UNIVERSIDAD DE GRANADA

Escuela Técnica Superior de Ingeniería Informática
Dept. de Ciencias de la Computación e Inteligencia Artificial

Group Decision Making With Incomplete Fuzzy Preference Relations

Memoria de Tesis presentada por

D. Sergio Alonso Burgos

para optar al grado de Doctor en Informática

Granada Abril 2006

Editor: Editorial de la Universidad de Granada
Autor: Sergio Alonso Burgos
D.L.: Gr. 947 - 2006
ISBN: 84-338-3884-9

Escuela Técnica Superior de Ingeniería Informática

Dept. de Ciencias de la Computación e Inteligencia Artificial

Group Decision Making With Incomplete Fuzzy Preference Relations

Memoria de Tesis presentada por

D. Sergio Alonso Burgos

para optar al grado de Doctor en Informática

Dr. Enrique Herrera Viedma
Dr. Francisco Herrera Triguero
Dr. Francisco Chiclana Parrilla
Directores

Fdo: Dr. Enrique Herrera Viedma

Dr. Francisco Herrera Triguero

Dr. Francisco Chiclana Parrilla

Sergio Alonso Burgos

Granada Abril 2006

La memoria titulada **Group Decision Making With Incomplete Fuzzy Preference Relations**, que presenta **D. Sergio Alonso Burgos** para optar al grado de Doctor en Informática, ha sido realizada en el **Departamento de Ciencias de la Computación e Inteligencia Artificial** de la Universidad de Granada bajo la dirección de los doctores **Enrique Herrera Viedma**, **Francisco Herrera Triguero** y **Francisco Chiclana Parrilla**.

Dr. Enrique Herrera Viedma
Director

Dr. Francisco Herrera Triguero
Director

Dr. Francisco Chiclana Parrilla
Director

Sergio Alonso Burgos
Doctorando

Abril, 21 2006

Me gustaría agradecerles a EP^2 (Enrique, Paco y Paco) por su apoyo y ayuda durante estos últimos años, así como a todos los demás miembros del grupo SCI^2S, prestando especial atención a Antonio y a ambos Carlos.

También estoy muy agradecido al Dr. Robert John que amablemente me aceptó para realizar una estancia en su centro en la De Montfort University (UK).

Y finalmente, a mi familia, amigos y Noe por su apoyo incondicional en todo momento.

Contents

Figure Index

Introducción

Planteamiento

La *Toma de Decisiones* es el proceso cognitivo en el que se seleccionan la mejor alternativa (o alternativas) de un conjunto dado de alternativas distintas. La toma de decisiones comienza cuando necesitamos hacer algo pero no sabemos qué. Por lo tanto, la toma de decisiones es un proceso de la razón, que puede ser racional o irracional, y que puede estar basado tanto en suposiciones explícitas (que normalmente se presentan con las alternativas) o suposiciones tácitas (las que no se nombran de manera explícita y que ni siquiera tienen que ser totalmente comprendidas por el que toma la decisión).

Las situaciones de toma de decisiones son muy normales en la vida diaria de las personas: ejemplos típicos incluyen hacer la compra, decidir qué comer, o decidir qué o a quién votar en unas elecciones o en un referéndum.

Sin embargo, la toma de decisiones no solo ocurre para individuos aislados. Existen numerosos problemas de decisión que deben resolverse por un *grupo* de personas (usualmente *expertos*), que tienen que decidir conjuntamente que alternativa de entre todas las posibles es mejor o preferible en una situación concreta. A este tipo de toma de decisiones con múltiples individuos se le denomina *toma de decisiones en grupo* (o toma de decisiones multipersona). El hecho de que en un proceso de toma de decisiones estén involucradas diversas personas implica algunas complicaciones adicionales que deben ser resueltas antes de obtener una solución adecuada. Por ejemplo, las opiniones sobre las alternativas de los distintos individuos pueden diferir en gran medida, y por lo tanto, durante el proceso de decisión, es necesario llegar a alcanzar algún tipo de acuerdo (o *consenso*) entre los expertos antes de la propiamente dicha *selección* de la mejor o mejores alternativas.

La toma de decisiones en grupo a veces se analiza por separado como un *proceso* y un *resultado*. El proceso se refiere a las interacciones entre los individuos que lleva a la elección de una línea de acción determinada. El resultado es la consecuencia de esa elección. El separar el proceso y el resultado es conveniente porque ayuda a explicar que un buen proceso de toma de decisiones no tiene por que desembocar en un buen resultado, y un buen resultado no presupone que se ha llevado a cabo un buen proceso de decisión. Por ejemplo, en el ámbito empresarial, los coordinadores que esten interesados en tomar buenas decisiones deben poner énfasis en la aplicación de buenos procesos de toma de decisiones. Aunque dichos procesos no garanticen un buen resultado, pueden inclinar la balanza a favor de conseguir buenos resultados. De hecho, un aspecto crítico para la toma de decisiones en grupo es tener la posibilidad de converger en una elección particular.

Para modelar correctamente las situaciones de toma de decisiones en grupo son varios los aspectos que tenemos que tener en cuenta:

- *El formato de respresentación* que pueden usar los expertos para expresar sus opiniones y preferencias. Dicho formato de representación puede afectar en gran manera al proceso de decisión. Por ejemplo, algunos formatos de representación, como la *selección de un subconjunto de alternativas* o los *ordenes de preferencias de las alternativas* son modelos de representación simples que los expertos que no están familiarizados con ellos pueden aprender a usarlos de manera efectiva fácilmente. Sin embargo, su simplicidad implica también que la cantidad de información que puede modelarse con ellos y la granularidad de la misma es escasa. Por otro lado, otros modelos de representación de preferencias como las *relaciones de preferencia* ofrecen una mayor expresividad, y por lo tanto se puede modelar mucha más información (y más precisa) con ellos.

- *Falta de información.* Aunque siempre es deseable que los expertos que se enfrentan a un problema de decisión tengan un conocimiento exhaustivo y amplio sobre todas las alternativas, esto no siempre se cumple. Existen numerosos factores culturales y personales que pueden llevar a situaciones donde existe falta de información para tomar una decisión correctamente. Por ejemplo, los expertos pueden no estar familiarizados con todas las alternativas (lo que suele ocurrir si el conjunto de alternativas posibles es grande), o quizás los expertos no son capaces de discriminar suficientemente algunas alternativas similares.

- *Falta de consistencia* (contradicción) en las preferencias expresadas por los expertos. Aunque la diversidad de opiniones entre los distintos expertos para resolver un problema de decisión es típicamente recomendable (incluso cuando las opiniones son antagónicas) ya que esto lleva a la discusión y el estudio profundo del problema a resolver, la contradicción en las opiniones individuales de los expertos no es util normalmente. De hecho, en cualquier situación real, si una persona expresa opiniones inconsistentes (opiniones contradictorias), esa persona suele ser ignorada por el resto.

Por lo tanto, el estudio de todos estos aspectos es un punto crítico para desarrollar modelos y procesos de decisión en grupo.

Objetivos

El principal objetivo del trabajo desarrollado en esta tesis es el desarrollo de modelos de toma de decisiones en grupo con información incompleta, que permitan abordar tanto los problemas de consenso como los de selección de alternativas.

Este objetivo principal puede desglosarse en los siguientes puntos:

- Desarrollar un procedimiento iterativo para estima la información incompleta en las relaciones de preferencia difusas que se base en propiedades de transitividad de las relaciones de preferencia difusas.
- Desarrollar un operador de agregación de relaciones de preferencia difusas que se base en las medidas de consistencia comentadas anteriormente: El operador *Induced Ordered Weighted Averaging* basado en consistencia aditiva (AC-IOWA).
- Desarrollar un *proceso de selección* que usando el procedimiento de estimación y el operador AC-IOWA permita resolver problemas de toma de decisiones en grupo con información incompleta.

- Para enriquecer y completar el proceso completo de toma de decisiones desarrollaremos un *modelo de consenso* que puede aplicarse antes del anteriormente mencionado proceso de selección. El desarrollo de este modelo de consenso implica:

  – Proponer medidas de consenso entre las preferencias expresadas por los expertos.

  – Desarrollar un operador de agregación para relaciones de preferencia incompletas que tenga en cuenta tanto las medidas de consistencia como de consenso: operador IOWA de Consistencia/Consenso. Este operador permitirá la obtención de una "relación de preferencia global consensuada" de todos los expertos como grupo.

  – Proponer algunas medidas de proximidad de las preferencias expresadas por los expertos a la preferencia colectiva previamente calculada.

  – Desarrollar un mecanismo de retroalimentación que use las medidas anteriores para dar consejo a los expertos sobre como deben cambiar sus preferencias para alcanzar una solución más consensuada en el caso de que sus opiniones no estén suficientemente cerca (esto es, que aún no hayan alcanzado un estado de suficiente consenso).

Estructura de la Memoria

Para desarrollar los contenidos previamente mencionados esta memoria está dividida en varios capítulos que se estructuran como se detalla a continuación:

- **Capítulo 1*:** haremos una introducción a la toma de decisiones, a los diferentes tipos de toma de decisiones y presentaremos las herramientas y modelos básicos que serán utilizados en los siguientes capítulos:
  - *Modelos de representación de preferencias.*
  - *Propiedades de las relaciones de preferencia difusas*: reciprocidad, transitividad y consistencia.
  - *Operadores de agregación*: Operadores *Ordered Weighted Averaging* (OWA) y *Induced Ordered Weighted Averaging* (IOWA).
- **Capítulo 2:** presentaremos un *proceso de selección* para resolver problemas de toma de decisiones con relaciones de preferencia difusas incompletas. Para ello desarrollaremos un procedimiento de estimación iterativo capaz de calcular la información perdida en las relaciones de preferencia difusas incompletas y el operador IOWA basado en Consistencia Aditiva que permite la agregación de las preferencias de los expertos.
- **Capítulo 3:** desarrollaremos un *modelo de consenso* que permitirá obtener una solución más consensuada en un problema de toma de decisiones en grupo. El modelo de consenso será capaz de generar

recomendaciones a los expertos para que puedan cambiar sus opiniones para llegar a una solución lo suficientemente consensuada. El modelo de consenso puede aplicarse antes de que se lleve a cabo el proceso de selección presentado en el *capítulo 2*.

- **Capítulo 4*:** en este capítulo haremos algunos comentarios finales de la tesis: señalaremos los resultados obtenidos en la tesis, algunas conclusiones que podemos extraer de los mismos y por último señalaremos algunos aspectos sobre trabajos futuros.

- **Apéndice A:** En este apéndice se aclara el uso de cuantificadores difusos para modelar el concepto de mayoría difusa.

- **Apéndice B:** En este apéndice se muestran todos los cálculos detallados para resolver el problema presentado como ejemplo en el *capítulo 3*.

* **Nota:** Para el debido cumplimiento de la normativa para la obtención de la mención de *"Doctorado Europeo"* los capítulos marcados con * y esta introducción se encuentran adicionalmente redactados en inglés.

Introduction

Approaching the Problem

Decision making is the cognitive process of selecting the best alternative (or alternatives) from among multiple different alternatives. It begins when we need to do something but we do not know what. Therefore decision making is a reasoning process which can be rational or irrational, and can be based on explicit assumptions (usually presented with the alternatives) or tacit assumptions (those which are not explicitly voiced nor necessarily understood by the decision maker).

Decision making situations are very common in every person's daily life: usual examples include shopping, deciding what to eat, and deciding whom or what to vote for in an election or referendum.

However, decision making not only occur for isolated individuals. Usually some decision problems have to be solved by a *group* of persons (usually *experts*), which together have to decide which alternative among the given ones is better or more preferrable in a particular situation. This kind of decision making with multiple individuals is called *group decision making* (or multiperson decision making). The existence of multiple persons in a decision process implies several additional difficulties that have to be solved. For example, opinions of the individuals about the alternatives can be very different, and thus, to reach some kind of agreement (or *consensus*) among experts in the decision process is necessary prior to the actual *selection* of the best alternative(s).

Group decision making is sometimes examined separately as *process* and *outcome*. Process refers to the interactions among individuals that lead to the choice of a particular course of action. An outcome is the consequence of that choice. Separating process and outcome is convenient because it helps to explain that a good decision making process does not guarantee a good outcome, and that a good outcome does not presuppose a good process. Thus, for example, managers interested in good decision making are encouraged to put good decision making processes in place. Although these good decision making processes do not guarantee good outcomes, they can tip the balance of chance in favor of good outcomes. In fact, a critical aspect for group decision making is the ability to converge on a choice.

To properly model group decision making situations several aspects have to be taken into account:

- *The preference representation formats* that experts can use to express their opinions and preferences. This representation format can greatly affect the whole decision process. For example, some representation formats as *selection sets of alternatives* or *preference orderings of the alternatives* are simple representation formats that experts which are not familiar with them can easily learn to use effectively. However, their simplicity usually implies that the amount of information that can be modelled using them and its granularity is quite small. On the contrary, other preference representation formats as *preference relations* offer a higher level expressivity, and thus, a lot more of information (and more complex information) can be modelled with them.

- *Lack of information.* Although it is desirable for experts who face a decision problem to have a wide and exhaustive knowledge about the different alternatives, this is a requirement that is not often fulfilled. Many different cultural and personal factors can lead to lack of information situations in decision making. For example, experts may not be familiar with some of the alternatives (specially if the set of feasible alternatives is large), or maybe they are not able to properly differentiate among some similar alternatives.

- *Lack of consistency* (contradiction) of the preferences expressed by experts. Although diversity in the opinions of the different experts to solve a decision problem is often desirable (even when the opinions are antagonic) because this leads to discussion and to a better study of the problem to solve, contradiction in the individual opinions of the experts is not usually useful. In fact, in every real life situation when a person expresses inconsistent opinions (self-contradictory opinions), this person tends to be ignored by the rest.

Thus, the study of those aspects is a key point to develop reliable and realistic group decision making models and processes.

Objectives

The main objective of the work developed in this thesis is to develop group decision making models with incomplete information which allow to tackle both the consensus and selection of alternatives problems.

This main objective can be detached into the following points:

- Develop an iterative procedure to estimate incomplete information in incomplete fuzzy preference relations based on transitivity properties of fuzzy preference relations.

- Develop an aggregation operator for fuzzy preference relations based on the previously mentioned consistency measures: Additive Consistency based Induced Ordered Weighted Averaging (AC-IOWA) operator.

- Develop a *selection process* that making use of the estimation procedure and the AC-IOWA operator allows the resolution of group decision making problems with incomplete information.

- To enrich and complete the whole decision making process we will develop a *consensus model* to be applied before the presented selection process. The development of the consensus model implies:

  - Propose some consensus degrees among the preferences expressed by experts.
  - Develop an aggregation operator for incomplete fuzzy preference relations taking into account both consistency and consensus degrees: Consistency/Consensus IOWA operator. This

operator will allow to obtain a ”consensued collective preference relation” for the experts as a group.

– Propose some proximity measures from the preferences expresed by experts to the collective preference.

– Develop a feedback mechanism which use the previous measures to advice experts on how should they change their preferences in order to reach a more consensued solution in the case that their opinions are not close enough (that is, they have not yet reached a consensued enough state).

Structure of this Thesis

To develop the previously mentioned contents this thesis is divided in several chapters which are structured as follows:

- **Chapter 1:** we make an introduction to decision making, different kinds of decision making and we present the basic tools and models that will be used in the next chapters:
  – *Preference representation models.*
  – *Fuzzy preference relations properties*: reciprocity, transitivity and consistency.
  – *Aggregation operators*: Ordered Weighted Averaging (OWA) and Induced Ordered Weighted Averaging (IOWA) operators.
- **Chapter 2:** we present a *selection process* to solve decision making problems with incomplete fuzzy preference relations. To do so, we

develop an iterative estimation procedure that is able to compute missing information in the incomplete fuzzy preference relations and the Additive Consistency based IOWA (AC-IOWA) operator that allows the aggregation of the experts' preferences.

- **Chapter 3:** we develop a *consensus model* that will allow to obtain a more consensued solution to a group decision making problem. The consensus model will be able to generate advice for the experts being able to change their opinions towards an enough consensued solution. The consensus model can be applied before the selection process presented in *chapter 2*.

- **Chapter 4:** in this chapter we present some final comments: we will point out the results of this thesis as well as some conclusions about them, and finally we introduce some aspects about future works.

- **Appendix A:** In this appendix we present fuzzy quantifiers and discuss their use to model the concept of fuzzy majority.

- **Appendix B:** In this appendix we present all the computations in great detail made for problem presented as an example in *chapter 3*.

Chapter 1

Preliminares: La Toma de Decisiones en Grupo

La *toma de decisiones* es el proceso cognitivo de seleccionar la mejor alternativa (o alternativas) de un conjunto de posibles alternativas. En nuestro contexto decimos que tenemos un conjunto finito de posibles alternativas en el problema $X = \{x_1, x_2, \ldots, x_n\}, n \geq 2$ de las cuales queremos obtener un conjunto solución de alternativas $S \mid S \subset X, S \neq \emptyset$ (las mejores soluciones para el problema).

Dado que la toma de decisiones está presente en casi cualquier actividad humana, existe un gran interés en el estudio de los modelos de decisión, no solo en el campo de Teoría de la Decisión, sino que también en otras areas y disciplinas como Inteligencia Artificial, Economía, Sociología, Ingeniería, etc. Sin embargo, los modelos de decisión básicos tienen poco en común con los modelos de decisión reales. Muchos procesos de toma de decisiones reales se desarrollan en ambientes donde los objetivos, restricciones y posibles alternativas no son conocidas con precisión o no están bien definidas. Por lo tanto es necesario estudiar y refinar esos modelos de decisión para ser capaces de modelar esa incertidumbre. Una manera práctica y poderosa para tratar dicha incertidumbre en el conocimiento humano fue propuesta por el profesor Zadeh en 1965: La Teoría de Conjuntos Difusos [78]. La aplicación de la Teoría de Conjuntos Difusos para resolver la incertidumbre en la información en los procesos de toma de decisiones fue propuesta por Bellman y Zadeh en 1970 [2], y desde ese momento se ha utilizado extensivamente debido a su utilidad. La Teoría de Conjuntos Difusos ha facilitado un marco de trabajo más flexible en el cual se pueden representar y manejar de manera sencilla la imprecisión de los juicios humanos.

Es normal que los problemas de decisión necesiten de un análisis de las diferentes alternativas y del problema al que nos enfrentamos. Sin embargo, no todo problema de decisión se resuelve por medio de un proceso completamente racional. De hecho, muchos factores externos y subjetivos afectan a los procesos de decisión, y por lo tanto, la solución final para un problema de decisión puede variar si las condiciones en las que se presenta el problema cambian. A continuación enumeraremos algunos ejemplos reales comunes de procesos de toma de decisiones y como su solución puede verse influenciada por factores externos o subjetivos:

- *Elegir lo que se va a comer.* Elegir entre varias comidas posibles cuando uno está hambriento es una situación común en nuestra vida diaria. Sin embargo, la elección de un tipo particular de comida o incluso la manera de cocinarla no depende exclusivamente de factores racionales (por ejemplo las necesidades corporales, las propiedades nutritivas del alimento, etc.), sino que otros factores externos y subjetivos afectan en gran manera a la decisión final, por ejemplo, gustos personales, el aspecto de los distintos platos (que no implica directamente buena calidad o sabor), etc.

- *Comprar*. Este es un típico ejemplo de toma de decisiones. Cuando queremos comprar un producto particular usualmente tenemos que elegir entre una gama de alternativas diferentes pero similares. Esta claro que existen factores externos que nos influencian en gran medida sobre que productos comprar, por ejemplo, el lugar donde los productos se encuentran situados en la tienda, o la ayuda que ofrece el vendedor al cliente son factores fundamentales que determinan que productos se venden bien y cuales no. Además de los factores externos, que pueden inflenciar mucho en la decisión final, este es un buen ejemplo donde nos enfrentamos al problema de la *falta de información*. No es extraño que cuando un cliente tiene que escoger entre diversos productos similares éste no posea información suficiente sobre las características particulares que los diferencian.

- *Votar en unas elecciones*. En unas elecciones los votantes tienen que elegir entre diversos candidatos. En este caso es facil percibir que factores muy subjetivos pueden influir muy seriamente en el resultado final.

1.1 Tipologías de Situaciones de Toma de Decisiones

Las situaciones de toma de decisiones pueden categorizarse en varios grupos distintos de acuerdo con ciertas características como la fuente (o fuentes) de la información y el formato de representación que se usan para resolver los problemas de decisión. En esta sección describiremos algunas tipologías posibles de situaciones de toma de decisiones.

1.1.1 Toma de Decisiones Monocriterio

La toma de decisiones *monocriterio* se refiere la situación de toma de decisiones donde solo existe una única fuente de información (o criterio) para obtener la solución al problema de decisión. Por lo tanto, en este tipo de situaciones la solución del problema viene directa y únicamente de la información provista. Obviamente, en este caso no hay necesidad de un proceso de consenso, e incluso en el proceso de selección para obtener la solución final al problema no se necesita ningún tipo de agregación de información. La solución final del problema se obtiene mediante la aplicación del paso de explotación dentro del proceso de selección.

1.1.2 Toma de Decisiones Multicriterio

La toma de decisiones *multicriterio* [22, 47, 48, 51, 72] incluye a todas las situaciones de toma de decisiones donde la información sobre las alternativas proviene de fuentes diferentes (o multiples criterios). Las fuentes de información pueden ser heterogéneas; por ejemplo, para resolver un problema de decisión particular podemos contar con las preferencias expresadas por un experto en la materia, la salida de algún dispositivo de diagnóstico y de datos históricos sobre el problema.

Normalmente, en estas situaciones no hay necesidad de un proceso de consenso, ya que la información proveniente de diferentes fuentes puedes ser inmutable, es decir, puede no ser posible cambiarla para producir una solución más consensuada (por ejemplo, las salidas del dispositivo de diagnóstico y los datos históricos no pueden cambiarse)..

Sin embargo, para solucionar este tipo de problemas es necesario contar con algún tipo de paso de agregación que permita combinar toda la información que tenemos de las diferentes fuentes antes de la aplicación de un paso de explotación donde se seleccionará la solución final del problema a partir de la información unificada.

1.1.3 Toma de Decisiones en Grupo

La toma de decisiones *en grupo* [20, 28, 26, 27, 29, 34, 30, 38, 42, 44, 63, 64, 68] se corresponde con un caso particular de toma de decisiones multicriterio donde los distintos criterios que deben tomarse en cuenta para resolver el problema son las preferencias de un grupo particular de personas, usualmente *expertos* en el campo del problema. A partir de ahora denotaremos $E = \{e_1, e_2, \ldots, e_m\}, m \geq 2$ al conjunto de expertos que expresan sus preferencias u opiniones para resolver el problema de toma de decisiones en grupo.

El hecho de que los múltiples criterios para resolver el problema sean opiniones de expertos provoca una necesidad natural de aplicar procesos de consenso que permitan obtener no solo buenas soluciones para el problema de decisión, sino también obtener un cierto nivel de acuerdo entre los expertos, esto es, para obtener una solución que maximice la satisfacción global de los expertos con la decisión final. Una vez que el proceso de consenso haya sido llevado a cabo podemos obtener la conjunto de alternativas solución consensuado mediante la aplicación de un proceso de selección.

1.1.4 Situaciones Homogéneas y Heterogéneas

Decimos que nos encontramos ante un problema de toma de decisiones en grupo *heterogéneo* cuando las opiniones de los diferentes expertos no son igualmente importantes. Por el contrario, si todas las opiniones de los expertos se trata igualitariamente diremos que nos encontramos ante una situación de toma de decisiones en grupo *homogénea*.

Una manera de implementar la heterogeneidad de los expertos es asignar un *peso* a cada uno de los individuos. Los pesos son valores cualitativos o cuantitativos y pueden ser asignados de diversas maneras: un moderador puede asignarlos directamente o los pesos pueden obtenerse automáticamente de las preferencias expresadas por los expertos (por ejemplo, los expertos más consistentes pueden obtener un peso mayor que los que sean inconsistentes). Los pesos pueden interpretarse como la *importancia* del experto dentro del grupo, o como de relevante es dicha persona en relación con el problema que se resuelve [16, 18]. Sin embargo, hemos de hacer notar que el peso debe actuar como una restricción adicional sobre las opiniones de los expertos en el proceso de resolución.

1.1.5 Formatos de Representación de Preferencias Homogéneos y Heterogéneos

En la investigación sobre el modelado de las situaciones de toma de decisiones en grupo es usual que todos los expertos utilicen un mismo formato de representación de preferencias para expresar sus opiniones. En esos casos decimos que tenemos un formato de representación de preferencias homogéneo entre los expertos. Sin embargo, en la práctica real esa modelización no siempre es posible ya que cada experto tiene sus propias características respecto al conocimiento, aptitudes, experiencia y personalidad, lo que implica que expertos distintos quieran expresar sus evaluaciones por medio de formatos de representación distintos (formatos de representación heterogéneos). De hecho, este punto ha atraido recientemente la atención de numerosos investigadores en el área de la toma de decisiones en grupo, y como resultado se han propuesto distintos acercamientos para integrar los distintos formatos de representación de preferencias [10, 9, 11, 20, 32, 34, 30, 83, 82].

Normalmente todas estas propuestas seleccionan un formato de representación particular que se usa como base para la integración de las diferentes estructuras de preferencia que se usen en el problema. Se han proporcinado numerosas razones para elegir las *relaciones de preferencia difusas* como el elemento base para dicha integración. Entre otras, es interesante resaltar que son una herramienta muy útil para la agregación de las preferencias de los expertos en relaciones de preferencia de grupo [13, 22, 33, 36, 63, 64].

1.2 Esquema General de los Modelos de Toma de Decisiones en Grupo

Los problemas de toma de decisiones en grupo aparecen cuando existe una cuestión que debe resolverse, existen una serie de alternativas posibles entre las que se puede elegir y un conjunto de personas (expertos) que dan sus opiniones o preferencias sobre las posibles alternativas. Los

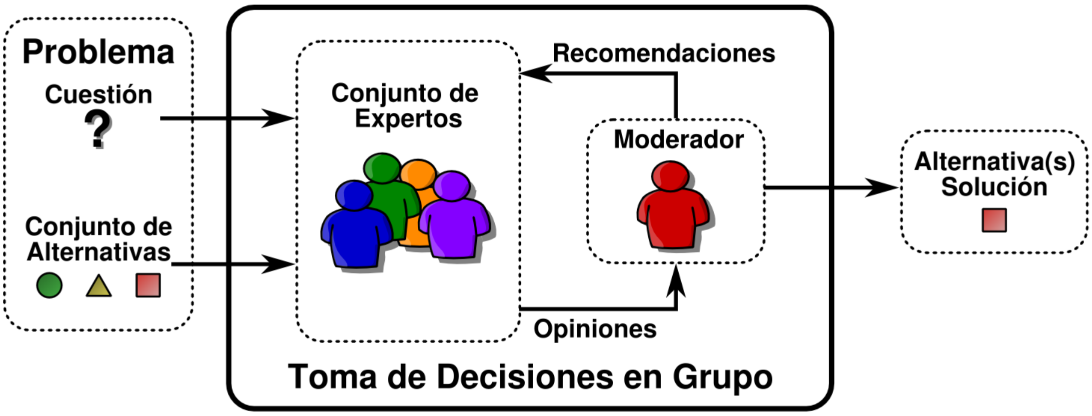

Figure 1.1: Aproximación al Problema de la Toma de Decisiones en Grupo

expertos deben tener la intención de llegar a una decisión colectiva sobre el problema. A veces existe una persona particular, llamada *moderador*, que se encarga de la dirigir el proceso de resolución completo hasta que los expertos lleguen a un acuerdo sobre la alternativa que debe elegirse como solución al problema (ver *figura 1.1*).

Para resolver correctamente un problema de toma de decisiones se deben llevar a cabo dos procesos distintos antes de de obtener una solución final [7]: el proceso de consenso y el proceso de selección. Ambos han sido estudiados de manera amplia por numerosos autores y en diferentes contextos de toma de decisiones en grupo [22, 41, 38, 43]. El primero se refiere a como obtener el mayor grado de consenso o acuerdo posible entre los expertos sobre el conjunto de alternativas. El segundo (también llamado *proceso de consenso algebraico*) se refiere a como obtener el conjunto solución de alternativas final a partir de las opiniones expresadas por los expertos.

Ambos procesos trabajan juntos de manera secuencial. Primero de

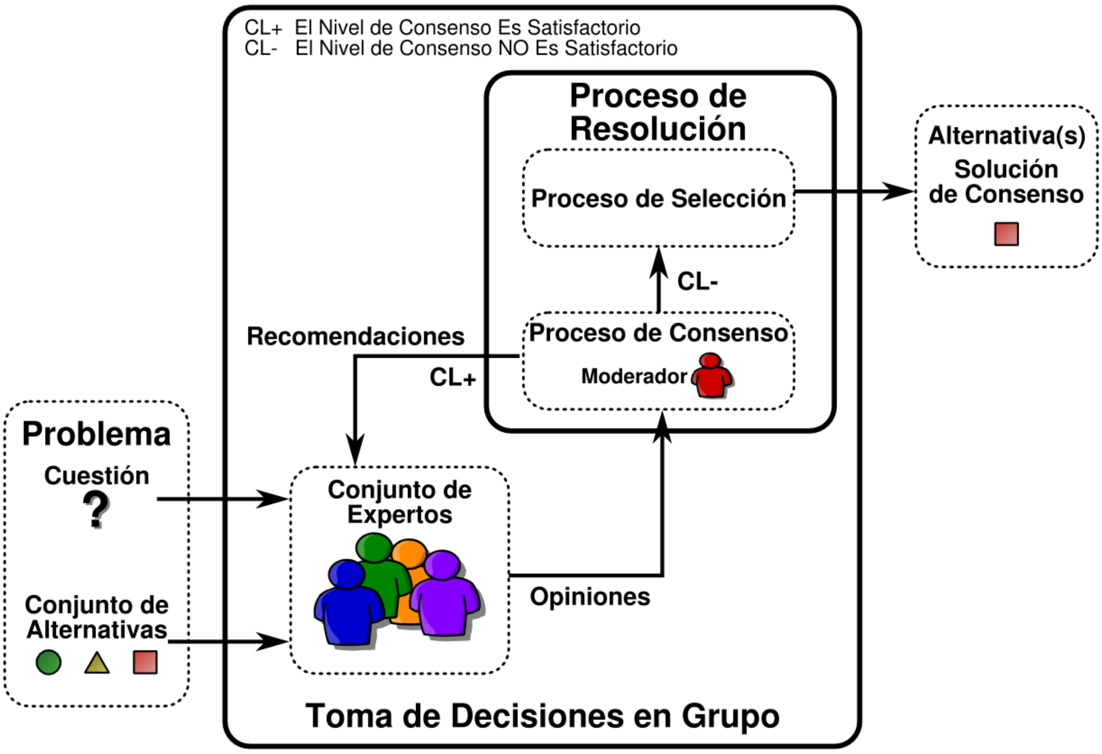

Figure 1.2: Esquema de un Proceso de Toma de Decisiones en Grupo

todo tiene lugar el proceso de consenso para alcanzar el mayor nivel posible de consenso entre las preferencias de los expertos. En cada paso del proceso se mide el grado de consenso actual, y si no alcanza un nivel deseado apropiado se incita a los expertos a discutir sus puntos de vista y consecuentemente cambiar sus opiniones para incrementar la proximidad de sus preferencias. Una vez que se ha alcanzado un cierto nivel de consenso se aplicará el proceso de selección y se obtendrá la solución final del problema.

Por lo tanto un proceso de toma de decisiones en grupo puede definirse como un proceso iterativo y dinámico en el cual los expertos cambian sus opiniones hasta que sus preferencias sobre la solución esten suficientemente próximas, y por lo tanto permitiendo obtener una solución consensuada mediante la aplicación del proceso de selección. Esto está representado gráficamente en la *figura 1.2*.

En esta sección describiremos ambos procesos en más detalle.

1.2.1 Proceso de Consenso

El *proceso de consenso* es el proceso iterativo que se compone de varias rondas de consenso donde los expertos aceptan cambiar sus preferencias siguiendo el consejo que les ofrece el moderador. El moderador conoce en todo momento el nivel de acuerdo que existe entre los expertos por medio de ciertas medidas de consenso que pueden ser calculadas. Como se comentó anteriormente, la mayoría de los modelos de consenso se guían y controlan por medio de medidas de consenso [5, 6, 7, 14, 19, 21, 26, 32, 37, 39, 40, 49, 60, 62, 81].

El proceso de consenso puede dividirse en varios pasos que están rep-

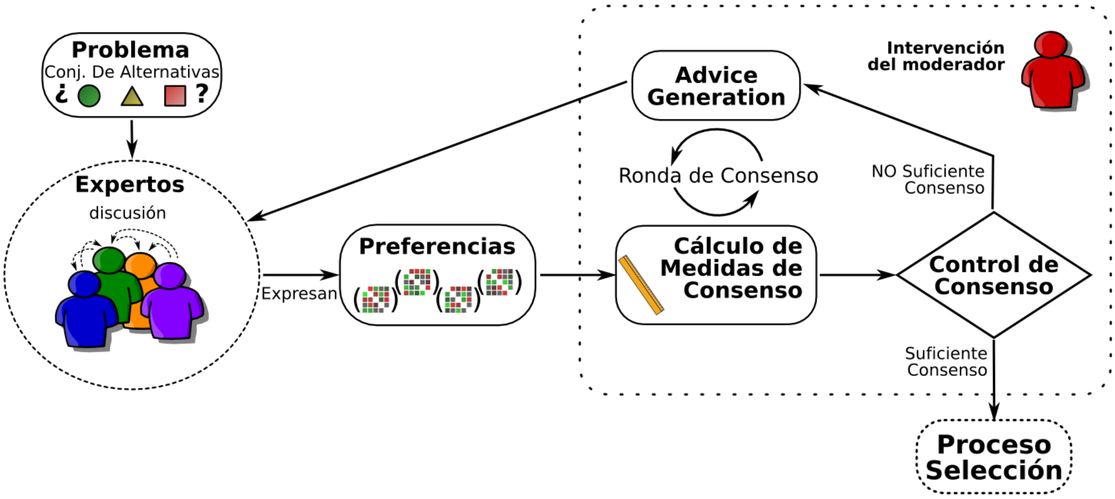

Figure 1.3: Esquema del Proceso de Consenso

resentados graficamente en la *figura 1.3*:

1. Lo primero que debe hacerse es presentar el problema a resolver a los expertos, junto con el conjunto de distintas alternativas sobre las cuales deben elegir la mejor (o mejores).

2. A continuación los expertos pueden discutir y compartir su conocimiento sobre el problema y las alternativas para facilitar el proceso de expresar sus opiniones.

3. Los expertos expresan sus proferencias sobre las alternativas en un formato de representación de preferencias particular.

4. El moderador recibe todas las preferencias de los expertos y calcula algunas medidas de consenso que le permitirán identificar si se ha alcanzado un estado de consenso suficiente o no.

5. Si se ha alcanzado un estado de suficiente consenso el proceso de consenso finaliza y comienza el proceso de selección. En otro caso,

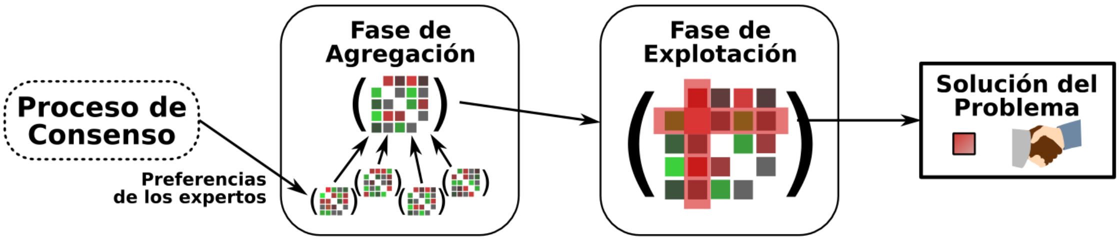

Figure 1.4: Esquema del Proceso de Selección

se puede aplicar un paso de generación de recomendaciones donde el moderador, con toda la información que posee (las preferencias expresadas por los expertos, medidas de consenso, etc.) puede preparar algunas recomendaciones o pistas para los expertos sobre como deben cambiar sus preferencias para alcanzar más fácilmente un estado de consenso. Hay que hacer notar que este paso es opcional y no tiene por tanto que estar presente en todos los modelos de consenso.

6. Por último, se les presenta a los expertos los consejos o recomendaciones del moderador y acaba la primera ronda de consenso. Otra vez los expertos deben discutir sobre sus opiniones y preferencias para acercar sus puntos de vista (paso 2).

1.2.2 Proceso de Selección

Una vez que el se ha llevado a cabo el proceso de consenso (esto es, las opiniones de los expertos están lo suficientemente cerca unas de otras) comienza el proceso de selección. El principal objetivo de este proceso es obtener el conjunto solución de alternativas para el problema teniendo en cuenta las opiniones de los distintos expertos. Este proceso se muestra en la *figura 1.4*.

El proceso de selección se puede dividir en dos fases distintas:

1. **Fase de Agregación.** En esta fase todas las preferencias dadas por los expertos deben agregarse en una sola estructura de preferencia. Esta agregación se suele llevar a cabo por medio de operadores de agregación que se definen usualmente para esta tarea. Este paso puede ser más complicado si nos encontramos ante una situación de toma de decisiones en grupo heterogénea (ya sea porque tengamos expertos con distinto grado de importancia o porque tengamos distintos formatos de representación de preferencias), ya que se hace necesario algún tipo de homogeneización que transforme todos los modelos de representación de preferencias en uno concreto que sirva como base para la agregación, y además el operador de agregación debe ser capaz de tratar adecuadamente con los pesos asignados a los expertos (esto es, dar más importancia a las preferencias de algunos expertos que las de otros).

2. **Fase de Explotación:** En este paso final se usa la información obtenida en la fase de agregación de preferencias para identificar el conjunto de alternativas solución para el problema. Para hacerlo se debe aplicar algún mecanismo que permita obtener un orden parcial de las alternativas y posteriormente seleccionar las mejor (o mejores) alternativas. Existen diversas maneras de conseguir esto, pero la más usual es asociar un cierto valor de utilidad a cada alternativa (basandonos en la información agregada), y por lo tanto produciendo un orden natural entre las alternativas.

1.3 Formatos de Representación de Preferencias

Como hemos mencionado previamente, existen diversos formatos de representación de preferencias que pueden ser usadas por los expertos para expresar sus opiniones sobre las alternativas en un problema de toma de decisiones en grupo. Adicionalmente en la literatura existen esfuerzos interesantes para crear modelos de decisión que permitan la expresión de preferencias en diversos formatos, ya que esto incrementa sobremanera la versatilidad en el uso de dichos modelos [10, 9, 11, 20, 32, 34, 30, 83, 82].

En esta sección describiremos brevemente algunos de los formatos de representación más comunes que han sido utilizados ampliamente en la literatura para finalmente comparar las ventajas y desventajas de cada uno de ellos.

1.3.1 Selección de un Subconjunto de Alternativas

Este es uno de los formatos de representación de preferencias más básicos.

Definition 1.1. Las preferencias de un experto $e_h \in E$ sobre un conjunto de posibles alternativas X se describen por medio de un subconjunto de alternativas $SS^h \subset X,\ SS^h \neq \emptyset$.

Básicamente el experto selecciona las alternativas que considera más relevantes para solucionar el problema. Por ejemplo, si el experto e_3 debe elegir entre cuatro alternativas distintas $X = \{x_1, x_2, x_3, x_4\}$ y piensa que las mejores alternativas para solucionar el problema son x_2 y x_3 daría el siguiente subconjunto de alternativas: $SS^3 = \{x_2, x_3\}$.

1.3.2 Órdenes de Preferencia

Definition 1.2. Las preferencias de un experto $e_h \in E$ sobre un conjunto de posibles alternativas X se describen como un orden de preferencia $O^h = \{o^h(1), \ldots, o^h(n)\}$ donde $o^h(\cdot)$ es una función de permutación sobre el conjunto de los índices $\{1, \ldots, n\}$ para ese experto [54, 61, 63].

Por lo tanto, un experto, de acuerdo con su punto de vista, da un vector ordenado de alternativas desde la mejor a la peor. Para cada orden de preferencia O^h suponemos, sin perdidad de generalidad, que cuanto menor sea la posición de la alternativa en el orden, ésta satisfará en mayor medida el criterio del experto. Por ejemplo, si un experto e_3 expresa sus preferencias sobre un conjunto de cuatro alternativas $X = \{x_1, x_2, x_3, x_4\}$ como el siguiente orden de preferencias (x_2, x_4, x_1, x_3) entonces $o^3(1) = 3, o^3(2) = 1, o^3(3) = 4, o^3(4) = 2$, lo que significa que la alternativa x_2 es la mejor para ese experto mientras que la alternativas x_3 es la peor.

1.3.3 Valores de Utilidad

Definition 1.3. Un experto $e_h \in E$ da sus preferencias sobre un conjunto de posibles alternativas X por medio de un conjunto de n valores de utilidad, $U^h = \{u_1^h, \ldots, u_n^h\}$, $u_i^h \in [0, 1]$. [15, 50, 64].

En este caso, el experto asocia un valor de utilidad a cada alternativa, lo que representa el nivel de cumplimiento desde su punto de vista de la alternativa. Para cada conjunto de valores de utilidad podemos suponer, sin perdida de generalidad, que cuanto más alto sea el valor para una alternativa, ésta satisface en mayor medida los objetivos del experto. Por ejemplo, si el experto e_3 expresa sus preferencias sobre un conjunto de cuatro alternativas posibles $X = \{x_1, x_2, x_3, x_4\}$ por medio del siguiente vector de utilidad: $U^3 = \{0.3, 0.7, 0.9, 0.4\}$ eso significaría que el piensa que la alternativa x_1 es la peor de todas y que x_3 es la mejor.

1.3.4 Relaciones de Preferencia

En la teoría matemática clásica, las preferencias sobre un conjunto de alternativas pueden modelarse por medio de una relación binaria R que se define como:

$$x_i R x_k \Leftrightarrow \text{"}x_i \text{ no es peor que } x_j\text{"}$$

Esta definición considera una relación binaria como una relación de preferencia débil e implica que dicha relación R es reflexiva. Partiendo de dicha definición es natural asociar un valor que se denota como $R(x_i, x_k) \in R$ que representa el grado de preferencia de la alternativa x_i sobre la alternativa x_k.

Se pueden usar diversos tipos de relaciones de preferencia según el dominio en el que se expresen las intensidades de las preferencias. Esto se expresa en la siguiente definición:

Definition 1.4. Una relación de preferencia P sobre un conjunto de alternativas X se caracteriza por una función $\mu_P \colon X \times X \longrightarrow D$, donde D es el dominio de representación de los grados de preferencia.

Cuando la cardinalidad de X es pequeña, la relación de preferencia puede representarse de manera conveniente como una matriz $n \times n$, $P = (p_{ij})$, donde $p_{ij} = \mu_P(x_i, x_j) \quad \forall i, j \in \{1, \ldots, n\}$.

1.3.4.1 Relaciones de Preferencia Difusas

Las relaciones de preferencia difusas han sido utilizadas de manera extensiva para modelar las preferencias en problemas de toma de decisiones. En este caso, la intensidad de las preferencias se mide usando una escala $[0,1]$ [10, 36, 55].

Definition 1.5. Una relación de preferencia difusa P sobre un conjunto de alternativas X es un conjunto difuso sobre el producto $\mathrm{X} \times \mathrm{X}$, esto es, está caracterizado por una función de pertenencia

$$\mu_P \colon \mathrm{X} \times \mathrm{X} \longrightarrow [0,1]$$

Cada valor p_{ik} en la matriz P representa el grado de preferencia o intensidad de preferencia de la alternativa x_i sobre x_j:

- $p_{ik} = 1/2$ indica que las alternativas x_i y x_k son indiferentes ($x_i \sim x_k$)
- $p_{ik} = 1$ indica que la alternativa x_i es completamente preferida a x_k
- $p_{ik} > 1/2$ indica que la alternativa x_i es preferida a x_k en un cierto nivel ($x_i \succ x_k$)

Por motivos de simplicidad notamos $p_{ii} = -,\ \forall i \in \{1, \ldots, n\}$ ya que una alternativa no puede compararse consigo misma.

Por ejemplo, si el experto e_3 expresa la siguiente relación de preferencia difusa cuando evalua un conjunto de cuatro alternativas distintas $X = \{x_1, x_2, x_3, x_4\}$:

$$P^3 = \begin{pmatrix} - & 0.0 & 0.4 & 0.4 \\ 1.0 & - & 0.7 & 0.5 \\ 0.6 & 0.3 & - & 0.75 \\ 0.6 & 0.5 & 0.25 & - \end{pmatrix}$$

quiere decir que, por ejemplo, que considera que $x_2 \sim x_4$ ya que $p_{24} = 0.5$, que piensa que x_2 es completamente mejor que x_1 porque $p_{21} = 1.0$ y que $x_3 \succ x_4$ ya que $p_{34} = 0.75$.

1.3.4.2 Relaciones de Preferencia Multiplicativas

En este caso la intensidad de las preferencias se representa como la razón entre las intensidades de preferencia entre las alternativas. De acuerdo con un estudio de Miller [52], Saaty sugiere medir cada valor de acuerdo con una escala multiplicativa, precisamente la escala 1-9 [57, 59].

Definition 1.6. Una relación de preferencia multiplicativa A sobre un conjunto de alternativas X se caracteriza por una función de pertenencia

$$\mu_A \colon X \times X \longrightarrow [1/9, 9]$$

Se asocian los siguientes significados con los números:

1	igualmente importantes
3	un poco más importante
5	bastante más importante
7	demostrablemente más importante o mucho más importante
9	completamente más importante
2,4,6,8	compromiso entre juicios ligeramente distintos

Por ejemplo, si el experto e_3 ofrece la siguiente relación de preferencia multiplicativa cuando evalua cuatro posibles alternativas $X = \{x_1, x_2, x_3, x_4\}$:

$$A^3 = \begin{pmatrix} - & 3 & 6 & 1/2 \\ 1/3 & - & 1 & 1/5 \\ 1/6 & 1 & - & 9 \\ 2 & 5 & 1/9 & - \end{pmatrix}$$

quiere decir, por ejemplo, que considera que $x_2 \sim x_3$ ya que $p_{23} = 1$, que piensa que x_3 is completamente más importante que x_4 porque $p_{34} = 9$ y que $x_1 \succ x_3$ ya que $p_{13} = 6$.

1.3.4.3 Relaciones de Preferencia Intervalares

Las relaciones de preferencia intervalares son usadas como alternativa a las relaciones de preferencia difusas cuando existe una dificultad en expresar las preferencias con valores numéricos exactos, pero sin embargo puede haber suficiente información como para poder hacer una estimación del intervalo en el que se encuentra dicha preferencia [4, 29, 62, 69].

Definition 1.7. Una relación de preferencia intervalar P sobre un conjunto de alternativas X se caracteriza por una función de pertenencia

$$\mu_P \colon X \times X \longrightarrow \mathcal{P}[0,1]$$

donde $\mathcal{P}[0,1] = \{[a,b] \mid a,b \in [0,1],\ a \leq b\}$ el es conjunto potencia de $[0,1]$.

Una relación de preferencia intervalar P puede verse como dos relaciones de preferencia difusas "independientes", la primera de ellas PL, que se corresponde con los extremos izquierdos de los intervalos y la segunda PR que se corresponde con los extremos derechos de los intervalos respectivamente:

$$P = (p_{ij}) = ([pl_{ij}, pr_{ij}]) \text{ with } PL = (pl_{ij}) \; PR = (pr_{ij}) \text{ and } pl_{ij} \leq pr_{ij} \;\forall i, j.$$

Obviamente es necesario definir algún tipo de operador de comparación para los valores intervalares para ser capaces de establecer un orden entre los elementos y por tanto ser capaces de interpretar correctamente cuando una alternativa es preferida a otra.

Por ejemplo, un experto x_3 puede dar la siguiente relación de preferencia intervalar cuando evalua cuatro alternativas distintas $X = \{x_1, x_2, x_3, x_4\}$:

$$P^3 = \begin{pmatrix} - & (0.0, 0.2) & (0.4, 0.6) & (0.4, 0.45) \\ (0, 8, 1.0) & - & (0.7, 0.9) & (0.5, 0.5) \\ (0.4, 0.6) & (0.1, 0.3) & - & (0.3, 0.55) \\ (0.55, 0.6) & (0.5, 0.5) & (0.45, 0.7) & - \end{pmatrix}$$

1.3.4.4 Relaciones de Preferencia Lingüísticas

Existen situaciones donde puede ser muy dificil para los expertos expresar valores de preferencia numéricos precisos o incluso intervalares. En esos casos pueden utilizarse valoraciones lingüísticas [25, 68, 79].

Definition 1.8. Una relación de preferencia lingüística P sobre un conjunto de alternativas X es un conjunto de términos lingüísticos de un conjunto de determinados términos lingüísticos $S = \{s_0, s_1, \ldots, s_{g-1}, s_g\}$

en el conjunto producto $X \times X$, esto es, está caracterizado por una función de pertenencia

$$\mu_P \colon X \times X \longrightarrow S.$$

Normalmente el conjunto de términos lingüísticos S tiene un número impar de elementos, y el elemento $s_{g/2}$ representa una etiqueta neutral (que significa *"igualmente preferido"*) y el resto de las etiquetas distribuidas de manera homogénea alrededor suya.

Por ejemplo, el experto e_3 puede dar la siguiente relación de preferencia lingüística cuando está evaluando cuatro alternativas distintas $X = \{x_1, x_2, x_3, x_4\}$ usando para ello el siguiente conjunto de etiquetas lingüísticas: $S = \{TW, MW, W, E, B, MB, TB\}$ con el siguiente significado:

TW = Totalmente Peor MW = Mucho Peor W = Peor E = Igualmente Preferido B = Mejor MB = Mucho Mejor Totalmente Mejor

$$P^3 = \begin{pmatrix} - & B & MW & MW \\ W & - & TB & E \\ MB & TW & - & B \\ MB & E & W & - \end{pmatrix}$$

1.3.5 Discusión

El uso de cada uno de los formatos de representación de preferencias que han sido presentados tiene una serie de ventajas e inconvenientes que van a ser presentadas y comparadas a continuación.

- Los *Subconjuntos de Selección* de alternativas es un formato muy sencillo de usar. Los expertos pueden entenderlo rápidamente, pero dada la inherente simplicidad del formato no ofrece una cantidad elevada de información. De hecho, como usa una evaluación binaria para las diferentes alternativas (relevante / no relevante) no permite diferenciar las preferencias del experto sobre las alternativas que considera relevantes.

- Los *Ordenes de Preferencia* ofrecen una evaluación de las alternativas a un nivel más fino y de hecho permite diferenciar un grado de preferencia entre cada para de alternativas. Sin embargo, ya que este formato necesita un orden total entre las alternativas no permite modelar situaciones típicas en toma de decisiones. Por ejemplo, un experto no es capaz usando este formato de expresar que tiene un grado de preferencia igual entre dos alternativas.

- Los *Valores de Utilidad* son un formato más fino que los anteriores por lo que los expertos pueden utilizarlos para representar correctamente sus preferencias sobre las alternativas. Sin embargo, su uso implica que el experto debe ser capaz de evaluar cada alternativa de manera global con respecto a las demás, lo que puede ser una tarea dificil.

- Las *Relaciones de Preferencia* resuelven el problema presentado por los valores de utilidad permitiendo la comparación de las alternativas por pares. Por lo tanto los expertos tienen mucha más libertad para expresar sus preferencias y por lo tanto con su uso pueden ganar en expresividad. La elección de un tipo particular de relación de preferencia depende de diversos factores. Las relaciones de preferencia difusas [10, 23, 36, 42, 55, 66] son unas de las más usadas por su gran expresividad, su efectividad como herramienta para modelar procesos de decisión y su utilidad y facilidad de uso cuando se quieren agregar las preferencias de los expertos en preferencias del grupo [33, 36, 63]. Aunque los otros tipos de relaciones de preferencia pueden ser una buena elección en determinados entornos, usualmente necesitan la definición de nuevos operadores que puedan manejarlas correctamente y el estudio de algunas de sus propiedades (que son bien conocidas en el caso de las relaciones de preferencia difusas), y por lo tanto no son tan usadas en los distintos modelos de decisión existentes.

1.4 Propiedades de las Relaciones de Preferencia Difusas

Las relaciones de preferencia difusas son un formato de representación de preferencias muy expresivo y poderoso y han sido usadas extensivamente en muchos modelos de toma de decisiones. Sin embargo, su gran expresividad puede llevar a situaciones donde las relaciones de preferencia no lleguen a representar realmente las preferencias de los expertos (porque los valores de preferencia pueden ser contradictorios). Así pues, es interesante estudiar algunas propiedades o restricciones que las relaciones de preferencia deben cumplir para que puedan considerarse realmente preferencias [64, 65].

1.4.1 Reciprocidad Aditiva

La *Reciprocidad Aditiva* es una de las restricciones que más comunmente se asume que las relaciones de preferencia difusas deben verificar [36]. Se describe como

$$p_{ik} + p_{ki} = 1 \;\; \forall i, j.$$

Sin embargo esta condición puede relajarse para ofrecer a los expertos un nivel de libertad más grande cuando expresan sus preferencias. Esta propiedad relajada se llama *Reciprocidad Débil*:

$$p_{ik} \geq 0.5 \Rightarrow p_{ki} \leq 0.5.$$

1.4.2 Transitividad

La *Transitividad* representa la idea de que un valor de preferencia que se obtiene comparando directamente dos alternativas debe ser igual o mayor que el valor de preferencia que se obtiene entre esas dos alternativas usando una cadena indirecta de alternativas. Esto puede expresarse en la siguiente definición:

Definition 1.9. Una relación de preferencia difusa P es T-transitiva con T una t-norma, si

$$p_{ik} \geq T(p_{ik}, p_{jk}) \forall i, j, k \in \{1, 2, ..., n\}$$

Siguiendo la definición existen multiples caracterizaciones posibles para la transitividad ya que existen diferentes funciones T. Aquí describiremos algunas de las que han sido más utilizadas en la literatura.

- **Transitividad Débil:**

$$min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq 0.5,\;\; \forall i, j, k$$

Esta es la condición mínima que una persona aplicaría racionalmente si no quiere expresar información inconsistente.

- **Transitividad MAX-MIN:**

$$p_{ik} \geq min\{p_{ij}, p_{jk}\},\ \forall i, j, k$$

Este tipo de transitividad ha sido tradicionalmente un requisito para caracterizar la consistencia de las relaciones de preferencia difusas.

- **Transitividad MAX-MAX:**

$$p_{ik} \geq max\{p_{ij}, p_{jk}\},\ foralli, j, k$$

Este tipo de transitividad representa un requisito más fuerte que la transitividad MAX-MIN.

- **Transitividad MAX-MIN Restringida:**

$$min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq min\{p_{ij}, p_{jk}\},\ \forall i, j, k$$

Esta es una condición más fuerte que la Transitividad Débil, pero más débil que la Transitividad MAX-MIN. Puede ser una suposición racional para considerar a una relación de preferencia difusa consistente el que verifique esta propiedad.

- **Transitividad MAX-MAX Restringida:**

$$min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq max\{p_{ij}, p_{jk}\},\ \forall i, j, k$$

Este es un concepto más fuerte que la Transitividad MAX-MIN Restringida, pero también puede ser una suposición racional para considerar consistente una relación de preferencia difusa el que tenga que verificar esta propiedad.

- **Transitividad Aditiva:** Como puede verse en [33], la transitividad aditiva para las relaciones de preferencia difusas puede verse como un concepto paralelo de la propiedad de consistencia de Saaty para las relaciones de preferencia miltiplicativas [58]. La formulación matemática de la transitividad aditiva fue propuesta por Tanino en [63]:

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5)\ \forall i, j, k \in \{1, \dots, n\} \quad (1.1)$$

Este tipo de transitividad tiene la siguiente interpretación: supongamos que queremos establecer un ranking entre tres alternativas x_i, x_j y x_k, y que la información disponible sobre dichas alternativas sugiere que nos encontramos ante una situación de indiferencia, es decir, $x_i \sim x_j \sim x_k$. Cuando expresemos nuestras preferencias esta situación se representaría como $p_{ij} = p_{jk} = p_{ik} = 0.5$. Supongamos a continuación que tenemos alguna información que dice que $x_i \prec x_j$, esto es, $p_{ij} < 0.5$. Esto significa que p_{jk} o p_{ik} deben cambiar, o si no incurririamos en una contradicción, porque tendríamos que $x_i \prec x_j \sim x_k \sim x_i$. Si suponemos que $p_{jk} = 0.5$ tendriamos la siguiente situación: x_j es preferida a x_i y no hay diferencia en la preferencia de x_j a x_k. Por lo tanto debemos concluir que x_k tiene que preferirse a x_i. Lo que es más, dado que $x_j \sim x_k$ entonces $p_{ij} = p_{ik}$, y por lo tanto $(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ij} - 0.5) = (p_{ik} - 0.5)$. Obtenemos la misma conclusión si $p_{ik} = 0.5$. En el caso de que $p_{jk} < 0.5$, tendríamos que x_k es preferida a x_j y esta a x_i, por lo que x_k debería preferirse a x_i. Por otro lado, el valor p_{ik} debe ser igual o menor que p_{ij}, siendo igual solo en el caso de que $p_{jk} = 0.5$ como ya hemos mostrado. Si interpretamos el valor $p_{ji} - 0.5$ como la intensidad de la preferencia de la alternativa x_j sobre x_i, entonces parece razonable suponer que la intensidad de la preferencia de x_i sobre x_k debería ser igual a la suma de las intensidades de las preferencias cuando se usa una alternativa intermedia x_j, esto es, $p_{ik} - 0.5 = (p_{ij} - 0.5) + (p_{jk} - 0.5)$. Podemos aplicar el mismo razonamiento en el caso de que $p_{jk} > 0.5$.

La transtividad aditiva implica la reciprocidad. De hecho, dado que $p_{ii} = 0.5\ \forall i$, si hacemos $k = i$ en la *expresión 1.1* tenemos que: $p_{ij} + p_{ji} = 1\ \ \forall i, j \in \{1, \dots, n\}$.

1.4.3 Consistencia Aditiva

La consistencia, esto es, la no existencia de contradicciones puede caracterizarse mediante la transitividad. Por ello, si una relación de preferencia difusa verifica cualquiera de las propiedades de transitividad presentadas con anteriodad, podemos decir que es consistente en esa manera particular. Por ejemplo, si una relación de preferencia difusa verifica la Transitividad MAX-MIN Restringida diremos que es consistente en el sentido MAX-MIN restringido.

No obstante, debido a sus buenas propiedades, la transitividad aditiva es la única propiedad que asumiremos a lo largo de esta memoria. De hecho, la *expresión 1.1* puede reescribirse como:

$$p_{ik} = p_{ij} + p_{jk} - 0.5\ \ \forall i, j, k \in \{1, \dots, n\} \qquad (1.2)$$

y por lo tanto, consideraremos que una relación de preferencia difusa es consistente de manera aditiva cuando para cada tres opciones en el problema $x_i, x_j, x_k \in X$ sus grados de preferencia asociados p_{ij}, p_{jk}, p_{ik}

cumplan la *expresión 1.2*. Llamaremos *consistente* a lo largo de esta memoria a cualquier relación de preferencia difusa que sea consistente de manera aditiva, ya que es la única propiedad de transitividad que estamos considerando.

1.5 Operadores de Agregación

La agregación es la operación que transforma un conjunto de elementos (conjuntos difusos, opiniones individuales sobre un conjunto de alternativas, etc.) en un único elemento que es representativo del conjunto al completo [17, 16, 72]. En toma de decisiones en grupo la agregación se lleva a cabo sobre las preferencias individuales de los expertos sobre el conjunto de alternativas para obtener una *preferencia global* que es un resumen de sus propiedades. El problema de la agregación de información se ha estudiado con gran profundidad. Existen montones de publicaciones al respecto, y entre ellas debemos citar [17, 16, 41, 72].

En esta sección presentamos dos familias de operadores de agregación diferentes pero relacionadas que serán utilizadas en los siguientes capítulos.

1.5.1 Operador OWA

El operador *Ordered Weighted Averaging* (OWA) fue propuesto por Yager en [72] y fue posteriormente estudiado en mayor profundidad y caracterizado en [73]. El operador OWA es conmutativo, idempotente, continuo, monotónico, neutral y estable para transformaciones lineales positivas. Un aspecto fundamental del operador OWA es la reordenación de los los argumentos que deben ser agregados, basada en la magnitud de sus valores respectivos:

Definition 1.10 ([72]). Un operador OWA de dimensión n es una función ϕ: $\Re^n \rightarrow \Re$, que posee unos pesos o vector de pesos asociado, $W = (w_1, ..., w_n)$, con $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$, y que se define para agregar una lista de valores $\{p_1, ..., p_n\}$ de acuerdo con la siguiente expresión,

$$\phi_W(p_1, ..., p_n) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)} \tag{1.3}$$

siendo σ: $\{1, ..., n\} \rightarrow \{1, ..., n\}$ una permutación tal que $p_{\sigma(i)} \geq p_{\sigma(i+1)}, \forall i = 1, ..., n-1$, esto es, $p_{\sigma(i)}$ es el i-esimo valor más grande en el conjunto $\{p_1, ..., p_n\}$.

Una pregunta natural en la definición del operador OWA es como obtener el vector de pesos asociado. En [72], Yager propuso dos maneras de obtenerlo. La primera propuesta es usar algún tipo de mecanismo de aprendizaje usando datos de ejemplo; mientras que la segunda propuesta trata de darle alguna semántica o significado a dichos pesos. La segunda posibilidad ha permitido desarrollar múltiples aplicaciones en el área que estamos estudiando, la agregacion guiada por cuantificadores [71].

En el proceso de agregación guiada por cuantificadores, dada una colección de n criterios representados como subconjuntos difusos de las alternativas X, el operador OWA se usa para implementar el concepto de mayoría difusa en la fase de agregación por medio de un *cuantificador lingüístico difuso* [80] que indica la proporción de criterios satisfechos *necesarios para una buena solución* [74] (ver el *apéndice A* para más detalles). Esta implementación se lleva a cabo usando el cuantificador para calcular los pesos del operador OWA. En el caso de un cuantificador monótono creciente regular (RIM) Q, el procedimiento para evaluar la satisfacción global de Q criterios (o expertos) (e_k) para la alternativa x_j se lleva a cabo calculando los pesos del operador OWA como sigue:

$$w_i = Q(i/n) - Q((i-1)/n), \quad i = 1, \ldots, n. \tag{1.4}$$

Cuando un cuantificador difuso Q se utiliza para calcular los pesos del operador OWA ϕ, entonces se simboliza como ϕ_Q. Debemos señalar que este tipo de agregación 'es altamente dependiente del vector de pesos que se use' [74], y por lo tanto también sobre la función que se haya usado para representar el cuantificador lingüístico difuso.

En [74] Yager propuso también un procedimiento para evaluar la satisfacción global de Q criterios de importancia (u_k) (o expertos) (e_k) para la alternativa x_j. En este procedimiento, una vez que los valores de satisfacción que deben ser agregados han sido ordenados, el vector de pesos asociado al operador OWA se calculan usando un cuantificador lingüístico Q utilizando la siguiente expresión

$$w_i = Q\left(\frac{\sum_{k=1}^{i} u_{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=1}^{i-1} u_{\sigma(k)}}{T}\right) \tag{1.5}$$

siendo $T = \sum_{k=1}^{n} u_k$ la suma total de importancia y σ la permutación usada para producir el orden de los valores que deben ser agregados. Esta propuesta de incluir los grados de importancia asocia una peso nulo a los expertos que tienen un nivel de importancia también nulo.

1.5.2 Operadores IOWA

Inspirándose en el trabajo de Mitchell y Estrakh [53], Yager y Filev en [76] definieron el operador *Induced Orderec Weighted Averaging* (IOWA) como una extensión del operador OWA para permitir un reordenamiento diferente de los valores que hay que agregar:

Definition 1.11 ([76]). Un operador IOWA de dimensión n es una función $\Phi_W : (\Re \times \Re)^n \rightarrow \Re$, a la cual se asocian unos pesos o vector de pesos, $W = (w_1, \ldots, w_n)$, with $w_i \in [0,1]$, $\Sigma_i w_i = 1$, y es definido para agregar el conjunto de segundos argumentos de una lista de n 2-tuplas $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$ de acuerdo con la siguiente expresión,

$$\Phi_W\left(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\right) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)} \tag{1.6}$$

donde σ es una permutación de $\{1, \ldots, n\}$ tal que $u_{\sigma(i)} \geq u_{\sigma(i+1)}, \forall i = 1, \ldots, n-1$, esto es, $\left\langle u_{\sigma(i)}, p_{\sigma(i)}\right\rangle$ es la 2-tupla con $u_{\sigma(i)}$ el i-ésimo valor más alto en el conjunto $\{u_1, \ldots, u_n\}$.

En la definición previa, el reordenamiento del conjunto de valores a agregar, $\{p_1, \ldots, p_n\}$, se induce por el reordenamiento del conjunto de valores $\{u_1, \ldots, u_n\}$ que tienen asociados, que está basado en su magnitud. Debido al uso del conjunto de valores $\{u_1, \ldots, u_n\}$, Yager y Filev los llamaron los valores de una variable de inducción de orden y a $\{p_1, \ldots, p_n\}$ los valores de la variable argumento [75, 76, 77].

Claramente los métodos anteriormente descritos para calcular el vector de pesos para un operador OWA puede ser aplicado en el caso de los operadores IOWA. Cuando un cuantificador lingüístico difuso Q se usa para calcular los pesos de un operador IOWA Φ, se simboliza como Φ_Q.

1.6 Información Incompleta

Uno de los principales problemas a los que debemos enfrentarnos cuando tratamos de resolver problemas de toma de decisiones en grupo es la falta de información. Como ya ha sido comentado, cada experto tiene sus propias experiencias conernientes al problema estudiado, lo cual puede implicar un grave inconveniente, que un experto no tenga un conocimiento perfecto sobre el problema a resolver [1, 46, 45, 44, 67]. Existen muchas causas posibles por las cuales un experto puede no ser capaz de expresar de manera eficiente todos los valores de preferencia que le son solicitados. Algunas de estas causas son:

- No tener suficiente conocimiento sobre las distintas alternativas. Especialmente si existe un número alto de alternativas distintas, los expertos pueden no estar familiarizados con todas ellas. Por ejemplo, si el problema al que se enfrentan los expertos es determinar cual de 10 aerolineas distintas es mejor, un experto concreto puede no tener conocimiento sobre una aerolinea concreta, pero puede tener una experiencia amplia y extensa con todas las demás. En ese caso parece obvio que ese experto no podrá expresar ningún tipo de preferencia sobre la aerolinea que desconoce.

- Un experto puede no ser capaz de discriminar el grado en el cual prefiere una alternativa sobre otra. Incluso si el experto posee un conocimiento profundo sobre las distintas alternativas quizás no sea capaz de comparar dos de ellas o de expresar de manera precisa un grado en el cual prefiere una alternativa sobre otra.

- A los expertos se les solicita usualmente que den información consistente, esto es, que sus preferencias no impliquen contradicciones. Por lo tanto, un experto puede preferir no dar todas las preferencias por las que se le preguntan para evitar introducir inconsistencias.

Por eso es muy importante ofrecer a los expertos herramientas que les permitan expresar esta falta de conocimiento en sus opiniones.

Tradicionalmente la falta de información se ha tratado como un tipo especial de incertidumbre. Es por eso que los conjuntos difusos, que han demostrado ser una herramienta poderosa para modelar y tratar incertidumbre han sido usados de manera extensiva para modelar este tipo de situaciones. En nuestro contexto particular, donde las preferencias de los expertos son modeladas por medio de relaciones de preferencia difusas, en las propuestas más simples se asumía que un valor de preferencia p_{ik} que no fuera dado por un experto podía asignarsele el valor 0.5, por lo tanto significando que al experto le son indiferentes las alternativas x_i y x_k entre sí. Nosotros creemos que esta manera de proceder no es correcta ya que que un experto no proporcione un valor de preferencia concreto puede ser el resultado de su incapacidad de cuantificar el grado de preferencia de una alternativa sobre la otra, en cuyo caso puede preferir no *adivinarlo* para mantener la consistencia de los valores que ya ha proporcionado. Debe quedar claro que cuando un experto no es capaz de expresar un valor concreto p_{ik} porque no tiene una idea clara sobre como de mejor es la alternativa x_i sobre la alternativa x_j esto no quiere decir que automáticamente el experto prefiera ambas opciones con la misma intensidad.

En el siguiente capítulo presentaremos una propuesta para modelar de manera correcta las situaciones de falta de información por medio de relaciones de preferencia difusas *incompletas*.

Chapter 1

Preliminaries: Group Decision Making

Decision making is the cognitive process of selecting the best alternative (or alternatives) from among multiple different alternatives. In our context we say that we have a finite set of feasible alternatives for the problem $X = \{x_1, x_2, \ldots, x_n\}, n \geq 2$ from where we want to obtain a solution set of alternatives $S \mid S \subset X, S \neq \emptyset$ (the best alternative(s) to solve the problem).

As decision making is present in almost every human activity, it provokes a great interest in the study of decision models, not only in Decision Theory, but also in other disciplines as Artificial Intelligence, Economy, Sociology, Engeneering and so on. However, basic decision models have little in common with real decision making. Many real decision making processes are developed in environments where objectives, restrictions and feasible options are not exactly known and defined. Thus, it is necessary to study and refine those decision models in order to be able to represent this uncertainty. A practical and powerful way to handle uncertainty in human knowledge was proposed by professor Zadeh in 1965:

Fuzzy Sets Theory [78]. The application of Fuzzy Sets Theory to solve information uncertainty in decision making was proposed by Bellman and Zadeh in 1970 [2] and since that moment it has been widely used because of its utility. Fuzzy Sets Theory has provided a much more flexible framework where it is possible to easily represent and tackle imprecision of human judgements.

Usually, decision problems require to make some analysis of the different alternatives and the problem that we face. However, not every decision problem is solved by means of a completely rational process. In fact, many external and subjective factors affect the decision processes, and thus, the final solution for a decision problem can change if the conditions in which the poblem is presented vary. Here we enumerate some common real life examples of decision making processes and how their solution may be influenced by external or subjective factors:

- *Choosing what to eat.* It is a common situation to have to choose among several different meals when a person is hungry. However, the election of the particular food or even the way to cook that food not only depends on rational assumptions (for example, corporal needs, nutrition properties of the food and so on), but other external and subjective factors affects greatly on the final decision, for example, personal tastes, good looking dishes (which does not directly imply good quality or taste) and so on.

- *Shopping.* This is a typical example of decision making. When we want to buy a particular item we usually have to choose among several similar but different alternatives. It is clear that external factors influence in a great manner which products are bought, for example, the place where the different items are placed in a shop or the guidance that the salesman gives to the customer are key factors that determine which products are sold and which products don't. In addition to external factors that can have great influence in the final decision, in this example it is also common to face the *lack of information* problem. It is not strange when we have to choose among several similar items to not have enough information about the particular characteristics which differentiate the items.

- *Vote in an election.* In an election the voters have to choose among several candidates. In this case it is easy to perceive that very subjective factors can seriously influenciate the final choice.

1.1 Typologies of Decision Making Situations

Decision making situations can be categorized in several different groups according to certain characteristics as the source (or sources) for the information and the preference representation formats that are used to solve the decision problem. In this section we describe some possible decision making typologies.

1.1.1 Single Criteria Decision Making

Single criteria decision making comprise every decision making situation where we have only one source of information (or criteria) to solve a decision problem. Thus, in this kind of situations, the solution of the problem comes directly and solely from the information provided. Obviously, there is no need of a consensus process, and even in the selection process to obtain the final solution of the problem we do not need to aggregate any information. Then, the final solution for the problem is found by applying the exploitation step of the selection process.

1.1.2 Multicriteria Decision Making

Multicriteria decision making [22, 47, 48, 51, 72] include the decision making situations where the information about the alternatives comes from different sources (or multiple criteria). Sources of information can be heterogeneous; for example, to solve a particular decision problem we can count with the preferences expressed by an expert in the field, some output from a diagnosis machine and historical data about the problem.

Usually, in these situations there is no need for a consensus process, since the information obtained by the different sources may not be possible to change to produce a more consensued solution (for example, both the output of the diagnosis machine and the historical data are inmutable).

However, to solve this kinds of problems we need some aggregation step that allows to combine all the information that we have from the different sources prior to the application of an exploitation step where we will select the solution of the problem.

1.1.3 Group Decision Making

Group decision making [20, 28, 26, 27, 29, 34, 30, 38, 42, 44, 63, 64, 68] is a particular case of multicriteria decision making where the different criteria that have to be taken into account to sol ve the decision problem are the preferences of a particular group of persons, usually *experts* in the field of the problem. We denote $E = \{e_1, e_2, \ldots, e_m\}, m \geq 2$ to the set of experts that will express their preferences or opinions to solve a group decision making problem.

The fact that the multiple criteria to solve the problem are opinions of experts provokes a natural need of consensus processes that allow to obtain not only good solutions for the decision problem, but also to obtain a certain level of agreement between experts, that is, to obtain a solution that maximizes the overall satisfaction of the experts with the final decision. After the consensus process has been carried out we can obtain the final consensued solution set of alternatives by means of the application of a selection process.

1.1.4 Homogeneous and Heterogeneous Situations

We say that we are in an *heterogeneuos* group decision making problem when the opinions of the different experts are not equally important. On the contrary, if every opinion is treated equally we say that we face an *homogeneous* group decision making problem.

A way to implement experts' heterogeneity aspect is to assign a *weight* to every individual. Weights are cualitative or cuantitative values that can be assigned in several different ways: a moderator can assign them directly, or the weights can be obtained automatically from the preferences expressed by experts (for example, the most consistent experts could receive a higher weight than inconsistent ones). The weights can be interpreted as the *importance* of the expert within the group, or how relevant is the person in relation with the problem to be solved [16, 18]. However, we must note that the weight must act as an adittional restriction over the opinions of the experts in the resolution process.

1.1.5 Homogeneous and Heterogeneous Preference Representation Formats

It has been common practice in research to model group decision making problems in which all the experts express their preferences using the same preference representation format. In those cases we have an homogeneus preference representation format among experts. However, in real practice this is not always possible because each expert has his/her unique characteristics with regard to knowledge, skills, experience and personality, which implies that different experts may express their evaluations by means of different preference representation formats (heterogeneous representation formats). In fact, this is an issue that recently has attracted the attention of many researchers in the area of group decision making, and as a result different approaches to integrating different preference representation formats have been proposed [10, 9, 11, 20, 32, 34, 30, 83, 82].

Those approaches usually select a particular representation format which is used as a base for the integration of the different preference structures in the problem. Many reasons are provided for *fuzzy preference relations* to be chosen as the base element of that integration. Amongst these reasons it is worth noting that they are a useful tool in the aggregation of experts' preferences into group preferences [13, 22, 33, 36, 63, 64].

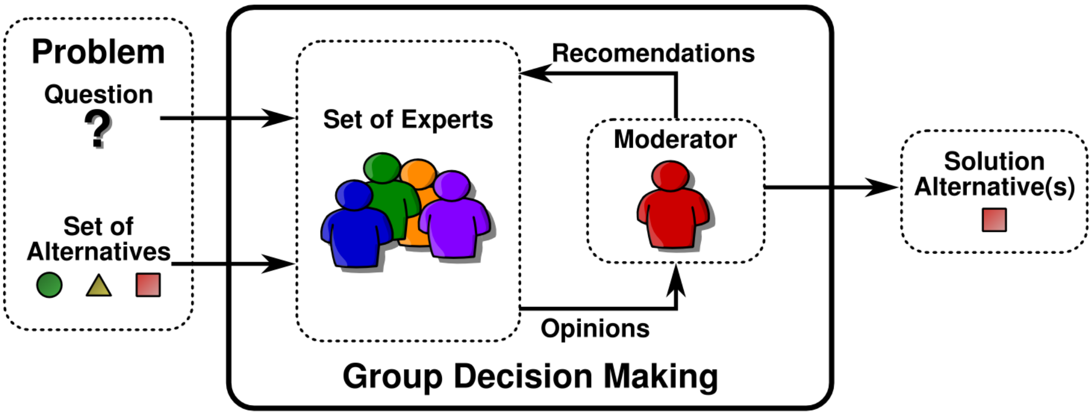

Figure 1.1: Group Decision Making Problem Approach

1.2 General Scheme of Group Decision Making Models

A group decision making problem appears when there is a question to be solved, a set of alternatives from where to choose and a set of persons (experts) which express their opionions or preferences about the available options. Experts should have the intention of reaching a collective decision about the problem. Sometimes there exists a particular person, called *moderator*, which is in charge of the direction of the whole resolution process until the experts reach an agreement about the solution to choose (see *figure 1.1*).

To correctly solve a group decision making problem two main different processes have to be carried out to obtain a solution [7]: the consensus process and the selection process. Both have been widely studied by different authors and in different group decision making contexts [22, 41, 38, 43]. The first one refers top how to obtain the highest consensus

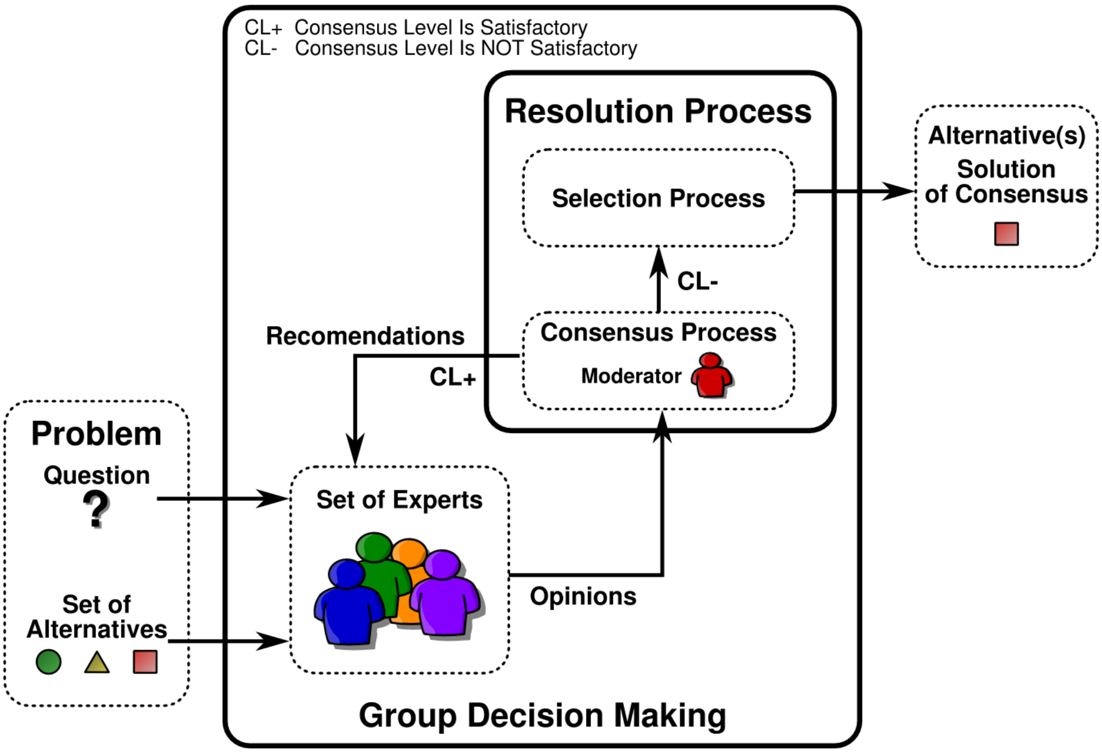

Figure 1.2: Group Decision Making Process Scheme

or agreement among experts about the set of alternatives. The second one (which is also called the *algebraic consensus process*) refers to how to obtain the final solution set of alternatives from the opinions expressed by experts.

Both processes work together secuentially. First of all, the consensus process is developed to reach the maximum consensus degree among experts' preferences. In every step of the process the current consensus degree is measured, and if it does not reach an acceptable level, experts are encouraged to discuss their points of view and change their opinions to increase the proximity of their preferences. Once a certain level of consensus have been reached the selection process is applied and the final solution is obtained.

Thus, a group decision making process can be defined as a dinamic and iterative process in which experts change their opinions until their preferences about the solution are close enough, therefore allowing the obtention of a consensued solution by means of the application of the selection process. This is graphically represented in *figure 1.2*.

In this section we will describe both processes with more detail.

1.2.1 Consensus Process

A *consensus process* is an iterate process which is composed by several consensus rounds, where the experts accept to change their preferences following the advice given by a moderator. The moderator knows the agreement in each moment of the consensus process by means of the computation of some consensus measures. As aforementioned, most of the consensus models are guided and controlled by means of consensus measures [5, 6, 7, 14, 19, 21, 26, 32, 37, 39, 40, 49, 60, 62, 81].

The consensus process can be divided in several steps which are graphically depicted in *figure 1.3*:

1. First of all, the problem to be solved is presented to the experts, along with the different alternatives among they have to choose the best one(s).

2. Then, experts can discuss and share their knowledge about the problem and alternatives in order to facilitate the process of laterly express their opinions.

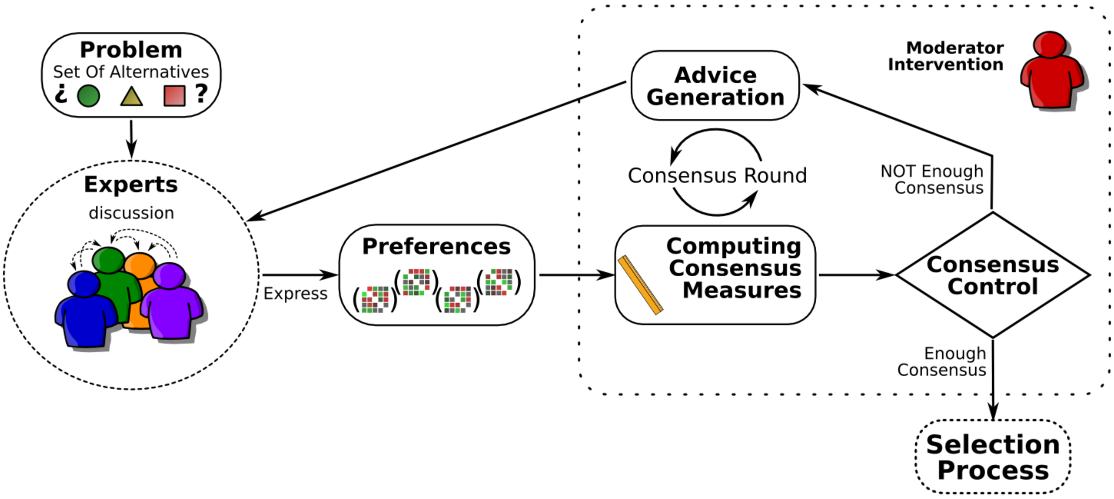

Figure 1.3: Consensus Process Scheme

3. Experts provide their preferences about the alternatives in a particular preference representation format.

4. The moderator receives all the experts' preferences and computes some consensus measures that will allow to identify if a consensus enough state has been reached or not.

5. If a consensued enough state has been reached the consensus process stops and the selection process begins. Otherwise, we can apply an advice generation step where the moderator, with all the information that he/she has (all preferences expressed by experts, consensus measures and so on) can prepare some guidance and advice for experts to more easily reach consensus. Note that this step is optional and is not present in every consensus model.

6. Finally, the advice is given to the experts and the first round of consensus is finished. Again, experts must discuss their opinions and preferences in order to approach their points of view (step 2).

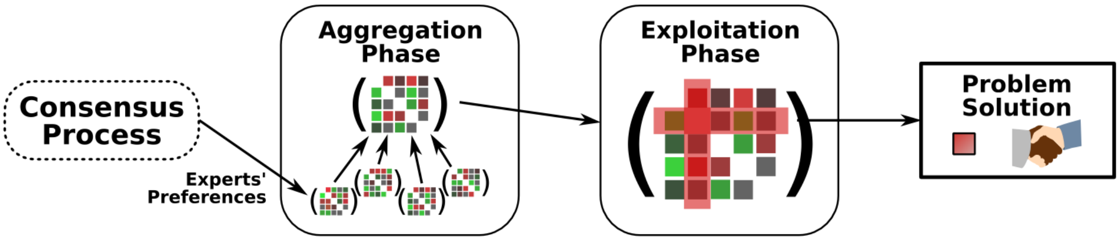

Figure 1.4: Selection Process Scheme

1.2.2 Selection Process

Once the consensus process has been carried out, (that is, experts' opinions are close enough) the selection process takes place. This process main aim is to select the final solution set of alternatives for the problem from the preferences given by the experts. This process is shown in *figure 1.4*.

The selection process can be splitted in two different phases:

1. **Aggregation Phase.** In this phase all preferences given by the experts must be aggregated into only one preference structure. This aggregation is usually carried out by means of particular aggregation operators that are usually defined for this purpose. This step can be more complicated if we have an heterogeneous decision making situation (not equally important experts or different preference representation formats), as some kind of homogeneization must be carried out to transform all different preference representantion formats into a particular one which acts as a base for the aggregation, and the aggregation operator must be able to handle the weights assigned to the experts (that is, giving more importance to some experts' preferences than others).

2. **Exploitation Phase:** This final step uses the information produced in the aggregation phase to identify the solution set of alternatives for the problem. To do so we must apply some mechanism to obtain a partial order of the alternatives and thus select the best alternative(s). There are several different ways to do this, but a usual one is to associate a certain utility value to each alternative (based on the aggregated information), thus producing a natural order of the alternatives.

1.3 Preference Representation Formats

As we have previously mentioned there are several different preference representations formats that can be used by experts to express their opinions about the alternatives in a group decision making problem. Moreover, there are some interesting efforts in the literature to create decision models that allow the expression of preferences in several different formats, because this can increase the versatility of those models [10, 9, 11, 20, 32, 34, 30, 83, 82].

In this section we will briefly describe some of the most common preference representation formats that have been widely used in the literature and we will finally compare the advantages and disadvantages of each one.

1.3.1 Selection Set

This is one of the most basic preference representation formats.

Definition 1.1. The preferences of an expert $e_h \in E$ about a set of feasible alternatives X are described by the selection set $SS^h \subset X,\ SS^h \neq \emptyset$.

Basically the expert selects the alternative(s) that he/she considers more relevant to solve the problem. For example, if expert e_3 has to choose among four different alternatives $X = \{x_1, x_2, x_3, x_4\}$ and he/she thinks that the best alternatives to solve the problem are x_2 and x_3 he would give the following selection set of alternatives: $SS^3 = \{x_2, x_3\}$.

1.3.2 Preference Orderings

Definition 1.2. The preferences of an expert $e_h \in E$ about a set of feasible alternatives X are described as a preference ordering $O^h = \{o^h(1), \ldots, o^h(n)\}$ where $o^h(\cdot)$ is a permutation function over the indexes set $\{1, \ldots, n\}$ for this expert [54, 61, 63].

Thus, an expert, according to his/her point of view, gives an ordered vector of alternatives from best to worst. For every preference ordering O^h we suppose, without loss of generalization, that as small is the position of an alternative in the ordering it will better satisfy the expert criteria. For example, if an expert e_3 expresses his preferences about a set of four alernatives $X = \{x_1, x_2, x_3, x_4\}$ as the following preference ordering (x_2, x_4, x_1, x_3) then $o^3(1) = 3, o^3(2) = 1, o^3(3) = 4, o^3(4) = 2$ which means that alternative x_2 is the best for that expert whilst alternative x_3 is the worst.

1.3.3 Utility Values

Definition 1.3. An expert $e_h \in E$ provides his/her preferences about a set of feasible alternatives X by means of a set of n utility values, $U^h = \{u_1^h, \ldots, u_n^h\}$, $u_i^h \in [0, 1]$. [15, 50, 64].

In this case, the expert associates an utility value to each alternative which represents the fulfilment degree from his/her point of view of the alternative. For every set of utility values we suppose, without loss of generalization, that the higher the value for an alternative, the better it satisfies experts' objective. For example, if he expert e_3 expresses his/her preferences about a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$ by means of the following utility vector: $U^3 = \{0.3, 0.7, 0.9, 0.4\}$ that would mean that he thinks that alternative x_1 is the worst one while alternative x_3 is the best one.

1.3.4 Preference Relations

In clasical mathematical theory, preferences about a set of alternatives can be modelled by means of a binary relation R defined as:

$$x_i R x_k \Leftrightarrow \text{"}x_i \text{ is not worst than } x_j\text{"}$$

This definition considers a binary relation as a weak preference relation and implies that relation R is reflexive. From that definition is natural to associate a value denoted $R(x_i, x_k) \in R$ which represents the preference degree of alternative x_i over alternative x_k.

Different types of preference relations can be used according to the domain used to evaluate the intensity of preference. This is expressed in the following definition:

Definition 1.4. A preference relation P on a set of alternatives X is characterized by a function $\mu_P \colon X \times X \longrightarrow D$, where D is the domain of representation of preference degrees.

When cardinality of X is small, the preference relation may be conveniently represented by an $n \times n$ matrix $P = (p_{ij})$, being $p_{ij} = \mu_P(x_i, x_j)\ \forall i, j \in \{1, \ldots, n\}$.

1.3.4.1 Fuzzy Preference Relations

Fuzzy preference relations have been widely used to model preferences for decision making problems. In this case, intensity of preference is usually measured using a difference scale $[0, 1]$ [10, 36, 55].

Definition 1.5. A fuzzy preference relation P on a set of alternatives X is a fuzzy set on the product set $\mathrm{X} \times \mathrm{X}$, i.e., it is characterized by a membership function

$$\mu_P \colon \mathrm{X} \times \mathrm{X} \longrightarrow [0, 1]$$

Every value p_{ik} in the matrix P represents the preference degree or intensity of preference of the alternative x_i over x_k:

- $p_{ik} = 1/2$ indicates indifference between alternatives x_i and x_k ($x_i \sim x_k$)
- $p_{ik} = 1$ indicates that alternative x_i is absolutely preferred to x_k
- $p_{ik} > 1/2$ indicates that alternative x_i is preferred to x_k ($x_i \succ x_k$)

For simplicity pourposes we will note $p_{ii} = -,\ \forall i \in \{1, \ldots, n\}$ as an alternative cannot be compared with itself.

For example, if expert e_3 gives the following fuzzy preference relation when he is evaluating four different alternatives $X = \{x_1, x_2, x_3, x_4\}$:

$$P = \begin{pmatrix} - & 0.0 & 0.4 & 0.4 \\ 1.0 & - & 0.7 & 0.5 \\ 0.6 & 0.3 & - & 0.75 \\ 0.6 & 0.5 & 0.25 & - \end{pmatrix}$$

he means, for example, that he considers that $x_2 \sim x_4$ as $p_{24} = 0.5$, that he thinks that x_2 is absolutelly preferred to x_1 because $p_{21} = 1.0$ and that $x_3 \succ x_4$ as $p_{34} = 0.75$.

1.3.4.2 Multiplicative Preference Relations

In this case, the intensity of preference represents the ratio of the preference intensity between the alternatives. According to Miller's study [52], Saaty suggests measuring every value using a ratio scale, precisely the 1-9 scale [57, 59].

Definition 1.6. A multiplicative preference relation A on a set of alternatives X is characterized by a membership function

$$\mu_A \colon X \times X \longrightarrow [1/9, 9]$$

The following meanings are associated to numbers:

1	equally important
3	weakly more important
5	strongly more important
7	demonstrably or very strongly more important
9	absolutely more important
2,4,6,8	compromise between slightly differing judgments

For example, if expert e_3 gives the following multiplicative preference relation when he is evaluating four different alternatives $X = \{x_1, x_2, x_3, x_4\}$:

$$A^3 = \begin{pmatrix} - & 3 & 6 & 1/2 \\ 1/3 & - & 1 & 1/5 \\ 1/6 & 1 & - & 9 \\ 2 & 5 & 1/9 & - \end{pmatrix}$$

he means, for example, that he considers that $x_2 \sim x_3$ as $p_{23} = 1$, that he thinks that x_3 is absolutelly preferred to x_4 because $p_{34} = 9$ and that $x_1 \succ x_3$ as $p_{13} = 6$.

1.3.4.3 Interval-Valued Preference Relations

Interval-valued preference relations are used as an alternative to fuzzy preference relations when there exists a difficulty in expressing the preferences with exact numerical values, but there is enough information as to estimate the intervals [4, 29, 62, 69].

Definition 1.7. An interval-valued preference relation P on a set of alternatives X is characterized by a membership function

$$\mu_P \colon X \times X \longrightarrow \mathcal{P}[0,1]$$

where $\mathcal{P}[0,1] = \{[a,b] \mid a,b \in [0,1],\ a \leq b\}$ is the power set of $[0,1]$.

An interval-valued preference relation P can be seen as two "independent" fuzzy preference relations, the first one PL corresponding to the left extremes of the intervals and the second one PR to the right extremes of the intervals, respectively:

$$P = (p_{ij}) = ([pl_{ij}, pr_{ij}]) \text{ with } PL = (pl_{ij}) \ PR = (pr_{ij}) \text{ and } pl_{ij} \leq pr_{ij} \ \forall i,j.$$

Obviouly it is necessary to define some comparison operators for the interval values to be able to stablish an order between elements and thus be able to properly interpret when an alternative is preferred over another.

For example, expert e_3 can give the following interval-valued preference relation when he is evaluating four different alternatives $X = \{x_1, x_2, x_3, x_4\}$:

$$P^3 = \begin{pmatrix} - & (0.0, 0.2) & (0.4, 0.6) & (0.4, 0.45) \\ (0,8, 1.0) & - & (0.7, 0.9) & (0.5, 0.5) \\ (0.4, 0.6) & (0.1, 0.3) & - & (0.3, 0.55) \\ (0.55, 0.6) & (0.5, 0.5) & (0.45, 0.7) & - \end{pmatrix}$$

1.3.4.4 Linguistic Preference Relations

There are situations where it could be very difficult for the experts to provide precise numerical or interval-valued preferences. In those cases it is possible to use linguistic assessments [25, 68, 79].

Definition 1.8. A linguistic preference relation P on a set of alternatives X is a set of linguistic terms from a certain linguistic term set $S = \{s_0, s_1, \ldots, s_{g-1}, s_g\}$ on the product set $X \times X$, i.e., it is characterized by a membership function

$$\mu_P \colon X \times X \longrightarrow S.$$

Usually the linguistic term set S has an odd number of elements, with $s_{g/2}$ being a neutral label (meaning *"equally preferred"*) and the rest of labels distributed homogeneously around it.

For example, expert e_3 can give the following linguistic preference relation when he is evaluating four different alternatives $X = \{x_1, x_2, x_3, x_4\}$ using the following set of linguistic labels: $S = \{TW, MW, W, E, B, MB, TB\}$ with the following meaning:

TW = Totally Worse MW = Much Worse W = Worse E = Equally Preferred B = Better MB = Much Better Totally Better

$$P^3 = \begin{pmatrix} - & B & MW & MW \\ W & - & TB & E \\ MB & TW & - & B \\ MB & E & W & - \end{pmatrix}$$

1.3.5 Discussion

The use of every preference representation format that has been presented has some advantages and disadvantages that will be now discussed.

- *Selection Sets* are a very easy to use format. It is very simple to understand by experts, but due to its inherent simplicity it does not give much information. As it uses a binary evaluation for the alternatives (relevant / not relevant) it does not allow to differentiate the experts' preferences about the relevant alternatives.

- *Preference Orderings* offer a more fine grained evaluation of the alternatives, and indeed it allows to differentiate a certain degree of preference between every pair of alternatives. However, as this format requires a total order among the alternatives, it does not allow to model some typical situations. For example, using this format an expert is not able to express that his preference degree between two of the alternatives is equal.

- *Utility Values* are a more fine grained format than the previous ones, and by using it an expert can properly represent his/her preferences about the alternatives. However, it implies that the expert must be able to evaluate every alternative against all the others as a whole, which can be a difficult task.

- *Preference Relations* solve the problem that the utility values presented by allowing the comparison of the alternatives in a pair by pair basis. Thus, experts have much more freedom when expressing their preferences and they can gain in expressivity. The choice of a particular kind of preference relation depends on several factors. Fuzzy preference relations [10, 23, 36, 42, 55, 66] are one of the most used because of their high expresivity, their effectiveness as a tool for modelling decision processes and their utility and easiness of use when we want to aggregate experts' preferences into group preferences [33, 36, 63]. Although the other presented types of preference relations can be a good choice in certain environments, they usually need the definition of new operators to properly handle them and the study of some of their properties (which are well known in the case of fuzzy preference relations), and thus, their are not so usually used in the different existing decision models.

1.4 Fuzzy Preference Relations Properties

Fuzzy preference relations are a very expresive and powerful preference representation format and it has been widely used in many decision making models. However, their high expressivity can lead to situations where the preference relations do not really reflect experts' preferences (because the preference values can be contradictory). Thus, it is interesting to study some properties or restrictions that fuzzy preference relations must comply to really be considered preferences [64, 65].

1.4.1 Additive Reciprocity

Additive reciprocity is one of the more usually assumed restrictions that a fuzzy preference relation must verify [36]. It is described as

$$p_{ik} + p_{ki} = 1 \quad \forall i, j.$$

However, this condition can be relaxed to offer a higher freedom degree to the experts when expressing their preferences. This relaxed property is called *Weak Reciprocity*:

$$p_{ik} \geq 0.5 \Rightarrow p_{ki} \leq 0.5.$$

1.4.2 Transitivity

Transitivity represents the idea that the preference value obtained by directly comparing two alternatives should be equal to or greater than the preference value between these two alternatives obtained using an indirect chain of alternatives. This is expressed in the following definition:

Definition 1.9. A fuzzy preference relation P is T-transitive, with T a t-norm, if

$$p_{ik} \geq T(p_{ik}, p_{jk}) \forall i, j, k \in \{1, 2, ..., n\}$$

Following the definition there exist multiple possible characterizations for transitivity as there exist different T functions. Here we describe some of them which have been widely used in the literature.

- **Weak Transitivity:**

  $$min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq 0.5, \ \forall i, j, k$$

  This is the minimum condition that a person would rationally apply if he/she does not want to express inconsistent information.

- **MAX-MIN Transitivity:**

  $$p_{ik} \geq min\{p_{ij}, p_{jk}\}, \ \forall i, j, k$$

  This kind of transitivity has been a traditional requirement to characterize consistency for fuzzy preference relations.

- **MAX-MAX Transitivity:**

$$p_{ik} \geq max\{p_{ij}, p_{jk}\}, \ forall i, j, k$$

This kind of transitivity represents a stronger requirement than the MAX-MIN Transitivity.

- **Restricted MAX-MIN Transitivity:**

$$min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq min\{p_{ij}, p_{jk}\}, \ \forall i, j, k$$

This is a stronger condition than the Weak Transitivity Condition, but weaker than MAX-MIN Trasitivity. It can be a rational assumption that a fuzzy preference relation should verify to be considered consistent.

- **Restricted MAX-MAX Transitivity:**

$$min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq max\{p_{ij}, p_{jk}\}, \ \forall i, j, k$$

This is a stronger concept than Restricted MAX-MIN Transitivity, but it can also be a rational assuption that a fuzzy preference relation should verify to be considered consistent.

- **Additive Transitivity:** As shown in [33], additive transitivity for fuzzy preference relations can be seen as the parallel concept of Saaty's consistency property for multiplicative preference relations [58]. The mathematical formulation of the additive transitivity was given by Tanino in [63]:

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5) \ \forall i, j, k \in \{1, \ldots, n\} \quad (1.1)$$

This kind of transitivity has the following interpretation: suppose we want to establish a ranking between three alternatives x_i, x_j and x_k, and that the information available about these alternatives suggests that we are in an indifference situation, i.e., $x_i \sim x_j \sim x_k$. When giving preferences this situation would be represented by $p_{ij} = p_{jk} = p_{ik} = 0.5$. Suppose now that we have a piece of information that says $x_i \prec x_j$, i.e., $p_{ij} < 0.5$. This means that p_{jk} or p_{ik} have to change, otherwise there would be a contradiction, because we would have $x_i \prec x_j \sim x_k \sim x_i$. If we suppose that $p_{jk} = 0.5$ then we have the situation: x_j is preferred to x_i and there is no difference in preferring x_j to x_k. We must then conclude that x_k has to be preferred to x_i. Furthermore, as $x_j \sim x_k$ then $p_{ij} = p_{ik}$, and so $(p_{ij}-0.5)+(p_{jk}-0.5) = (p_{ij}-0.5) = (p_{ik}-0.5)$. We have the same conclusion if $p_{ik} = 0.5$. In the case of $p_{jk} < 0.5$, then we have that x_k is preferred to x_j and this to x_i, so x_k should be preferred to x_i. On the other hand, the value p_{ik} has to be equal to or lower than p_{ij}, being equal only in the case of $p_{jk} = 0.5$ as we have already shown. Interpreting the value $p_{ji} - 0.5$ as the intensity of preference of alternative x_j over x_i, then it seems reasonable to suppose that the intensity of preference of x_i over x_k should be equal to the sum of the intensities of preferences when using an intermediate alternative x_j, that is, $p_{ik} - 0.5 = (p_{ij} - 0.5) + (p_{jk} - 0.5)$. The same reasoning can be applied in the case of $p_{jk} > 0.5$.

Additive transitivity implies additive reciprocity. Indeed, because $p_{ii} = 0.5\ \forall i$, if we make $k = i$ in *expression 1.1* then we have: $p_{ij} + p_{ji} = 1\ \ \forall i, j \in \{1, \ldots, n\}$.

1.4.3 Additive Consistency

Consistency, that is, no contradiction can be characterized by transitivity. Thus, if a fuzzy preference relation verify any of the previous transitivity properties we can say that it is consistent in that particular way. For example, if a fuzzy preference relation verifies Restricted MAX-MIN Transitivity we will say that it is restricted MAX-MIN consistent.

However, due to its good properties, additive transitivity is the only property that we will assume throughout this dissertation. In fact, *expression 1.1* can be rewritten as:

$$p_{ik} = p_{ij} + p_{jk} - 0.5 \quad \forall i, j, k \in \{1, \ldots, n\} \tag{1.2}$$

and thus, we will consider a fuzzy preference relation to be *additive consistent* when for every three options in the problem $x_i, x_j, x_k \in X$ their associated preference degrees p_{ij}, p_{jk}, p_{ik} fulfil *expression 1.2.* An additive consistent fuzzy preference relation will be referred as *consistent* throughout the dissertation, as this is the only transitivity property we are considering.

1.5 Aggregation Operators

Aggregation is the operation that transforms a set of elements (fuzzy sets, individual opinions about a set of alternatives, and so on) in a single element which is representative of the whole set [17, 16, 72]. In group decision making aggregation is made over individual experts' preferences about the set of alternatives to obtain a *global preference* which is a summary of their properties. The information aggregation problem has been widely studied. There exist lots of publications about this topic, and we must cite [17, 16, 41, 72].

In this section we present two different but related families of aggregation operators which will be used in the next chapters.

1.5.1 Ordered Weighted Averaging Operator

Ordered Weighted Averaging (OWA) operator was proposed by Yager in [72] and laterly have been more insightfully studied and characterized in [73]. OWA operator is commutative, idempotent, continuous, monotonic, neutral, compensative and stable for positive linear transformations. A fundamental aspect of the OWA operator is the reordering of the arguments to be aggregated, based upon the magnitude of their respective values:

Definition 1.10 ([72]). An OWA operator of dimension n is a function $\phi\colon \Re^n \to \Re$, that has a set of weights or weighting vector associated with it, $W = (w_1, ..., w_n)$, with $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$, and it is defined to aggregate a list of values $\{p_1, ..., p_n\}$ according to the following expression,

$$\phi_W(p_1, ..., p_n) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)} \tag{1.3}$$

being $\sigma\colon \{1, ..., n\} \to \{1, ..., n\}$ a permutation such that $p_{\sigma(i)} \geq p_{\sigma(i+1)}, \forall i = 1, ..., n-1$, i.e., $p_{\sigma(i)}$ is the i-highest value in the set $\{p_1, ..., p_n\}$.

A natural question in the definition of the OWA operator is how to obtain the associated weighting vector. In [72], Yager proposed two ways to obtain it. The first approach is to use some kind of learning mechanism using some sample data; while the second one tries to give some semantics or meaning to the weights. The latter possibility has allowed multiple applications in the area we are interested in, quantifier-guided aggregation [71].

In the process of quantifier guided aggregation, given a collection of n criteria represented as fuzzy subsets of the alternatives X, the OWA operator is used to implement the concept of fuzzy majority in the aggregation phase by means of a *fuzzy linguistic quantifier* [80] which indicates the proportion of satisfied criteria *necessary for a good solution* [74] (see *appendix A* for more details). This implementation is done by using the quantifier to calculate the OWA weights. In the case of a regular increasing monotone (RIM) quantifier Q, the procedure to evaluate the overall satisfaction of Q criteria (or experts) (e_k) by the alternative x_j is carried out calculating the OWA weights as follows:

$$w_i = Q(i/n) - Q((i-1)/n), \quad i = 1, \ldots, n. \tag{1.4}$$

When a fuzzy quantifier Q is used to compute the weights of the OWA operator ϕ, then it is symbolized by ϕ_Q. We must note that this type of aggregation 'is very strongly dependent upon the weighting vector used' [74], and consequently also upon the function expression used to represent the fuzzy linguistic quantifier.

In [74], Yager also proposed a procedure to evaluate the overall satisfaction of Q importance (u_k) criteria (or experts) (e_k) by the alternative x_j. In this procedure, once the satisfaction values to be aggregated have been ordered, the weighting vector associated with an OWA operator using a linguistic quantifier Q are calculated following the expression

$$w_i = Q\left(\frac{\sum_{k=1}^{i} u_{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=1}^{i-1} u_{\sigma(k)}}{T}\right) \tag{1.5}$$

being $T = \sum_{k=1}^{n} u_k$ the total sum of importance, and σ the permutation used to produce the ordering of the values to be aggregated. This approach for the inclusion of importance degrees associates a zero weight to those experts with a zero importance degree.

1.5.2 Induced Ordered Weighted Averaging Operators

Inspired by the work of Mitchell and Estrakh [53], Yager and Filev in [76] defined the *Induced Orderec Weighted Averaging* (IOWA) operator as an extension of the OWA operator to allow a different reordering of the values to be aggregated:

Definition 1.11 ([76]). An IOWA operator of dimension n is a function $\Phi_W\colon (\Re \times \Re)^n \rightarrow \Re$, to which a set of weights or weighting vector is associated, $W = (w_1, \ldots, w_n)$, with $w_i \in [0,1]$, $\Sigma_i w_i = 1$, and it is defined to aggregate the set of second arguments of a list of n 2-tuples $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$ according to the following expression,

$$\Phi_W\left(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\right) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)} \tag{1.6}$$

being σ a permutation of $\{1, \ldots, n\}$ such that $u_{\sigma(i)} \geq u_{\sigma(i+1)}, \forall i = 1, \ldots, n-1$, i.e., $\left\langle u_{\sigma(i)}, p_{\sigma(i)}\right\rangle$ is the 2-tuple with $u_{\sigma(i)}$ the i-th highest value in the set $\{u_1, \ldots, u_n\}$.

In the above definition, the reordering of the set of values to be aggregated, $\{p_1, \ldots, p_n\}$, is induced by the reordering of the set of values $\{u_1, \ldots, u_n\}$ associated with them, which is based upon their magnitude. Due to this use of the set of values $\{u_1, \ldots, u_n\}$, Yager and Filev called them the values of an order inducing variable and $\{p_1, \ldots, p_n\}$ the values of the argument variable [75, 76, 77].

Clearly, the aforementioned approaches to calculate the weighting vector of an OWA operator can also be applied to the case of IOWA operators. When a fuzzy linguistic quantifier Q is used to compute the weights of the IOWA operator Φ, then it is symbolized by Φ_Q.

1.6 Incomplete Information

One of the main problems that have to be faced when dealing with group decision making problems is lack of information. As aforementioned, each expert has his/her own experience concerning the problem being studied, which may imply a major drawback, that of an expert not having a perfect knowledge of the problem to be solved [1, 46, 45, 44, 67]. There are multiple possible causes for which an expert may not be able to efficiently express all the preference values that they are required. Some of those causes are:

- Not having enough knowledge about the different alternatives. Specially when there is a high number of alternatives, experts may not be familiar with all of them. For example, if the problem that the experts face is to find which of 10 different airlines are better, a particular expert may not have any knowledge about a particular airline, but can have a broad and extensive experience with the other airlines. In those cases, it is obvious that this particular expert will not be able to express any preference about his/her unknown airline.

- An expert might not be able to discriminate the degree to which some options are better than others. Even if an expert has a deep knowledge about the different alternatives, he might not be able to compare two of them or to precisely express a degree on which he/she prefers one alternative over another.

- Experts are usually required to provide consistent information, that is, their preferences might not imply contradiction. So, an expert could prefer to not give all the preferences that he is asked for to avoid introducing inconsistencies.

Therefore, it would be of great importance to provide the experts with tools that allow them to express this lack of knowledge in their opinions.

Traditionally, lack of information has been tackled as a certain kind of uncertainty. Thus, fuzzy sets, which have been a very powerful tool to model an tackle uncertainty, have been widely used to model this kind of situations. In our particular context, where experts' preferences are modelled by means of fuzzy preference relations, usual simple approches assumed that a preference value p_{ik} that was not given by an expert could be set to 0.5, thus meaning that the expert is indifferent between alternatives x_i and x_k. We think that this approach is not correct, as an expert not providing a particular preference value can be the result of the incapacity of the expert to quantify the degree of preference of one alternative over another, in which case he/she may decide not to *guess* to maintain the consistency of the values already provided. It must be clear then that when an expert is not able to express the particular value p_{ij}, because he/she does not have a clear idea of how better alternative x_i is over alternative x_j, this does not automatically mean that he/she prefers both options with the same intensity.

In the next chapter we will present an approach to correctly model this sitations of lack of information by means of *incomplete* fuzzy preference relations.

Chapter 2

A Selection Process For Group Decision Making With Incomplete Fuzzy Preference Relations

2.1 Incomplete Fuzzy Preference Relations

As we have already mentioned, missing information is a problem that we have to deal with because usual decision-making procedures assume that experts are able to provide preference degrees between any pair of possible alternatives, which is not always possible. We note that a missing value in a fuzzy preference relation is not equivalent to a lack of preference of one alternative over another. A missing value can be the result of the incapacity of an expert to quantify the degree of preference of one alternative over another, in which case he/she may decide not to *guess* to maintain the consistency of the values already provided. It must be clear then that when an expert is not able to express the particular value p_{ij}, because he/she does not have a clear idea of how better alternative x_i is over alternative x_j, this does not automatically mean that he/she prefers both options with the same intensity.

In order to model these situations, in the following definitions we express the concept of an incomplete fuzzy preference relation:

Definition 2.1. A function $f\colon X \longrightarrow Y$ is *partial* when not every element in the set X necessarily maps onto an element in the set Y. When every element from the set X maps onto one element of the set Y then we have a *total* function.

Definition 2.2. An *incomplete fuzzy preference relation* P on a set of alternatives X is a fuzzy set on the product set $X \times X$ that is characterized by a *partial* membership function.

As per this definition, we call a fuzzy preference relation complete when its membership function is a total one. Clearly, *definition 1.5* includes both definitions of complete and incomplete fuzzy preference relations. However, as there is no risk of confusion between a complete and an incomplete fuzzy preference relation, in this paper we refer to the first type as simply fuzzy preference relation.

2.2 Basic Estimation Of Missing Preference Values Based On Additive Consistency

As consistency is a desirable property for fuzzy preference relations, expression (1.2) can be used to calculate the value of a preference degree using other preference degrees in a fuzzy preference relation maintaining its internal consistency. Indeed, the preference value p_{ik} $(i \neq k)$ can be estimated using an intermediate alternative x_j in three different ways:

1. From $p_{ik} = p_{ij} + p_{jk} - 0.5$ we obtain the estimate

$$(cp_{ik})^{j1} = p_{ij} + p_{jk} - 0.5 \tag{2.1}$$

2. From $p_{jk} = p_{ji} + p_{ik} - 0.5$ we obtain the estimate

$$(cp_{ik})^{j2} = p_{jk} - p_{ji} + 0.5 \tag{2.2}$$

3. From $p_{ij} = p_{ik} + p_{kj} - 0.5$ we obtain the estimate

$$(cp_{ik})^{j3} = p_{ij} - p_{kj} + 0.5 \tag{2.3}$$

The overall estimated value cp_{ik} of p_{ik} is obtained as the average of all possible $(cp_{ik})^{j1}$, $(cp_{ik})^{j2}$ and $(cp_{ik})^{j3}$ values:

$$cp_{ik} = \frac{\sum_{j=1; i \neq k \neq j}^{n} (cp_{ik})^{j1} + (cp_{ik})^{j2} + (cp_{ik})^{j3}}{3(n-2)} \tag{2.4}$$

2.3 Consistency Measures

When the information provided is completely consistent then $(cp_{ik})^{jl} = p_{ik}\ \forall j, l$. However, because experts are not always fully consistent, the information given by an expert may not verify expression (1.2). Thus, it is necessary to determine the error between the estimated values and the given values to give a measure of how consistent is a particular fuzzy preference relation. Note that until this moment we could only check if a fuzzy preference relation is completely consistent or not. Now we are going to introduce some *consistency measures* that can help to identify which experts are being more inconsistent, and even in which particular preference values they are expressing a higher amount of contradiction.

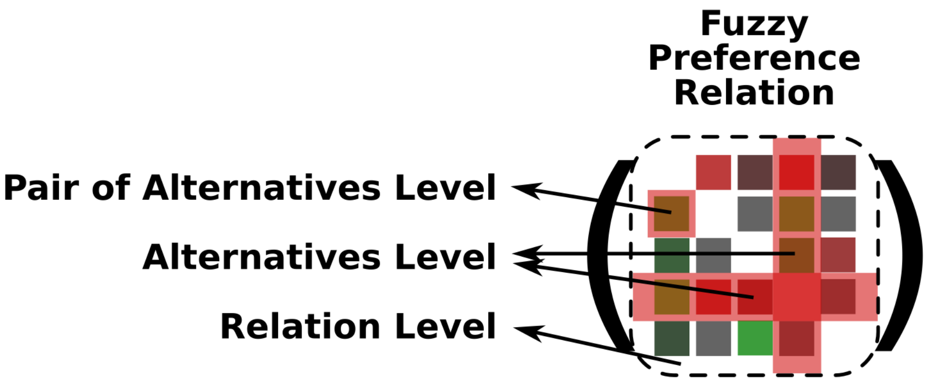

Figure 2.1: 3 Levels in Which Consistency Measures Are Defined

To do so, we will introduce the consistency measures for complete fuzzy preference relations at three different levels: *pairs of alternatives*, *alternatives* and *relation* levels (see *figure 2.1*). Laterly we will redefine them to be able to measure consistency in incomplete fuzzy preference relations.

2.3.1 Level of Pairs of Alternatives

As we have previously mentioned, experts are not always fully consistent and thus, the information given by an expert may not verify *expression 1.2* and some of the estimated preference degree values $(cp_{ik})^{jl}$ may not belong to the unit interval $[0, 1]$. We note, from *expressions 2.1–2.3*, that the maximum value of any of the preference degrees $(cp_{ik})^{jl}$ ($l \in \{1, 2, 3\}$) is 1.5 while the minimum one is -0.5. Taking this into account, we define the error between a preference value and its estimated one as follows:

Definition 2.3. The error between a preference value and its estimated one in $[0, 1]$ is computed as:

$$\varepsilon p_{ik} = \frac{2}{3} \cdot |cp_{ik} - p_{ik}| \tag{2.5}$$

This error can be used to define the consistency level between the preference degree p_{ik} and the rest of preference values of the fuzzy preference relation. Thus, it can be used to define the consistency level between the preference degree p_{ik} and the rest of the preference values of the fuzzy preference relation.

Definition 2.4. The consistency level associated to a preference value p_{ik} is defined as

$$cl_{ik} = 1 - \varepsilon p_{ik} \tag{2.6}$$

When $cl_{ik} = 1$ then $\varepsilon p_{ik} = 0$ and there is no inconsistency at all. The lower the value of cl_{ik}, the higher the value of εp_{ik} and the more inconsistent is p_{ik} with respect to the rest of information.

2.3.2 Level of Alternatives

We define the consistency measures for particular alternatives as:

Definition 2.5. The consistency measure associated to a particular alternative x_i of a fuzzy preference relation P is defined as

$$cl_i = \frac{\sum_{\substack{k=1 \\ i \neq k}}^{n} cl_{ik} + cl_{ki}}{2(n-1)} \tag{2.7}$$

with $cl_i \in [0, 1]$.

When $cl_i = 1$ all the preference values involving the alternative x_i are fully consistent, otherwise, the lower cl_i the more inconsistent this preference values are.

2.3.3 Level of Relation

Finally we define a consistency level for a whole fuzzy preference relation:

Definition 2.6. The consistency level of a fuzzy preference relation P is defined as follows:

$$cl = \frac{\sum_{i=1}^{n} cl_i}{n} \tag{2.8}$$

with $cl \in [0, 1]$.

When $cl = 1$ the preference relation P is fully consistent, otherwise, the lower cl the more inconsistent P.

Example 2.1. Suppose the following complete fuzzy preference relation

$$P = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix}$$

The computation of the consistency level of the preference value p_{43} is as follows:

$$
\begin{aligned}
(cp_{43})^{11} &= p_{41} + p_{13} - 0.5 = 0.6 + 0.6 - 0.5 = 0.7 \\
(cp_{43})^{21} &= p_{42} + p_{23} - 0.5 = 0.3 + 0.9 - 0.5 = 0.7 \\
(cp_{43})^{12} &= p_{13} - p_{14} + 0.5 = 0.6 - 0.4 + 0.5 = 0.7 \\
(cp_{43})^{22} &= p_{23} - p_{24} + 0.5 = 0.9 - 0.7 + 0.5 = 0.7 \\
(cp_{43})^{13} &= p_{41} - p_{31} + 0.5 = 0.6 - 0.4 + 0.5 = 0.7 \\
(cp_{43})^{23} &= p_{42} - p_{32} + 0.5 = 0.3 - 0.1 + 0.5 = 0.7
\end{aligned}
$$

$$
cp_{43} = 0.7 \Rightarrow \varepsilon p_{34} = \frac{2}{3} \cdot |cp_{34} - p_{34}| = 0 \Rightarrow cl_{34} = 1 - \varepsilon p_{34} = 1
$$

The same consistency value 1 is obtained for all the preference values of this fuzzy preference relation, which means that it is a completely additive consistent fuzzy preference relation.

2.3.4 Consistency Measures For Incomplete Fuzzy Preference Relations

When working with an incomplete fuzzy preference relation, expression (2.4) cannot be used to obtain the estimate of a known preference value, and therefore the above definitions of cl_{ik} has to be extended.

If expert e_h provides an incomplete fuzzy preference relation P^h, the following sets are defined [31]:

$$
\begin{aligned}
&A = \{(i,j) \mid i,j \in \{1,\ldots,n\} \wedge i \neq j\}\\
&MV^h = \left\{(i,j) \in A \mid p^h_{ij} \ is \ unknown\right\}\\
&EV^h = A \setminus MV_h\\
&H^{h1}_{ik} = \left\{j \neq i,k \mid (i,j),(j,k) \in EV^h\right\}\\
&H^{h2}_{ik} = \left\{j \neq i,k \mid (j,i),(j,k) \in EV^h\right\}\\
&H^{h3}_{ik} = \left\{j \neq i,k \mid (i,j),(k,j) \in EV^h\right\}\\
&EV^h_i = \{(a,b) \mid (a,b) \in EV^h \ \wedge \ (a = i \vee b = i)\}
\end{aligned}
$$

MV^h is the set of pairs of alternatives whose preference degrees are not given by expert e_h, EV^h is the set of pairs of alternatives whose preference degrees are given by the expert e_h; H^{h1}_{ik}, H^{h2}_{ik}, H^{h3}_{ik} are the sets of intermediate alternative x_j $(j \neq i,k)$ that can be used to estimate the preference value p^h_{ik} $(i \neq k)$ using equations (2.1), (2.2), (2.3) respectively; and EV^h_i is the set of pairs of alternatives whose preference degrees involving the alternative x_i are given by the expert e_h.

The estimated value of a particular preference degree p^h_{ik} $((i,k) \in EV^h)$ can be calculated only when $\#(H^{h1}_{ik} \cup H^{h2}_{ik} \cup H^{h3}_{ik}) \neq 0$:

$$cp^h_{ik} = \frac{\sum_{j \in H^{h1}_{ik}} (cp^h_{ik})^{j1} + \sum_{j \in H^{h2}_{ik}} (cp^h_{ik})^{j2} + \sum_{j \in H^{h3}_{ik}} (cp^h_{ik})^{j3}}{(\#H^{h1}_{ik} + \#H^{h2}_{ik} + \#H^{h3}_{ik})} \tag{2.9}$$

In the case of being $(\#H^{h1}_{ik} + \#H^{h2}_{ik} + \#H^{h3}_{ik}) = 0$ then the preference value p^h_{ik} $((i,k) \in EV^h)$ cannot be estimated using the rest of known values. In these cases we define $\varepsilon p^h_{ik} = 0$. Additionally if $p^h_{ik} \in MV^h$ we also consider that $\varepsilon p^h_{ik} = 0$ because there is no possible error in the estimation (we don't have a particular value to compare the estimation).

In decision-making situations with incomplete information, the notion of completeness is also an important factor to take into account when analyzing the consistency. Clearly, the higher the number of preference values provided by an expert the higher the chance of inconsistency. Therefore, a degree of completeness associated to the number of preference values provided should also be taken into account to produce a fairer measure of consistency of an incomplete fuzzy preference relation.

Given an incomplete fuzzy preference relation P^h, we can easily characterize *the completeness level of an alternative*, C_i^h, as the ratio between the actual number of preference values known for x_i, $\#EV_i^h$, and the total number of possible preference values in which x_i is involved with a different alternative, $2(n-1)$:

$$C_i^h = \frac{\#EV_i^h}{2(n-1)} \tag{2.10}$$

Definition 2.7 ([31]). The consistency level cl_{ik}^h, associated to a preference value $p_{ik}^h,\ (i,k) \in EV^h$, is defined as a linear combination of its associated error, εp_{ik}^h, and the average of the completeness values associated to the two alternatives involved in that preference degree, C_i^h and C_k^h:

$$cl_{ik}^h = (1-\alpha_{ik}^h)\cdot(1-\varepsilon p_{ik}^h) + \alpha_{ik}^h \cdot \frac{C_i^h + C_k^h}{2} \quad ; \ \alpha_{ik}^h \in [0,1] \tag{2.11}$$

The parameter α_{ik}^h controls the influence of completeness in the evaluation of the consistency levels. Thus, parameter α_{ik}^h should decrease with respect to the number of preference values known, in such a way that it takes the value of 0 when all the preference values in which x_i and x_k are involved are known, in which case the completeness concept lacks any meaning and, therefore, should not be taken into account; and it takes the value of 1 when no values are known.

The total number of different preference values involving the alternatives x_i and x_k is equal to $4(n-1)-2$: the total number of possible preference values involving x_i ($2(n-1)$) plus the the total number of possible preference values involving x_k ($2(n-1)$) minus the common preference value involving x_i and x_k, p^h_{ik} and p^h_{ki}. The number of different preference values known for x_i and x_k is $\#EV_i^h + \#EV_k^h - \#(EV_i^h \cap EV_k^h)$. Thus, we claim that $\alpha^h_{ik} = f(\#EV_i^h + \#EV_k^h - \#(EV_i^h \cap EV_k^h))$, being f a decreasing function with $f(0) = 1$ and $f(4(n-1)-2) = 0$. The simple linear solution could be used to obtain the corresponding parameter α^h_{ik} [31]:

$$\alpha^h_{ik} = 1 - \frac{\#EV_i^h + \#EV_k^h - \#(EV_i^h \cap EV_k^h)}{4(n-1)-2} \tag{2.12}$$

Clearly, expression (2.11) is an extension of expression (2.6), because when P is complete both EV and A coincide and $\alpha_{ik} = 0\ \forall i, k$.

The consistency measures for alternatives and the whole preference relation are computed in the same way as they were defined for complete fuzzy preference relations (*expressions 2.7 and 2.8*).

2.4 Iterative Procedure To Estimate Missing Values In Fuzzy Preference Relations Based in Consistency Properties

As we have already mentioned, missing information is a problem that has to be addressed because experts are not always able to provide preference degrees between every pair of possible alternatives. Nevertheless, in this section we will show that these values can be estimated from the existing information.

Usual procedures for GDM problems correct this lack of knowledge of a particular expert using the information provided by the rest of experts together with aggregation procedures [44]. These kind of approaches have several disadvantages. Amongst them we can cite the requirement of multiple experts in order to estimate the missing value of a particular expert. Another drawback is that these procedures do not usually take into account the differences between experts' preferences, which could lead to the estimation of a missing value that would not naturally be compatible with the rest of the preference values given by that expert. Finally, some of these missing information retrieval procedures are interactive, that is, they need experts to collaborate in *real time*, an option which is not always possible.

Our proposal is quite different to the above approaches. We put forward a procedure that estimates missing information in an expert's incomplete fuzzy preference relation using only the rest of the preference values provided by that particular expert. By doing this, we assure that the reconstruction of the incomplete fuzzy preference relation is compatible with the rest of the information provided by the expert.

In order to develop the iterative procedure to estimate missing values two different tasks have to be carried out:

A) Establish the elements that can be estimated in each step of the procedure, and

B) produce the particular expression that will be used to estimate a particular missing value.

2.4.1 Elements To Be Estimated In Every Iteration

Given an incomplete fuzzy preference relation P^h, the subset of missing values MV^h that can be estimated in step t is denoted by EMV_t^h and defined as follows:

$$EMV_t^h = \left\{ (i,k) \in MV^h \setminus \bigcup_{l=0}^{t-1} EMV_l^h \mid i \neq k \ \wedge \ \exists j \in \{H_{ik}^{h1} \cup H_{ik}^{h2} \cup H_{ik}^{h3}\} \right\} \tag{2.13}$$

and $EMV_0^h = \emptyset$ (by definition). When $EMV_{maxIter}^h = \emptyset$ with $maxIter > 0$ the procedure will stop as there will not be any more missing values to be estimated. Moreover, if $\bigcup_{l=0}^{maxIter} EMV_l^h = MV^h$ then all missing values are estimated, and consequently, the procedure is said to be successful in the completion of the incomplete fuzzy preference relation.

2.4.2 Expression To Estimate a Particular Value

In order to estimate a particular value p_{ik}^h with $(i,k) \in EMV_t^h$, the following function $estimate\_p(h,i,k)$ is used function estimate_p(h,i,k)

1. $(cp_{ik}^h)^1 = 0,\ (cp_{ik}^h)^2 = 0,\ (cp_{ik}^h)^3 = 0$
2. $if \ \ \#H_{ik}^{h1} \neq 0 \ \ then \ \ (cp_{ik}^h)^1 = \sum_{j \in H_{ik}^{h1}} (cp_{ik}^h)^{j1}$
3. $if \ \ \#H_{ik}^{h2} \neq 0 \ \ then \ \ (cp_{ik}^h)^2 = \sum_{j \in H_{ik}^{h2}} (cp_{ik}^h)^{j2}$
4. $if \ \ \#H_{ik}^{h3} \neq 0 \ \ then \ \ (cp_{ik}^h)^3 = \sum_{j \in H_{ik}^{h3}} (cp_{ik}^h)^{j3}$
5. Calculate $cp_{ik}^h = \dfrac{(cp_{ik}^h)^1 + (cp_{ik}^h)^2 + (cp_{ik}^h)^3}{(\#H_{ik}^{h1} + \#H_{ik}^{h2} + \#H_{ik}^{h3})}$

end function

The function $estimate\_p(i, k)$ computes the final estimated value of the missing value, cp_{ik}, as the average of all possible estimated values that can be computed using all the possible intermediate alternatives x_j and using the three possible expressions (2.1–2.3).

We should point out that some estimated values of an incomplete fuzzy preference relation could lie outside the unit interval, i.e. for some (i, k) we may have $cp_{ik} < 0$ or $cp_{ik} > 1$. In order to normalize the expression domains in the decision model the following function is used

$$f(y) = \begin{cases} 0 & if \ y < 0 \\ 1 & if \ y > 1 \\ y & otherwise \end{cases}$$

We also point out that the consistency level, cl_{ik} of an estimated preference value cp_{ik}, if necessary, may be easily computed inside the function $estimate\_p(i, k)$.

2.4.3 Iterative Estimation Procedure Pseudo-Code

The complete iterative estimation procedure is the following

ITERATIVE ESTIMATION PROCEDURE

0. $EMV_0^h = \emptyset$
1. $t = 1$
2. while $EMV_t^h \neq \emptyset$ {
3. for every $(i,k) \in EMV_t^h$ {
4. estimate_p(h,i,k)
5. }
6. $t++$
7. }

2.4.4 Example of Application of the Estimation Procedure

Suppose the following incomplete fuzzy preference relation

$$P^1 = \begin{pmatrix} - & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.4} \\ x & - & x & x \\ x & x & - & x \\ x & x & x & - \end{pmatrix}$$

The application of the estimation procedure is provided:

Step 1 (t=1): The set of elements that can be estimated are:

$$EMV_1^1 = \{(2,3),(2,4),(3,2),(3,4),(4,2),(4,3)\}\}$$

After these elements have been estimated, we have:

$$P^1 = \begin{pmatrix} - & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.4} \\ x & - & 0.9 & 0.7 \\ x & 0.1 & - & 0.3 \\ x & 0.3 & 0.7 & - \end{pmatrix}$$

As an example, to estimate p^1_{43} the procedure is as follows:

$$\begin{array}{lll} H^{11}_{43} = \emptyset & \Rightarrow & (cp^1_{43})^1 = 0 \\ H^{12}_{43} = \{1\} & \Rightarrow & (cp^1_{43})^{12} = p^1_{13} - p^1_{14} + 0.5 = 0.6 - 0.4 + 0.5 = 0.7 \Rightarrow \\ & \Rightarrow & (cp^1_{43})^2 = 0.7 \\ H^{13}_{43} = \emptyset & \Rightarrow & (cp^1_{43})^3 = 0 \end{array}$$

$$cp^1_{43} = \frac{0 + 0.7 + 0}{1} = 0.7$$

Step 2 (t=2): The set of elements that can be estimated are:

$$EMV^1_2 = \{(2,1),(3,1),(4,1)\}\}$$

After these elements have been estimated, we have the following completed fuzzy preference relation:

$$P^1 = \begin{pmatrix} - & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.4} \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix}$$

As an example, to estimate p^1_{41} the procedure is as follows:

$$\begin{array}{lll} H^{11}_{41} = \emptyset & \Rightarrow & (cp^1_{41})^1 = 0 \\ H^{12}_{41} = \emptyset & \Rightarrow & (cp^1_{41})^2 = 0 \\ H^{13}_{41} = \{2,3\} & \Rightarrow & \left\{ \begin{array}{l} (cp^1_{41})^{23} = p^1_{42} - p^1_{12} + 0.5 = 0.6 \\ (cp^1_{41})^{33} = p^1_{43} - p^1_{13} + 0.5 = 0.6 \end{array} \right\} \Rightarrow (cp^1_{41})^3 = 1.2 \end{array}$$

$$cp_{41}^{1} = \frac{0 + 0 + 1.2}{2} = 0.6$$

2.4.5 Sufficient Conditions To Estimate All Missing Values

It is very important to establish conditions that guarantee that all the missing values of an incomplete fuzzy preference relation can be estimated. We assume that experts provide their judgements freely by means of incomplete fuzzy preference relations with preferences degrees $p_{ik} \in [0, 1]$ and $p_{ii} = 0.5$, without any other restriction, as for example, that of imposing the additive reciprocity property.

In the following, we provide sufficient conditions that guarantee the success of the above iterative estimation procedure.

1. If for all $p_{ik} \in MV$ $(i \neq k)$ there exists at least a $j \in \{H_{ik}^1 \cup H_{ik}^2 \cup H_{ik}^3\}$ then all missing preference values can be estimated in the first iteration of the procedure $(EMV_1 = MV)$.

2. Under the assumption of additive consistency property, a different sufficient condition was given in [33]. This condition states that any incomplete fuzzy preference relation can be completed when the set of $n - 1$ preference values $\{p_{12}, p_{23}, \ldots, p_{(n-1)n}\}$ is known.

3. A more general condition than the previous one which includes that when a complete row or column of preference values is known is given in the following proposition.

Proposition 2.1. *An incomplete fuzzy preference relation can be completed if a set of $n-1$ non-leading diagonal preference values, where each one of the alternatives is compared at least once, is known.*

Proof. Proof by induction on the number of alternatives will be used:

1. Basis: For $n = 3$, we suppose that two preference degrees involving the three alternatives are known. These degrees can be provided in three different ways:

   (a) p_{ij} and p_{jk} $(i \neq j \neq k)$ are given.

   In this first case, all the possible combinations of the two preference values are: $\{p_{12}, p_{23}\}$, $\{p_{13}, p_{32}\}$, $\{p_{21}, p_{13}\}$, $\{p_{23}, p_{31}\}$, $\{p_{31}, p_{12}\}$ and $\{p_{32}, p_{21}\}$. In any of these cases, we can find the remaining preference degrees of the relation $\{p_{ik}, p_{kj}, p_{ji}, p_{ki}\}$ as follows:

   $$p_{ik} = p_{ij} + p_{jk} - 0.5 \quad ; \quad p_{kj} = p_{ik} - p_{ij} + 0.5$$
   $$p_{ji} = p_{jk} - p_{ik} + 0.5 \quad ; \quad p_{ki} = p_{kj} - p_{ij} + 0.5$$

   (b) p_{ji} and p_{jk} $(i \neq j \neq k)$ are given.

   In this second case, all the possible combinations of the two preference values are: $\{p_{21}, p_{23}\}$, $\{p_{31}, p_{32}\}$ and $\{p_{12}, p_{13}\}$. In any of these cases, we can find the remaining preference degrees of the relation $\{p_{ik}, p_{ki}, p_{kj}, p_{ij}\}$ as follows:

   $$p_{ik} = p_{jk} - p_{ji} + 0.5 \quad ; \quad p_{ki} = p_{ji} - p_{jk} + 0.5$$
   $$p_{kj} = p_{ki} - p_{ji} + 0.5 \quad ; \quad p_{ij} = p_{kj} - p_{ki} + 0.5$$

   (c) p_{ij} and p_{kj} $(i \neq j \neq k)$ are given.

   In this third case, all the possible combinations of the two preference values are: $\{p_{12}, p_{32}\}$, $\{p_{13}, p_{23}\}$ and $\{p_{21}, p_{31}\}$. In any of these cases, we can find the remaining preference degrees of the relation $\{p_{ik}, p_{ki}, p_{ji}, p_{jk}\}$ as follows:

$$p_{ik} = p_{ij} - p_{kj} + 0.5 \quad ; \quad p_{ki} = p_{kj} - p_{ij} + 0.5$$
$$p_{ji} = p_{ki} - p_{kj} + 0.5 \quad ; \quad p_{jk} = p_{ik} - p_{ij} + 0.5$$

2. Induction hypothesis: Let us assume that the proposition is true for $n = q - 1$.

3. Induction step: Let us suppose that the expert provides only $q - 1$ preference degrees where each one of the q alternatives is compared at least once.

   In this case, we can select a set of $q - 2$ preference degrees where $q - 1$ different alternatives are involved. Without loss of generality, we can assume that these $q-1$ alternatives are $x_1, x_2, \ldots, x_{q-1}$, and therefore the remaining preference degree involving the alternative x_q could be p_{qi} $(i \in \{1, \ldots, q-1\})$ or p_{iq} $(i \in \{1, \ldots, q-1\})$.

   By the induction hypothesis we can estimate all the preference values of the fuzzy preference relation of order $(q-1)\times(q-1)$ associated with the set of alternatives $\{x_1, x_2, \ldots, x_{q-1}\}$. Therefore, we have estimates for the following set of preference degrees

$$\{p_{ij},\ i, j = 1, \ldots, q-1,\ i \neq j\}.$$

   If the value we know is $p_{qi},\ i \in \{1, \ldots, q-1\}$ then we can estimate $\{p_{qj},\ j = 1, \ldots, q-1,\ i \neq j\}$ and $\{p_{jq},\ j = 1, \ldots, q-1\}$ using $p_{qj} = p_{qi} + p_{ij} - 0.5,\ \forall j$ and $p_{jq} = p_{ji} - p_{qi} + 0.5,\ \forall j$, respectively.

   If the value we know is $p_{iq},\ i \in \{1, \ldots, q-1\}$ then $\{p_{qj},\ j = 1, \ldots, q-1\}$ and $\{p_{jq},\ j = 1, \ldots, q-1,\ i \neq j\}$, are estimated by means of $p_{qj} = p_{ij} - p_{iq} + 0.5, \ \forall j$ and $p_{jq} = p_{ji} + p_{iq} - 0.5, \ \forall j$, respectively.

□

2.4.6 Strategies To Manage Ignorance Situations

In this section we will study what an ignorance situations is in decision making problems with fuzzy preference relations, and will present several strategies to manage these situations. The strategies aim is to complete the information which was not provided by the experts.

We have divided the strategies in two main groups: *ad-hoc strategies* and *criteria guided strategies*.

As we have already mentioned, missing information situations can be quite common in decision problems. Particularly, if the experts are allowed to use incomplete fuzzy preference relations to express their opinions about the alternatives, some information (preference values) can be missing.

Previous studies have shown different approaches to solve these situations [1, 31, 67]. However, not every missing information situation can be solved in the same way, and situations where experts do not provide *any* information about particular alternatives are difficult to solve. For example, in section 2.4.3 we have outlined a procedure that allows the computation of the missing values in an incomplete fuzzy preference relation, but this procedure cannot correctly handle incomplete fuzzy preference relation which does not have any information about a particular alternative. We will say that there is an “ignorance situation” when an incomplete fuzzy preference relation does not provide any information about an alternative:

Definition 4. In a decision making problem with a set of alternatives $X = \{x_1, ..., x_n\}$ and some experts $E = \{e_1, ..., e_m\}$ which provide their incomplete fuzzy preference relations $P_1, ..., P_m$ we will have a *ignorance situation* if $\exists\ (h, i) \mid EV_h^i = \emptyset$, that is, at least one of the experts (e_h) does not provide any preference value involving a particular alternative (x_i). We will call x_i the "unknown alternative" for the expert e_h.

2.4.6.1 Ad-hoc Strategies

In this section we present some direct approaches to solve ignorance situati ons with fuzzy preference relations. This strategies do not need any external source of information, and they are not guided by any specific criteria or property of the preference relations.

Strategy 1: Assume Indifference on the Unknown Alternative

The first strategy consists on assuming that the expert does not have any kind of preference over the unknown alternative, that is, it completes every missing value for the unknown alternative with a 0.5 value:

Estimation Procedure 1: If an incomplete fuzzy preference relation P_h has an unknown alternative x_i, this strategy will compute every missing value as:

$$p_{ik}^h = 0.5\ ;\ p_{ki}^h = 0.5\ \ \forall k \in \{1, ..., n\}, k \neq i.$$

Example 1: We have to solve a decision making problem to find the best of 4 different alternatives: $X = \{x_1, x_2, x_3, x_4\}$. An expert gives the following incomplete fuzzy preference relation

$$P = \begin{pmatrix} - & 0.7 & x & 0.68 \\ 0.4 & - & x & 0.7 \\ x & x & - & x \\ 0.6 & 0.75 & x & - \end{pmatrix},$$

that is, he gives no information about alternative x_3, and thus, we are in a *ignorance situation*. The first estimation procedure assumes that the expert is indifferent with respect to x_3, and the reconstructed fuzzy preference relation is:

$$P = \begin{pmatrix} - & 0.7 & \mathbf{0.5} & 0.68 \\ 0.4 & - & \mathbf{0.5} & 0.7 \\ \mathbf{0.5} & \mathbf{0.5} & - & \mathbf{0.5} \\ 0.6 & 0.75 & \mathbf{0.5} & - \end{pmatrix}.$$

Strategy 2: Random Preference Values for the Unknown Alternative

This strategy computes the missing values for the unknown alternative as random values limited by the maximum and minimum preference values provided by the expert on the known alternatives:

Estimation Procedure 2: If an incomplete fuzzy preference relation P_h has an unknown alternative x_i, this strategy will compute every missing value as:

$$p^h_{ik} = rand(min(\{p^h_{jk}\}), max(\{p^h_{jk}\}))$$
$$p^h_{ki} = rand(min(\{p^h_{kj}\}), max(\{p^h_{kj}\}))$$
$$\forall j, k \in \{1, ..., n\}, j \neq k \neq i$$

where $rand(a, b)$ means a random value between a and b and $max(...)$ and $min(...)$ are the usual maximum and minimum operators.

Example 2: We part from the previously presented problem (in example 1). In this case, the estimation procedure reconstructs the missing values with random values between the maximum and minimum preference degrees provided by the expert. For example, $p_{13} \in [0.68, 0.7]$ and $p_{32} \in [0.7, 0.75]$. An example of a possible reconstructed preference relation is:

$$P = \begin{pmatrix} - & 0.7 & \mathbf{0.69} & 0.68 \\ 0.4 & - & \mathbf{0.47} & 0.7 \\ \mathbf{0.53} & \mathbf{0.71} & - & \mathbf{0.7} \\ 0.6 & 0.75 & \mathbf{0.72} & - \end{pmatrix}.$$

2.4.6.2 Criteria Guided Strategies

The following strategies apply some criteria or property about the incomplete preference relations in order to reconstruct the ignored information. We distinguish among three different approaches: *consistency based strategies*, *consensus and proximity guided strategies* and *mixed guided strategies*.

Consistency Guided Strategies

These strategies use the estimation procedure presented in section 2.4.3, which was able to estimate missing values taking into account the consistency expressed by the expert. Thus, these strategies do use the consistency property of the incomplete fuzzy preference relation. As it was previously mentioned that procedure needs at least a preference value involving the unknown alternative in order to estimate the other preference values.

Strategy 3: Based On An Indifferent Seed Value

Similarly as we did on the strategy 1, we can assume some initial indifference on the preference values for the unknown alternative, and then apply the estimation procedure to complete the missing values. In fact, the procedure can be simplified if we directly apply the expressions derived from the additive transitivity property:

Estimation Procedure 3: We have an incomplete fuzzy preference relation P with an unknown alternative x_i. We assume that $p_{ij} = 0.5$ for a $j \in \{1, \ldots, n\}$ (initial indifference). Then the preference degrees $\{p_{ik}\}, \forall k \neq i \neq j$ can be estimated via the alternative x_j by means of two of the three possible estimation equations (2.1–2.3):

$$cp_{ik}^{j1} = p_{ij} + p_{jk} - 0.5 \;\; \text{and} \;\; p_{ij} = cp_{ik}^{j3} + p_{kj} - 0.5,$$

which result in

$$cp_{ik}^{j1} = p_{jk} \;\; \text{and} \;\; cp_{ik}^{j3} = 1 - p_{kj},$$

respectively. Because the indifference of a preference value can be assumed for any of the possible values of $j \in \{1, \ldots, n\}$ with $j \neq i \neq k$, then the final estimate values for the i-th row of the incomplete fuzzy preference relation are:

$$cp_{ik} = \frac{1}{2}\left(\frac{\sum_{j=1;j\neq i\neq k}^{n} cp_{ik}^{j1}}{n-2} + \frac{\sum_{j=1;j\neq i\neq k}^{n} cp_{ik}^{j3}}{n-2}\right) =$$

$$\frac{1}{2}\left(\frac{\sum_{j=1\;;\;j\neq i\neq k}^{n} p_{jk}}{n-2} + \frac{\sum_{j=1\;;\;j\neq i\neq k}^{n} (1 - p_{kj})}{n-2}\right) =$$

$$0.5 + \frac{SC_k - SR_k}{2}$$

with SC_k and SR_k representing the average of the k-th column and k-th row of the complete $(n-1) \times (n-1)$ fuzzy preference relation that is obtained without taking into account the alternative x_i. The parallel application of the above assumption for the preference values p_{ki} provides the following estimate of the values of the i-th column:

$$cp_{ki} = 0.5 + \frac{SR_k - SC_k}{2}$$

Example 3: If we apply this strategy to the previously mentioned problem (examples 1 and 2), we obtain the following values for p_{13} and p_{32}:

$$p_{13} = 0.5 + \frac{(0.7 + 0.68)/2 - (0.4 + 0.6)/2}{2} = 0.6$$

and

$$p_{32} = 0.5 + \frac{(0.7 + 0.75)/2 - (0.4 + 0.7)/2}{2} = 0.59$$

In this case, the complete reconstructed preference relation is:

$$P = \begin{pmatrix} - & 0.7 & \mathbf{0.6} & 0.68 \\ 0.4 & - & \mathbf{0.41} & 0.7 \\ \mathbf{0.4} & \mathbf{0.59} & - & \mathbf{0.51} \\ 0.6 & 0.75 & \mathbf{0.49} & - \end{pmatrix}.$$

Strategy 4: Based On a Random Seed Value

This strategy is based on obtaining just one random seed value (calculated as in strategy 2), and then to apply the procedure to estimate the rest of missing values for the unknown alternative.

Estimation Procedure 4: We have an incomplete fuzzy preference relation P_h with an unknown alternative x_i. We apply the following scheme:

1. *do* {
2. $k = irand(1, n)$ // Choose random k
3. } $while(k \neq i)$
4. *if* $(rand(0, 1) < 0.5)$ { // Place it in missing row
5. $p^h_{ik} = rand(min(\{p^h_{jk}\}), max(\{p^h_{jk}\}))$
   $\forall j \in \{1, ..., n\}, j \neq k \neq i$
6. } *else* { // Place it in missing column
7. $p^h_{ki} = rand(min(\{p^h_{kj}\}), max(\{p^h_{kj}\}))$
   $\forall j \in \{1, ..., n\}, j \neq k \neq i$
8. }
9. Apply the estimation procedure

where $irand(a, b)$ means an integer random value between a and b.

Example 4: From the problem presented in the previous examples, we are going to apply this strategy to reconstruct the missing values. First of all, we obtain a random $k \neq i$. For example $k = 2$. We obtain a random value between $[0, 1]$ to determine if we are going to calculate a seed value for p_{32} or p_{23}. Suppose that the random value is 0.34, so we are going to obtain a random value for $p_{32} \in [0.7, 0.75]$, for example, $p_{32} = 0.74$. Then, we apply the estimation procedure:

$$\begin{pmatrix} - & 0.7 & x & 0.68 \\ 0.4 & - & x & 0.7 \\ x & 0.74 & - & x \\ 0.6 & 0.75 & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 0.7 & \mathbf{0.46} & 0.68 \\ 0.4 & - & x & 0.7 \\ \mathbf{0.59} & \mathbf{0.74} & - & \mathbf{0.61} \\ 0.6 & 0.75 & \mathbf{0.51} & - \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} - & 0.7 & \mathbf{0.46} & 0.68 \\ 0.4 & - & \mathbf{0.42} & 0.7 \\ \mathbf{0.59} & \mathbf{0.74} & - & \mathbf{0.61} \\ 0.6 & 0.75 & \mathbf{0.51} & - \end{pmatrix}$$

Proximity And Consensus Guided Strategies

The strategies presented in this section make use of additional information (not only the own preference relations) to complete the missing information about the unknown alternatives.

Depending on the problem, some different sources or criteria may be available.

Strategy 5: Proximity Of The Alternatives

In some decision problems there exist information which relates the different alternatives between them. For example, for a particular problem, some of the alternatives could be very similar. If this is the case, the preference values from an unknown alternative may be computed as some small random changes from the values provided for a similar alternative:

Estimation Procedure 5: If an incomplete fuzzy preference relation P_h has an unknown alternative x_i, and we know that alternative x_i is very similar to alternative x_j, this strategy will compute every missing value as:

$$\begin{aligned} p^h_{ik} &= rand(p^h_{jk} - \delta, p^h_{jk} + \delta) \\ p^h_{ki} &= rand(p^h_{kj} - \delta, p^h_{kj} + \delta) \\ &\forall k \in \{1, ..., n\}, k \neq i \neq j, \\ p^h_{ij} &= rand(0.5 - \delta, 0.5 + \delta) \\ p^h_{ji} &= rand(0.5 - \delta, 0.5 + \delta) \end{aligned}$$

where δ is a small factor (for example 0.1) that determines the magnitude on the change from the preference values of the similar alternative.

We must note that the similarities between the alternatives is additional external information about the problem, and it is not available in the proper preference relations. That is, if many experts choose very similar preference values for alternatives x_i and x_j, thay cannot be considered as if they were similar options. It would just mean that both options are more or less equally preferred, and thus, this strategy cannot be applied on the unknown alternative.

Example 5: We part from the problem presented in the previous examples. If we know that alternative x_3 is similar to x_2 and we assume that $\delta = 0.1$ we can estimate the unknown values following the *estimation procedure 5*. For example: $p_{13} = rand(p_{12} - 0.1, p_{12} + 0.1) = rand(0.6, 0.8)$. A possible result of the application of the procedure is:

$$\begin{pmatrix} - & 0.7 & x & 0.68 \\ 0.4 & - & x & 0.7 \\ x & x & - & x \\ 0.6 & 0.75 & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 0.7 & \mathbf{0.76} & 0.68 \\ 0.4 & - & \mathbf{0.52} & 0.7 \\ \mathbf{0.41} & \mathbf{0.49} & - & \mathbf{0.63} \\ 0.6 & 0.75 & \mathbf{0.71} & - \end{pmatrix}$$

Strategy 6: Proximity Among Experts

This approach studies the similarities between experts to find an expert similar to the expert which has not provided preference values for an unknown alternative. If sufficiently simmilar opinions from a different expert are found, then his / her preference values about the unknown alternative are used:

Estimation Procedure 6: If an incomplete fuzzy preference relation P_h given by an expert e_h has an unknown alternative x_i, we apply the following scheme:

1. For every expert $e_v \in E,\ e_v \neq e_h$ {
2. Compute $d_v = distance(e_v, e_h)$
3. }
4. e_{sel} = expert with minimum distance
5. if $(d_{sel} < \gamma)$ {
6. $p^h_{ik} = p^{sel}_{ik}$; $p^h_{ki} = p^{sel}_{ki}$ $\forall k \in \{1, ..., n\}, k \neq i.$
7. }

where $distance(e_a, e_b)$ computes a distance function between the two experts e_a and e_b and γ is a minimum distance threshold to avoid using preferences from an expert which is not near enough to e_h (which can be far away from all the other experts). Note that we do not provide a particular distance function as there are plenty that can be suitable to a particular problem.

Additionally, if there exist other measures about experts opinions (consistency measures [31], for example) this strategy can be improved to use them and choose not only an expert which is near, but which also has a high consistency level associated.

Example 6: Lets suppose that expert e_1 provides the incomplete preference relation of the previous examples (P_1). Additionally, experts e_2 and e_3 give the following preference relations:

$$P_2 = \begin{pmatrix} - & 0.6 & 0.4 & 0.7 \\ 0.4 & - & 0.7 & 0.4 \\ 0.6 & 0.35 & - & 0.6 \\ 0.3 & 0.7 & 0.4 & - \end{pmatrix} \; ; \; P_3 = \begin{pmatrix} - & 0.3 & 0.6 & 0.25 \\ 0.7 & - & 0.55 & 0.5 \\ 0.4 & 0.45 & - & 0.7 \\ 0.8 & 0.5 & 0.3 & - \end{pmatrix}$$

We compute a distance between the experts, for example, as a mean of the error between the preference values given by each expert:

$$d_2 = distance(e_2, e_1) = 0.13 \;\; ; \;\; d_3 = distance(e_3, e_1) = 0.3$$

Clearly, expert e_2 has more similar opinions about the alternatives than expert e_3. If we suppose that $\gamma = 0.15$, the incomplete preference relation P_1 would be reconstructed with the following values:

$$P_1 = \begin{pmatrix} - & 0.7 & \mathbf{0.4} & 0.68 \\ 0.4 & - & \mathbf{0.7} & 0.7 \\ \mathbf{0.6} & \mathbf{0.35} & - & \mathbf{0.6} \\ 0.6 & 0.75 & \mathbf{0.4} & - \end{pmatrix}$$

Strategy 7: Consensus Among Experts

If the resolution process for the decision problem involves obtaining a global (or consensued) opinion (preference relation) [31, 55, 62, 64] (by means of an aggregation function over the different preference relations, for example) and it can be applied even if there exist missing values (aggregating only the given preference values, for example), its preference values about the unknown alternative can be used:

Estimation Procedure 7: If an incomplete fuzzy preference relation P_h has an unknown alternative x_i, and we have a global preference relation P^c which represents the current global or consensued opinion of all the experts on the problem, we can complete every missing value as:

$$p_{ik}^h = p_{ik}^c \; ; \; p_{ki}^h = p_{ki}^c \quad \forall k \in \{1, ..., n\}, k \neq i.$$

Example 7: Lets suppose that expert e_1 provides the incomplete preference relation of the previous examples (P_1). Additionally, we know that the global (consensued) preference relation P^c is as follows:

$$P_1 = \begin{pmatrix} - & 0.7 & x & 0.68 \\ 0.4 & - & x & 0.7 \\ x & x & - & x \\ 0.6 & 0.75 & x & - \end{pmatrix} \; ; \; P^c = \begin{pmatrix} - & 0.43 & 0.57 & 0.42 \\ 0.5 & - & 0.61 & 0.55 \\ 0.38 & 0.5 & - & 0.44 \\ 0.67 & 0.5 & 0.33 & - \end{pmatrix}$$

Then we can complete P_1 taking the unknown preference values from P^c:

$$P_1 = \begin{pmatrix} - & 0.7 & \mathbf{0.57} & 0.68 \\ 0.4 & - & \mathbf{0.61} & 0.7 \\ \mathbf{0.38} & \mathbf{0.5} & - & \mathbf{0.44} \\ 0.6 & 0.75 & \mathbf{0.33} & - \end{pmatrix}$$

2.4.6.3 Mixed Guided Strategies

To solve ignorance problems in decision making with incomplete fuzzy preference relations we also have the possibility of using mixed strategies, that is, strategies that use more than one criteria. The amount of combinations can be very high, and thus, we present only three interesting strategies that make use of both kinds of information.

Strategy 8: Consistency and Proximity Among Alternatives

In this case we mix both strategies 4 and 5. Instead of choosing a random value as a seed for the estimation procedure, we choose a preference value from an alternative that we know is similar to unknown one:

Estimation Procedure 8: We have an incomplete fuzzy preference relation P_h with an unknown alternative x_i, and we know that alternative x_i is very similar to alternative x_j. We apply the following scheme:

1. *do* {
2. $\quad k = irand(1, n)$
3. } $while(k \neq i \neq j)$
4. $if\ (rand(0,1) < 0.5)$ { $p^h_{ik} = p^h_{jk}$ }
5. $\quad else$ { $p^h_{ki} = p^h_{kj}$ }
6. Apply the estimation procedure

Example 8: We part from the incomplete preference relation of the previous examples. We also know that alternative x_2 is similar to alternative x_3. Then, we pick one of the preference values that involved alternative x_2 (in this case $p_{31} = p_{21}$ and from it we apply the estimation procedure):

$$\begin{pmatrix} - & 0.7 & x & 0.68 \\ 0.4 & - & x & 0.7 \\ x & x & - & x \\ 0.6 & 0.75 & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 0.7 & x & 0.68 \\ 0.4 & - & x & 0.7 \\ \mathbf{0.4} & x & - & x \\ 0.6 & 0.75 & x & - \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} - & 0.7 & \mathbf{0.7} & 0.68 \\ 0.4 & - & \mathbf{0.5} & 0.7 \\ \mathbf{0.4} & \mathbf{0.55} & - & \mathbf{0.44} \\ 0.6 & 0.75 & \mathbf{0.7} & - \end{pmatrix}$$

Strategy 9: Consistency and Proximity Among Experts

This strategy combines both strategies 4 and 6. Instead of choosing a random seed for the estimation procedure it looks for another expert whose opinions are similar and chooses one of the preference values for the unknown alternative as seed:

Estimation Procedure 9: If an incomplete fuzzy preference relation P_h given by an expert e_h has an unknown alternative x_i, we apply the following scheme:

1. For every expert $e_v \in E,\ e_v \neq e_h$ {
2. Compute $d_v = distance(e_v, e_h)$
3. }
4. e_{sel} = expert with minimum distance
5. if $(d_{sel} < \gamma)$ {
6. $do\ \{\ k = irand(1, n)\ \}\ while(k \neq i)$
7. $if\ (rand(0,1) < 0.5)\ \{\ p^h_{ik} = p^{sel}_{jk}\ \}$
8. $else\ \{\ p^h_{ki} = p^{sel}_{kj}\ \}$
9. Apply the estimation procedure
10. }

Example 9: We part from *example 6* where expert e_1 gave an incomplete fuzzy preference relation P_1 and experts e_2 and e_3 expressed their preferences about the alternatives as the fuzzy preference relations P_2 and P_3. As expert e_2 have more similar preferences to e_1 and $distance(e_2, e_1) < \gamma$ we randomly pick a preference value involving alternative x_3 from P_2 and use it as a seed for the estimation procedure:

$$\begin{pmatrix} - & 0.7 & x & 0.68 \\ 0.4 & - & x & 0.7 \\ x & x & - & x \\ 0.6 & 0.75 & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 0.7 & x & 0.68 \\ 0.4 & - & \mathbf{0.7} & 0.7 \\ x & x & - & x \\ 0.6 & 0.75 & x & - \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} - & 0.7 & \mathbf{0.85} & 0.68 \\ 0.4 & - & \mathbf{0.7} & 0.7 \\ \mathbf{0.2} & \mathbf{0.44} & - & \mathbf{0.5} \\ 0.6 & 0.75 & \mathbf{0.73} & - \end{pmatrix}$$

Strategy 10: Consistency and Consensus Among Experts

In this case we mix strategies 4 and 7. The random seed value is chosen among the preference values from a global preference relation on the unknown alternative:

Estimation Procedure 10: We have an incomplete fuzzy preference relation P_h with an unknown alternative x_i, and we also have a global preference relation P^c which represents the current global or consensued opinion of all the experts on the problem. We apply the following scheme:

1. *do* {
2. $\quad k = irand(1, n)$
3. } $while(k \neq i)$
4. $if\ (rand(0,1) < 0.5)$ { $p^h_{ik} = p^c_{jk}$ }
5. $\quad else$ { $p^h_{ki} = p^c_{kj}$ }
6. Apply the estimation procedure

Example 10: We part from *example 7* where expert e_1 gave an incomplete fuzzy preference relation P_1 and we knew that the global (consensued) preference relation P^c is as follows:

$$P_1 = \begin{pmatrix} - & 0.7 & x & 0.68 \\ 0.4 & - & x & 0.7 \\ x & x & - & x \\ 0.6 & 0.75 & x & - \end{pmatrix} \; ; \; P^c = \begin{pmatrix} - & 0.43 & 0.57 & 0.42 \\ 0.5 & - & 0.61 & 0.55 \\ 0.38 & 0.5 & - & 0.44 \\ 0.67 & 0.5 & 0.33 & - \end{pmatrix}$$

To estimate the missing values in P_1 we randomly pick a preference value involving alternative x_3 from P^c and use it as a seed value for the estimation procedure:

$$\begin{pmatrix} - & 0.7 & x & 0.68 \\ 0.4 & - & x & 0.7 \\ x & x & - & x \\ 0.6 & 0.75 & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 0.7 & x & 0.68 \\ 0.4 & - & x & 0.7 \\ x & x & - & \mathbf{0.44} \\ 0.6 & 0.75 & x & - \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} - & 0.7 & \mathbf{0.74} & 0.68 \\ 0.4 & - & \mathbf{0.76} & 0.7 \\ \mathbf{0.4} & \mathbf{0.47} & - & \mathbf{0.44} \\ 0.6 & 0.75 & \mathbf{0.74} & - \end{pmatrix}$$

2.4.6.4 Additional Possible Improvements

We must note that the presented strategies work independently from the experts, that is, experts provide their incomplete preference relations and the system that implements the decision process applies one of the strategies in order to complete the ignored information. However, the presented approaches do not take into account the satisfaction or agreement of the experts with the completed information, and thus, it is possible to find experts who do not accept the preference values that have been estimated.

There are some possible improvements to the strategies that can be used to minize this issue. For example, if the system is prepared to carry out a consensus process, it usually implies some feedback from the system to the experts. We can use that feedback to give some guidance to the experts on how to provide some information about the unknown alternatives.

An easy way to do this for strategies 5 to 10 would be to present the proximity and/or consensus information to the expert and allow him/her to update his/her preference relation according to that information. That would allow the system to avoid some estimated value that would not be easily accepted by the experts.

2.4.6.5 Analisys of the Advantages and Disadvantages of Each Strategy

Depending on the information available about the particular decision making problem that we are solving we should apply some of the previous strategies to deal with ignorance situations. In this section we will discuss the advantages and disadvantages of each possible strategy, showing some example situations where some of the strategies may be more adecuate than others.

- *Strategy 1* is a very simple approach to solve ignorance situations. Although it is not always adequate to asume that not giving preference values for one alternative implies indifference between the unknown alternative and the rest of them, in some situations could be an acceptable option. In fact, its easiness of application can be a very appealing factor to use it, speacially in problems where there are no other sources of information (neither information about the alternatives or other experts). Particularly, decision making problems with only one expert or criterion are good candidates to apply this strategy.

- *Strategy 2* is also a simple approach, but it can produce a higher level of diversity in the opinions given by the experts. However, it is important to remark that this strategy can produce a decrease in the consistency of the fuzzy preference relations, because the random values will not usually comply with any kind of transitivity property. This strategy can be a good one to apply in decision problems with a high number of experts or criterions which do not differ too much between them (because it can introduce some diversity in the problem).

- *Strategy 3*: This strategy improves strategy 1, as it adjusts the estimated preference values to make the preference relation more consistent with the previously existing information. Moreover, the initial indifference supposed for every preference value for the unknown alternative is softened according to the existing information in the preference relation. This approach is interesting when there are no external sources of information about the problem and when a high consistency level is required in the experts' preference relations.

- *Strategy 4* tries to unify the advantages of strategies 2 and 3: it tries to maintain a high consistency degree in the fuzzy preference relations (with the aplication of the estimation procedure) whilst it gives a slightly higher level of diversity than strategy 3 (with the generation of the random seed for the estimation procedure).

- *Strategy 5* implies knowing some external information about the alternatives of the problem. To obtain this information is not usually an easy task, specially because it is difficult to quantify the similarity degrees between the different options. This information usually requires some study about the problem previous to the start of the decision making process. However, this kind of study will not always be succesful in obtaining clear relations between the alternatives. This strategy can be useful, for example, in decision problems where the alternatives are products that have to be evaluated. Probably some of these products have very similar characteristics (similar models), and information about these similarities can be helpful to avoid ignorance situations when an expert is not familiar with one of the products, but have enough knowledge about a similar one.

- *Strategy 6*: This approach allows a certain level of collaboration and communication between the experts. In situations when some experts have similar points of view about the problem they could "share" some information and thus, increase the knowledge level about the unknown alternatives. It is important to remark that it is not always possible to apply this kind of approaches, since there are no guarantees that the opinions of the different experts are close enough to assume that they have similar points of view about the problem. Decision processes with a very high number of experts can benefit from this approach, since the probabilities of finding a similar expert are greater.

- *Strategy 7*: Since lots of decision models imply to obtain a global opinion from all the individual preferences, and there exist some consensus models that try to find a consensued enough solution to the problem, the kind of information that this strategy requires is usually available. Moreover, this strategy could help to improve the consensus level of the model, making the opinions of the experts closer to each other (since the unknown alternatives are completed with already global or consensued information). Thus, this kind of approach is very useful in problems where a higher consensus level is desired.

- *Strategies 8, 9 and 10* try to achieve the advantages of the strategy 4 (obtain a high degree of consistency on the relations), but also taking into account existing information about the problem. Particularly, *strategy 8* is useful in situations where there exist information about the alternatives and it is interesting to improve experts consistency. *Strategy 9* could help to establish some communication between different experts without degrading the consistency level of the preferences, and finally, *strategy 10* could help to improve both consensus and consistency aspects in the decision process, so it can be a particularly good approach to problems which require high levels of both consensus and consistency.

2.4.7 Extending The Procedure: An Interactive Support System To Aid Experts To Express Consistent Information

In previous sections we have presented a procedure to compute the missing values of an incomplete FPR taking into account the expert consistency level. Nevertheless, that procedure could not deal with the initial contradiction that the expert could have introduced in his/her preferences, and what could be worse, the expert might not accept the estimated values (even if they increase the overall consistency level).

Thus, when designing a computer driven model to deal with group decision making problems where the information is given in the form of fuzzy preference relations, software tools to aid the experts to express their preferences avoiding the mentioned problems should be implemented. As experts might not be familiar with preference relations, the aiding tools should be easy enough to use and they should follow the general principles of interface design [8].

In this section we present an *interactive support system* to aid experts to express their preferences using fuzzy preference relations in a consistent way. The system will give recommendations to the expert while he/she is providing the preference values in order to maintain a high level of consistency in the preferences, as well as trying to avoid missing information. Also, the system will provide measures of the current level of consistency and completeness that the expert has achieved, which can be used to avoid situations of self contradiction. The system has been programmed using Java technologies, which allows its integration in web-based applications which are increasingly being used in GDM and Decision Support environments [3, 56].

Firstly we will enumerate all the design goals and requirements that we have taken into account and secondly we will describe the actual implementation of every requirement in the system.

2.4.7.1 Design Goals and Requirements

Our design goals and requirements could be split in two different parts: *Interface Requirements*, and *Logical Goals*.

Interface Requirements: These requirements deal with the visual representation of the information and the different controls in the system. We want our system to comply the so called *"Eight Golden Rules"*[8] for interface design:

- **GR 1.** Strive for consistency.
- **GR 2.** Enable frequent users to use shortcuts.
- **GR 3.** Offer informative feedback.
- **GR 4.** Design dialogues to yield closure.
- **GR 5.** Offer simple error handling.
- **GR 6.** Permit easy reversal of actions (undo action).
- **GR 7.** Support internal focus of control (user is in charge).
- **GR 8.** Reduce short-term memory load of the user.

Logical Goals:

- **Goal 1.** Offer recommendations to the expert to guide him toward a highly consistent and complete fuzzy preference relation.

- **Goal 2.** Recommendations must be given interactively.
- **Goal 3.** Recommendations must be simple to understand and to apply.
- **Goal 4.** The user must be able to refuse recommendations.
- **Goal 5.** The system must provide indicators of the consistency and completeness level achieved in every step.
- **Goal 6.** The system should be easy to adapt to other types of preference relations.
- **Goal 7.** The system should be easy to incorporate to Web-based GDM models and decision support systems[3, 56]

2.4.7.2 Actual Implementation

We will now detail how we have dealt with every requirement and goal that we have presented in the previous section. To do so we will make use of a snapshot of the system (*figure 2.2*) where we will point out every implementation solution.

Implementation of the Interface Requirements:

- **GR 1.** The interface has been homogenised in order to present an easy to understand view of the process which is being carried. We have introduced 3 main areas: In area number **(1)** we present the *fuzzy preference relation* that the expert is introducing, as well as a brief description and photo of every alternative. Area number **(2)** contains several global controls to activate/deactivate certain functions (*"show recommendations"* and *"mark highly inconsistent*

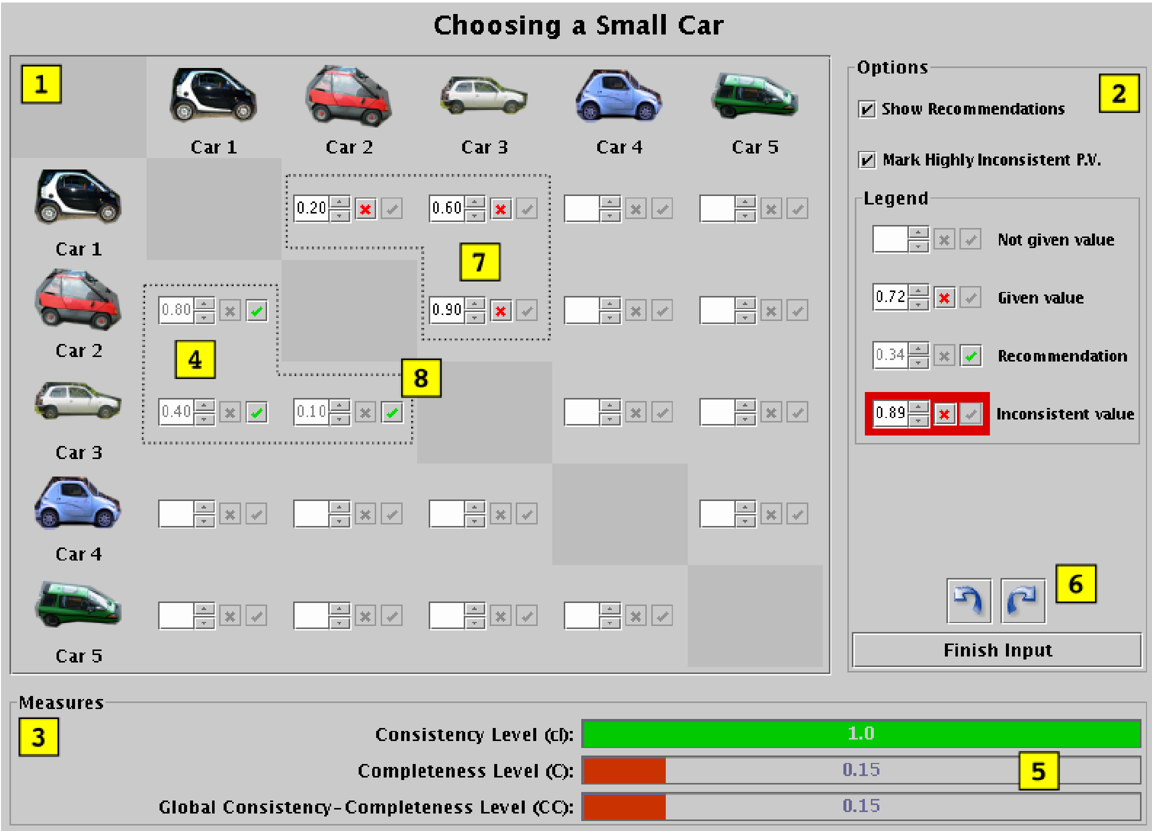

Figure 2.2: Snapshot of the Support System

preference values"), as well as to *"finish the input process"*, *undo* and *redo* actions **(6)**. Area number **(3)** contains different measures that show the overall progress (see below).

- **GR 2.** Shortcuts have been added to the most frequent options and the input text areas for the preference values have been ordered to access to them easily using the keyboard.

- **GR 3.** Our systems provides recommendations to the expert **(4)** as well as it presents consistency and completeness measures **(5)** (see below). Additionally, all controls have tooltips to better explain their purpose.

- **GR 4.** With every change that the user makes to his/her preferences the system interactively provides new recommendations and measures to help him to complete the preference matrix in a consistent way.

- **GR 5.** Although it is difficult to produce errors in the system, all invalid inputs are correctly handled or prompt by means of error dialogs.

- **GR 6.** All the actions can be reversed by means of the use of the *undo* and *redo* buttons **(6)**.

- **GR 7.** The user can choose at every moment which preference value wants to give or update, as well as enabling/disabling options as the system does not impose any order on the completion of the preference relations.

- **GR 8.** The user does not need to remind any data as every value that he/she has provided and the recommendations and measures are presented in a single screen.

Logical Goals:

- **Goal 1.** To offer recommendations, the system computes all the missing values that could be estimated by using *equation 2.9* and it presents them in area **(1)**. As the values are computed taking into account the additive transitivity property, the recommendations should tend to increment the overall consistency level. They are presented in a different color (gray) **(4)** to be easily distinguishable from the proper expert values **(7)**. Thus, if the expert follows the recommendations, he/she should be able easily provide a complete a highly consistent fuzzy preference relation. However we must note that the recommendations can be completetely ignored if the user does not think that they are correct. Additionally, the recommendations can be turned off if the expert does not need them or does not want to be influenciated by them by means of the *"Show Recommendations"* option.

- **Goal 2.** All the recommendations are recomputed and presented when the expert changes any single preference value in the fuzzy preferece relation. Thus, the recommendations do take into account every piece of information that has been given by the expert.

- **Goal 3.** As the recommendations are given in the same manner as the user inputs his/her preferences they are intuitive and easy to understand. We have also incorporated a button that enables the user to accept or validate a given recommendation **(8)** and so, accepting a recommendation becomes an easy task for the user.

- **Goal 4.** The recommendations can be ignored by the user by just introducing a particular preference value. Additionally, an expert can turned off by means of the *"Show Recommendations"* option.

- **Goal 5.** In area **(3)** the system provides information about completeness and consistency of the preference relation which can help the expert to know if the preferences that he/she is introducing are being consistent or not.

- **Goal 6.** As the system is programmed following the principles of Object Oriented Programming, to adapt it to new kinds of preference relations is an easy task.

- **Goal 7.** As the system is Java based, it is easy to incorporate it into a web-based environment or any other kind of program / application. Additionally, when the input process if finished, the given fuzzy preference relation can be exported using an XML based language, which helps to easily integrate it into different applications.

2.4.7.3 Example of Usage

In this section we present an example of usage of the support system. We part from the following situation: an expert has to express his/her preferences in form of fuzzy preference relations about a set of 5 different cars $X = \{x_1, x_2, x_3, x_4, x_5\}$ in order to finally obtain which car is the best one. To do so he is going to use the presented support system. In this example we will represent every value which is given by the expert in

bold letters, every recommendation given by the system as *emphasized text*, the missing values (still not given by the expert) with letter x and a highly inconsistent preference value will be marked with a `typewriter font`. For example:

$$P = \begin{pmatrix} - & \mathbf{0.2} & \mathit{0.6} \\ x & - & \mathtt{0.4} \\ x & x & - \end{pmatrix}$$

In the example, 0.2 would be a value given by the expert, 0.6 would be a recommendation of the system and 0.4 a highly inconsistent preference value.

Thus, the expert parts from an empty fuzzy preference relation:

$$P = \begin{pmatrix} - & x & x & x & x \\ x & - & x & x & x \\ x & x & - & x & x \\ x & x & x & - & x \\ x & x & x & x & - \end{pmatrix}$$

As he prefers x_3 over x_2 he sets $p_{32} = 0.65$, and as he al knows that he prefers x_3 over x_5 (even more than x_2) he sets $p_{35} = 0.75$:

$$\begin{pmatrix} - & x & x & x & x \\ x & - & x & x & x \\ x & x & - & x & x \\ x & x & x & - & x \\ x & x & x & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & x & x & x & x \\ x & - & x & x & x \\ x & \mathbf{0.65} & - & x & x \\ x & x & x & - & x \\ x & x & x & x & - \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} - & x & x & x & x \\ x & - & x & x & \mathit{0.60} \\ x & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ x & \mathit{0.40} & x & x & - \end{pmatrix}$$

At this point the system has provided 2 recommendations for p_{25} and p_{52}. He decides that those values are appropiate and he accepts them:

$$\begin{pmatrix} - & x & x & x & x \\ x & - & x & x & \mathit{0.60} \\ x & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ x & \mathit{0.40} & x & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & x & x & x & x \\ x & - & x & x & \mathit{0.60} \\ x & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ x & \mathbf{0.40} & \mathit{0.25} & x & - \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} - & x & x & x & x \\ x & - & \mathit{0.35} & x & \mathbf{0.60} \\ x & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ x & \mathbf{0.40} & \mathit{0.25} & x & - \end{pmatrix}$$

Once he has accepted those values the system suggests another 2 values: p_{32} and p_{53}. Note that as the expert has been completely consistent (the *consistency level* is 1.0) until this moment those values are the reciprocals from p_{23} and p_{35} respectively. He decides to accept them:

$$\begin{pmatrix} - & x & x & x & x \\ x & - & \mathit{0.35} & x & \mathbf{0.60} \\ x & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ x & \mathbf{0.40} & \mathit{0.25} & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & x & x & x & x \\ x & - & \mathbf{0.35} & x & \mathbf{0.60} \\ x & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ x & \mathbf{0.40} & \mathit{0.25} & x & - \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} - & x & x & x & x \\ x & - & \mathbf{0.35} & x & \mathbf{0.60} \\ x & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ x & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix}$$

Now the expert sets $p_{13} = 0.55$ as he thinks that alternative x_1 is slightly better than alternative x_3:

$$\begin{pmatrix} - & x & x & x & x \\ x & - & \mathbf{0.35} & x & \mathbf{0.60} \\ x & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ x & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & \mathit{0.70} & \mathbf{0.55} & x & \mathit{0.80} \\ \mathit{0.30} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ x & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ \mathit{0.20} & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix}$$

The system then provides some recommendations that link the alternative x_1 with alternatives x_3 and x_5, which were previously compared. However, the expert does not like the recommendation given for p_{51} as he thinks that alternative x_5 is better than alternative x_1. Thus, he sets $p_{51} = 0.7$:

$$\begin{pmatrix} - & \mathit{0.70} & \mathbf{0.55} & x & \mathit{0.80} \\ \mathit{0.30} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ x & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ \mathit{0.20} & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & \mathit{0.53} & \mathtt{0.55} & x & \mathit{0.80} \\ \mathit{0.63} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ \mathit{0.95} & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ \mathtt{0.70} & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix}$$

As it can be seen, the system has marked both p_{51} and p_{13} as highly inconsistent. That indicates that the previously introduced preference values are contradictory. Thus, the expert can think about the values that he/she has introduced and study the possible contradiction on them. In fact, as $x_1 \succ x_3$ ($p_{13} = 0.55$) and $x_3 \succ x_5$ ($p_{35} = 0.75$) the expert realizes that $x_1 \succ x_5$, and accordingly, $p_{51} < 0.5$. Thus he decides to delete the given p_{51} value and set it again with a value which is nearer to the suggested one: $p_{51} = 0.3$:

$$\begin{pmatrix} - & \mathit{0.53} & \mathtt{0.55} & x & \mathit{0.80} \\ \mathit{0.63} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ \mathit{0.95} & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ \mathtt{0.70} & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & \mathit{0.70} & \mathbf{0.55} & x & \mathit{0.80} \\ \mathit{0.30} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ x & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ \mathit{0.20} & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} - & \mathit{0.67} & \mathbf{0.55} & x & \mathit{0.80} \\ \mathit{0.37} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ \mathit{0.55} & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix}$$

At this point the *consistency level* has decreased a little bit (0.97), but it is acceptable. The expert now accepts the suggestions for p_{12} and p_{21} as he realizes that x_1 is slightly preferred to x_3 ($p_{13} = 0.55$) and x_3 is preferred to x_2 ($p_{32} = 0.65$), and thus, x_1 should be preferred to x_2 in a slighly higher degree than x_3 to x_2:

$$\begin{pmatrix} - & \mathit{0.67} & \mathbf{0.55} & x & \mathit{0.80} \\ \mathit{0.37} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ \mathit{0.55} & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & \mathit{0.67} & \mathbf{0.55} & x & \mathit{0.78} \\ \mathbf{0.37} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ \mathit{0.54} & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & x & \mathit{0.77} \\ \mathbf{0.37} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ \mathit{0.52} & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix}$$

Now the expert focus his attention to the fourth alternative x_4 and he realizes that it is worse than x_2. He decides to set $p_{42} = 0.25$:

$$\begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & x & \mathit{0.77} \\ \mathbf{0.37} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ \mathit{0.52} & \mathbf{0.65} & - & x & \mathbf{0.75} \\ x & x & x & - & x \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathit{0.92} & \mathit{0.77} \\ \mathbf{0.37} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ \mathit{0.52} & \mathbf{0.65} & - & \mathit{0.90} & \mathbf{0.75} \\ \mathit{0.10} & \mathbf{0.25} & \mathit{0.10} & - & \mathit{0.35} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathit{0.65} & - \end{pmatrix}$$

The new recommendations show that alternative x_4 is the worst, as every recommended $p_{4j} < 0.5,\ j \in \{1, 3, 5\}$. The expert recognizes that this fact is true, and he7she assigns simmilar values to the recommended ones for p_{41}, p_{43} and p_{45}:

$$\begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathit{0.92} & \mathit{0.77} \\ \mathbf{0.37} & - & \mathbf{0.35} & x & \mathbf{0.60} \\ \mathit{0.52} & \mathbf{0.65} & - & \mathit{0.90} & \mathbf{0.75} \\ \mathit{0.10} & \mathbf{0.25} & \mathit{0.10} & - & \mathit{0.35} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathit{0.65} & - \end{pmatrix} \rightarrow \begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathit{0.92} & \mathit{0.77} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathit{0.72} & \mathbf{0.60} \\ \mathit{0.52} & \mathbf{0.65} & - & \mathit{0.90} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathit{0.12} & - & \mathit{0.37} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathit{0.65} & - \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathit{0.94} & \mathit{0.77} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathit{0.74} & \mathbf{0.60} \\ \mathit{0.52} & \mathbf{0.65} & - & \mathit{0.90} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathbf{0.10} & - & \mathit{0.35} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathit{0.65} & - \end{pmatrix} \rightarrow \begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathit{0.94} & \mathit{0.76} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathit{0.74} & \mathbf{0.60} \\ \mathit{0.52} & \mathbf{0.65} & - & \mathit{0.90} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathbf{0.10} & - & \mathbf{0.36} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathit{0.65} & - \end{pmatrix}$$

At this point the expert sets $p_{14} = 0.85$ as he knows that x_1 is much better than x_4, but not as better as the system suggests (0.94):

$$\begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathit{0.94} & \mathit{0.76} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathit{0.74} & \mathbf{0.60} \\ \mathit{0.52} & \mathbf{0.65} & - & \mathit{0.90} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathbf{0.10} & - & \mathbf{0.36} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathit{0.65} & - \end{pmatrix} \rightarrow \begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathbf{0.85} & \mathit{0.75} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathit{0.71} & \mathbf{0.60} \\ \mathit{0.52} & \mathbf{0.65} & - & \mathit{0.85} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathbf{0.10} & - & \mathbf{0.36} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathit{0.65} & - \end{pmatrix}$$

Finally, the expert accepts all given recommendations for the missing values:

$$\begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathbf{0.85} & \mathit{0.75} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathit{0.71} & \mathbf{0.60} \\ \mathit{0.52} & \mathbf{0.65} & - & \mathit{0.85} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathbf{0.10} & - & \mathbf{0.36} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathit{0.65} & - \end{pmatrix} \rightarrow \begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathbf{0.85} & \mathit{0.75} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathit{0.71} & \mathbf{0.60} \\ \mathbf{0.52} & \mathbf{0.65} & - & \mathit{0.85} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathbf{0.10} & - & \mathbf{0.36} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathit{0.65} & - \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathbf{0.85} & \mathit{0.75} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathbf{0.71} & \mathbf{0.60} \\ \mathbf{0.52} & \mathbf{0.65} & - & \mathit{0.86} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathbf{0.10} & - & \mathbf{0.36} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathit{0.63} & - \end{pmatrix} \rightarrow \begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathbf{0.85} & \mathit{0.75} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathbf{0.71} & \mathbf{0.60} \\ \mathbf{0.52} & \mathbf{0.65} & - & \mathbf{0.86} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathbf{0.10} & - & \mathbf{0.36} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathit{0.63} & - \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathbf{0.85} & \mathit{0.75} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathbf{0.71} & \mathbf{0.60} \\ \mathbf{0.52} & \mathbf{0.65} & - & \mathbf{0.86} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathbf{0.10} & - & \mathbf{0.36} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathbf{0.63} & - \end{pmatrix} \rightarrow \begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathbf{0.85} & \mathbf{0.75} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathbf{0.71} & \mathbf{0.60} \\ \mathbf{0.52} & \mathbf{0.65} & - & \mathbf{0.86} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathbf{0.10} & - & \mathbf{0.36} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathbf{0.63} & - \end{pmatrix}$$

At the end of the input process, the expert has provided the following fuzzy preference relation about the five alternatives in the problem:

$$P = \begin{pmatrix} - & \mathbf{0.67} & \mathbf{0.55} & \mathbf{0.85} & \mathbf{0.75} \\ \mathbf{0.37} & - & \mathbf{0.35} & \mathbf{0.71} & \mathbf{0.60} \\ \mathbf{0.52} & \mathbf{0.65} & - & \mathbf{0.86} & \mathbf{0.75} \\ \mathbf{0.15} & \mathbf{0.25} & \mathbf{0.10} & - & \mathbf{0.36} \\ \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.25} & \mathbf{0.63} & - \end{pmatrix}$$

Additionally, the expert is sure that he has provided highly consistent preferences as the *consistency level* is 0.98. Moreover, during the input process, the system has helped him to avoid a high inconsistency situation where the preference values that he gave were very contradictory.

2.5 Additive Consistency Based IOWA Operator [AC-IOWA]

Definition 1.11 allows the construction of many different operators. Indeed, the set of consistency levels of the relations, $\{cl^1, \ldots, cl^m\}$, or the set of consistency levels of the preference values, $\{cl^1_{ik}, \ldots, cl^m_{ik}\}$, may be used not just to associate *importance* values to the experts $E = \{e_1, \ldots, e_m\}$ but also to define an IOWA operator, i.e, the ordering of the preference values to be aggregated $\{p^1_{ik}, \ldots, p^m_{ik}\}$ can be induced by ordering the experts from the most to the least consistent one. In this case, we obtain an IOWA operator that we call the additive-consistency IOWA (AC-IOWA) operator and denote it as Φ^{AC}_W. This new operator can be viewed as an extension of the Consistency IOWA operator (C-IOWA) defined in [12].

One of the main advantages of this operator is that treats the information provided by the experts in an heterogeneus way, that is, assigning more importance to the most consistent ones. This approach allows to avoid contradictions in the selection process, because the higher the inconsistency, the lower is the effect that it has over the aggregated information.

Definition 2.8. The AC-IOWA operator of dimension m, Φ^{AC}_W, is an IOWA operator whose set of order inducing values is $\{cl^1, \ldots, cl^m\}$ or

$\{cl_{ik}^1, \ldots, cl_{ik}^m\}$.

Although the aggregation can be made using the set of consistency levels of the whole preference relations as the inducing variable, we think that using the consistency levels for every preference value is a better approach because it handles inconsistencies in a finer grain way: As an expert may be consistent in some of his preferences and inconsistent in others, our aggregation process is carried out using an AC-IOWA operator guided by the set of consistency levels of preference values, i.e. $\{cl_{ik}^1, \ldots, cl_{ik}^m\}$. Therefore, the collective fuzzy preference relation is obtained as follows:

$$p_{ik}^c = \Phi_Q^{AC}\left(\left\langle CL_{ik}^1, p_{ik}^1\right\rangle, \ldots, \langle CL_{ik}^m, p_{ik}^m\rangle\right) \tag{2.14}$$

where Q is the fuzzy quantifier used to implement the fuzzy majority concept and, using (1.5), to compute the weighting vector of the AC-IOWA operator Φ_Q^{AC}.

2.6 The Selection Process

The selection process that we propose to solve group decision making problems with incomplete fuzzy preference relation requires three different steps to be carried out, which are represented in *figure 2.3*:

1. *Estimation of missing information.* In this step, incomplete fuzzy preference relations are completed. To do this, we use the previously presented estimation procedure.

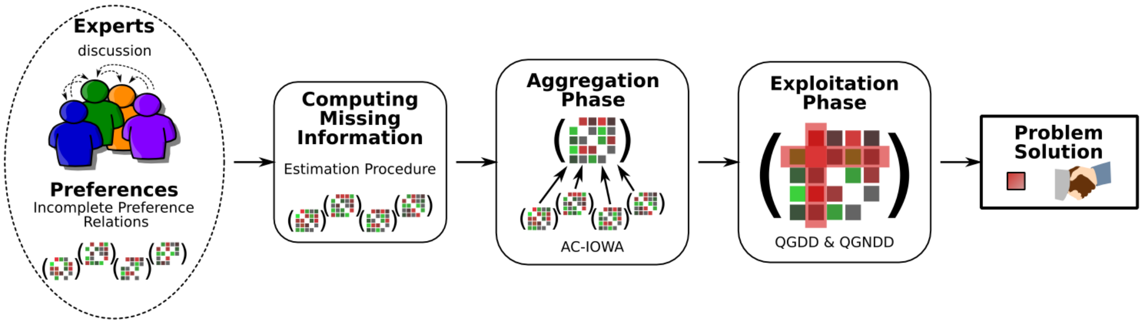

Figure 2.3: Selection Process for a Group Decision Making Problem With Incomplete Fuzzy Preference Relations

2. *Aggregation phase.* A collective fuzzy preference relation is obtained by aggregating all completed individual fuzzy preference relations. This aggregation is carried out by applying the AC-IOWA operator.

3. *Exploitation phase.* Using the concept of fuzzy majority (of alternatives), two choice degrees of alternatives are used: the *quantifier-guided dominance degree (QGDD)* and the *quantifier-guided non-dominance degree (QGNDD)* [10]. These choice degrees will act over the collective preference relation resulting in a global ranking of the alternatives, from which the set of solution alternatives will be obtained.

2.6.1 Aggregation: The Collective Fuzzy Preference Relation

Once we have estimated all the missing values in every incomplete fuzzy preference relation, we have a set of m individual fuzzy preference relations $\{P^1, ..., P^m\}$. From this set a collective fuzzy preference relation $P^c = (p^c_{ik})$ must be derived by means of an aggregation procedure. In our case, each value $p^c_{ik} \in [0, 1]$ will represent the preference of alternative

x_i over alternative x_k according to the majority of the most consistent experts' opinions.

Clearly, a rational assumption in the resolution process of a group decision making problem is that of associating more importance to the experts who provide the most *consistent* information. This assumption implies that group decision making problems should be viewed as heterogeneous. Indeed, in any group decision making problem with incomplete fuzzy preference relations, each expert e_h can have an importance degree associated with him/her which, for example, can be his/her own consistency level of the relation cl^h or consistency levels of the preference values cl^h_{ik} in each preference value p_{ik}.

Usually, procedures for the inclusion of these importance values in the aggregation process involve the transformation of the preference values, p^h_{ik}, under the importance degree I^h, to generate a new value, $\overline{p}^h_{ik}$ [24, 27]. This activity is carried out by means of a transformation function g, $\overline{p}^h_{ik} = g\left(p^h_{ik}, I^h\right)$. Examples of functions g used in these cases include the minimum operator [27], the exponential function $g(x, y) = x^y$ [70], or generally any t-norm operator. In our case, we apply an alternative approach which consists of using the previously presented AC-IOWA operator which takes into account the consistency degrees of the preference values when aggregating the preference relations.

2.6.2 Exploitation: Choosing The Best Alternative(s)

At this point, in order to select the alternative(s) *best* acceptable for the majority (Q) of the most consistent experts, we propose two quantifier-guided choice degrees of alternatives, a dominance and a non-dominance degree, which can be applied according to different selection policies.

2.6.2.1 Choice degrees of alternatives

$QGDD_i$: The quantifier-guided dominance degree quantifies the dominance that one alternative has over all the others in a fuzzy majority sense and is defined as follows:

$$QGDD_i = \phi_Q(p^c_{i1}, p^c_{i2}, ..., p^c_{i(i-1)}, p^c_{i(i+1)}, ..., p^c_{in}) \tag{2.15}$$

$QGNDD_i$: The quantifier-guided non-dominance degree gives the degree in which each alternative is not dominated by a fuzzy majority of the remaining alternatives. Its expression being:

$$\begin{aligned} QGNDD_i = \; & \phi_Q(1 - p^s_{1i}, 1 - p^s_{2i}, ..., 1 - p^s_{(i-1)i}, \\ & 1 - p^s_{(i+1)i}, ..., 1 - p^s_{ni}) \end{aligned} \tag{2.16}$$

where $p^s_{ji} = max\{p^c_{ji} - p^c_{ij}, 0\}$, represents the degree in which x_i is strictly dominated by x_j. When the fuzzy quantifier represents the statement *all*, whose algebraic aggregation corresponds to the conjunction operator *min*, this non-dominance degree coincides with Orlovski's non-dominated alternative concept [55].

2.6.2.2 Selection policies

The application of the above choice degrees of alternatives over X may be carried out according to two different policies.

1. *Sequential policy.* One of the choice degrees is selected and applied to X according to the preference of the experts, obtaining a selection set of alternatives. If there is more than one alternative in this

selection set, then the other choice degree is applied to select the alternative of this set with the best second choice degree.

2. *Conjunctive policy.* Both choice degrees are applied to X, obtaining two selection sets of alternatives. The final selection set of alternatives is obtained as the intersection of these two selection sets of alternatives.

The latter conjunction selection process is more restrictive than the former sequential selection process because it is possible to obtain an empty selection set. Therefore, in a complete selection process the choice degrees can be applied in three steps:

- *Step 1:* The application of each choice degree of alternatives over X to obtain the following sets of alternatives:

$$X^{QGDD} = \{x_i \in X \mid QGDD_i = \sup_{x_j \in X} QGDD_j\} \tag{2.17}$$

$$X^{QGNDD} = \{x_i \in X \mid QGNDD_i = \sup_{x_j \in X} QGNDD_j\} \tag{2.18}$$

whose elements are called maximum dominance elements on the fuzzy majority of X quantified by Q and maximal non-dominated elements by the fuzzy majority of X quantified by Q, respectively.

- *Step 2:* The application of the conjunction selection policy, obtaining the following set of alternatives:

$$X^{QGCP} = X^{QGDD} \bigcap X^{QGNDD} \tag{2.19}$$

If $X^{QGCP} \neq \emptyset$, then End.

Otherwise continue.

- *Step 3:* The application of the one of the two sequential selection policies, according to either a dominance or non-dominance criterion, i.e.,

  – *Dominance based sequential selection process QG-DD-NDD.* To apply the quantifier guided dominance degree over X, and obtain X^{QGDD}. If $\#(X^{QGDD}) = 1$ then End, and this is the solution set. Otherwise, continue obtaining

  $$X^{QG-DD-NDD} = \{x_i \in X^{QGDD} \mid QGNDD_i = \sup_{x_j \in X^{QGDD}} QGNDD_j\} \tag{2.20}$$

  This is the selection set of alternatives.

  – *Non-dominance based sequential selection process QG-NDD-DD.* To apply the quantifier guided non-dominance degree over X, and obtain X^{QGNDD}. If $\#(X^{QGNDD}) = 1$ then End, and this is the solution set. Otherwise, continue obtaining

  $$X^{QG-NDD-DD} = \{x_i \in X^{QGNDD} \mid QGDD_i = \sup_{x_j \in X^{QGNDD}} QGDD_j\} \tag{2.21}$$

  This is the selection set of alternatives.

2.7 Illustrative Example

For the sake of simplicity we will assume a low number of experts and alternatives. Let us suppose that four different experts $\{e_1, e_2, e_3, e_4\}$ provide the following fuzzy preference relations over a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$:

$$P^1 = \begin{pmatrix} - & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.4} \\ x & - & x & x \\ x & x & - & x \\ x & x & x & - \end{pmatrix}$$

$$P^2 = \begin{pmatrix} - & x & \mathbf{0.7} & x \\ \mathbf{0.4} & - & x & \mathbf{0.6} \\ \mathbf{0.3} & x & - & x \\ x & \mathbf{0.4} & x & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & \mathbf{0.3} & \mathbf{0.5} & \mathbf{0.75} \\ \mathbf{0.7} & - & \mathbf{0.7} & \mathbf{0.9} \\ \mathbf{0.5} & \mathbf{0.3} & - & \mathbf{0.7} \\ \mathbf{0.25} & \mathbf{0.1} & \mathbf{0.3} & - \end{pmatrix}$$

$$P^4 = \begin{pmatrix} - & x & \mathbf{0.6} & \mathbf{0.3} \\ \mathbf{0.4} & - & \mathbf{0.4} & \mathbf{0.3} \\ \mathbf{0.4} & \mathbf{0.6} & - & \mathbf{0.3} \\ \mathbf{0.7} & \mathbf{0.7} & \mathbf{0.7} & - \end{pmatrix}$$

2.7.1 Estimation of missing values

Three given preference relations are incomplete $\{P^1, P^2, P^4\}$. For P^1 there are just 3 known values; because they involve all 4 alternatives then all the missing values can be successfully estimated:

Step 1: The set of elements that can be estimated are:

$$EMV_1^1 = \{(2,3),(2,4),(3,2),(3,4),(4,2),(4,3)\}\}$$

After these elements have been estimated, we have:

$$P^1 = \begin{pmatrix} - & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.4} \\ x & - & 0.9 & 0.7 \\ x & 0.1 & - & 0.3 \\ x & 0.3 & 0.7 & - \end{pmatrix}$$

As an example, to estimate p^1_{43} the procedure is as follows:

$$\begin{aligned} H^{11}_{43} = \emptyset \quad & \Rightarrow \quad (cp^1_{43})^1 = 0 \\ H^{12}_{43} = \{1\} \quad & \Rightarrow \quad (cp^1_{43})^{12} = p^1_{13} - p^1_{14} + 0.5 = 0.6 - 0.4 + 0.5 = 0.7 \Rightarrow \\ & \Rightarrow \quad (cp^1_{43})^2 = 0.7 \\ H^{13}_{43} = \emptyset \quad & \Rightarrow \quad (cp^1_{43})^3 = 0 \end{aligned}$$

$$cp^1_{43} = \frac{0 + 0.7 + 0}{1} = 0.7$$

Step 2: The set of elements that can be estimated are:

$$EMV^1_2 = \{(2,1), (3,1), (4,1)\}\}$$

After these elements have been estimated, we have the following completed fuzzy preference relation:

$$P^1 = \begin{pmatrix} - & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.4} \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix}$$

As an example, to estimate p^1_{41} the procedure is as follows:

$$\begin{aligned} H^{11}_{41} = \emptyset \quad & \Rightarrow \quad (cp^1_{41})^1 = 0 \\ H^{12}_{41} = \emptyset \quad & \Rightarrow \quad (cp^1_{41})^2 = 0 \\ H^{13}_{41} = \{2,3\} \quad & \Rightarrow \quad \left\{ \begin{array}{l} (cp^1_{41})^{23} = p^1_{42} - p^1_{12} + 0.5 = 0.6 \\ (cp^1_{41})^{33} = p^1_{43} - p^1_{13} + 0.5 = 0.6 \end{array} \right\} \Rightarrow (cp^1_{41})^3 = 1.2 \end{aligned}$$

$$cp_{41}^1 = \frac{0+0+1.2}{2} = 0.6$$

The corresponding consistency level matrix associated with the incomplete fuzzy preference relation P^1 is calculated as follows:

$$\begin{array}{l} EV_1^1 = \{(1,2),(1,3),(1,4)\};\ EV_2^1 = \{(1,2)\} \\ EV_3^1 = \{(1,3)\};\ EV_4^1 = \{(1,4)\} \\ C_1^1 = 3/6\ ;\ C_2^1 = C_3^1 = C_4^1 = 1/6 \\ \alpha_{12}^1 = \alpha_{13}^1 = \alpha_{14}^1 = 1 - \frac{3+1-1}{10} = 0.7 \end{array}$$

For p_{12} we have that there is no intermediate alternative to calculate an estimated value and consequently we have:

$$\varepsilon p_{12} = 0 \Rightarrow cl_{12}^1 = (1-0.7)\cdot(1-0) + 0.7 \cdot \frac{\frac{3}{6}+\frac{1}{6}}{2} \approx 0.53$$

The same result is obtained for p_{13} and p_{14}, i.e., $cl_{12}^1 = cl_{13}^1 = cl_{14}^1 \approx 0.53$. This means that the consistency level for each one of the estimated values is also 0.53, as they are calculated as the average of the consistency values used to estimate them. Consequently, we have:

$$CL^1 = \begin{pmatrix} - & 0.53 & 0.53 & 0.53 \\ 0.53 & - & 0.53 & 0.53 \\ 0.53 & 0.53 & - & 0.53 \\ 0.53 & 0.53 & 0.53 & - \end{pmatrix}$$

For P^2, P^3 and P^4 we get:

$$P^2 = \begin{pmatrix} - & 0.6 & \mathbf{0.7} & 0.7 \\ \mathbf{0.4} & - & 0.6 & \mathbf{0.6} \\ \mathbf{0.3} & 0.4 & - & 0.5 \\ 0.3 & \mathbf{0.4} & 0.5 & - \end{pmatrix}$$

$$CL^2 = \begin{pmatrix} - & 0.62 & 0.59 & 0.67 \\ 0.75 & - & 0.64 & 0.59 \\ 0.59 & 0.67 & - & 0.62 \\ 0.64 & 0.59 & 0.62 & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & \mathbf{0.3} & \mathbf{0.5} & \mathbf{0.75} \\ \mathbf{0.7} & - & \mathbf{0.7} & \mathbf{0.9} \\ \mathbf{0.5} & \mathbf{0.3} & - & \mathbf{0.7} \\ \mathbf{0.25} & \mathbf{0.1} & \mathbf{0.3} & - \end{pmatrix}$$

$$CL^3 = \begin{pmatrix} - & 0.98 & 0.98 & 0.97 \\ 0.98 & - & 1.0 & 0.98 \\ 0.98 & 1.0 & - & 0.98 \\ 0.97 & 0.98 & 0.98 & - \end{pmatrix}$$

$$P^4 = \begin{pmatrix} - & 0.6 & \mathbf{0.6} & \mathbf{0.3} \\ \mathbf{0.4} & - & \mathbf{0.4} & \mathbf{0.3} \\ \mathbf{0.4} & \mathbf{0.6} & - & \mathbf{0.3} \\ \mathbf{0.7} & \mathbf{0.7} & \mathbf{0.7} & - \end{pmatrix}$$

$$CL^4 = \begin{pmatrix} - & 0.93 & 0.93 & 0.93 \\ 0.92 & - & 0.93 & 0.93 \\ 0.93 & 0.93 & - & 0.93 \\ 0.93 & 0.93 & 0.93 & - \end{pmatrix}$$

As an example of how the iterative estimation procedure has worked over P^4 we show the estimate for p_{12}, the only missing value in this preference relation:

$$H_{12}^{41} = \{3,4\} \;\Rightarrow \left\{ \begin{array}{c} (cp_{12}^{4})^{31} = 0.7 \\ (cp_{12}^{4})^{41} = 0.5 \end{array} \right\} \Rightarrow (cp_{12}^{4})^{1} = 1.2$$
$$H_{12}^{42} = \{3,4\} \;\Rightarrow \left\{ \begin{array}{c} (cp_{12}^{4})^{32} = 0.7 \\ (cp_{12}^{4})^{42} = 0.5 \end{array} \right\} \Rightarrow (cp_{12}^{4})^{2} = 1.2$$
$$H_{12}^{43} = \{3,4\} \;\Rightarrow \left\{ \begin{array}{c} (cp_{12}^{4})^{33} = 0.7 \\ (cp_{12}^{4})^{43} = 0.5 \end{array} \right\} \Rightarrow (cp_{12}^{4})^{3} = 1.2$$

$$cp_{12}^{4} = \frac{1.2 + 1.2 + 1.2}{6} = 0.6$$

The following calculations are needed to obtain the consistency level cl_{21}^{4}

- $\varepsilon p_{21}^{4} = \dfrac{0.2}{3}$
- Computation of C_1^4, C_2^4 and α_{21}^4:

$$EV_1^4 = \{(1,3),(1,4),(2,1),(3,1),(4,1)\} \Rightarrow C_1^4 = \tfrac{5}{6}$$
$$EV_2^4 = \{(2,1),(2,3),(2,4),(3,2),(4,2)\} \Rightarrow C_2^4 = \tfrac{5}{6}$$
$$EV_1^4 \cap EV_2^4 = \{(2,1)\} \Rightarrow \alpha_{21} = 1 - \tfrac{5+5-1}{10} = \tfrac{1}{10}$$

- Computation of CL_{21}:

$$cl_{21}^{4} = \left(1 - \frac{1}{10}\right) \cdot \left(1 - \frac{0.2}{3}\right) + \frac{1}{10} \cdot \frac{\frac{5}{6} + \frac{5}{6}}{2} \approx 0.92$$

We should point out that because the third and fourth experts have been very consistent when expressing their preferences and have provided an almost complete preference relation, the consistency levels for their preference values are very high.

2.7.2 Aggregation Phase

Once the fuzzy preference relations are completed we aggregate them by means of the AC-IOWA operator and using the consistency level of the preference values as the order inducing variable. We make use of the linguistic quantifier *most of*, represented by the RIM quantifier $Q(r) = r^{1/2}$ (see Appendix), which applying (1.5), generates a weighting vector of four values to obtain each collective preference value p^c_{ik} .

As an example, the collective preference value p^c_{12} with 2 decimal places is obtained as follows:

$$cl^1_{12} = 0.53, cl^2_{12} = 0.62, cl^3_{12} = 0.98, cl^4_{12} = 0.93$$

$$p^1_{12} = 0.2, p^2_{12} = 0.6, p^3_{12} = 0.3, p^4_{12} = 0.6$$

$$\sigma(1) = 3, \sigma(2) = 4, \sigma(3) = 2, \sigma(4) = 1$$

$$T = cl^1_{12} + cl^2_{12} + cl^3_{12} + cl^4_{12}$$

$$Q(0) = 0$$

$$Q\left(\frac{cl^4_{12}}{T}\right) = 0.57$$

$$Q\left(\frac{cl^4_{12} + cl^3_{12}}{T}\right) = 0.79$$

$$Q\left(\frac{cl^4_{12} + cl^3_{12} + cl^2_{12}}{T}\right) = 0.91$$

$$Q\left(\frac{cl^4_{12} + cl^3_{12} + cl^2_{12} + cl^1_{12}}{T}\right) = Q(1) = 1$$

$$w_1 = 0.57;\ w_2 = 0.22;\ w_3 = 0.12;\ w_4 = 0.09$$

$$\begin{aligned} p_{12}^c &= w_1 \cdot p_{12}^3 + w_2 \cdot p_{12}^4 + w_3 \cdot p_{12}^2 + w_4 \cdot p_{12}^1 \\ &= 0.57 \cdot 0.3 + 0.22 \cdot 0.6 + 0.12 \cdot 0.6 + 0.09 \cdot 0.2 \\ &= 0.39 \end{aligned}$$

Finally, the collective fuzzy preference relation is:

$$P^c = \begin{pmatrix} - & 0.39 & 0.55 & 0.61 \\ 0.6 & - & 0.64 & 0.71 \\ 0.45 & 0.36 & - & 0.55 \\ 0.39 & 0.29 & 0.45 & - \end{pmatrix}$$

2.7.3 Exploitation Phase

Using again the same fuzzy quantifier *most of*, and (1.4), we obtain the weighting vector $W = (w_1, w_2, w_3)$:

$$w_1 = Q\left(\frac{1}{3}\right) - Q(0) = 0.58 - 0 = 0.58$$

$$w_2 = Q\left(\frac{2}{3}\right) - Q\left(\frac{1}{3}\right) = 0.82 - 0.58 = 0.24$$

$$w_3 = Q(1) - Q\left(\frac{2}{3}\right) = 1 - 0.82 = 0.18$$

Then we compute the following quantifier guided dominance and non-dominance degrees of all the alternatives:

	x_1	x_2	x_3	x_4
$QGDD_i$	0.57	0.67	0.49	0.41
$QGNDD_i$	0.96	1.00	0.93	0.81

Clearly, the maximal sets are:

$$X^{QGDD} = \{x_2\} \quad \text{and} \quad X^{QGNDD} = \{x_2\}.$$

Finally, applying the conjunction selection policy we obtain:

$$X^{QGCP} = X^{QGDD} \cap X^{QGNDD} = \{x_2\}$$

which means that alternative x_2 is the best alternative according to *most of* the most consistent experts.

Chapter 3

A Consensus Model For Group Decision Making With Incomplete Fuzzy Preference Relations

In this chapter we focus on the development of a consensus model for group decision making problems under incomplete fuzzy preference relations. This consensus model is not only based on consensus measures but also on consistency measures. We consider that both criteria are important to guide the consensus process in an incomplete decision framework. In such a way, we get that experts change their opinions toward agreement positions in a consistent way, which is desirable to achieve consistent and consensus solutions.

Hence, we present a consensus model guided by two kinds of measures: consistency and consensus measures. We have designed it trying to obtain the maximum possible consensus level while trying to achieve a high level of consistency in experts' preferences. We should point out that the consistency search often leads to reduce the consensus level and viceversa. Thus, we try to maintain a balance between both. Moreover, the consensus model not only is able to achieve a solution with certain consensus and consistency degrees simultaneously, but also it is able to deal with incomplete fuzzy preference relations, giving advice to the experts on how to complete them.

As in [26], we use two kinds of consensus measures to guide the consensus reaching processes, consensus degrees (to evaluate the agreement of all the experts) and proximity degrees (to evaluate the agreement between the experts' individual preferences and the group preference). To compute them, first we estimate all missing values of the incomplete fuzzy preference relations using the estimation procedure based on consistency defined in *chapter 2*. Afterwards, we compute some consistency measures for each expert. We use both consensus measures and consistency measures to design a feedback mechanism that generates advice to the experts on how they should change and complete their fuzzy preference relations to obtain a solution with a high consensus degree (making experts' opinions closer), but also maintaining a certain consistency level on their fuzzy preference relations (avoiding self contradiction). This feedback mechanism is able to substitute the actions of the moderator.

Figure 3.1 depicts this consensus model. We assume that experts provide their opinions on a set of alternatives by means of incomplete fuzzy preference relations. Then, the consensus model uses the estimation procedure of missing values [31] to complete each incomplete fuzzy preference relation. Later, consistency and consensus measures are computed from completed fuzzy preference relations. These measures are used in a consistency/consensus control step to determine if an appropriate consistency/consensus level has been reached. If so, the consensus reaching process finishes and a selection process should be

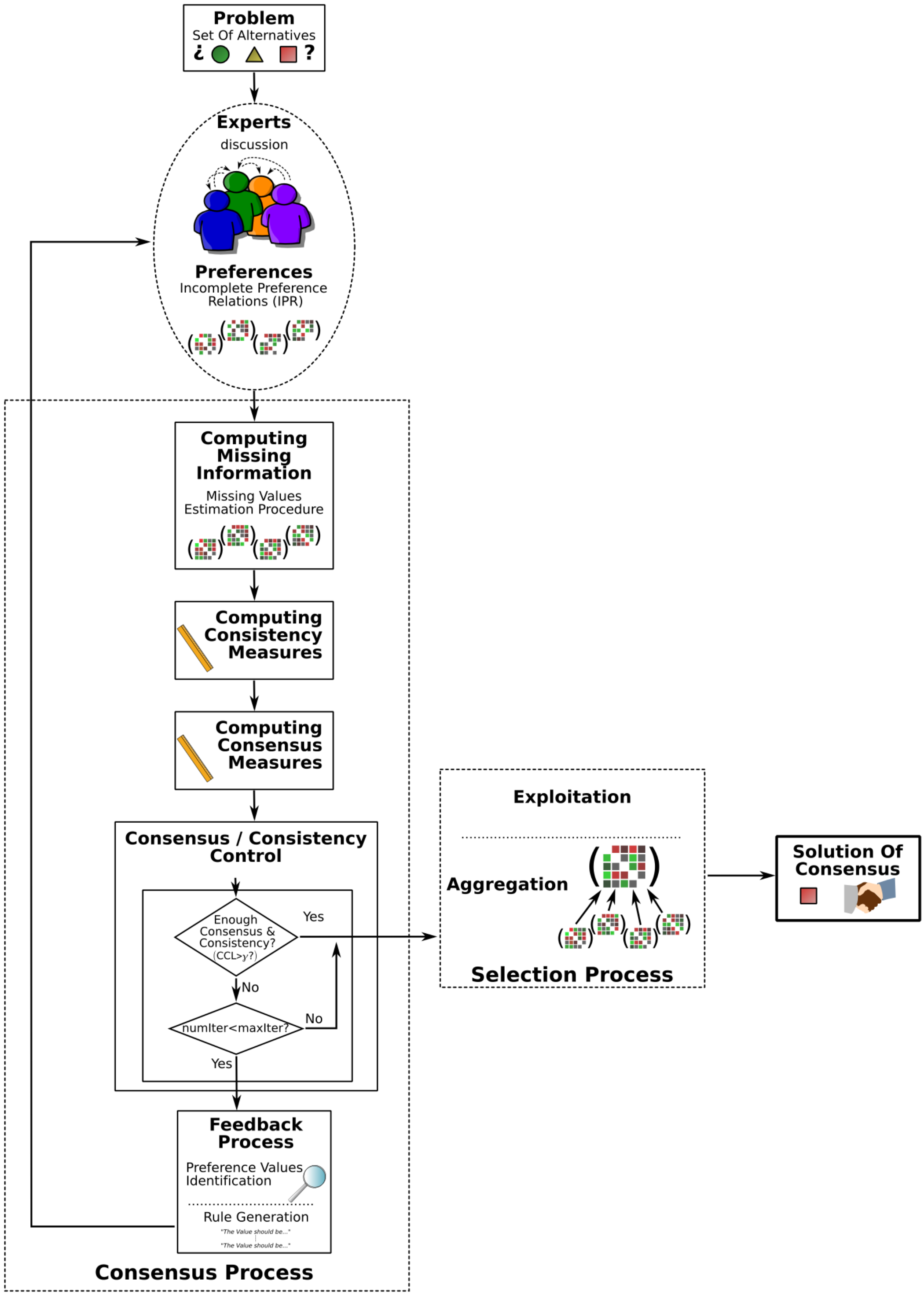

Figure 3.1: Consensus Model for Group Decision Making Problems With Incomplete Fuzzy Preference Relations

applied to obtain the solution. Otherwise, the consensus reaching process activates a feedback mechanism, where the preference values which are not contributing to obtain a high consensus / consistency level are detected and some easy rules about how to alter them are generated to help the experts to change and complete their opinions to guide them towards a more consensued solution (maintaining a reasonable level of consistency).

The steps of this consensus model are the following:

1. Computing Missing Information

2. Computing Consistency Measures

3. Computing Consensus Measures

4. Controlling the Consistency/Consensus State

5. Feedback Mechanism

They are presented in detail in the following subsections, along with a step-by-step example which illustrates the computations that are being carried out (in *appendix B* we provide an exhaustive description of all the computations made for this example). In the example, for the sake of simplicity, we will assume a low number of experts and alternatives. Concretely, let us suppose that four different experts $\{e_1, e_2, e_3, e_4\}$ provide the following incomplete fuzzy preference relations over a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$:

$$P^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ x & - & x & x \\ x & x & - & x \\ x & x & x & - \end{pmatrix}; P^2 = \begin{pmatrix} - & x & 0.7 & x \\ 0.4 & - & x & 0.7 \\ 0.3 & x & - & x \\ x & 0.4 & x & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.3 & x & 0.75 \\ 0.6 & - & x & x \\ x & x & - & x \\ 0.3 & 0.4 & x & - \end{pmatrix}; P^4 = \begin{pmatrix} - & x & 0.6 & 0.3 \\ 0.4 & - & 0.4 & 0.2 \\ 0.5 & 0.6 & - & 0.3 \\ 0.7 & 0.7 & 0.7 & - \end{pmatrix}$$

3.1 Computing Missing Information

In this first step each incomplete fuzzy preference relation is estimated by means of the estimation procedure described in *chapter 2*. Then, for each incomplete fuzzy preference relation P^h we obtain its respective complete fuzzy preference relation $\overline{P}^h$.

Example

The incomplete fuzzy preference relations P^1, P^2, P^3 and P^4 are reconstructed by applying the iterative procedure and we obtain the following complete fuzzy preference relations:

$$\overline{P}^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix} ; \overline{P}^2 = \begin{pmatrix} - & 0.62 & 0.7 & 0.8 \\ 0.4 & - & 0.6 & 0.7 \\ 0.3 & 0.4 & - & 0.57 \\ 0.25 & 0.4 & 0.45 & - \end{pmatrix}$$

$$\overline{P}^3 = \begin{pmatrix} - & 0.3 & 0.54 & 0.75 \\ 0.6 & - & 0.69 & 0.87 \\ 0.46 & 0.31 & - & 0.73 \\ 0.3 & 0.4 & 0.27 & - \end{pmatrix} ; \overline{P}^4 = \begin{pmatrix} - & 0.6 & 0.6 & 0.3 \\ 0.4 & - & 0.4 & 0.2 \\ 0.5 & 0.6 & - & 0.3 \\ 0.7 & 0.7 & 0.7 & - \end{pmatrix}$$

3.2 Computing Consistency Measures

To compute consistency measures, firstly, we compute for each $\overline{P}^h$ their respective consistent fuzzy preference relation $CP^h = \left(cp_{ik}^h\right)$ according to expression (2.4).

Example

The respective consistent fuzzy preference relations for P^1, P^2, P^3 and P^4 are:

$$CP^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix}; CP^2 = \begin{pmatrix} - & 0.62 & 0.72 & 0.77 \\ 0.4 & - & 0.6 & 0.67 \\ 0.3 & 0.43 & - & 0.57 \\ 0.25 & 0.35 & 0.45 & - \end{pmatrix}$$

$$CP^3 = \begin{pmatrix} - & 0.45 & 0.51 & 0.69 \\ 0.61 & - & 0.61 & 0.89 \\ 0.48 & 0.42 & - & 0.64 \\ 0.34 & 0.1 & 0.41 & - \end{pmatrix}; CP^4 = \begin{pmatrix} - & 0.6 & 0.5 & 0.35 \\ 0.4 & - & 0.45 & 0.2 \\ 0.5 & 0.55 & - & 0.3 \\ 0.65 & 0.8 & 0.7 & - \end{pmatrix}$$

Secondly, from every $\overline{P}^h$ and CP^h we apply expressions (2.6)–(2.8) to compute the consistency measures cl^h_{ik}, cl^h_i, cl^h $\forall i, k \in \{1, ..., n\}$.

Example

We compute the consistency measures every pair of alternatives in the experts' preferences:

$$CL^1 = \begin{pmatrix} - & 1.0 & 1.0 & 1.0 \\ 1.0 & - & 1.0 & 1.0 \\ 1.0 & 1.0 & - & 1.0 \\ 1.0 & 1.0 & 1.0 & - \end{pmatrix} ; CL^2 = \begin{pmatrix} - & 1.0 & 0.98 & 0.98 \\ 1.0 & - & 1.0 & 0.97 \\ 1.0 & 0.98 & - & 1.0 \\ 1.0 & 0.95 & 1.0 & - \end{pmatrix}$$

$$CL^3 = \begin{pmatrix} - & 0.85 & 0.98 & 0.94 \\ 0.99 & - & 0.92 & 0.97 \\ 0.99 & 0.89 & - & 0.91 \\ 0.96 & 0.7 & 0.86 & - \end{pmatrix} ; CL^4 = \begin{pmatrix} - & 1.0 & 0.9 & 0.95 \\ 1.0 & - & 0.95 & 1.0 \\ 1.0 & 0.95 & - & 1.0 \\ 0.95 & 0.9 & 1.0 & - \end{pmatrix}$$

and from them we compute consistency measure that each expert presents in his/her preferences:

$$cl^1 = 1.0 \; ; \; cl^2 = 0.99 \; ; \; cl^3 = 0.91 \; ; \; cl^4 = 0.97$$

Additionally, we define a global consistency measure among all experts to control the global consistency situation.

Definition 3.1. The global consistency measure is computed as follows:

$$CL = \frac{\sum_{h=1}^{m} cl^h}{m} \tag{3.1}$$

Example

The global consistency level is:

$$CL = \frac{1.0 + 0.99 + 0.91 + 0.97}{4} = 0.97$$

3.3 Computing Consensus Measures

We compute several consensus measures for the different fuzzy preference relations. In fact, as in [26, 35] we compute two different kinds of measures: consensus degrees and proximity measures. Consensus degrees are used to measure the actual level of consensus in the process, whilst the proximity measures give information about how near to the collective solution every expert is. These measures were given on three different levels for a fuzzy preference relation: pairs of alternatives, alternatives and relations. This measure structure allows us to find out the consensus state of the process at different levels, for example, we are able to identify which experts are close to the consensus solution, or in which alternatives the experts are having more trouble to reach consensus.

3.3.1 Consensus Degrees

Firstly, for each pair of experts e_h, e_l $(h < l)$ we define a similarity matrix $SM^{hl} = \left(sm_{ik}^{hl}\right)$ where

$$sm_{ik}^{hl} = 1 - |\overline{p}_{ik}^{h} - \overline{p}_{ik}^{l}| \tag{3.2}$$

Then, a whole similarity matrix, $SM = (sm_{ik})$ is obtained by aggregating all the similarity matrices using the arithmetic mean as the aggregation function ϕ:

$$sm_{ik} = \phi(sm_{ik}^{hl}) \ ; \ \forall h, l = 1, ..., m \mid h < l. \tag{3.3}$$

Once the similarity matrix are computed we proceed to calculate the consensus degrees in the three different levels:

1. **Level 1.** *Consensus degree on pairs of alternatives.* The consensus degree on a pair of alternatives (x_i, x_k), called cop_{ik} is defined to measure the consensus degree amongst all the experts on that pair of alternatives:

$$cop_{ik} = sm_{ik} \tag{3.4}$$

2. **Level 2.** *Consensus degree on alternatives.* The consensus degree on an alternative x_i, called ca_i is defined to measure the consensus degree amongst all the experts on that alternative:

$$ca_i = \frac{\sum_{k=1;k\neq i}^{n}(cop_{ik} + cop_{ki})}{2(n-1)} \tag{3.5}$$

3. **Level 3.** *Consensus degree on the relation.* The consensus degree on the relation, called CR is defined to measure the global consensus degree amongst all the experts' opinions:

$$CR = \frac{\sum_{i=1}^{n} ca_i}{n} \tag{3.6}$$

Example

Firstly, we compute similarity matrix for each pair of experts (in this case we obtain 6 matrices that are not included by simplicity), and then we obtain the following whole similarity matrix:

$$SM = \begin{pmatrix} - & 0.74 & 0.92 & 0.69 \\ 0.77 & - & 0.73 & 0.67 \\ 0.89 & 0.73 & - & 0.74 \\ 0.73 & 0.8 & 0.74 & - \end{pmatrix}$$

and thus, from SM we obtain the following consensus degree on the relation

$$CR = 0.76.$$

3.3.2 Proximity Measures

To compute proximity measures for each expert we need to obtain the collective fuzzy preference relation, P^c, which summarizes preferences given by all the experts. To obtain P^c we introduce a new IOWA operator, the *Consistency/Consensus IOWA* operator, which is able to aggregate fuzzy preference relations inducing the order of the arguments to be aggregated according to consistency and consensus criteria.

Definition 3.2. The Consistency/Consensus IOWA (CC-IOWA) operator to aggregate fuzzy preference relations is defined as follows:

$$p^c_{ik} = \Phi_W(\langle z^1_{ik}, p^1_{ik} \rangle, \cdots, \langle z^m_{ik}, p^m_{ik} \rangle) \quad (3.7)$$

where the set of values of the inducing variable $\{z^1_{ik}, \ldots, z^m_{ik}\}$ are computed as

$$z^h_{ik} = (1 - \delta) \cdot cl^h_{ik} + \delta \cdot co^h_{ik}, \tag{3.8}$$

being co^h_{ik} a consensus measure for the preference value p_{ik} expressed by expert e_h and $\delta \in [0, 1]$ a parameter to control the weight of both consistency and consensus criteria in the inducing variable. Usually $\delta > 0.5$ will be used to give more importance to the consensus criterion.

We should note that in our framework, each value co^h_{ik} used to calculate $\{z^1_{ik}, \ldots, z^m_{ik}\}$ is defined as

$$co^h_{ik} = \frac{\sum_{l=h+1}^{n} sm^{hl}_{ik} + \sum_{l=1}^{h-1} sm^{lh}_{ik}}{n - 1} \tag{3.9}$$

Example

To compute proximity measures it is necessary to obtain the values of the inducing variable of the CC-IOWA operator. To do so, firstly we compute consensus values co^h_{ik}:

$$CO^1 = \begin{pmatrix} - & 0.69 & 0.95 & 0.72 \\ 0.67 & - & 0.66 & 0.78 \\ 0.91 & 0.66 & - & 0.77 \\ 0.75 & 0.8 & 0.77 & - \end{pmatrix}; CO^2 = \begin{pmatrix} - & 0.74 & 0.88 & 0.68 \\ 0.8 & - & 0.8 & 0.78 \\ 0.85 & 0.8 & - & 0.77 \\ 0.72 & 0.87 & 0.77 & - \end{pmatrix}$$

$$CO^3 = \begin{pmatrix} - & 0.76 & 0.9 & 0.72 \\ 0.8 & - & 0.8 & 0.67 \\ 0.91 & 0.8 & - & 0.66 \\ 0.75 & 0.87 & 0.65 & - \end{pmatrix}; CO^4 = \begin{pmatrix} - & 0.76 & 0.95 & 0.65 \\ 0.8 & - & 0.67 & 0.44 \\ 0.89 & 0.67 & - & 0.77 \\ 0.68 & 0.67 & 0.77 & - \end{pmatrix}$$

In our case, the consistency/consensus values, $\{z_{ik}^1, z_{ik}^2, \ldots, z_{ik}^m\}$, may be considered as importance degrees associated to the values to be aggregated, $\{p_{ik}^1, p_{ik}^2, \ldots, p_{ik}^m\}$, and therefore the weighting vector in the above CC-IOWA operator will be calculated using expression (1.5).

Example

With the co_{ik}^h and cl_{ik}^h we obtain the set of values of the inducing variable of the CC-IOWA operator for each expert as (we assume that $\delta = 0.75$):

$$Z^1 = \begin{pmatrix} - & 0.77 & 0.96 & 0.79 \\ 0.75 & - & 0.75 & 0.83 \\ 0.93 & 0.75 & - & 0.82 \\ 0.81 & 0.85 & 0.83 & - \end{pmatrix}; Z^2 = \begin{pmatrix} - & 0.81 & 0.9 & 0.76 \\ 0.85 & - & 0.85 & 0.83 \\ 0.88 & 0.85 & - & 0.82 \\ 0.79 & 0.89 & 0.83 & - \end{pmatrix}$$

$$Z^3 = \begin{pmatrix} - & 0.78 & 0.92 & 0.77 \\ 0.85 & - & 0.83 & 0.74 \\ 0.93 & 0.82 & - & 0.72 \\ 0.8 & 0.82 & 0.71 & - \end{pmatrix}; Z^4 = \begin{pmatrix} - & 0.82 & 0.93 & 0.73 \\ 0.85 & - & 0.74 & 0.58 \\ 0.92 & 0.74 & - & 0.82 \\ 0.75 & 0.73 & 0.83 & - \end{pmatrix}$$

Using the following fuzzy linguistic quantifier *"most of"* Q:

$$Q(r) = \begin{cases} 0 & \text{if } r < 0.3 \\ \frac{r-0.3}{0.8-0.3} & \text{if } 0.3 \leq r < 0.8 \\ 1 & \text{if } r \geq 0.8 \end{cases}$$

to compute the weighting vector in the CC-IOWA we obtain the collective fuzzy preference relation P^c:

$$P^c = \begin{pmatrix} - & 0.43 & 0.58 & 0.74 \\ 0.52 & - & 0.77 & 0.78 \\ 0.47 & 0.23 & - & 0.44 \\ 0.31 & 0.37 & 0.57 & - \end{pmatrix}$$

Once we have computed P^c using a CC-IOWA operator we can compute proximity measures in each level of a fuzzy preference relation:

1. **Level 1.** *Proximity measure on pairs of alternatives.* The proximity measure of an expert e_h on a pair of alternatives (x_i, x_k) to the group one, called pp_{ik}^h, is calculated as

$$pp_{ik}^h = 1 - |\overline{p}_{ik}^h - p_{ik}^c| \tag{3.10}$$

2. **Level 2.** *Proximity measure on alternatives.* The proximity measure of an expert e_h on an alternative x_i to the group one, called pa_i^h, is calculated as:

$$pa_i^h = \frac{\sum_{k=1;k\neq i}^{n}(pp_{ik}^h + pp_{ki}^h)}{2(n-1)} \tag{3.11}$$

3. **Level 3.** *Proximity measure on the relation.* The proximity measure of an expert e_h on his/her preference relation to the group one, called pr^h, is calculated as:

$$pr^h = \frac{\sum_{i=1}^{n} pa_i^h}{n} \tag{3.12}$$

Example

We compute all the proximity measures for each expert

$$PP^1 = \begin{pmatrix} - & 0.77 & 0.98 & 0.66 \\ 0.72 & - & 0.87 & 0.92 \\ 0.93 & 0.87 & - & 0.86 \\ 0.71 & 0.93 & 0.87 & - \end{pmatrix} ; PP^2 = \begin{pmatrix} - & 0.8 & 0.88 & 0.94 \\ 0.88 & - & 0.83 & 0.92 \\ 0.83 & 0.83 & - & 0.87 \\ 0.94 & 0.97 & 0.88 & - \end{pmatrix}$$

$$PP^3 = \begin{pmatrix} - & 0.87 & 0.96 & 0.99 \\ 0.92 & - & 0.92 & 0.91 \\ 0.99 & 0.92 & - & 0.71 \\ 0.99 & 0.97 & 0.7 & - \end{pmatrix} ; PP^4 = \begin{pmatrix} - & 0.83 & 0.98 & 0.56 \\ 0.88 & - & 0.63 & 0.42 \\ 0.97 & 0.63 & - & 0.86 \\ 0.61 & 0.67 & 0.87 & - \end{pmatrix}$$

$$pa^1 = \begin{pmatrix} 0.8 & 0.85 & 0.9 & 0.82 \end{pmatrix} ; pa^2 = \begin{pmatrix} 0.88 & 0.87 & 0.85 & 0.92 \end{pmatrix}$$

$$PA^3 = \begin{pmatrix} 0.96 & 0.92 & 0.87 & 0.88 \end{pmatrix} ; PA^4 = \begin{pmatrix} 0.8 & 0.67 & 0.82 & 0.66 \end{pmatrix}$$

$$pr^1 = 0.84 \ ; \ pr^2 = 0.88 \ ; \ pr^3 = 0.91 \ ; \ pr^4 = 0.74$$

3.4 Controlling Consistency/Consensus State

The consistency/consensus state control process will be used to decide when the feedback mechanism should be applied to give advice to the experts or when the consensus reaching process has to come to an end. It should take into account both the consensus and consistency measures.

To do that, we define a new measure or level of satisfaction, called *consistency/consensus level (CCL)*, which is used as a control parameter:

$$CCL = (1 - \delta) \cdot CL + \delta \cdot CR$$

with δ the same value used in (3.8). When CCL satisfies a minimum satisfaction threshold value $\gamma \in [0, 1]$, then the consensus reaching process finishes and the selection process can be applied.

Additionally, the system should avoid stagnation, that is, situations in which consensus and consistency measures never reach an appropriate satisfaction value. To do so, a maximum number of iterations $maxIter$ should be fixed and compared to the actual number of iterations of the consensus process $numIter$.

The consensus/consistency control routine follows the schema shown in Figure 3.2: first the consistency/consensus level is checked against the minimum satisfaction threshold value. If $CCL > \gamma$ the consensus reaching process finishes. Otherwise, it will check if maximum number of iterations have been reached. If so, the consensus reaching process finishes, and if not it activates the feedback mechanism.

Example

The consistency/consensus level at this moment is

$$CCL = (1 - 0.75) \cdot 0.97 + 0.75 \cdot 0.76 = 0.81$$

If we fix a minimum threshold value $\gamma = 0.85$, then the system must continue with the feedback mechanism. Otherwise, the consensus model should finish and continue to the decision process with its corresponding selection process.

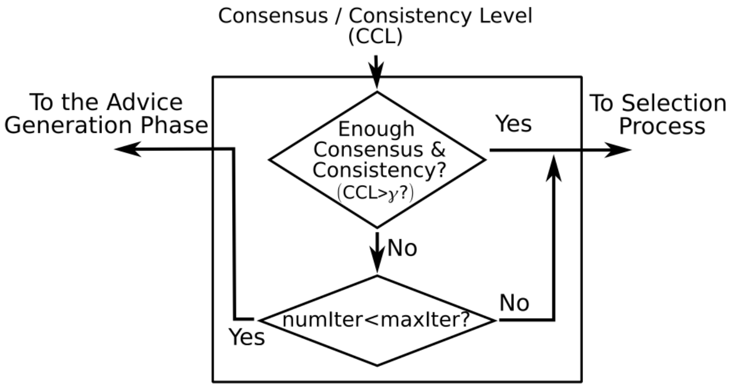

Figure 3.2: Consensus / Consistency Status Control Routine

3.5 Feedback Mechanism

The feedback mechanism generates advice to the experts according to the consistency and consensus criteria. It helps experts to change their preferences and to complete their missing values. This activity is carried out in two steps: *Identification of the preference values* that should be changed and *Generation of advice*.

3.5.1 Identification of the Preference Values

We must identify preference values provided by the experts that are contributing less to reach a high consensus/consistency state. To do that, we define the set APS that contains 3-tuples (h, i, k) symbolizing preference degrees p_{ik}^h that should be changed because they affect badly to that consistency/consensus state. To compute APS, we apply a three step identification process that uses the proximity and consistency measures previously defined.

Step 1. We identify the set of experts $EXPCH$ that should receive advice on how to change some of their preference values. The experts that should change their opinions are those whose preference relation level of satisfaction is lower than the satisfaction threshold γ, i.e.,

$$EXPCH = \{h \mid (1-\delta) \cdot cl^h + \delta \cdot pr^h < \gamma\} \tag{3.13}$$

Step 2. We identify the alternatives that the above experts should consider to change. This set of alternatives is denoted as ALT. To do this, we select the alternatives with a level of satisfaction lower than the satisfaction threshold γ, i.e.,

$$ALT = \{(h,i) \mid (1-\delta) \cdot cl_i^h + \delta \cdot pa_i^h < \gamma \ and \ e_h \in EXPCH\} \tag{3.14}$$

Step 3. Finally, we identify preference values for every alternative and expert (x_i ; $e_h \mid (h,i) \in ALT$) that should be changed according to their proximity and consistency measures on the pairs of alternatives, i.e.,

$$APS = \{(h,i,k) \mid (h,i) \in ALT \ and \ (1-\delta) \cdot cl_{ik}^h + \delta \cdot pp_{ik}^h < \gamma\} \tag{3.15}$$

Additionally the feedback process must provide rules for missing preference values. To do so, it has to take into account in APS all missing values that were not provided by the experts, i.e.,

$$APS' = APS \cup \{(h,i,k) \mid p_{ik}^h \in MV_h\} \tag{3.16}$$

Example

Firstly, we compute the set of 3-tuples APS that experts should change:

$APS = \{(4, 1, 4), (4, 2, 3), (4, 2, 4), (4, 4, 1), (4, 4, 2)\}$

Then, we include in APS those 3-tuples that symbolize the preference degrees that some experts have not provided, i.e.,

$APS' = \{(1, 2, 1), (1, 2, 3), (1, 2, 4), (1, 3, 1), (1, 3, 2), (1, 3, 4), (1, 4, 1), (1, 4, 2), (1, 4, 3), (2, 1, 2), (2, 1, 4), (2, 2, 3), (2, 3, 2), (2, 3, 4), (2, 4, 1), (2, 4, 3), (3, 1, 3), (3, 2, 3), (3, 2, 4), (3, 3, 1), (3, 3, 2), (3, 3, 4), (3, 4, 3), (4, 1, 2), (4, 1, 4), (4, 2, 3), (4, 2, 4), (4, 4, 1), (4, 4, 2)\}$

Note that there are too many 3-tuples in APS' because there were many missing values in the incomplete fuzzy preference relations provided by the experts.

3.5.2 Generation of Advice

In this step, the feedback mechanism generates recommendations to help experts to change their fuzzy preference relations. These recommendations are based on easy recommendation rules. The rules not only tell experts which preference values should they change, but also they provide them particular values for each preference to reach a higher consistency/consensus state.

To calculate these particular values to recommend we use consistent preference values cp^h_{ik} and collective preference values p^c_{ik}:

$$rp^h_{ik} = (1 - \delta) \cdot cp^h_{ik} + \delta \cdot p^c_{ik}, \tag{3.17}$$

where rp^h_{ik} is the value to recommend to the expert e_h as the new preference degree of alternatives x_i over alternative x_k. As previously mentioned, with $\delta > 0.5$ the consensus model leads the experts towards a consensus solution rather than towards an increase on their own consistency levels.

Finally, we should distinguish two cases: if the rule has to be given because a preference value is far from the consensus/consistency state or because the expert did not provide the preference value. Therefore, there are two kinds of recommendation rules:

1. If $p_{ik}^h \in EV_h$ the recommendation generated for the expert e_h is: *"You should change your preference value (i,k) to a value close to rp_{ik}^h."*

2. If $p_{ik}^h \in MV_h$ the recommendation generated for the expert e_h is: *"You should provide a value for (i,k) close to rp_{ik}^h."*

For each 3-tuple using the recommendation rules we generate a recommendation:

Example

To expert e_1: You should provide a value for (2, 1) near to 0.59

To expert e_1: You should provide a value for (2, 3) near to 0.8

To expert e_1: You should provide a value for (2, 4) near to 0.76

To expert e_1: You should provide a value for (3, 1) near to 0.45

To expert e_1: You should provide a value for (3, 2) near to 0.2

To expert e_1: You should provide a value for (3, 4) near to 0.41

To expert e_1: You should provide a value for (4, 1) near to 0.38

To expert e_1: You should provide a value for (4, 2) near to 0.35

To expert e_1: You should provide a value for (4, 3) near to 0.6

To expert e_2: You should provide a value for (1, 2) near to 0.48

To expert e_2: You should provide a value for (1, 4) near to 0.75

To expert e_2: You should provide a value for (2, 3) near to 0.73

To expert e_2: You should provide a value for (3, 2) near to 0.28

To expert e_2: You should provide a value for (3, 4) near to 0.47

To expert e_2: You should provide a value for (4, 1) near to 0.29

To expert e_2: You should provide a value for (4, 3) near to 0.54

To expert e_3: You should provide a value for (1, 3) near to 0.56

To expert e_3: You should provide a value for (2, 3) near to 0.73

To expert e_3: You should provide a value for (2, 4) near to 0.81

To expert e_3: You should provide a value for (3, 1) near to 0.47

To expert e_3: You should provide a value for (3, 2) near to 0.28

To expert e_3: You should provide a value for (3, 4) near to 0.49

To expert e_3: You should provide a value for (4, 3) near to 0.53

To expert e_4: You should provide a value for (1, 2) near to 0.47

To expert e_4: You should change your preference value (1, 4) near to 0.64

To expert e_4: You should change your preference value (2, 3) near to 0.69

To expert e_4: You should change your preference value (2, 4) near to 0.64

To expert e_4: You should change your preference value (4, 1) near to 0.39

To expert e_4: You should change your preference value (4, 2) near to 0.48

Once experts receive the recommendations, another round of the consensus process should start, with the experts giving new fuzzy preference relations closer to a consensus solution and with higher levels of consistency.

Finishing the Example: Second Consensus Round

We suppose that the experts follow the given advice, and thus, the new consensus round begins with the following fuzzy preference relations:

$$P^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.52 & - & 0.8 & 0.76 \\ 0.44 & 0.2 & - & 0.4 \\ 0.38 & 0.35 & 0.6 & - \end{pmatrix}; P^2 = \begin{pmatrix} - & 0.48 & 0.7 & 0.75 \\ 0.4 & - & 0.73 & 0.7 \\ 0.3 & 0.28 & - & 0.49 \\ 0.29 & 0.4 & 0.54 & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.3 & 0.56 & 0.75 \\ 0.6 & - & 0.73 & 0.81 \\ 0.46 & 0.28 & - & 0.5 \\ 0.3 & 0.4 & 0.53 & - \end{pmatrix}; P^4 = \begin{pmatrix} - & 0.47 & 0.6 & 0.3 \\ 0.4 & - & 0.69 & 0.64 \\ 0.5 & 0.6 & - & 0.3 \\ 0.39 & 0.48 & 0.7 & - \end{pmatrix}$$

If we follow the same process (which will not be detailed here) we obtain the following global consistency and consensus levels:

$$CL = 0.91 \text{ and } CR = 0.88.$$

Obviously, the consistency level has decreased a little bit, since our process gives more importance to the consensus criteria than the consistency one. However, the consensus level has been increased. Finally, as the consistency/consensus level satisfies the minimum threshold value, i.e.,

$$CCL = 0.89 > \gamma = 0.85,$$

then the consensus process finishes.

Chapter 4

Comentarios Finales

Este último capítulo está dedicado presentar los resultados obtenidos en la presente tesis, así como algunas conclusiones que se derivan de los mismos, comentar algunos trabajos futuros y presentar las publicaciones realizadas sobre los temas que se presentan en esta tesis.

Resultados Obtenidos y Conclusiones

Resolver problemas de toma de decisiones en grupo en situaciones donde hay falta de información implica el estudio de tres aspectos fundamentales: (i) modelar correctamente las situaciones de falta de información, (ii) el desarrollo de un proceso de selección que permita el manejo de la información incompleta, y (iii) el desarrollo de procesos de consenso apropiados para este tipo de situaciones. Teniendo en cuenta estos aspectos, los resultados presentados en esta memoria y algunas conclusiones sobre los mismos se exponen en los siguientes puntos:

1. **Sobre el modelado de situaciones de falta de información.**

   En esta memoria hemos presentado las *relaciones de preferencia di-*

fusas incompletas como modelo de representación de preferencias así como medidas de consistencia, completitud y consenso que nos permiten modelar situaciones de falta de información de manera apropiada. Sobre este aspecto hemos obtenido las siguientes conclusiones:

(a) Los modelos usuales de representación de preferencias no están preparados para tratar correctamente las situaciones de falta de información. De hecho, para solucionar este tipo de problemas, tradicionalmente se confunde el concepto de falta de información con el de incertidumbre en la información. Sim embargo, con las relaciones de preferencia difusas incompletas hemos comprobado que se pueden modelizar de manera mucho más correcta este tipo de situaciones, diferenciando de manera mucho más clara la incertidumbre de la falta de información.

(b) Un factor crítico que determinará en gran medida la calidad de las soluciones en los problemas de toma de decisiones es la consistencia de la información que ofrecen los expertos, ya que usualmente la información contradictoria no nos proporcionará soluciones coherentes. Es por tanto necesario el estudio de propiedades de consistencia que ayuden a evitar los problemas de inconsistencia.

(c) Otro factor de gran importancia en la toma de decisiones en grupo es el consenso. Para que los expertos acaben satifechos con la solución obtenida esta debe tener un cierto nivel de consenso.

2. **Sobre el proceso de selección.**

Hemos desarrollado un proceso de selección nuevo que permite manejar correctamente las situaciones de falta de información en entornos de toma de decisiones en grupo heterogéneos. Para ello usa las relaciones de preferencia difusas incompletas como formato de representación de preferencias y un nuevo *procedimiento de estimación* que es capaz de calcular los valores perdidos en las relaciones de preferencia incompletas. Además, el proceso de selección tiene en cuenta el nivel de consistencia de los expertos a la hora de agregar sus opiniones. Del desarrollo de este proceso de selección podemos extraer las siguientes conclusiones:

(a) Los procesos de selección desarrollados hasta ahora no poseian ningún mecanismo eficiente que permitiera tratar situaciones de falta de información. Sin embargo, mediante el uso de las relaciones de preferencia difusas incompletas y algunas medidas y propiedades de consistencia hemos verificado que las situaciones de falta de información pueden ser solucionadas eficientemente.

(b) Las medidas de consistencia son un factor interesante a tener en cuenta cuando agregamos las relaciones de preferencia de los distintos expertos, ya que nos permiten establecer un ranking entre los mismos y prestar distinto grado de atención a los expertos que expresan opiniones consistentes de los que no.

(c) Es muy recomendable desarrollar *sistemas de soporte a la decisión* que permitan expresar de manera consistente las opiniones de los expertos, ya que con algunos formatos de representación de preferencias los expertos pueden introducir inconsistencias en el proceso.

3. **Sobre el proceso de consenso.**

   Hemos desarrollado un proceso de consenso nuevo que permite acercar las opiniones de los expertos sobre las alternativas en un problema de toma de decisiones en grupo con relaciones de preferencia difusas incompletas. El proceso de consenso tiene en cuenta tanto la consistencia de los expertos como el nivel de consenso obtenido en cada momento. Además, el modelo de consenso se ha complementado con un mecanismo de retroalimentación que genera recomendaciones a los expertos sobre como deberían cambiar sus preferencias para conseguir una solución de consenso sin perder mucha consistencia en sus propias opiniones. Del desarrollo de este proceso de consenso podemos extraer las siguientes conclusiones:

   (a) Los modelos de consenso típicos no tratan de manera correcta las situaciones de falta de información. Sin embargo, con el uso del procedimiento de estimación previamente presentado podemos manejar de manera apropiada este tipo de problemas.

   (b) Típicamente los procesos de consenso solo tienen en cuenta ciertas medidas de consenso entre los expertos. Esto induce que los expertos cambien sus opiniones para llegar a una solución consensuada, pero olvidando cualquier consideración respecto a la consistencia de la información que expresan. Es por lo tanto obvio que la inclusión de alguna propiedad de consistencia en el proceso de consenso ayudará a obtener mejores soluciones en los dos sentidos: soluciones consensuadas y consistentes.

(c) Para facilitar a los expertos el llegar a una solución del problema es recomendable disponer de algún mecanismo de retroalimentación que les ayude a alcanzar una solución de consenso sin perder por ello demasiada consistencia en las opiniones que expresan.

Trabajos Futuros

En el mundo real existen numerosas situaciones que pueden considerarse como problemas de toma de decisiones en grupo. Por lo tanto sería interesante aplicar los modelos de selección y consenso presentados a dichas situaciones. Para llevarlo a cabo nuestros futuros esfuerzos iran encaminados en los siguientes caminos distintos:

- **Teóricos:**
  - Para incrementar el número de situaciones de toma de decisiones en grupo del mundo real que puedan ser modeladas estudiaremos los problemas donde la información incompleta pueda ser expresada por los expertos con distintos formatos de representación de preferencias (otros tipos de relaciones de preferencia incompletas, valores de utilidad incompletos, etc.).
  - Para modelar procesos de consenso reales más complejos, se tendrán que desarrollar diversos cambios y adiciones a los modelos actuales. Por ejemplo, en los procesos de consenso reales es usualmente posible añadir y quitar alternativas (lo cual ocurre

durante las discusiones de los expertos en las diversas rondas de consenso). Además, conseguir un mejor balance entre la consistencia de las preferencias de los expertos y el nivel de consenso global puede conseguirse cambiando dinámicamente su importancia respectiva. Parece lógico que en las primeras rondas de consenso la consistencia sea el factor que prime, pero según vayan pasando las distintas rondas, el consenso debe ser el factor más importante, y de hecho, en los procesos de consenso reales, cuando la gente comienza a cambiar sus opiniones para alcanzar un estado de consenso, la probabilidad de expresar opiniones inconsistentes son mayores.

- **Prácticos:**

  – Es importante implementar todos los modelos presentados en esta tesis para ser capaces de usarlos en diversos contextos. Debemos aprovechar las poderosas ventajas de comunicación que nos ofrece la World Wide Web hoy en día. Por eso, implementar estos modelos usando las nuevas tecnologías de Internet permitirá llevar a cabo procesos de toma de decisiones en grupo en cualquier país del mundo, incluso cuando los expertos esten lejos unos de otros.

  – Desarrollaremos nuevas herramientas de ayuda y soporte a la decisión para ayudar a los expertos a alcanzar buenas soluciones. Como ejemplo de estas herramientas podemos comentar la creación de interfaces gráficos que permitan comprender de manera sencilla el estado de consenso actual (quién está lejos de quién, cuantas facciones hay en el grupo de discusión, etc.).

– Estudiaremos el desarrollo de mecanismos de feedback más eficientes que produzcan recomendaciones útiles para los expertos, y por tanto ayuden a resolver los problemas de decisión de manera más eficiente.

Publicaciones Derivadas de esta Tesis

Para finalizar debemos citar que las partes principales del trabajo desarrollado en esta memoria han sido publicadas en diferentes revistas y conferencias tanto internacionales como nacionales así como un capítulo de libro:

- Artículos en Revistas Internacionales:

1. F. Chiclana, E. Herrera-Viedma, F. Herrera, S. Alonso. Induced Ordered Weighted Geometric Operators and Their Use in the Aggregation of Multiplicative Preference Relations. *International Journal of Intelligent Systems*, 19 (2004) 233–255.

2. S. Alonso, F. Chiclana F. Herrera, E. Herrera-Viedma, J. Alcalá, C. Porcel. A Consistency Based Procedure to Estimate Missing Pairwise Preference Values. *International Journal of Intelligent Systems*, (2006). In press.

3. E. Herrera-Viedma, F. Chiclana, F. Herrera, and S. Alonso. A Group Decision-Making Model with Incomplete Fuzzy Preference Relations Based on Additive Consistency. *IEEE Transac-*

tions On Systems Man And Cybernetics, Part B, (2006). To appear.

- Contribuciones a Congresos Internacionales:

  1. S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma. A Learning Procedure to Estimate Missing Values in Fuzzy Preference Relations Based on Additive Consistency. *Modeling Decisions for Artificial Intelligence Conference (MDAI-2004)*, Barcelona (Spain), August 2004. *LNCS/LNAI* 3131, 227–238.
  2. F. Chiclana, S. Alonso, F. Herrera, E. Herrera-Viedma, A.G. López-Herrera. Additive Consistency as a Tool to Solve Group Decision Making Problems. *Proceedings of the 5th International Conference on Recent Advances in Soft Computing (RASC 2004)*, Nottingham (UK), December 2004, 237–242.
  3. S. Alonso, E. Herrera-Viedma, F. Chiclana, F. Herrera. An Approach to Consensus Reaching in Multiperson Decision Making Problems with Incomplete Information. *Proceedings of the 4th International Conference of the European Society for Fuzzy Logic and Technology and LFA (EUSFLAT-LFA 2005)*, Barcelona (Spain), September 2005, 904–909.
  4. S. Alonso, E. Herrera-Viedma, F. Herrera, F. Chiclana, C. Porcel. Strategies to Manage Ignorance Situations in Multiperson Decision Making Problems. *Modeling Decisions for Artificial Intelligence Conference (MDAI-2006)*, Tarragona (Spain), April 2006. *LNCS/LNAI 3885*, 34–45.

- Contribuciones a Congresos Nacionales:

1. S. Alonso, E. Herrera-Viedma, F. Chiclana, F. Herrera, Managing Incomplete Information in Consensus Processes. In Proceedings of the I Congreso Español de Informática (CEDI 2005). Simposio sobre Lógica Fuzzy and Soft Computing (LFSC 2005), 13–16 September, Granada (Spain), 175–182, 2005.
2. An IOWA Operator Based On Additive Consistency For Group Decision Making. Proceedings of the XII Congreso Español sobre Tecnología y Lógica Fuzzy (ESTYLF 2004), Jaén (Spain), 443–448, 2004.

- Capítulos de Libros:

1. S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma, Group Decision Making With Incomplete Information. In: Herrera-Viedma, E. (ed.): Procesos de Toma de Decisiones, Modelado y Agregación de Preferencias. Granada (2005) 21–30. I.S.B.N.: 84-934654-1-0.

Chapter 4

Final Comments

In this last chapter we present the results obtained in this thesis and some conclusions derived from them, as well as some future works and a list of published papers about the topics in this thesis.

Obtained Results and Conclusions

To solve group decision making problems in situations of lack of information implies the study of three main aspects: (i) to correctly model lack of information situations, (ii) the development of a selection process that allows the handling of missing information, and (iii) the development of appropiate consensus processes. Paying attention to these aspects, the results presented in this thesis and some conclusions about them are presented in the following points:

1. **About modelling lack of information situations.**

   In this thesis we have presented the *fuzzy preference relations* as a preference representation format, as well as some consistency, completeness and consensus measures which allow to model lack of information situations properly. We have obtained the following conclusions about this aspect:

(a) Usual models to represent preferences are not well prepared to properly tackle lack of information situations. In fact, traditionally, to solve this kind of problems the concepts of lack of information an uncertainty in the information are often confused. However, we have shown that with incomplete fuzzy preference relations we can model in a much more proper way this kind of situations, distinguishing uncertainty and lack of information.

(b) A critic factor which determines the quality of the solutions for a group decision making problem is the consistency of the information that experts give. Contradictory preferences lead the decision process to non coherent solutions. Thus, the study of consistency properties is necessary to solve inconsistency problems.

(c) Another factor of great importance in group decision making is consensus. The experts will usually not be satisfied with the solution of the problem if it does not guarantee a certain level of consensus among experts.

2. **About the selection process.**

We have developed a new selection process that properly handles lack of information situations in heterogeneous group decision making environments. To do so, it uses incomplete fuzzy preference relations as the preference representation format and a new *esti-*

mation procedure that is able to compute the missing values in an incomplete preference relation. Additionally, we take into account the experts' consistnecy level in the selection process when aggregating their preferences. From the development of this selection process we extract the following conslusions: .

(a) Current selection processes do not have any efficient mechanism to tackle lack of information situations. However, with the use of incomplete fuzzy preference relations and some consistency measures and properties we have verified that lack of information situations can be efficiently solved.

(b) Consistency measures are an important factor when aggregating experts' preferences as they allow to rank the experts according to their consistency and so, to offer a greater attention degree to the most consistent ones.

(c) It is also very conveniente to develop *decision support systems* to help the experts to express consistent information, as some of the preference representation formats the experts could introduce some inconsistencies in the process.

3. **About the consensus process.**

We have developed a new consensus process that allow experts to bring near their opinions about the alternatives in a group decision making problem with incomplete fuzzy preference relations. The process takes into account both consistency and consensus measures. Additionally, the consensus model is complemented with a feedback mechanism that generates advice for the experts about how to change their preferences in order to obtain a more consensued solution without loosing too much consistency in their opinions. From the development of this consensus process we extract the following conslusions:

(a) Usual consensus models do not properly handle lack of information situations. However, with the use of the previously presented estimation procedure is possible to correctly tackle thi kind of situations.

(b) Usually, consensus processes only take into account some consensus measures among experts. This induces the experts to change their opinions towards a consensued solution, usually ignoring the consistency issues. Then it is obvious that the inclusion of some consistency properties in the consensus process will help to obtain better solutions in both senses: consensued solutions and consistent ones.

(c) To allow the experts to reach a solution for their problem it is also desirable to have a feedback mechanism which helps them to reach a consensued solution without loosing too much consistency in the preferences that they express.

Future Works

In the real world there are lots of situations that can be considered as a group decision making problem. Thus, it would be interesting to be able to apply the presented consensus and selection processes in those situations. To do so, our future efforts follow to different paths:

- **Theoretical:**
  - To increment the number of real world group decision making situations that can be modelled we will study problems where incomplete information could be expressed by experts in different preference representation formats (other incomplete preference relations, incomplete utility values, and so on).
  - To model more complex real consensus processes, several changes and additions have to be developed. For example, in real consensus processes it is possible to add or remove alternatives (this happens when experts discuss in the different consensus rounds). Also, a better balance between consistency of the preferences of the experts and the consensus degree can be achieved by dinamically changing their respective importance. It seems logical that in the first rounds of the consensus process, consistency is necessary, but as more and more rounds pass, consensus must be a much more important factor, and in fact, in real word consensus processes, when people begin to change their opinions to reach a consensus state, the chances of express inconsistent opinions are greater.
- **Practical:**
  - It is important to implement all the models presented to be able to use them in different contexts. We have to take advantage of the great power of communication that the World Wide Wed provides today. To do so, implement these models using Internet technologies will allow to carry out group decision making

processes in every country of the world, even when the different experts are far away from each other.

– We will develop new aiding tools and decision support systems to help experts to reach good solutions. Examples of these tools can be graphical interfaces to easily understand the current consensus state (who is far from whom, how many different factions are in the discussion group, and so on).

– We will also study the development of more powerful feedback mechanisms to produce useful recommendations for the experts and thus help them to solve the decision problem more efficiently.

Publications Derived from this Tesis

To conclude we must cite that the main parts of this work have been published in different international and national journals and conferences as well as a book chapter:

- Papers in International Journals:

1. F. Chiclana, E. Herrera-Viedma, F. Herrera, S. Alonso. Induced Ordered Weighted Geometric Operators and Their Use in the Aggregation of Multiplicative Preference Relations. *International Journal of Intelligent Systems*, 19 (2004) 233–255.

2. S. Alonso, F. Chiclana F. Herrera, E. Herrera-Viedma, J. Alcalá, C. Porcel. A Consistency Based Procedure to Estimate

Missing Pairwise Preference Values. *International Journal of Intelligent Systems*, (2006). In press.

3. E. Herrera-Viedma, F. Chiclana, F. Herrera, and S. Alonso. A Group Decision-Making Model with Incomplete Fuzzy Preference Relations Based on Additive Consistency. *IEEE Transactions On Systems Man And Cybernetics, Part B*, (2006). To appear.

- Papers in International Conferences:

1. S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma. A Learning Procedure to Estimate Missing Values in Fuzzy Preference Relations Based on Additive Consistency. *Modeling Decisions for Artificial Intelligence Conference (MDAI-2004)*, Barcelona (Spain), August 2004. *LNCS/LNAI* 3131, 227–238.
2. F. Chiclana, S. Alonso, F. Herrera, E. Herrera-Viedma, A.G. Lopez-Herrera. Additive Consistency as a Tool to Solve Group Decision Making Problems. *Proceedings of the 5th International Conference on Recent Advances in Soft Computing (RASC 2004)*, Nottingham (UK), December 2004, 237–242.
3. S. Alonso, E. Herrera-Viedma, F. Chiclana, F. Herrera. An Approach to Consensus Reaching in Multiperson Decision Making Problems with Incomplete Information. *Proceedings of the 4th International Conference of the European Society for Fuzzy Logic and Technology and LFA (EUSFLAT-LFA 2005)*, Barcelona (Spain), September 2005, 904–909.

4. S. Alonso, E. Herrera-Viedma, F. Herrera, F. Chiclana, C. Porcel. Strategies to Manage Ignorance Situations in Multiperson Decision Making Problems. *Modeling Decisions for Artificial Intelligence Conference (MDAI-2006)*, Tarragona (Spain), April 2006. *LNCS/LNAI 3885*, 34–45.

- Papers in National Conferences:

1. S. Alonso, E. Herrera-Viedma, F. Chiclana, F. Herrera, Managing Incomplete Information in Consensus Processes. In Proceedings of the I Congreso Español de Informática (CEDI 2005). Simposio sobre Lógica Fuzzy and Soft Computing (LFSC 2005), 13–16 September, Granada (Spain), 175–182, 2005.
2. An IOWA Operator Based On Additive Consistency For Group Decision Making. Proceedings of the XII Congreso Español sobre Tecnología y Lógica Fuzzy (ESTYLF 2004), Jaén (Spain), 443–448, 2004.

- Book Chapter:

1. S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma, Group Decision Making With Incomplete Information. In: Herrera-Viedma, E. (ed.): Procesos de Toma de Decisiones, Modelado y Agregación de Preferencias. Granada (2005) 21–30. I.S.B.N.: 84-934654-1-0.

Appendix A

Fuzzy Quantifiers and Their Use to Model Fuzzy Majority

The majority is traditionally defined as a threshold number of individuals. Fuzzy majority is a soft majority concept expressed by a fuzzy quantifier, which is manipulated via a fuzzy logic-based calculus of linguistically quantified propositions. Therefore, using fuzzy majority guided aggregation operators we can incorporate the concept of majority into the computation of the solution.

Quantifiers can be used to represent the amount of items satisfying a given predicate. Classic logic is restricted to the use of the two quantifiers, *there exists* and *for all*, that are closely related, respectively, to the *or* and *and* connectives. Human discourse is much richer and more diverse in its quantifiers, e.g. *about 5*, *almost all*, *a few*, *many*, *most of*, *as many as possible*, *nearly half*, *at least half*. In an attempt to bridge the gap between formal systems and natural discourse and, in turn, provide a more flexible knowledge representation tool, Zadeh introduced the concept of fuzzy quantifiers [80].

Zadeh suggested that the semantics of a fuzzy quantifier can be captured by using fuzzy subsets for its representation. He distinguished between two types of fuzzy quantifiers: *absolute* and *relative*. Absolute quantifiers are used to represent amounts that are absolute in nature such as *about 2* or *more than 5*. These absolute linguistic quantifiers are closely related to the concept of the count or number of elements. He defined these quantifiers as fuzzy subsets of the non-negative real numbers, $\Re^+$. In this approach, an absolute quantifier can be represented by a fuzzy subset Q, such that for any $r \in \Re^+$ the membership degree of r in Q, $Q(r)$, indicates the degree in which the amount r is compatible with the quantifier represented by Q. Relative quantifiers, such as *most*, *at least half*, can be represented by fuzzy subsets of the unit interval, $[0, 1]$. For any $r \in [0, 1]$, $Q(r)$ indicates the degree in which the proportion r is compatible with the meaning of the quantifier it represents. Any quantifier of natural language can be represented as a relative quantifier or, given the cardinality of the elements considered, as an absolute quantifier.

A relative quantifier, $Q : [0, 1] \to [0, 1]$, satisfies:

$$Q(0) = 0, \text{ and } \exists r \in [0, 1] \text{ such that } Q(r) = 1.$$

Yager in [74] distinguishes two categories of these relative quantifiers: regular increasing monotone (RIM) quantifiers such as *all, most, many, at least* α; and regular decreasing monotone (RDM) quantifiers such as *at most one, few, at most* α.

A regular increasing monotone (RIM) quantifier satisfies:

$$\forall a, b \text{ if } a > b \text{ then } Q(a) \geq Q(b).$$

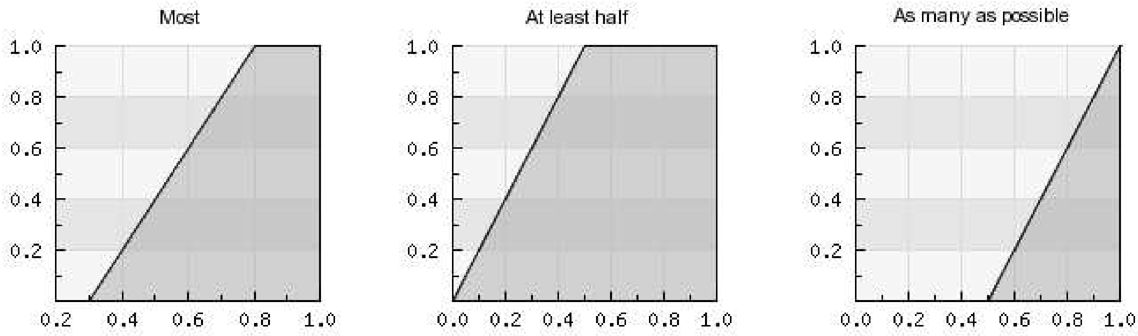

Figure A.1: Examples of relative fuzzy linguistic quantifiers

A widely used membership function for RIM quantifiers is [36]

$$Q(r) = \begin{cases} 0 & \text{if } r < a \\ \frac{r-a}{b-a} & \text{if } a \leq r < b \\ 1 & \text{if } r \geq b \end{cases} \tag{A.1}$$

with $a, b, r \in [0, 1]$. Some examples of relative quantifiers are shown in Fig. A.1, where the parameters, (a, b) are $(0.3, 0.8)$, $(0, 0.5)$ and $(0.5, 1)$, respectively.

The particular RIM function with parameters $(0.3, 0.8)$ used to represent the fuzzy linguistic quantifier *most of* when applied with an OWA or IOWA operator associates a *low weighting* value to the most important/consistent experts because it assigns a value of 0 to the first 30% of experts. To overcome this problem, a different RIM function to represent the fuzzy linguistic quantifier *most of* should be used. To guarantee that all the important/consistent experts have associated a non-zero weighting value, and therefore all of them contribute to the final aggregated value, a strictly increasing RIM function should be used. On the other hand, in order to associate a high weighting value to those values with a high consistency level, a RIM function with a rate of increase in the

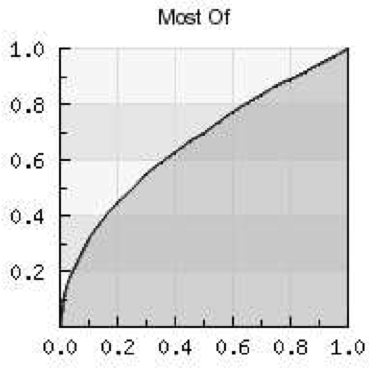

Figure A.2: The relative fuzzy linguistic quantifier *most of*

unit interval inversely proportional to the value of the variable r seems to be adequate.

Yager in [74] considers the parameterized family of RIM quantifiers

$$Q(r) = r^a, \quad a \geq 0$$

and the particular function with $a = 2$ to represent fuzzy linguistic quantifier *most of*. This function is strictly increasing but, when used with an OWA or IOWA operators, associates high weighting values to low consistent values. In order to overcome this drawback, two approaches could be adopted:

i) the values are ordered using the opposite criteria, i.e. the first one being the one with lowest consistency degree, or

ii) a RIM function with $a < 1$ is used.

We have opted for the second one, and in particular in this paper we use RIM function $Q(r) = r^{1/2}$ given in *figure A.2* to represent fuzzy linguistic quantifier *most of*.

Appendix B

Complete Computations for the Example in Chapter 3

In this section we present all the computations made in the example in *chapter 3* for the first round of consensus.

We part from the following fuzzy preference relations expressed by four experts $E = \{e_1, e_2, e_3, e_4\}$ about a four different alternatives $X = \{x_1, x_2, x_3, x_4\}$:

$$P^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ x & - & x & x \\ x & x & - & x \\ x & x & x & - \end{pmatrix}; P^2 = \begin{pmatrix} - & x & 0.7 & x \\ 0.4 & - & x & 0.7 \\ 0.3 & x & - & x \\ x & 0.4 & x & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.3 & x & 0.75 \\ 0.6 & - & x & x \\ x & x & - & x \\ 0.3 & 0.4 & x & - \end{pmatrix}; P^4 = \begin{pmatrix} - & x & 0.6 & 0.3 \\ 0.4 & - & 0.4 & 0.2 \\ 0.5 & 0.6 & - & 0.3 \\ 0.7 & 0.7 & 0.7 & - \end{pmatrix}$$

B.1 Computing Missing Information

$$P^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ x & - & x & x \\ x & x & - & x \\ x & x & x & - \end{pmatrix}$$

Reconstruction of P^1, iteration 1:

Computation of cp^1_{21}:

$cp^1_{21} =?$; Cannot be computed at this iteration

Computation of cp^1_{23}:

$$(cp^1_{23})^{12} = p^1_{13} - p^1_{12} + 0.5 = 0.6 - 0.2 + 0.5 = 0.9$$
$$(cp^1_{23})^2 = ((cp^1_{23})^{12})/1.0 = 0.9/1.0 = 0.9$$
$cp^1_{23} = 0.9/1 = 0.9$

Computation of cp^1_{24}:

$$(cp^1_{24})^{12} = p^1_{14} - p^1_{12} + 0.5 = 0.4 - 0.2 + 0.5 = 0.7$$
$$(cp^1_{24})^2 = ((cp^1_{24})^{12})/1.0 = 0.7/1.0 = 0.7$$
$cp^1_{24} = 0.7/1 = 0.7$

Computation of cp^1_{31}:

$cp^1_{31} =?$; Cannot be computed at this iteration

Computation of cp^1_{32}:

$$(cp^1_{32})^{12} = p^1_{12} - p^1_{13} + 0.5 = 0.2 - 0.6 + 0.5 = 0.1$$
$$(cp^1_{32})^2 = ((cp^1_{32})^{12})/1.0 = 0.1/1.0 = 0.1$$
$cp^1_{32} = 0.1/1 = 0.1$

Computation of cp^1_{34}:

$$(cp^1_{34})^{12} = p^1_{14} - p^1_{13} + 0.5 = 0.4 - 0.6 + 0.5 = 0.3$$
$$(cp^1_{34})^2 = ((cp^1_{34})^{12})/1.0 = 0.3/1.0 = 0.3$$
$cp^1_{34} = 0.3/1 = 0.3$

Computation of cp^1_{41}:

$cp^1_{41} =?$; Cannot be computed at this iteration

Computation of cp^1_{42}:

$$(cp^1_{42})^{12} = p^1_{12} - p^1_{14} + 0.5 = 0.2 - 0.4 + 0.5 = 0.3$$
$$(cp^1_{42})^2 = ((cp^1_{42})^{12})/1.0 = 0.3/1.0 = 0.3$$

$cp^1_{42} = 0.3/1 = 0.3$

Computation of cp^1_{43}*:*

$$(cp^1_{43})^{12} = p^1_{13} - p^1_{14} + 0.5 = 0.6 - 0.4 + 0.5 = 0.7$$
$$(cp^1_{43})^{2} = ((cp^1_{43})^{12})/1.0 = 0.7/1.0 = 0.7$$
$$cp^1_{43} = 0.7/1 = 0.7$$

$$Partial\ \overline{P}^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ x & - & 0.9 & 0.7 \\ x & 0.1 & - & 0.3 \\ x & 0.3 & 0.7 & - \end{pmatrix}$$

Reconstruction of P^1, iteration 2:

Computation of cp^1_{21}*:*

$$(cp^1_{21})^{33} = p^1_{23} - p^1_{13} + 0.5 = 0.9 - 0.6 + 0.5 = 0.8$$
$$(cp^1_{21})^{43} = p^1_{24} - p^1_{14} + 0.5 = 0.7 - 0.4 + 0.5 = 0.8$$
$$(cp^1_{21})^{3} = ((cp^1_{21})^{33} + (cp^1_{21})^{43})/2.0 = 1.6/2.0 = 0.8$$
$$cp^1_{21} = 0.8/1 = 0.8$$

Computation of cp^1_{31}*:*

$$(cp^1_{31})^{23} = p^1_{32} - p^1_{12} + 0.5 = 0.1 - 0.2 + 0.5 = 0.4$$
$$(cp^1_{31})^{43} = p^1_{34} - p^1_{14} + 0.5 = 0.3 - 0.4 + 0.5 = 0.4$$
$$(cp^1_{31})^{3} = ((cp^1_{31})^{23} + (cp^1_{31})^{43})/2.0 = 0.8/2.0 = 0.4$$
$$cp^1_{31} = 0.4/1 = 0.4$$

Computation of cp^1_{41}*:*

$$(cp^1_{41})^{23} = p^1_{42} - p^1_{12} + 0.5 = 0.3 - 0.2 + 0.5 = 0.6$$
$$(cp^1_{41})^{33} = p^1_{43} - p^1_{13} + 0.5 = 0.7 - 0.6 + 0.5 = 0.6$$
$$(cp^1_{41})^{3} = ((cp^1_{41})^{23} + (cp^1_{41})^{33})/2.0 = 1.2/2.0 = 0.6$$
$$cp^1_{41} = 0.6/1 = 0.6$$

$$\overline{P}^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix}$$

$$P^2 = \begin{pmatrix} - & x & 0.7 & x \\ 0.4 & - & x & 0.7 \\ 0.3 & x & - & x \\ x & 0.4 & x & - \end{pmatrix}$$

Reconstruction of P^2, iteration 1:

Computation of cp^2_{12}:

$cp^2_{12} =?$; Cannot be computed at this iteration

Computation of cp^2_{14}:

$$(cp^2_{14})^{22} = p^2_{24} - p^2_{21} + 0.5 = 0.7 - 0.4 + 0.5 = 0.8$$
$$(cp^2_{14})^{2} = ((cp^2_{14})^{22})/1.0 = 0.8/1.0 = 0.8$$
$cp^2_{14} = 0.8/1 = 0.8$

Computation of cp^2_{23}:

$$(cp^2_{23})^{11} = p^2_{21} + p^2_{13} - 0.5 = 0.4 + 0.7 - 0.5 = 0.6$$
$$(cp^2_{23})^{1} = ((cp^2_{23})^{11})/1.0 = 0.6/1.0 = 0.6$$
$$(cp^2_{23})^{13} = p^2_{21} - p^2_{31} + 0.5 = 0.4 - 0.3 + 0.5 = 0.6$$
$$(cp^2_{23})^{3} = ((cp^2_{23})^{13})/1.0 = 0.6/1.0 = 0.6$$
$cp^2_{23} = 1.2/2 = 0.6$

Computation of cp^2_{32}:

$$(cp^2_{32})^{13} = p^2_{31} - p^2_{21} + 0.5 = 0.3 - 0.4 + 0.5 = 0.4$$
$$(cp^2_{32})^{3} = ((cp^2_{32})^{13})/1.0 = 0.4/1.0 = 0.4$$
$cp^2_{32} = 0.4/1 = 0.4$

Computation of cp^2_{34}:

$cp^2_{34} =?$; Cannot be computed at this iteration

Computation of cp^2_{41}:

$$(cp^2_{41})^{21} = p^2_{42} + p^2_{21} - 0.5 = 0.4 + 0.4 - 0.5 = 0.3$$
$$(cp^2_{41})^{1} = ((cp^2_{41})^{21})/1.0 = 0.3/1.0 = 0.3$$
$$(cp^2_{41})^{22} = p^2_{21} - p^2_{24} + 0.5 = 0.4 - 0.7 + 0.5 = 0.2$$

$(cp_{41}^2)^2 = ((cp_{41}^2)^{22})/1.0 = 0.2/1.0 = 0.2$
$cp_{41}^2 = 0.5/2 = 0.25$

Computation of cp_{43}^2*:*
$cp_{43}^2 =?$; Cannot be computed at this iteration

$$Partial\ \overline{P}^2 = \begin{pmatrix} - & x & 0.7 & 0.8 \\ 0.4 & - & 0.6 & 0.7 \\ 0.3 & 0.4 & - & x \\ 0.25 & 0.4 & x & - \end{pmatrix}$$

Reconstruction of P^2, iteration 2:

Computation of cp_{12}^2*:*

$(cp_{12}^2)^{31} = p_{13}^2 + p_{32}^2 - 0.5 = 0.7 + 0.4 - 0.5 = 0.6$
$(cp_{12}^2)^{41} = p_{14}^2 + p_{42}^2 - 0.5 = 0.8 + 0.4 - 0.5 = 0.7$
$(cp_{12}^2)^1 = ((cp_{12}^2)^{31} + (cp_{12}^2)^{41})/2.0 = 1.3/2.0 = 0.65$
$(cp_{12}^2)^{32} = p_{32}^2 - p_{31}^2 + 0.5 = 0.4 - 0.3 + 0.5 = 0.6$
$(cp_{12}^2)^{42} = p_{42}^2 - p_{41}^2 + 0.5 = 0.4 - 0.25 + 0.5 = 0.65$
$(cp_{12}^2)^2 = ((cp_{12}^2)^{32} + (cp_{12}^2)^{42})/2.0 = 1.25/2.0 = 0.63$
$(cp_{12}^2)^{33} = p_{13}^2 - p_{23}^2 + 0.5 = 0.7 - 0.6 + 0.5 = 0.6$
$(cp_{12}^2)^{43} = p_{14}^2 - p_{24}^2 + 0.5 = 0.8 - 0.7 + 0.5 = 0.6$
$(cp_{12}^2)^3 = ((cp_{12}^2)^{33} + (cp_{12}^2)^{43})/2.0 = 1.2/2.0 = 0.6$
$cp_{12}^2 = 1.87/3 = 0.62$

Computation of cp_{34}^2*:*

$(cp_{34}^2)^{11} = p_{31}^2 + p_{14}^2 - 0.5 = 0.3 + 0.8 - 0.5 = 0.6$
$(cp_{34}^2)^{21} = p_{32}^2 + p_{24}^2 - 0.5 = 0.4 + 0.7 - 0.5 = 0.6$
$(cp_{34}^2)^1 = ((cp_{34}^2)^{11} + (cp_{34}^2)^{21})/2.0 = 1.2/2.0 = 0.6$
$(cp_{34}^2)^{12} = p_{14}^2 - p_{13}^2 + 0.5 = 0.8 - 0.7 + 0.5 = 0.6$
$(cp_{34}^2)^{22} = p_{24}^2 - p_{23}^2 + 0.5 = 0.7 - 0.6 + 0.5 = 0.6$
$(cp_{34}^2)^2 = ((cp_{34}^2)^{12} + (cp_{34}^2)^{22})/2.0 = 1.2/2.0 = 0.6$
$(cp_{34}^2)^{13} = p_{31}^2 - p_{41}^2 + 0.5 = 0.3 - 0.25 + 0.5 = 0.55$
$(cp_{34}^2)^{23} = p_{32}^2 - p_{42}^2 + 0.5 = 0.4 - 0.4 + 0.5 = 0.5$
$(cp_{34}^2)^3 = ((cp_{34}^2)^{13} + (cp_{34}^2)^{23})/2.0 = 1.05/2.0 = 0.52$
$cp_{34}^2 = 1.72/3 = 0.57$

Computation of cp_{43}^2*:*

$$(cp_{43}^2)^{11} = p_{41}^2 + p_{13}^2 - 0.5 = 0.25 + 0.7 - 0.5 = 0.45$$
$$(cp_{43}^2)^{21} = p_{42}^2 + p_{23}^2 - 0.5 = 0.4 + 0.6 - 0.5 = 0.5$$
$$(cp_{43}^2)^{1} = ((cp_{43}^2)^{11} + (cp_{43}^2)^{21})/2.0 = 0.95/2.0 = 0.48$$
$$(cp_{43}^2)^{12} = p_{13}^2 - p_{14}^2 + 0.5 = 0.7 - 0.8 + 0.5 = 0.4$$
$$(cp_{43}^2)^{22} = p_{23}^2 - p_{24}^2 + 0.5 = 0.6 - 0.7 + 0.5 = 0.4$$
$$(cp_{43}^2)^{2} = ((cp_{43}^2)^{12} + (cp_{43}^2)^{22})/2.0 = 0.8/2.0 = 0.4$$
$$(cp_{43}^2)^{13} = p_{41}^2 - p_{31}^2 + 0.5 = 0.25 - 0.3 + 0.5 = 0.45$$
$$(cp_{43}^2)^{23} = p_{42}^2 - p_{32}^2 + 0.5 = 0.4 - 0.4 + 0.5 = 0.5$$
$$(cp_{43}^2)^{3} = ((cp_{43}^2)^{13} + (cp_{43}^2)^{23})/2.0 = 0.95/2.0 = 0.48$$

$cp_{43}^2 = 1.35/3 = 0.45$

$$\overline{P}^2 = \begin{pmatrix} - & 0.62 & 0.7 & 0.8 \\ 0.4 & - & 0.6 & 0.7 \\ 0.3 & 0.4 & - & 0.57 \\ 0.25 & 0.4 & 0.45 & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.3 & x & 0.75 \\ 0.6 & - & x & x \\ x & x & - & x \\ 0.3 & 0.4 & x & - \end{pmatrix}$$

Reconstruction of P^3, iteration 1:

Computation of cp_{13}^3*:*

$cp_{13}^3 =?$; Cannot be computed at this iteration

Computation of cp_{23}^3*:*

$cp_{23}^3 =?$; Cannot be computed at this iteration

Computation of cp_{24}^3*:*

$$(cp_{24}^3)^{11} = p_{21}^3 + p_{14}^3 - 0.5 = 0.6 + 0.75 - 0.5 = 0.85$$
$$(cp_{24}^3)^{1} = ((cp_{24}^3)^{11})/1.0 = 0.85/1.0 = 0.85$$

$$(cp^3_{24})^{12} = p^3_{14} - p^3_{12} + 0.5 = 0.75 - 0.3 + 0.5 = 0.95$$
$$(cp^3_{24})^2 = ((cp^3_{24})^{12})/1.0 = 0.95/1.0 = 0.95$$
$$(cp^3_{24})^{13} = p^3_{21} - p^3_{41} + 0.5 = 0.6 - 0.3 + 0.5 = 0.8$$
$$(cp^3_{24})^3 = ((cp^3_{24})^{13})/1.0 = 0.8/1.0 = 0.8$$
$cp^3_{24} = 2.6/3 = 0.87$

Computation of cp^3_{31}*:*
cp^3_{31} =? ; Cannot be computed at this iteration

Computation of cp^3_{32}*:*
cp^3_{32} =? ; Cannot be computed at this iteration

Computation of cp^3_{34}*:*
cp^3_{34} =? ; Cannot be computed at this iteration

Computation of cp^3_{43}*:*
cp^3_{43} =? ; Cannot be computed at this iteration

$$Partial\ \overline{P}^3 = \begin{pmatrix} - & 0.3 & x & 0.75 \\ 0.6 & - & x & 0.87 \\ x & x & - & x \\ 0.3 & 0.4 & x & - \end{pmatrix}$$

We are in an *Ignorance Situation*. We apply *Strategy Number 3* to completely reconstruct P^3:
Computation of cp^3_{13}*:*
$$SC_1 = 0.6 + 0.3 = 0.45$$
$$SR_1 = 0.3 + 0.75 = 0.53$$
$cp^3_{13} = 0.5 + (0.53 - 0.45)/2 = 0.54$

Computation of cp^3_{23}*:*
$$SC_2 = 0.3 + 0.4 = 0.35$$
$$SR_2 = 0.6 + 0.87 = 0.73$$
$cp^3_{23} = 0.5 + (0.73 - 0.35)/2 = 0.69$

Computation of cp_{31}^3:

$SC_1 = 0.6 + 0.3 = 0.45$

$SR_1 = 0.3 + 0.75 = 0.53$

$cp_{31}^3 = 0.5 + (0.45 - 0.53)/2 = 0.46$

Computation of cp_{32}^3:

$SC_2 = 0.3 + 0.4 = 0.35$

$SR_2 = 0.6 + 0.87 = 0.73$

$cp_{32}^3 = 0.5 + (0.35 - 0.73)/2 = 0.31$

Computation of cp_{34}^3:

$SC_4 = 0.75 + 0.87 = 0.81$

$SR_4 = 0.3 + 0.4 = 0.35$

$cp_{34}^3 = 0.5 + (0.81 - 0.35)/2 = 0.73$

Computation of cp_{43}^3:

$SC_4 = 0.75 + 0.87 = 0.81$

$SR_4 = 0.3 + 0.4 = 0.35$

$cp_{43}^3 = 0.5 + (0.35 - 0.81)/2 = 0.27$

$$\overline{P}^3 = \begin{pmatrix} - & 0.3 & 0.54 & 0.75 \\ 0.6 & - & 0.69 & 0.87 \\ 0.46 & 0.31 & - & 0.73 \\ 0.3 & 0.4 & 0.27 & - \end{pmatrix}$$

$$P^4 = \begin{pmatrix} - & x & 0.6 & 0.3 \\ 0.4 & - & 0.4 & 0.2 \\ 0.5 & 0.6 & - & 0.3 \\ 0.7 & 0.7 & 0.7 & - \end{pmatrix}$$

Reconstruction of P^4, iteration 1:

Computation of cp_{12}^4:

$(cp_{12}^4)^{31} = p_{13}^4 + p_{32}^4 - 0.5 = 0.6 + 0.6 - 0.5 = 0.7$

$$(cp^4_{12})^{41} = p^4_{14} + p^4_{42} - 0.5 = 0.3 + 0.7 - 0.5 = 0.5$$
$$(cp^4_{12})^1 = ((cp^4_{12})^{31} + (cp^4_{12})^{41})/2.0 = 1.2/2.0 = 0.6$$
$$(cp^4_{12})^{32} = p^4_{32} - p^4_{31} + 0.5 = 0.6 - 0.5 + 0.5 = 0.6$$
$$(cp^4_{12})^{42} = p^4_{42} - p^4_{41} + 0.5 = 0.7 - 0.7 + 0.5 = 0.5$$
$$(cp^4_{12})^2 = ((cp^4_{12})^{32} + (cp^4_{12})^{42})/2.0 = 1.1/2.0 = 0.55$$
$$(cp^4_{12})^{33} = p^4_{13} - p^4_{23} + 0.5 = 0.6 - 0.4 + 0.5 = 0.7$$
$$(cp^4_{12})^{43} = p^4_{14} - p^4_{24} + 0.5 = 0.3 - 0.2 + 0.5 = 0.6$$
$$(cp^4_{12})^3 = ((cp^4_{12})^{33} + (cp^4_{12})^{43})/2.0 = 1.3/2.0 = 0.65$$
$$cp^4_{12} = 1.8/3 = 0.6$$

$$\overline{P}^4 = \begin{pmatrix} - & 0.6 & 0.6 & 0.3 \\ 0.4 & - & 0.4 & 0.2 \\ 0.5 & 0.6 & - & 0.3 \\ 0.7 & 0.7 & 0.7 & - \end{pmatrix}$$

B.2 Computing Consistent Fuzzy Preference Relations

Computing CP^1:

Computation of cp^1_{12}:

$$(cp^1_{12})^{31} = p^1_{13} + p^1_{32} - 0.5 = 0.6 + 0.1 - 0.5 = 0.2$$
$$(cp^1_{12})^{41} = p^1_{14} + p^1_{42} - 0.5 = 0.4 + 0.3 - 0.5 = 0.2$$
$$(cp^1_{12})^1 = ((cp^1_{12})^{31} + (cp^1_{12})^{41})/2.0 = 0.4/2.0 = 0.2$$
$$(cp^1_{12})^{32} = p^1_{32} - p^1_{31} + 0.5 = 0.1 - 0.4 + 0.5 = 0.2$$
$$(cp^1_{12})^{42} = p^1_{42} - p^1_{41} + 0.5 = 0.3 - 0.6 + 0.5 = 0.2$$
$$(cp^1_{12})^2 = ((cp^1_{12})^{32} + (cp^1_{12})^{42})/2.0 = 0.4/2.0 = 0.2$$
$$(cp^1_{12})^{33} = p^1_{13} - p^1_{23} + 0.5 = 0.6 - 0.9 + 0.5 = 0.2$$
$$(cp^1_{12})^{43} = p^1_{14} - p^1_{24} + 0.5 = 0.4 - 0.7 + 0.5 = 0.2$$
$$(cp^1_{12})^3 = ((cp^1_{12})^{33} + (cp^1_{12})^{43})/2.0 = 0.4/2.0 = 0.2$$
$$cp^1_{12} = 0.6/3 = 0.2$$

Computation of cp^1_{13}:

$$(cp^1_{13})^{21} = p^1_{12} + p^1_{23} - 0.5 = 0.2 + 0.9 - 0.5 = 0.6$$
$$(cp^1_{13})^{41} = p^1_{14} + p^1_{43} - 0.5 = 0.4 + 0.7 - 0.5 = 0.6$$
$$(cp^1_{13})^{1} = ((cp^1_{13})^{21} + (cp^1_{13})^{41})/2.0 = 1.2/2.0 = 0.6$$
$$(cp^1_{13})^{22} = p^1_{23} - p^1_{21} + 0.5 = 0.9 - 0.8 + 0.5 = 0.6$$
$$(cp^1_{13})^{42} = p^1_{43} - p^1_{41} + 0.5 = 0.7 - 0.6 + 0.5 = 0.6$$
$$(cp^1_{13})^{2} = ((cp^1_{13})^{22} + (cp^1_{13})^{42})/2.0 = 1.2/2.0 = 0.6$$
$$(cp^1_{13})^{23} = p^1_{12} - p^1_{32} + 0.5 = 0.2 - 0.1 + 0.5 = 0.6$$
$$(cp^1_{13})^{43} = p^1_{14} - p^1_{34} + 0.5 = 0.4 - 0.3 + 0.5 = 0.6$$
$$(cp^1_{13})^{3} = ((cp^1_{13})^{23} + (cp^1_{13})^{43})/2.0 = 1.2/2.0 = 0.6$$
$$cp^1_{13} = 1.8/3 = 0.6$$

Computation of cp^1_{14}*:*

$$(cp^1_{14})^{21} = p^1_{12} + p^1_{24} - 0.5 = 0.2 + 0.7 - 0.5 = 0.4$$
$$(cp^1_{14})^{31} = p^1_{13} + p^1_{34} - 0.5 = 0.6 + 0.3 - 0.5 = 0.4$$
$$(cp^1_{14})^{1} = ((cp^1_{14})^{21} + (cp^1_{14})^{31})/2.0 = 0.8/2.0 = 0.4$$
$$(cp^1_{14})^{22} = p^1_{24} - p^1_{21} + 0.5 = 0.7 - 0.8 + 0.5 = 0.4$$
$$(cp^1_{14})^{32} = p^1_{34} - p^1_{31} + 0.5 = 0.3 - 0.4 + 0.5 = 0.4$$
$$(cp^1_{14})^{2} = ((cp^1_{14})^{22} + (cp^1_{14})^{32})/2.0 = 0.8/2.0 = 0.4$$
$$(cp^1_{14})^{23} = p^1_{12} - p^1_{42} + 0.5 = 0.2 - 0.3 + 0.5 = 0.4$$
$$(cp^1_{14})^{33} = p^1_{13} - p^1_{43} + 0.5 = 0.6 - 0.7 + 0.5 = 0.4$$
$$(cp^1_{14})^{3} = ((cp^1_{14})^{23} + (cp^1_{14})^{33})/2.0 = 0.8/2.0 = 0.4$$
$$cp^1_{14} = 1.2/3 = 0.4$$

Computation of cp^1_{21}*:*

$$(cp^1_{21})^{31} = p^1_{23} + p^1_{31} - 0.5 = 0.9 + 0.4 - 0.5 = 0.8$$
$$(cp^1_{21})^{41} = p^1_{24} + p^1_{41} - 0.5 = 0.7 + 0.6 - 0.5 = 0.8$$
$$(cp^1_{21})^{1} = ((cp^1_{21})^{31} + (cp^1_{21})^{41})/2.0 = 1.6/2.0 = 0.8$$
$$(cp^1_{21})^{32} = p^1_{31} - p^1_{32} + 0.5 = 0.4 - 0.1 + 0.5 = 0.8$$
$$(cp^1_{21})^{42} = p^1_{41} - p^1_{42} + 0.5 = 0.6 - 0.3 + 0.5 = 0.8$$
$$(cp^1_{21})^{2} = ((cp^1_{21})^{32} + (cp^1_{21})^{42})/2.0 = 1.6/2.0 = 0.8$$
$$(cp^1_{21})^{33} = p^1_{23} - p^1_{13} + 0.5 = 0.9 - 0.6 + 0.5 = 0.8$$
$$(cp^1_{21})^{43} = p^1_{24} - p^1_{14} + 0.5 = 0.7 - 0.4 + 0.5 = 0.8$$
$$(cp^1_{21})^{3} = ((cp^1_{21})^{33} + (cp^1_{21})^{43})/2.0 = 1.6/2.0 = 0.8$$

$cp^1_{21} = 2.4/3 = 0.8$

Computation of cp^1_{23}*:*

$$(cp^1_{23})^{11} = p^1_{21} + p^1_{13} - 0.5 = 0.8 + 0.6 - 0.5 = 0.9$$
$$(cp^1_{23})^{41} = p^1_{24} + p^1_{43} - 0.5 = 0.7 + 0.7 - 0.5 = 0.9$$
$$(cp^1_{23})^{1} = ((cp^1_{23})^{11} + (cp^1_{23})^{41})/2.0 = 1.8/2.0 = 0.9$$
$$(cp^1_{23})^{12} = p^1_{13} - p^1_{12} + 0.5 = 0.6 - 0.2 + 0.5 = 0.9$$
$$(cp^1_{23})^{42} = p^1_{43} - p^1_{42} + 0.5 = 0.7 - 0.3 + 0.5 = 0.9$$
$$(cp^1_{23})^{2} = ((cp^1_{23})^{12} + (cp^1_{23})^{42})/2.0 = 1.8/2.0 = 0.9$$
$$(cp^1_{23})^{13} = p^1_{21} - p^1_{31} + 0.5 = 0.8 - 0.4 + 0.5 = 0.9$$
$$(cp^1_{23})^{43} = p^1_{24} - p^1_{34} + 0.5 = 0.7 - 0.3 + 0.5 = 0.9$$
$$(cp^1_{23})^{3} = ((cp^1_{23})^{13} + (cp^1_{23})^{43})/2.0 = 1.8/2.0 = 0.9$$

$cp^1_{23} = 2.7/3 = 0.9$

Computation of cp^1_{24}*:*

$$(cp^1_{24})^{11} = p^1_{21} + p^1_{14} - 0.5 = 0.8 + 0.4 - 0.5 = 0.7$$
$$(cp^1_{24})^{31} = p^1_{23} + p^1_{34} - 0.5 = 0.9 + 0.3 - 0.5 = 0.7$$
$$(cp^1_{24})^{1} = ((cp^1_{24})^{11} + (cp^1_{24})^{31})/2.0 = 1.4/2.0 = 0.7$$
$$(cp^1_{24})^{12} = p^1_{14} - p^1_{12} + 0.5 = 0.4 - 0.2 + 0.5 = 0.7$$
$$(cp^1_{24})^{32} = p^1_{34} - p^1_{32} + 0.5 = 0.3 - 0.1 + 0.5 = 0.7$$
$$(cp^1_{24})^{2} = ((cp^1_{24})^{12} + (cp^1_{24})^{32})/2.0 = 1.4/2.0 = 0.7$$
$$(cp^1_{24})^{13} = p^1_{21} - p^1_{41} + 0.5 = 0.8 - 0.6 + 0.5 = 0.7$$
$$(cp^1_{24})^{33} = p^1_{23} - p^1_{43} + 0.5 = 0.9 - 0.7 + 0.5 = 0.7$$
$$(cp^1_{24})^{3} = ((cp^1_{24})^{13} + (cp^1_{24})^{33})/2.0 = 1.4/2.0 = 0.7$$

$cp^1_{24} = 2.1/3 = 0.7$

Computation of cp^1_{31}*:*

$$(cp^1_{31})^{21} = p^1_{32} + p^1_{21} - 0.5 = 0.1 + 0.8 - 0.5 = 0.4$$
$$(cp^1_{31})^{41} = p^1_{34} + p^1_{41} - 0.5 = 0.3 + 0.6 - 0.5 = 0.4$$
$$(cp^1_{31})^{1} = ((cp^1_{31})^{21} + (cp^1_{31})^{41})/2.0 = 0.8/2.0 = 0.4$$
$$(cp^1_{31})^{22} = p^1_{21} - p^1_{23} + 0.5 = 0.8 - 0.9 + 0.5 = 0.4$$
$$(cp^1_{31})^{42} = p^1_{41} - p^1_{43} + 0.5 = 0.6 - 0.7 + 0.5 = 0.4$$
$$(cp^1_{31})^{2} = ((cp^1_{31})^{22} + (cp^1_{31})^{42})/2.0 = 0.8/2.0 = 0.4$$

$$(cp_{31}^1)^{23} = p_{32}^1 - p_{12}^1 + 0.5 = 0.1 - 0.2 + 0.5 = 0.4$$
$$(cp_{31}^1)^{43} = p_{34}^1 - p_{14}^1 + 0.5 = 0.3 - 0.4 + 0.5 = 0.4$$
$$(cp_{31}^1)^3 = ((cp_{31}^1)^{23} + (cp_{31}^1)^{43})/2.0 = 0.8/2.0 = 0.4$$
$$cp_{31}^1 = 1.2/3 = 0.4$$

Computation of cp_{32}^1*:*

$$(cp_{32}^1)^{11} = p_{31}^1 + p_{12}^1 - 0.5 = 0.4 + 0.2 - 0.5 = 0.1$$
$$(cp_{32}^1)^{41} = p_{34}^1 + p_{42}^1 - 0.5 = 0.3 + 0.3 - 0.5 = 0.1$$
$$(cp_{32}^1)^1 = ((cp_{32}^1)^{11} + (cp_{32}^1)^{41})/2.0 = 0.2/2.0 = 0.1$$
$$(cp_{32}^1)^{12} = p_{12}^1 - p_{13}^1 + 0.5 = 0.2 - 0.6 + 0.5 = 0.1$$
$$(cp_{32}^1)^{42} = p_{42}^1 - p_{43}^1 + 0.5 = 0.3 - 0.7 + 0.5 = 0.1$$
$$(cp_{32}^1)^2 = ((cp_{32}^1)^{12} + (cp_{32}^1)^{42})/2.0 = 0.2/2.0 = 0.1$$
$$(cp_{32}^1)^{13} = p_{31}^1 - p_{21}^1 + 0.5 = 0.4 - 0.8 + 0.5 = 0.1$$
$$(cp_{32}^1)^{43} = p_{34}^1 - p_{24}^1 + 0.5 = 0.3 - 0.7 + 0.5 = 0.1$$
$$(cp_{32}^1)^3 = ((cp_{32}^1)^{13} + (cp_{32}^1)^{43})/2.0 = 0.2/2.0 = 0.1$$
$$cp_{32}^1 = 0.3/3 = 0.1$$

Computation of cp_{34}^1*:*

$$(cp_{34}^1)^{11} = p_{31}^1 + p_{14}^1 - 0.5 = 0.4 + 0.4 - 0.5 = 0.3$$
$$(cp_{34}^1)^{21} = p_{32}^1 + p_{24}^1 - 0.5 = 0.1 + 0.7 - 0.5 = 0.3$$
$$(cp_{34}^1)^1 = ((cp_{34}^1)^{11} + (cp_{34}^1)^{21})/2.0 = 0.6/2.0 = 0.3$$
$$(cp_{34}^1)^{12} = p_{14}^1 - p_{13}^1 + 0.5 = 0.4 - 0.6 + 0.5 = 0.3$$
$$(cp_{34}^1)^{22} = p_{24}^1 - p_{23}^1 + 0.5 = 0.7 - 0.9 + 0.5 = 0.3$$
$$(cp_{34}^1)^2 = ((cp_{34}^1)^{12} + (cp_{34}^1)^{22})/2.0 = 0.6/2.0 = 0.3$$
$$(cp_{34}^1)^{13} = p_{31}^1 - p_{41}^1 + 0.5 = 0.4 - 0.6 + 0.5 = 0.3$$
$$(cp_{34}^1)^{23} = p_{32}^1 - p_{42}^1 + 0.5 = 0.1 - 0.3 + 0.5 = 0.3$$
$$(cp_{34}^1)^3 = ((cp_{34}^1)^{13} + (cp_{34}^1)^{23})/2.0 = 0.6/2.0 = 0.3$$
$$cp_{34}^1 = 0.9/3 = 0.3$$

Computation of cp_{41}^1*:*

$$(cp_{41}^1)^{21} = p_{42}^1 + p_{21}^1 - 0.5 = 0.3 + 0.8 - 0.5 = 0.6$$
$$(cp_{41}^1)^{31} = p_{43}^1 + p_{31}^1 - 0.5 = 0.7 + 0.4 - 0.5 = 0.6$$
$$(cp_{41}^1)^1 = ((cp_{41}^1)^{21} + (cp_{41}^1)^{31})/2.0 = 1.2/2.0 = 0.6$$

$$(cp^1_{41})^{22} = p^1_{21} - p^1_{24} + 0.5 = 0.8 - 0.7 + 0.5 = 0.6$$
$$(cp^1_{41})^{32} = p^1_{31} - p^1_{34} + 0.5 = 0.4 - 0.3 + 0.5 = 0.6$$
$$(cp^1_{41})^{2} = ((cp^1_{41})^{22} + (cp^1_{41})^{32})/2.0 = 1.2/2.0 = 0.6$$
$$(cp^1_{41})^{23} = p^1_{42} - p^1_{12} + 0.5 = 0.3 - 0.2 + 0.5 = 0.6$$
$$(cp^1_{41})^{33} = p^1_{43} - p^1_{13} + 0.5 = 0.7 - 0.6 + 0.5 = 0.6$$
$$(cp^1_{41})^{3} = ((cp^1_{41})^{23} + (cp^1_{41})^{33})/2.0 = 1.2/2.0 = 0.6$$
$$cp^1_{41} = 1.8/3 = 0.6$$

Computation of cp^1_{42}*:*

$$(cp^1_{42})^{11} = p^1_{41} + p^1_{12} - 0.5 = 0.6 + 0.2 - 0.5 = 0.3$$
$$(cp^1_{42})^{31} = p^1_{43} + p^1_{32} - 0.5 = 0.7 + 0.1 - 0.5 = 0.3$$
$$(cp^1_{42})^{1} = ((cp^1_{42})^{11} + (cp^1_{42})^{31})/2.0 = 0.6/2.0 = 0.3$$
$$(cp^1_{42})^{12} = p^1_{12} - p^1_{14} + 0.5 = 0.2 - 0.4 + 0.5 = 0.3$$
$$(cp^1_{42})^{32} = p^1_{32} - p^1_{34} + 0.5 = 0.1 - 0.3 + 0.5 = 0.3$$
$$(cp^1_{42})^{2} = ((cp^1_{42})^{12} + (cp^1_{42})^{32})/2.0 = 0.6/2.0 = 0.3$$
$$(cp^1_{42})^{13} = p^1_{41} - p^1_{21} + 0.5 = 0.6 - 0.8 + 0.5 = 0.3$$
$$(cp^1_{42})^{33} = p^1_{43} - p^1_{23} + 0.5 = 0.7 - 0.9 + 0.5 = 0.3$$
$$(cp^1_{42})^{3} = ((cp^1_{42})^{13} + (cp^1_{42})^{33})/2.0 = 0.6/2.0 = 0.3$$
$$cp^1_{42} = 0.9/3 = 0.3$$

Computation of cp^1_{43}*:*

$$(cp^1_{43})^{11} = p^1_{41} + p^1_{13} - 0.5 = 0.6 + 0.6 - 0.5 = 0.7$$
$$(cp^1_{43})^{21} = p^1_{42} + p^1_{23} - 0.5 = 0.3 + 0.9 - 0.5 = 0.7$$
$$(cp^1_{43})^{1} = ((cp^1_{43})^{11} + (cp^1_{43})^{21})/2.0 = 1.4/2.0 = 0.7$$
$$(cp^1_{43})^{12} = p^1_{13} - p^1_{14} + 0.5 = 0.6 - 0.4 + 0.5 = 0.7$$
$$(cp^1_{43})^{22} = p^1_{23} - p^1_{24} + 0.5 = 0.9 - 0.7 + 0.5 = 0.7$$
$$(cp^1_{43})^{2} = ((cp^1_{43})^{12} + (cp^1_{43})^{22})/2.0 = 1.4/2.0 = 0.7$$
$$(cp^1_{43})^{13} = p^1_{41} - p^1_{31} + 0.5 = 0.6 - 0.4 + 0.5 = 0.7$$
$$(cp^1_{43})^{23} = p^1_{42} - p^1_{32} + 0.5 = 0.3 - 0.1 + 0.5 = 0.7$$
$$(cp^1_{43})^{3} = ((cp^1_{43})^{13} + (cp^1_{43})^{23})/2.0 = 1.4/2.0 = 0.7$$
$$cp^1_{43} = 2.1/3 = 0.7$$

$$CP^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix}$$

Computing CP^2:

Computation of cp^2_{12}:

$$(cp^2_{12})^{31} = p^2_{13} + p^2_{32} - 0.5 = 0.7 + 0.4 - 0.5 = 0.6$$
$$(cp^2_{12})^{41} = p^2_{14} + p^2_{42} - 0.5 = 0.8 + 0.4 - 0.5 = 0.7$$
$$(cp^2_{12})^{1} = ((cp^2_{12})^{31} + (cp^2_{12})^{41})/2.0 = 1.3/2.0 = 0.65$$
$$(cp^2_{12})^{32} = p^2_{32} - p^2_{31} + 0.5 = 0.4 - 0.3 + 0.5 = 0.6$$
$$(cp^2_{12})^{42} = p^2_{42} - p^2_{41} + 0.5 = 0.4 - 0.25 + 0.5 = 0.65$$
$$(cp^2_{12})^{2} = ((cp^2_{12})^{32} + (cp^2_{12})^{42})/2.0 = 1.25/2.0 = 0.63$$
$$(cp^2_{12})^{33} = p^2_{13} - p^2_{23} + 0.5 = 0.7 - 0.6 + 0.5 = 0.6$$
$$(cp^2_{12})^{43} = p^2_{14} - p^2_{24} + 0.5 = 0.8 - 0.7 + 0.5 = 0.6$$
$$(cp^2_{12})^{3} = ((cp^2_{12})^{33} + (cp^2_{12})^{43})/2.0 = 1.2/2.0 = 0.6$$

$cp^2_{12} = 1.87/3 = 0.62$

Computation of cp^2_{13}:

$$(cp^2_{13})^{21} = p^2_{12} + p^2_{23} - 0.5 = 0.62 + 0.6 - 0.5 = 0.73$$
$$(cp^2_{13})^{41} = p^2_{14} + p^2_{43} - 0.5 = 0.8 + 0.45 - 0.5 = 0.75$$
$$(cp^2_{13})^{1} = ((cp^2_{13})^{21} + (cp^2_{13})^{41})/2.0 = 1.48/2.0 = 0.74$$
$$(cp^2_{13})^{22} = p^2_{23} - p^2_{21} + 0.5 = 0.6 - 0.4 + 0.5 = 0.7$$
$$(cp^2_{13})^{42} = p^2_{43} - p^2_{41} + 0.5 = 0.45 - 0.25 + 0.5 = 0.7$$
$$(cp^2_{13})^{2} = ((cp^2_{13})^{22} + (cp^2_{13})^{42})/2.0 = 1.4/2.0 = 0.7$$
$$(cp^2_{13})^{23} = p^2_{12} - p^2_{32} + 0.5 = 0.62 - 0.4 + 0.5 = 0.72$$
$$(cp^2_{13})^{43} = p^2_{14} - p^2_{34} + 0.5 = 0.8 - 0.57 + 0.5 = 0.73$$
$$(cp^2_{13})^{3} = ((cp^2_{13})^{23} + (cp^2_{13})^{43})/2.0 = 1.45/2.0 = 0.73$$

$cp^2_{13} = 2.16/3 = 0.72$

Computation of cp^2_{14}:

$$(cp^2_{14})^{21} = p^2_{12} + p^2_{24} - 0.5 = 0.62 + 0.7 - 0.5 = 0.82$$

$$(cp^2_{14})^{31} = p^2_{13} + p^2_{34} - 0.5 = 0.7 + 0.57 - 0.5 = 0.77$$
$$(cp^2_{14})^{1} = ((cp^2_{14})^{21} + (cp^2_{14})^{31})/2.0 = 1.6/2.0 = 0.8$$
$$(cp^2_{14})^{22} = p^2_{24} - p^2_{21} + 0.5 = 0.7 - 0.4 + 0.5 = 0.8$$
$$(cp^2_{14})^{32} = p^2_{34} - p^2_{31} + 0.5 = 0.57 - 0.3 + 0.5 = 0.77$$
$$(cp^2_{14})^{2} = ((cp^2_{14})^{22} + (cp^2_{14})^{32})/2.0 = 1.57/2.0 = 0.79$$
$$(cp^2_{14})^{23} = p^2_{12} - p^2_{42} + 0.5 = 0.62 - 0.4 + 0.5 = 0.72$$
$$(cp^2_{14})^{33} = p^2_{13} - p^2_{43} + 0.5 = 0.7 - 0.45 + 0.5 = 0.75$$
$$(cp^2_{14})^{3} = ((cp^2_{14})^{23} + (cp^2_{14})^{33})/2.0 = 1.48/2.0 = 0.74$$
$$cp^2_{14} = 2.32/3 = 0.77$$

Computation of cp^2_{21}*:*

$$(cp^2_{21})^{31} = p^2_{23} + p^2_{31} - 0.5 = 0.6 + 0.3 - 0.5 = 0.4$$
$$(cp^2_{21})^{41} = p^2_{24} + p^2_{41} - 0.5 = 0.7 + 0.25 - 0.5 = 0.45$$
$$(cp^2_{21})^{1} = ((cp^2_{21})^{31} + (cp^2_{21})^{41})/2.0 = 0.85/2.0 = 0.43$$
$$(cp^2_{21})^{32} = p^2_{31} - p^2_{32} + 0.5 = 0.3 - 0.4 + 0.5 = 0.4$$
$$(cp^2_{21})^{42} = p^2_{41} - p^2_{42} + 0.5 = 0.25 - 0.4 + 0.5 = 0.35$$
$$(cp^2_{21})^{2} = ((cp^2_{21})^{32} + (cp^2_{21})^{42})/2.0 = 0.75/2.0 = 0.38$$
$$(cp^2_{21})^{33} = p^2_{23} - p^2_{13} + 0.5 = 0.6 - 0.7 + 0.5 = 0.4$$
$$(cp^2_{21})^{43} = p^2_{24} - p^2_{14} + 0.5 = 0.7 - 0.8 + 0.5 = 0.4$$
$$(cp^2_{21})^{3} = ((cp^2_{21})^{33} + (cp^2_{21})^{43})/2.0 = 0.8/2.0 = 0.4$$
$$cp^2_{21} = 1.2/3 = 0.4$$

Computation of cp^2_{23}*:*

$$(cp^2_{23})^{11} = p^2_{21} + p^2_{13} - 0.5 = 0.4 + 0.7 - 0.5 = 0.6$$
$$(cp^2_{23})^{41} = p^2_{24} + p^2_{43} - 0.5 = 0.7 + 0.45 - 0.5 = 0.65$$
$$(cp^2_{23})^{1} = ((cp^2_{23})^{11} + (cp^2_{23})^{41})/2.0 = 1.25/2.0 = 0.63$$
$$(cp^2_{23})^{12} = p^2_{13} - p^2_{12} + 0.5 = 0.7 - 0.62 + 0.5 = 0.58$$
$$(cp^2_{23})^{42} = p^2_{43} - p^2_{42} + 0.5 = 0.45 - 0.4 + 0.5 = 0.55$$
$$(cp^2_{23})^{2} = ((cp^2_{23})^{12} + (cp^2_{23})^{42})/2.0 = 1.13/2.0 = 0.56$$
$$(cp^2_{23})^{13} = p^2_{21} - p^2_{31} + 0.5 = 0.4 - 0.3 + 0.5 = 0.6$$
$$(cp^2_{23})^{43} = p^2_{24} - p^2_{34} + 0.5 = 0.7 - 0.57 + 0.5 = 0.63$$
$$(cp^2_{23})^{3} = ((cp^2_{23})^{13} + (cp^2_{23})^{43})/2.0 = 1.23/2.0 = 0.61$$
$$cp^2_{23} = 1.8/3 = 0.6$$

Computation of cp^2_{24}*:*

$(cp^2_{24})^{11} = p^2_{21} + p^2_{14} - 0.5 = 0.4 + 0.8 - 0.5 = 0.7$
$(cp^2_{24})^{31} = p^2_{23} + p^2_{34} - 0.5 = 0.6 + 0.57 - 0.5 = 0.67$
$(cp^2_{24})^{1} = ((cp^2_{24})^{11} + (cp^2_{24})^{31})/2.0 = 1.37/2.0 = 0.69$
$(cp^2_{24})^{12} = p^2_{14} - p^2_{12} + 0.5 = 0.8 - 0.62 + 0.5 = 0.68$
$(cp^2_{24})^{32} = p^2_{34} - p^2_{32} + 0.5 = 0.57 - 0.4 + 0.5 = 0.67$
$(cp^2_{24})^{2} = ((cp^2_{24})^{12} + (cp^2_{24})^{32})/2.0 = 1.35/2.0 = 0.68$
$(cp^2_{24})^{13} = p^2_{21} - p^2_{41} + 0.5 = 0.4 - 0.25 + 0.5 = 0.65$
$(cp^2_{24})^{33} = p^2_{23} - p^2_{43} + 0.5 = 0.6 - 0.45 + 0.5 = 0.65$
$(cp^2_{24})^{3} = ((cp^2_{24})^{13} + (cp^2_{24})^{33})/2.0 = 1.3/2.0 = 0.65$
$cp^2_{24} = 2.01/3 = 0.67$

Computation of cp^2_{31}*:*

$(cp^2_{31})^{21} = p^2_{32} + p^2_{21} - 0.5 = 0.4 + 0.4 - 0.5 = 0.3$
$(cp^2_{31})^{41} = p^2_{34} + p^2_{41} - 0.5 = 0.57 + 0.25 - 0.5 = 0.32$
$(cp^2_{31})^{1} = ((cp^2_{31})^{21} + (cp^2_{31})^{41})/2.0 = 0.63/2.0 = 0.31$
$(cp^2_{31})^{22} = p^2_{21} - p^2_{23} + 0.5 = 0.4 - 0.6 + 0.5 = 0.3$
$(cp^2_{31})^{42} = p^2_{41} - p^2_{43} + 0.5 = 0.25 - 0.45 + 0.5 = 0.3$
$(cp^2_{31})^{2} = ((cp^2_{31})^{22} + (cp^2_{31})^{42})/2.0 = 0.6/2.0 = 0.3$
$(cp^2_{31})^{23} = p^2_{32} - p^2_{12} + 0.5 = 0.4 - 0.62 + 0.5 = 0.28$
$(cp^2_{31})^{43} = p^2_{34} - p^2_{14} + 0.5 = 0.57 - 0.8 + 0.5 = 0.27$
$(cp^2_{31})^{3} = ((cp^2_{31})^{23} + (cp^2_{31})^{43})/2.0 = 0.55/2.0 = 0.28$
$cp^2_{31} = 0.89/3 = 0.3$

Computation of cp^2_{32}*:*

$(cp^2_{32})^{11} = p^2_{31} + p^2_{12} - 0.5 = 0.3 + 0.62 - 0.5 = 0.42$
$(cp^2_{32})^{41} = p^2_{34} + p^2_{42} - 0.5 = 0.57 + 0.4 - 0.5 = 0.47$
$(cp^2_{32})^{1} = ((cp^2_{32})^{11} + (cp^2_{32})^{41})/2.0 = 0.9/2.0 = 0.45$
$(cp^2_{32})^{12} = p^2_{12} - p^2_{13} + 0.5 = 0.62 - 0.7 + 0.5 = 0.42$
$(cp^2_{32})^{42} = p^2_{42} - p^2_{43} + 0.5 = 0.4 - 0.45 + 0.5 = 0.45$
$(cp^2_{32})^{2} = ((cp^2_{32})^{12} + (cp^2_{32})^{42})/2.0 = 0.88/2.0 = 0.44$
$(cp^2_{32})^{13} = p^2_{31} - p^2_{21} + 0.5 = 0.3 - 0.4 + 0.5 = 0.4$

$$(cp_{32}^2)^{43} = p_{34}^2 - p_{24}^2 + 0.5 = 0.57 - 0.7 + 0.5 = 0.37$$
$$(cp_{32}^2)^3 = ((cp_{32}^2)^{13} + (cp_{32}^2)^{43})/2.0 = 0.77/2.0 = 0.39$$
$$cp_{32}^2 = 1.27/3 = 0.43$$

Computation of cp_{34}^2*:*

$$(cp_{34}^2)^{11} = p_{31}^2 + p_{14}^2 - 0.5 = 0.3 + 0.8 - 0.5 = 0.6$$
$$(cp_{34}^2)^{21} = p_{32}^2 + p_{24}^2 - 0.5 = 0.4 + 0.7 - 0.5 = 0.6$$
$$(cp_{34}^2)^1 = ((cp_{34}^2)^{11} + (cp_{34}^2)^{21})/2.0 = 1.2/2.0 = 0.6$$
$$(cp_{34}^2)^{12} = p_{14}^2 - p_{13}^2 + 0.5 = 0.8 - 0.7 + 0.5 = 0.6$$
$$(cp_{34}^2)^{22} = p_{24}^2 - p_{23}^2 + 0.5 = 0.7 - 0.6 + 0.5 = 0.6$$
$$(cp_{34}^2)^2 = ((cp_{34}^2)^{12} + (cp_{34}^2)^{22})/2.0 = 1.2/2.0 = 0.6$$
$$(cp_{34}^2)^{13} = p_{31}^2 - p_{41}^2 + 0.5 = 0.3 - 0.25 + 0.5 = 0.55$$
$$(cp_{34}^2)^{23} = p_{32}^2 - p_{42}^2 + 0.5 = 0.4 - 0.4 + 0.5 = 0.5$$
$$(cp_{34}^2)^3 = ((cp_{34}^2)^{13} + (cp_{34}^2)^{23})/2.0 = 1.05/2.0 = 0.52$$
$$cp_{34}^2 = 1.72/3 = 0.57$$

Computation of cp_{41}^2*:*

$$(cp_{41}^2)^{21} = p_{42}^2 + p_{21}^2 - 0.5 = 0.4 + 0.4 - 0.5 = 0.3$$
$$(cp_{41}^2)^{31} = p_{43}^2 + p_{31}^2 - 0.5 = 0.45 + 0.3 - 0.5 = 0.25$$
$$(cp_{41}^2)^1 = ((cp_{41}^2)^{21} + (cp_{41}^2)^{31})/2.0 = 0.55/2.0 = 0.28$$
$$(cp_{41}^2)^{22} = p_{21}^2 - p_{24}^2 + 0.5 = 0.4 - 0.7 + 0.5 = 0.2$$
$$(cp_{41}^2)^{32} = p_{31}^2 - p_{34}^2 + 0.5 = 0.3 - 0.57 + 0.5 = 0.23$$
$$(cp_{41}^2)^2 = ((cp_{41}^2)^{22} + (cp_{41}^2)^{32})/2.0 = 0.43/2.0 = 0.21$$
$$(cp_{41}^2)^{23} = p_{42}^2 - p_{12}^2 + 0.5 = 0.4 - 0.62 + 0.5 = 0.28$$
$$(cp_{41}^2)^{33} = p_{43}^2 - p_{13}^2 + 0.5 = 0.45 - 0.7 + 0.5 = 0.25$$
$$(cp_{41}^2)^3 = ((cp_{41}^2)^{23} + (cp_{41}^2)^{33})/2.0 = 0.53/2.0 = 0.26$$
$$cp_{41}^2 = 0.75/3 = 0.25$$

Computation of cp_{42}^2*:*

$$(cp_{42}^2)^{11} = p_{41}^2 + p_{12}^2 - 0.5 = 0.25 + 0.62 - 0.5 = 0.38$$
$$(cp_{42}^2)^{31} = p_{43}^2 + p_{32}^2 - 0.5 = 0.45 + 0.4 - 0.5 = 0.35$$
$$(cp_{42}^2)^1 = ((cp_{42}^2)^{11} + (cp_{42}^2)^{31})/2.0 = 0.73/2.0 = 0.36$$
$$(cp_{42}^2)^{12} = p_{12}^2 - p_{14}^2 + 0.5 = 0.62 - 0.8 + 0.5 = 0.32$$

$(cp^2_{42})^{32} = p^2_{32} - p^2_{34} + 0.5 = 0.4 - 0.57 + 0.5 = 0.33$
$(cp^2_{42})^2 = ((cp^2_{42})^{12} + (cp^2_{42})^{32})/2.0 = 0.65/2.0 = 0.33$
$(cp^2_{42})^{13} = p^2_{41} - p^2_{21} + 0.5 = 0.25 - 0.4 + 0.5 = 0.35$
$(cp^2_{42})^{33} = p^2_{43} - p^2_{23} + 0.5 = 0.45 - 0.6 + 0.5 = 0.35$
$(cp^2_{42})^3 = ((cp^2_{42})^{13} + (cp^2_{42})^{33})/2.0 = 0.7/2.0 = 0.35$
$cp^2_{42} = 1.04/3 = 0.35$

Computation of cp^2_{43}*:*
$(cp^2_{43})^{11} = p^2_{41} + p^2_{13} - 0.5 = 0.25 + 0.7 - 0.5 = 0.45$
$(cp^2_{43})^{21} = p^2_{42} + p^2_{23} - 0.5 = 0.4 + 0.6 - 0.5 = 0.5$
$(cp^2_{43})^1 = ((cp^2_{43})^{11} + (cp^2_{43})^{21})/2.0 = 0.95/2.0 = 0.48$
$(cp^2_{43})^{12} = p^2_{13} - p^2_{14} + 0.5 = 0.7 - 0.8 + 0.5 = 0.4$
$(cp^2_{43})^{22} = p^2_{23} - p^2_{24} + 0.5 = 0.6 - 0.7 + 0.5 = 0.4$
$(cp^2_{43})^2 = ((cp^2_{43})^{12} + (cp^2_{43})^{22})/2.0 = 0.8/2.0 = 0.4$
$(cp^2_{43})^{13} = p^2_{41} - p^2_{31} + 0.5 = 0.25 - 0.3 + 0.5 = 0.45$
$(cp^2_{43})^{23} = p^2_{42} - p^2_{32} + 0.5 = 0.4 - 0.4 + 0.5 = 0.5$
$(cp^2_{43})^3 = ((cp^2_{43})^{13} + (cp^2_{43})^{23})/2.0 = 0.95/2.0 = 0.48$
$cp^2_{43} = 1.35/3 = 0.45$

$$CP^2 = \begin{pmatrix} - & 0.62 & 0.72 & 0.77 \\ 0.4 & - & 0.6 & 0.67 \\ 0.3 & 0.43 & - & 0.57 \\ 0.25 & 0.35 & 0.45 & - \end{pmatrix}$$

Computing CP^3**:**

Computation of cp^3_{12}*:*
$(cp^3_{12})^{31} = p^3_{13} + p^3_{32} - 0.5 = 0.54 + 0.31 - 0.5 = 0.35$
$(cp^3_{12})^{41} = p^3_{14} + p^3_{42} - 0.5 = 0.75 + 0.4 - 0.5 = 0.65$
$(cp^3_{12})^1 = ((cp^3_{12})^{31} + (cp^3_{12})^{41})/2.0 = 1.0/2.0 = 0.5$
$(cp^3_{12})^{32} = p^3_{32} - p^3_{31} + 0.5 = 0.31 - 0.46 + 0.5 = 0.35$
$(cp^3_{12})^{42} = p^3_{42} - p^3_{41} + 0.5 = 0.4 - 0.3 + 0.5 = 0.6$
$(cp^3_{12})^2 = ((cp^3_{12})^{32} + (cp^3_{12})^{42})/2.0 = 0.95/2.0 = 0.47$
$(cp^3_{12})^{33} = p^3_{13} - p^3_{23} + 0.5 = 0.54 - 0.69 + 0.5 = 0.35$

$$(cp^3_{12})^{43} = p^3_{14} - p^3_{24} + 0.5 = 0.75 - 0.87 + 0.5 = 0.38$$
$$(cp^3_{12})^{3} = ((cp^3_{12})^{33} + (cp^3_{12})^{43})/2.0 = 0.73/2.0 = 0.36$$
$$cp^3_{12} = 1.34/3 = 0.45$$

Computation of cp^3_{13}*:*

$$(cp^3_{13})^{21} = p^3_{12} + p^3_{23} - 0.5 = 0.3 + 0.69 - 0.5 = 0.49$$
$$(cp^3_{13})^{41} = p^3_{14} + p^3_{43} - 0.5 = 0.75 + 0.27 - 0.5 = 0.52$$
$$(cp^3_{13})^{1} = ((cp^3_{13})^{21} + (cp^3_{13})^{41})/2.0 = 1.01/2.0 = 0.51$$
$$(cp^3_{13})^{22} = p^3_{23} - p^3_{21} + 0.5 = 0.69 - 0.6 + 0.5 = 0.59$$
$$(cp^3_{13})^{42} = p^3_{43} - p^3_{41} + 0.5 = 0.27 - 0.3 + 0.5 = 0.47$$
$$(cp^3_{13})^{2} = ((cp^3_{13})^{22} + (cp^3_{13})^{42})/2.0 = 1.06/2.0 = 0.53$$
$$(cp^3_{13})^{23} = p^3_{12} - p^3_{32} + 0.5 = 0.3 - 0.31 + 0.5 = 0.49$$
$$(cp^3_{13})^{43} = p^3_{14} - p^3_{34} + 0.5 = 0.75 - 0.73 + 0.5 = 0.52$$
$$(cp^3_{13})^{3} = ((cp^3_{13})^{23} + (cp^3_{13})^{43})/2.0 = 1.01/2.0 = 0.51$$
$$cp^3_{13} = 1.54/3 = 0.51$$

Computation of cp^3_{14}*:*

$$(cp^3_{14})^{21} = p^3_{12} + p^3_{24} - 0.5 = 0.3 + 0.87 - 0.5 = 0.67$$
$$(cp^3_{14})^{31} = p^3_{13} + p^3_{34} - 0.5 = 0.54 + 0.73 - 0.5 = 0.77$$
$$(cp^3_{14})^{1} = ((cp^3_{14})^{21} + (cp^3_{14})^{31})/2.0 = 1.43/2.0 = 0.72$$
$$(cp^3_{14})^{22} = p^3_{24} - p^3_{21} + 0.5 = 0.87 - 0.6 + 0.5 = 0.77$$
$$(cp^3_{14})^{32} = p^3_{34} - p^3_{31} + 0.5 = 0.73 - 0.46 + 0.5 = 0.77$$
$$(cp^3_{14})^{2} = ((cp^3_{14})^{22} + (cp^3_{14})^{32})/2.0 = 1.53/2.0 = 0.77$$
$$(cp^3_{14})^{23} = p^3_{12} - p^3_{42} + 0.5 = 0.3 - 0.4 + 0.5 = 0.4$$
$$(cp^3_{14})^{33} = p^3_{13} - p^3_{43} + 0.5 = 0.54 - 0.27 + 0.5 = 0.77$$
$$(cp^3_{14})^{3} = ((cp^3_{14})^{23} + (cp^3_{14})^{33})/2.0 = 1.17/2.0 = 0.58$$
$$cp^3_{14} = 2.07/3 = 0.69$$

Computation of cp^3_{21}*:*

$$(cp^3_{21})^{31} = p^3_{23} + p^3_{31} - 0.5 = 0.69 + 0.46 - 0.5 = 0.65$$
$$(cp^3_{21})^{41} = p^3_{24} + p^3_{41} - 0.5 = 0.87 + 0.3 - 0.5 = 0.67$$
$$(cp^3_{21})^{1} = ((cp^3_{21})^{31} + (cp^3_{21})^{41})/2.0 = 1.32/2.0 = 0.66$$
$$(cp^3_{21})^{32} = p^3_{31} - p^3_{32} + 0.5 = 0.46 - 0.31 + 0.5 = 0.65$$

$$(cp^3_{21})^{42} = p^3_{41} - p^3_{42} + 0.5 = 0.3 - 0.4 + 0.5 = 0.4$$
$$(cp^3_{21})^{2} = ((cp^3_{21})^{32} + (cp^3_{21})^{42})/2.0 = 1.05/2.0 = 0.53$$
$$(cp^3_{21})^{33} = p^3_{23} - p^3_{13} + 0.5 = 0.69 - 0.54 + 0.5 = 0.65$$
$$(cp^3_{21})^{43} = p^3_{24} - p^3_{14} + 0.5 = 0.87 - 0.75 + 0.5 = 0.62$$
$$(cp^3_{21})^{3} = ((cp^3_{21})^{33} + (cp^3_{21})^{43})/2.0 = 1.27/2.0 = 0.64$$
$$cp^3_{21} = 1.82/3 = 0.61$$

Computation of cp^3_{23}*:*

$$(cp^3_{23})^{11} = p^3_{21} + p^3_{13} - 0.5 = 0.6 + 0.54 - 0.5 = 0.64$$
$$(cp^3_{23})^{41} = p^3_{24} + p^3_{43} - 0.5 = 0.87 + 0.27 - 0.5 = 0.64$$
$$(cp^3_{23})^{1} = ((cp^3_{23})^{11} + (cp^3_{23})^{41})/2.0 = 1.28/2.0 = 0.64$$
$$(cp^3_{23})^{12} = p^3_{13} - p^3_{12} + 0.5 = 0.54 - 0.3 + 0.5 = 0.74$$
$$(cp^3_{23})^{42} = p^3_{43} - p^3_{42} + 0.5 = 0.27 - 0.4 + 0.5 = 0.37$$
$$(cp^3_{23})^{2} = ((cp^3_{23})^{12} + (cp^3_{23})^{42})/2.0 = 1.11/2.0 = 0.55$$
$$(cp^3_{23})^{13} = p^3_{21} - p^3_{31} + 0.5 = 0.6 - 0.46 + 0.5 = 0.64$$
$$(cp^3_{23})^{43} = p^3_{24} - p^3_{34} + 0.5 = 0.87 - 0.73 + 0.5 = 0.64$$
$$(cp^3_{23})^{3} = ((cp^3_{23})^{13} + (cp^3_{23})^{43})/2.0 = 1.27/2.0 = 0.64$$
$$cp^3_{23} = 1.83/3 = 0.61$$

Computation of cp^3_{24}*:*

$$(cp^3_{24})^{11} = p^3_{21} + p^3_{14} - 0.5 = 0.6 + 0.75 - 0.5 = 0.85$$
$$(cp^3_{24})^{31} = p^3_{23} + p^3_{34} - 0.5 = 0.69 + 0.73 - 0.5 = 0.92$$
$$(cp^3_{24})^{1} = ((cp^3_{24})^{11} + (cp^3_{24})^{31})/2.0 = 1.77/2.0 = 0.89$$
$$(cp^3_{24})^{12} = p^3_{14} - p^3_{12} + 0.5 = 0.75 - 0.3 + 0.5 = 0.95$$
$$(cp^3_{24})^{32} = p^3_{34} - p^3_{32} + 0.5 = 0.73 - 0.31 + 0.5 = 0.92$$
$$(cp^3_{24})^{2} = ((cp^3_{24})^{12} + (cp^3_{24})^{32})/2.0 = 1.87/2.0 = 0.94$$
$$(cp^3_{24})^{13} = p^3_{21} - p^3_{41} + 0.5 = 0.6 - 0.3 + 0.5 = 0.8$$
$$(cp^3_{24})^{33} = p^3_{23} - p^3_{43} + 0.5 = 0.69 - 0.27 + 0.5 = 0.92$$
$$(cp^3_{24})^{3} = ((cp^3_{24})^{13} + (cp^3_{24})^{33})/2.0 = 1.72/2.0 = 0.86$$
$$cp^3_{24} = 2.68/3 = 0.89$$

Computation of cp^3_{31}*:*

$$(cp^3_{31})^{21} = p^3_{32} + p^3_{21} - 0.5 = 0.31 + 0.6 - 0.5 = 0.41$$

$$(cp^3_{31})^{41} = p^3_{34} + p^3_{41} - 0.5 = 0.73 + 0.3 - 0.5 = 0.53$$
$$(cp^3_{31})^{1} = ((cp^3_{31})^{21} + (cp^3_{31})^{41})/2.0 = 0.94/2.0 = 0.47$$
$$(cp^3_{31})^{22} = p^3_{21} - p^3_{23} + 0.5 = 0.6 - 0.69 + 0.5 = 0.41$$
$$(cp^3_{31})^{42} = p^3_{41} - p^3_{43} + 0.5 = 0.3 - 0.27 + 0.5 = 0.53$$
$$(cp^3_{31})^{2} = ((cp^3_{31})^{22} + (cp^3_{31})^{42})/2.0 = 0.94/2.0 = 0.47$$
$$(cp^3_{31})^{23} = p^3_{32} - p^3_{12} + 0.5 = 0.31 - 0.3 + 0.5 = 0.51$$
$$(cp^3_{31})^{43} = p^3_{34} - p^3_{14} + 0.5 = 0.73 - 0.75 + 0.5 = 0.48$$
$$(cp^3_{31})^{3} = ((cp^3_{31})^{23} + (cp^3_{31})^{43})/2.0 = 0.99/2.0 = 0.49$$
$$cp^3_{31} = 1.43/3 = 0.48$$

Computation of cp^3_{32}*:*

$$(cp^3_{32})^{11} = p^3_{31} + p^3_{12} - 0.5 = 0.46 + 0.3 - 0.5 = 0.26$$
$$(cp^3_{32})^{41} = p^3_{34} + p^3_{42} - 0.5 = 0.73 + 0.4 - 0.5 = 0.63$$
$$(cp^3_{32})^{1} = ((cp^3_{32})^{11} + (cp^3_{32})^{41})/2.0 = 0.89/2.0 = 0.45$$
$$(cp^3_{32})^{12} = p^3_{12} - p^3_{13} + 0.5 = 0.3 - 0.54 + 0.5 = 0.26$$
$$(cp^3_{32})^{42} = p^3_{42} - p^3_{43} + 0.5 = 0.4 - 0.27 + 0.5 = 0.63$$
$$(cp^3_{32})^{2} = ((cp^3_{32})^{12} + (cp^3_{32})^{42})/2.0 = 0.89/2.0 = 0.45$$
$$(cp^3_{32})^{13} = p^3_{31} - p^3_{21} + 0.5 = 0.46 - 0.6 + 0.5 = 0.36$$
$$(cp^3_{32})^{43} = p^3_{34} - p^3_{24} + 0.5 = 0.73 - 0.87 + 0.5 = 0.36$$
$$(cp^3_{32})^{3} = ((cp^3_{32})^{13} + (cp^3_{32})^{43})/2.0 = 0.73/2.0 = 0.36$$
$$cp^3_{32} = 1.25/3 = 0.42$$

Computation of cp^3_{34}*:*

$$(cp^3_{34})^{11} = p^3_{31} + p^3_{14} - 0.5 = 0.46 + 0.75 - 0.5 = 0.71$$
$$(cp^3_{34})^{21} = p^3_{32} + p^3_{24} - 0.5 = 0.31 + 0.87 - 0.5 = 0.68$$
$$(cp^3_{34})^{1} = ((cp^3_{34})^{11} + (cp^3_{34})^{21})/2.0 = 1.39/2.0 = 0.69$$
$$(cp^3_{34})^{12} = p^3_{14} - p^3_{13} + 0.5 = 0.75 - 0.54 + 0.5 = 0.71$$
$$(cp^3_{34})^{22} = p^3_{24} - p^3_{23} + 0.5 = 0.87 - 0.69 + 0.5 = 0.68$$
$$(cp^3_{34})^{2} = ((cp^3_{34})^{12} + (cp^3_{34})^{22})/2.0 = 1.39/2.0 = 0.69$$
$$(cp^3_{34})^{13} = p^3_{31} - p^3_{41} + 0.5 = 0.46 - 0.3 + 0.5 = 0.66$$
$$(cp^3_{34})^{23} = p^3_{32} - p^3_{42} + 0.5 = 0.31 - 0.4 + 0.5 = 0.41$$
$$(cp^3_{34})^{3} = ((cp^3_{34})^{13} + (cp^3_{34})^{23})/2.0 = 1.07/2.0 = 0.54$$
$$cp^3_{34} = 1.92/3 = 0.64$$

Computation of cp^3_{41}*:*

$$(cp^3_{41})^{21} = p^3_{42} + p^3_{21} - 0.5 = 0.4 + 0.6 - 0.5 = 0.5$$
$$(cp^3_{41})^{31} = p^3_{43} + p^3_{31} - 0.5 = 0.27 + 0.46 - 0.5 = 0.23$$
$$(cp^3_{41})^{1} = ((cp^3_{41})^{21} + (cp^3_{41})^{31})/2.0 = 0.73/2.0 = 0.37$$
$$(cp^3_{41})^{22} = p^3_{21} - p^3_{24} + 0.5 = 0.6 - 0.87 + 0.5 = 0.23$$
$$(cp^3_{41})^{32} = p^3_{31} - p^3_{34} + 0.5 = 0.46 - 0.73 + 0.5 = 0.23$$
$$(cp^3_{41})^{2} = ((cp^3_{41})^{22} + (cp^3_{41})^{32})/2.0 = 0.47/2.0 = 0.23$$
$$(cp^3_{41})^{23} = p^3_{42} - p^3_{12} + 0.5 = 0.4 - 0.3 + 0.5 = 0.6$$
$$(cp^3_{41})^{33} = p^3_{43} - p^3_{13} + 0.5 = 0.27 - 0.54 + 0.5 = 0.23$$
$$(cp^3_{41})^{3} = ((cp^3_{41})^{23} + (cp^3_{41})^{33})/2.0 = 0.83/2.0 = 0.42$$
$$cp^3_{41} = 1.02/3 = 0.34$$

Computation of cp^3_{42}*:*

$$(cp^3_{42})^{11} = p^3_{41} + p^3_{12} - 0.5 = 0.3 + 0.3 - 0.5 = 0.1$$
$$(cp^3_{42})^{31} = p^3_{43} + p^3_{32} - 0.5 = 0.27 + 0.31 - 0.5 = 0.08$$
$$(cp^3_{42})^{1} = ((cp^3_{42})^{11} + (cp^3_{42})^{31})/2.0 = 0.18/2.0 = 0.09$$
$$(cp^3_{42})^{12} = p^3_{12} - p^3_{14} + 0.5 = 0.3 - 0.75 + 0.5 = 0.05$$
$$(cp^3_{42})^{32} = p^3_{32} - p^3_{34} + 0.5 = 0.31 - 0.73 + 0.5 = 0.08$$
$$(cp^3_{42})^{2} = ((cp^3_{42})^{12} + (cp^3_{42})^{32})/2.0 = 0.13/2.0 = 0.06$$
$$(cp^3_{42})^{13} = p^3_{41} - p^3_{21} + 0.5 = 0.3 - 0.6 + 0.5 = 0.2$$
$$(cp^3_{42})^{33} = p^3_{43} - p^3_{23} + 0.5 = 0.27 - 0.69 + 0.5 = 0.08$$
$$(cp^3_{42})^{3} = ((cp^3_{42})^{13} + (cp^3_{42})^{33})/2.0 = 0.28/2.0 = 0.14$$
$$cp^3_{42} = 0.29/3 = 0.1$$

Computation of cp^3_{43}*:*

$$(cp^3_{43})^{11} = p^3_{41} + p^3_{13} - 0.5 = 0.3 + 0.54 - 0.5 = 0.34$$
$$(cp^3_{43})^{21} = p^3_{42} + p^3_{23} - 0.5 = 0.4 + 0.69 - 0.5 = 0.59$$
$$(cp^3_{43})^{1} = ((cp^3_{43})^{11} + (cp^3_{43})^{21})/2.0 = 0.93/2.0 = 0.46$$
$$(cp^3_{43})^{12} = p^3_{13} - p^3_{14} + 0.5 = 0.54 - 0.75 + 0.5 = 0.29$$
$$(cp^3_{43})^{22} = p^3_{23} - p^3_{24} + 0.5 = 0.69 - 0.87 + 0.5 = 0.32$$
$$(cp^3_{43})^{2} = ((cp^3_{43})^{12} + (cp^3_{43})^{22})/2.0 = 0.61/2.0 = 0.31$$
$$(cp^3_{43})^{13} = p^3_{41} - p^3_{31} + 0.5 = 0.3 - 0.46 + 0.5 = 0.34$$

$$(cp^3_{43})^{23} = p^3_{42} - p^3_{32} + 0.5 = 0.4 - 0.31 + 0.5 = 0.59$$
$$(cp^3_{43})^3 = ((cp^3_{43})^{13} + (cp^3_{43})^{23})/2.0 = 0.93/2.0 = 0.46$$
$$cp^3_{43} = 1.24/3 = 0.41$$

$$CP^3 = \begin{pmatrix} - & 0.45 & 0.51 & 0.69 \\ 0.61 & - & 0.61 & 0.89 \\ 0.48 & 0.42 & - & 0.64 \\ 0.34 & 0.1 & 0.41 & - \end{pmatrix}$$

Computing CP^4:

Computation of cp^4_{12}:

$$(cp^4_{12})^{31} = p^4_{13} + p^4_{32} - 0.5 = 0.6 + 0.6 - 0.5 = 0.7$$
$$(cp^4_{12})^{41} = p^4_{14} + p^4_{42} - 0.5 = 0.3 + 0.7 - 0.5 = 0.5$$
$$(cp^4_{12})^1 = ((cp^4_{12})^{31} + (cp^4_{12})^{41})/2.0 = 1.2/2.0 = 0.6$$
$$(cp^4_{12})^{32} = p^4_{32} - p^4_{31} + 0.5 = 0.6 - 0.5 + 0.5 = 0.6$$
$$(cp^4_{12})^{42} = p^4_{42} - p^4_{41} + 0.5 = 0.7 - 0.7 + 0.5 = 0.5$$
$$(cp^4_{12})^2 = ((cp^4_{12})^{32} + (cp^4_{12})^{42})/2.0 = 1.1/2.0 = 0.55$$
$$(cp^4_{12})^{33} = p^4_{13} - p^4_{23} + 0.5 = 0.6 - 0.4 + 0.5 = 0.7$$
$$(cp^4_{12})^{43} = p^4_{14} - p^4_{24} + 0.5 = 0.3 - 0.2 + 0.5 = 0.6$$
$$(cp^4_{12})^3 = ((cp^4_{12})^{33} + (cp^4_{12})^{43})/2.0 = 1.3/2.0 = 0.65$$
$$cp^4_{12} = 1.8/3 = 0.6$$

Computation of cp^4_{13}:

$$(cp^4_{13})^{21} = p^4_{12} + p^4_{23} - 0.5 = 0.6 + 0.4 - 0.5 = 0.5$$
$$(cp^4_{13})^{41} = p^4_{14} + p^4_{43} - 0.5 = 0.3 + 0.7 - 0.5 = 0.5$$
$$(cp^4_{13})^1 = ((cp^4_{13})^{21} + (cp^4_{13})^{41})/2.0 = 1.0/2.0 = 0.5$$
$$(cp^4_{13})^{22} = p^4_{23} - p^4_{21} + 0.5 = 0.4 - 0.4 + 0.5 = 0.5$$
$$(cp^4_{13})^{42} = p^4_{43} - p^4_{41} + 0.5 = 0.7 - 0.7 + 0.5 = 0.5$$
$$(cp^4_{13})^2 = ((cp^4_{13})^{22} + (cp^4_{13})^{42})/2.0 = 1.0/2.0 = 0.5$$
$$(cp^4_{13})^{23} = p^4_{12} - p^4_{32} + 0.5 = 0.6 - 0.6 + 0.5 = 0.5$$
$$(cp^4_{13})^{43} = p^4_{14} - p^4_{34} + 0.5 = 0.3 - 0.3 + 0.5 = 0.5$$
$$(cp^4_{13})^3 = ((cp^4_{13})^{23} + (cp^4_{13})^{43})/2.0 = 1.0/2.0 = 0.5$$
$$cp^4_{13} = 1.5/3 = 0.5$$

Computation of cp_{14}^4*:*

$$(cp_{14}^4)^{21} = p_{12}^4 + p_{24}^4 - 0.5 = 0.6 + 0.2 - 0.5 = 0.3$$
$$(cp_{14}^4)^{31} = p_{13}^4 + p_{34}^4 - 0.5 = 0.6 + 0.3 - 0.5 = 0.4$$
$$(cp_{14}^4)^{1} = ((cp_{14}^4)^{21} + (cp_{14}^4)^{31})/2.0 = 0.7/2.0 = 0.35$$
$$(cp_{14}^4)^{22} = p_{24}^4 - p_{21}^4 + 0.5 = 0.2 - 0.4 + 0.5 = 0.3$$
$$(cp_{14}^4)^{32} = p_{34}^4 - p_{31}^4 + 0.5 = 0.3 - 0.5 + 0.5 = 0.3$$
$$(cp_{14}^4)^{2} = ((cp_{14}^4)^{22} + (cp_{14}^4)^{32})/2.0 = 0.6/2.0 = 0.3$$
$$(cp_{14}^4)^{23} = p_{12}^4 - p_{42}^4 + 0.5 = 0.6 - 0.7 + 0.5 = 0.4$$
$$(cp_{14}^4)^{33} = p_{13}^4 - p_{43}^4 + 0.5 = 0.6 - 0.7 + 0.5 = 0.4$$
$$(cp_{14}^4)^{3} = ((cp_{14}^4)^{23} + (cp_{14}^4)^{33})/2.0 = 0.8/2.0 = 0.4$$
$$cp_{14}^4 = 1.05/3 = 0.35$$

Computation of cp_{21}^4*:*

$$(cp_{21}^4)^{31} = p_{23}^4 + p_{31}^4 - 0.5 = 0.4 + 0.5 - 0.5 = 0.4$$
$$(cp_{21}^4)^{41} = p_{24}^4 + p_{41}^4 - 0.5 = 0.2 + 0.7 - 0.5 = 0.4$$
$$(cp_{21}^4)^{1} = ((cp_{21}^4)^{31} + (cp_{21}^4)^{41})/2.0 = 0.8/2.0 = 0.4$$
$$(cp_{21}^4)^{32} = p_{31}^4 - p_{32}^4 + 0.5 = 0.5 - 0.6 + 0.5 = 0.4$$
$$(cp_{21}^4)^{42} = p_{41}^4 - p_{42}^4 + 0.5 = 0.7 - 0.7 + 0.5 = 0.5$$
$$(cp_{21}^4)^{2} = ((cp_{21}^4)^{32} + (cp_{21}^4)^{42})/2.0 = 0.9/2.0 = 0.45$$
$$(cp_{21}^4)^{33} = p_{23}^4 - p_{13}^4 + 0.5 = 0.4 - 0.6 + 0.5 = 0.3$$
$$(cp_{21}^4)^{43} = p_{24}^4 - p_{14}^4 + 0.5 = 0.2 - 0.3 + 0.5 = 0.4$$
$$(cp_{21}^4)^{3} = ((cp_{21}^4)^{33} + (cp_{21}^4)^{43})/2.0 = 0.7/2.0 = 0.35$$
$$cp_{21}^4 = 1.2/3 = 0.4$$

Computation of cp_{23}^4*:*

$$(cp_{23}^4)^{11} = p_{21}^4 + p_{13}^4 - 0.5 = 0.4 + 0.6 - 0.5 = 0.5$$
$$(cp_{23}^4)^{41} = p_{24}^4 + p_{43}^4 - 0.5 = 0.2 + 0.7 - 0.5 = 0.4$$
$$(cp_{23}^4)^{1} = ((cp_{23}^4)^{11} + (cp_{23}^4)^{41})/2.0 = 0.9/2.0 = 0.45$$
$$(cp_{23}^4)^{12} = p_{13}^4 - p_{12}^4 + 0.5 = 0.6 - 0.6 + 0.5 = 0.5$$
$$(cp_{23}^4)^{42} = p_{43}^4 - p_{42}^4 + 0.5 = 0.7 - 0.7 + 0.5 = 0.5$$
$$(cp_{23}^4)^{2} = ((cp_{23}^4)^{12} + (cp_{23}^4)^{42})/2.0 = 1.0/2.0 = 0.5$$
$$(cp_{23}^4)^{13} = p_{21}^4 - p_{31}^4 + 0.5 = 0.4 - 0.5 + 0.5 = 0.4$$
$$(cp_{23}^4)^{43} = p_{24}^4 - p_{34}^4 + 0.5 = 0.2 - 0.3 + 0.5 = 0.4$$

$(cp_{23}^4)^3 = ((cp_{23}^4)^{13} + (cp_{23}^4)^{43})/2.0 = 0.8/2.0 = 0.4$
$cp_{23}^4 = 1.35/3 = 0.45$

Computation of cp_{24}^4*:*

$(cp_{24}^4)^{11} = p_{21}^4 + p_{14}^4 - 0.5 = 0.4 + 0.3 - 0.5 = 0.2$
$(cp_{24}^4)^{31} = p_{23}^4 + p_{34}^4 - 0.5 = 0.4 + 0.3 - 0.5 = 0.2$
$(cp_{24}^4)^1 = ((cp_{24}^4)^{11} + (cp_{24}^4)^{31})/2.0 = 0.4/2.0 = 0.2$
$(cp_{24}^4)^{12} = p_{14}^4 - p_{12}^4 + 0.5 = 0.3 - 0.6 + 0.5 = 0.2$
$(cp_{24}^4)^{32} = p_{34}^4 - p_{32}^4 + 0.5 = 0.3 - 0.6 + 0.5 = 0.2$
$(cp_{24}^4)^2 = ((cp_{24}^4)^{12} + (cp_{24}^4)^{32})/2.0 = 0.4/2.0 = 0.2$
$(cp_{24}^4)^{13} = p_{21}^4 - p_{41}^4 + 0.5 = 0.4 - 0.7 + 0.5 = 0.2$
$(cp_{24}^4)^{33} = p_{23}^4 - p_{43}^4 + 0.5 = 0.4 - 0.7 + 0.5 = 0.2$
$(cp_{24}^4)^3 = ((cp_{24}^4)^{13} + (cp_{24}^4)^{33})/2.0 = 0.4/2.0 = 0.2$
$cp_{24}^4 = 0.6/3 = 0.2$

Computation of cp_{31}^4*:*

$(cp_{31}^4)^{21} = p_{32}^4 + p_{21}^4 - 0.5 = 0.6 + 0.4 - 0.5 = 0.5$
$(cp_{31}^4)^{41} = p_{34}^4 + p_{41}^4 - 0.5 = 0.3 + 0.7 - 0.5 = 0.5$
$(cp_{31}^4)^1 = ((cp_{31}^4)^{21} + (cp_{31}^4)^{41})/2.0 = 1.0/2.0 = 0.5$
$(cp_{31}^4)^{22} = p_{21}^4 - p_{23}^4 + 0.5 = 0.4 - 0.4 + 0.5 = 0.5$
$(cp_{31}^4)^{42} = p_{41}^4 - p_{43}^4 + 0.5 = 0.7 - 0.7 + 0.5 = 0.5$
$(cp_{31}^4)^2 = ((cp_{31}^4)^{22} + (cp_{31}^4)^{42})/2.0 = 1.0/2.0 = 0.5$
$(cp_{31}^4)^{23} = p_{32}^4 - p_{12}^4 + 0.5 = 0.6 - 0.6 + 0.5 = 0.5$
$(cp_{31}^4)^{43} = p_{34}^4 - p_{14}^4 + 0.5 = 0.3 - 0.3 + 0.5 = 0.5$
$(cp_{31}^4)^3 = ((cp_{31}^4)^{23} + (cp_{31}^4)^{43})/2.0 = 1.0/2.0 = 0.5$
$cp_{31}^4 = 1.5/3 = 0.5$

Computation of cp_{32}^4*:*

$(cp_{32}^4)^{11} = p_{31}^4 + p_{12}^4 - 0.5 = 0.5 + 0.6 - 0.5 = 0.6$
$(cp_{32}^4)^{41} = p_{34}^4 + p_{42}^4 - 0.5 = 0.3 + 0.7 - 0.5 = 0.5$
$(cp_{32}^4)^1 = ((cp_{32}^4)^{11} + (cp_{32}^4)^{41})/2.0 = 1.1/2.0 = 0.55$
$(cp_{32}^4)^{12} = p_{12}^4 - p_{13}^4 + 0.5 = 0.6 - 0.6 + 0.5 = 0.5$
$(cp_{32}^4)^{42} = p_{42}^4 - p_{43}^4 + 0.5 = 0.7 - 0.7 + 0.5 = 0.5$

$(cp^4_{32})^2 = ((cp^4_{32})^{12} + (cp^4_{32})^{42})/2.0 = 1.0/2.0 = 0.5$
$(cp^4_{32})^{13} = p^4_{31} - p^4_{21} + 0.5 = 0.5 - 0.4 + 0.5 = 0.6$
$(cp^4_{32})^{43} = p^4_{34} - p^4_{24} + 0.5 = 0.3 - 0.2 + 0.5 = 0.6$
$(cp^4_{32})^3 = ((cp^4_{32})^{13} + (cp^4_{32})^{43})/2.0 = 1.2/2.0 = 0.6$
$cp^4_{32} = 1.65/3 = 0.55$

Computation of cp^4_{34}*:*
$(cp^4_{34})^{11} = p^4_{31} + p^4_{14} - 0.5 = 0.5 + 0.3 - 0.5 = 0.3$
$(cp^4_{34})^{21} = p^4_{32} + p^4_{24} - 0.5 = 0.6 + 0.2 - 0.5 = 0.3$
$(cp^4_{34})^1 = ((cp^4_{34})^{11} + (cp^4_{34})^{21})/2.0 = 0.6/2.0 = 0.3$
$(cp^4_{34})^{12} = p^4_{14} - p^4_{13} + 0.5 = 0.3 - 0.6 + 0.5 = 0.2$
$(cp^4_{34})^{22} = p^4_{24} - p^4_{23} + 0.5 = 0.2 - 0.4 + 0.5 = 0.3$
$(cp^4_{34})^2 = ((cp^4_{34})^{12} + (cp^4_{34})^{22})/2.0 = 0.5/2.0 = 0.25$
$(cp^4_{34})^{13} = p^4_{31} - p^4_{41} + 0.5 = 0.5 - 0.7 + 0.5 = 0.3$
$(cp^4_{34})^{23} = p^4_{32} - p^4_{42} + 0.5 = 0.6 - 0.7 + 0.5 = 0.4$
$(cp^4_{34})^3 = ((cp^4_{34})^{13} + (cp^4_{34})^{23})/2.0 = 0.7/2.0 = 0.35$
$cp^4_{34} = 0.9/3 = 0.3$

Computation of cp^4_{41}*:*
$(cp^4_{41})^{21} = p^4_{42} + p^4_{21} - 0.5 = 0.7 + 0.4 - 0.5 = 0.6$
$(cp^4_{41})^{31} = p^4_{43} + p^4_{31} - 0.5 = 0.7 + 0.5 - 0.5 = 0.7$
$(cp^4_{41})^1 = ((cp^4_{41})^{21} + (cp^4_{41})^{31})/2.0 = 1.3/2.0 = 0.65$
$(cp^4_{41})^{22} = p^4_{21} - p^4_{24} + 0.5 = 0.4 - 0.2 + 0.5 = 0.7$
$(cp^4_{41})^{32} = p^4_{31} - p^4_{34} + 0.5 = 0.5 - 0.3 + 0.5 = 0.7$
$(cp^4_{41})^2 = ((cp^4_{41})^{22} + (cp^4_{41})^{32})/2.0 = 1.4/2.0 = 0.7$
$(cp^4_{41})^{23} = p^4_{42} - p^4_{12} + 0.5 = 0.7 - 0.6 + 0.5 = 0.6$
$(cp^4_{41})^{33} = p^4_{43} - p^4_{13} + 0.5 = 0.7 - 0.6 + 0.5 = 0.6$
$(cp^4_{41})^3 = ((cp^4_{41})^{23} + (cp^4_{41})^{33})/2.0 = 1.2/2.0 = 0.6$
$cp^4_{41} = 1.95/3 = 0.65$

Computation of cp^4_{42}*:*
$(cp^4_{42})^{11} = p^4_{41} + p^4_{12} - 0.5 = 0.7 + 0.6 - 0.5 = 0.8$
$(cp^4_{42})^{31} = p^4_{43} + p^4_{32} - 0.5 = 0.7 + 0.6 - 0.5 = 0.8$

$$(cp_{42}^4)^1 = ((cp_{42}^4)^{11} + (cp_{42}^4)^{31})/2.0 = 1.6/2.0 = 0.8$$
$$(cp_{42}^4)^{12} = p_{12}^4 - p_{14}^4 + 0.5 = 0.6 - 0.3 + 0.5 = 0.8$$
$$(cp_{42}^4)^{32} = p_{32}^4 - p_{34}^4 + 0.5 = 0.6 - 0.3 + 0.5 = 0.8$$
$$(cp_{42}^4)^2 = ((cp_{42}^4)^{12} + (cp_{42}^4)^{32})/2.0 = 1.6/2.0 = 0.8$$
$$(cp_{42}^4)^{13} = p_{41}^4 - p_{21}^4 + 0.5 = 0.7 - 0.4 + 0.5 = 0.8$$
$$(cp_{42}^4)^{33} = p_{43}^4 - p_{23}^4 + 0.5 = 0.7 - 0.4 + 0.5 = 0.8$$
$$(cp_{42}^4)^3 = ((cp_{42}^4)^{13} + (cp_{42}^4)^{33})/2.0 = 1.6/2.0 = 0.8$$
$$cp_{42}^4 = 2.4/3 = 0.8$$

Computation of cp_{43}^4*:*

$$(cp_{43}^4)^{11} = p_{41}^4 + p_{13}^4 - 0.5 = 0.7 + 0.6 - 0.5 = 0.8$$
$$(cp_{43}^4)^{21} = p_{42}^4 + p_{23}^4 - 0.5 = 0.7 + 0.4 - 0.5 = 0.6$$
$$(cp_{43}^4)^1 = ((cp_{43}^4)^{11} + (cp_{43}^4)^{21})/2.0 = 1.4/2.0 = 0.7$$
$$(cp_{43}^4)^{12} = p_{13}^4 - p_{14}^4 + 0.5 = 0.6 - 0.3 + 0.5 = 0.8$$
$$(cp_{43}^4)^{22} = p_{23}^4 - p_{24}^4 + 0.5 = 0.4 - 0.2 + 0.5 = 0.7$$
$$(cp_{43}^4)^2 = ((cp_{43}^4)^{12} + (cp_{43}^4)^{22})/2.0 = 1.5/2.0 = 0.75$$
$$(cp_{43}^4)^{13} = p_{41}^4 - p_{31}^4 + 0.5 = 0.7 - 0.5 + 0.5 = 0.7$$
$$(cp_{43}^4)^{23} = p_{42}^4 - p_{32}^4 + 0.5 = 0.7 - 0.6 + 0.5 = 0.6$$
$$(cp_{43}^4)^3 = ((cp_{43}^4)^{13} + (cp_{43}^4)^{23})/2.0 = 1.3/2.0 = 0.65$$
$$cp_{43}^4 = 2.1/3 = 0.7$$

$$CP^4 = \begin{pmatrix} - & 0.6 & 0.5 & 0.35 \\ 0.4 & - & 0.45 & 0.2 \\ 0.5 & 0.55 & - & 0.3 \\ 0.65 & 0.8 & 0.7 & - \end{pmatrix}$$

B.3 Computing Consistency Levels

Computing CL^1:

$$CL^1 = (1) - |\overline{P}^1 - CP^1| = (1) - \left| \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix} - \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 1.0 & 1.0 & 1.0 \\ 1.0 & - & 1.0 & 1.0 \\ 1.0 & 1.0 & - & 1.0 \\ 1.0 & 1.0 & 1.0 & - \end{pmatrix}$$

Computing CL^2**:**

$$CL^2 = (1) - |\overline{P}^2 - CP^2| = (1) - \left| \begin{pmatrix} - & 0.62 & 0.7 & 0.8 \\ 0.4 & - & 0.6 & 0.7 \\ 0.3 & 0.4 & - & 0.57 \\ 0.25 & 0.4 & 0.45 & - \end{pmatrix} - \begin{pmatrix} - & 0.62 & 0.72 & 0.77 \\ 0.4 & - & 0.6 & 0.67 \\ 0.3 & 0.43 & - & 0.57 \\ 0.25 & 0.35 & 0.45 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 1.0 & 0.98 & 0.98 \\ 1.0 & - & 1.0 & 0.97 \\ 1.0 & 0.98 & - & 1.0 \\ 1.0 & 0.95 & 1.0 & - \end{pmatrix}$$

Computing CL^3**:**

$$CL^3 = (1) - |\overline{P}^3 - CP^3| = (1) - \left| \begin{pmatrix} - & 0.3 & 0.54 & 0.75 \\ 0.6 & - & 0.69 & 0.87 \\ 0.46 & 0.31 & - & 0.73 \\ 0.3 & 0.4 & 0.27 & - \end{pmatrix} - \begin{pmatrix} - & 0.45 & 0.51 & 0.69 \\ 0.61 & - & 0.61 & 0.89 \\ 0.48 & 0.42 & - & 0.64 \\ 0.34 & 0.1 & 0.41 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 0.85 & 0.98 & 0.94 \\ 0.99 & - & 0.92 & 0.97 \\ 0.99 & 0.89 & - & 0.91 \\ 0.96 & 0.7 & 0.86 & - \end{pmatrix}$$

Computing CL^4**:**

$$CL^4 = (1) - |\overline{P}^4 - CP^4| = (1) - \left| \begin{pmatrix} - & 0.6 & 0.6 & 0.3 \\ 0.4 & - & 0.4 & 0.2 \\ 0.5 & 0.6 & - & 0.3 \\ 0.7 & 0.7 & 0.7 & - \end{pmatrix} - \begin{pmatrix} - & 0.6 & 0.5 & 0.35 \\ 0.4 & - & 0.45 & 0.2 \\ 0.5 & 0.55 & - & 0.3 \\ 0.65 & 0.8 & 0.7 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 1.0 & 0.9 & 0.95 \\ 1.0 & - & 0.95 & 1.0 \\ 1.0 & 0.95 & - & 1.0 \\ 0.95 & 0.9 & 1.0 & - \end{pmatrix}$$

Computing cl_i^1 and cl^1:

$cl_1^1 = 1.0$; $cl_2^1 = 1.0$; $cl_3^1 = 1.0$; $cl_4^1 = 1.0$

$cl^1 = 1.0$

Computing cl_i^2 and cl^2:

$cl_1^2 = 0.99$; $cl_2^2 = 0.98$; $cl_3^2 = 0.99$; $cl_4^2 = 0.98$

$cl^2 = 0.99$

Computing cl_i^3 and cl^3:

$cl_1^3 = 0.95$; $cl_2^3 = 0.89$; $cl_3^3 = 0.92$; $cl_4^3 = 0.89$

$cl^3 = 0.91$

Computing cl_i^4 and cl^4:

$cl_1^4 = 0.97$; $cl_2^4 = 0.97$; $cl_3^4 = 0.97$; $cl_4^4 = 0.97$

$cl^4 = 0.97$

Computing the global consistency level (CL):

$$CL = \frac{cl^1 + cl^2 + cl^3 + cl^4}{4} = 0.97$$

B.4 Computing Similarity Matrix

$$SM^{12} = (1) - |P^1 - P^2| = (1) - \left| \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix} - \begin{pmatrix} - & 0.62 & 0.7 & 0.8 \\ 0.4 & - & 0.6 & 0.7 \\ 0.3 & 0.4 & - & 0.57 \\ 0.25 & 0.4 & 0.45 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 0.58 & 0.9 & 0.6 \\ 0.6 & - & 0.7 & 1.0 \\ 0.9 & 0.7 & - & 0.73 \\ 0.65 & 0.9 & 0.75 & - \end{pmatrix}$$

$$SM^{13} = (1) - |P^1 - P^3| = (1) - \left| \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix} - \begin{pmatrix} - & 0.3 & 0.54 & 0.75 \\ 0.6 & - & 0.69 & 0.87 \\ 0.46 & 0.31 & - & 0.73 \\ 0.3 & 0.4 & 0.27 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 0.9 & 0.94 & 0.65 \\ 0.8 & - & 0.79 & 0.83 \\ 0.94 & 0.79 & - & 0.57 \\ 0.7 & 0.9 & 0.57 & - \end{pmatrix}$$

$$SM^{14} = (1) - |P^1 - P^4| = (1) - \left| \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix} - \begin{pmatrix} - & 0.6 & 0.6 & 0.3 \\ 0.4 & - & 0.4 & 0.2 \\ 0.5 & 0.6 & - & 0.3 \\ 0.7 & 0.7 & 0.7 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 0.6 & 1.0 & 0.9 \\ 0.6 & - & 0.5 & 0.5 \\ 0.9 & 0.5 & - & 1.0 \\ 0.9 & 0.6 & 1.0 & - \end{pmatrix}$$

$$SM^{23} = (1) - |P^2 - P^3| = (1) - \left| \begin{pmatrix} - & 0.62 & 0.7 & 0.8 \\ 0.4 & - & 0.6 & 0.7 \\ 0.3 & 0.4 & - & 0.57 \\ 0.25 & 0.4 & 0.45 & - \end{pmatrix} - \begin{pmatrix} - & 0.3 & 0.54 & 0.75 \\ 0.6 & - & 0.69 & 0.87 \\ 0.46 & 0.31 & - & 0.73 \\ 0.3 & 0.4 & 0.27 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 0.68 & 0.84 & 0.95 \\ 0.8 & - & 0.91 & 0.83 \\ 0.84 & 0.91 & - & 0.85 \\ 0.95 & 1.0 & 0.82 & - \end{pmatrix}$$

$$SM^{24} = (1) - |P^2 - P^4| = (1) - \left| \begin{pmatrix} - & 0.62 & 0.7 & 0.8 \\ 0.4 & - & 0.6 & 0.7 \\ 0.3 & 0.4 & - & 0.57 \\ 0.25 & 0.4 & 0.45 & - \end{pmatrix} - \begin{pmatrix} - & 0.6 & 0.6 & 0.3 \\ 0.4 & - & 0.4 & 0.2 \\ 0.5 & 0.6 & - & 0.3 \\ 0.7 & 0.7 & 0.7 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 0.98 & 0.9 & 0.5 \\ 1.0 & - & 0.8 & 0.5 \\ 0.8 & 0.8 & - & 0.73 \\ 0.55 & 0.7 & 0.75 & - \end{pmatrix}$$

$$SM^{34} = (1) - |P^3 - P^4| = (1) - \left| \begin{pmatrix} - & 0.3 & 0.54 & 0.75 \\ 0.6 & - & 0.69 & 0.87 \\ 0.46 & 0.31 & - & 0.73 \\ 0.3 & 0.4 & 0.27 & - \end{pmatrix} - \begin{pmatrix} - & 0.6 & 0.6 & 0.3 \\ 0.4 & - & 0.4 & 0.2 \\ 0.5 & 0.6 & - & 0.3 \\ 0.7 & 0.7 & 0.7 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 0.7 & 0.94 & 0.55 \\ 0.8 & - & 0.71 & 0.33 \\ 0.96 & 0.71 & - & 0.57 \\ 0.6 & 0.7 & 0.57 & - \end{pmatrix}$$

$$SM = \phi(SM^{12}, SM^{13}, SM^{14}, SM^{23}, SM^{24}, SM^{34}) = \begin{pmatrix} - & 0.74 & 0.92 & 0.69 \\ 0.77 & - & 0.73 & 0.67 \\ 0.89 & 0.73 & - & 0.74 \\ 0.73 & 0.8 & 0.74 & - \end{pmatrix}$$

$$CR = \frac{0.74 + 0.92 + 0.69 + 0.77 + 0.73 + 0.67 + 0.89 + 0.73 + 0.74 + 0.73 + 0.8 + 0.74}{12} = 0.76$$

B.5 Computing Consensus Values CO^h

$$CO^1 = \frac{+sm^{12} + sm^{13} + sm^{14}}{3} =$$

$$= \frac{+\begin{pmatrix} - & 0.58 & 0.9 & 0.6 \\ 0.6 & - & 0.7 & 1.0 \\ 0.9 & 0.7 & - & 0.73 \\ 0.65 & 0.9 & 0.75 & - \end{pmatrix} + \begin{pmatrix} - & 0.9 & 0.94 & 0.65 \\ 0.8 & - & 0.79 & 0.83 \\ 0.94 & 0.79 & - & 0.57 \\ 0.7 & 0.9 & 0.57 & - \end{pmatrix} + \begin{pmatrix} - & 0.6 & 1.0 & 0.9 \\ 0.6 & - & 0.5 & 0.5 \\ 0.9 & 0.5 & - & 1.0 \\ 0.9 & 0.6 & 1.0 & - \end{pmatrix}}{3} =$$

$$= \begin{pmatrix} - & 0.69 & 0.95 & 0.72 \\ 0.67 & - & 0.66 & 0.78 \\ 0.91 & 0.66 & - & 0.77 \\ 0.75 & 0.8 & 0.77 & - \end{pmatrix}$$

$$CO^2 = \frac{+sm^{23} + sm^{24} + sm^{12}}{3} =$$

$$= \frac{+\begin{pmatrix} - & 0.68 & 0.84 & 0.95 \\ 0.8 & - & 0.91 & 0.83 \\ 0.84 & 0.91 & - & 0.85 \\ 0.95 & 1.0 & 0.82 & - \end{pmatrix} + \begin{pmatrix} - & 0.98 & 0.9 & 0.5 \\ 1.0 & - & 0.8 & 0.5 \\ 0.8 & 0.8 & - & 0.73 \\ 0.55 & 0.7 & 0.75 & - \end{pmatrix} + \begin{pmatrix} - & 0.58 & 0.9 & 0.6 \\ 0.6 & - & 0.7 & 1.0 \\ 0.9 & 0.7 & - & 0.73 \\ 0.65 & 0.9 & 0.75 & - \end{pmatrix}}{3} =$$

$$= \begin{pmatrix} - & 0.74 & 0.88 & 0.68 \\ 0.8 & - & 0.8 & 0.78 \\ 0.85 & 0.8 & - & 0.77 \\ 0.72 & 0.87 & 0.77 & - \end{pmatrix}$$

$$CO^3 = \frac{+sm^{34} + sm^{13} + sm^{23}}{3} =$$

$$= \frac{+\begin{pmatrix} - & 0.7 & 0.94 & 0.55 \\ 0.8 & - & 0.71 & 0.33 \\ 0.96 & 0.71 & - & 0.57 \\ 0.6 & 0.7 & 0.57 & - \end{pmatrix} + \begin{pmatrix} - & 0.9 & 0.94 & 0.65 \\ 0.8 & - & 0.79 & 0.83 \\ 0.94 & 0.79 & - & 0.57 \\ 0.7 & 0.9 & 0.57 & - \end{pmatrix} + \begin{pmatrix} - & 0.68 & 0.84 & 0.95 \\ 0.8 & - & 0.91 & 0.83 \\ 0.84 & 0.91 & - & 0.85 \\ 0.95 & 1.0 & 0.82 & - \end{pmatrix}}{3} =$$

$$= \begin{pmatrix} - & 0.76 & 0.9 & 0.72 \\ 0.8 & - & 0.8 & 0.67 \\ 0.91 & 0.8 & - & 0.66 \\ 0.75 & 0.87 & 0.65 & - \end{pmatrix}$$

$$CO^4 = \frac{+sm^{14} + sm^{24} + sm^{34}}{3} =$$

$$= \frac{+\begin{pmatrix} - & 0.6 & 1.0 & 0.9 \\ 0.6 & - & 0.5 & 0.5 \\ 0.9 & 0.5 & - & 1.0 \\ 0.9 & 0.6 & 1.0 & - \end{pmatrix} + \begin{pmatrix} - & 0.98 & 0.9 & 0.5 \\ 1.0 & - & 0.8 & 0.5 \\ 0.8 & 0.8 & - & 0.73 \\ 0.55 & 0.7 & 0.75 & - \end{pmatrix} + \begin{pmatrix} - & 0.7 & 0.94 & 0.55 \\ 0.8 & - & 0.71 & 0.33 \\ 0.96 & 0.71 & - & 0.57 \\ 0.6 & 0.7 & 0.57 & - \end{pmatrix}}{3} =$$

$$= \begin{pmatrix} - & 0.76 & 0.95 & 0.65 \\ 0.8 & - & 0.67 & 0.44 \\ 0.89 & 0.67 & - & 0.77 \\ 0.68 & 0.67 & 0.77 & - \end{pmatrix}$$

B.6 Computing Order Inducing Variables

$$\delta = 0.75$$

$$Z^1 = (1 - \delta) \cdot CL^1 + \delta \cdot CO^1 =$$

$$(1-0.75)\cdot\begin{pmatrix} - & 1.0 & 1.0 & 1.0 \\ 1.0 & - & 1.0 & 1.0 \\ 1.0 & 1.0 & - & 1.0 \\ 1.0 & 1.0 & 1.0 & - \end{pmatrix} + 0.75\cdot\begin{pmatrix} - & 0.69 & 0.95 & 0.72 \\ 0.67 & - & 0.66 & 0.78 \\ 0.91 & 0.66 & - & 0.77 \\ 0.75 & 0.8 & 0.77 & - \end{pmatrix} =$$

$$= \begin{pmatrix} - & 0.77 & 0.96 & 0.79 \\ 0.75 & - & 0.75 & 0.83 \\ 0.93 & 0.75 & - & 0.82 \\ 0.81 & 0.85 & 0.83 & - \end{pmatrix}$$

$$Z^2 = (1-\delta)\cdot CL^2 + \delta\cdot CO^2 =$$

$$(1-0.75)\cdot\begin{pmatrix} - & 1.0 & 0.98 & 0.98 \\ 1.0 & - & 1.0 & 0.97 \\ 1.0 & 0.98 & - & 1.0 \\ 1.0 & 0.95 & 1.0 & - \end{pmatrix} + 0.75\cdot\begin{pmatrix} - & 0.74 & 0.88 & 0.68 \\ 0.8 & - & 0.8 & 0.78 \\ 0.85 & 0.8 & - & 0.77 \\ 0.72 & 0.87 & 0.77 & - \end{pmatrix} =$$

$$= \begin{pmatrix} - & 0.81 & 0.9 & 0.76 \\ 0.85 & - & 0.85 & 0.83 \\ 0.88 & 0.85 & - & 0.82 \\ 0.79 & 0.89 & 0.83 & - \end{pmatrix}$$

$$Z^3 = (1-\delta)\cdot CL^3 + \delta\cdot CO^3 =$$

$$(1-0.75)\cdot\begin{pmatrix} - & 0.85 & 0.98 & 0.94 \\ 0.99 & - & 0.92 & 0.97 \\ 0.99 & 0.89 & - & 0.91 \\ 0.96 & 0.7 & 0.86 & - \end{pmatrix} + 0.75\cdot\begin{pmatrix} - & 0.76 & 0.9 & 0.72 \\ 0.8 & - & 0.8 & 0.67 \\ 0.91 & 0.8 & - & 0.66 \\ 0.75 & 0.87 & 0.65 & - \end{pmatrix} =$$

$$= \begin{pmatrix} - & 0.78 & 0.92 & 0.77 \\ 0.85 & - & 0.83 & 0.74 \\ 0.93 & 0.82 & - & 0.72 \\ 0.8 & 0.82 & 0.71 & - \end{pmatrix}$$

$$Z^4 = (1-\delta) \cdot CL^4 + \delta \cdot CO^4 =$$

$$(1-0.75) \cdot \begin{pmatrix} - & 1.0 & 0.9 & 0.95 \\ 1.0 & - & 0.95 & 1.0 \\ 1.0 & 0.95 & - & 1.0 \\ 0.95 & 0.9 & 1.0 & - \end{pmatrix} + 0.75 \cdot \begin{pmatrix} - & 0.76 & 0.95 & 0.65 \\ 0.8 & - & 0.67 & 0.44 \\ 0.89 & 0.67 & - & 0.77 \\ 0.68 & 0.67 & 0.77 & - \end{pmatrix} =$$

$$= \begin{pmatrix} - & 0.82 & 0.93 & 0.73 \\ 0.85 & - & 0.74 & 0.58 \\ 0.92 & 0.74 & - & 0.82 \\ 0.75 & 0.73 & 0.83 & - \end{pmatrix}$$

B.7 Computing the Collective Fuzzy Preference Relation

$$Q(r) = \begin{cases} 0 & \text{if } r < 0.3 \\ \frac{r-0.3}{0.8-0.3} & \text{if } 0.3 \leq r < 0.8 \\ 1 & \text{if } r \geq 0.8 \end{cases}$$

$$p_{12}^c = \Phi_Q((p_{12}^1, z_{12}^1), (p_{12}^2, z_{12}^2), (p_{12}^3, z_{12}^3), (p_{12}^4, z_{12}^4)) =$$

$$= \Phi_Q((0.2, 0.77), (0.62, 0.81), (0.3, 0.78), (0.6, 0.82)) =$$

$$= 0.0 \cdot 0.6 + 0.42 \cdot 0.62 + 0.49 \cdot 0.3 + 0.08 \cdot 0.2 = \mathbf{0.43}$$

$$(W = \{0.0, 0.42, 0.49, 0.08\})$$

$$p_{13}^c = \Phi_Q((p_{13}^1, z_{13}^1), (p_{13}^2, z_{13}^2), (p_{13}^3, z_{13}^3), (p_{13}^4, z_{13}^4)) =$$

$$= \Phi_Q((0.6, 0.96), (0.7, 0.9), (0.54, 0.92), (0.6, 0.93)) =$$

$$= 0.0 \cdot 0.6 + 0.42 \cdot 0.6 + 0.5 \cdot 0.54 + 0.09 \cdot 0.7 = \mathbf{0.58}$$

$$(W = \{0.0, 0.42, 0.5, 0.09\})$$

$$p^c_{14} = \Phi_Q((p^1_{14}, z^1_{14}), (p^2_{14}, z^2_{14}), (p^3_{14}, z^3_{14}), (p^4_{14}, z^4_{14})) =$$

$$= \Phi_Q((0.4, 0.79), (0.8, 0.76), (0.75, 0.77), (0.3, 0.73)) =$$

$$= 0.0 \cdot 0.4 + 0.43 \cdot 0.75 + 0.5 \cdot 0.8 + 0.08 \cdot 0.3 = \mathbf{0.74}$$

$$(W = \{0.0, 0.43, 0.5, 0.08\})$$

$$p^c_{21} = \Phi_Q((p^1_{21}, z^1_{21}), (p^2_{21}, z^2_{21}), (p^3_{21}, z^3_{21}), (p^4_{21}, z^4_{21})) =$$

$$= \Phi_Q((0.8, 0.75), (0.4, 0.85), (0.6, 0.85), (0.4, 0.85)) =$$

$$= 0.0 \cdot 0.4 + 0.43 \cdot 0.4 + 0.51 \cdot 0.6 + 0.05 \cdot 0.8 = \mathbf{0.52}$$

$$(W = \{0.0, 0.43, 0.51, 0.05\})$$

$$p^c_{23} = \Phi_Q((p^1_{23}, z^1_{23}), (p^2_{23}, z^2_{23}), (p^3_{23}, z^3_{23}), (p^4_{23}, z^4_{23})) =$$

$$= \Phi_Q((0.9, 0.75), (0.6, 0.85), (0.69, 0.83), (0.4, 0.74)) =$$

$$= 0.0 \cdot 0.6 + 0.46 \cdot 0.69 + 0.47 \cdot 0.9 + 0.07 \cdot 0.4 = \mathbf{0.77}$$

$$(W = \{0.0, 0.46, 0.47, 0.07\})$$

$$p^c_{24} = \Phi_Q((p^1_{24}, z^1_{24}), (p^2_{24}, z^2_{24}), (p^3_{24}, z^3_{24}), (p^4_{24}, z^4_{24})) =$$

$$= \Phi_Q((0.7, 0.83), (0.7, 0.83), (0.87, 0.74), (0.2, 0.58)) =$$

$$= 0.0 \cdot 0.7 + 0.51 \cdot 0.7 + 0.49 \cdot 0.87 + 0.0 \cdot 0.2 = \mathbf{0.78}$$

$$(W = \{0.0, 0.51, 0.49, 0.0\})$$

$$p^c_{31} = \Phi_Q((p^1_{31}, z^1_{31}), (p^2_{31}, z^2_{31}), (p^3_{31}, z^3_{31}), (p^4_{31}, z^4_{31})) =$$

$$= \Phi_Q((0.4, 0.93), (0.3, 0.88), (0.46, 0.93), (0.5, 0.92)) =$$

$$= 0.0 \cdot 0.4 + 0.42 \cdot 0.46 + 0.5 \cdot 0.5 + 0.08 \cdot 0.3 = \mathbf{0.47}$$

$$(W = \{0.0, 0.42, 0.5, 0.08\})$$

$$p_{32}^c = \Phi_Q((p_{32}^1, z_{32}^1), (p_{32}^2, z_{32}^2), (p_{32}^3, z_{32}^3), (p_{32}^4, z_{32}^4)) =$$

$$= \Phi_Q((0.1, 0.75), (0.4, 0.85), (0.31, 0.82), (0.6, 0.74)) =$$

$$= 0.0 \cdot 0.4 + 0.46 \cdot 0.31 + 0.47 \cdot 0.1 + 0.07 \cdot 0.6 = \mathbf{0.23}$$

$$(W = \{0.0, 0.46, 0.47, 0.07\})$$

$$p_{34}^c = \Phi_Q((p_{34}^1, z_{34}^1), (p_{34}^2, z_{34}^2), (p_{34}^3, z_{34}^3), (p_{34}^4, z_{34}^4)) =$$

$$= \Phi_Q((0.3, 0.82), (0.57, 0.82), (0.73, 0.72), (0.3, 0.82)) =$$

$$= 0.0 \cdot 0.3 + 0.43 \cdot 0.57 + 0.52 \cdot 0.3 + 0.05 \cdot 0.73 = \mathbf{0.44}$$

$$(W = \{0.0, 0.43, 0.52, 0.05\})$$

$$p_{41}^c = \Phi_Q((p_{41}^1, z_{41}^1), (p_{41}^2, z_{41}^2), (p_{41}^3, z_{41}^3), (p_{41}^4, z_{41}^4)) =$$

$$= \Phi_Q((0.6, 0.81), (0.25, 0.79), (0.3, 0.8), (0.7, 0.75)) =$$

$$= 0.0 \cdot 0.6 + 0.42 \cdot 0.3 + 0.5 \cdot 0.25 + 0.08 \cdot 0.7 = \mathbf{0.31}$$

$$(W = \{0.0, 0.42, 0.5, 0.08\})$$

$$p_{42}^c = \Phi_Q((p_{42}^1, z_{42}^1), (p_{42}^2, z_{42}^2), (p_{42}^3, z_{42}^3), (p_{42}^4, z_{42}^4)) =$$

$$= \Phi_Q((0.3, 0.85), (0.4, 0.89), (0.4, 0.82), (0.7, 0.73)) =$$

$$= 0.0 \cdot 0.4 + 0.46 \cdot 0.3 + 0.5 \cdot 0.4 + 0.04 \cdot 0.7 = \mathbf{0.37}$$

$$(W = \{0.0, 0.46, 0.5, 0.04\})$$

$$p_{43}^c = \Phi_Q((p_{43}^1, z_{43}^1), (p_{43}^2, z_{43}^2), (p_{43}^3, z_{43}^3), (p_{43}^4, z_{43}^4)) =$$

$$= \Phi_Q((0.7, 0.83), (0.45, 0.83), (0.27, 0.71), (0.7, 0.83)) =$$

$$= 0.0 \cdot 0.7 + 0.44 \cdot 0.45 + 0.52 \cdot 0.7 + 0.04 \cdot 0.27 = \mathbf{0.57}$$

$$(W = \{0.0, 0.44, 0.52, 0.04\})$$

$$P^c = \begin{pmatrix} - & 0.43 & 0.58 & 0.74 \\ 0.52 & - & 0.77 & 0.78 \\ 0.47 & 0.23 & - & 0.44 \\ 0.31 & 0.37 & 0.57 & - \end{pmatrix}$$

B.8 Computing Proximity Measures

$$PP^1 = (1) - |\overline{P}^1 - P^c| = (1) - \left| \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix} - \begin{pmatrix} - & 0.43 & 0.58 & 0.74 \\ 0.52 & - & 0.77 & 0.78 \\ 0.47 & 0.23 & - & 0.44 \\ 0.31 & 0.37 & 0.57 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 0.77 & 0.98 & 0.66 \\ 0.72 & - & 0.87 & 0.92 \\ 0.93 & 0.87 & - & 0.86 \\ 0.71 & 0.93 & 0.87 & - \end{pmatrix}$$

$$PP^2 = (1) - |\overline{P}^2 - P^c| = (1) - \left| \begin{pmatrix} - & 0.62 & 0.7 & 0.8 \\ 0.4 & - & 0.6 & 0.7 \\ 0.3 & 0.4 & - & 0.57 \\ 0.25 & 0.4 & 0.45 & - \end{pmatrix} - \begin{pmatrix} - & 0.43 & 0.58 & 0.74 \\ 0.52 & - & 0.77 & 0.78 \\ 0.47 & 0.23 & - & 0.44 \\ 0.31 & 0.37 & 0.57 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 0.8 & 0.88 & 0.94 \\ 0.88 & - & 0.83 & 0.92 \\ 0.83 & 0.83 & - & 0.87 \\ 0.94 & 0.97 & 0.88 & - \end{pmatrix}$$

$$PP^3 = (1) - |\overline{P}^3 - P^c| = (1) - \left| \begin{pmatrix} - & 0.3 & 0.54 & 0.75 \\ 0.6 & - & 0.69 & 0.87 \\ 0.46 & 0.31 & - & 0.73 \\ 0.3 & 0.4 & 0.27 & - \end{pmatrix} - \begin{pmatrix} - & 0.43 & 0.58 & 0.74 \\ 0.52 & - & 0.77 & 0.78 \\ 0.47 & 0.23 & - & 0.44 \\ 0.31 & 0.37 & 0.57 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 0.87 & 0.96 & 0.99 \\ 0.92 & - & 0.92 & 0.91 \\ 0.99 & 0.92 & - & 0.71 \\ 0.99 & 0.97 & 0.7 & - \end{pmatrix}$$

$$PP^4 = (1) - |\overline{P}^4 - P^c| = (1) - \left| \begin{pmatrix} - & 0.6 & 0.6 & 0.3 \\ 0.4 & - & 0.4 & 0.2 \\ 0.5 & 0.6 & - & 0.3 \\ 0.7 & 0.7 & 0.7 & - \end{pmatrix} - \begin{pmatrix} - & 0.43 & 0.58 & 0.74 \\ 0.52 & - & 0.77 & 0.78 \\ 0.47 & 0.23 & - & 0.44 \\ 0.31 & 0.37 & 0.57 & - \end{pmatrix} \right| =$$

$$= \begin{pmatrix} - & 0.83 & 0.98 & 0.56 \\ 0.88 & - & 0.63 & 0.42 \\ 0.97 & 0.63 & - & 0.86 \\ 0.61 & 0.67 & 0.87 & - \end{pmatrix}$$

$$pa_1^1 = \frac{pp_{12}^1 + pp_{21}^1 + pp_{13}^1 + pp_{31}^1 + pp_{14}^1 + pp_{41}^1}{6} =$$

$$= \frac{0.77 + 0.72 + 0.98 + 0.93 + 0.66 + 0.71}{6} = \mathbf{0.8}$$

$$pa_2^1 = \frac{pp_{21}^1 + pp_{12}^1 + pp_{23}^1 + pp_{32}^1 + pp_{24}^1 + pp_{42}^1}{6} =$$

$$= \frac{0.72 + 0.77 + 0.87 + 0.87 + 0.92 + 0.93}{6} = \mathbf{0.85}$$

$$pa_3^1 = \frac{pp_{31}^1 + pp_{13}^1 + pp_{32}^1 + pp_{23}^1 + pp_{34}^1 + pp_{43}^1}{6} =$$

$$= \frac{0.93 + 0.98 + 0.87 + 0.87 + 0.86 + 0.87}{6} = \mathbf{0.9}$$

$$pa_4^1 = \frac{pp_{41}^1 + pp_{14}^1 + pp_{42}^1 + pp_{24}^1 + pp_{43}^1 + pp_{34}^1}{6} =$$

$$= \frac{0.71 + 0.66 + 0.93 + 0.92 + 0.87 + 0.86}{6} = \mathbf{0.82}$$

$$PA^1 = \begin{pmatrix} 0.8 & 0.85 & 0.9 & 0.82 \end{pmatrix}$$

$$pa_1^2 = \frac{pp_{12}^2 + pp_{21}^2 + pp_{13}^2 + pp_{31}^2 + pp_{14}^2 + pp_{41}^2}{6} =$$

$$= \frac{0.8 + 0.88 + 0.88 + 0.83 + 0.94 + 0.94}{6} = \mathbf{0.88}$$

$$pa_2^2 = \frac{pp_{21}^2 + pp_{12}^2 + pp_{23}^2 + pp_{32}^2 + pp_{24}^2 + pp_{42}^2}{6} =$$

$$= \frac{0.88 + 0.8 + 0.83 + 0.83 + 0.92 + 0.97}{6} = \mathbf{0.87}$$

$$pa_3^2 = \frac{pp_{31}^2 + pp_{13}^2 + pp_{32}^2 + pp_{23}^2 + pp_{34}^2 + pp_{43}^2}{6} =$$

$$= \frac{0.83 + 0.88 + 0.83 + 0.83 + 0.87 + 0.88}{6} = \mathbf{0.85}$$

$$pa_4^2 = \frac{pp_{41}^2 + pp_{14}^2 + pp_{42}^2 + pp_{24}^2 + pp_{43}^2 + pp_{34}^2}{6} =$$

$$= \frac{0.94 + 0.94 + 0.97 + 0.92 + 0.88 + 0.87}{6} = \mathbf{0.92}$$

$$PA^2 = \begin{pmatrix} 0.88 & 0.87 & 0.85 & 0.92 \end{pmatrix}$$

$$pa_1^3 = \frac{pp_{12}^3 + pp_{21}^3 + pp_{13}^3 + pp_{31}^3 + pp_{14}^3 + pp_{41}^3}{6} =$$

$$= \frac{0.87 + 0.92 + 0.96 + 0.99 + 0.99 + 0.99}{6} = \mathbf{0.96}$$

$$pa_2^3 = \frac{pp_{21}^3 + pp_{12}^3 + pp_{23}^3 + pp_{32}^3 + pp_{24}^3 + pp_{42}^3}{6} =$$

$$= \frac{0.92 + 0.87 + 0.92 + 0.92 + 0.91 + 0.97}{6} = \mathbf{0.92}$$

$$pa_3^3 = \frac{pp_{31}^3 + pp_{13}^3 + pp_{32}^3 + pp_{23}^3 + pp_{34}^3 + pp_{43}^3}{6} =$$

$$= \frac{0.99 + 0.96 + 0.92 + 0.92 + 0.71 + 0.7}{6} = \mathbf{0.87}$$

$$pa_4^3 = \frac{pp_{41}^3 + pp_{14}^3 + pp_{42}^3 + pp_{24}^3 + pp_{43}^3 + pp_{34}^3}{6} =$$

$$= \frac{0.99 + 0.99 + 0.97 + 0.91 + 0.7 + 0.71}{6} = \mathbf{0.88}$$

$$PA^3 = \begin{pmatrix} 0.96 & 0.92 & 0.87 & 0.88 \end{pmatrix}$$

$$pa_1^4 = \frac{pp_{12}^4 + pp_{21}^4 + pp_{13}^4 + pp_{31}^4 + pp_{14}^4 + pp_{41}^4}{6} =$$

$$= \frac{0.83 + 0.88 + 0.98 + 0.97 + 0.56 + 0.61}{6} = \mathbf{0.8}$$

$$pa_2^4 = \frac{pp_{21}^4 + pp_{12}^4 + pp_{23}^4 + pp_{32}^4 + pp_{24}^4 + pp_{42}^4}{6} =$$

$$= \frac{0.88 + 0.83 + 0.63 + 0.63 + 0.42 + 0.67}{6} = \mathbf{0.67}$$

$$pa_3^4 = \frac{pp_{31}^4 + pp_{13}^4 + pp_{32}^4 + pp_{23}^4 + pp_{34}^4 + pp_{43}^4}{6} =$$

$$= \frac{0.97 + 0.98 + 0.63 + 0.63 + 0.86 + 0.87}{6} = \mathbf{0.82}$$

$$pa_4^4 = \frac{pp_{41}^4 + pp_{14}^4 + pp_{42}^4 + pp_{24}^4 + pp_{43}^4 + pp_{34}^4}{6} =$$

$$= \frac{0.61 + 0.56 + 0.67 + 0.42 + 0.87 + 0.86}{6} = \mathbf{0.66}$$

$$PA^4 = \begin{pmatrix} 0.8 & 0.67 & 0.82 & 0.66 \end{pmatrix}$$

$$pr^1 = \frac{pa_1^1 + pa_2^1 + pa_3^1 + pa_4^1}{4} = \frac{0.8 + 0.85 + 0.9 + 0.82}{4} = \mathbf{0.84}$$

$$pr^2 = \frac{pa_1^2 + pa_2^2 + pa_3^2 + pa_4^2}{4} = \frac{0.88 + 0.87 + 0.85 + 0.92}{4} = \mathbf{0.88}$$

$$pr^3 = \frac{pa_1^3 + pa_2^3 + pa_3^3 + pa_4^3}{4} = \frac{0.96 + 0.92 + 0.87 + 0.88}{4} = \mathbf{0.91}$$

$$pr^4 = \frac{pa_1^4 + pa_2^4 + pa_3^4 + pa_4^4}{4} = \frac{0.8 + 0.67 + 0.82 + 0.66}{4} = \mathbf{0.74}$$

B.9 Controlling Consistency/Consensus State

$$CCL = (1 - \delta) \cdot CL + \delta \cdot CR = (1 - 0.75) \cdot 0.97 + 0.75 \cdot 0.76 = 0.81$$

B.10 Identification of the Preference Values

(1, 1, 2):

$$(1 - \delta) \cdot cl^1 + \delta \cdot pr^1 = (1 - 0.75) \cdot 1.0 + 0.75 \cdot 0.84 = 0.88$$

$$> \gamma, DO\ NOT\ ADD\ TO\ APS'$$

(1, 1, 3):

$$(1 - \delta) \cdot cl^1 + \delta \cdot pr^1 = (1 - 0.75) \cdot 1.0 + 0.75 \cdot 0.84 = 0.88$$

$$> \gamma, DO\ NOT\ ADD\ TO\ APS'$$

(1, 1, 4):

$$(1-\delta)\cdot cl^1 + \delta \cdot pr^1 = (1-0.75)\cdot 1.0 + 0.75\cdot 0.84 = 0.88$$

$$> \gamma, DO\ NOT\ ADD\ TO\ APS'$$

(1, 2, 1):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(1, 2, 3):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(1, 2, 4):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(1, 3, 1):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(1, 3, 2):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(1, 3, 4):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(1, 4, 1):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(1, 4, 2):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(1, 4, 3):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(2, 1, 2):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(2, 1, 3):

$$(1-\delta)\cdot cl^2+\delta\cdot pr^2=(1-0.75)\cdot 0.99+0.75\cdot 0.88=0.91$$

$$>\gamma, DO\ NOT\ ADD\ TO\ APS'$$

(2, 1, 4):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(2, 2, 1):

$$(1-\delta)\cdot cl^2+\delta\cdot pr^2=(1-0.75)\cdot 0.99+0.75\cdot 0.88=0.91$$

$$>\gamma, DO\ NOT\ ADD\ TO\ APS'$$

(2, 2, 3):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(2, 2, 4):

$$(1-\delta)\cdot cl^2+\delta\cdot pr^2=(1-0.75)\cdot 0.99+0.75\cdot 0.88=0.91$$

$$>\gamma, DO\ NOT\ ADD\ TO\ APS'$$

(2, 3, 1):

$$(1-\delta)\cdot cl^2+\delta\cdot pr^2=(1-0.75)\cdot 0.99+0.75\cdot 0.88=0.91$$

$$>\gamma, DO\ NOT\ ADD\ TO\ APS'$$

(2, 3, 2):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(2, 3, 4):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(2, 4, 1):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(2, 4, 2):

$$(1-\delta)\cdot cl^2+\delta\cdot pr^2=(1-0.75)\cdot 0.99+0.75\cdot 0.88=0.91$$
$$>\gamma, DO\ NOT\ ADD\ TO\ APS'$$

(2, 4, 3):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(3, 1, 2):

$$(1-\delta)\cdot cl^3+\delta\cdot pr^3=(1-0.75)\cdot 0.91+0.75\cdot 0.91=0.91$$
$$>\gamma, DO\ NOT\ ADD\ TO\ APS'$$

(3, 1, 3):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(3, 1, 4):

$$(1-\delta)\cdot cl^3+\delta\cdot pr^3=(1-0.75)\cdot 0.91+0.75\cdot 0.91=0.91$$
$$>\gamma, DO\ NOT\ ADD\ TO\ APS'$$

(3, 2, 1):

$$(1-\delta)\cdot cl^3+\delta\cdot pr^3=(1-0.75)\cdot 0.91+0.75\cdot 0.91=0.91$$

$$> \gamma, DO\ NOT\ ADD\ TO\ APS'$$

(3, 2, 3):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(3, 2, 4):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(3, 3, 1):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(3, 3, 2):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(3, 3, 4):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(3, 4, 1):

$$(1-\delta) \cdot cl^3 + \delta \cdot pr^3 = (1-0.75) \cdot 0.91 + 0.75 \cdot 0.91 = 0.91$$
$$> \gamma, DO\ NOT\ ADD\ TO\ APS'$$

(3, 4, 2):

$$(1-\delta) \cdot cl^3 + \delta \cdot pr^3 = (1-0.75) \cdot 0.91 + 0.75 \cdot 0.91 = 0.91$$
$$> \gamma, DO\ NOT\ ADD\ TO\ APS'$$

(3, 4, 3):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(4, 1, 2):

$$MISSING\ VALUE,\ ADD\ TO\ APS'$$

(4, 1, 3):

$$(1-\delta)\cdot cl^4+\delta\cdot pr^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.8$$
$$(1-\delta)\cdot cl_1^4+\delta\cdot pa_1^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.84$$
$$(1-\delta)\cdot cl_{13}^4+\delta\cdot pp_{13}^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.96$$
$$>\gamma, DO\ NOT\ ADD\ TO\ APS'$$

(4, 1, 4):

$$(1-\delta)\cdot cl^4+\delta\cdot pr^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.8$$
$$(1-\delta)\cdot cl_1^4+\delta\cdot pa_1^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.84$$
$$(1-\delta)\cdot cl_{14}^4+\delta\cdot pp_{14}^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.66$$
$$<\gamma,\ ADDED\ TO\ APS$$

(4, 2, 1):

$$(1-\delta)\cdot cl^4+\delta\cdot pr^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.8$$
$$(1-\delta)\cdot cl_2^4+\delta\cdot pa_2^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.75$$
$$(1-\delta)\cdot cl_{21}^4+\delta\cdot pp_{21}^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.91$$
$$>\gamma, DO\ NOT\ ADD\ TO\ APS'$$

(4, 2, 3):

$$(1-\delta)\cdot cl^4+\delta\cdot pr^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.8$$
$$(1-\delta)\cdot cl_2^4+\delta\cdot pa_2^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.75$$
$$(1-\delta)\cdot cl_{23}^4+\delta\cdot pp_{23}^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.71$$
$$<\gamma,\ ADDED\ TO\ APS$$

(4, 2, 4):

$$(1-\delta)\cdot cl^4+\delta\cdot pr^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.8$$
$$(1-\delta)\cdot cl_2^4+\delta\cdot pa_2^4=(1-0.75)\cdot 0.97+0.75\cdot 0.74=0.75$$

$$(1-\delta)\cdot cl^4_{24} + \delta \cdot pp^4_{24} = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.56$$
$$< \gamma,\ \textit{ADDED TO APS}$$

(4, 3, 1):

$$(1-\delta)\cdot cl^4 + \delta \cdot pr^4 = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.8$$
$$(1-\delta)\cdot cl^4_3 + \delta \cdot pa^4_3 = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.86$$
$$> \gamma, \textit{DO NOT ADD TO APS}'$$

(4, 3, 2):

$$(1-\delta)\cdot cl^4 + \delta \cdot pr^4 = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.8$$
$$(1-\delta)\cdot cl^4_3 + \delta \cdot pa^4_3 = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.86$$
$$> \gamma, \textit{DO NOT ADD TO APS}'$$

(4, 3, 4):

$$(1-\delta)\cdot cl^4 + \delta \cdot pr^4 = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.8$$
$$(1-\delta)\cdot cl^4_3 + \delta \cdot pa^4_3 = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.86$$
$$> \gamma, \textit{DO NOT ADD TO APS}'$$

(4, 4, 1):

$$(1-\delta)\cdot cl^4 + \delta \cdot pr^4 = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.8$$
$$(1-\delta)\cdot cl^4_4 + \delta \cdot pa^4_4 = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.74$$
$$(1-\delta)\cdot cl^4_{41} + \delta \cdot pp^4_{41} = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.69$$
$$< \gamma,\ \textit{ADDED TO APS}$$

(4, 4, 2):

$$(1-\delta)\cdot cl^4 + \delta \cdot pr^4 = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.8$$
$$(1-\delta)\cdot cl^4_4 + \delta \cdot pa^4_4 = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.74$$
$$(1-\delta)\cdot cl^4_{42} + \delta \cdot pp^4_{42} = (1-0.75)\cdot 0.97 + 0.75\cdot 0.74 = 0.73$$

$$< \gamma,\ ADDED\ TO\ APS$$

(4, 4, 3):

$$(1-\delta)\cdot cl^4 + \delta \cdot pr^4 = (1-0.75)\cdot 0.97 + 0.75 \cdot 0.74 = 0.8$$

$$(1-\delta)\cdot cl^4_4 + \delta \cdot pa^4_4 = (1-0.75)\cdot 0.97 + 0.75 \cdot 0.74 = 0.74$$

$$(1-\delta)\cdot cl^4_{43} + \delta \cdot pp^4_{43} = (1-0.75)\cdot 0.97 + 0.75 \cdot 0.74 = 0.9$$

$$> \gamma, DO\ NOT\ ADD\ TO\ APS'$$

$$APS = \{(1,2,1),(1,2,3),(1,2,4),(1,3,1),(1,3,2),(1,3,4),(1,4,1),(1,4,2),(1,4,3),$$

$$(2,1,2),(2,1,4),(2,2,3),(2,3,2),(2,3,4),(2,4,1),(2,4,3),(3,1,3),(3,2,3),(3,2,4),$$

$$(3,3,1),(3,3,2),(3,3,4),(3,4,3),(4,1,2),(4,1,4),(4,2,3),(4,2,4),(4,4,1),(4,4,2)\}$$

B.11 Generation of Advice

(1, 2, 1):

$$rp^1_{21} = (1-\delta)\cdot cp^1_{21} + \delta \cdot p^c_{21} = (1-0.75)\cdot 0.8 + 0.75 \cdot 0.52 = \mathbf{0.59} \quad ; \quad p^1_{21} \in MV^1$$

To expert e_1: You should provide a value for (2, 1) near to 0.59

(1, 2, 3):

$$rp^1_{23} = (1-\delta)\cdot cp^1_{23} + \delta \cdot p^c_{23} = (1-0.75)\cdot 0.9 + 0.75 \cdot 0.77 = \mathbf{0.8} \quad ; \quad p^1_{23} \in MV^1$$

To expert e_1: You should provide a value for (2, 3) near to 0.8

(1, 2, 4):

$$rp^1_{24} = (1-\delta)\cdot cp^1_{24} + \delta \cdot p^c_{24} = (1-0.75)\cdot 0.7 + 0.75 \cdot 0.78 = \mathbf{0.76} \quad ; \quad p^1_{24} \in MV^1$$

To expert e_1: You should provide a value for (2, 4) near to 0.76

(1, 3, 1):

$$rp^1_{31} = (1-\delta)\cdot cp^1_{31} + \delta \cdot p^c_{31} = (1-0.75)\cdot 0.4 + 0.75 \cdot 0.47 = \mathbf{0.45} \quad ; \quad p^1_{31} \in MV^1$$

To expert e_1: You should provide a value for (3, 1) near to 0.45

(1, 3, 2):

$$rp^1_{32} = (1-\delta)\cdot cp^1_{32} + \delta \cdot p^c_{32} = (1-0.75)\cdot 0.1 + 0.75 \cdot 0.23 = \mathbf{0.2} \;\; ; \;\; p^1_{32} \in MV^1$$

To expert e_1: You should provide a value for (3, 2) near to 0.2

(1, 3, 4):

$$rp^1_{34} = (1-\delta)\cdot cp^1_{34} + \delta \cdot p^c_{34} = (1-0.75)\cdot 0.3 + 0.75 \cdot 0.44 = \mathbf{0.41} \;\; ; \;\; p^1_{34} \in MV^1$$

To expert e_1: You should provide a value for (3, 4) near to 0.41

(1, 4, 1):

$$rp^1_{41} = (1-\delta)\cdot cp^1_{41} + \delta \cdot p^c_{41} = (1-0.75)\cdot 0.6 + 0.75 \cdot 0.31 = \mathbf{0.38} \;\; ; \;\; p^1_{41} \in MV^1$$

To expert e_1: You should provide a value for (4, 1) near to 0.38

(1, 4, 2):

$$rp^1_{42} = (1-\delta)\cdot cp^1_{42} + \delta \cdot p^c_{42} = (1-0.75)\cdot 0.3 + 0.75 \cdot 0.37 = \mathbf{0.35} \;\; ; \;\; p^1_{42} \in MV^1$$

To expert e_1: You should provide a value for (4, 2) near to 0.35

(1, 4, 3):

$$rp^1_{43} = (1-\delta)\cdot cp^1_{43} + \delta \cdot p^c_{43} = (1-0.75)\cdot 0.7 + 0.75 \cdot 0.57 = \mathbf{0.6} \;\; ; \;\; p^1_{43} \in MV^1$$

To expert e_1: You should provide a value for (4, 3) near to 0.6

(2, 1, 2):

$$rp^2_{12} = (1-\delta)\cdot cp^2_{12} + \delta \cdot p^c_{12} = (1-0.75)\cdot 0.62 + 0.75 \cdot 0.43 = \mathbf{0.48} \;\; ; \;\; p^2_{12} \in MV^2$$

To expert e_2: You should provide a value for (1, 2) near to 0.48

(2, 1, 4):

$$rp^2_{14} = (1-\delta)\cdot cp^2_{14} + \delta \cdot p^c_{14} = (1-0.75)\cdot 0.77 + 0.75 \cdot 0.74 = \mathbf{0.75} \;\; ; \;\; p^2_{14} \in MV^2$$

To expert e_2: You should provide a value for (1, 4) near to 0.75

(2, 2, 3):

$$rp^2_{23} = (1-\delta)\cdot cp^2_{23} + \delta \cdot p^c_{23} = (1-0.75)\cdot 0.6 + 0.75 \cdot 0.77 = \mathbf{0.73} \quad ; \quad p^2_{23} \in MV^2$$

To expert e_2: You should provide a value for (2, 3) near to 0.73

(2, 3, 2):

$$rp^2_{32} = (1-\delta)\cdot cp^2_{32} + \delta \cdot p^c_{32} = (1-0.75)\cdot 0.43 + 0.75 \cdot 0.23 = \mathbf{0.28} \quad ; \quad p^2_{32} \in MV^2$$

To expert e_2: You should provide a value for (3, 2) near to 0.28

(2, 3, 4):

$$rp^2_{34} = (1-\delta)\cdot cp^2_{34} + \delta \cdot p^c_{34} = (1-0.75)\cdot 0.57 + 0.75 \cdot 0.44 = \mathbf{0.47} \quad ; \quad p^2_{34} \in MV^2$$

To expert e_2: You should provide a value for (3, 4) near to 0.47

(2, 4, 1):

$$rp^2_{41} = (1-\delta)\cdot cp^2_{41} + \delta \cdot p^c_{41} = (1-0.75)\cdot 0.25 + 0.75 \cdot 0.31 = \mathbf{0.29} \quad ; \quad p^2_{41} \in MV^2$$

To expert e_2: You should provide a value for (4, 1) near to 0.29

(2, 4, 3):

$$rp^2_{43} = (1-\delta)\cdot cp^2_{43} + \delta \cdot p^c_{43} = (1-0.75)\cdot 0.45 + 0.75 \cdot 0.57 = \mathbf{0.54} \quad ; \quad p^2_{43} \in MV^2$$

To expert e_2: You should provide a value for (4, 3) near to 0.54

(3, 1, 3):

$$rp^3_{13} = (1-\delta)\cdot cp^3_{13} + \delta \cdot p^c_{13} = (1-0.75)\cdot 0.51 + 0.75 \cdot 0.58 = \mathbf{0.56} \quad ; \quad p^3_{13} \in MV^3$$

To expert e_3: You should provide a value for (1, 3) near to 0.56

(3, 2, 3):

$$rp^3_{23} = (1-\delta)\cdot cp^3_{23} + \delta \cdot p^c_{23} = (1-0.75)\cdot 0.61 + 0.75 \cdot 0.77 = \mathbf{0.73} \quad ; \quad p^3_{23} \in MV^3$$

To expert e_3: You should provide a value for (2, 3) near to 0.73

(3, 2, 4):

$$rp^3_{24} = (1-\delta)\cdot cp^3_{24} + \delta \cdot p^c_{24} = (1-0.75)\cdot 0.89 + 0.75 \cdot 0.78 = \mathbf{0.81} \quad ; \quad p^3_{24} \in MV^3$$

To expert e_3: You should provide a value for (2, 4) near to 0.81

(3, 3, 1):

$$rp_{31}^3 = (1-\delta) \cdot cp_{31}^3 + \delta \cdot p_{31}^c = (1-0.75) \cdot 0.48 + 0.75 \cdot 0.47 = \mathbf{0.47} \quad ; \quad p_{31}^3 \in MV^3$$

To expert e_3: You should provide a value for (3, 1) near to 0.47

(3, 3, 2):

$$rp_{32}^3 = (1-\delta) \cdot cp_{32}^3 + \delta \cdot p_{32}^c = (1-0.75) \cdot 0.42 + 0.75 \cdot 0.23 = \mathbf{0.28} \quad ; \quad p_{32}^3 \in MV^3$$

To expert e_3: You should provide a value for (3, 2) near to 0.28

(3, 3, 4):

$$rp_{34}^3 = (1-\delta) \cdot cp_{34}^3 + \delta \cdot p_{34}^c = (1-0.75) \cdot 0.64 + 0.75 \cdot 0.44 = \mathbf{0.49} \quad ; \quad p_{34}^3 \in MV^3$$

To expert e_3: You should provide a value for (3, 4) near to 0.49

(3, 4, 3):

$$rp_{43}^3 = (1-\delta) \cdot cp_{43}^3 + \delta \cdot p_{43}^c = (1-0.75) \cdot 0.41 + 0.75 \cdot 0.57 = \mathbf{0.53} \quad ; \quad p_{43}^3 \in MV^3$$

To expert e_3: You should provide a value for (4, 3) near to 0.53

(4, 1, 2):

$$rp_{12}^4 = (1-\delta) \cdot cp_{12}^4 + \delta \cdot p_{12}^c = (1-0.75) \cdot 0.6 + 0.75 \cdot 0.43 = \mathbf{0.47} \quad ; \quad p_{12}^4 \in MV^4$$

To expert e_4: You should provide a value for (1, 2) near to 0.47

(4, 1, 4):

$$rp_{14}^4 = (1-\delta) \cdot cp_{14}^4 + \delta \cdot p_{14}^c = (1-0.75) \cdot 0.35 + 0.75 \cdot 0.74 = \mathbf{0.64} \quad ; \quad p_{14}^4 \in EV^4$$

To expert e_4: You should change your preference value (1, 4) near to 0.64

(4, 2, 3):

$$rp_{23}^4 = (1-\delta) \cdot cp_{23}^4 + \delta \cdot p_{23}^c = (1-0.75) \cdot 0.45 + 0.75 \cdot 0.77 = \mathbf{0.69} \quad ; \quad p_{23}^4 \in EV^4$$

To expert e_4: You should change your preference value (2, 3) near to 0.69

(4, 2, 4):

$$rp^4_{24} = (1-\delta)\cdot cp^4_{24} + \delta \cdot p^c_{24} = (1-0.75)\cdot 0.2 + 0.75 \cdot 0.78 = \mathbf{0.64} \quad ; \quad p^4_{24} \in EV^4$$

To expert e_4: You should change your preference value (2, 4) near to 0.64

(4, 4, 1):

$$rp^4_{41} = (1-\delta)\cdot cp^4_{41} + \delta \cdot p^c_{41} = (1-0.75)\cdot 0.65 + 0.75 \cdot 0.31 = \mathbf{0.39} \quad ; \quad p^4_{41} \in EV^4$$

To expert e_4: You should change your preference value (4, 1) near to 0.39

(4, 4, 2):

$$rp^4_{42} = (1-\delta)\cdot cp^4_{42} + \delta \cdot p^c_{42} = (1-0.75)\cdot 0.8 + 0.75 \cdot 0.37 = \mathbf{0.48} \quad ; \quad p^4_{42} \in EV^4$$

To expert e_4: You should change your preference value (4, 2) near to 0.48

Bibliography

[1] Efe A.Ok., *Utility representation of an incomplete preference relation*, Journal of Economic Theory **104** (2002), 429–449.

[2] R.E. Bellman and Zadeh L.A., *Decision-making in a fuzzy environment*, Managment Science **17** (1970), 53–79.

[3] H.K. Bhargava, D.J. Power, and D. Sun, *Progress in web-based decision support technologies*, Decision Support Systems **In Press**.

[4] T. Bilgiç, *Interval-valued preference structures*, European Journal of Operational Research **105** (1998), 162–183.

[5] G. Bordogna, M. Fedrizzi, and G. Pasi, *A linguistic modeling of consensus in group decision making based on owa operators*, IEEE Transactions on Systems, Man and Cybernetics, Part A **27** (1997), 126–133.

[6] N. Bryson, *Group decision-making and the analytic hierarchy process: exploring the consensus-relevant information content*, European Journal of Operational Research **23** (1996), 27–35.

[7] C. Carlsson, D. Ehrenberg, P. Eklund, M. Fedrizzi, P. Gustafsson, P. Lindholm, G. Merkuryeva, T. Riissanen, and A.G.S. Ventre, *Consensus in distributed soft environments*, European Journal of Operational Research **61** (1992), 165–185.

[8] Z. Chen, *Interacting with software components*, Decision Support Systems **14** (1995), 349–357.

[9] F. Chiclana, F. Herrera, and E. Herrera-Viedma, *Integrating multiplicative preference relations in a multiplicative decision making model based on fuzzy preference relations*, Fuzzy Sets and Systems **112** (1998), 277–291.

[10] F. Chiclana, F. Herrera, and E. Herrera-Viedma, *Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations*, Fuzzy Sets and Systems **97** (1998), 33–48.

[11] ———, *A note on the internal consistency of various preference representations*, Fuzzy Sets and Systems **131:1** (2002), 75–78.

[12] ———, *Rationality of induced ordered weighted operators based on the reliability of the source of information in group decision-making*, Kybernetika **40** (2004), 121–142.

[13] F. Chiclana, F. Herrera, E. Herrera-Viedma, and L. Martínez, *A note on the reciprocity in the aggregation of fuzzy preference relations using owa operators*, Fuzzy Sets Systems **137:1** (2003), 71–83.

[14] A.K. Choudhury, R. Shankar, and M.K. Tiwari, *Consensus-based intelligent group decision-making model for the selection of advanced technology*, Decision Support Systems (In press).

[15] J. Dombi, *Fuzzy logic and soft computing*, ch. A General Framework for the Utility-Based and Outranking Methods, pp. 202–

208, Bouchon, B.B. et al. (Eds.) (World Ccientific), London, 1995.

[16] D. Dubois and J.L. Koning, *Social choice axioms for fuzzy set aggregation*, Fuzzy Sets and Systems **43** (1991), 257–274.

[17] D. Dubois and H. Prade, *A review of fuzzy sets aggregation connectives*, Information Sciences **36** (1985), 85–121.

[18] D. Dubois, H. Prade, and C. Testemale, *Weighted fuzzy patter matching*, Fuzzy Sets and Systems **28** (1988), 313–331.

[19] E. Ephrati and J.S. Rosenschein, *Deriving consensus in multiagent systems*, Artificial Intelligence **87** (1996), 21–74.

[20] Z.-P. Fan, S.-H. Xiao, and G.-F. Hu, *An optimization method for integrating two kinds of preference information in group decision-making*, Computers & Industrial Engineering **46** (2004), 329–335.

[21] M. Fedrizzi and R.A. Marques Pereira, *Soft consensus and network dynamics in group decision making*, International Journal of Intelligent Systems **14:1** (1999), 63–77.

[22] J. Fodor and M. Roubens, *Fuzzy preference modelling and multicriteria decision support*, Kluwer Academic Publishers, Dordrecht, 1994.

[23] J. González-Pachón, D. Gómez, J. Montero, and J. Yáñez, *Searching for the dimension of valued preference relations*, International Journal of Approximate Reasoning **33:2** (2003), 133–157.

[24] F. Herrera and E. Herrera-Viedma, *Aggregation operators for linguistic weighted information*, IEEE Transactions on Systems,

Man, and Cybernetics Part. A: Systems and Humans **27** (1997), 646–656.

[25] ———, *Choice functions and mechanisms for linguistic preference relations*, European Journal of Operational Research **120** (2000), 144–161.

[26] F. Herrera, E. Herrera-Viedma, and Verdegay J.L., *A model of consensus in group decision making under linguistic assessments*, Fuzzy Sets and Systems **78** (1996), 73–87.

[27] ———, *Choice processes for non-homogeneous group decision making in linguistic setting*, Fuzzy Sets and Systems **94** (1998), 287–308.

[28] F. Herrera, E. Herrera-Viedma, and J.L. Verdegay, *Direct approach processes in group decision making using linguistic owa operators*, Fuzzy Sets and Systems **79** (1996), 175–190.

[29] F. Herrera, L. Martínez, and P. J. Sánchez, *Managing non-homogeneous information in group decision making*, European Journal of Operational Research **166** (2005), 115–132.

[30] F. Herrera, L. Martinez, and P.J. Sánchez, *Managing non-homogeneous information in group decision making*, European Journal of Operational Research **166** (2005), 115–132.

[31] E. Herrera-Viedma, F. Chiclana, F. Herrera, and S. Alonso, *A group decision making model with incomplete fuzzy preference relations based on additive consistency*, Technical Report #SCI2S-2004-11, University of Granada

http://sci2s.ugr.es/publications/ficheros/TechnicalReportSCI2S-2004-11.pdf (2004).

[32] E. Herrera-Viedma, F. Herrera, and F. Chiclana, *A consensus model for multiperson decision making with different preference structures*, IEEE Transactions on Systems, Man and Cybernetics. Part A: Systems and Humans **32:3** (2002), 394–402.

[33] E. Herrera-Viedma, F. Herrera, F. Chiclana, and M. Luque, *Some issues on consistency of fuzzy preference relations*, European Journal of Operational Research **154** (2004), 98–109.

[34] E. Herrera-Viedma, L. Martinez, F. Mata, and F. Chiclana, *A consensus support system model for group decision-making problems with multi-granular linguistic preference relations*, IEEE Transactions on Fuzzy Systems **In Press** (2005).

[35] E. Herrera-Viedma, F. Mata, L. Martínez, and F. Chiclana, *A consensus support system model for group decision-making problems with multi-granular linguistic preference relations*, IEEE Transactions on Fuzzy Systems **13:5** (2005), 644–658.

[36] J. Kacprzyk, *Group decision making with a fuzzy linguistic majority*, Fuzzy Sets and Systems **18** (1986), 105–118.

[37] J. Kacprzyk and M. Fedrizzi, *A "soft" measure of consensus in the setting of partial (fuzzy) preferences*, European Journal of Operational Research **34** (1988), 316–325.

[38] ———, *Multiperson decision making models using fuzzy sets and possibility theory*, Kluwer Academic Publishers, Dordrecht, 1990.

[39] J. Kacprzyk, M. Fedrizzi, and H. Nurmi, *Group decision making and consensus under fuzzy preferences and fuzzy majority*, Fuzzy Sets and Systems **49** (1992), 21–31.

[40] J. Kacprzyk, H. Nurmi, and M. (Eds.) Fedrizzi, *Consensus under fuzziness*, Kluwer Academic Publishers, Massachusetts, 1997.

[41] J. Kacprzyk and M. Roubens, *Non-conventional preference relations in decision making*, Springer-Verlag, 1988.

[42] C. Kahraman, D. Ruan, and I. Dogan, *Fuzzy group decision-making for facility location selection*, Infomation Sciences **157** (2003), 135–153.

[43] W.J.M. Kickert, *Fuzzy theories on decision making*, Nijhoff, 1978.

[44] S. H. Kim, S. H. Choi, and J. K. Kim, *An interactive procedure for multiple attribute group decision making with incomplete information: range-based approach*, European Journal of Operational Research **118** (1999), 139–152.

[45] S.H. Kim and B.S. Ahn, *Interactive group decision making procedure under incomplete information*, European Journal of Operational Research **116** (1999), 498–507.

[46] S.H. Kim and Ahn B.S., *Group decision making procedure considering preference strenght under incomplete information*, Computers and Operations Research **24** (1997), 1101–1112.

[47] L. Kitainick, *Fuzzy decision procedures with binary relations, towards an unified theory*, Kluwer Academic Publishers, Dordrecht, 1993.

[48] ______, *Fuzzy decision procedures with binary relations*, Kluwer Academic Publishers, Dordrecht, 1995.

[49] L.I. Kuncheva, *Five measures of consensus in group decision making using fuzzy sets*, Proceedings IFSA, 1991, pp. 141–144.

[50] R.D. Luce and P. Suppes, *Handbook of mathematical psychology, vol. iii*, ch. Preferences, Utility and Subject Probability, pp. 249–410, Luce, R.D. et al. (Eds.) (Wiley), New York, 1965.

[51] C. Macharis, J. Springael, K. De Brucker, and A. Verbeke, *Promethee and ahp: The design of operational synergies in multicriteria analysis: Strengthening promethee with ideas of ahp*, European Journal of Operational Research **153:2** (2004), 307–317.

[52] G.A. Miller, *The magical number seven plus or minus two: some limits on our capacity of processing information*, Psychological Review **63** (1956), 81–97.

[53] H.B. Mitchell and D.D. Estrakh, *A modified owa operator and its use in lossless dpcm image compression*, International Journal of Uncertainty and Fuzzy Knowledge-Based Systems **5** (1997), 429–436.

[54] H. Nurmi, *Non-conventional preference relations in decision making*, ch. Assumptions of Individual Preferences in the Theory of Voting Procedures, pp. 142–155, Kacprzyk, J. and Roubens, M. (Eds.) (Springer-Verlag), Berlin, 1988.

[55] S.A. Orlovski, *Decision-making with fuzzy preference relations*, Fuzzy Sets and Systems **1** (1978), 155–167.

[56] Zhang S. and S. Goddard, *A software architecture and framework for web-based distributed decision support systems*, Decision Support Systems **In Press**.

[57] Th.L. Saaty, *The analytic hierarchy process*, MacGraw-Hill, New York, 1980.

[58] ______, *Fundamentals of decision making and priority theory with the ahp*, RWS Publications, Pittsburg, 1994.

[59] Th.L. Saaty and L. Vargas, *Models, methods, concepts and applications of the analytic hierarchy process*, Kluwer, Boston, 2000.

[60] S. Saint and J. R. Lawson, *Rules for reaching consensus. a modern approach to decision making*, Jossey-Bass, San Francisco, 1994.

[61] F. Seo and M. Sakawa, *Fuzzy multiattribute utility analysis for collective choice*, IEEE Transactions on Systems, Man, and Cybernetics **15** (1985), 45–53.

[62] E. Szmidt and J. Kacprzyk, *A consensus-reaching process under intuitionistic fuzzy preference relations*, International Journal of Intelligent Systems **18** (2003), 837–852.

[63] T. Tanino, *Fuzzy preference orderings in group decision making*, Fuzzy Sets and Systems **12** (1984), 117–131.

[64] ______, *Non-conventional preference relations in decision making*, ch. Fuzzy preference relations in group decision making, pp. 54–71, Kacprzyk, J. and Roubens, M. (Eds.) (Springer-Verlag), Berlin, 1988.

[65] ______, *Multiperson decision making using fuzzy sets and possibility theory*, ch. On Group Decision Making Under Fuzzy Preferences,

pp. 172–185, Kacprzyk, J. and Fedrizzi, M. (Eds.) (Kluwer Academic Publishers), 1990.

[66] Z. Witalski, *General transitivity conditions for fuzzy reciprocal preference matrices*, Fuzzy Sets and Systems **137:1** (2003), 85–100.

[67] Z.S. Xu, *Goal programming models for obtaining the priority vector of incomplete fuzzy preference relation*, International Journal of Approximate Reasoning **36:3** (2004), 261–270.

[68] ______, *A method based on linguistic aggregation operators for group decision making with linguistic preference relations*, Information Sciences **166:1-4** (2004), 19–30.

[69] ______, *On compatibility of interval fuzzy preference relations*, Fuzzy Optimization and Decision Making **3** (2004), 217–225.

[70] R.R. Yager, *Fuzzy decision making including unequal objectives*, Fuzzy Sets and Systems **1** (1978), 87–95.

[71] ______, *Quantifiers in the formulation of multiple objective decision functions*, Information Sciences **31** (1983), 107–139.

[72] ______, *On ordered weighted averaging aggregation operators in multicriteria decision making*, IEEE Transactions on Systems, Man and Cybernetics **18:1** (1988), 183–190.

[73] ______, *Families of owa operators*, Fuzzy Sets and Systems **59** (1993), 125–148.

[74] ______, *Quantifier guided aggregation using owa operators*, International Journal of Intelligent Systems **11** (1996), 49–73.

[75] ______, *Induced aggregation operators*, Fuzzy Sets and Systems **137** (2003), 59–69.

[76] R.R. Yager and D.P. Filev, *Operations for granular computing: mixing words and numbers*, Proceedings of the FUZZ-IEEE World Congress on Computational Intelligence, Anchorage, 1998, pp. 123–128.

[77] ______, *Induced ordered weighted averaging operators*, IEEE Transactions on Systems, Man and Cybernetics **29** (1999), 141–150.

[78] L.A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), no. 3, 338–353.

[79] ______, *The concept of a linguistic variable and its applications to approximate reasoning, part i*, Information Sciences **8** (1975), 199–249.

[80] ______, *A computational approach to fuzzy quantifiers in natural languages*, Computers & Mathematics with Applications **9** (1983), 149–184.

[81] S. Zadrozny, *An approach to the consensus reaching support in fuzzy environment*, ch. Stemming Algorithms, pp. 83–109, Kacprzyk, J. and Nurmi, H. and Fedrizzi, M., Eds. (Kluwer Academic Publishers), Massachusetts, 1997.

[82] Q. Zhang, J.C.H. Chen, and P.P. Chong, *Decision consolidation: criteria weight determination using multiple preference formats*, Decision Support Systems **38** (2004), 247–258.

[83] Q. Zhang, J.C.H. Chen, Y.-Q. He, J. Ma, and D.-N. Zhou, *Multiple attribute decision making: approach integrating subjective and ibective information*, International Journal of Manufacturing Technology and Management **5** (2003), 338–361.