q(v) = g(v,v)$ $\forall v \in V$.*

To every metric g on V there is an associated quadratic form given by $q_g(v) := g(v,v)$ $\forall v \in V$. More importantly, the converse is also true.

Proposition A.3. *Given a quadratic form q on V, there is a unique metric g_q such that $q(v) = g_q(v,v)$ $\forall v \in V$.*

Proof. By definition, there exists a metric g on V such that $q(v) = g(v,v)$. Then

$$q(u+v) = g(u+v,u+v) = g(u,u) + g(u,v) + g(v,u) + g(v,v) = q(u) + q(v) + 2g(u,v).$$

Therefore

$$g(u,v) = \frac{1}{2}(q(u+v) - q(u) - q(v)) \tag{A.1}$$

and q determines g uniquely, so $g_q = g$. □

Equation A.1 is called the *polarization formula* and the metric g_q the *polarization* of q. Thus, equipping V with a metric g is equivalent to equipping it with its associated quadratic form q and vice versa. We will call the space (V,g) a *quadratic space*.

Definition A.4. *Let (V,g) be a quadratic space, we will say that g is*

(i) positive (negative) definite if $q(v) > 0$ ($q(v) < 0$) $\forall v \in V \setminus \{0\}$,

(ii) positive (negative) semidefinite if $q(v) \geq 0$ ($q(v) \leq 0$) $\forall v \in V \setminus \{0\}$,

(iii) indefinite if it is neither positive nor negative semidefinite,

(iv) non-degenerate if $g(u,v) = 0$ $\forall v \in V \implies u = 0$. Otherwise, we will say that g is degenerate, and the subspace $N = \{u \in V : g(u,v) = 0\ \forall v \in V\}$ will be called the radical of g.

The same terminology applies to the associated quadratic form q.

Definition A.5. *Let $e_1, ..., e_n$ be a basis of V and call $g_{ij} := g(e_i, e_j)$. The rank of g is defined as the rank of the matrix $(g_{ij})_{1\leq i,j\leq n}$, which is invariant under a change of basis. Then, g is non-degenerate if and only if the matrix $(g_{ij})_{1\leq i,j\leq n}$ is regular, i.e., it has rank n.*

Definition A.6. *If (V,g) is a quadratic space, we define the length or norm of a vector $v \in V$ as $|v| := \sqrt{|q(v)|}$. When $|v| = 1$, we say that it is a unit vector.*

Definition A.7. *Let (V, g) be a quadratic space. We say that two vectors $u, v \in V$ are orthogonal if $g(u, v) = 0$ and denote it by $u \perp v$. Similarly, for $A, B \subseteq V$ we say that they are orthogonal if $a \perp b$ $\forall a \in A, b \in B$ and denote it by $A \perp B$. We also define the orthogonal complement of $A \subseteq V$ as*

$$A^{\perp} := \{v \in V : v \perp a \ \forall a \in A\}.$$

Moreover, we say that a subspace $W \leq V$ is non-degenerate if $q\big|_W$ is non-degenerate, which is equivalent to having $W \cap W^{\perp} = \{0\}$.

Definition A.8. *A basis $e_1, ..., e_n$ of the quadratic space (V, g) is said to be orthogonal if $e_i \perp e_j$ $\forall i \neq j$. If, in addition, all the basis vectors are also unit vectors we say that the basis is orthonormal (o.n.).*

Proposition A.9. *Let (V, g) be a non-degenerate quadratic space. If $W \leq V$, then*

(i) $\dim W + \dim W^{\perp} = \dim V$,

(ii) $(W^{\perp})^{\perp} = W$,

(iii) $V = W + W^{\perp} \iff W \cap W^{\perp} = \{0\}$.

Proof. (i) Let $e_1, ..., e_k, e_{k+1}, ..., e_n$ be a basis of V where $e_1, ..., e_k$ is a basis of W. If $v = \sum v^i e_i$, then

$$v \in W^{\perp} \iff g(v, e_i) = 0 \text{ for } i = 1, ..., k \iff \sum g_{ij} v^j = 0 \text{ for } i = 1, ..., k.$$

Since g is non-degenerate, the matrix $(g_{ij})^{1 \leq i \leq k}_{1 \leq j \leq n}$ has rank k. Therefore, $W^{\perp}$ is the solution space of a system of k independent linear equations with n variables, so $\dim W^{\perp} = n - k$.

(ii) The inclusion $(W^{\perp})^{\perp} \supseteq W$ follows from the fact that $u \perp v \iff v \perp u \quad \forall u, v \in V$. By (i), $\dim W + \dim W^{\perp} = V = \dim W^{\perp} + \dim(W^{\perp})^{\perp}$, so $\dim W = \dim(W^{\perp})^{\perp}$ and we obtain (ii).

(iii) The result follows from (i) and the fact that $\dim W + \dim W^{\perp} = \dim(W + W^{\perp}) + \dim(W \cap W^{\perp})$.

□

As a consequence of the previous proposition we will prove the following important theorem, which generalizes the notion of an orthonormal basis to arbitrary quadratic spaces.

Theorem A.10 (Sylvester's Law of Inertia)**.** *Let (V, g) be a quadratic space. There exists a basis $e_1, ..., e_n$ of V and integers $p, m, z \in \mathbb{N}_0$ satisfying*

$$\begin{aligned} g(e_i, e_j) &= 0 \textit{ for } i \neq j, \\ q(e_i) &= 1 \textit{ for } 1 \leq i \leq p, \\ q(e_i) &= -1 \textit{ for } p + 1 \leq i \leq p + m = r, \\ q(e_i) &= 0 \textit{ for } r + 1 \leq i \leq r + z = n. \end{aligned}$$

Such a basis will be called a Sylvester basis of (V, g). In the case where g is non-degenerate, i.e., when $r = p + m = n$, a Sylvester basis is an orthonormal basis. Moreover, the integers (p, m, z) do not depend on the chosen Sylvester basis.

Proof. We will prove the first part of theorem in two steps.

(i) **Regular case.**

We prove it by induction. If $\dim V = 0$ the result is vacuously true. Assume now that $\dim V = n > 0$ and that the result is true for spaces of dimension $k < n$. Since g is non-degenerate there exists $v \in V$ such that $q(v) \neq 0$, so $W = \text{span}_{\mathbb{R}}(\{v\})$ is a non-degenerate subspace of dimension 1 $\implies$ $W \cap W^{\perp} = \{0\} = W^{\perp} \cap (W^{\perp})^{\perp}$, where in the second equation we have used part (ii) of the above proposition. Therefore $W^{\perp}$ is also a non-degenerate subspace of V. On the other hand, by part (i) of Proposition A.9 we have that $W^{\perp}$ has dimension $n-1$. Using the induction hypothesis we deduce the existence of an o.n. basis $e_1, ..., e_{n-1}$ of $W^{\perp}$. Finally, $e_1, ..., e_{n-1}, v/|v|$ constitutes an o.n. basis of V.

(ii) **General case.**

From g we can define a non-degenerate metric $\overline{g}$ on the quotient space V/N setting $\overline{g}([u],[v]) := g(u,v)$ $\forall u, v \in V$. Using case 1 above, we obtain an o.n. basis $[e_1], ..., [e_r]$ of V/N and choose a basis $e_{r+1}, ..., e_{r+z}$ of N. The vectors $e_1, ..., e_r, e_{r+1}, ..., e_{r+z}$ constitute a Sylvester basis for V.

To prove the second part of the theorem, consider two Sylvester bases $e_1, ..., e_n$ and $e'_1, ..., e'_n$ of V, with associated integers (p, m, z) and (p', m', z'), respectively. We set $U := \text{span}\{e_1, ..., e_l\}$ and $W := \text{span}\{e'_{l'+1}, ..., e'_n\}$. The restriction of g to U is positive definite, while its restriction to W is negative semidefinite, so $U \cap W = \{0\}$. Therefore

$$p + (n - p') = \dim U + \dim W = \dim(U + W) \leq \dim V = n$$

which implies $0 \leq p \leq p' \leq 0$. Thus, $p = p'$ and since the rank of g is an invariant quantity, we also get $m = m'$ from $p + m = r = p' + m'$ and $z = z'$ from $p + m + z = n = p' + m' + z'$.

□

Definition A.11. *The triple (p, m, z) is called the signature of g (or q). If $e_1, ..., e_n$ is a Sylvester basis of V and we define $\varepsilon_i := q(e_i)$ for $i = 1, ..., n$, one can also write the signature of g as the list* $(\varepsilon_1, ..., \varepsilon_n) = (\underbrace{+, ..., +}_{l \text{ times}}, \underbrace{-, ..., -}_{m \text{ times}}, \underbrace{0, ..., 0}_{z \text{ times}})$.

B Covering spaces

The following results are known from the Topology II course. Therefore, we refer to [23] for their proofs, except for Proposition B.7 and Lemma B.8, which are proven explicitly. Some of the definitions are taken from [11].

In the following, X and $\tilde{X}$ will denote two topological spaces.

Definition B.1. *Let $p : \tilde{X} \to X$ be a surjective and continuous map. An open subset $U \subseteq X$ is said to be **evenly covered** by p if $p^{-1}(U) = \bigsqcup_\alpha V_\alpha$, where each set $V_\alpha \subseteq \tilde{X}$ is open in $\tilde{X}$ and $p|_{V_\alpha} : V_\alpha \to U$ is an homeomorphism.*

Definition B.2. *Let $p : \tilde{X} \to X$ be surjective and continuous. We say that p is a **covering map** and $\tilde{X}$ a **covering space** of X if every point $x \in X$ has an open neighborhood $U \subseteq X$ that is evenly covered by p.*

Definition B.3. *A covering map $p : \tilde{X} \to X$ is said to be **trivial** if X itself is evenly covered (i.e., $\tilde{X} = \bigsqcup_\alpha V_\alpha$ where each V_α is homeomorphic to X) and **non-trivial** otherwise.*

Definition B.4. *When the cardinal $|p^{-1}(x)|$ does not depend on the point $x \in X$ it is called the **number of sheets** of p. In addition, if $|p^{-1}(x)| = n \in \mathbb{N}$ is finite, we say that p is an **n-covering** or **n-sheeted covering**.*

Definition B.5. *Let G be a group and X a topological space. An **action** of G on X (on the left) is a map $\alpha : G \times X \to X$, $(g, x) \mapsto g \cdot x := \alpha(g, x)$ such that:*

(i) $e \cdot x = x$ *for all* $x \in X$*, where* e *denotes the identity element of* G*.*

(ii) $g \cdot (h \cdot x) = (gh) \cdot x$ *for all* $g, h \in G$ *and* $x \in X$*.*

(iii) For all $g \in G$*, the map* $l_g : X \to X$*,* $l_g(x) := g \cdot p$ *is an homeomorphism. In particular, from* (i) *and* (ii) *we have that* $l_e = Id_X$ *and* $(l_g)^{-1} = l_{g^{-1}}$*.*

The group G is then said to act on X (on the left).

Every action $\alpha : G \times X \to X$ induces an equivalence relation on X as follows

$$x \sim y \iff \exists g \in G : g \cdot x = y.$$

Given $x \in X$, the equivalence class $[x] = \{g \cdot x : g \in G\} = G \cdot x \subseteq X$ is called *the orbit* of x and the quotient space $X/G := X/\sim = \{[x] : x \in X\}$ is called the *space of orbits*. We always equip X/G with the quotient topology, so that the projection map $p : X \to X/G$ becomes continuous.

Definition B.6. *We say that an action $\alpha : G \times X \to X$ is **properly discontinuous** or that G **acts properly discontinuously** on X if every $x \in X$ has an open neighborhood $U \subseteq X$ such that $U \cap l_g(U) = \emptyset$ for all $g \in G \setminus \{e\}$.*

Proposition B.7. *Let X be a topological space and G a group which acts properly discontinuously on X. Then the projection map $p : X \to X/G$ is a covering map.*

Proof. Let $[x] \in X/G$ with $x \in X$. Since the action is properly discontinuous, there exists an open subset $U \subseteq X$ such that $x \in U$ and $l_f(U) \cap l_g(U) \neq \emptyset$ for all $f, g \in G$ with

$f \neq g$. Since the projection map $p : X \to X/G$ is an open map, we have that $p(U)$ is open in X/G with $[x] = p(x) \in p(U)$. It suffices to check that $p(U)$ is evenly covered. First, since l_f is an homeomorphism we have that $l_f(U)$ is open for all $f \in G$ and

$$p^{-1}(p(U)) = \bigcup_{x \in U} [x] = \bigcup_{x \in U} \{l_f(x) : f \in G\} = \bigcup_{f \in G} l_f(U) = \bigsqcup_{f \in G} l_f(U).$$

Now, we check that $p|_{l_f(u)} : l_f(U) \to p(U)$ is an homeomorphism. First of all, it is well-defined and surjective because $p(l_f(x)) = [l_f(x)] = [x] = p(x)$ for all $x \in X$. Next, we show that it is injective. Indeed, take $y_1 = l_f(x_1), y_2 = l_f(x_2) \in l_f(U)$ such that $p(y_1) = p(y_2)$. Using again that $[x_1] = p(l_f(x_1)) = p(l_f(x_2)) = [x_2]$, we conclude that there exists $g \in G$ such that $x_2 = l_g(x_1)$. Therefore, $l_g(U) \cap U \neq \emptyset$ and since the action is properly discontinuous we conclude that $l_g = Id_X$. Thus, $x_2 = x_1$ and $y_2 = y_1$. Finally, we have that $p|_{l_f(u)}$ is continuous and open, since it is the restriction of a continuous and open map to the open set $l_f(U) \subseteq X$. Hence, it is an homeomorphism and we are done. □

Lemma B.8. *Let X be a Hausdorff topological space and G a finite group acting on X. If for every $g \in G \setminus \{e\}$ the homeomorphism l_g has no fixed points, then the action is properly discontinuous.*

Proof. ej.5 c), relación 2, topo II. Fix $x \in X$ and take any $f \in G \setminus \{e\}$. Since, by hypothesis, $l_f \neq Id_X$ has no fixed points, $l_f(x) \neq x$. Therefore, using that X is Hausdorff, we can find open sets $U, V \subseteq X$ such that $x \in U$, $l_f(x) \in V$ and $U \cap V = \emptyset$. However, $x \in U_f := l_f^{-1}(V) \cap U \neq \emptyset$ and U_f is open (it is a finite intersection of open sets). Moreover, since $l_f(U_f) \subseteq V$ and $U_f \subseteq U$, we must have $l_f(U_f) \cap U_f = \emptyset$. If we repeat this procedure for every $f \in G \setminus \{e\}$, we can construct $W := \cap_{f \in G \setminus \{e\}} U_f$, which is open since it is a finite intersection of open sets (recall that G is finite by hypothesis). Finally, we have that $x \in W$ by construction and

$$l_g(W) \bigcap W \subseteq \Big(\bigcap_{f \in G \setminus \{e\}} l_g(U_f)\Big) \bigcap \Big(\bigcap_{f \in G \setminus \{e\}} U_f\Big) \subseteq l_g(U_g) \bigcap U_g = \emptyset$$

for all $g \in G \setminus \{e\}$, as we wanted to show. □

Let us also recall the relationship between covering maps and fundamental groups.

Proposition B.9. *If $f : X \to Y$ is continuous, then the map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ defined by $f([\alpha]) := [f \circ \alpha]$ is well defined and is a group homomorphism. In addition:*

(i) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $(g \circ f)_* = g_* \circ f_*$.

(ii) $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X, x_0)}$.

Proof. See the definition at the end of p. 333 and Theorem 52.4. on p. 334 in [23]. □

Proposition B.10 (Path lifting)**.** *Let $\rho : \tilde{X} \to X$ be a covering map and $\gamma : [0,1] \to X$ a (continuous) path in X. If $\gamma(0) = x_0 \in X$, then for any $\tilde{x}_0 \in \tilde{X}$ such that $\rho(\tilde{x}_0) = x_0$ there is a unique path $\tilde{\gamma} : [0,1] \to \tilde{X}$ such that $\tilde{\gamma}(0) = \tilde{x}_0$ and $\rho \circ \tilde{\gamma} = \gamma$.*

Proof. See [23, Lemma 54.1., p.342]. □

Theorem B.11 (Homotopy lifting)**.** *Let $\rho : \tilde{X} \to X$ be a covering map, $\alpha, \beta : [0,1] \to X$ two paths with $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1) = y_0$, and let $\tilde{\alpha}, \tilde{\beta} : [0,1] \to \tilde{X}$ be their respective liftings beginning at $\tilde{x}_0 \in \rho^{-1}(x_0)$. If α and β are path homotopic, then $\tilde{\alpha}(1) = \tilde{\beta}(1)$ and $\tilde{\alpha}, \tilde{\beta}$ are also path homotopic.*

Proof. See [23, Thm. 54.3, p. 344]. □

Definition B.12. *Let $\rho : \tilde{X} \to X$ be a covering map, $x_0 \in X$ and $\tilde{x}_0 \in \rho^{-1}(x_0)$. Given an element $[\alpha] \in \pi_1(X, x_0)$, let $\tilde{\alpha}$ be the (unique) lifting of α such that $\tilde{\alpha}(0) = \tilde{x}_0$. Then, the map $\Psi : \pi_1(X, x_0) \to \rho^{-1}(x_0)$, $[\alpha] \mapsto \Psi([\alpha]) := \tilde{\alpha}(1)$ is well-defined and is called the* ***lifting correspondence****.*

Theorem B.13. *Let $\rho : \tilde{X} \to X$ be a covering map, $x_0 \in X$ and $\tilde{x}_0 \in \rho^{-1}(x_0)$.*

(i) The homomorphism $\rho_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is a monomorphism.

(ii) Let $H = \rho_*(\pi_1(\tilde{X}, \tilde{x}_o))$. The lifting correspondence Ψ induces an injective map $\pi_1(X, x_0)/H \to \rho^{-1}(x_0)$, which is bijective if $\tilde{X}$ is path-connected.

Proof. See [23, Theorem 54.6, p. 346]. □