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Teoría de Representación de Coálgebras. Localización en Coálgebras

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Teoría de Representación de Coálgebras. Localización en Coálgebras

por

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V. B.
El director.

El aspirante al grado.

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Introducción

El problema de clasificar un objeto matemático aparece en todas las áreas de las Matemáticas. Normalmente, la clasificación se obtiene de una forma más sencilla cuando representamos dicho objeto por uno más simple; éste es el caso de la teoría de representación de álgebras. Muchos de los progresos actuales en teoría de representación de álgebras (finito dimensionales sobre un cuerpo algebraicamente cerrado) usan las técnicas teóricas sobre quivers formuladas por P. Gabriel y su escuela en los años setenta, ver por ejemplo las referencias: [ASS05], [ARS95] y [GR92]. El origen de este método se puede situar en el conocido teorema de Gabriel: *Toda álgebra basica de dimensión finita, sobre un cuerpo algebraicamente cerrado K, es isomorfa a un cociente KQ/I, donde KQ es el álgebra de caminos del quiver Q e I es un ideal admisible de KQ.*

Este resultado nos da una descripción explícita de no sólo cualquier álgebra finito dimensional, y así como de su categoría de módulos finitamente generados mediante representaciones lineales del quiver asociado al álgebra.

Sin embargo una importante restricción condiciona el teorema anterior: el álgebra debe ser de dimensión finita. Por tanto, de manera natural se plantea la siguiente pregunta: +es posible generalizar este resultado para cualquier álgebra (de dimensión infinita)? Una ligera comprobación nos convence que no es posible utilizar la misma demostración: los vértices del quiver corresponden con los idempotentes primitivos del álgebra; y si el quiver tiene un número infinito de ellos, entonces el álgebra de caminos no tiene elemento unidad.

En este contexto, la teoría de coalgebras aparece en un nivel intermedio de dificultad. La estructura de coálgebra se obtiene invirtiendo las aplicaciones que definen la estructura de álgebra, esto es, una coálgebra C verifica los siguientes diagramas conmutativos

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes I"] \\
C \otimes C \arrow[r, "I \otimes \Delta"] & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C \otimes C \arrow[dl, "I \otimes \epsilon"'] \arrow[dr, "\epsilon \otimes I"] & \\
C \otimes K & & K \otimes C \\
 & C \arrow[ul, "\cong"] \arrow[uu, "\Delta"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

(Coasociatividad) (Counidad)

donde Δ y ϵ (la comultiplicación y la counidad, respectivamente) son aplicaciones lineales que corresponden con las nociones duales de multiplicación y unidad en álgebras.

Se debería esperar que el espacio dual de un álgebra fuera una coálgebra y vice-versa. Esto es cierto si los espacios vectoriales son finito dimensionales (aunque no es cierto en general), por lo que la categoría de álgebras finito dimensionales es equivalente a la categoría de coálgebras de dimensión finita. Por el teorema de estructura de las coálgebras, toda coálgebra es una unión directa de sus subcoálgebras finito dimensionales, es decir, es una unión directa de álgebras de dimensión finita. Por lo que la teoría de representación de coálgebras podría pensarse como un paso intermedio entre el estudio de las álgebras de dimensión finita y las álgebras de dimensión infinita.

La categoría de comódulos sobre una coálgebra es una categoría abeliana localmente finita y, por tanto, es posible utilizar en ella ciertas herramientas que no son validas en una categoría de módulos general. En particular, es de tipo finito y entonces se puede pensar en encontrar una teoría para coálgebras análoga a la existente para álgebras de dimensión finita. Este trabajo está dedicado a desarrollar este objetivo, es decir, describir coálgebras y su categoría de comódulos mediante quivers y representaciones lineales de quivers. Para conseguir este propósito, el primer paso debe ser obtener una versión del teorema de Gabriel, antes mencionado, para coálgebras.

Siguiendo esta idea, podemos dotar a un álgebra de caminos KQ con una estructura de K-coálgebra graduada con comultiplicación inducida por la descomposición de caminos, esto es, si $p = \alpha_m \cdots \alpha_1$ es un camino no trivial desde un vértice i a un vértice j, entonces

$$\Delta(p) = e_j \otimes p + p \otimes e_i + \sum_{i=1}^{m-1} \alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 = \sum_{\eta\tau = p} \eta \otimes \tau$$

y $\Delta(e_i) = e_i \otimes e_i$ para un camino trivial e_i. La counidad de KQ se define como

$$\epsilon(\alpha) = \begin{cases} 1 & \text{si } \alpha \text{ es un vértice,} \\ 0 & \text{si } \alpha \text{ es un camino no trivial.} \end{cases}$$

Esta coálgebra es conocida como la *coálgebra de caminos* del quiver Q.

En [Woo97], el autor demuestra que toda coálgebra punteada es isomorfa a una subcoálgebra de una coálgebra de caminos. Además, contiene la subcoálgebra generada por todos los vértices y todas las flechas, esto es, es una subcoálgebra admisible. Más tarde, en [Sim01], se define la noción de coálgebra de caminos de un quiver con relaciones (Q, Ω) como el subespacio de KQ dado por

$$C(Q, \Omega) = \{a \in KQ \mid \langle a, \Omega \rangle = 0\},$$

donde $\langle -, - \rangle : KQ \times KQ \longrightarrow K$ es la aplicación bilinear definida por $\langle v, w \rangle = \delta_{v,w}$ (la delta de Kronecker) para cualesquiera dos caminos v, w en Q.

Una de las razones expuestas en [Sim01] y [Sim05] para escribir una coálgebra básica C de la forma $C(Q, \Omega)$ es el hecho de que, en este caso, existe una equivalencia K-lineal de la categoría $\mathcal{M}_f^C$ de los C-comódulos derecha de dimensión finita con la categoría $\mathrm{nilrep}_K^{lf}(Q, \Omega)$ de las representaciones K-lineales nilpotentes de longitud finita del quiver con relaciones (Q, Ω) (ver [Sim01, p. 135] y [Sim05, Theorem 3.14]). Entonces, esta definición es consistente con la teoría clásica y reduce el estudio de la categoría $\mathcal{M}^C$ al estudio de las representaciones lineales del quiver con relaciones (Q, Ω).

Así pues, se plantea el siguiente problema: *¿es toda coálgebra básica, sobre un cuerpo algebraicamente cerrado, isomorfa a una coálgebra de caminos de un quiver con relaciones?*

En el Capítulo 2 tratamos este problema. La clase de todas las subcoálgebras admisibles de una coálgebra de caminos puede dividirse en dos subclases, dependiendo de si la coálgebra está generada por caminos o no. Obviamente, la primera clase es fácil de estudiar y entonces en necesario centrarse únicamente en la segunda. Para ello, establecemos un ambiente más general considerando la topología débil* en el álgebra dual para tratar el problema en un contexto elemental. En particular, la coálgebra de caminos de un quiver con relaciones (Q, Ω) es descrita como el espacio ortogonal $\Omega^{\perp}$ del ideal Ω. Entonces el anterior problema puede ser formulado como sigue: *¿existe para toda subcoálgebra admisible $C \leq KQ$, un ideal con relaciones Ω del álgebra KQ tal que $\Omega^{\perp} = C$?* Desgraciadamente, esto no es cierto y mostramos un ejemplo de una subcoálgebra admisible que no es el espacio ortogonal de ningún ideal de KQ. Ante esta situación, se hace necesario un criterio que permita decidir cuando una subcoálgebra admisible es la coálgebra de caminos de un quiver con relaciones. En la última sección del Capítulo 2 probamos el siguiente resultado:

Criterio (2.5.11). *Sea C una subcoálgebra admisible de una coálgebra de caminos KQ. Entonces C no es la coálgebra de caminos de un quiver con relaciones si, y sólo si, existe un número infinito de caminos $\{\gamma_i\}_{i\in\mathbb{N}}$ en Q verificando las siguientes condiciones:*

(a) *Todos tienen mismo origen y mismo final.*

(b) *Ninguno de ellos pertenece a C.*

(c) *Existen escalares $a_j^n \in K$ para todo $j, n \in \mathbb{N}$ tal que el conjunto $\{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$ está contenido en C.*

Es bien conocido que la clase de las álgebras de dimensión finita, sobre un cuerpo algebraicamente cerrado, es unión disjunta de dos clases de álgebras: las álgebras tame y las álgebras wild. Esto se conoce como la dicotomía tame-wild, ver [Dro79]. La idea de tales clases es que la categoría de módulos finitamente generados sobre un álgebra wild es tan grande que contiene la categoría de módulos finitamente generados de cualquier álgebra de dimensión finita. Por tanto, no es razonable el propósito de describir completamente su categoría de módulos y la teoría se restringe únicamente a coálgebras tame. De forma dual D. Simson define en [Sim01] y [Sim05] los conceptos análogos para una coálgebra básica (punteada). Además, demuestra la versión débil de la dicotomía tame-wild (la versión completa es todavía un problema abierto): *Sea K un cuerpo algebraicamente cerrado. Entonces toda K-coálgebra tame no es wild.*

El contraejemplo dado en el Capítulo 2, que muestra que no toda subcoálgebra admisible es una coálgebra de caminos de un quiver con relaciones, es una coálgebra wild. Más aún, por el criterio anterior, una coálgebra con dicha propiedad necesita que el quiver asociado tenga un número infinito de caminos entre dos puntos, y entonces parece cercana a ser wild. Por tanto, debemos reformular el problema de la siguiente manera: *¿toda coálgebra básica, sobre un cuerpo algebraicamente cerrado, que no es wild, es isomorfa a una coálgebra de caminos de un quiver con relaciones?*. En caso de ser cierto, esto implicaría que *toda coálgebra básica tame, sobre un cuerpo algebraicamente cerrado, es isomorfa a una coálgebra de caminos de un quiver con relaciones*. Podemos decir más, puesto que, en tal caso, la demostración de la dicotomía tame-wild quedaría reducida a coálgebras de caminos de quivers con relaciones.

Para atender este problema, en el Capítulo 3 desarrollamos la noción de localización en coálgebras. La categoría de comódulos a derecha $\mathcal{M}^C$ sobre una coálgebra C es una categoría de Grothendieck localmente finita en la que la teoría de localización descrita por Gabriel en [Gab62] puede ser aplicada. La idea principal es que, para cualquier subcategoría densa $\mathcal{T} \subseteq \mathcal{M}^C$, podemos construir una categoría cociente, $\mathcal{M}^C/\mathcal{T}$, y un funtor exacto, $T : \mathcal{M}^C \to \mathcal{M}^C/\mathcal{T}$, verificando una propiedad universal. En [Gab72] se prueba que la categoría cociente es de nuevo una categoría de comódulos. Si el funtor T tiene un adjunto por la derecha, S, entonces se dice que la subcategoría es localizante. Dualmente, una categoría se dice colocalizante si T tiene un adjunto por la izquierda H. Los funtores S y H son un embebimiento exacto a izquierda y un embebimiento exacto a derecha, respectivamente, por lo que parece factible una relación entre el tipo de comódulos de la coálgebra C y de la categoría cociente.

La versión para coálgebras está desarrollada principalmente en [Gre76], [Lin75], [NT94] y [NT96]. En estos artículos son estudiadas las subcategorías localizantes y las relaciones existentes con otros conceptos como coálgebras coidempotentes, comódulos inyectivos o comódulos simples. En [CGT02] y [JMNR06], se define una correspondencia biyectiva entre la subcategorías localizantes y las clases de equivalencia de elementos idempotentes del álgebra dual. Este hecho nos permite describir la categoría cociente como la categoría de comódulos sobre la coálgebra eCe cuya estructura viene dada por

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e \quad \text{y} \quad \epsilon_{eCe}(exe) = \epsilon_C(x)$$

para todo $x \in C$, donde $\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, con la notación de [Swe69]. También se prueba que los funtores asociados a la localización son $T = e(-) = -\square_C eC = \mathrm{Cohom}_C(Ce, -)$, $S = -\square_{eCe} Ce$ and

$H = \mathrm{Cohom}_{eCe}(eC, -)$. Esto es usado frecuentemente en la última sección del Capítulo 3 para describir la localización de subcoálgebras admisibles de una coálgebra de caminos. En esta dirección, definimos las células y las colas y probamos lo siguiente:

Teorema (3.6.3 y 3.6.9)**.** *Sea C una subcoálgebra admisible de una coálgebra de caminos KQ de un quiver Q. Sea e un elemento idempotente de C^* asociado a un conjunto de vértices $X \subseteq Q_0$. Entonces:*

(a) *La coálgebra localizada eCe es una subcoálgebra admisible de la coálgebra de caminos KQ^e, donde Q^e es el quiver cuyo conjunto de vértices es $(Q^e)_0 = X$, y el número de flechas de un vértice x a un vértice y es $\dim_K K\mathcal{C}ell_X^Q(x, y) \cap C$ para todo $x, y \in X$.*

(b) *La subcategoría localizante $\mathcal{T}_X$ de $\mathcal{M}^C$ es colocalizante si, y sólo si, $\dim_K K\mathcal{T}ail_X^Q(x) \cap C$ es finita para todo $x \in X$.*

En cualquier caso, puesto que tratamos de relacionar las teoría de representación de una coálgebra y sus coálgebras localizadas, el Capítulo 3 está dedicado principalmente a estudiar el comportamiento, a través de los funtores de localización, de ciertas clases de comódulos como simples, inyectivos, indescomponibles y finitamente generados. Para ello, una gran cantidad de propiedades y ejemplos son expuestos. Estos serán utilizados para obtener algunos resultados inesperados en la Sección 5 del Capítulo 3. El más importante de ellos describe las subcategorías estables desde diferentes puntos de vista. En particular, se demuestra que las categorías estables corresponden con los idempotentes semicentrales a derecha definidos por Birkenmeier en [Bir83].

Teorema (3.5.2)**.** *Sea C una coálgebra y $\mathcal{T}_e \subseteq \mathcal{M}^C$ una subcategoría localizante asociada a un elemento idempotente $e \in C^*$. Las siguientes condiciones son equivalentes:*

(a) *$\mathcal{T}_e$ es una subcategoría estable.*

(b) $T(E_x) = \begin{cases} \overline{E}_x & \textit{si } x \in I_e, \\ 0 & \textit{si } x \notin I_e. \end{cases}$

(c) *No existen flechas $S_x \to S_y$ en Γ_C tales que $T(S_x) = S_x$ y $T(S_y) = 0$.*

(d) *e es un idempotente semicentral a derecha en C^*.*

Si T_e es una subcategoría colocalizante, estas condiciones son equivalentes a:

(e) $H(S_x) = S_x$ *para todo* $x \in I_e$.

En el Capítulo 4, conjugamos los resultados obtenidos en los capítulos anteriores, para demostrar la versión acíclica del teorema de Gabriel anteriormente expuesto:

Teorema (4.4.3). *Sea Q un quiver acíclico. Entonces toda subcoalgebra admisible tame de KQ es la coálgebra de caminos de un quiver con relaciones.*

Para probar este resultado, primero necesitamos relacionar la propiedad de ser tame, o wild, de una coálgebra, con sus coálgebras localizadas. El inconveniente a la hora de estudiar la localización de coálgebras tame radica en el hecho de que el funtor sección no preserva comómulos de dimensión finita, o equivalentemente, la imagen de un simple no es necesariamente un comódulo finito dimensional. Por tanto, siguiendo este camino, no aparece, de forma natural, ningún funtor entre las categorías de comódulos finitamente generados. Una vez que asumimos esta condición en S, la pregunta que se plantea es si el proceso de localización preserva coálgebras tame. Para analizar este problema es conveniente empezar con un caso sencillo. Supongamos que S preserva comódulos simples. Es fácil de provar que, en tal caso, para un eCe-comódulo N tal que $\mathrm{length}\, N = v = (v_i)_{i \in I_e}$, se verifica que

$$\mathrm{length}\, S(N) = \overline{v} = \begin{cases} v_i, & \text{si } i \in I_e \\ 0, & \text{si } i \in I_C \backslash I_e \end{cases}$$

y entonces el hecho de que C sea tame para el vector $\overline{v}$ implica que eCe es tame para el vector v. Sin embargo, este resultado puede generalizarse. El razonamiento propuesto en la demostración parte de la idea de que al ser posible controlar los C-comódulos cuya longitud está asociada a v mediante S, entonces es posible controlar los eCe-comódulos cuya longitud es v. Obviamente, el problema aparecería si existe un número infinito de eCe-comódulos $\{N_i\}_{i \in I}$, con longitud v, tales que $\mathrm{length}\, S(N_i) \neq \mathrm{length}\, S(N_j)$ para $i \neq j$. En ese caso, el número de $K[t]$-eCe-bimódulos que se obtienen de ser C tame podría ser infinito. Por tanto, si suponemos que $\Omega_v = \{\mathrm{length}\, S(N)$, donde N es tal que $\mathrm{length}\, N = v\}$ es un conjunto finito, es posible utilizar la misma demostración. Pero esto se verifica si S preserva comódulos de dimensión finita; entonces obtenemos los siguiente:

Teorema (4.2.10)**.** *Sea C una coálgebra y $e \in C^*$ un elemento idempotente tal que el funtor sección preserva comódulos de dimensión finita. Si C es tame entonces eCe es tame.*

En particular, S preserva comódulos finito dimensionales para cualquier idempotente, si C es pura semisimple a derecha.

Teorema (4.2.11)**.** *Sea C una coálgebra tame pura semisimple a derecha. Entonces eCe es tame para cualquier idempotente $e \in C^*$.*

El estudio de las coálgebras wild es bastante más complicado. El problema proviene del hecho de que, a priori, no existen funtores exactos de $\mathcal{M}^{eCe}$ a $\mathcal{M}^C$. Entonces tenemos que suponer que el funtor sección o el funtor colocalización son exactos, esto es, la subcategoría (co)localizante es (co)localizante perfecta. Un caso particular es estudiado. Cuando la coálgebra eCe es una subcoálgebra de C. En este caso probamos que esta situación corresponde con la localización por un idempotente escindido (ver [Lam]). Entonces se hace necesario estudiar una descripción de dichos idempotentes. De hecho, probamos el siguiente resultado para coálgebras punteada:

Proposición (4.3.6)**.** *Sea Q un quiver y C una subcoálgebra admisible de KQ. Sea $e \in C^*$ un elemento idempotente asociado a un conjunto de vértices $X \subseteq Q_0$. Entonces e es escindido en C^* si, y sólo si, $I_p \subseteq X$ para cualquier camino p en $\mathrm{PSupp}(eCe)$.*

Finalmente, el Capítulo 5 está dedicado a presentar ejemplos relacionados con los conceptos de los capítulos previos. Para este propósito se analizan ciertas clases de coálgebras cuya existencia viene motivada por el concepto análogo en la categoría de álgebras finito dimensionales. El ejemplo central son las coálgebras hereditarias. Esta es una clase de coálgebras bien conocida y que ha sido estudiada en diferentes artículos con resultados satisfactorios, ver [Chi02], [JLMS06], [JMNR06] y [NTZ96]. El caso de las coálgebras punteadas hereditarias, es decir, coálgebras de caminos de un quiver, es estudiado exhaustivamente. En particular, describimos la localización de coálgebras de caminos mediante células y colas. Para terminar, presentamos una clase de coálgebras íntimamente relacionada con las coálgebras hereditarias: las coálgebras localmente hereditarias. Estas coálgebras vienen definidas por la propiedad de que todo morfismo no nulo entre injectivos indescomponibles es sobreyectivo y ,por tanto, constituyen una generalización de las coálgebras hereditarias.

Introduction

The problem of classifying a mathematical object appears in all areas of mathematics. Commonly, the classification is eased by representing that object by a simpler one; that is the case of the representation theory of algebras. Many of the present developments of the representation theory of finite dimensional algebras over an algebraically closed field use the quiver-theoretical techniques formulated by P. Gabriel and his school in the seventies, see for example [ASS05], [ARS95] and [GR92]. The well-known Gabriel's theorem can be considered as the origin of that method: *every basic finite dimensional algebra A, over an algebraically closed field K, is isomorphic to a quotient KQ/I, where KQ is the path algebra of the quiver Q and I is an admissible ideal of KQ.* This result allows us to give an explicit description of not only any finite dimensional algebra but also of the category of its finitely generated modules by means of linear representations of the quiver associated to the algebra.

Nevertheless an important restriction appears in the former theorem: the algebra must be of finite dimension. Therefore a natural question is raised: is it possible to generalized this result to any (infinite dimensional) algebra? With a quick look at the proof one is convinced that it is not possible to do it directly: the vertices of the quiver correspond to the primitive idempotents of the algebra; hence if the quiver has an infinite number of them, the path algebra has no identity.

In this framework the theory of coalgebras appears in a middle state of difficulty. The coalgebra structure is obtained by reversing the maps in the algebra structure, that is, in a coalgebra C the following diagrams are commutative

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes I"] \\ C \otimes C \arrow[r, "I\otimes\Delta"] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} & C \otimes C \arrow[dl, "I\otimes\epsilon"'] \arrow[dr, "\epsilon\otimes I"] & \\ C \otimes K & & K \otimes C \\ & C \arrow[ul, "\cong"] \arrow[uu, "\Delta"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

(Coassociativity) (Counit)

where Δ and ϵ are linear maps which correspond to the dual notion of multiplication and unit in algebras (the comultiplication and the counit, respectively). Then one should expect that the dual space of a coalgebra is an algebra and vice versa. This is true if the vector spaces are finite dimensional (although it is not in general), so the category of finite dimensional algebras and the category of finite dimensional coalgebras are equivalent. Now, by the Fundamental Coalgebra Structure Theorem, any coalgebra is a directed union of its finite dimensional subcoalgebras, i.e., it is a directed union of finite dimensional algebras. Thus the representation theory of coalgebras could be expected to be an intermediate step between the study of finite dimensional algebras and infinite dimensional algebras.

The category of comodules over a coalgebra is a locally finite abelian category and therefore it has many more useful properties than a module category. In particular, it is of finite type and then it is conceivable that one can find a theory for coalgebras analogous to the one for of finite dimensional algebras. This work is devoted to developing that aim, i.e., to describe coalgebras and their comodule category by means of quivers and representations of quivers. In this context, the first step must be to obtain a version of Gabriel's theorem for coalgebras.

Following this idea, one can endow the path algebra KQ a structure of graded K-coalgebra with comultiplication induced by the decomposition of paths, that is, if $p = \alpha_m \cdots \alpha_1$ is a non-trivial path from a vertex i to a vertex j, then

$$\Delta(p) = e_j \otimes p + p \otimes e_i + \sum_{i=1}^{m-1} \alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 = \sum_{\eta\tau=p} \eta \otimes \tau$$

and $\Delta(e_i) = e_i \otimes e_i$ for a trivial path e_i. The counit of KQ is defined by the formula

$$\epsilon(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is a vertex,} \\ 0 & \text{if } \alpha \text{ is a non-trivial path.} \end{cases}$$

This coalgebra is called the *path coalgebra* of the quiver Q.

In [Woo97], the author proves that every pointed coalgebra is isomorphic to a subcoalgebra of a path coalgebra (the path coalgebra of its Gabriel quiver). Furthermore, it contains the subcoalgebra generated by all vertices and all arrows, that is, it is an admissible subcoalgebra. Later, in [Sim01], it is introduced the notion of path coalgebra of a quiver with relations (Q, Ω) as the subspace of KQ given by

$$C(Q, \Omega) = \{a \in KQ \mid \langle a, \Omega \rangle = 0\},$$

where $\langle -, - \rangle : KQ \times KQ \longrightarrow K$ is the bilinear map defined by $\langle v, w \rangle = \delta_{v,w}$ (the Kronecker delta) for any two paths v, w in Q.

One of the main motivations given in [Sim01] and [Sim05] for presenting a basic coalgebra C in the form $C(Q, \Omega)$ is the fact that, in this case, there is a K-linear equivalence of the category $\mathcal{M}_f^C$ of finite dimensional right C-comodules with the category $\mathrm{nilrep}_K^{lf}(Q, \Omega)$ of nilpotent K-linear representations of finite length of the quiver with relations (Q, Ω) (see [Sim01, p. 135] and [Sim05, Theorem 3.14]). Then that definition is consistent with the classical theory and reduces the study of the category $\mathcal{M}^C$ to the study of linear representations of a bound quiver (Q, Ω).

Therefore the following question is raised: *is any basic coalgebra, over an algebraically closed field, isomorphic to the path coalgebra of a quiver with relations?*

In Chapter 2 we consider this problem. We separate the admissible subcoalgebras of a path coalgebra into two classes depending on whether the coalgebra is generated by paths or not. Obviously the first class is easy to study and we focus our efforts on the second one. For this purpose, we establish a general framework using the weak* topology on the dual algebra to treat the problem in an elementary context. In particular, we describe the path coalgebra of a quiver with relations (Q, Ω) as the orthogonal space $\Omega^\perp$ of the ideal Ω. Then the former problem may be rewritten as follows: *for any admissible subcoalgebra $C \leq KQ$, is there a relation ideal Ω of the algebra KQ such that $\Omega^\perp = C$?* Unfortunately, this is not true and we show an example of an admissible subcoalgebra which is not the orthogonal of an ideal of KQ. Then, one should ask for a criterion to decide whether or not an admissible subcoalgebra is the path coalgebra of a quiver with relations. In the last section of Chapter 2 we prove the following result:

Criterion (2.5.11). *Let C be an admissible subcoalgebra of a path coalgebra KQ. Then C is not the path coalgebra of a quiver with relations if and only if there exist infinite different paths $\{\gamma_i\}_{i\in\mathbb{N}}$ in Q such that:*

(a) *All of them have common source and common sink.*

(b) *None of them is in C.*

(c) *There exist elements $a_j^n \in K$ for all $j, n \in \mathbb{N}$ such that the set $\{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$ is contained in C.*

It is well known that the category of finite dimensional algebras over an algebraically closed field is the disjoint union of two classes: the class of all tame algebras and the class of all wild algebras. This is known as the *tame-wild dichotomy*, see [Dro79]. The idea of such classes is that the category of finitely generated modules over a wild coalgebra is so large that it contains the category of finite dimensional modules over any finite dimensional algebra. Therefore it is not a realistic aim to get a description of its representation theory. Hence we exclude them from our study and the theory is restricted only to tame algebras. Symmetrically, Simson defines in [Sim01] and [Sim05] the analogous concepts for a basic (pointed) coalgebra. Moreover, he proves the weak version of the tame-wild dichotomy (the full version is still an open problem): *let K be an algebraically closed field. Then every K-coalgebra of tame comodule type is not of wild comodule type.*

In order to show that not every admissible coalgebra is a path coalgebra of a quiver with relations, the example we give is of wild comodule type. Furthermore, by the above criterion, a coalgebra with such property needs that its quiver have infinite paths between two vertices and then it seems close to be wild. Consequently, we should reformulate the problem as the following question: *is every basic non-wild coalgebra, over an algebraically closed field, isomorphic to the path coalgebra of a quiver with relations*? In particular, this implies that *every basic tame coalgebra, over an algebraically closed field, is isomorphic to the path coalgebra of a quiver with relations.* Moreover, if the statement holds, this will reduce the proof of the tame-wild dichotomy to path coalgebras of quivers with relations.

In order to attempt this problem, in Chapter 3 we develop the notion of localization in coalgebras. The category $\mathcal{M}^C$ of right comodules over a coalgebra C is a locally finite Grothendieck category in which the theory of localization as described by Gabriel in [Gab62] can be applied. The main thought is that, for any dense subcategory $\mathcal{T} \subseteq \mathcal{M}^C$, we can consider a quotient category $\mathcal{M}^C/\mathcal{T}$ and an exact functor $T : \mathcal{M}^C \to \mathcal{M}^C/\mathcal{T}$ verifying a universal condition. It is proved in [Gab72] that the quotient category is again a category of comodules. If the functor T has a right adjoint functor S then the subcategory is called localizing. Dually, the subcategory is said to be colocalizing if T has a left adjoint functor H. The functors S and H are a left exact embedding and a right exact embedding, respectively, so one can imagine the possibility of a relation between the comodule type of C and of the quotient category.

The version for coalgebras is developed mainly in [Gre76], [Lin75], [NT94] and [NT96]. In these references, there is a very well founded theory about the localizing subcategories and the existing relationships with other concepts as coidempotent coalgebras, injective comodules or simple comodules. Following [CGT02] and [JMNR06], we found a bijective correspondence between localizing subcategories of $\mathcal{M}^C$ and equivalence classes of idempotent elements of the dual algebra. That fact allows us to describe the quotient category as the category of comodules over the coalgebra eCe whose structure is given by

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e \quad \text{and} \quad \epsilon_{eCe}(exe) = \epsilon_C(x)$$

for any $x \in C$, where $\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, using the sigma-notation of [Swe69]. It is also proved that the functors of the localization are $T = e(-) = -\square_C eC = \mathrm{Cohom}_C(Ce, -)$, $S = -\square_{eCe} Ce$ and $H = \mathrm{Cohom}_{eCe}(eC, -)$. That is used frequently in the last section of Chapter 3 in order to describe the localization of admissible subcoalgebras of a path coalgebra. In this direction, we introduce cells and tails of a quiver and prove the following:

Theorem (3.6.3 y 3.6.9). *Let C be an admissible subcoalgebra of a path coalgebra KQ of a quiver Q. Let e be the idempotent element of C^* associated to a subset of vertices X. The following statements hold:*

(a) *The localized coalgebra eCe is an admissible subcoalgebra of the path coalgebra KQ^e, where Q^e is the quiver whose vertices $(Q^e)_0 = X$ and the number of arrows from a vertex x to a vertex y is $\dim_K K\mathcal{C}ell_X^Q(x,y) \cap C$ for all $x, y \in X$.*

(b) *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^C$ is colocalizing if and only if $\dim_K K\mathcal{T}ail_X^Q(x) \cap C$ is finite for all $x \in X$.*

Anyhow, since we wish to relate the representation theory of a coalgebra and its localized coalgebras, Chapter 3 is mainly devoted to the study of the behavior, through the localization functors, of some classes of comodules as simple, injective, indecomposable and finitely generated. Many properties and examples are given there. From these we obtain in Section 5 of Chapter 3 some unexpected results. The main of them describes stable subcategories from different points of view. In particular, it asserts that stable subcategories correspond with the left semicentral idempotents in the dual algebra introduced by Birkenmeier in [Bir83].

Theorem (3.5.2)**.** *Let C be a coalgebra and $\mathcal{T}_e \subseteq \mathcal{M}^C$ be a localizing subcategory associated to an idempotent element $e \in C^*$. The following conditions are equivalent:*

(a) *$\mathcal{T}_e$ is a stable subcategory.*

(b) $T(E_x) = \begin{cases} \overline{E}_x & \text{if } x \in I_e, \\ 0 & \text{If } x \notin I_e. \end{cases}$

(c) *$\mathcal{K} = \{S \in (\Gamma_C)_0 \mid eS = S\}$ is a right link-closed subset of $(\Gamma_C)_0$, i.e., there is no arrow $S_x \to S_y$ in Γ_C, where $T(S_x) = S_x$ and $T(S_y) = 0$.*

(d) *e is a left semicentral idempotent in C^*.*

If T_e is a colocalizing subcategory these are also equivalent to

(e) *$H(S_x) = S_x$ for any $x \in I_e$.*

In Chapter 4 we conjugate the results obtained in the previous chapters in order to prove the acyclic version of Gabriel's theorem for coalgebras:

Theorem (4.4.3)**.** *Let Q be an acyclic quiver. Then any tame admissible subcoalgebra of KQ is the path coalgebra of a quiver with relations.*

In order to do that, first we need to relate the tameness and wildness of a coalgebra and its localized coalgebras. The drawback of treating the tameness from a general point of view lies in the fact that the section functor does not preserve finite dimensional comodules, or equivalently, see Lemma 4.2.9, it does not preserve the finite dimension of the simple comodules. Therefore there are no functors between the categories of finite dimensional comodules defined in a natural way. Once we assume the above condition for S, the question is whether the localization process preserves tameness. We start with a simple case and suppose that S preserves simple comodules. It is easy to prove that, in such a case, for an eCe-comodule N such that $\mathrm{length}\, N = v = (v_i)_{i\in I_e}$, then

$$\mathrm{length}\, S(N) = \overline{v} = \begin{cases} v_i, & \text{if } i \in I_e \\ 0, & \text{if } i \in I_C \backslash I_e \end{cases}$$

and therefore the tameness of C in $\overline{v}$ implies the tameness of eCe in v. Nevertheless that result can be generalized. The underlying idea of the proof is that if we control the C-comodules whose length vector is associated to v through S, then we can control the eCe-comodules of length v. Obviously, the problem appears when there are infinite eCe-comodules $\{N_i\}_{i\in I}$ with length v such that $\mathrm{length}\, S(N_i) \neq \mathrm{length}\, S(N_j)$ for $i \neq j$. In that case, the number of $K[t]$-eCe-bimodules obtained from the tameness of C could be infinite. Therefore if $\Omega_v = \{\mathrm{length}\, S(N) \text{ such that } \mathrm{length}\, N = v\}$ is a finite set, we may use the same proof. But this holds if S preserves finite dimensional comodules, so we obtain the following:

Theorem (4.2.10)**.** *Let C be a coalgebra and $e \in C^*$ an idempotent element such that S preserves finite dimensional comodules. If C is tame then eCe is tame.*

In particular this is verified for any idempotent if C is right pure semisimple.

Theorem (4.2.11)**.** *Let C be a right pure semisimple coalgebra of tame comodule type. Then eCe is of tame comodule type for each idempotent $e \in C^*$.*

Wildness is much more complicated to study. The problem comes from the fact that, a priori, there is no exact functor from $\mathcal{M}^{eCe}$ to $\mathcal{M}^C$. Therefore we have to assume that the section functor or the colocalizing functor are exact, that is, the (co)localizing subcategory is also perfect (co)localizing. A particular case is studied. When the coalgebra eCe is a subcoalgebra of C. We prove that the situation corresponds to the localization by a split idempotent (see [Lam]). Therefore we attempt the description of that kind of idempotents. In fact we prove the following result in pointed coalgebras.

Proposition (4.3.6)**.** *Let Q be a quiver and C be an admissible subcoalgebra of KQ. Let $e \in C^*$ be the idempotent element associated to a subset of vertices X. Then e is split in C^* if and only if $I_p \subseteq X$ for any path p in $\mathrm{PSupp}(eCe)$.*

Finally, Chapter 5 is devoted to the presentation of examples related to the topics considered in the previous chapters. To that end we use some classes of coalgebras whose existence are motivated by the analogous classes in the category of finite dimensional algebras. The main example for us shall be the hereditary coalgebras. This is a well-known kind of coalgebras which have been studied with satisfactory results in many papers, see [Chi02], [JLMS06], [JMNR06] and [NTZ96]. The case of a pointed hereditary coalgebra, that is, a path coalgebra of a quiver, is studied extensively. In particular we describe the localization of a path coalgebra by means of the cells and tails of its quiver. Lastly, we also introduce a class of coalgebras related to the hereditary coalgebras: the locally hereditary coalgebras. That kind of coalgebras can be defined by the property that every non-zero morphism between indecomposable injective comodules is surjective, and thus, these are a generalization of the hereditary case.

Chapter 1

Preliminaries

This chapter contains some of the background material that will be used throughout this work. Namely, after a few categorical remarks, we introduce the notation and terminology on coalgebras and we recall some basic facts about their representation theory. We assume that the reader is familiar with elementary category theory and ring theory, and some homological concepts such as injective and projective objects; anyhow we refer to [AF91], [Mac71], [Pop73] and [Wis91] for questions on these subjects. All rings considered have identity and modules are unitary. By a field we will mean a commutative division ring.

1.1 Some categorical remarks

This section is devoted to establish some categorical definitions and properties which we will assume for a category of comodules in what follows. For further information see, for example, [Mac71], [Pop73] or [Wis91].

A category $\mathcal{C}$ is said to be *abelian* if the following conditions are satisfied:

(a) There exists the direct sum of any finite set of objects of $\mathcal{C}$.

(b) For each pair of objects X and Y of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is equipped with an abelian group structure such that the composition of morphisms in $\mathcal{C}$ is bilinear.

(c) $\mathcal{C}$ has a zero object.

(d) Each morphism $f : X \to Y$ in $\mathcal{C}$ admits a kernel $(\mathrm{Ker}\, f, u)$ and a cokernel $(\mathrm{Coker}\, f, p)$, and the unique morphism $\overline{f}$ making commutative the diagram

$$\begin{tikzcd} \mathrm{Ker}\, f \arrow[r, "u"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "p"] & \mathrm{Coker}\, f \\ & \mathrm{Coker}\, u \arrow[r, "\overline{f}"] & \mathrm{Ker}\, p \arrow[u] & \end{tikzcd}$$

is an isomorphism.

Throughout this work we fix a field K. We say that $\mathcal{C}$ is a K-*category* if, for each pair of objects X and Y of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is equipped with a K-vector space structure such that the composition of morphisms in $\mathcal{C}$ is a K-bilinear map.

An abelian category is said to be a *Grothendieck category* if it has arbitrary direct sums, a set of generators and direct limits are exact. Moreover, if each object of the set of generators has finite length then $\mathcal{C}$ is known as a *locally finite category*.

Proposition 1.1.1. *[Gab62] Let $\mathcal{C}$ be a locally finite abelian K-category. Then it verifies the following assertions:*

(a) *$\mathcal{C}$ has injective envelopes.*

(b) *The direct sum of injective objects is injective.*

(c) *Each object of $\mathcal{C}$ is an essential extension of its socle (the sum of all its simple subobjects).*

(d) *An injective object E of $\mathcal{C}$ is indecomposable if and only if its socle is simple.*

(e) *If $\{S_i\}_{i\in I}$ is a complete set of isomorphism classes of simple objects of $\mathcal{C}$ and E_i is the injective envelope of S_i for each $i \in I$, then $\{E_i\}_{i\in I}$ is a complete set of indecomposable injective objects of $\mathcal{C}$.*

(f) *Each injective object E of $\mathcal{C}$ is isomorphic to a direct sum $\bigoplus_{i\in I} E_i^{\alpha_i}$, where each α_i is a non-negative integer. Furthermore, this sum is uniquely determined by the set $\{\alpha_i\}_{i\in I}$.*

(g) *$E = \bigoplus_{i\in I} E_i^{\alpha_i}$ is an injective cogenerator of $\mathcal{C}$ if and only if $\alpha_i > 0$ for all $i \in I$.*

Let $\mathcal{C}$ be a locally finite abelian K-category. We say $\mathcal{C}$ is of *finite type* if, for each pair of objects X and Y of finite length of $\mathcal{C}$, the vector space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has finite dimension over K.

Proposition 1.1.2. *[Tak77] Let $\mathcal{C}$ be a locally finite abelian K-category. The following conditions are equivalent:*

(a) *$\mathcal{C}$ is of finite type.*

(b) *For each simple object S of $\mathcal{C}$, the vector space $\mathrm{Hom}_{\mathcal{C}}(S, S)$ is finite dimensional over K.*

An object F of an abelian K-category of finite type $\mathcal{C}$ is said to be *quasi-finite* if, for each object X of finite length, the vector space $\mathrm{Hom}_{\mathcal{C}}(X, F)$ has finite dimension over K.

Proposition 1.1.3. *[Tak77] Let $\mathcal{C}$ be an abelian K-category of finite type and F be an object of $\mathcal{C}$. The following sentences are equivalent:*

(a) *F is quasi-finite.*

(b) *For each simple object S of $\mathcal{C}$, the space $\mathrm{Hom}_{\mathcal{C}}(S, F)$ is finite dimensional over K.*

(c) *The socle of F is isomorphic to $\bigoplus_{i\in I} S_i^{\alpha_i}$ where the non-negative integers α_i are finite for all $i \in I$.*

Corollary 1.1.4. *[Tak77] Let $\mathcal{C}$ be an abelian K-category of finite type then $\bigoplus_{i\in I} E_i$ is a quasi-finite injective cogenerator of $\mathcal{C}$.*

1.2 The category of comodules

Let us now define the main object of our study, that is, coalgebras and their category of comodules. Recall that the category of comodules of a coalgebra is a particular case of a category of finite type so all definitions and results of the last section are valid here.

Following [Abe77] and [Swe69], by a *K-coalgebra* we mean a triple (C, Δ, ϵ), where C is a K-vector space and $\Delta : C \to C \otimes C$ and $\epsilon : C \to K$ are linear maps, called *comultiplication* and *counit*, such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes I"] \\ C \otimes C \arrow[r, "I\otimes\Delta"] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} & C \otimes C \arrow[dl, "I\otimes\epsilon"'] \arrow[dr, "\epsilon\otimes I"] & \\ C \otimes K & & K \otimes C \\ & C \arrow[ul, "\cong"] \arrow[uu, "\Delta"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

(Coassociativity) (Counit)

In what follows we shall refer the coalgebra (C, Δ, ϵ) simply by C.

A K-vector subspace V of C is a *subcoalgebra* of C if $\Delta(V) \subseteq V \otimes V$. It is a *right* (resp. *left*) *coideal* if $\Delta(V) \subseteq V \otimes C$ (resp. $\Delta(V) \subseteq C \otimes V$) and it is a *coideal* if $\Delta(V) \subseteq V \otimes C + C \otimes V$ and $\epsilon(V) = 0$. Note that a right and left coideal is not a coideal but a subcoalgebra. If S is a subset of a coalgebra, the *subcoalgebra generated* by S is the intersection of all subcoalgebras containing S.

Theorem 1.2.1. *[Swe69]*

(a) *The intersection of subcoalgebras is again a subcoalgebra.*

(b) *A subcoalgebra generated by a finite set is finite dimensional.*

(c) *A simple subcoalgebra of a coalgebra is finite dimensional.*

Given two K-coalgebras C and D, a *morphism* of K-coalgebras $f : C \to D$ is a linear map such that the following diagrams are commutative:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f\otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\epsilon_C"'] & & D \arrow[dl, "\epsilon_D"] \\ & K & \end{tikzcd}$$

If $f : C \to D$ is a morphism of coalgebras, it is easy to prove that $\operatorname{Ker} f$ is a coideal of C and $\operatorname{Im} f$ is a subcoalgebra of D.

The following result is often called the *Fundamental Coalgebra Structure Theorem* and it show us the locally finite nature of a coalgebra, see [Mon93] and [Swe69].

Theorem 1.2.2. *Any K-coalgebra is a directed union of its finite dimensional subcoalgebras.*

Let C be a K-coalgebra. A right C-*comodule* is a pair (M,ω) where M is a K-vector space and $\omega : M \to C \otimes M$ is a linear map making commutative the following diagrams:

$$\begin{tikzcd} M \arrow[r, "\omega"] \arrow[d, "\omega"'] & M \otimes C \arrow[d, "\omega\otimes I"] \\ M \otimes C \arrow[r, "I\otimes\Delta"] & M \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\omega"] \arrow[dr, "\cong"'] & M \otimes C \arrow[d, "I\otimes\epsilon"] \\ & M \otimes K \end{tikzcd}$$

In what follows we shall refer the right C-comodule (M,ω) simply by M, or by M_C.

Given two right C-comodules M and N, a *morphism* of right C-comodules $f : M \to N$ is a linear map such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\omega_M"'] & N \arrow[d, "\omega_N"] \\ M \otimes C \arrow[r, "f\otimes I"] & N \otimes C \end{tikzcd}$$

From now on we will identify every comodule with the identity map defined on it, so we will use the notation $f \otimes M$, $f \otimes 1_M$ or simply $f \otimes I$, it doesn't matter which. We will denote by $\mathcal{M}^C$ the category of right C-comodules and morphisms of right C-comodules and by $\mathcal{M}^C_{qf}$ and $\mathcal{M}^C_f$ the full subcategories of $\mathcal{M}^C$ whose objects are the quasi-finite right C-comodules and the finite dimensional right C-comodules, respectively. Analogously we may define and denote the category of left C-comodules.

Example 1.2.3. *Let C be a coalgebra, V be a vector space and M be a right C-comodule. Then $V \otimes M$ has an structure of right C-comodule with comultiplication $I \otimes \omega_M$. It is easy to prove that we have an isomorphism $\operatorname{Hom}_C(V \otimes M, N) \cong \operatorname{Hom}_K(V, \operatorname{Hom}_C(M,N))$ for any right C-comodule N.*

Let C and D be K-coalgebras. A (C,D)-*bicomodule* is a K-vector space M with an structure of left C-comodule (M,ω) and an structure of right D-comodule (M,ρ) verifying a property of compatibility between both given by the commutativity of the following diagram

$$\begin{tikzcd} M \arrow[r, "\omega"] \arrow[d, "\rho"'] & C \otimes M \arrow[d, "I\otimes\rho"] \\ M \otimes D \arrow[r, "\omega\otimes I"] & C \otimes M \otimes D \end{tikzcd}$$

The reader should note that this means that ρ is a morphism of left C-modules, or equivalently, ω is a morphism of right D-modules.

Here we list some important properties of the category of comodules in the sense of the last section, see [Mon93] and [Swe69] for details.

Proposition 1.2.4. *Let C be a K-coalgebra. Then:*

(a) *$\mathcal{M}^C$ is a abelian K-category of finite type.*

(b) *$\mathcal{M}^C_f$ is a skeletally small abelian Krull-Schmidt K-category.*

(c) *$\mathcal{M}^C$ has enough injective objects.*

(d) *The coalgebra C, viewed as a right C-comodule, is a quasi-finite injective cogenerator in $\mathcal{M}^C$.*

(e) *A direct sum of indecomposable right C-comodules is injective if and only if each direct summand is injective.*

(f) *Every right C-comodule is the directed union of its finite dimensional subcomodules.*

(g) *Each simple right C-comodule has finite dimension.*

Remark. In general, the category $\mathcal{M}^C$ has no enough projectives and sometimes it has no non-zero projective objects.

Throughout we denote by $\{S_i\}_{i\in I_C}$ a complete set of pairwise non-isomorphic simple right C-comodules and by $\{E_i\}_{i\in I_C}$ a complete set of pairwise non-isomorphic indecomposable injective right C-comodules.

Let (M,ρ) be a right C-comodule. There exists a unique minimal subcoalgebra $\mathrm{cf}(M)$ of C such that $\rho(M)\subseteq M\otimes \mathrm{cf}(M)$, that is, such that M is a right $\mathrm{cf}(M)$-comodule. This coalgebra $\mathrm{cf}(M)$ is called the *coefficient space* of M.

Proposition 1.2.5. *[Gre76] Let C be a coalgebra and $m_i = \dim_K S_i$ for any $i \in I_C$. Then:*

(a) *Each simple subcoalgebra of C is isomorphic to $\mathrm{cf}(S_i)$ for some $i \in I_C$.*

(b) $\mathrm{cf}(S_i) = S_i \oplus \cdots \oplus S_i = S_i^{m_i}$ *for each $i \in I_C$.*

(c) $\mathrm{Corad}(C) = C_0 = \oplus_{i \in I_C} \mathrm{cf}(S_i) = \oplus_{i \in I_C} S_i^{m_i}$.

We finish this section giving an important characterization of the categories of comodules.

Theorem 1.2.6. *[Tak77] Let $\mathcal{C}$ be an abelian K-category. $\mathcal{C}$ is K-linearly equivalent to $\mathcal{M}^C$ for some K-coalgebra C if and only if $\mathcal{C}$ is of finite type.*

1.3 Cotensor product

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian K-categories. A functor $T : \mathcal{A} \to \mathcal{B}$ is said to be *K-linear* if the map $T_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(T(X), T(Y))$ defined by $T_{X,Y}(f) = T(f)$ is linear for any objects X and Y of $\mathcal{A}$.

Let now $S : \mathcal{A} \to \mathcal{B}$ and $T : \mathcal{B} \to \mathcal{A}$ be two functors. We say that S is *left adjoint* to T or T is *right adjoint* to S if there exists a natural isomorphism $\mathrm{Hom}_{\mathcal{A}}(S(-), -) \backsimeq \mathrm{Hom}_{\mathcal{B}}(-, T(-))$. In this case, S is right exact and preserves colimits and T is left exact and preserves limits.

In the particular case of categories of modules over a K-algebra, we have an important example of adjoint functors: the tensor functor and the Hom functor.

Suppose R is a K-algebra, M is a right R-module and N is a left R-module. Then we may introduce the tensor product $M_R \otimes_R {}_R N$ as the cokernel of the maps

$$M \otimes_K R \otimes_K N \overset{\mu_M \otimes I}{\underset{I \otimes \mu_N}{\rightrightarrows}} M \otimes_K N \dashrightarrow M \otimes_R N,$$

where μ_M and μ_N are the structure maps of M and N as R-modules. Furthermore, if S is other K-algebra and N is a R-S-bimodule then $M \otimes_R N$ has an structure of right S-module. Thus we can define a functor $- \otimes_R N : \mathrm{Mod}_R \to \mathrm{Mod}_S$ which is left adjoint to $\mathrm{Hom}_S(N, -)$, that is, $\mathrm{Hom}_R(M, \mathrm{Hom}_S(N, T)) \backsimeq \mathrm{Hom}_S(M \otimes_R N, T)$ for any right R-module M and any right S-module T.

Let us come back to coalgebras. We would like to obtain a situation similar to above, i.e., a functor between categories of comodules over different coalgebras with adjoint properties. Let C be a K-coalgebra, M be a right C-comodule and N be a left C-comodule. Then we may define the *cotensor product* of M and N, $M_C \square_C {}_C N$, as the kernel of the maps

$$M\square_C N \dashrightarrow M \otimes_K N \underset{I\otimes\omega_N}{\overset{\omega_M\otimes I}{\rightrightarrows}} M \otimes_K C \otimes_K N,$$

where ω_M and ω_N are the structure maps of M and N as right C-comodule and left C-comodule, respectively.

We collect here some properties of the cotensor product.

Proposition 1.3.1. *[Tak77] Let C be a coalgebra, M be a right C-comodule and N be a left C-comodule. Then:*

(a) *If $C = K$ then $M\square_C N = M \otimes_K N$.*

(b) *The cotensor product is associative.*

(c) *The functors $M\square_C -$ and $-\square_C N$ are left exact and preserve direct sums.*

(d) *We have $M\square_C(N \otimes_K W) \cong (M\square_C N) \otimes_K W$ and $(W \otimes_K M)\square_C N \cong W \otimes_K (M\square_C N)$ for any K-vector space W.*

(e) *The functor $M\square_C -$ (resp. $-\square_C N$) is exact if and only if M (resp. N) is an injective right (resp. left) C-comodule .*

(f) *$M\square_C C \cong M$ and $C\square_C N \cong N$.*

Let now D and E be two coalgebras, M be a (E, C)-bicomodule and N be a (C, D)-bicomodule. Then $M\square_C N$ acquires a structure of (E, D)-bicomodule with structure maps

$$\rho_M\square I : M\square_C N \to (E \otimes_K M)\square_C N \cong E \otimes_K (M\square_C N)$$

and

$$I\square\rho_N : M\square_C N \to M\square_C(N \otimes_K D) \cong (M\square_C N) \otimes_K D.$$

Therefore we may consider a functor $-\square_C N : \mathcal{M}^C \to \mathcal{M}^D$. Unfortunately, in general, $-\square_C N$ does not have a left adjoint functor.

Theorem 1.3.2. *[Tak77] Let C and D be two coalgebras and M be a (D, C)-bicomodule. Then the functor $-\square_D M$ has a left adjoint functor if and only if M is a quasi-finite right C-comodule.*

If M is a quasi-finite right C-comodule, we will denote the left adjoint functor of $-\square_D M$ by $\mathrm{Cohom}_C(M, -)$. The functor $\mathrm{Cohom}_C(M, -)$ has a behavior similar to the usual Hom functor of algebras.

Proposition 1.3.3. *[Tak77] Let C, D and E be three coalgebras. Let M and N be a (D, C)-bicomodule and a (E, C)-bicomodule, respectively, such that M is quasi-finite as right C-comodule. Then:*

(a) *We have $\mathrm{Cohom}_C(M, N) = \varinjlim \mathrm{Hom}_C(N_\lambda, M)^*$, where $N = \varinjlim N_\lambda$ with $\{N_\lambda\}_\lambda$ the set of finite dimensional subcomodules of N.*

(b) *The vector space $\mathrm{Cohom}_C(M, N)$ is a (E, D)-bicomodule.*

(c) *The functor $\mathrm{Cohom}_C(M, -)$ is right exact and preserves direct sums.*

(d) *The functor $\mathrm{Cohom}_C(M, -)$ is exact if and only if M is injective as right C-comodule.*

Remark. The set $\mathrm{Coend}_C(M) = \mathrm{Cohom}_C(M, M)$ has an structure of K-coalgebra and then M becomes a $(\mathrm{Coend}_C(M), C)$-bicomodule, see [Tak77] for details.

Symmetrically, ${}_DM_C$ is quasi-finite as left D-comodule if and only if the functor $M\square_C - : {}^C\mathcal{M} \to {}^D\mathcal{M}$ has a left adjoint functor. In this case we denote by $\mathrm{Cohom}_D(-, M)$ that functor.

As a consequence we may prove the Krull-Remak-Schmidt-Azumaya theorem for comodules. Before we need the following lemmata:

Lemma 1.3.4. *Let E be an indecomposable injective right C-comodule. Then $\mathrm{Hom}_C(E, E) = \mathrm{End}_C(E)$ is a local ring.*

Proof. Let $f \in \mathrm{End}_C(E)$. It holds that $\mathrm{Ker}\, f \cap \mathrm{Ker}\,(id_E - f) = 0$. Since E is indecomposable, $\mathrm{Ker}\, f = 0$ or $\mathrm{Ker}\,(id_E - f) = 0$.

If f is injective then there exists a map g such that

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "id_E"'] & E \arrow[dl, dashed, "g"] \\ E & \end{tikzcd}$$

is commutative and then $E \xrightarrow{f} E \longrightarrow \mathrm{Coker}\, f$ splits. Therefore $E = E \oplus \mathrm{Coker}\, f$. Thus $\mathrm{Coker}\, f = 0$ and f is bijective.

On the other case, proceeding as before, $id_E - f$ is bijective so it is quasi-regular. Then f is in the radical. This proves that $\mathrm{End}_C(E)$ is local. $\square$

Let M be a quasi-finite right C-comodule we denote by $\mathrm{add}\, M$ the category of direct summands of arbitrary direct sums of copies of M. Let us consider the coalgebra $D = \mathrm{Cohom}_C(M, M)$. Then the functor Cohom and the cotensor functor can be restricted to $\mathrm{Cohom}_C(M, -) : \mathrm{add}\ M \to \mathrm{add}\ D$ and $-\square_D M : \mathrm{add}\ D \to \mathrm{add}\ M$

Lemma 1.3.5. *[CKQ02] Let M be a quasi-finite right C-comodule and let D be the coalgebra $\mathrm{Cohom}_C(M, M)$. Then the functors*

$$\mathrm{add}\ M \underset{-\square_D M}{\overset{\mathrm{Cohom}_C(M,-)}{\rightleftarrows}} \mathrm{add}\ D$$

are inverse equivalences of categories.

Corollary 1.3.6. *Let M be a quasi-finite indecomposable right C-comodule then $\mathrm{End}_C(M)$ is a local ring.*

Proof. By Lemma 1.3.5, $\mathrm{Cohom}_C(M, -) : \mathrm{add}\ M \to \mathrm{add}\ D$ is an equivalence. Since D is quasi-finite then $\mathrm{add}\ D = \mathcal{I}^D$, the category of quasi-finite injective right D-comodules. Therefore it is enough to prove it for injective indecomposable comodules. But this is exactly Lemma 1.3.4. $\square$

Theorem 1.3.7 (Krull-Remak-Schmidt-Azumaya Theorem). *Let C be a coalgebra and M be a quasi-finite right C-comodule. Then two decompositions of M as direct sum of indecomposable right C-comodules are essentially the same, that is, if $M = \bigoplus_{i\in I} M_i = \bigoplus_{j\in J} N_j$, where all $M_i's$ and $N_j's$ are indecomposable right C-comodules, then $I = J$ and there exists a bijective correspondence $\sigma : I \to J$ such that $M_i \cong N_{\sigma(i)}$ for all $i \in I$.*

Proof. See [Gab62, I.6, Theorem 1]. $\square$

1.4 Equivalence between categories of comodules

Let C and D be two coalgebras and M and N be a (C, D)-bicomodule and a (D, C)-bicomodule, respectively. Suppose $f : C \to M\square_D N$ and $g : D \to N\square_C M$ are two bicomodule maps. We say that (C, D, M, N, f, g) is a *Morita-Takeuchi context* if the following diagrams commute:

$$\begin{array}{ccc} M & \xrightarrow{\cong} & M\square_D D \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle I\square g} \\ C\square_C M & \xrightarrow{f\square I} & M\square_D N\square_C M \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{\cong} & N\square_C C \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle I\square f} \\ D\square_D N & \xrightarrow{g\square I} & N\square_C M\square_D N \end{array}$$

The Morita-Takeuchi context (C, D, M, N, f, g) is said to be *injective* if f is injective. If f and g are isomorphisms then we say that C and D are *Morita-Takeuchi equivalent.*

Proposition 1.4.1. *[Tak77] Let (C, D, M, N, f, g) be an injective Morita-Takeuchi context. Then:*

(a) *f is an isomorphism.*

(b) *M_D and ${}_DN$ are quasi-finite and injective.*

(c) *${}_CM$ and N_C are cogenerators.*

(d) *$\mathrm{Cohom}_D(M, D) \cong N$ as (D, C)-bicomodules and $\mathrm{Cohom}_D(N, D) \cong M$ as (C, D)-bicomodules.*

(e) *$\mathrm{Coend}_D(M) \cong C$ and $\mathrm{Coend}_D(N) \cong C$ as coalgebras.*

Example 1.4.2. *Suppose D is a coalgebra and M is a quasi-finite right D-comodule. Denote by C the coalgebra $\mathrm{Coend}_D(M)$ and by N the (D, C)-bicomodule $\mathrm{Cohom}_D(M, D)$. Then we have the adjoint equivalence $\mathrm{Hom}_D(D, N\Box_C M) \simeq \mathrm{Hom}_C(N, N)$. Now, consider $g : D \longrightarrow N\Box_C M$ the associated morphism to id_Y via the equivalence, and let f be the morphism*

$$f : C \cong \mathrm{Cohom}_D(M, M\Box_D D) \to M\Box_D \mathrm{Cohom}_D(M, D) = M\Box_D N.$$

Then (C, D, X, Y, f, g) is a Morita-Takeuchi context which is usually known as the Morita-Takeuchi context associated to M_D.

Clearly, f is injective if and only if M_D is injective and g is injective if and only if M_D is a cogenerator.

Proposition 1.4.3. *[Tak77] Let M_D be a quasi-finite D-comodule and let (C, D, X, Y, f, g) be the Morita-Takeuchi context associated to M. Then C and D are Morita-Takeuchi equivalent if and only if M is an injective cogenerator of the category $\mathcal{M}^D$.*

We may use Morita-Takeuchi contexts in order to know whenever two categories of comodules are equivalent.

Theorem 1.4.4. *[Tak77] Let M be a (C, D)-bicomodule which is quasi-finite as right D-comodule. The following conditions are equivalent:*

(a) *The functor $-\Box_C M : \mathcal{M}^C \to \mathcal{M}^D$ is an equivalence of categories.*

(b) *The functor $M\Box_D- :{}^D\mathcal{M} \to{}^C\mathcal{M}$ is an equivalence of categories.*

(c) M_D *is a quasi-finite injective cogenerator and* $\mathrm{Coend}_D(M) \cong C$ *as coalgebras.*

(d) ${}_C M$ *is a quasi-finite injective cogenerator and* $\mathrm{Coend}_C(M) \cong D$ *as coalgebras.*

(e) *There exists a Morita-Takeuchi context* (C, D, M, N, f, g), *where* f *and* g *are injective.*

(f) *There exists a Morita-Takeuchi context* (D, C, N', M, f', g'), *where* f' *and* g' *are injective.*

If these conditions hold, there is an isomorphism between the (C, D)*-bicomodules* $\mathrm{Cohom}_D(M, D)$ *and* $\mathrm{Cohom}_C(M, C)$. *If we denote it by* N *then* $-\Box_D N$ *and* $N\Box_C-$ *are the quasi-inverse functors of* $-\Box_C M$ *and* $M\Box_D-$, *respectively.*

Corollary 1.4.5. *Two coalgebras are Morita-Takeuchi equivalent if and only if their categories of right comodules are equivalent.*

The reader could ask about what happen when two categories of comodules are equivalent but the functor is not of the form $-\Box_C M$ where M is a bicomodule. The answer is simple: that situation cannot appear.

Theorem 1.4.6. *[Tak77] Let* $T : \mathcal{M}^C \to \mathcal{M}^D$ *be a* K*-linear functor. If* T *is left exact and preserves direct sums then there is a* (C, D)*-bicomodule* M *such that* $T \cong -\Box_C M$.

We have seen that quasi-finite injective cogenerators play an important rôle on equivalence between categories of comodules. We recall from Section 1.1 that this kind of comodules has an easy description.

Proposition 1.4.7. *Let* C *be a coalgebra and* $\{E_i\}_{i\in I_C}$ *be a complete set of pairwise non-isomorphic indecomposable injective right* C*-comodules. A right* C*-comodule* E *is a quasi-finite injective cogenerator of* $\mathcal{M}^C$ *if and only if* $E = \bigoplus_{i\in I} E_i^{\alpha_i}$, *where* α_i *is a finite cardinal number greater than zero for all* $i \in I$.

1.5 Basic and pointed coalgebras

Any coalgebra C is a quasi-finite injective cogenerator of its category $\mathcal{M}^C$ of right C-comodules. Then, by the last section, $C =$

$\bigoplus_{i\in I_C} E_i^{\alpha_i}$, where each α_i is a finite positive integer, that is, its socle $\operatorname{Soc} C = \bigoplus_{i\in I_C} S_i^{\alpha_i}$. The coalgebra C is called *basic* if $\alpha_i = 1$ for all $i \in I_C$, i.e., if $\operatorname{Soc} C = \bigoplus_{i\in I_C} S_i$, where $S_i \not\cong S_j$ for $i \neq j$. Following this definition, we may obtain an immediate consequence:

Proposition 1.5.1. *The following conditions are equivalent:*

(a) *C is basic.*

(b) *$C = \bigoplus_{i\in I_C} E_i$.*

(c) *C is a minimal injective cogenerator of the category $\mathcal{M}^C$.*

The main reason to study basic coalgebras comes from the fact that in order to classify coalgebras by means of its category of comodules it is enough to consider only this kind of coalgebras, see for example [Sim01].

Theorem 1.5.2. *Let C be a coalgebra then there exits an unique (up to isomorphism) basic coalgebra D such that $\mathcal{M}^C \cong \mathcal{M}^D$.*

Proof. Suppose that $C = \bigoplus_{i\in I_C} E_i^{\alpha_i}$. We consider the comodule $E = \bigoplus_{i\in I_C} E_i$. By Proposition 1.4.7, E is a quasi-finite injective cogenerator and, by Theorem 1.4.4, the functor $-\square_D E$ defines an equivalence between the categories $\mathcal{M}^D$ and $\mathcal{M}^C$, where $D = \operatorname{Cohom}_C(E, E)$. Thus we only need to prove that D is a basic coalgebra.

Let $\{E'_i\}_{i\in I_D}$ be a complete set of indecomposable injective right D-comodules. Since $-\square_D E$ is an equivalence, we may number them in order to do that $E'_i\square_D E = E_i$ for all $i \in I_C = I_D$. Now, suppose that $D = \bigoplus_{i\in I_D} E_i'^{t_i}$. Then $E \cong D\square_D E \cong \bigoplus_{i\in I_D} E_i'^{t_i}\square_D E \cong \bigoplus_{i\in I_D}(E'_i\square_D E)^{t_i} = \bigoplus_{i\in I_D} E_i^{t_i}$ and therefore, by Krull-Remak-Schmidt-Azumaya Theorem, $t_i = 1$ for all $i \in I_C$.

Let now H be another basic coalgebra such that $\mathcal{M}^C \cong \mathcal{M}^H$. Then there exists an equivalence $-\square_{DD}Q_H : \mathcal{M}^D \to \mathcal{M}^H$, where Q is a quasi-finite injective cogenerator of $\mathcal{M}^H$. Since the equivalences preserve the minimal quasi-finite injective cogenerator then $Q \cong D\square_D Q = H$ because D and H are basic. Then, by Theorem 1.4.4, $\operatorname{Cohom}_H(H, H) \cong D$ as coalgebras. Consider the inverse equivalence and we obtain $\operatorname{Cohom}_D(D, D) \cong H$. Finally, if $D = \varinjlim D_\gamma$, where $\{D_\gamma\}_\gamma$ is the set of its finite dimensional subcoalgebras, then $H \cong \operatorname{Cohom}_D(D, D) = \varinjlim \operatorname{Hom}_D(D_\gamma, D)^* \cong \varinjlim \operatorname{Hom}_H(D_\gamma\square_D H, H)^* = \operatorname{Cohom}_H(H, H) \cong D$. $\square$

Corollary 1.5.3. *Any coalgebra is Morita-Takeuchi equivalent to a basic coalgebra.*

If K is an algebraically closed field we can say more. Suppose that S is a simple right C-comodule. Then S has finite dimension as K-vector space. A coalgebra is said to be *pointed* if every simple comodule is one dimensional.

Proposition 1.5.4. *Every pointed coalgebra is basic.*

Proof. Let C be a coalgebra such that $\operatorname{Soc} C = \bigoplus_{i \in I_C} S_i^{t_i}$. Since $S_i^* = \operatorname{Hom}_K(S_i, K) \cong \operatorname{Hom}_C(S_i, C) \cong \operatorname{Hom}_C(S_i, \operatorname{Soc} C) \cong \operatorname{Hom}_C(S_i, S_i)^{t_i} \cong \operatorname{End}_C(S_i)^{t_i}$ then $\dim_K S_i = \dim_K S_i^* = t_i \dim_K \operatorname{End}_C(S_i)$ because $\dim_K S_i$ is finite. Therefore $t_i = \frac{\dim_K S_i}{\dim_K \operatorname{End}_C(S_i)}$. Now, if C is pointed then $\dim_K S_i = 1$ and thus $\dim_K \operatorname{End}_C(S_i) = t_i = 1$. □

Corollary 1.5.5. *Let K be an algebraically closed field and C be a K-coalgebra. Then C is basic if and only if C is pointed.*

Proof. If C is basic then $t_i = 1$. Now, every K-division algebra is one dimensional so $\dim_K \operatorname{End}_C(S_i) = 1$. Thus $dim_K S_i = 1$. □

Corollary 1.5.6. *Every coalgebra over an algebraically closed field is Morita-Takeuchi equivalent to a pointed coalgebra.*

1.6 Path coalgebras

In representation theory of coalgebras, an important rôle is played by path coalgebras. This is the analogous case to the path algebra associated to a quiver (see [ASS05], [ARS95] and [GR92]). In this section we will give a brief approach to them and they will be studied deeper in the next chapters.

Following [Gab72], by a *quiver*, Q, we mean a quadruple (Q_0, Q_1, s, t), where Q_0 is the set of *vertices* (or *points*), Q_1 is the set of *arrows* and, for each arrow $\alpha \in Q_1$, the vertices $s(\alpha)$ and $t(\alpha)$ are the *source* (or *start point* or *origin*) and the *sink* (or *end point* or *tail*) of α, respectively. We denote an arrow α such that $s(\alpha) = i$ and $t(\alpha) = j$ as $\alpha : i \to j$ or $i \xrightarrow{\alpha} j$. If $i = j$ we say that α is a *loop*.

If i and j are vertices, an (oriented) *path* in Q of length m from i to j is a formal composition of arrows

$$p = \alpha_m \cdots \alpha_2 \alpha_1,$$

where $s(\alpha_1) = i$, $t(\alpha_m) = j$ and $t(\alpha_{k-1}) = s(\alpha_k)$ for $k = 2, \ldots, m$. To any vertex $i \in Q_0$, we attach a *trivial path* of length 0, say e_i or simply i, starting and ending at i such that $\alpha e_i = \alpha$ (resp. $e_j\beta = \beta$) for any arrow α (resp. β) with $s(\alpha) = i$ (resp. $t(\beta) = i$). We identify the set of vertices and the set of trivial paths. A *cycle* is a path which starts and ends at the same vertex.

Let KQ be the K-vector space generated by the set of all paths in Q. Then KQ can be endowed with a structure of (non necessarily unitary) K-algebra with multiplication induced by the concatenation of paths, that is,

$$(\alpha_m \cdots \alpha_2\alpha_1)(\beta_n \cdots \beta_2\beta_1) = \begin{cases} \alpha_m \cdots \alpha_2\alpha_1\beta_n \cdots \beta_2\beta_1 & \text{if } t(\beta_n) = s(\alpha_1), \\ 0 & \text{otherwise;} \end{cases}$$

KQ is the *path algebra* of the quiver Q. The algebra KQ can be graded by

$$KQ = KQ_0 \oplus KQ_1 \oplus \cdots \oplus KQ_m \oplus \cdots,$$

where Q_m is the set of all paths of length m; Q_0 is a complete set of primitive orthogonal idempotents of KQ. If Q_0 is finite then KQ is unitary and it is clear that KQ has finite dimension if and only if Q is finite and has no cycles.

An ideal $\Omega \subseteq KQ$ is called an *ideal of relations* or a *relation ideal* if $\Omega \subseteq KQ_2 \oplus KQ_3 \oplus \cdots = KQ_{\geq 2}$. An ideal $\Omega \subseteq KQ$ is *admissible* if it is a relation ideal and there exists a positive integer, m, such that $KQ_m \oplus KQ_{m+1} \oplus \cdots = KQ_{\geq m} \subseteq \Omega$.

By a *quiver with relations* we mean a pair (Q, Ω), where Q is a quiver and Ω a relation ideal of KQ. If Ω is admissible then (Q, Ω) is said to be a *bound quiver* (for more details see [ASS05] and [ARS95]).

The path algebra KQ can be viewed as a graded K-coalgebra with comultiplication induced by the decomposition of paths, that is, if $p = \alpha_m \cdots \alpha_1$ is a path from the vertex i to the vertex j, then

$$\Delta(p) = e_j \otimes p + p \otimes e_i + \sum_{i=1}^{m-1} \alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 = \sum_{\eta\tau = p} \eta \otimes \tau$$

and for a trivial path, e_i, we have $\Delta(e_i) = e_i \otimes e_i$. The counit of KQ is defined by the formula

$$\epsilon(\alpha) = \begin{cases} 1 & \text{if } \alpha \in Q_0, \\ 0 & \text{if } \alpha \text{ is a path of length } \geq 1. \end{cases}$$

The coalgebra (KQ, Δ, ϵ) is the *path coalgebra* of the quiver Q.

Proposition 1.6.1. *Let Q be a quiver and KQ be the associated path coalgebra. Then:*

(a) $KQ = KQ_0 \oplus KQ_1 \oplus \cdots \oplus KQ_n \oplus \cdots$ *is a graded K-coalgebra.*

(b) *The subcoalgebras $KQ_0 \subseteq KQ_0 \oplus KQ_1 \subseteq KQ_0 \oplus KQ_1 \oplus KQ_2 \subseteq \cdots$ give the coradical filtration of KQ.*

(c) *Every simple right KQ-comodule is isomorphic to Ke_i for some trivial path e_i.*

(d) KQ *is pointed.*

(e) $Soc(KQ) = \bigoplus_{i \in Q_0} Ke_i$.

(f) *For each $i \in Q_0$, the injective envelope of S_i is generated by the set of all paths ending at e_i.*

We introduce path coalgebras in another way which allow us to relate any pointed coalgebra with a path coalgebra.

Following [Nic78], let C be a coalgebra and M be a (C, C)-bicomodule. Then we may construct the *cotensor coalgebra*

$$CT_C(M) = C \oplus M \oplus M \square_C M \oplus M \square_C M \square_C M \oplus \cdots$$

Since the cotensor product of M n-times is usually denoted by $M^{\square n}$, then we shall write $CT_C(M) = \bigoplus_n M^{\square n}$.

We may define a comultiplication in $CT_C(M)$ given by

$$\begin{aligned}&\Delta(m_1 \otimes m_2 \otimes \cdots \otimes m_n) = \omega^l(m_1) \otimes m_2 \otimes \cdots \otimes m_n + \\ &+ \sum_{i=1}^{n-1} (m_1 \otimes \cdots \otimes m_i) \otimes (m_{i+1} \otimes \cdots \otimes m_n) + m_1 \otimes \cdots \otimes m_{n-1} \otimes \omega^r(m_n)\end{aligned}$$

where ω^l and ω^r are the structure maps of M as left and right C-comodule; and a counit given by $\epsilon = \epsilon_C \circ \pi$ where π is the projection onto C.

Example 1.6.2. *Let Q be a quiver. Then it is easy to prove from the definition that $KQ \cong CT_{KQ_0}(KQ_1)$. Furthermore, each piece $(KQ_1)^{\square n} \cong KQ_n$.*

An element $x \in C$ is said to be a *group-like element* if $\Delta_C(x) = x \otimes x$. It is not hard to prove that the set of group-like elements, $\mathcal{G}(C)$, is bijective with the set of one dimensional subcoalgebras (which are simple) by the map $x \mapsto Kx$. If C is pointed then all simple subcoalgebras are 1-dimensional so the group-like elements generate the coradical.

Let x and y be two group-like elements. We say that $c \in C$ is a (x, y)-*primitive element* if $\Delta_C(c) = y \otimes c + c \otimes x$. We denote the vector space of (x, y)-primitive elements of C by $P^C_{x,y}$. Note that the vector space $T^C_{x,y} = K(x - y) \subseteq P^C_{x,y}$. These elements are called the trivial (x, y)-primitive elements. We will denote the vector space formed by the non-trivial (x, y)-primitive elements $P^C_{x,y}/T^C_{x,y}$ by $P'^C_{x,y}$.

Lemma 1.6.3. *[Mon93] Let C be a pointed coalgebra and*

$$C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n \subseteq \cdots$$

be its coradical filtration. Then $C_1 = \bigoplus_{x,y\in\mathcal{G}(C)} P^C_{x,y}$. Consequently, $C_1/C_0 = \bigoplus_{x,y\in\mathcal{G}(C)} P'^C_{x,y}$.

Observe that C_0 is a coalgebra and C_1/C_0 is a (C_0, C_0)-bicomodule with structure maps $\omega^l(c) = y \otimes c$ and $\omega^r(\alpha) = c \otimes x$ for each $c \in P'^C_{x,y}$. Thus, for each coalgebra C, we may associate its cotensor coalgebra $CT_{C_0}(C_1/C_0)$.

Proposition 1.6.4. *Every pointed coalgebra is a subcoalgebra of its cotensor coalgebra.*

Proof. By [Nic78], a cotensor coalgebra $CT_{C_0}(C_1/C_0)$ verifies that if C' is a coalgebra, $h : C' \to C_0$ and $q : C \to C'$ are coalgebras maps, and $f : C \to C_1/C_0$ is a (C, C)-bicomodule map with $f(\operatorname{Soc} C) = 0$ then there exists a unique coalgebra map $F : C \to CT_{C_0}(C_0/C_1)$ such that the diagrams

$$\begin{tikzcd} CT_C(M) \arrow[d, "\pi"'] & D' \arrow[l, "F"'] \arrow[d, "q"] \\ C & C' \arrow[l, "h"] \end{tikzcd} \qquad \begin{tikzcd} CT_C(M) \arrow[d, "p"'] & \\ M & D' \arrow[l, "f"] \arrow[ul, "F"'] \end{tikzcd}$$

are commutative, where π and p are projections. Furthermore, the map F is exactly $h \circ q + \sum_{n\geq 0} T_n(f)\Delta_{n-1}$.
In this case, we choose $C' = C_0$, $h = id$, q the projection from $C = C_0 \oplus I$ onto C_0 and $f : C = C_0 \oplus I \to C_1/C_0$ the linear projection from I to C_1/C_0 extended to C_0 taking $f(C_0) = 0$. Then $F|_{C_1} = id$ and thus F is injective (see [Nic78], [Rad78] and [Mon93] for details). $\square$

Given a pointed coalgebra, C, we can construct a quiver Q in the following way: Q_0 will be the set of group-like elements and, for each $x, y \in Q_0$, the number of arrows from x to y equals $dim_K P'^C_{x,y}$. This quiver is called the *Gabriel quiver* of C. Also it is known as the *Ext-quiver* of C because of the vector space $P'^C_{x,y} \cong Ext^1_C(Kx, Ky)$.

Lemma 1.6.5. *Let C be pointed coalgebra and Q be the quiver associated to C. Then $CT_{C_0}(C_1/C_0) \cong KQ$.*

Proof. We have that $KQ_0 \cong C_0$ as coalgebras and $KQ_1 \cong C_1/C_0$ as (C_0, C_0)-bicomodules. Thus $CT_{C_0}(C_1/C_0) \cong CT_{KQ_0}(KQ_1) \cong KQ$. $\square$

As a consequence of Proposition 1.6.4 and Lemma 1.6.5, we obtain the main result of this section.

Theorem 1.6.6. *[Woo97] Let C be a pointed coalgebra. Then C is isomorphic to a subcoalgebra of the path coalgebra of its Gabriel quiver. Furthermore, C contains the subcoalgebra generated by all vertices and all arrows.*

A subcoalgebra of a path coalgebra is said to be *admissible* if it contains the subcoalgebra generated by all vertices and all arrows, that is, $KQ_0 \oplus KQ_1$ (see [Woo97]). A subcoalgebra C of a path coalgebra KQ is called a *relation subcoalgebra* (see [Sim05]) if C satisfies the following conditions:

(a) C is admissible.

(b) $C = \bigoplus_{x,y \in Q_0} C \cap KQ(x, y)$, where $KQ(x, y)$ is the subspace generated by all paths starting at x and ending at y.

Chapter 2

Path Coalgebras of Quivers with Relations

Path algebras of bound quivers are one of the major tools in the representation theory of finite dimensional algebras. Indeed, a very well-known result of Gabriel (see for instance [ASS05], [ARS95], [GR92] and the references given there) asserts that any finite dimensional basic algebra is isomorphic to a quotient of the path algebra of its Gabriel quiver modulo an admissible ideal. The main aim of this chapter is to study the possibility of an analogous result for coalgebras, through the notion of the path coalgebra of a quiver with relations defined by Simson in [Sim01]. For this purpose we establish a general framework using the weak* topology on the dual algebra to treat the problem in an elementary context. Next, a result of [JMR] allows us to obtain a more manageable basis of a relation coalgebra which we use in order to give a criterion for deciding whether or not a relation subcoalgebra is the path coalgebra of a quiver with relations.

2.1 Duality

One can see from the definition of the coalgebra structure that there should be some kind of duality between algebras and coalgebras (the structure of coalgebra is obtained reversing the maps in the algebra structure). The aim of this section is to recall this duality and some known facts involving it in order to apply them throughout this work.

Let (C, Δ, ϵ) be a K-coalgebra, then we equip the dual K-vector space $C^* = \mathrm{Hom}_K(C, K)$ with a K-algebra structure in the following way:

- The product m is the composition of the maps

$$\begin{tikzcd} C^* \otimes C^* \arrow[r, "\rho"] \arrow[rr, bend right, "m"'] & (C \otimes C)^* \arrow[r, "\Delta^*"] & C^*, \end{tikzcd}$$

  where ρ is defined by $\rho(f \otimes g)(v \otimes w) = f(v)g(w)$ for any $f, g \in C^*$ and $u, v \in C$. That is, for each $f, g \in C^*$, $m(f \otimes g) = (f \otimes g) \circ \Delta$. This product is known as the *convolution product*. We shall denote $m(f \otimes g)$ by $f * g$ or simply by fg.

- The unit is $u = \epsilon^* : K \to C^*$.

Proposition 2.1.1. *[Tak77] (C^*, m, u) is a K-algebra called the* dual algebra *of C.*

We can relate the subspaces of C and its dual algebra. Let $c \in C$. The *orthogonal space* to c is the vector space $c^\perp = \{f \in C^* \mid f(c) = 0\}$. More generally, for any subset $S \subseteq C$, we may define the orthogonal space to S to be the space

$$S^\perp = \{\ f \in C^* \mid f(S) = 0\}.$$

On the other hand, for any subset $T \subseteq C^*$, the *orthogonal space* to T in C is defined by the formula

$$T^\perp = \{\ c \in C \mid f(c) = 0 \text{ for all } f \in T\}.$$

We say that $T \subseteq C^*$ is *closed* if $T^{\perp\perp} = T$.

Proposition 2.1.2. *[Swe69]*

(a) *If $D \subseteq C$ is a subcoalgebra then $D^\perp$ is an ideal of C^*.*

(b) *If $I \subseteq C^*$ is an ideal then $I^\perp$ is a subcoalgebra of C.*

(c) *$D \subseteq C$ is a subcoalgebra if and only if $D^\perp$ is an ideal of C^*. In this case $C^*/D^\perp \cong D^*$ as algebras.*

Proposition 2.1.3. *[Swe69]*

(a) *If $J \subseteq C$ is a right (left) coideal then $J^\perp$ is a right (left) ideal of C^*.*

(b) *If $I \subseteq C^*$ is a right (left) ideal then $I^\perp$ is a right (left) coideal of C.*

(c) *$J \subseteq C$ is a right (left) coideal if and only if $J^\perp$ is a right (left) ideal of C^*.*

Proposition 2.1.4. (a) *If $J \subseteq C$ is a coideal then $D^\perp$ is a subalgebra of C^*.*

(b) *If $I \subseteq C^*$ is a subalgebra then $I^\perp$ is a coideal of C.*

(c) *$J \subseteq C$ is a coideal if and only if $D^\perp$ is a subalgebra of C^*.*

In general, if (A, m, u) is a K-algebra then its dual vector space, A^*, does not have to be a K-coalgebra. That fact comes true if A is finite dimensional since, in that case, the map ρ defined above is bijective. Therefore we take $\Delta = \rho^{-1} \circ m^*$ and $\epsilon = u^*$ and then (A^*, Δ, ϵ) is a coalgebra. Thus we get an equivalence between the category of finite dimensional coalgebras and finite dimensional algebras over a field.

$$\mathcal{F}in\mathcal{D}im\mathcal{A}lg_K \xleftrightarrow{(-)^*} \mathcal{F}in\mathcal{D}im\mathcal{C}oalg_K$$

Now, we know that every coalgebra is direct limit of its finite dimensional subcoalgebras so we can see a coalgebra as direct limit of finite dimensional algebras. For that reason coalgebras might be considered as an intermediate structure between finite dimensional and infinite dimensional algebras.

A coalgebra C can be endowed with a right and left C^*-module structure using the actions $\leftharpoonup$ and $\rightharpoonup$ defined by

$$c \leftharpoonup f = \sum_{(c)} f(c_{(1)})c_{(2)} \quad \text{and} \quad f \rightharpoonup c = \sum_{(c)} f(c_{(2)})c_{(1)},$$

where $f \in C^*$ and $c \in C$ such that $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ using the sigma-notation of Sweedler (see [Swe69]). For simplicity we will write cf and fc instead of $c \leftharpoonup f$ and $f \rightharpoonup c$.

A right C-comodule (M, ω) can acquire a structure of left C^*-module (M, ρ) (which is called the *rational* C^* structure) where ρ is the composition

$$C^* \otimes M \xrightarrow{I \otimes \omega} C^* \otimes M \otimes C \xrightarrow{T \otimes I} M \otimes C^* \otimes C \xrightarrow{I \otimes e} M \otimes K \cong M,$$

where $T : C^* \otimes M \to M \otimes C^*$ is the *flip map* defined by $T(f \otimes m) = m \otimes f$ for any $f \in C^*$ and $m \in M$, and e is the evaluation map. That is, using the sigma-notation

$$fm = \rho(f \otimes m) = \sum_{(m)} f(m_{(1)})m_{(0)},$$

where $f \in C^*$ and $m \in M$ such that $\omega(m) = \sum m_{(0)} \otimes m_{(1)}$. Observe that if $M = C$, we obtain the structure defined above. Analogously, for a left C-comodule we can define a right C^*-module structure.

The reader should consider the question of which modules arise in the above fashion from comodules. The solution comes from the so-called rational modules (or discrete modules in the terminology of [Sim01]).

Let (M, ρ) be a left C^*-module and $\omega : M \to \mathrm{Hom}_{C^*}(C^*, M)$ be the linear map defined by $\omega(m)(f) = \rho(f \otimes m)$ for any $f \in C^*$ and $m \in M$. There exist the following injective maps:

$$\begin{array}{lllll} M \otimes C \longrightarrow M \otimes C^{**} & \xrightarrow{f} & \mathrm{Hom}_{C^*}(C^*, M) & & \\ m \otimes c \longmapsto m \otimes c^{**} & \longmapsto & f_{m\otimes c^{**}} : C^{**} & \longrightarrow & M \\ & & c^* & \longmapsto & f_{m\otimes c^{**}}(c^*) = c^{**}(c^*)m \end{array}$$

Then a C^*-module is called *rational* if $\omega(M) \subseteq M \otimes C$.

Proposition 2.1.5. *Let (M, ρ) be a rational left C^*-module. Then (M, ω) is a right C-comodule.*

This produces an equivalence of categories, $\mathcal{M}^C \cong \mathrm{Rat}(C^*)$, between the category of right C-comodules and the category of rational left C^*-modules.

2.2 Pairings and weak* topology

This is a technical section devoted to developing some basic facts on topologies induced by pairing of vector spaces which will be useful in what follows. For further information see [Abe77], [HR73], [Rad74a] and [Rad74b].

Let V and W be vector spaces over a field K. A *pairing* (V, W) of V and W is a bilinear map $\langle -, - \rangle : V \times W \to K$.

A pairing $\langle -, - \rangle$ is *non degenerate* if the following properties hold

$$\begin{cases} \text{if } \langle v, w \rangle = 0 \text{ for all } v \in V, \text{ then } w = 0, \\ \text{if } \langle v, w \rangle = 0 \text{ for all } w \in W, \text{ then } v = 0. \end{cases}$$

This means that the linear maps $\sigma : V \longrightarrow W^*$ and $\tau : W \longrightarrow V^*$ defined by $\sigma(v)(w) = \langle v, w\rangle$ and $\tau(w)(v) = \langle v, w\rangle$ for all $v \in V$ and $w \in W$ are injective.

Throughout this section all pairings will be non-degenerate.

A well-known example of a non degenerate pairing is the dual pairing, (V, V^*), given by the evaluation map $\langle v, f\rangle = f(v)$ for all $v \in V$, $f \in V^*$.

Given a pairing, (V, W), we can relate subspaces of V and W through the dual pairing, compare with last section. Let $v \in V$. The *orthogonal complement* to v is the set $v^\perp = \{f \in V^* \mid f(v) = 0\}$. More generally, for any subset $S \subseteq V$, we may define the orthogonal complement to S to be the space

$$S^\perp = \{\ f \in V^* \mid f(S) = 0\}.$$

Since W can be embebed in V^* by the pairing, we may consider the orthogonal subspace to S in W

$$S^{\perp_W} = S^\perp \cap W = \{w \in W \mid \langle S, w\rangle = 0\}.$$

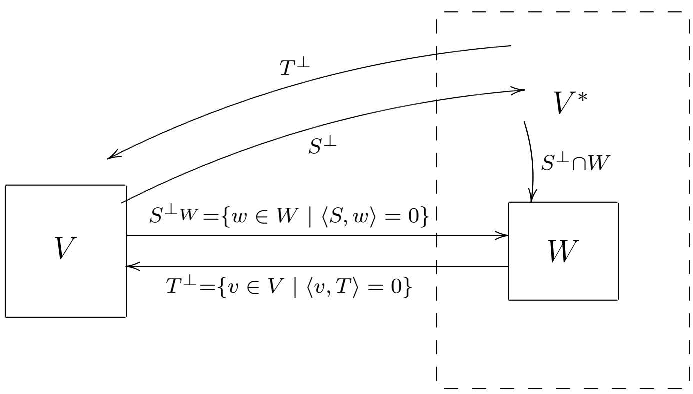
On the other hand, for any subset $T \subseteq V^*$, the *orthogonal complement* to T in V is defined by the formula

$$T^{\perp_V} = \{\ v \in V \mid f(v) = 0 \text{ for all } f \in T\},$$

and if $T \subseteq W$, then we write $T^{\perp_V} = \{\ v \in V \mid \langle v, w\rangle = 0 \text{ for all } w \in T\}$.

For simplicity we write $\perp$ instead of $\perp_V$.

The following diagram summarizes the above discussion:



The following lemma gives a neighbourhood subbasis and a neighbourhood basis of a topology on V^*. We call it the *weak* topology* on V^*, see [Abe77], [Rad74a] and [Rad74b].

Lemma 2.2.1. *Let f be a linear map in V^*.*

(a) *The set $\mathcal{U}_f = \{\, f + v^\perp \mid v \in V\}$ is a neighbourhood subbasis of f for a topology on V^*.*

(b) *The sets $\mathcal{B}^f_{x_1,\ldots,x_n} = \{g \in V^* \mid g(x_i) = f(x_i)\ \forall i = 1, \ldots, n\} \subseteq V^*$, for any $x_1, \ldots, x_n \in V$ and $n \in \mathbb{N}^*$, form a neighbourhood basis at f for the topology on V^* defined in (a).*

Proof. (a) This is straightforward.

(b) The finite intersections of elements of a neighbourhood subbasis form a neighbourhood basis and it is easy to check that

$$f + x^\perp = \{g \in V^* \mid g(x) = f(x)\},$$

for any $x \in V$.

□

If we view W as a subspace of the vector space V^*, the induced topology on W is called the *V-topology*.

In the next proposition we collect some properties of the weak* topology which we shall need.

Proposition 2.2.2. *Let (V, W) be a pairing of K-vector spaces.*

(a) *The weak* topology is the weakest topology on V^* which makes continuous the elements of V, that is, it is the initial topology for the elements of V.*

(b) *The closed subspaces on the weak* topology are $S^\perp$, where S is a subspace of V.*

(c) *The closure of a subspace T of V^* (in the weak* topology) is $T^{\perp\perp}$.*

(d) *The closed subspaces on the V-topology are $S^{\perp_W}$, where S is a subspace of V.*

(e) *The closure of a subspace T of W (in the V-topology) is $T^{\perp\perp_W}$.*

(f) *Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of subspaces of V. Then*

$$\left(\sum_{\lambda\in\Lambda} A_\lambda\right)^{\perp} = \bigcap_{\lambda\in\Lambda} A_\lambda^{\perp} \quad \text{and} \quad \left(\sum_{\lambda\in\Lambda} A_\lambda\right)^{\perp_W} = \bigcap_{\lambda\in\Lambda} A_\lambda^{\perp_W}.$$

(g) *Any finite dimensional subspace of* W *is closed.*

Proof. (a) Let $\mathcal{T}$ be the initial topology for the elements of V, and $\mathcal{W}$ the weak* topology on V^*. Let $k \in K$ and $\mathrm{ev}_y \in V$ be the evaluation on y. Then

$$(\mathrm{ev}_y)^{-1}(k) = \{f \in V^* \mid f(y) = k\}.$$

But given $g \in (\mathrm{ev}_y)^{-1}(k)$ we obtain $g \in g + y^{\perp} \subseteq (\mathrm{ev}_y)^{-1}(k)$ so $(\mathrm{ev}_y)^{-1}(k)$ is an open set in weak* topology and thus $\mathcal{T} \subseteq \mathcal{W}$. Conversely, given $f \in V^*$ and $x \in V$, a neighbourhood of f in weak* topology is $f + x^{\perp} = \mathrm{ev}_x^{-1}(f(x))$, which is open in $\mathcal{T}$ and thus $\mathcal{W} \subseteq \mathcal{T}$.

(b) Let $S \subseteq V$, if $f \notin S^{\perp}$ then there exists $x \in S$ such that $f(x) \neq 0$. Thus $(f + x^{\perp}) \cap S^{\perp} = \emptyset$ and $f \notin \overline{S^{\perp}}$. Conversely, let T be a closed subspace; it suffices to prove that $T^{\perp\perp} \subseteq T$. Fix $f \in T^{\perp\perp}$ and $x \in V$; if $x \in T^{\perp}$ then $f(x) = 0$, hence $0 \in (f + x^{\perp}) \cap T$. If, on the contrary, $x \notin T^{\perp}$ then there exists $g \in T$ such that $g(x) \neq 0$, therefore $\frac{f(x)}{g(x)} g \in (f + x^{\perp}) \cap T$. This shows that $f \in \overline{T} = T$.

(c) $T^{\perp\perp}$ is a closed set satisfying $T \subseteq T^{\perp\perp}$, therefore $\overline{T} \subseteq T^{\perp\perp}$. We can now proceed analogously to the proof of (b) to show $T^{\perp\perp} \subseteq \overline{T}$.

(d) The V-topology on W is induced by the weak* topology on V^* so $S^{\perp_W} = S^{\perp} \cap W$ is closed. If T is closed, then $T = \overline{T}^W = \overline{T} \cap W = T^{\perp\perp} \cap W = T^{\perp\perp_W}$.

(e) The proof is straightforward from (d).

(f) We have

$$\begin{aligned} f \in \bigcap_{\lambda \in \Lambda} A_\lambda^{\perp} \quad &\Leftrightarrow \quad f(A_\lambda) = 0 \quad \forall \lambda \in \Lambda, \\ &\Leftrightarrow \quad f(\textstyle\sum_{\lambda \in \Lambda} A_\lambda) = 0, \\ &\Leftrightarrow \quad f \in \left(\textstyle\sum_{\lambda \in \Lambda} A_\lambda\right)^{\perp}. \end{aligned}$$

(g) See [Abe77, Chapter 2].

$\square$

Finally, from the point of view of subspaces of V we have

Lemma 2.2.3. *Let* (V, W) *be a pairing of* K*-vector spaces.*

(a) *Let A be a subspace of V. Then $A^{\perp\perp} = A$.*

(b) *Let A be a finite dimensional subspace of V. Then $A^{\perp_W \perp} = A$.*

(c) *Let $\{T_i\}_{i\in I}$ be a family of subspaces of V^*. Then*

$$\left(\sum_{i\in I} T_i\right)^{\perp} = \bigcap_{i\in I} T_i^{\perp}.$$

Proof. (a) $f(A) = 0$ for each $f \in A^{\perp}$ and so $A \subseteq A^{\perp\perp}$. Converselly, let $v \notin A \subsetneq V$. There exists $f \in V^*$ such that $f(A) = 0$ and $f(v) \neq 0$. By Proposition 2.2.2, $A^{\perp}$ is closed so $A^{\perp\perp\perp} = A^{\perp}$ and therefore, $\forall g \in V^*$, $g(A) = 0 \Leftrightarrow g(A^{\perp\perp}) = 0$, which implies that $v \notin A^{\perp\perp}$.

(b) See, for instance, [Abe77, Theorem 2.2.1].

(c) We have

$$\begin{aligned} v \in \bigcap_{i\in I} T_i^{\perp} &\Leftrightarrow f(v) = 0 \quad \forall f \in T_i \ \forall i \in I, \\ &\Leftrightarrow f(v) = 0 \ \forall f \in \sum_{i\in I} T_i, \\ &\Leftrightarrow v \in \left(\sum_{i\in I} T_i\right)^{\perp}. \end{aligned}$$

□

2.3 Basis of a relation subcoalgebra

The aim of this section is to obtain a more manageable basis for a relation subcoalgebra of a path coalgebra. For more information and technical properties of subcoalgebras see [JMR].

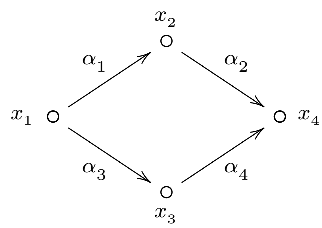
Let $Q = (Q_0, Q_1)$ be a quiver and C a subcoalgebra of KQ. Fix a path $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$ in Q; a *subpath* of p is a path, q, such that either q is a vertex of p or q is a non-trivial path $\alpha_i \alpha_{i+1} \cdots \alpha_j$, where $1 \leq j \leq i \leq n$.

Lemma 2.3.1. *Let $C \subseteq KQ$ be a subcoalgebra, and p be a path in C. Then all subpaths of p are in C.*

Proof. See [JMR]. □

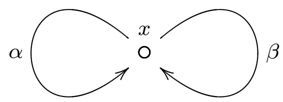
This result could lead the reader to ask if any subcoalgebra could be generated by a set of paths. Unfortunately this is not true as the next examples show.

Example 2.3.2. *Let Q be the quiver*



The subspace generated by $\{e_{x_1}, e_{x_2}, e_{x_3}, e_{x_4}, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2\alpha_1+\alpha_4\alpha_3\}$ *is a subcoalgebra of KQ which cannot be generated by paths.*

Example 2.3.3. *Let Q be the quiver*



The subcoalgebra $C = K\{e_x, \alpha+\beta\}$ is not generated by paths.

One may observe that, in the preceding examples, the basic elements which are not paths have the common property of being a linear combination of paths with the same source and the same sink. The next proposition asserts that, in general, every subcoalgebra of a path coalgebra has this property.

Proposition 2.3.4. *Let Q be a quiver and $C \subseteq KQ$ a subcoalgebra. Then there exists a K-linear basis of C such that each basic element is a linear combination of paths with common source and common sink.*

Proof. See [JMR, Proposition 2.8]. □

Corollary 2.3.5. *Any admissible subcoalgebra of a path coalgebra is a relation subcoalgebra.*

Proposition 2.3.4 is the key-tool which allows us to give a more precise description of the basis of a relation subcoalgebra. Throughout, we assume that C is a relation subcoalgebra and $\mathcal{B}$ is a K-linear basis of C as in Proposition 2.3.4. By definition, C contains the set of all vertices, $V = \{e_i\}_{i \in Q_0}$, and the set of all arrows,

$F = \{\alpha\}_{\alpha \in Q_1}$, therefore we rearrange the elements of the basis $\mathcal{B}$ as follows:

$$\mathcal{B} = V \cup F \cup \{G_{ij}^{\tau} \mid \tau \in \mathrm{T}_{ij} \text{ and } i, j \in Q_0\},$$

where, for all $\tau \in \mathrm{T}_{ij}$, the element G_{ij}^{τ} is a K-linear combination of paths with length greater than one which start at i and end at j.

We now assume that $D = \{p_\lambda\}_{\lambda \in \Lambda}$ is the set of all paths of length greater than one in C. Proceeding as before we can be write

$$\mathcal{B} = V \cup F \cup D \cup \{R_{ij}^{\upsilon} \mid \upsilon \in \mathrm{U}_{ij} \text{ and } i, j \in Q_0\},$$

where, for all $\upsilon \in \mathrm{U}_{ij}$, the element R_{ij}^{υ} is a K-linear combination of at least two paths of length greater than one which start at i and end at j. Obviously, the paths involved in the linear combinations R_{ij}^{υ} are not in C, for any $\upsilon \in \mathrm{U}_{ij}$ and $i, j \in Q_0$.

For the convenience we introduce some notation. We denote by $\mathcal{Q} = Q_0 \cup Q_1 \cup \cdots \cup Q_n \cup \cdots$ the set of all paths in Q. Let a be an element of KQ. Then we can write $a = \sum_{p \in \mathcal{Q}} a_p p$, for some $a_p \in K$. We define *the path support* of a to be $\mathrm{PSupp}(a) = \{p \in \mathcal{Q} \mid a_p \neq 0\}$. In this way, for any set $S \subseteq KQ$, we define $\mathrm{PSupp}(S) = \bigcup_{a \in S} \mathrm{PSupp}(a)$.

Definition 2.3.6. *Let S be a set in KQ. S is called* connected *if* $\mathrm{PSupp}(S_1) \cap \mathrm{PSupp}(S_2) \neq \emptyset$ *for any subsets $S_1, S_2 \subseteq S$ such that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$. A subset $S' \subset S$ is a* connected component *of S when S' is connected and* $\mathrm{PSupp}(S') \cap \mathrm{PSupp}(S \backslash S') = \emptyset$.

Therefore we can break down each set $S_{ij} = \{R_{ij}^{\upsilon}\}_{\upsilon \in \mathrm{U}_{ij}}$ into its connected components and then write the basis $\mathcal{B}$ of C as

$$\mathcal{B} = V \cup F \cup D \cup \bigcup_{\phi \in \Phi} \Upsilon_\phi,$$

where, for any $\phi \in \Phi$, the set Υ_ϕ is a connected set of K-linear combinations of at least two paths such that $\mathrm{PSupp}(\Upsilon_\phi) \subset KQ_{\geq 2}$ and $\mathrm{PSupp}(\Upsilon_{\phi_1}) \cap \mathrm{PSupp}(\Upsilon_{\phi_2}) = \emptyset \Leftrightarrow \phi_1 \neq \phi_2$.

As a final reduction, it will be useful to distinguish those sets Υ_ϕ which are finite. Thus the basis $\mathcal{B}$ of C can be written as

$$\mathcal{B} = V \cup F \cup D \cup \bigcup_{\gamma \in \Gamma} \Pi_\gamma \cup \bigcup_{\beta \in B} \Sigma_\beta,$$

where Π_γ is a finite set for all $\gamma \in \Gamma$ and Σ_β is infinite for all $\beta \in B$.

2.4 Path coalgebras of quivers with relations

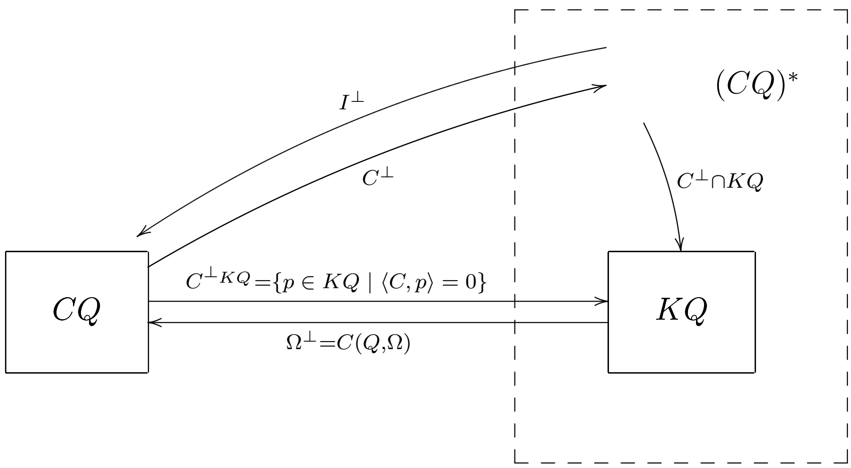
In this section we study the notion of the path coalgebra of a quiver with relations introduced by Simson in [Sim01] and [Sim05]. For the convenience of the reader we shall denote by CQ and by KQ the path coalgebra and the path algebra associated to a quiver Q, respectively (despite that the underlying vector space is the same).

Definition 2.4.1. *Let* (Q, Ω) *be a quiver with relations. The* path coalgebra *of* (Q, Ω) *is defined by the subspace of* CQ*,*

$$C(Q, \Omega) = \{a \in CQ \mid \langle a, \Omega \rangle = 0\}$$

where $\langle -, - \rangle : CQ \times KQ \longrightarrow K$ *is the bilinear map defined by* $\langle v, w \rangle = \delta_{v,w}$ *(the Kronecker delta) for any two paths* $v, w \in \mathcal{Q}$*.*

This notion may be reformulated in the notation of the Section 2.2. It is clear that $\langle -, - \rangle$ is a non-degenerate pairing between CQ and KQ, therefore we have the following picture:



First we prove the following result.

Lemma 2.4.2. *If* Q *is any quiver, then the injective morphism* $KQ \hookrightarrow (CQ)^*$ *defined by the pairing* $\langle -, - \rangle$ *of* 2.4.1 *is a morphism of algebras.*

Proof. Recall that in the dual algebra $(CQ)^* := \mathrm{Hom}_K(CQ, K)$ the (convolution) product is defined by

$$(f * g)(p) = \sum_{p = p_2 p_1} f(p_2) g(p_1) \text{ for any } f, g \in (CQ)^* \text{ and any } p \in \mathcal{Q}.$$

Fix $p \in \mathcal{Q}$ and let $p^* : CQ \to K$ be the linear map defined by $p^*(q) = \delta_{p,q}$ for any $q \in \mathcal{Q}$. It is enough to prove that $(pq)^* = p^* * q^*$ for any two paths $p, q \in \mathcal{Q}$. To prove this, let r be a path in Q. Then:

$$(p^* * q^*)(r) = \sum_{r=r_2r_1} \delta_{p,r_2}\delta_{q,r_1} = \begin{cases} 0 & \text{if } r \neq pq \\ 1 & \text{if } r = pq \end{cases} = (pq)^*(r),$$

and so $(pq)^* = p^* * q^*$. □

It may be helpful to point out that the algebra KQ does not need to have a unit (it has unit if and only if Q_0 is finite) and then the morphism defined above is an injective morphism of algebras without unit. Therefore the situation is the following:

$$KQ \hookrightarrow (KQ)_1 \hookrightarrow (CQ)^*,$$

where $(KQ)_1 = KQ \oplus K \cdot 1$ is the unification of KQ.

Lemma 2.4.3. *KQ is dense in $(CQ)^*$ in the weak* topology on $(CQ)^*$. Consequently, KQ is dense in $(KQ)_1$ and $(KQ)_1$ is dense in $(CQ)^*$ in the weak* topology on $(CQ)^*$.*

Proof. This is a particular case of Lemma 2.5.2 that we shall prove later. It is enough to consider $C = 0$, obviously, $(KQ)^\perp = 0$ and then $0^\perp \cap KQ = KQ$ is dense in $0^\perp = (CQ)^*$. □

From now on we will make no distinction between elements of KQ and linear maps $f : CQ \to K$ with finite path support, that is, $f(p) = 0$, for almost all p in $\mathcal{Q}$. On the other hand, it is convenient to note that any element $g \in (CQ)^*$ can be written as a formal sum $g = \sum_{p \in \mathcal{Q}} a_p p$, where $a_p = g(p) \in K$.

Corollary 2.4.4. *Let Q be a quiver and C an admissible subcoalgebra of CQ. Then $C^{\perp_{KQ}}$ is a relation ideal of KQ.*

Proof. Since $C^\perp$ is an ideal of $(CQ)^*$, $C^\perp \cap KQ = C^{\perp_{KQ}}$ is an ideal of KQ by Lemma 2.4.2. If $c \in KQ_0 \oplus KQ_1$, then $c \in C$ since C is an admissible subcoalgebra. Therefore $\langle c, C\rangle \neq 0$, so $c \notin C^{\perp_{KQ}}$, which completes the proof. □

The following result, proved in [Sim05], justifies the preceding definition of the path coalgebra of a quiver with relations.

Proposition 2.4.5. *Let Q be a quiver and Ω a relation ideal of KQ, then $C(Q,\Omega) = \Omega^{\perp}$ is a relation subcoalgebra of CQ.*

A *K-linear representation* of a quiver Q is a system

$$X = (X_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1},$$

where X_i is a K-vector space and $\varphi_\alpha : X_i \to X_j$ is a K-linear map for any $\alpha : i \to j$. A morphism $f : (X_i, \varphi_\alpha) \to (Y_i, \psi_\alpha)$ of representations of Q is a system $f = (f_i)_{i \in Q_0}$ of K-linear maps $f_i : X_i \to Y_i$ for any $i \in Q_0$ such that $f_j \varphi_\alpha = \psi_\alpha f_i$ for all $\alpha : i \to j$ in Q_1. We denote by $\mathrm{Rep}_K(Q)$ the Grothendieck K-category of K-linear representations of Q. A representation X of Q is said to be of *finite length* if X_i is a finite dimensional vector space for all $i \in Q_0$ and $X_i = 0$ for almost all indices i, we will denote that subcategory by $\mathrm{rep}_K^{lf}(Q)$. A representation X is *nilpotent* if there exists a $m \geq 2$ such that the composed linear map

$$X_{i_0} \xrightarrow{\varphi_{\alpha_1}} X_{i_1} \xrightarrow{\varphi_{\alpha_2}} X_{i_2} \dashrightarrow \cdots \dashrightarrow X_{i_{m-1}} \xrightarrow{\varphi_{\alpha_m}} X_{i_m}$$

is zero for any path $\alpha_m \alpha_{m-1} \cdots \alpha_1$ in Q of length m. We denote by $\mathrm{Rep}_K^{lf}(Q) \supseteq \mathrm{Rep}_K^{lnlf}(Q)$ the full subcategory of $\mathrm{Rep}_K(Q)$ formed by all locally and locally nilpotent representations of finite length, respectively, and by $\mathrm{nilrep}_K^{lf}(Q)$ the subcategory of all nilpotent representations of finite length.

Given a quiver with relations (Q,Ω), a linear representation of (Q,Ω) is a linear representation $X = (X_i, \varphi_\alpha)$ of Q which verifies that if $p = \sum_{i=1}^n \lambda_i \alpha^i_{m_i} \cdots \alpha^i_1$ is in Ω then $\sum_{i=1}^n \lambda_i \varphi_{\alpha^i_{m_i}} \cdots \varphi_{\alpha^i_1} = 0$. Then, analogously, we may define the categories $\mathrm{Rep}_K(Q,\Omega)$, $\mathrm{rep}_K^{lf}(Q,\Omega)$, $\mathrm{Rep}_K^{lnlf}(Q,\Omega)$, $\mathrm{Rep}_K^{lf}(Q,\Omega)$ and $\mathrm{nilrep}_K^{lf}(Q,\Omega)$.

Theorem 2.4.6 ([Sim05], Theorem3.5)**.** *Let (Q,Ω) be a quiver with relations. There are category isomorphisms*

$$\mathcal{M}_f^{C(Q,\Omega)} \cong \mathrm{nilrep}_K^{lf}(Q,\Omega) \ \textit{and} \ \mathcal{M}^{C(Q,\Omega)} \cong \mathrm{Rep}_K^{lnlf}(Q,\Omega)$$

Then, this definition is consistent with the representation theory of algebras and reduces the study of the category $\mathcal{M}^C$ to the study of linear representations of a quiver with relations.

2.5 When is a coalgebra a path coalgebra of a quiver with relations?

It is well-known that, over a algebraically closed field, a finite dimensional algebra, A, is isomorphic to KQ_A/Ω, where Q_A is the Gabriel quiver of A and Ω is an admissible ideal of KQ. In [Sim01], it is suggested, as an open problem, to relate the admissible subcoalgebras of a path coalgebra CQ and the relation ideals of the path algebra KQ, through the above-mentioned notion of path coalgebra of a quiver with relations. That is, for any admissible subcoalgebra $C \leq CQ$, is there a relation ideal $\Omega \leq KQ$ such that $C = C(Q, \Omega)$? In other words, in the notation of Section 2, for any admissible subcoalgebra $C \leq CQ$, is there a relation ideal Ω of KQ such that $\Omega^{\perp} = C$?

Note that if C has finite dimension, then, by Lemma 2.2.3, $(C^{\perp_{KQ}})^{\perp} = C$ and the result follows. This yields a reduction of the problem:

Problem 2.5.1. *Verify the relation $\Omega^{\perp} = C$ for the ideal $\Omega = C^{\perp_{KQ}}$.*

Lemma 2.5.2. *Let Q be a quiver and C a vector subspace of CQ. Then the following conditions are equivalent.*

(a) *There exists a subspace Ω of KQ such that $\Omega^{\perp} = C$.*

(b) *$C^{\perp_{KQ}}$ is dense in $C^{\perp}$ in the weak* topology on $(CQ)^*$.*

(c) *$(C^{\perp_{KQ}})^{\perp} = C$.*

Proof. $(a) \Rightarrow (b)$. Since $C = \Omega^{\perp}$, it follows that $C^{\perp} = \Omega^{\perp\perp}$ is the closure of Ω in weak* topology by Proposition 2.2.2. Thus $\Omega \subset C^{\perp} \cap KQ = C^{\perp_{KQ}} \subset C^{\perp}$ and, by Proposition 2.2.3, $C = C^{\perp\perp} \subset (C^{\perp_{KQ}})^{\perp} \subset \Omega^{\perp} = C$. Therefore $C = (C^{\perp_{KQ}})^{\perp}$ and thus $C^{\perp} = (C^{\perp_{KQ}})^{\perp\perp} = \overline{C^{\perp_{KQ}}}$.

$(b) \Rightarrow (c)$. Since $C^{\perp} = (C^{\perp_{KQ}})^{\perp\perp}$, we have $C^{\perp\perp} = (C^{\perp_{KQ}})^{\perp\perp\perp}$ and, by Proposition 2.2.3, $C = (C^{\perp_{KQ}})^{\perp}$.

$(c) \Rightarrow (a)$. It is trivial. □

We now assume that C is an admissible subcoalgebra of CQ. If we consider the basis of C,

$$\mathcal{B} = V \cup F \cup D \cup \bigcup_{\gamma \in \Gamma} \Pi_{\gamma} \cup \bigcup_{\beta \in B} \Sigma_{\beta},$$

built in Section 3, then we have

$$C = KV \oplus KF \oplus KD \oplus \left(\bigoplus_{\gamma \in \Gamma} K\Pi_\gamma \right) \oplus \left(\bigoplus_{\beta \in B} K\Sigma_\beta \right) \tag{2.1}$$

as K-vector space. Since the subsets into which we have partitioned $\mathcal{B}$ have disjoint path supports, it is easily seen that $\Omega^\perp = C$ if and only if each direct summand C_i of (2.1) is the orthogonal complement $\Omega_i^\perp$ of a subspace Ω_i and, in this case, $\Omega = \bigcap \Omega_i$.

There are two trivial cases:

CASE 1. It is immediate that $KV = K(\mathcal{Q} \backslash V)^\perp$, $KF = K(\mathcal{Q} \backslash F)^\perp$ and $KD = K(\mathcal{Q} \backslash D)^\perp$.

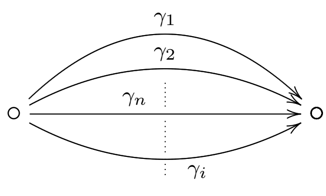
CASE 2. For each $\gamma \in \Gamma$, $K\Pi_\gamma$ is a finite dimensional subspace and so, by Lemma 2.2.3, $K\Pi_\gamma = ((K\Pi_\gamma)^{\perp_{KQ}})^\perp$. As a consequence we get:

Corollary 2.5.3. *With the above notation, $C = \Omega^\perp$ if and only if $\Sigma_\beta = (\Sigma_\beta)^{\perp_{KQ}\perp}$ for each $\beta \in B$.*

In particular, this implies the following proposition proved in [Sim05].

Proposition 2.5.4. *Let Q be a quiver without cycles such that the set of paths in Q from i to j is finite, for all $i, j \in Q_0$. Then the map $C \longmapsto C^{\perp_{KQ}}$ define a bijection between the set of all relation subcoalgebras of CQ and the set of all admissible ideals of KQ. The inverse map is defined by $\Omega \longmapsto \Omega^\perp$, for any relation ideal Ω of KQ.*

Therefore, we can reduce Problem 2.5.1 to the situation of a quiver Q with the following structure

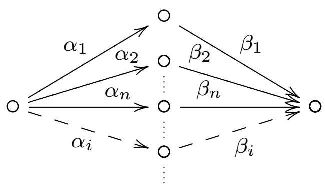
$Q \equiv$ 

length(γ_i)> 1, $i \in I$, I infinite (2.2)

and C an admissible subcoalgebra generated, as vector space, by $V \cup F \cup D \cup \Sigma$, where Σ is an infinite connected set with $\mathrm{PSupp}(\Sigma) = \{\gamma_i\}_{i \in I}$. We may assume that $\gamma_i \notin C$ for all $i \in I$. Then the question is: when the equality $\Sigma = \Sigma^{\perp_{KQ}\perp}$ holds?

Let us first show that, at least, there is an example of an admissible subcoalgebra $C \subseteq CQ$ such that C is not of the form $C = \Omega^\perp$, where Ω is a relation ideal of KQ.

Example 2.5.5. *Let Q be the quiver*



$$\gamma_i = \beta_i\alpha_i \text{ for all } i \in \mathbb{N} \tag{2.3}$$

and let H be the admissible subcoalgebra of CQ as in (2.2) *with $\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i\in\mathbb{N}}$.*

Assume that $x = \sum_{i\geq 1} a_i\gamma_i$ belongs to $H^\perp$ and $a_i = 0$ for $i \geq n$ we have some $n \in \mathbb{N}$. Then $\langle\gamma_i - \gamma_{i+1}, x\rangle = a_i - a_{i+1} = 0$ for all $i \in \mathbb{N}$, so $a_i = a_{i+1}$ for all $i \in \mathbb{N}$. But $a_n = 0$ and it follows that $x = 0$. Hence $H^{\perp_{KQ}} = 0$.

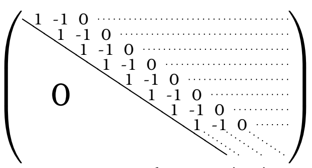
By a similar argument $H^\perp = \langle f\rangle$, where $f(\gamma_i) = 1$ for all $i \in \mathbb{N}$. That is, $f \equiv \sum_{i\geq 1}\gamma_i$. Obviously, $H^{\perp_{KQ}}$ is not dense in $H^\perp$.

Here we present a positive example

Example 2.5.6. *Let Q be the quiver of* (2.3), *and C the admissible subcoalgebra generated by $\Sigma = \{\gamma_{2n-1} + \gamma_{2n} + \gamma_{2n+1}\}_{n\geq 1}$. A straightforward calculation shows that $\Omega^\perp = C$, where $\Omega = \langle\{\gamma_1 - \gamma_2, \{\gamma_{2n} - \gamma_{2n+1} + \gamma_{2n+2}\}_{n\geq 1}\}\rangle$.*

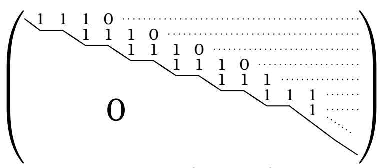
We now analyze them deeply to provide a criterium which allows us to know, when an admissible relation subcoalgebra of CQ is the path coalgebra $C(Q, \Omega)$ of a quiver with relations.

First, it is convenient to see Examples 2.5.5 and 2.5.6 from a more graphic point of view. We write the elements of Σ in matrix form. Thus we have the associated infinite matrices



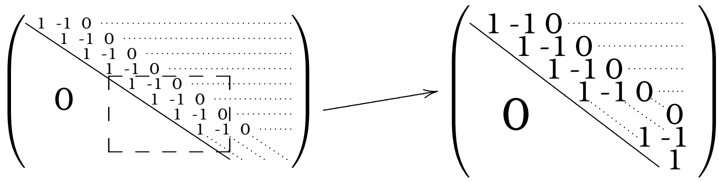
Example 2.5.5

and



Example 2.5.6

We can observe that Example 2.5.5 has an infinite diagonal of non zero elements. Let $h \in H^{\perp_{KQ}}$. Then h must have finite path support, and so, if we want to know h, we only have to solve a finite linear system of equations with associated matrix



but zero is the unique solution.

In this way we obtain a class of admissible subcoalgebras which are not path coalgebras of quivers with relations:

Definition 2.5.7. *Let Q be a quiver as in (2.2), C be an admissible subcoalgebra generated by a connected set Σ with $\mathrm{PSupp}(\Sigma) = \{\gamma_i\}_{i\in I}$ and $\gamma_i \notin C$ for all $i \in I$. We say that C has the* infinite diagonal property *(IDP for short) if there exists a subset $\Sigma' \subseteq \Sigma$ with $\mathrm{PSupp}(\Sigma') = \{\gamma_n\}_{n\in\mathbb{N}}$ such that by means of elementary transformations, Σ' can be reduced to $\{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$, where $a_j^n \in K$ for all $j, n \in \mathbb{N}$.*

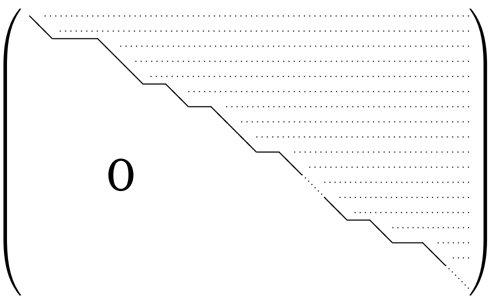
Proposition 2.5.8. *Let Q be a quiver as in (2.2) and C be an admissible subcoalgebra generated by a connected set Σ with $\mathrm{PSupp}(\Sigma) = \{\gamma_i\}_{i\in I}$. Suppose that $\gamma_i \notin C$, for each $i \in I$. If C has IDP, then there is no relation ideal $\Omega \subseteq KQ$ such that $C = C(Q,\Omega)$.*

Proof. Let $\Sigma' = \{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}} \subseteq \Sigma$. Assume that the assertion is not true, i.e., there is a relation ideal $\Omega \subseteq KQ$ such that $C = C(Q,\Omega)$. By Lemma 2.5.2, $C^{\perp_{KQ}}$ is dense in $C^{\perp}$. Since $\gamma_1 \notin C$, there exists a linear map $g \in C^{\perp}$ such that $g(\gamma_1) \neq 0$. By the density of $C^{\perp_{KQ}}$ in $C^{\perp}$, there exists a linear map h with finite path support such that $h(\gamma_1) = g(\gamma_1)$. Defining $x_i := h(\gamma_i)$, for any $i \in \mathbb{N}$, we obtain that $h(\Sigma') = 0$ is the infinite system of linear equations $\{x_n + \sum_{j>n} a_j x_j = 0\}_{n\in\mathbb{N}}$. Since h has finite path support, there exists an integer m such that $x_k = 0$, for $k \geq m$. Hence $x_1, \ldots x_m$ satisfy the finite system of linear equations

$$\begin{aligned} x_1 + a_2^1 x_2 + \cdots + a_m^1 x_m &= 0 \\ x_2 + \cdots + a_m^2 x_m &= 0 \\ &\vdots \\ x_m &= 0 \end{aligned}$$

which has the unique solution $x_m = x_{m-1} = \cdots = x_1 = h(\gamma_1) = 0$, and we get a contradiction. $\square$

We claim that Example 2.5.6 does not have IDP. This means that for any infinite countable subset $\Sigma' \subseteq \Sigma$, the associated matrix can be reduced to a matrix of a ”staircase” form



(2.4)

that is, for any positive integer n, the first n rows have at least n variables and there is an integer $m > n$ such that the first m rows have more than m variables. We can prove that for any linear map $f \in C^{\perp}$ and any finite set $\{\gamma_1, \dots, \gamma_n\}$ of paths in Q, we obtain a linear map $g \in C^{\perp}$ such that $f(\gamma_i) = g(\gamma_i)$ for all $i = 1, \dots, n$. That is, $C^{\perp_{KQ}}$ is dense on $C^{\perp}$.

Proposition 2.5.9. *Under the assumptions of Proposition* 2.5.8*, if* C *fails IDP, then there exists a relation ideal* Ω *such that* $C = C(Q, \Omega)$*.*

Proof. It suffices to show that $\Sigma^{\perp_{KQ}}$ is dense in $\Sigma^{\perp}$, that is, given $f \in \Sigma^{\perp}$ and $\gamma_1, \dots, \gamma_n \in \mathrm{PSupp}(\Sigma)$ there exists $h \in \Sigma^{\perp}$, with finite path support, such that $h(\gamma_i) = f(\gamma_i)$ for all $i = 1, \dots, n$. We give the proof only for $n = 1$; the general case is analogous and left to the reader.

We know that $h(\Sigma) = 0$ produces an infinite system of linear equations with variables $\{h(\gamma_i) = x_i\}_{i \in I}$. We rewrite the system in the following way:

STEP 1. Fix an equation, say E_1, such that the coefficient of x_1 is not zero. We may assume that it is the only one with this property. Suppose that

$$E_1 \equiv x_1 + a_2^1 x_2 + \cdots + a_{r_1}^1 x_{r_1} + \cdots + a_m^1 x_m = 0,$$

where $a_2^1, \dots, a_m^1$ are non zero and $x_1, \dots, x_{r_1 - 1}$ do not appear in any other equation of the system.

STEP 2. We take now x_{r_1}. There is at least one equation, say E_2, different from E_1, in which the coefficient of x_{r_1} is not zero. We eliminate it from the remaining equations different from E_1. Choose variables $x_{r_1+1}, \dots, x_{r_2-1}$ which only appear in E_1 or E_2, and the system starts as

$$\begin{aligned} x_1 + a_2^1 x_2 + \cdots + a_{r_1}^1 x_{r_1} + \cdots + a_m^1 x_m &= 0 \\ x_{r_1} + \cdots + a_m^2 x_m + \cdots + a_l^2 x_l &= 0. \end{aligned}$$

STEP 3. We do the same with x_{r_2} to obtain

$$\begin{aligned} x_1 + \cdots + a_{r_1}^1 x_{r_1} + \cdots + a_{r_2}^1 x_{r_2} + \cdots + a_m^1 x_m &= 0 \\ x_{r_1} + \cdots + a_{r_2}^2 x_{r_2} + \cdots\cdots + a_l^2 x_l &= 0 \\ x_{r_2} + a_{r_2+1}^3 x_{r_2+1} + \cdots + a_h^3 x_h &= 0. \end{aligned}$$

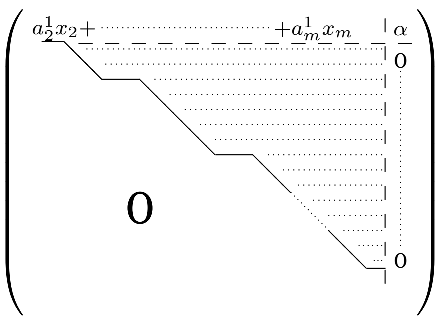
STEP 4. We continue in this fashion. When we finish with the variables of E_1, we proceed with the variables of E_2 and so on. The reader should observe that the variables $x_1, \ldots, x_{r_1}, x_{r_1+1}, \ldots, x_{r_i}$ only appear in the equations $E_1, E_2, \ldots, E_{i+1}$, for all $i \in \mathbb{N}$.

There are two cases to consider:

CASE 1. This process stops after a finite number of steps. Then we consider $x_\alpha = 0$, for all variables outside the finite subsystem which we have obtained. Since any equation has at least two variables, the subsystem has more variables than equations and maximal range. This follows that there is a solution for $x_1 = -f(\gamma_1)$.

CASE 2. This process is infinite. Then we stop after finding a variable x_{r_k} where r_k is the minimal integer such that $r_k > n$ and $r_{k+1} - r_k > 1$ (it is possible because C fails IDP). Roughly speaking, this means that we stop this process on the first 'step' (horizontal segments in (2.4)) after processing the variables of E_1.

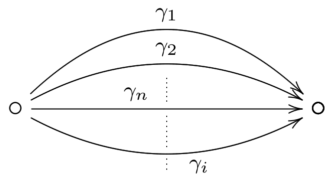
We consider $x_i = 0$, for all $i \neq 1, \ldots, r_k+1$, and therefore it suffices to prove that the finite system of $k+1$ equations and r_k variables



has a solution, where $\alpha = -f(\gamma_1)$. But this is clearly true, because $r_k \geq k+1$ and the matrix of coefficients has maximal range. ■ □

Let Q be a quiver as in (2.2) and C be an admissible subcoalgebra as in the assumption of Proposition 2.5.8. Let us suppose that there exists a subset $\Sigma' \subseteq \Sigma$ such that $\Sigma' = \{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$, where $a_j^n \in K$ for all $j, n \in \mathbb{N}$, and $\gamma_i = \alpha_{n_i}^i \alpha_{n_i-1}^i \cdots \alpha_2\alpha_1$, for all $i \in \mathbb{N}$. We may consider the subquiver $Q' = (Q'_0, Q'_1)$, where $Q'_0 = \{t(\alpha_j^i), s(\alpha_j^i)\}_{j=1,\ldots,n_i}^{i\in\mathbb{N}}$ and $Q'_1 = \{\alpha_j^i\}_{j=1,\ldots,n_i}^{i\in\mathbb{N}}$. Then C contains the admissible subcoalgebra of CQ' generated by Σ'.

Therefore we turn to the case of a quiver Q with the following structure:

$Q \equiv$  length$(\gamma_i) > 1$, $i \in \mathbb{N}$ (2.5)

and C an admissible subcoalgebra of CQ generated by an infinite countable connected set $\Sigma = \{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$, where $a_j^n \in K$ for all $j, n \in \mathbb{N}$. We may suppose that $\gamma_i \notin C$ for all $i \in \mathbb{N}$.

Under these conditions, we denote by $\mathbb{H}_Q^n$ the class of admissible subcoalgebras of CQ such that $\dim_K(\langle \mathrm{PSupp}(\Sigma)\rangle / \langle \Sigma \rangle) = n$ and by $\mathbb{H}_Q$ the class of admissible subcoalgebras of CQ such that the dimension$\dim_K(\langle \mathrm{PSupp}(\Sigma)\rangle / \langle \Sigma \rangle) = \infty$. Finally, we set

$$\mathbb{H}_Q^\infty = \mathbb{H}_Q \cup \bigcup_{n\in\mathbb{N}} \mathbb{H}_Q^n$$

Theorem 2.5.10. *[JMN05] Let Q be any quiver and C be an admissible subcoalgebra of CQ. There exists a relation ideal Ω of KQ such that $C = C(Q, \Omega)$ if and only if there is no subquiver Γ of Q such that C contains a subcoalgebra in $\mathbb{H}_\Gamma^\infty$.*

Proof. This follows from Proposition 2.5.8 and 2.5.9, and the arguments mentioned above. □

Corollary 2.5.11 (Criterion). *Let C be an admissible subcoalgebra of a path coalgebra CQ. Then C is not the path coalgebra of a quiver with relations if and only if there exist an infinite number of different paths $\{\gamma_i\}_{i\in\mathbb{N}}$ in Q such that:*

(a) *All of them have common source and common sink.*

(b) *None of them is in C.*

(c) *There exist elements $a_j^n \in K$ for all $j, n \in \mathbb{N}$ such that the set $\{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$ is contained in C.*

Remark. The reader could ask if an admissible subcoalgebra, C, of CQ_C, which contains a subcoalgebra in $\mathbb{H}_\Gamma^\infty$ can be written as $C(Q',\Omega')$, where Q' is a quiver which is not the Gabriel quiver of C.

We know that there exists an injective map $f : C \longrightarrow CQ$ such that $f|_{C_1} = id$. If there is a quiver Q' and an inclusion $C \overset{i}{\hookrightarrow} CQ'$, the following diagram conmutes:

$$\begin{tikzcd} C_1 \arrow[r, leftrightarrow, "id"] \arrow[dr, hook, "i"'] & CQ_1 \arrow[d, hook, dashed, "i"] \\ & CQ'_1 \end{tikzcd}$$

We need the following lemma to finish our remark.

Lemma 2.5.12. *Let $f : C \to D$ be a morphism of coalgebras.*

(a) *If e is a group-like element of C then $f(e)$ is a group-like element of D.*

(b) *If f is injective and x is a non-trivial (e,d)-primitive element of C then $f(x)$ is a non-trivial $(f(e),f(d))$-primitive element of D.*

Thus, since CQ_1 and CQ'_1 are generated by the set of all vertices and arrows of Q and Q', respectively, using Lemma 2.5.12, we conclude that Q is a subquiver of Q'; so it contains some coalgebra in $\mathbb{H}_\Gamma^\infty$.

As a consequence, we get a negative answer to the following open problem considered by Simson in [Sim01] and [Sim05]: *Is any basic coalgebra, over an algebraically closed field, isomorphic to the path coalgebra of a quiver with relations?*

Chapter 3

Localization in Coalgebras

The category $\mathcal{M}^C$ of right comodules over a coalgebra C is a locally finite Grothendieck category in which the theory of localization described by Gabriel in [Gab62] can be applied. The localizing subcategories of $\mathcal{M}^C$ have been studied in several papers with satisfactory results, see [Gre76], [Lin75], [NT94] and [NT96]. In this context, the idempotent elements of the dual algebra play an important rôle to permit us to give an explicit description of the elements of a situation of localization and to characterize some important classes of localizing subcategories. In particular, we shall consider the stable localizations of $\mathcal{M}^C$ and characterize them by left semicentral idempotent elements. This will be obtained as a consequence of the study of the behavior of injective and simple comodules under the action of the localization functors. Lastly, we shall contemplate the particular case of admissible subcoalgebras of path coalgebras.

3.1 Some categorical remarks about localization

Let $\mathcal{C}$ be an abelian category. A full subcategory $\mathcal{A}$ of $\mathcal{C}$ is said to be *dense* if, for each exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, we have that X belongs to $\mathcal{A}$ if and only if X' and X'' belong to $\mathcal{A}$.

For any dense subcategory $\mathcal{A}$ of $\mathcal{C}$, there exists an abelian category $\mathcal{C}/\mathcal{A}$ and an exact functor $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$ such that $T(X) = 0$, for each $X \in \mathcal{A}$, satisfying the following universal property: for any exact functor $H : \mathcal{C} \to \mathcal{C}'$ such that $H(X) = 0$ for each $X \in \mathcal{A}$, there exists a unique functor $\overline{H} : \mathcal{C}/\mathcal{A} \to \mathcal{C}'$ such that $H = \overline{H}T$. This category $\mathcal{C}/\mathcal{A}$ is called the *quotient category* of $\mathcal{C}$ with respect to $\mathcal{A}$. See [Gab62].

A dense subcategory $\mathcal{A}$ of $\mathcal{C}$ is called *localizing* if the quotient functor $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$ has a right adjoint functor, namely $S : \mathcal{C}/\mathcal{A} \to \mathcal{C}$. The functor S is called the *section* functor of T.

Lemma 3.1.1. *[Gab62] In the above situation, we have that:*

(a) T *is an exact functor.*

(b) S *is a left exact functor.*

(c) S *is a fully faithful functor.*

(d) *The equality* $TS = 1_{\mathcal{C}/\mathcal{A}}$ *holds.*

Conversely, if $T : \mathcal{C} \to \mathcal{C}'$ is an exact functor between abelian categories and $S : \mathcal{C}' \to \mathcal{C}$ is a full and faithful right adjoint functor of T, the dense subcategory $\mathrm{Ker}\,(T)$, whose object class is $\{X \in \mathcal{C} \mid T(X) = 0\}$, is a localizing subcategory of $\mathcal{C}$ and $\mathcal{C}'$ is equivalent to $\mathcal{C}/\mathrm{Ker}\,(T)$, see [Pop73, 4.4.9].

In the particular case in which $\mathcal{C}$ is a Grothendieck category we can say more.

Proposition 3.1.2. *[Gab62] A dense subcategory $\mathcal{A}$ of a Grothendieck category $\mathcal{C}$ is localizing if and only if it is closed under direct sums, or equivalently, if each object $X \in \mathcal{C}$ contains a subobject $\mathcal{A}(X)$ which is maximal among the subobjects of X belonging to $\mathcal{A}$.*

We say that a localizing subcategory is *perfect localizing* if the composition functor $Q = ST : \mathcal{C} \to \mathcal{C}$ is exact, or equivalently, by [Gab62, Chapter III, Corollary 3], if the section functor S is exact.

There exists a dual notion of localizing subcategory. Indeed, if $\mathcal{C}$ is an abelian category, a dense subcategory $\mathcal{A}$ of $\mathcal{C}$ is said to be *colocalizing* if the functor $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$ has a left adjoint functor $H : \mathcal{C}/\mathcal{A} \to \mathcal{C}$, see [NT96]. H is called the colocalizing functor.

Lemma 3.1.3. *[NT96] Let $\mathcal{A}$ be a colocalizing subcategory of a Grothendieck category $\mathcal{C}$. Then $\mathcal{A}$ is a localizing subcategory of $\mathcal{C}$.*

Lemma 3.1.4. *[NT96] Let $\mathcal{A}$ be a colocalizing subcategory of $\mathcal{C}$. Then*

(a) $X \in \mathcal{A}$ *if and only if* $\mathrm{Hom}_{\mathcal{C}}(H(Y), X) = 0$ *for any* $Y \in \mathcal{C}/\mathcal{A}$.

(b) *The colocalizing functor H is a fully faithful and right exact functor.*

(c) *The equality* $TH = 1_{\mathcal{C}/\mathcal{A}}$ *holds.*

A colocalizing subcategory $\mathcal{A}$ of $\mathcal{C}$ is said to be *perfect colocalizing* if the colocalization functor $H : \mathcal{C}/\mathcal{A} \to \mathcal{C}$ is exact.

3.2 Localizing subcategories of a category of comodules

Let us restrict our attention to the localization of categories of comodules. In the literature there is a very well founded theory about the localizing subcategories and about the relationships that there exist with other concepts. We recall briefly some of them:

Coidempotent coalgebras. A subcoalgebra A of C is said to be *coidempotent* if $A \wedge A = A$. In [NT94], a bijective correspondence between localizing subcategories of $\mathcal{M}^C$ and coidempotent subcoalgebras of C is established. Indeed, the authors associate to every localizing subcategory $\mathcal{T}$ the subcoalgebra $\mathcal{T}(C) = \sum_{M \in \mathcal{T}} \mathrm{cf}(M)$ and to every coidempotent subcoalgebra A of C the closed subcategory $\mathcal{T}_A$ whose class of objects is $\{M \in \mathcal{M}^C \mid \mathrm{cf}(M) \subseteq A\}$.

Equivalence classes of injective comodules. From the general theory of localizing subcategories in a Grothendieck category, $\mathcal{C}$, it is well known that there exists a bijective correspondence between localizing subcategories of $\mathcal{C}$ and equivalence classes of injective objects. Two injective objets E_1 and E_2 are *equivalent* if E_i can be embedded in a direct product of copies of E_j for $i,\ j \in \{1,2\}$. The above correspondence associates to any injective object E the localizing subcategory $\mathcal{T}_E = \{M \in \ \mathcal{C} \mid \ \mathrm{Hom}_{\mathcal{C}}(M,E) = 0\}$. When we apply this to a comodule category $\mathcal{M}^C$, for any localizing subcategory $\mathcal{T}$ of $\mathcal{M}^C$, the inverse maps $\mathcal{T}$ to the injective right C-comodule $E = S(D)$, where D is an injective cogenerator of $\mathcal{M}^C/\mathcal{T}$.

Sets of indecomposable injective comodules. Since two injective right C-comodules are equivalent if and only if in their decompositions, as a direct sum of indecomposable injective right C-comodules, appear the same indecomposable injective comodules, maybe with different multiplicity, every equivalence class of injective right C-comodules is uniquely determined by a set of isomorphism classes of indecomposable injective right C-comodules.

Sets of simple comodules. To any indecomposable injective right

C-comodule, we can attach a simple right C-comodule defined by its socle. Conversely, given a simple comodule, its injective envelope is an indecomposable injective comodule. Therefore we have a bijective correspondence between sets of indecomposable injective comodules and sets of simple comodules.

Let us give a description of the localizing functors using Morita-Takeuchi contexts.

Theorem 3.2.1. *[JMNR06] Let $\mathcal{T}$ be a localizing subcategory of $\mathcal{M}^C$ and X be a injective quasifinite right C-comodule such that $\mathcal{T} = \mathcal{T}_X$. Consider the injective Morita-Takeuchi context (D, C, X, Y, f, g) defined by X. Then the functors*

$$T = -\square_C Y : \mathcal{M}^C \to \mathcal{M}^D \text{ and } S = -\square_D X : \mathcal{M}^D \to \mathcal{M}^C$$

define a localization of $\mathcal{M}^C$ with respect to the localizing subcategory $\mathcal{T}$. In particular, $\mathcal{M}^C/\mathcal{T}$ is equivalent to $\mathcal{M}^D$.

Proof. Since X_C is injective and quasifinite, the funtor $S = -\square_D X$ has an exact left adjoint functor $\mathrm{Cohom}_C(X, -)$. This functor preserves direct sums so, for every $N \in \mathcal{M}^C$, there is an isomorphism $\mathrm{Cohom}_C(X, N) \cong N\square_C \mathrm{Cohom}_C(X, C) = N\square_C Y$. Therefore, we obtain a natural isomorphism $\mathrm{Cohom}_C(X, -) \cong -\square_C Y = T$ and thus S is right adjoint of T.

Now, we have to show that $\mathrm{Ker}\,(T) = \mathcal{T}_X$. Let us point out that $X = S(D)$. Then, by the adjunction, for every $M \in \mathcal{M}^C$, There is a bijection $\mathrm{Hom}_C(M, X) \longleftrightarrow \mathrm{Hom}_D(T(M), D)$. Thus $M \in \mathcal{T}_X$ if and only if $\mathrm{Hom}_C(M, X) = 0$ if and only if $\mathrm{Hom}_D(T(M), D) = 0$ if and only if $T(M) = 0$. $\square$

Equivalence classes of idempotents of the dual algebra. This is the most important relation for us and we shall give a complete description of the elements of a localization, see [CGT02], [JMNR06] and [Woo97] for details. Given two idempotent elements $f, g \in C^*$, we say that f is *equivalent* to g if the injective right C-comodules Cf and Cg are equivalent in the sense defined above. On the other hand, it is easy to see that every injective right C-comodule E is of the form $E = Ce$ for some idempotent $e \in C^*$. Therefore there exists a bijective correspondence between equivalence classes of injective comodules and equivalence classes of idempotent elements of the dual algebra. Given an idempotent element $e \in C^*$, we will denote the localizing subcategory associated to e by $\mathcal{T}_e$ and by $I_e \subseteq I_C$

the indices of the subset of simple (or indecomposable injective) comodules associated to this idempotent element.

Let e be an idempotent element of C^* and $\mathcal{T}_e$ be the localizing subcategory associated to e. From above, $\mathcal{M}^C/\mathcal{T}_e$ is an abelian category of finite type and therefore $\mathcal{M}^C/\mathcal{T}_e \cong \mathcal{M}^D$ for some coalgebra D. We may give an explicit description of D.

Let us consider the subspace $eCe \subseteq C$. Then it can be endowed with a structure of coalgebra given by

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e \quad \text{and} \quad \epsilon_{eCe}(exe) = \epsilon_C(x)$$

for any $x \in C$, where $\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ using the sigma-notation of [Swe69].

Lemma 3.2.2. *[Woo97] There exists a equivalence between the categories $\mathcal{M}^C/\mathcal{T}_e$ and $\mathcal{M}^{eCe}$.*

If M is a right C-comodule, the vector space eM has a natural structure of right eCe-comodule given by

$$\omega_{eM}(em) = \sum_{(m)} em_{(0)} \otimes em_{(1)}e,$$

for any $m \in M$, where $\omega_M(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}$ in the sigma-notation.

There is a natural right eCe-comodule isomorphism $eM \cong M\square_C eC$, defined by $ex \mapsto x_{(0)} \otimes ex_{(1)}$. This means that the functor $-\square_C eC$ is naturally isomorphic to the functor from $\mathcal{M}^C$ to $\mathcal{M}^{eCe}$ defined by $M \mapsto eM$. Observe that when we take $M = Ce$, we obtain a natural isomorphism $eCe \cong Ce\square_C eC$, see [CGT02] for details.

Since Ce is a quasifinite injective right C-comodule, we may consider the injective Morita-Takeuchi context associated to Ce, which, by [CGT02], is (eCe, C, Ce, eC, f, g), where $f : eCe \cong Ce\square_C eC$ is the aforementioned isomorphism and $g : C \to eC\square_{eCe} Ce$ is defined by $g(x) = \sum ex_{(1)} \otimes x_{(2)}e$ for any $x \in C$. Hence there exist isomorphisms $eC \cong \mathrm{Cohom}_C(Ce, C)$ and $eCe \cong \mathrm{Coend}_C(Ce)$.

Thus we can rewrite Theorem 3.2.1 as follows:

Theorem 3.2.3. *The functors*

$$T = -\square_C eC = e(-) : \mathcal{M}^C \to \mathcal{M}^{eCe} \text{ and } S = -\square_{eCe} Ce : \mathcal{M}^{eCe} \to \mathcal{M}^C$$

define a localization of $\mathcal{M}^C$ with respect to the localizing subcategory $\mathcal{T}_e$.

Corollary 3.2.4. *[CGT02] The functor T is equivalent to the functor* $\mathrm{Cohom}_C(Ce, -)$.

Corollary 3.2.5. *A localizing subcategory $\mathcal{T}_e$ is perfect localizing if and only if Ce is an injective left eCe-comodule.*

Note that, as a consequence of Theorem 3.2.3, we obtain an easy description of the localizing subcategory:

$$\mathcal{T}_e = \mathrm{Ker}\,(T) = \{M \in \mathcal{M}^C \mid M\square_C eC = 0\} = \{M \in \mathcal{M}^C \mid eM = 0\}.$$

Remark. If $e \in C^*$ is an idempotent element, for a simple right C-comodule S, we have exactly two possibilities:

(1) $eS = 0$, in this case $e \cdot \mathrm{cf}(S) = 0$ and $e_{|\mathrm{cf}(S)} = 0$, or

(2) $eS = S$, in this case $e \cdot \mathrm{cf}(S) = \mathrm{cf}(S)$ and $e_{|\mathrm{cf}(S)} = \epsilon_{|\mathrm{cf}(S)}$.

Thus the class $\mathcal{T}_e$ is the localizing subcategory of $\mathcal{M}^C$ determined by the subset $I_e = \{i \in I_C \mid eS_i = S_i\}$. It is not difficult to see that the coidempotent subcoalgebra determined by $\mathcal{T}_e$ is the biggest subcoalgebra of C annihilated by e. (Note that, for any subcoalgebra A of C, $eA = 0$ if and only if $Ae = 0$).

We now turn to colocalizing subcategories. From Theorem 3.2.3 we may deduce easily the following result.

Proposition 3.2.6. *Let $e \in C^*$ be an idempotent element and $\mathcal{T}_e$ be its associated localizing subcategory in $\mathcal{M}^C$. Then $\mathcal{T}_e$ is a colocalizing subcategory if and only if eC is a quasi-finite right eCe-comodule.*

Proof. By Theorem 1.3.2, the functor $T = -\square_C eC : \mathcal{M}^C \to \mathcal{M}^{eCe}$ has a left adjoint functor if and only if eC is quasi-finite as right eCe-comodule. □

In the same direction we may characterize, also in terms of idempotent elements, perfect colocalizing subcategories.

Proposition 3.2.7. *Let $e \in C^*$ be an idempotent element and let $\mathcal{T}_e$ be the associated localizing subcategory in $\mathcal{M}^C$. Then $\mathcal{T}_e$ is a perfect colocalizing subcategory if and only if eC is a quasifinite injective right eCe-comodule.*

Proof. Observe that the left adjoint of $T = -\square_C eC : \mathcal{M}^C \to \mathcal{M}^{eCe}$ is $H = \mathrm{Cohom}_{eCe}(eC, -)$. By Proposition 1.3.3, H is exact if and only if eC is an injective right eCe-comodule. □

The reader can compare the last two propositions with [NT96, Proposition 3.1] and [NT96, Proposition 4.1], where colocalizing and perfect colocalizing subcategories of $\mathcal{M}^C$ are characterized in terms of the biggest subcoalgebra of C annihilated by e.

Proposition 3.2.8. *Let C be a coalgebra and $\mathcal{T}$ be a perfect colocalizing subcategory of $\mathcal{M}^C$. Then $\mathcal{T}$ is a perfect localizing subcategory of $\mathcal{M}^C$.*

Proof. If $\mathcal{T} = \mathcal{T}_e$ is a a perfect colocalizing subcategory of $\mathcal{M}^C$, eC is an quasifinite injective right eCe-comodule. Thus, the functor $-\square_C eC$ has an exact left adjoint, namely $\mathrm{Cohom}_{eCe}(eC, -)$. On the other hand, since Ce is a quasifinite injective right C-comodule, $-\square_{eCe} Ce$ admits an exact left adjoint, namely $\mathrm{Cohom}_C(Ce, -)$. Then the composed functor $-\square_C(eC\square_{eCe}Ce) = (-\square_{eCe}Ce)\circ(-\square_C eC)$ has an exact left adjoint functor $\mathrm{Cohom}_{eCe}(eC, -)\circ \mathrm{Cohom}_C(Ce, -)$. Therefore $eC\square_{eCe}Ce$ is an quasifinite injective right C-comodule and $\mathcal{T}_e$ is a a perfect localizing subcategory of $\mathcal{M}^C$. $\square$

A symmetric version of all this section may be done for left comodules. In particular, the localization by means of idempotents is described as follows:

For each localizing subcategory $\mathcal{T}'$ of ${}^C\mathcal{M}$, there exists a unique (up to equivalence) idempotent element e in C^* such that the localizing functors are equivalent to

$$
{}^C\mathcal{M} \underset{S=-\square_{eCe}eC}{\overset{T=(-)e=-\square_C Ce}{\rightleftarrows}} {}^C\mathcal{M}/\mathcal{T}' ,
$$

where ${}^C\mathcal{M}/\mathcal{T}'$ is equivalent to ${}^{eCe}\mathcal{M}$.

3.3 The Ext-quiver

To any coalgebra C, we may associate a quiver Γ_C known as the (right) Ext-quiver of C, see [Mon95]. We recall that the set of vertices of Γ_C is the set of pairwise non-isomorphic simple right C-comodules $\{S_x\}_{x\in I_C}$ and, for two vertices S_x and S_y, there exists an arrow $S_x \to S_y$ if and only if $\mathrm{Ext}^1_C(S_x, S_y) \neq 0$.

Let us take into consideration some geometric properties of Γ_C. Given a vertex S_x, we say that the vertex S_y is an *immediate predecessor* (respectively, a *predecessor*) of S_x if there exists an arrow $S_y \to S_x$ in Γ_C (respectively, a path from S_y to S_x in Γ_C).

Lemma 3.3.1. *S_y is an immediate predecessor of S_x if and only if $S_y \subseteq \mathrm{Soc}\,(E_x/S_x)$.*

Proof. Let us consider the short exact sequence $S_x \hookrightarrow E_x \to E_x/S_x$. Then we obtain the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_C(S_y,S_x) \arrow[r] & \mathrm{Hom}_C(S_y,E_x) \arrow[r] & {} \\ {} \arrow[r] & \mathrm{Hom}_C(S_y,E_x/S_x) \arrow[r] & \mathrm{Ext}_1^C(S_y,S_x) \arrow[r] & 0 \end{tikzcd}$$

Since $\mathrm{Hom}_C(S_y,S_x) \cong \mathrm{Hom}_C(S_y,E_x)$ then $\mathrm{Hom}_C(S_y,E_x/S_x) \cong \mathrm{Ext}^1_C(S_y,S_x)$ and the result follows. $\square$

The following result gives a necessary condition for having an arrow between two vertices of the Ext-quiver.

Lemma 3.3.2. *For each two simple right C-comodules S_x and S_y, if $\mathrm{Hom}_C(E_y,E_x)=0$ then $\mathrm{Ext}^1_C(S_x,S_y)=0$.*

Proof. By the proof of Lemma 3.3.1, $Ext^1_C(S_x,S_y) \cong \mathrm{Hom}_C(S_x,E_y/S_y)$ for each simple right comodules S_x and S_y. Now, since E_x is the injective envelope of S_x, for each non-zero morphism $f: S_x \hookrightarrow E_y/S_y$, there exists a non-zero map $g: E_y/S_y \to E_X$ making commutative the diagram

$$\begin{tikzcd} S_x \arrow[r, hook, "f"] \arrow[d, hook, "i"'] & E_y/S_y \arrow[dl, dashed, "g"] \\ E_x & \end{tikzcd}$$

Then the composition $E_y \xrightarrow{p} E_y/S_y \xrightarrow{g} E_x$ is a non-zero morphism in $\mathrm{Hom}_C(E_y,E_x)$. $\square$

It is easy to see that each morphism f in $\mathrm{Hom}_C(E_y,E_x)$, obtained from a nonzero element in $Ext^1_C(S_x,S_y)$ by means of the construction of the former lemma, verifies the following condition: If f decomposes through two morphisms $t: E_y \to E_z$ and $h: E_z \to E_x$, where E_z is an indecomposable injective right C-comodule, then t is an isomorphism or h is an isomorphism. To prove that assume the contrary and then we have the following commutative diagram

$$\begin{tikzcd} S_x \arrow[r, hook, "f"] \arrow[dr, hook, "i"'] & E_y/S_y \arrow[d, "g"] & E_y \arrow[l, "p"'] \arrow[d, "t"] \\ & E_x & E_z \arrow[l, "h"] \end{tikzcd}$$

If t is not an isomorphism then it decomposes through the projection, i.e., we have the diagram

$$\begin{tikzcd}
S_x \arrow[r, hook, "f"] \arrow[dr, hook, "i"'] & E_y/S_y \arrow[d, "g"] \arrow[dr, "t'"] & E_y \arrow[l, "p"'] \arrow[d, "t"] \\
 & E_x & E_z \arrow[l, "h"]
\end{tikzcd}$$

Furthermore, $p^*(ht') = ht'p = ht = gp = p^*(g)$ implies, by the injectivity of p^*, that $ht' = g$ and then the diagram is commutative. Now, since h is not an isomorphism, $h(S_x) = 0$ and then $i = ht'f = 0$ and we get a contradiction.

That property suggests that if we consider the extensions between simple comodules as arrows, the morphisms between indecomposable injective comodules should be the paths in the Ext-quiver. Unfortunately, in general, it is not true.

Example 3.3.3. *Let Q be the quiver $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ and C be the subcoalgebra of KQ generated by $\{x, y, z, \alpha, \beta\}$. Then the quiver Γ_C is*

$$S_x \longrightarrow S_y \longrightarrow S_z.$$

Obviously, there is a path from S_x to S_z but any morphism

$$f : E_z =< z, \beta >\longrightarrow E_x =< x >$$

is zero.

On the other hand, if C is the coalgebra KQ, the Ext-quiver Γ_C is also the former quiver but, in this case, we may obtain a map

$$f : E_z =< z, \beta, \beta\alpha >\longrightarrow E_x =< x >$$

defined by $f(\beta\alpha) = x$ and zero otherwise. Observe that f is the composition of the morphisms

$$g : E_z =< z, \beta, \beta\alpha >\longrightarrow E_y =< y, \alpha >$$

which maps $z \mapsto 0$, $\beta \mapsto y$ and $\beta\alpha \mapsto \alpha$, and

$$h : E_y =< y, \alpha >\longrightarrow E_x =< x >$$

which maps $y \mapsto 0$ and $\alpha \mapsto x$.

Remark. Observe that the second coalgebra is a hereditary coalgebra (see Chapter 5) and the morphisms between indecomposable injective comodules are surjective. Hence the composition of two non-zero morphisms is a non-zero morphism. Thus if there is a path (non necessarily of length one) in Γ_C from S_x to S_y, we obtain that $\mathrm{Hom}_C(E_y, E_x) \neq 0$.

Therefore the Ext-quiver provides us information about the extension groups of the simple comodules (i.e., the arrows) but it is not exact at all about the morphisms between the injective envelopes (we would like to say "the paths").

3.4 Injective and simple comodules

Given an arbitrary coalgebra C, many properties of its category of comodules are given by means of the simple objects or, since it is a Grothendieck category, their injective envelopes, i.e., the indecomposable injective comodules. Therefore we are interested in knowing how the localizing functors map these classes of comodules.

Let us consider an idempotent element $e \in C^*$, $\mathcal{T}_e$ its localizing subcategory and the localizing functors:

$$\mathcal{M}^C \underset{S=-\square_{eCe}Ce}{\overset{T=e(-)=-\square_C eC}{\rightleftarrows}} \mathcal{M}^{eCe}.$$

We recall that there exists a torsion theory on $\mathcal{M}^C$ associated to the functor T, where a right C-comodule M is a torsion comodule if $T(M) = 0$. If M is not torsion, we denote by $t(M)$ the torsion subcomodule of M.

We know that, by the remark of Theorem 3.2.3, for a simple right C-comodule S_x, $T(S_x) = S_x$ if $x \in I_e$ and zero otherwise (therefore any simple comodule is torsion or torsion-free). From that fact we obtain the following result:

Lemma 3.4.1. *Let M be a right C-comodule then* $T(\mathrm{Soc}\, M) \subseteq \mathrm{Soc}\, T(M)$.

Proof. Let us suppose that $\mathrm{Soc}\, M = \oplus_{i\in I} S_i \oplus (\oplus_{j\in J} T_j)$, where S_i and T_j are simple right C-comodules such that $T(S_i) = S_i$ and $T(T_j) = 0$ for all $i \in I$ and $j \in J$. Since $\mathrm{Soc}\, M \subseteq M$, $T(\mathrm{Soc}\, M) = \oplus_{i\in I} S_i \subseteq T(M)$. $\square$

Let us study the behavior of the injective comodules under the action of the section functor. Indeed, we shall prove that S preserves indecomposable injective comodules and, consequently, injective envelopes. In what follows we will denote by $\{\overline{E}_x\}_{x \in I_e}$ a complete set of pairwise non-isomorphic indecomposable injective right eCe-comodules.

Proposition 3.4.2. *In the above situation, the following properties hold:*

(a) *The functor S preserves injective comodules.*

(b) *If N is a quasi-finite indecomposable right eCe-comodule then $S(N)$ is indecomposable.*

(c) *The functor S preserves indecomposable injective comodules.*

(d) *If S_x is a simple eCe-comodule then $\mathrm{Soc}\, S(S_x) = S_x$.*

(e) *If S_x is a simple eCe-comodule then $S(S_x)$ is torsion-free.*

(f) *We have $S(\overline{E}_x) = E_x$ for all $x \in I_e$.*

(g) *The functor S preserves quasi-finite comodules.*

(h) *The functor $S : \mathcal{M}^{eCe} \to \mathcal{M}^C$ restricts to a fully faithful functor $S : \mathcal{M}^{eCe}_{qf} \to \mathcal{M}^C_{qf}$ between the categories of quasi-finite comodules which preserves indecomposables comodules and respects isomorphism classes.*

Proof. (a) The functor T is exact and left adjoint of S so, by [Ste75, Proposition 9.5], the result follows.

(b) If N is quasi-finite and indecomposable then the ring of endomorphism $\mathrm{End}_{eCe}(N) \cong \mathrm{End}_C(S(N))$ is a local ring. Thus $S(N)$ is indecomposable.

(c) It follows from (a) and (b).

(d) Suppose that $\mathrm{Soc}\, S(S_x) = \oplus_{i \in I} S_i \oplus (\oplus_{j \in J} T_j)$, where S_i and T_j are simple right C-comodules such that $T(S_i) = S_i$ and $T(T_j) = 0$ for all $i \in I$ and $j \in J$. By Lemma 3.4.1, $\oplus_{i \in I} S_i = T(\mathrm{Soc}\, S(S_x)) \subseteq \mathrm{Soc}\, TS(S_x) = \mathrm{Soc}\, S_x = S_x$. Since S is left exact and preserves indecomposable injective comodules, $S_x \subseteq \mathrm{Soc}\, S(S_x) \subseteq \mathrm{Soc}\, S(\overline{E}_x) = S_y$. Then $S_y = S_x = \mathrm{Soc}\, S(S_x)$.

(e) If $M \subseteq S(S_x)$ is a non-zero torsion subcomodule of $S(S_x)$ then there exists a simple C-comodule R contained in M such that $T(R) = 0$. But $\operatorname{Soc} S(S_x) = S_x$ so $S_x = R$ and we get a contradiction.

(f) It is easy to see from (c) and (d).

(g) Let M be a quasi-finite right eCe-comodule. The injective envelope of M is a quasi-finite injective comodule $M \hookrightarrow E = \oplus \overline{E}_x^{n_x}$. Since S is left exact then $S(M) \hookrightarrow S(E) = \oplus E_x^{n_x}$. Thus $S(M)$ is quasi-finite.

(h) It is a consequence of the above assertions and the equality $TS = 1_{\mathcal{M}^{eCe}}$.

$\square$

Corollary 3.4.3. *S preserves injective envelopes.*

After proving Proposition 3.4.2, one should ask if the behavior of the simple comodules is analogous to the injective ones, that is, if S preserves simple comodules and, consequently, in view of Proposition 3.4.2(c), $S(S_x) = S_x$ for all $x \in I_e$. Unfortunately, in general, this is not true and we can only say that $S(S_x)$ is a subcomodule of E_x which contains S_x.

Example 3.4.4. *This example shows that $S(S_x)$ does not have to be S_x for every $x \in I_e$. Consider the quiver Q*

$$\underset{y}{\circ} \xrightarrow{\alpha} \underset{x}{\circ},$$

$C = KQ$ and the idempotent $e \in C^*$ associated to the set $\equiv \{y\}$. Then, the localized coalgebra eCe is S_x and

$$S(S_x) = S_x \square_{eCe} Ce = eCe \square_{eCe} Ce \cong Ce \cong < x, \alpha > \neq S_x.$$

The reader should observe that $S(S_x)$ could be an infinite dimensional right C-comodule. Therefore, in general, S cannot be restricted to a functor between the categories of finite dimensional comodules.

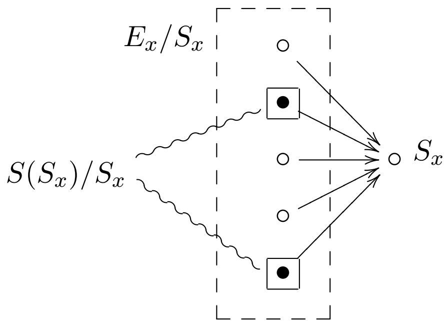
Example 3.4.5. *Consider the quiver Q*

$$\cdots \xrightarrow{\alpha_{n+1}} \underset{n+1}{\circ} \xrightarrow{\alpha_n} \underset{n}{\circ} \xrightarrow{\alpha_{n-1}} \underset{n-1}{\circ} - \cdots - \underset{3}{\circ} \xrightarrow{\alpha_2} \underset{2}{\circ} \xrightarrow{\alpha_1} \underset{1}{\circ},$$

$C = KQ$ and the idempotent $e \in C^*$ associated to the set $\equiv \{1\}$. Then the localized coalgebra eCe is S_1 and

$$S(S_1) = S_1 \square_{eCe} Ce = eCe \square_{eCe} Ce \cong Ce \cong < x, \{\alpha_1 \cdots \alpha_{n-1}\alpha_n\}_{n\geq 1} > .$$

In order to characterize the simple comodules invariant under the functor S we need the following technical lemma. It asserts that the torsion predecessors of a torsion-free vertex S_x in Γ_C are the simple C-comodules contained in the socle of $S(S_x)/S_x$. In the following picture the torsion-free vertices are represented by white points.



Lemma 3.4.6. *Let S_y be a simple C-comodule. Then we have that $S_y \subseteq \mathrm{Soc}\,(S(S_x)/S_x)$ if and only if $S_y \subseteq \mathrm{Soc}\,(E_x/S_x)$ and $T(S_y) = 0$.*

Proof. Consider the short exact sequence

$$S_x \hookrightarrow S(S_x) \twoheadrightarrow S(S_x)/S_x \tag{3.1}$$

Since $S_x = T(S_x) = TS(S_x)$, $S(S_x)/S_x$ is a torsion subcomodule of E_x/S_x. Therefore if $S_y \subseteq \mathrm{Soc}\,(S(S_x)/S_x)$ then $S_y \subseteq \mathrm{Soc}\,(E_x/S_x)$ and $T(S_y) = 0$.

For the converse, first we prove that $\mathrm{Ext}^1_C(S_y, S(S_x)) = 0$. We apply the functor S to the exact sequence

$$S_x \xrightarrow{i} \overline{E}_x \xrightarrow{p} \overline{E}_x/S_x$$

and we obtain the following commutative (and exact) diagram:

$$\begin{tikzcd}
S(S_x) \arrow[r, "i"] & E_x \arrow[rr, "S(p)"] \arrow[dr, "\mathrm{Coker}\, i"'] & & S(\overline{E}_x/S_x) \arrow[r] & \mathrm{Coker} \\
& & E_x/S(S_x) \arrow[ur, hook] \arrow[r] & 0 &
\end{tikzcd}$$

Therefore we have that $\mathrm{Hom}_C(S_y, E_x/S(S_x))$ is included the set of morphisms $\mathrm{Hom}_C(S_y, S(\overline{E}_x/S_x)) \cong \mathrm{Hom}_{eCe}(T(S_y), \overline{E}_x/S_x) = 0$. Consider now the short exact sequence

$$S(S_x) \hookrightarrow E_x \twoheadrightarrow E_x/S(S_x)$$

which produces the exact sequence

$$0 = \mathrm{Hom}_C(S_y, E_x) \to \mathrm{Hom}_C(S_y, E_x/S(S_x)) \to \mathrm{Ext}_1^C(S_y, S(S_x)) \to 0$$

and then $0 = \mathrm{Hom}_C(S_y, E_x/S(S_x)) \cong \mathrm{Ext}_1^C(S_y, S(S_x))$.

Let us now apply the functor $\mathrm{Hom}_C(S_y, -)$ to (3.1) and we obtain the exactness of the sequence

$$\mathrm{Hom}_C(S_y, S(S_x)/S_x) \xrightarrow{\cong} \mathrm{Ext}_1^C(S_y, S_x) \neq 0 \longrightarrow \mathrm{Ext}_1^C(S_y, S(S_x)) = 0\,.$$

Then the result follows. □

Corollary 3.4.7. *Let S_x be a simple eCe-comodule. The following conditions are equivalent:*

(a) *E_x/S_x is torsion-free.*

(b) *There is no arrow in Γ_C from a torsion vertex S_y to S_x.*

(c) *$S(S_x) = S_x$.*

Let us now analyse the quotient functor. We start with an example which shows that, in general, T does not preserve injective comodules.

Example 3.4.8. *Let Q be the quiver*

$$\underset{x}{\circ} \xrightarrow{\alpha} \underset{y}{\circ} \xrightarrow{\beta} \underset{z}{\circ},$$

C be the subcoalgebra of KQ generated by $\{x, y, z, \alpha, \beta\}$ and $I_e = \{x, y\}$. The injective right C-comodule E_z is generated by $< z, \beta >$ and $T(E_z) =< \beta >\cong S_y \neq E_y$.

Proposition 3.4.9. *The following statements hold:*

(a) *$T(E_x) = \overline{E}_x$ for any $x \in I_e$.*

(b) *If E is an injective torsion-free right C-comodule then $T(E)$ is an injective right eCe-comodule.*

(c) *If M is a torsion-free right C-comodule then* $\mathrm{Soc}\, M = \mathrm{Soc}\, T(M) = T(\mathrm{Soc}\, M)$.

(d) *The functor $T : \mathcal{M}^C \to \mathcal{M}^{eCe}$ restricts to a functor $T : \mathcal{M}^C_{qf} \to \mathcal{M}^{eCe}_{qf}$ and a functor $T : \mathcal{M}^C_f \to \mathcal{M}^{eCe}_f$ between the categories of quasi-finite and finite dimensional comodules, respectively.*

Proof. (a) By Proposition 3.4.2, $E_x = S(\overline{E}_x)$ for any $x \in I_e$. Then $T(E_x) = TS(\overline{E}_x) = \overline{E}_x$.

(b) It follows from (a).

(c) Consider the chain $\oplus_{x\in I} S_x = \mathrm{Soc}\, M \subseteq M \subseteq E(M) = \oplus_{x\in I} E_x$. Since M is torsion-free then $I \subseteq I_e$. Therefore $\mathrm{Soc}\, M = \oplus_{x\in I} S_x = \oplus_{x\in I} T(S_x) = T(\mathrm{Soc}\, M) \subseteq T(M) \subseteq T(E(M)) = \oplus_{x\in I} T(E_x) = \oplus_{x\in I} \overline{E}_x$ and the result follows.

(d) It is easy to see.

□

Example 3.4.10. *In general, the functor T is not full. Let Q be the quiver*

$$\underset{x}{\circ} \xrightarrow{\ \alpha\ } \underset{y}{\circ},$$

$C = KQ$ and $e \in C^*$ be the idempotent associated to the set $\{x\}$. Then $\dim_K \mathrm{Hom}_C(S_x, C) = \dim_K \mathrm{End}(S_x) = 1$ and $\dim_K \mathrm{Hom}_{eCe}(S_x, eC) = 2$. Therefore the map $T_{S_x,C}$ cannot be surjective.

Example 3.4.11. *In general, the functor T does not preserve indecomposable comodules. Let KQ be the path coalgebra of the quiver*

$$\begin{tikzcd} x\circ \arrow[dr, "\alpha"] & \\ & \underset{z}{\circ} \\ y\circ \arrow[ur, "\beta"'] & \end{tikzcd}$$

and $e \in C^*$ be the idempotent associated to the set $\{x, y\}$. Then T maps the indecomposable injective right C-comodule $E_z = < z, \alpha, \beta >$ to the right eCe-comodule $S_x \oplus S_y$.

Nevertheless, it is easy to see that T preserves indecomposable torsion-free comodules. Since $T(S_y) = 0$ for each torsion simple C-comodule, one could expect the analogous property for their injective envelopes.

Example 3.4.12. *In general, it is not true that* $T(E_x) = 0$ *for any* $x \notin I_e$. *Let* KQ *be the path coalgebra of the quiver*

$$\underset{x}{\circ} \xrightarrow{\alpha} \underset{y}{\circ},$$

$C = KQ$ *and* $e \in C^*$ *be the idempotent associated to the set* $\{x\}$. *Then* $T(E_y) = T(< y, \alpha >) \cong S_x \neq 0$.

Proposition 3.4.13. *Let* E_y *be an indecomposable injective* C*-comodule with* $y \notin I_e$. *We have* $T(E_y) = 0$ *if and only if* $\mathrm{Hom}_C(E_y, E_x) = 0$ *for all* $x \in I_e$.

Proof. $\Rightarrow$) Since S is left adjoint to T then we have the following $\mathrm{Hom}_C(E_y, E_x) = \mathrm{Hom}_C(E_y, S(\overline{E}_x)) \cong \mathrm{Hom}_{eCe}(T(E_y), \overline{E}_x) = 0$ for all $x \in I_e$.
$\Leftarrow$) By hypothesis, for all $x \in I_e$, $0 = \mathrm{Hom}_C(E_y, E_x) = \mathrm{Hom}_C(E_y, S(\overline{E}_x)) = \mathrm{Hom}_{eCe}(T(E_y), \overline{E}_x)$. Then $\mathrm{Hom}_{eCe}(T(E_y), eCe) = 0$ and thus $T(E_y) = 0$ □

Let us finish the study of the quotient functor by giving an approach to the image of an indecomposable injective comodule associated to a torsion simple comodules.

Lemma 3.4.14. *Let* S_y *be a torsion simple right* C*-comodule and* $\{S_x, T_z\}_{x \in I, z \in J}$ *be the set of all immediate predecessors of* S_y *in* Γ_C, *where* S_x *is torsion-free for all* $x \in I$ *and* T_z *is torsion for all* $z \in J$. *Then*

$$\mathrm{Soc}\, T(E_y) \subseteq (\bigoplus_{x \in I} S_x) \bigoplus (\bigoplus_{z \in J} \mathrm{Soc}\, T(E_z)).$$

Proof. By Lemma 3.3.1, $\mathrm{Soc}\,(E_y/S_y) = (\oplus_{x \in I} S_x) \oplus (\oplus_{z \in J} T_z)$ and, consequently, $E_y/S_y \subseteq (\oplus_{x \in I} E_x) \oplus (\oplus_{z \in J} E_z)$. Then $T(E_y) \cong T(E_y/S_y) \subseteq (\oplus_{x \in I} \overline{E}_x) \oplus (\oplus_{z \in J} T(E_z))$ and then $\mathrm{Soc}\, T(E_y) \subseteq (\oplus_{x \in I} S_x) \oplus (\oplus_{z \in J} \mathrm{Soc}\, T(E_z))$. □

It is not possible to prove the equality of Lemma 3.4.14. Consider the quiver of Example 3.4.8, the coalgebra generated by the set $< x, y, z, \alpha, \beta >$ and $I_e = \{x\}$. Then $\mathrm{Soc}\, T(E_z) = 0$ and $\mathrm{Soc}\, T(E_y) = S_x$.

Until the end of the section we assume that $\mathcal{T}_e$ is a colocalizing subcategory of $\mathcal{M}^C$. Then the quotient functor T has a left adjoint functor $H : \mathcal{M}^{eCe} \to \mathcal{M}^C$.

Proposition 3.4.15. *Under the above conditions, we have that:*

(a) H *preserves projective comodules.*

(b) H *preserves finite dimensional comodules.*

(c) H *preserves finite dimensional indecomposable comodules.*

(d) *The functor* $H : \mathcal{M}^{eCe} \to \mathcal{M}^C$ *restricts to a fully faithful functor* $H : \mathcal{M}_f^{eCe} \to \mathcal{M}_f^C$ *between the categories of finite-dimensional comodules which preserves indecomposable comodules and respects isomorphism classes.*

Proof. (a) It is symmetric to the proof of Proposition 3.4.2(a) .

(b) Let N be a finite dimensional right eCe-comodule. Then $H(N) = \mathrm{Cohom}_{eCe}(eC, N) = \varinjlim \mathrm{Hom}_{eCe}(N_\lambda, eC)^* = \mathrm{Hom}_{eCe}(N, eC)^*$. Now, since eC is a quasi-finite right eCe-comodule, $\mathrm{Hom}_{eCe}(N, eC)$ has finite dimension.

(c) Let N be a finite dimensional indecomposable right eCe-comodule. Since H is fully faithful then $\mathrm{End}_{eCe}(N) \cong \mathrm{End}_C(S(N))$ is a local ring. Now, by (b), $S(N)$ is finite dimensional and then $S(N)$ is indecomposable.

(d) It is straightforward from (b), (c) and the equality $TH = 1_{\mathcal{M}^{eCe}}$.

□

Analogously to the study of the section functor, let us characterize the simple comodules which are invariant under the functor H. For that purpose we need the following lemma:

Lemma 3.4.16. *Let* S_x *be a simple* eCe*-comodule. Then* $H(S_x) = S_x$ *if and only if* $\mathrm{Hom}_{eCe}(S_x, T(E_y)) = 0$ *for all* $y \notin I_e$*.*

Proof. We have that $H(S_x) = \mathrm{Cohom}_{eCe}(eC, S_x) = \mathrm{Cohom}_{eCe}(eCe, S_x) \oplus \mathrm{Cohom}_{eCe}(eC(1-e), S_x) \cong S_x \oplus \mathrm{Cohom}_{eCe}(eC(1-e), S_x)$ and therefore

$$\begin{aligned} H(S_x) = S_x \ &\Leftrightarrow \mathrm{Cohom}_{eCe}(eC(1-e), S_x) = 0 \\ &\Leftrightarrow \mathrm{Hom}_{eCe}(S_x, eC(1-e)) = 0 \\ &\Leftrightarrow \mathrm{Hom}_{eCe}(S_x, \oplus_{y \notin I_e} T(E_y)) = 0 \\ &\Leftrightarrow \oplus_{y \notin I_e} \mathrm{Hom}_{eCe}(S_x, T(E_y)) = 0 \\ &\Leftrightarrow \mathrm{Hom}_{eCe}(S_x, T(E_y)) = 0 \ \text{ for all } \ y \notin I_e. \end{aligned}$$

□

Corollary 3.4.17. *Let* E_x *be an indecomposable injective* C*-comodule with* $x \in I_e$*. If* $\mathrm{Hom}_C(E_y, E_x) = 0$ *for all* $y \notin I_e$ *then* $H(S_x) = S_x$*.*

Proof. Let us suppose that there exists a nonzero morphism $g \in \mathrm{Hom}_{eCe}(S_x, T(E_y))$ for some $y \notin I_e$. Since $\overline{E}_x$ is injective, there exists a non-zero morphism f that makes commutative the following diagram

$$\begin{tikzcd} S_x \arrow[r, "g"] \arrow[d, hook, "i"'] & T(E_y) \arrow[dl, dashed, "f"] \\ \overline{E}_x & \end{tikzcd}$$

Hence $\mathrm{Hom}_C(E_y, E_x) \simeq \mathrm{Hom}_{eCe}(T(E_y), \overline{E}_x) \neq 0$. The result follows applying Lemma 3.4.16. $\square$

Proposition 3.4.18. *Let S_x be a simple eCe-comodule. $H(S_x) = S_x$ if and only if $\mathrm{Ext}^1_C(S_x, S_y) = 0$ for all $y \notin I_e$, i.e., there is no arrow $S_x \to S_y$ in Γ_C, where S_y is a torsion simple C-comodule.*

Proof. By Lemma 3.4.16, it is enough to prove that $\mathrm{Ext}^1_C(S_x, S_y) = 0$ for all $y \notin I_e$ if and only if $\mathrm{Hom}_{eCe}(S_x, T(E_y)) = 0$ for all $y \notin I_e$.

$\Leftarrow$) Suppose that $\mathrm{Ext}^1_C(S_x, S_y) \neq 0$ for some $y \notin I_e$. By Lemma 3.3.1, $S_x \subseteq \mathrm{Soc}\,(E_y/S_y)$. Then

$$S_x = T(S_x) \subseteq T(\mathrm{Soc}\; E_y/S_y) \subseteq \mathrm{Soc}\, T(E_y/S_y) = \mathrm{Soc}\, T(E_y)$$

and therefore $\mathrm{Hom}_{eCe}(S_x, T(E_y)) \neq 0$.

$\Rightarrow$) For each $z \notin I_e$, we consider the set $\{S^z_\lambda\}_{\lambda \in \Lambda_z}$ of all torsion-free immediate predecessors in Γ_C of S_z. Then, by Lemma 3.4.14, it is verified that the simple comodules contained in $T(E_y)$ are in the set $\{S^z_\lambda\}_{\lambda \in \Lambda_z, z \notin I_e} = P$ for any $y \notin I_e$.

Now, if $\mathrm{Ext}^1_C(S_x, S_y) = 0$ for all $y \notin I_e$, $S_x \notin P$ and then $S_x \nsubseteq T(E_y)$ for any $y \notin I_e$. Thus $\mathrm{Hom}_{eCe}(S_x, T(E_y)) = 0$ for any $y \notin I_e$. $\square$

3.5 Stable subcategories

The bijective correspondence between localizing subcategories of $\mathcal{M}^C$ and equivalence classes of idempotent elements in C^* is interesting because we may parameterize some classes of localizing subcategories using well known classes of idempotent elements.

The first interesting case appears when we consider central idempotent elements. Which localizing subcategories correspond to central idempotent elements?

The answer to that question is given, for example, in [GJM99]. If $\mathcal{T}$ is a localizing subcategory such that the associated idempotent element is central, then $\mathcal{T}$ is closed under left and right links.

Following these ideas, we may consider more general classes of idempotent elements and the corresponding localizing subcategories. Using the results of the last section we shall deal with semicentral idempotent elements in C^* and see that they define a special, and well known, class of localizing subcategories.

Let us recall that a localizing subcategory $\mathcal{T}$ of $\mathcal{M}^C$ which is closed for essential extensions is called *stable*. The first result we consider on stable localization subcategories appears in [NT94]. There it is proved that the localizing subcategory of $\mathcal{M}^C$ defined by a coidempotent subcoalgebra A of C is stable if and only if A is an injective right C-comodule.

In order to characterize stable localizing subcategories of $\mathcal{M}^C$ in terms of idempotent elements, first we recall some definitions from the theory of idempotent elements. Following [Bir83], an idempotent element e of a ring R is said to be *left semicentral* if $(1-e)Re = 0$. Right semicentral idempotents are defined in an analogous way. The following characterizations of semicentral idempotent are well known and easy to prove:

$$\begin{aligned} e \text{ is left semicentral in R} &\Leftrightarrow eRe = Re \\ &\Leftrightarrow ere = re \text{ for all } r \in R \\ &\Leftrightarrow eR \text{ is an ideal of } R \\ &\Leftrightarrow R(1-e) \text{ is an ideal of } R \end{aligned}$$

As a consequence of these equivalences, an idempotent element $e \in C^*$ is left semicentral if and only if $1-e$ is right semicentral.

Let us give some extra characterizations of a left semicentral idempotent element in the dual algebra C^* of a coalgebra C.

Lemma 3.5.1. *Let C be a coalgebra and e be an idempotent element in C^*. The following conditions are equivalent:*

(a) e *is a left semicentral idempotent in* C^*.

(b) $eCe = eC$.

(c) $C(1-e)$ *is a subcoalgebra of* C.

(d) eC *is a subcoalgebra of* C.

(e) eM *is a subcomodule of* M *for every right* C*-comodule* M.

Proof. $(a) \Rightarrow (b)$ For any element $x \in C$ and any $g \in C^*$, we have $g(ex) = (g * e)(x) = (e * g * e)(x) = g(exe)$. Therefore $exe = ex$ and thus $eCe = eC$.
$(b) \Rightarrow (a)$ Given $g \in C^*$, we have that for every $x \in C$, $(g * e)(x) = g(ex) = g(exe) = (e * g * e)(x)$. Therefore $e * g * e = g * e$ for every $g \in C^*$ and e is left semicentral in C^*.
$(a) \Leftrightarrow (c)$ It is easy to see that $(C(1-e))^{\perp} = eC^*$, so $C(1-e)$ is a subcoalgebra of C if, and only if eC^* is an ideal of C^* if and only if e is a left semicentral idempotent in C^*.
$(a) \Rightarrow (e)$ Let M be a right C-comodule and g an arbitrary element in C^*. Then, for every $x \in eM$, we have $gx = g(ex) = (g*e)x = (e*g*e)x = e(g * ex) \in eM$. Therefore eM is left C^*-submodule of M and thus it is a right C-subcomodule.
$(e) \Rightarrow (d)$ It is trivial
$(d) \Leftrightarrow (a)$ As before, $(eC)^{\perp} = C^*(1-e)$, thus eC is a subcoalgebra of C if and only if $C^*(1-e)$ is an ideal of C^* if and only if e is a left semicentral idempotent in C^*. □

The following theorem is the main result of this section. In it, we describe stable subcategories from different points of view. A proof of some equivalences is given in [JMNR06, Theorem 4.3]. We recall that, for a subset Λ of the vertex set $(\Gamma_C)_0$, we say that Λ is *right link-closed* if it satisfies that, for each arrow $S \to T$ in Γ_C, if $S \in \Lambda$ then $T \in \Lambda$.

Theorem 3.5.2. *Let C be a coalgebra and $\mathcal{T}_e \subseteq \mathcal{M}^C$ be a localizing subcategory associated to an idempotent element $e \in C^*$. The following conditions are equivalent:*

(a) $\mathcal{T}_e$ *is a stable subcategory.*

(b) $T(E_x) = 0$ *for any* $x \notin I_e$.

(c) $T(E_x) = \begin{cases} \overline{E}_x & \text{if } x \in I_e, \\ 0 & \text{if } x \notin I_e. \end{cases}$

(d) $\operatorname{Hom}_C(E_y, E_x) = 0$ *for all* $x \in I_e$ *and* $y \notin I_e$.

(e) $\mathcal{K} = \{S \in (\Gamma_C)_0 \mid eS = S\}$ *is a right link-closed subset of* $(\Gamma_C)_0$, *i.e., there is no arrow* $S_x \to S_y$ *in* Γ_C, *where* $T(S_x) = S_x$ *and* $T(S_y) = 0$.

(f) *There is no path in* Γ_C *from a vertex* S_x *to a vertex* S_y *such that* $T(S_x) = S_x$ *and* $T(S_y) = 0$.

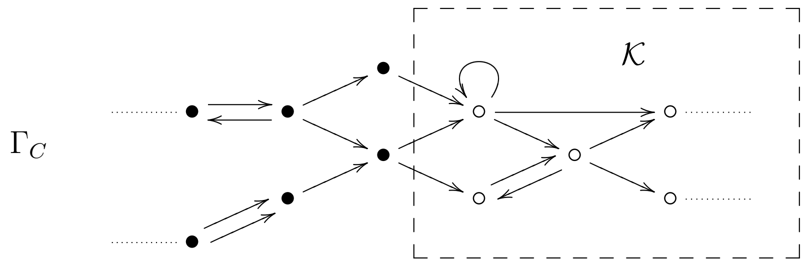
(g) e *is a left semicentral idempotent in* C^*.

If T_e *is a colocalizing subcategory this is also equivalent to*

(h) $H(S_x) = S_x$ *for any* $x \in I_e$.

Proof. $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f)$ follows from the definition and from Proposition 3.4.13.
$(f) \Rightarrow (e)$. Trivial.
$(e) \Rightarrow (c)$. By hypothesis, the set P defined in the proof of Proposition 3.4.18 is zero. Therefore $\operatorname{Soc} T(E_y) = 0$ for all $y \notin I_e$. Then $T(E_y) = 0$ for all $t \notin I_e$.
$(c) \Rightarrow (a)$. Let M be a torsion right C-comodule such that its injective envelope is $\oplus_{i \in J} E_i$. Then $S_i \subseteq M$ is torsion for all $i \in J$ and, by hypothesis, $T(E_i) = 0$ for all $i \in J$. Thus $T(\oplus_{i\in J} E_i) = 0$.
$(c) \Leftrightarrow (g)$. We have $C = \oplus_{x \in I_C} E_x$ then $T(C) = \oplus_{x\in I_e} \overline{E}_x \oplus \oplus_{y \notin I_e} T(E_y)$. On the other hand, $eCe = \oplus_{x \in I_e} \overline{E}_x$. Therefore if (c) holds then $eCe = T(C)$. Conversely, if $eCe = T(C)$ then $\oplus_{x\in I_e} \overline{E}_x = \oplus_{x\in I_e} \overline{E}_x \oplus \oplus_{y \notin I_e} T(E_y)$. Since eCe is quasi-finite, by Krull-Remak-Schmidt-Azumaya theorem, $T(E_y) = 0$ for all $y \notin I_e$.
$(e) \Leftrightarrow (h)$. It is Proposition 3.4.18. $\square$

Then we could say that the vertices which determine a stable localization are placed "on the right side" of the Ext-quiver. In the following picture we denote the vertices in $\mathcal{K}$ by white points.



As a direct consequence of Theorem 3.5.2, we obtain an alternative proof of the following fact:

Corollary 3.5.3. *[NT96, 4.6] Any stable localizing subcategory of* $\mathcal{M}^C$ *is a perfect colocalizing subcategory*

Proof. If the localizing subcategory $\mathcal{T}$ is stable, $\mathcal{T}$ is associated to a left semicentral idempotent element $e \in C^*$ and hence $Ce = eCe$ is certainly injective and quasifinite as right eCe-comodule. Therefore, by Corollary 3.2.7, $\mathcal{T}$ is a perfect colocalizing subcategory of $\mathcal{M}^C$. $\square$

Let us point out the following remarks.

Remark. It is well known that each stable localizing subcategory $\mathcal{T}$ is a TTF class, that is, the torsion class $\mathcal{T}$ is the torsionfree class for another localizing subcategory. If $\mathcal{T}_e$ is stable then $\mathcal{T}_e = \mathcal{F}_{1-e}$ the torsionfree class associated to the localizing subcategory $\mathcal{T}_{1-e}$. Indeed, using Theorem 3.5.2 and Lemma 3.5.1, for every right C-comodule M, we have that eM is a subcomodule of M, therefore eM is precisely the torsion of M for the localizing subcategory $\mathcal{T}_{1-e}$. Then $M \in \mathcal{F}_{1-e}$ if and only if $eM = 0$ if and only if $M \in T_e$.

Remark. For an idempotent $e \in C^*$, we can consider also the localizing subcategory $\mathcal{T}_e'$ of the category ${}^C\mathcal{M}$ of left C-comodules, determined by e, that is, $\mathcal{T}_e' = \{M \in {}^C\mathcal{M} \mid Me = 0\}$. Using Theorem 3.5.2 and its left version, we obtain that the localizing subcategory $\mathcal{T}_e'$ of ${}^C\mathcal{M}$ is stable if, and only if the localizing subcategory $\mathcal{T}_{1-e}$ of $\mathcal{M}^C$ is stable.

We may find an analogous result to Theorem 3.5.2 for right semicentral idempotents:

Proposition 3.5.4. *Let C be a coalgebra and $\mathcal{T}_e \subseteq \mathcal{M}^C$ be a localizing subcategory associated to an idempotent element $e \in C^*$. The following conditions are equivalent:*

(a) *$\mathcal{T}_{1-e}$ is a stable subcategory.*

(b) *$T(E_x) = E_x$ for any $x \in I_e$.*

(c) *There is no path in Γ_C from a vertex S_y to a vertex S_x such that $T(S_x) = S_x$ and $T(S_y) = 0$.*

(d) *$\mathcal{K} = \{S \in (\Gamma_C)_0 \mid eS = S\}$ is a left link-closed subset of $(\Gamma_C)_0$, i.e., there is no arrow $S_y \to S_x$ in Γ_C, where $T(S_x) = S_x$ and $T(S_y) = 0$.*

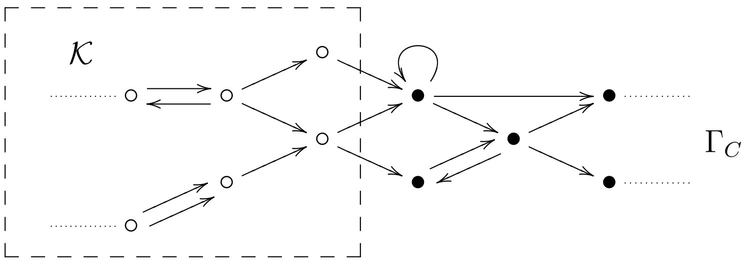
(e) *e is a right semicentral idempotent in C^*.*

(f) *The torsion subcomodule of a right C-comodule M is $(1-e)M$*

(g) *$S(S_x) = S_x$ for all $x \in I_e$.*

Proof. By the above remarks and Proposition 3.5.2, it is easy to prove $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$. $(d) \Leftrightarrow (g)$ is Corollary 3.4.7. $\square$

And now, we could say that the vertices of the localization are placed "on the left side" of the Ext-quiver.



As a consequence, we get the following immediate result:

Corollary 3.5.5. *The following are equivalent:*

(a) e *is a central idempotent.*

(b) *For each arrow* $S_x \to S_y$ *in* Γ_C, $T(S_x) = 0$ *if and only if* $T(S_y) = 0$.

(c) *For each connected component of* Γ_C, *either all vertices are torsion or all vertices are torsion-free.*

(d) $T(E_x) = 0$ *for any* $x \notin I_e$ *and* $S(S_x) = S_x$ *for all* $x \in I_e$.

Corollary 3.5.6. *Let* e *be a central idempotent in* C^* *and* $\{\Sigma_C^t, \Delta_C^s\}_{t,s}$ *be the connected components of* Γ_C, *where the vertices of each* Σ_C^t *are torsion-free and the vertices of each* Δ_C^s *are torsion. Then* $\{\Sigma_C^t\}_t$ *are the connected components of* Γ_{eCe}.

3.6 Localization in pointed coalgebras

We finish the chapter studying the localization in pointed coalgebras. Remember that if the base field is algebraically closed, every coalgebra is Morita-Takeuchi equivalent to a pointed coalgebra and therefore this is a very large class of coalgebras.

In order to obtain an easier description of the theory we need the following notation:

Let Q be a quiver and $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$ be a path in Q. We denote by I_p the subset of vertices $\{s(\alpha_1), t(\alpha_1), t(\alpha_2), \ldots, t(\alpha_n)\}$. Given a subset of vertices $X \subseteq Q_0$, we say that p is a *cell* in Q relative to X (shortly a cell) if $I_p \cap X = \{s(p), t(p)\}$ and $t(\alpha_i) \notin X$ for all $i = 1, \ldots, n-1$. Given $x, y \in X$, we denote by $\mathcal{C}ell_X^Q(x, y)$ the set of all cells from x to y. We will denote the set of all cells in Q relative to X by $\mathcal{C}ell_X^Q$.

Lemma 3.6.1. *Let Q be a quiver and $X \subseteq Q_0$ a subset of vertices. Given a path p in Q such that $s(p)$ and $t(p)$ are in X, then p has a unique decomposition $p = q_r \cdots q_1$, where each q_i is a cell in Q relative to X.*

Proof. It is straightforward. □

Let p be a non-trivial path in Q which starts and ends at vertices in $X \subseteq Q_0$. We shall call the *cellular decomposition* of p relative to X to the decomposition given in the above lemma.

By Theorem 1.6.6, every pointed coalgebra C is isomorphic to a subcoalgebra of a path coalgebra KQ of a quiver Q. Furthermore, C is an admissible subcoalgebra (contains the subspace generated by the set of all vertices and all arrows) and then, by Corollary 2.3.5, it is a relation subcoalgebra in the sense of Simson [Sim05]. Thus C has a decomposition, as vector space, $C = \bigoplus_{a,b \in Q_0} C \cap KQ(a,b)$, where $Q(a,b)$ is the set of all paths in Q from a to b. That is, C has a basis in which every basic element is a linear combination of paths with common source and common sink.

Lemma 3.6.2. *Let C be an admissible subcoalgebra of a path coalgebra KQ of a quiver Q. There exists a bijective correspondence between localizing subcategories of $\mathcal{M}^C$ and subsets of vertices of Q.*

Proof. The set of simple C-comodules is $\{Kx\}_{x \in Q_0}$ and therefore there is a bijection between the subsets of simple comodules and the subsets of vertices of Q. By the arguments of Section 2, the result follows. □

Let X be a subset of vertices of Q. We will denote by $\mathcal{T}_X$ the localizing subcategory of $\mathcal{M}^C$ associated to X.

Given an admissible subcoalgebra C of KQ, we can say more about the idempotent elements of its dual algebra and the bijection between them and the vertices of the quiver. For any idempotent element e in C^* and any vertex x in Q, we have that either $e(x) = 0$ or $e(x) = 1$. Hence two idempotent elements $e,\ f \in C^*$ are equivalent if and only if $e_{|Q_0} = f_{|Q_0}$. In this way, we obtain that every localizing subcategory of $\mathcal{M}^C$ is associated to an idempotent element $e \in C^*$ such that $e(p) = 0$ for any non trivial path p. Therefore, for an idempotent $e \in C^*$, we may consider the subset of vertices $\{x \in Q_0$ such that $e(x) = 1\}$ and, conversely, for a subset of vertices X, it is associated the idempotent $e \in C^*$ such that $e(x) = 1$ if $x \in X$, and zero otherwise. In what follows, by the idempotent associated to a subset of vertices, we shall mean the idempotent described above.

For an idempotent element $e \in C^*$, the localized coalgebra eCe has a decomposition $eCe = \bigoplus_{a,b \in X} C \cap KQ(a,b)$, that is, the elements of eCe are linear combinations of paths which start and end at vertices in the set of vertices associated to e. Also we have that eCe is a pointed coalgebra so, by Theorem 1.6.6, there exists a quiver Q^e such that eCe is an admissible subcoalgebra of KQ^e. Let us analyze the elements of the quiver Q^e:

Vertices. We know that Q_0 equals the set of group-like elements $\mathcal{G}(C)$ of C, therefore $(Q^e)_0 = \mathcal{G}(eCe) = e\mathcal{G}(C)e = eQ_0e = X$.

Arrows. Let x and y be vertices in X. An element $p \in eCe$ is a non-trivial (x,y)-primitive element if and only if $p \notin KX$ and $\Delta_{eCe}(p) = y \otimes p + p \otimes x$. Without loss of generality we may assume that $p = \sum_{i=1}^{n} \lambda_i p_i$ is an element in eCe such that each path p_i is not trivial, and $p_i = \alpha_{n_i}^i \cdots \alpha_2^i \alpha_1^i$ and $p_i = q_{r_i}^i \cdots q_1^i$ are the decomposition of p_i in arrows of Q and the cellular decomposition of p_i relative to X, respectively, for all $i = 1, \ldots, n$. Then

$$\Delta_C(p) = \sum_{i=1}^{n} \lambda_i p_i \otimes h(p_i) + \sum_{i=1}^{n} \lambda_i s(p_i) \otimes p_i + \sum_{i=1}^{n} \lambda_i \sum_{j=2}^{n_i} \alpha_{n_i}^i \cdots \alpha_j^i \otimes \alpha_{j-1}^i \cdots \alpha_1^i$$

and therefore,

$$\begin{aligned}\Delta_{eCe}(p) = \sum_{i=1}^{n} \lambda_i (e\, p_i\, e \otimes e\, h(p_i)\, e) + \sum_{i=1}^{n} \lambda_i (e\, s(p_i)\, e \otimes e\, p_i\, e) + \\ + \sum_{i=1}^{n} \lambda_i \sum_{j=2}^{n_i} e(\alpha_{n_i}^i \cdots \alpha_j^i) e \otimes e(\alpha_{j-1}^i \cdots \alpha_1^i) e.\end{aligned}$$

We have that, for each path q in Q, $eqe = q$ if q starts and ends at vertices in X, and zero otherwise. Thus,

$$\begin{aligned}\Delta_{eCe}(p) = \sum_{i=1}^{n} \lambda_i (p_i \otimes h(p_i)) + \sum_{i=1}^{n} \lambda_i (s(p_i) \otimes p_i) + \\ + \sum_{i=1}^{n} \lambda_i \sum_{j=2}^{r_i} q_{r_i}^i \cdots q_j^i \otimes q_{j-1}^i \cdots q_1^i.\end{aligned}$$

Now, this is a linear combination of linearly independent vectors of the vector space $eCe \otimes eCe$, so $\Delta_{eCe}(p) = y \otimes p + p \otimes x$ if and only if we have that

(a) $h(p_i) = x$ for all $i = 1, \ldots, n$;

(b) $s(p_i) = y$ for all $i = 1, \ldots, n$;

(c) $\sum_{i=1}^{n} \lambda_i \sum_{j=2}^{r_i} q_{r_i}^i \cdots q_j^i \otimes q_{j-1}^i \cdots q_1^i = 0$.

Condition (c) is satisfied if and only if $r_i = 1$ for all $i = 1, \ldots, n$. Therefore $\Delta_{eCe}(p) = y \otimes p + p \otimes x$ if and only if p_i is a cell from x to y for all $i = 1, \ldots, n$. Thus the vector space of all non-trivial (x, y)-primitive elements is $K\mathcal{C}ell_X^Q(x, y) \cap C$.

Proposition 3.6.3. *Let C be an admissible subcoalgebra of a path coalgebra KQ of a quiver Q. Let e be the idempotent element of C^* associated to a subset of vertices X. Then the localized coalgebra eCe is an admissible subcoalgebra of the path coalgebra KQ^e, where Q^e is the quiver whose set of vertices is $(Q^e)_0 = X$ and the number of arrows from x to y is $\dim_K K\mathcal{C}ell_X^Q(x, y) \cap C$ for all $x, y \in X$.*

Corollary 3.6.4. *Let Q be a quiver and e be the idempotent element of $(KQ)^*$ associated to a subset of vertices X. Then the localized coalgebra $e(KQ)e$ is an admissible subcoalgebra of the path coalgebra KQ^e, where $Q^e = (X, \mathcal{C}ell_X^Q)$.*

If C is a path coalgebra then we can say more:

Proposition 3.6.5. *Let Q be a quiver and e be the idempotent element of $(KQ)^*$ associated to a subset of vertices X. Then the localized coalgebra $e(KQ)e$ is isomorphic to the path coalgebra KQ^e, where $Q^e = (X, \mathcal{C}ell_X^Q)$.*

Proof. Consider the morphism of coalgebras $f : e(KQ)e \to KQ^e$ defined in the following way: $f(x) = x$ for any vertex $x \in X$, and for any non-trivial path $p = \alpha_n \cdots \alpha_1$ such that $s(\alpha_1), t(\alpha_n) \in X$, we choose $f(p) = p_r \cdots p_1$, where $p_r \cdots p_1$ is the cellular decomposition of p. This is a bijective morphism of coalgebras. □

Examples of the former proposition are given in Section 1 of Chapter 5. Nevertheless, these are not the unique examples of localized coalgebras which are path coalgebras as we show in the following example:

Example 3.6.6. *Let Q be the quiver*

$$\begin{tikzcd} & \overset{x_2}{\bullet} \arrow[dr, "\alpha_2"] & \\ x_1\ \circ \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_3"'] & & \circ\ x_4 \\ & \underset{x_3}{\circ} \arrow[ur, "\alpha_4"'] & \end{tikzcd}$$

and C be the admissible subcoalgebra generated by $\alpha_2\alpha_1 + \alpha_4\alpha_3$. Let us consider $X = \{x_1, x_3, x_4\}$. Then eCe is the path coalgebra of the quiver

$$Q^e \equiv \begin{tikzcd} \underset{x_1}{\circ} \arrow[r, "\overline{\alpha_3}"] & \underset{x_3}{\circ} \arrow[r, "\overline{\alpha_4}"] & \underset{x_4}{\circ} \end{tikzcd}$$

Here, the element $\alpha_2\alpha_1 + \alpha_4\alpha_3$ corresponds to the composition of the arrows $\overline{\alpha_3}$ and $\overline{\alpha_4}$ of Q^e.

On the other hand, if $C = KQ$, the quiver Q_e is the following

$$Q_e \equiv \begin{tikzcd} \underset{x_1}{\circ} \arrow[r, "\overline{\alpha_3}"'] \arrow[rr, bend left, "\beta\equiv\alpha_2\alpha_1"] & \underset{x_3}{\circ} \arrow[r, "\overline{\alpha_4}"'] & \underset{x_4}{\circ} \end{tikzcd}$$

And $\alpha_2\alpha_1 + \alpha_4\alpha_3$ corresponds to the element $\beta + \overline{\alpha_4}\,\overline{\alpha_3}$.

As in the previous example, it is worth pointing out that if C is a proper admissible subcoalgebra of a path coalgebra KQ, then we may consider two quivers: the quiver Q^e defined above and the quiver Q_e such that $e(KQ)e \cong KQ_e$. Clearly Q^e is a subquiver of Q_e (differences are in the set of arrows). Then we may relate both quivers by means of a morphism $g : KQ^e \to KQ_e$ defined in the following way: we choose $g(x) = x$, for any element $x \in X$, and given an arrow $\beta \in (Q^e)_1$ from x to y, β corresponds to a basic element of $KCell_X^Q(x, y) \cap C$, $p = \sum_{i=1}^n \lambda_i p_i$, where p_i is a cell from x to y, then we define $g(\beta) = p$. By [Nic78, Proposition 1.4.2], g extend to a morphism of coalgebras and by [Mon93, Theorem 5.3.1], g is injective.

Thus eCe can be viewed as an admissible subcoalgebra of KQ_e. We denote by F the composition $eCe \xrightarrow{h} KQ^e \xrightarrow{g} KQ_e$, where h is the inclusion.

Corollary 3.6.7. *$F(eCe)$ is a subcoalgebra of KQ_e isomorphic to eCe such that each element $x \in F(eCe)$ can be written as $\sum_{i=1}^n \lambda_i c^i_{r_i} \cdots c^i_1$, where any $c^i_j \in F(eCe)$ and it is a linear combination of cells with common source and common sink.*

Remark. The reader should observe that we have the following diagram:

$$\begin{tikzcd} eCe \arrow[r, hook, "i"] \arrow[dr, "h"'] & e(KQ)e \arrow[r, "f"] & KQ_e \\ & KQ^e \arrow[ur, "g"'] & \end{tikzcd}$$

where i is the inclusion. This diagram is not always commutative although F equals f in $(eCe)_1$ (compare with [Woo97, 4.1]). Consider Example 3.6.6, then $F(\alpha_2\alpha_2+\alpha_4\alpha_3) = \overline{\alpha_4}\,\overline{\alpha_3}$ and $f(\alpha_2\alpha_2+\alpha_4\alpha_3) = \overline{\alpha_4}\,\overline{\alpha_3} + \beta$.

Let us now restrict our attention to the colocalizing subcategories of $\mathcal{M}^C$. For the convenience of the reader we introduce the following notation:

We say $p = \alpha_n \cdots \alpha_2\alpha_1$ is a $s(p)$-*tail* in Q relative to X if $I_p \cap X = \{s(p)\}$ and $t(\alpha_i) \notin X$ for all $i = 1, \ldots, n$. If there is no confusion we simply say that p is a tail. Given a vertex $x \in X$ we denote by $\mathcal{T}ail_X^Q(x)$ the set of all x-tails in Q relative to X.

Lemma 3.6.8. *Let Q be a quiver and $X \subseteq Q_0$ be a subset of vertices. Given a path p in Q such that $s(p) \in X$ and $t(p) \notin X$, then p has a unique decomposition $p = cq_r \cdots q_1$, where $c, q_1, \ldots, q_r$ are subpaths of p such that $c \in \mathcal{T}ail_X^Q(s(c))$ and $q_i \in \mathcal{C}ell_X^Q$ for all $i = 1, \ldots, r$.*

Proof. It is straightforward. $\square$

Let p be a path in Q such that $s(p) \in X \subseteq Q_0$ and $t(p) \notin X$, we shall call the *tail decomposition* of p relative to X to the decomposition given in the above lemma. We say that c is the tail of p relative to X if $p = cq_r \cdots q_1$ is the tail decomposition of p relative to X.

Let $\{S_x\}_{x\in Q_0}$ be a complete set of pairwise non isomorphic indecomposable simple right C-comodules. We know that a right C-comodule M is *quasifinite* if and only if $\mathrm{Hom}_C(S_x, M)$ has finite dimension for all $x \in Q_0$. Let $x \in Q_0$ and f be a linear map in $\mathrm{Hom}_C(S_x, M)$. Then $\rho_M \circ f = (f \otimes I) \circ \rho_{S_x}$, where ρ_M and ρ_{S_x} are the structure maps of M and S_x as right C-comodules, respectively. In order to describe f, since $S_x = Kx$, it is enough to choose an image for x. Suppose that $f(x) = m \in M$. Since $(\rho_M f)(x) = ((f \otimes I)\rho_{S_x})(x)$, we obtain that $\rho_M(m) = m \otimes x$. Therefore

$$\mathrm{Hom}_C(S_x, M) \cong \{m \in M \text{ such that } \rho_M(m) = m \otimes x\} = M_x,$$

as K-vector spaces, and M is quasifinite if and only if M_x has finite dimension for all $x \in Q_0$.

Our aim now is to establish when a localizing subcategory $\mathcal{T}_e$ is colocalizing, or equivalently, by Proposition 3.2.6, when eC is a quasifinite right eCe-comodule. Following the sigma-notation of [Swe69], the structure of eC as right eCe-comodule is given by $\rho_{eC}(p) = \sum_{(p)} ep_{(1)} \otimes ep_{(2)}e$ if $\Delta_{KQ}(p) = \sum_{(p)} p_{(1)} \otimes p_{(2)}$, for all $p \in eC$. It is easy to see that eC has a decomposition $eC = \bigoplus_{a\in X, b\in Q_0} C \cap KQ(a,b)$, as vector space, that is, the elements of eC are linear combinations of paths which start at vertices in X.

Proposition 3.6.9. *Let C be an admissible subcoalgebra of a path coalgebra KQ. Let e be an idempotent element in C^* associated to a subset of vertices X. The following conditions are equivalent:*

(a) *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^C$ is colocalizing.*

(b) *eC is a quasifinite right eCe-comodule.*

(c) *$\dim_K K\mathcal{T}ail_X^Q(x) \cap C$ is finite for all $x \in X$.*

Proof. By the arguments mentioned above, it is enough to prove that $(eC)_x = K\mathcal{T}ail_X^Q(x) \cap C$.

Let $p = \sum_{i=1}^n \lambda_i c_i \in C$ be a K-linear combination of x-tail such that $c_i = \alpha_{r_i}^i \cdots \alpha_1^i$ ends at y_i for all $i = 1, \ldots n$. Then,

$$\Delta_{KQ}(p) = p \otimes x + \sum_{i=1}^n y_i \otimes \lambda_i c_i + \sum_{i=1}^n \lambda_i \sum_{j=2}^{r_i} \alpha_{r_i}^i \cdots \alpha_j^i \otimes \alpha_{j-1}^i \cdots \alpha_1^i,$$

and then,

$$\begin{aligned}\rho_{eC}(p) = e\,p \otimes e\,x\,e + \sum_{i=1}^n \lambda_i e\,y_i \otimes e\,c_i\,e + \\ + \sum_{i=1}^n \lambda_i \sum_{j=2}^{r_i} e(\alpha_{r_i}^i \cdots \alpha_j^i) \otimes e(\alpha_{j-1}^i \cdots \alpha_1^i)e = p \otimes x\end{aligned}$$

because α_j^i ends at a point not in X for all $j = 1, \ldots, r_i$ and $i = 1, \ldots n$. Thus $p \in (eC)_x$.

Conversely, consider an element $p = \sum_{i=1}^n \lambda_i p_i + \sum_{k=1}^m \mu_k q_k \in (eC)_x$, where $t(p_i) \in X$ for all $i = 1, \ldots, n$, and $t(q_k) \notin X$ for all $k = 1, \ldots, m$.

Moreover, let us suppose that $p_i = \overline{p}^i_{r_i} \cdots \overline{p}^i_1$ is the cellular decomposition of p_i relative to X for all $i = 1, \ldots, n$, and $q_k = c_k \overline{q}^k_{s_k} \cdots \overline{q}^k_1$ is the tail decomposition of q_k relative to X for all $k = 1, \ldots, m$. Then,

$$\rho_{eC}(p) = \sum_{i=1}^{n} \lambda_i \sum_{j=2}^{r_i} \overline{p}^i_{r_i} \cdots \overline{p}^i_j \otimes \overline{p}^i_{j-1} \cdots \overline{p}^i_1 + \sum_{i=1}^{n} \lambda_i t(p_i) \otimes p_i + \sum_{i=1}^{n} \lambda_i p_i \otimes s(p_i) +$$
$$+ \sum_{k=1}^{m} \mu_k c_k \otimes q_k + \sum_{k=1}^{m} \mu_k \sum_{l=2}^{s_k} c_k \overline{q}^k_{s_k} \cdots \overline{q}^k_l \otimes \overline{q}^k_{l-1} \cdots \overline{q}^k_1 + \sum_{k=1}^{m} \mu_k q_k \otimes s(q_k).$$

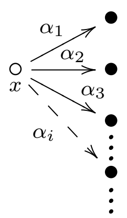
A straightforward calculation proves that if $\rho_{eC}(p) = p \otimes x$ then $n = 0$, $s(q_k) = x$ and $s_k = 0$ for all $k = 1, \ldots, m$. Therefore $p \in K\mathcal{T}ail^Q_X(x) \cap C$ and the proof is finished. $\square$

Corollary 3.6.10. *Let Q be a quiver and e be the idempotent element in $(KQ)^*$ associated to a subset $X \subseteq Q_0$. Then the following conditions are equivalent:*

(a) *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^C$ is colocalizing.*

(b) *$\mathcal{T}ail^Q_X(x)$ is a finite set for all $x \in X$. That is, roughly speaking, there are at most a finite number of paths starting at the same point whose only vertex in X is the first one.*

Example 3.6.11. *Consider the quiver Q*



where $i \in \mathbb{N}$

and the idempotent $e \in (KQ)^*$ associated to the subset $X = \{x\}$. Then $\mathcal{T}ail^Q_X(x) = \{\alpha_i\}_{i \in \mathbb{N}}$ is an infinite set and the localizing subcategory $\mathcal{T}_X$ is not colocalizing.

Remark. Observe that if the set $Q_0 \backslash X$ is finite and Q has no cycles, or if C is finite dimensional, then every localizing subcategory is colocalizing.

Chapter 4

Tame and Wild Coalgebras

In the category of coalgebras, over a fixed algebraically closed field, we may distinguish between two disjoint classes: the tame coalgebras and the wild coalgebras, see [Sim05]. The idea of such classes is that the category of comodules over a wild coalgebra is so large that it contains the representation theory of any finite dimensional algebra. Therefore it is not a realistic aim to get a description of all its comodules and we exclude it from our study. Moreover, it is expected that each coalgebra is either tame or wild (tame-wild dichotomy) and hence, from that point of view, we should restrict the theory to tame coalgebras.

In Chapter 2, we saw that there exist admissible subcoalgebras which are not path coalgebras of a quiver with relations and then we cannot find, in this way, an analogous result for coalgebras of the famous Gabriel's Theorem: *every basic finite dimensional algebra, over an algebraically closed field, is the path algebra of a quiver with relations*. Nevertheless these examples are of wild comodule type and, furthermore, a coalgebra with such property seems closed to be wild. Therefore the problem should be reformulated as the following question: *is any basic tame coalgebra, over an algebraically closed field, isomorphic to the path coalgebra of a quiver with relations?*.

In this chapter we consider this problem. In particular, after relating tameness and wildness of a coalgebra and its localized coalgebras, we shall prove the following theorem: "Let Q be an acyclic quiver. Then any tame admissible subcoalgebra of KQ is the path coalgebra of a quiver with relations".

4.1 Comodule types of coalgebras

Throughout this chapter we shall assume that K is an algebraically closed field. It is well known that the category of finite dimensional K-algebras is the disjoint union of two classes: tame algebras and wild algebras. This is known as the *tame-wild dichotomy*, see [Dro79] or [Sim92]. The idea of such classes is that the category of finite dimensional modules over a wild algebra is so large (because it contains the finite dimensional modules over the polynomial algebra of two non-commuting variables) that it is not a realistic aim to study its representation theory. Therefore the theory is restricted to tame algebras. In this section we recall from [Sim01] and [Sim05] the analogous concepts for a basic (pointed) coalgebra.

Let C be a basic coalgebra such that $C_0 = \oplus_{i\in I_C} S_i$. For every finite dimensional right C-comodule M we may consider the *length vector* of M, $\operatorname{length} M = (m_i)_{i\in I_C} \in \mathbb{Z}^{I_c}$, where $m_i \in \mathbb{N}$ is the number of simple composition factors of M isomorphic to S_i. In [Sim01] it is proved that the map $M \mapsto \operatorname{length} M$ extends to a group isomorphism $K_0(C) \longrightarrow \mathbb{Z}^{I_C}$, where $K_0(C)$ is the Grothendieck group of C. Recall that the Grothendieck group of a coalgebra (or of the category $\mathcal{M}_f^C$) is the quotient of the free abelian group generated by the set of isomorphism classes $[M]$ of modules M in $\mathcal{M}_f^C$ modulo the subgroup generated by the elements $[M]-[N]-[L]$ corresponding to all exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{M}_f^C$.

Let R be a K-algebra. By a R-C-bimodule we mean a K-vector space L endowed with a left R-module structure $\cdot : R\otimes L \to L$ and a right C-comodule structure $\rho : L \to L\otimes C$ such that $\rho_L(r\cdot x) = r\cdot\rho_L(x)$, i.e., the following diagram is commutative

$$\begin{tikzcd} R\otimes L \arrow[r, "\cdot"] \arrow[d, "I\otimes\rho_L"'] & L \arrow[d, "\rho_L"] \\ R\otimes L\otimes C \arrow[r, "\cdot\otimes I"] & L\otimes C \end{tikzcd}$$

We denote by ${}_R\mathcal{M}^C$ the category of R-C-bimodules.

Example 4.1.1. *Let L be a R-C-bimodule and e be an idempotent element in $\in C^*$. Then eL is a R-eCe-bimodule. That is, we have a functor*

$$T = e(-) :_R\mathcal{M}^C \longrightarrow_R\mathcal{M}^{eCe}.$$

From the above diagram we obtain the following equalities:

$$\sum_{(r\cdot x)}(r\cdot x)_{(0)}\otimes(r\cdot x)_{(1)}=\rho_L(r\cdot x)=r\cdot\rho_L(x)=\sum_x r\cdot x_{(0)}\otimes x_{(1)}, \tag{4.1}$$

for each element $r\in R$ *and* $x\in L$. *Now, we have that*

$$\begin{array}{rll} r\cdot(e\cdot x) & =r\cdot(\sum_{(x)}x_{(0)}e(x_{(1)})) & \\ & =\sum_{(x)}r\cdot x_{(0)}e(x_{(1)}) & \\ & =(I\otimes e)(\sum_{(x)}r\cdot x_{(0)}\otimes x_{(1)}) & (4.1)\\ & =(I\otimes e)\rho_L(r\cdot x) & \\ & =e\cdot(r\cdot x) & \end{array}$$

Then eL *has an structure of left* R*-module and right* eCe*-comodule. Let us see the compatibility property.*

$$\begin{array}{rll} r\cdot\rho_{eL}(e\cdot x) & =r\cdot(\sum_{(x)}e\cdot x_{(0)}\otimes e\cdot x_{(1)}\cdot e) & \\ & =\sum_{(x)}r\cdot(e\cdot x_{(0)})\otimes e\cdot x_{(1)}\cdot e & \\ & =\sum_{(x)}e\cdot(r\cdot x_{(0)})\otimes e\cdot x_{(1)}\cdot e & (4.1)\\ & =\sum_{(r\cdot x)}e\cdot(r\cdot x)_{(0)}\otimes e\cdot(r\cdot x)_{(1)}\cdot e & \\ & =\rho_{eL}(e\cdot(r\cdot x)) & \\ & =\rho_{eL}(r\cdot(e\cdot x)) & \end{array}$$

Following [Sim01] and [Sim05], let us recall that a K-coalgebra C is said to be of *tame comodule type* (tame for short) if for every $v\in K_0(C)$ there exist $K[t]$-C-bimodules $L^{(1)},\dots,L^{(r_v)}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable right C-comodules M with $\mathrm{length}\, M=v$ are of the form $M\cong K^1_\lambda\otimes_{K[t]}L^{(s)}$, where $s\leq r_v$, $K^1_\lambda=K[t]/(t-\lambda)$ and $\lambda\in K$. If there is a common bound for the numbers r_v for all $v\in K_0(C)$, then C is called *domestic*.

If C is a tame coalgebra then there exists a *growth function* $\mu^1_C: K_0(C)\to\mathbb{N}$ defined as $\mu^1_C(v)$ to be the minimal number r_v of $K[t]$-C-bimodules $L^{(1)},\dots,L^{(r_v)}$ satisfying the above conditions, for each $v\in K_0(C)$. C is said to be of *polynomial growth* if there exists a formal power series

$$G(t)=\sum_{m=1}^{\infty}\sum_{j_1,\dots,j_m\in I_C}g_{j_1,\dots,j_m}t_{j_1,\dots,j_m}$$

with $t=(t_j)_{j\in I_C}$ and non-negative coefficients $g_{j_1,\dots,j_m}\in\mathbb{Z}$ such that $\mu^1_C(v)\leq G(v)$ for all $v=(v(j))_{j\in I_C}\in K_0(C)\cong\mathbb{Z}^{(I_C)}$ such that $\|v\|:=$

$\sum_{j \in I_C} v(j) \geq 2$. If $G(t) = \sum_{j \in I_C} g_j t_j$, where $g_j \in \mathbb{N}$, then C is called of *linear growth*. If μ_C^1 is zero we say that C is of *discrete* comodule type. Observe that domestic coalgebras are of linear growth.

Let Q be the quiver $\circ \overset{\textstyle\rightarrow}{\underset{\textstyle\rightarrow}{\rightarrow}} \circ$, KQ the path algebra of the quiver Q and $\mathcal{M}^f_{KQ}$ the category of finite dimensional right KQ-comodules. A K-coalgebra C is of *wild comodule type*(wild for short) if there exists an exact and faithful K-linear functor $F : \mathcal{M}^f_{KQ} \to \mathcal{M}^C_f$ that respects isomorphism classes and carries indecomposables right KQ-modules to indecomposable right C-comodules. If, in addition, the functor F is fully faithful, we will say that C is of *fully wild comodule type*. This definition can be done in an equivalent way if we take the quiver Q consisting on one vertex and two loops.

Let us collect from [Sim01] and [Sim05] some properties of wild and tame comodule type.

Proposition 4.1.2. (a) *The tame, polynomial growth, linear growth, discrete, domestic and wild comodule type are invariant under Morita-Takeuchi equivalence of coalgebras.*

(b) *The notion of wild comodule type is left-right symmetric.*

(c) *If there exist a pair S, S' of simple right C-comodules such that $\dim_K \mathrm{Ext}^1_C(S, S') \geq 3$ then C is of wild comodule type.*

(d) *The following conditions are equivalent:*

  (i) *C is of wild comodule type.*

  (ii) *There exists a finite dimensional subcoalgebra H of C of wild comodule type.*

  (iii) *The coalgebra C is a direct union of finite dimensional subcoalgebras of wild comodule type.*

(e) *If C is tame then each finite dimensional subcoalgebra of C is also tame.*

Corollary 4.1.3. *Let C be a K-coalgebra and D be a subcoalgebra of C of wild comodule type. Then C is of wild comodule type.*

As a consequence of the former proposition, Simson proves in [Sim05] the *weak tame-tild dichotomy for coalgebras*.

Corollary 4.1.4 (*Weak tame-wild dichotomy for coalgebras*)**.** *Let K be an algebraically closed field. Then every K-coalgebra of tame comodule type is not of wild comodule type.*

We hope that the following tame-wild dichotomy holds.

Conjecture 4.1.5. *[Sim05][*Tame-wild dichotomy for coalgebras*] Let K be an algebraically closed field and C be a K-coalgebra. Then C is either of tame comodule type, or of wild comodule type, and these types are mutually exclusive.*

4.2 Localization and tame comodule type

This section and the subsequent one are devoted to study the relation between the comodule type of a coalgebra and its localized coalgebras.

Let us analyze the behavior of the length vector under the action of the quotient functor.

Lemma 4.2.1. *Let C be a coalgebra and $e \in C^*$ be the idempotent element associated to a set of simple right C-comodules $\mathcal{K} = \{S_i\}_{i \in I_e}$. Let L be a finite dimensional right C-comodule, then $(\text{length } L)_i = (\text{length } eL)_i$ for all $i \in I_e$.*

Proof. Let $0 \subset L_1 \subset L_2 \subset \cdots \subset L_{n-1} \subset L_n$ be a composition series for L. Then, we obtain the inclusions $0 \subseteq eL_1 \subseteq eL_2 \subseteq \cdots \subseteq eL_{n-1} \subseteq eL_n$. Since $e(-)$ is an exact functor, $eL_j/eL_{j-1} \cong e(L_j/L_{j-1}) = eS_j$, where S_j is a simple C-comodule for all $j = 1, \ldots, n$. But $eS_j = S_j$ if $S_j \in \mathcal{K}$ and zero otherwise. Thus $(\text{length } L)_i = (\text{length } eL)_i$ for all $i \in I_e$. □

Corollary 4.2.2. *$\ell(eM) = \sum_{i \in I_e} (\text{length } eM)_i \leq \sum_{i \in I_C} (\text{length } M)_i = \ell(M)$.*

Therefore the following diagram is commutative

$$\begin{tikzcd} \mathcal{M}_f^C \arrow[r, "e(-)"] \arrow[d, "\text{length}"'] & \mathcal{M}_f^{eCe} \arrow[d, "\text{length}"] \\ K_0(C) \arrow[r, "f"] & K_0(eCe) \end{tikzcd}$$

where f is the projection from $K_0(C) \cong \mathbb{Z}^{I_C}$ onto $K_0(eCe) \cong \mathbb{Z}^{I_e}$.

Let us now consider the opposite direction, that is, if N is a right eCe-comodule whose length vector is known, which is the length vector of $S(N)$? In general, S does not preserve finite dimensional comodules and then we have to assume some conditions. We start with a simple case.

Lemma 4.2.3. *Let N be a finite dimensional right eCe-comodule with* $\text{length}\, N = (v_i)_{i\in I_e}$. *Suppose that $S(S_i) = S_i$ for all $i \in I_e$ such that $v_i \neq 0$. Then*

$$\text{length}\, S(N) = \begin{cases} v_i, & \textit{if}\, i \in I_e \\ 0, & \textit{if}\, i \in I_C \backslash I_e \end{cases}$$

Proof. Let $0 \subset N_1 \subset N_2 \subset \cdots \subset N_{n-1} \subset N_n = N$ be a composition series for N. Since S is left exact, we have the chain of right C-comodules

$$0 \subset S(N_1) \subset S(N_2) \subset \cdots \subset S(N_{n-1}) \subset S(N_n) = S(N).$$

Now, for all $j = 0, \ldots, n-1$, we consider the short exact sequence

$$0 \longrightarrow N_j \xrightarrow{i} N_{j+1} \xrightarrow{p} S_{j+1} \longrightarrow 0$$

and applying the functor S we have

$$0 \longrightarrow S(N_j) \xrightarrow{S(i)} S(N_{j+1}) \xrightarrow{S(p)} S(S_{j+1}) = S_{j+1}$$

This sequence is exact since $S(p)$ is non-zero (otherwise $S(i)$ is bijective and then so is i). Thus $S(N_{j+1})/S(N_j) \cong S_{j+1}$ and the chain is a composition series of $S(N)$. $\square$

Lemma 4.2.4. *Let C be a K-coalgebra and R be a K-algebra. Suppose that N is a R-C-bimodule, M is a right R-module and f is an idempotent element in C^*. Then $f(M \otimes_R N) = M \otimes_R fN$.*

Proof. Let us suppose that the right C-comodule structure of N is given by the map ρ_N. Then $M \otimes_R N$ is endowed with a structure of right C-comodule given by the map $I \otimes_R \rho_N : M \otimes_R N \to M \otimes_R N \otimes C$ defined as

$$m \otimes_R n \mapsto m \otimes_R \left(\sum_{(n)} n_{(0)} \otimes n_{(1)} \right) = \sum_{(n)} (m \otimes_R n_{(0)} \otimes n_{(1)})$$

for all $m \in M$ and $n \in N$.

Therefore $f \cdot (m \otimes_R n) = \sum_{(n)} m \otimes_R n_{(0)} \otimes f(n_{(1)}) = \sum_{(n)} m \otimes_R n_{(0)} f(n_{(1)}) = m \otimes_R \sum_{(n)} n_{(0)} f(n_{(1)}) = m \otimes_R f \cdot n$ for all $m \in M$ and $n \in N$. Thus $f(M \otimes_R N) = M \otimes_R fN$. $\square$

Proposition 4.2.5. *Let $v = (v_i)_{i \in I_e} \in K_0(eCe)$ such that $S(S_i) = S_i$ for all $i \in I_e$ with $v_i \neq 0$. If C satisfies the tameness condition for*

$$\overline{v} = \begin{cases} v_i, & \text{if } i \in I_e \\ 0, & \text{if } i \in I_C \backslash I_e \end{cases}$$

then eCe satisfies the tameness condition for v.

Proof. By hypothesis, there exist $K[t]$-C-bimodules $L^{(1)}, L^{(2)}, \ldots, L^{(r_{\overline{v}})}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable right C-comodules M with $\operatorname{length} M = \overline{v}$ are of the form $M \cong K^1_\lambda \otimes_{K[t]} L^{(s)}$, where $s \leq r_{\overline{v}}$, $K^1_\lambda = K[t]/(t-\lambda)$ and $\lambda \in K$. Consider the $K[t]$-eCe-bimodules $eL^{(1)}, \ldots, eL^{(r_{\overline{v}})}$. Obviously, they are finitely generated free as left $K[t]$-modules. Let now N be a right eCe comodule with $\operatorname{length} N = v$. By Lemma 4.2.3, $\operatorname{length} S(N) = \overline{v}$ and therefore $S(N) \cong K^1_\lambda \otimes_{K[t]} L^{(s)}$ for some $s \leq r_{\overline{v}}$ and some $\lambda \in K$ (since S is an embedding, there are only finitely many eCe-comodules N such that $S(N)$ is not of the above form). Then, by the previous lemma, $eS(N) \cong N \cong K^1_\lambda \otimes_{K[t]} eL^{(s)}$. Thus eCe satisfies the tameness condition for v. □

Corollary 4.2.6. *Under the conditions of Proposition 4.2.5, we have that $\mu^1_{eCe}(v) \leq \mu^1_C(\overline{v})$.*

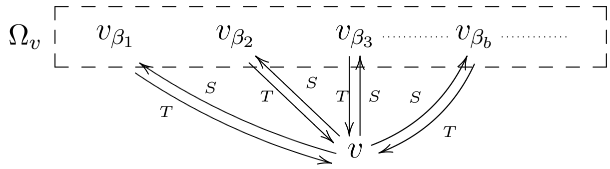
Corollary 4.2.7. *Let C be a coalgebra and $e \in C^*$ be a right semicentral idempotent. If C is tame (of polynomial growth, of linear growth, domestic, discrete) then eCe is tame (of polynomial growth, of linear growth, domestic, discrete).*

Proof. It is clear from Proposition 3.5.4 and the above results. □

The underlying idea of the proof of Proposition 4.2.5 is that if we control the C-comodules whose length vector is obtained from v under the action of S (in Proposition 4.2.5 there is only one vector), then we may control the eCe-comodules of length v. Obviously, a problem appears if there are infinite eCe-comodules $\{N_i\}_{i \in I}$ with length v such that $\operatorname{length} S(N_i) \neq \operatorname{length} S(N_j)$ for $i \neq j$. Then, the number of $K[t]$-eCe-bimodules obtained could be infinite. Therefore the result may be generalized using that method. For the convenience of the reader we introduce the following notation.

To any vector $v \in K_0(eCe) \cong \mathbb{Z}^{I_e}$ we shall associate the set $\Omega_v = \{v_\beta\}_{\beta \in B}$ of all vectors in $K_0(C) \cong \mathbb{Z}^{I_C}$ such that each $v_\beta = \operatorname{length} S(N)$

for some eCe-comodule N such that $\operatorname{length} N = v$.



Proposition 4.2.8. *Let $v \in K_0(eCe)$ such that Ω_v is finite. If C satisfies the tameness condition for each $v_\beta \in \Omega_v$ then eCe satisfies the tameness condition for v.*

Proof. Consider the set of all $K[t]$-C-bimodules associated to all vectors $v_\beta \in \Omega_v$, namely $\mathcal{L} = \{L_\beta^{(j)}\}_{\beta\in\Omega_v, j=1,\ldots,r_\beta}$. By hypothesis, this is a finite set and then so is $T(\mathcal{L})$. We proceed analogously to Proposition 4.2.5 and the result follows. $\square$

Given two vectors $v = (v_i)_{i\in I_C}, w = (w_i)_{i\in I_C} \in K_0(C)$, we will say that $v \leq w$ if $v_i \leq w_i$ for all $i \in I_C$.

Lemma 4.2.9. *Let $v = (v_i)_{i\in I_e} \in K_0(eCe)$ which verifies that $S(S_i)$ is a finite dimensional right C-comodule for all $i \in I_e$ such that $v_i \neq 0$. Then Ω_v is a finite set.*

Proof. Let N be a right eCe-comodule such that $\operatorname{length} N = v$. Consider a composition series for N, $0 \subset N_1 \subset N_2 \subset \cdots \subset N_{n-1} \subset N_n = N$. Since S is left exact, we have the chain of right C-comodules

$$0 \subset S(N_1) \subset S(N_2) \subset \cdots \subset S(N_{n-1}) \subset S(N_n) = S(N).$$

Then, for each $j = 1, \ldots, n$, we have a sequence

$$0 \longrightarrow S(N_{j-1}) \longrightarrow S(N_j) \longrightarrow S(S_j)\ ,$$

where S_j is a simple eCe-comodule.

Since $S(S_j)$ is finite dimensional, it has a composition series $0 \subset S_j \subset S(S_j)_2 \subset \cdots \subset S(S_j)_{r-1} \subset S(S_j)$. Then we can complete the following commutative diagram taking the pullback P_i for $i = 1, \ldots, r-1$.

$$\begin{tikzcd}
S(N_j) \arrow[r, tail] \arrow[d, equal] & P_1 \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\blacksquare"] & S_j \arrow[d, tail] & \\
S(N_j) \arrow[r, tail] \arrow[d, equal] & P_2 \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\blacksquare"] & S(S_j)_2 \arrow[r] \arrow[d, tail] & \mathrm{Coker} \\
\vdots \arrow[d, equal] & \vdots \arrow[d, tail] & \vdots \arrow[d, tail] & \vdots \\
S(N_j) \arrow[r, tail] \arrow[d, equal] & P_{r-1} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\blacksquare"] & S(S_j)_{r-1} \arrow[r] \arrow[d, tail] & \mathrm{Coker} \\
S(N_j) \arrow[r, tail] & S(N_{j+1}) \arrow[r] & S(S_j) \arrow[r] & \mathrm{Coker}
\end{tikzcd}$$

Consider two consecutive rows and their quotient sequence

$$\begin{tikzcd}
S(N_j) \arrow[r, tail] \arrow[dd, equal] & P_t \arrow[rr, "g_t"] \arrow[dr] \arrow[dd, tail, "i"] \arrow[r, phantom, "\blacksquare"'] & & S(S_j)_t \arrow[r] \arrow[dd, tail, "i"] & \mathrm{Coker} \\
& & \mathrm{Im}\ g_t \arrow[ur] \arrow[r] \arrow[dd, tail, "i"] & 0 & \\
S(N_j) \arrow[r, tail] \arrow[dd] & P_{t+1} \arrow[rr, "g_{t+1}"] \arrow[dr] \arrow[dd, two heads, "p"] & & S(S_j)_{t+1} \arrow[r] \arrow[dd, two heads, "p"] & \mathrm{Coker} \\
& & \mathrm{Im}\ g_{t+1} \arrow[ur] \arrow[r] \arrow[dd, two heads, "p"] & 0 & \\
0 \arrow[r] & P_{t+1}/P_t \arrow[rr, "\overline{g}"] \arrow[dr] & & S_k \cong S(S_j)_{t+1}/S(S_j)_t & \\
& & \mathrm{Im}\ g_t/\mathrm{Im}\ g_{t+1} \cong \mathrm{Im}\ \overline{g} \arrow[ur] \arrow[r] & 0 &
\end{tikzcd}$$

Suppose that $P_{t+1} \neq P_t$, then $P_{t+1}/P_t \cong \mathrm{Im}\ \overline{g} \hookrightarrow S_k$, and thus $P_{t+1}/P_t \cong S_k$.

Hence we have obtained a chain

$$0 \subset P_1^1 \subseteq \cdots \subseteq P_{r_1}^1 = S(N_1) \subseteq \cdots \subseteq S(N_{n-1}) \subseteq P_1^n \subseteq \cdots \subseteq P_{r_n}^n = S(N),$$

where the quotient of two consecutive comodules is zero or a simple comodule.

Therefore $\mathrm{length}\ S(N) \leq \sum_{j=1}^n \mathrm{length}\ S(S_j)$ for any right eCe-comodule N whose $\mathrm{length}\ N = v$. Thus Ω_v is a finite set. $\square$

Theorem 4.2.10. *Let C be a coalgebra and $e \in C^*$ be an idempotent element such that $S(S_i)$ is a finite dimensional right C-comodule for all $i \in I_e$. If C is tame then eCe is tame.*

Proof. It is straightforward from the above results. □

In particular, the conditions of Theorem 4.2.10 are satisfied for any idempotent if C is pure semisimple or, moreover, if it is left semiperfect. A coalgebra is said to be *right pure semisimple* if every indecomposable right comodule is finite dimensional. It is *left semiperfect* if every finite dimensional left comodule has a finite dimensional projective cover, or equivalently, if any indecomposable injective right comodule is finite dimensional.

Corollary 4.2.11. *Let C be a right pure semisimple or a left semiperfect coalgebra and $e \in C^*$ be an idempotent element. If C is of tame comodule type then eCe is of tame comodule type.*

Unfortunately, the proof is not valid in the general case and is still an open problem.

Problem 4.2.12. *Let C be a coalgebra of tame comodule type and e be an idempotent element in C^*. Then eCe is of tame comodule type.*

It is also interesting to study if the localization process preserves polynomial growth, linear growth, discrete comodule type or domestic coalgebras. It is clear that the converse result is not true as the following example shows.

Example 4.2.13. *Let us consider the quiver*

$$
\begin{tikzcd}
 & & & \circ \arrow[d] & & & & & & \\
Q: & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ
\end{tikzcd}
$$

Since its underlying graph is neither a Dynkin diagram nor an Euclidean graph then KQ is wild, see for example [ASS05]. But it is easy to see that eCe is of finite representation type for each nontrivial idempotent element $e \in C^*$.

4.3 Split idempotents

Let us study the wildness of a coalgebra and its localized coalgebras. Directly from the definition we may prove the following proposition.

Proposition 4.3.1. *Let C be a coalgebra and $e \in C^*$ be an idempotent element which defines a perfect colocalization. If eCe is wild then C is wild.*

Proof. By hypothesis, there is an exact and faithful functor $F : \mathcal{M}^f_{KQ} \to \mathcal{M}^{eCe}_f$, where Q is the quiver formed by two points and three arrows between them, which respects isomorphism classes and preserves indecomposables. Consider the restriction to finite dimensional comodules of the colocalization functor $H : \mathcal{M}^{eCe}_f \to \mathcal{M}^C_f$; by Proposition 3.4.15, the composition $HF : \mathcal{M}^f_{KQ} \to \mathcal{M}^C_f$ is an exact and faithful functor that preserves indecomposables and respects isomorphism classes. Thus C is wild. $\square$

An analogous result can be obtained using the section functor if the subcategory is perfect localizing and S preserves finite dimensional comodules. For example, if C is pure semisimple or semiperfect.

Proposition 4.3.2. *Let C be a right pure semisimple or a left semiperfect coalgebra and $e \in C^*$ be an idempotent element which defines a perfect localization. If eCe is wild then C is wild.*

Proof. It is similar to the former proposition. We only have to prove that if C is left semiperfect then S preserves finite dimensional comodules. Let M be a finite dimensional right eCe-comodule. Then $\operatorname{Soc} M \subseteq M$ is finite dimensional. Suppose that $\operatorname{Soc} M = S_1 \oplus \cdots \oplus S_n$, then $\overline{E}_1 \oplus \cdots \oplus \overline{E}_n$ is the injective envelope of M. Therefore $E_1 \oplus \cdots \oplus E_n = E$ is the injective envelope of $S(M)$. By hypothesis, E is finite dimensional and thus so is $S(M)$. $\square$

Let us now consider the following question: when is the coalgebra eCe a subcoalgebra of C? This is interesting for us because, by Corollary 4.1.3, in such a case, we have the following.

Proposition 4.3.3. *Let C be a coalgebra and $e \in C^*$ be an idempotent such that eCe is a subcoalgebra of C. If eCe is wild then C is wild.*

In general, we always have the inclusion $eCe \subseteq C$, nevertheless the structures may be different. This is not the case if, for instance, e is a left semicentral idempotent. In that case, by [JMNR06], $eC = eCe$ is a subcoalgebra of C. The same result holds if e is a right semicentral or a central idempotent.

An idempotent element $e \in C^*$ is said to be *split* if in the decomposition $C^* = eC^*e \oplus eC^*f \oplus fC^*e \oplus fC^*f$, where $e + f = 1$, the direct summand $H_e := eC^*f \oplus fC^*e \oplus fC^*f$ is a twosided ideal of C^*. These elements were used by Lam in [Lam]. The main result there, see [Lam, Theorem 4.5], assures that the following statements are equivalent:

(a) H_e is a twosided ideal of C^*.

(b) $e(C^*fC^*)e = 0$.

(c) $exeye = exye$ for any x, $y \in C^*$.

As a consequence, every left or right semicentral idempotent element in C^* is split. Let us characterize when eCe is a subcoalgebra of C.

Theorem 4.3.4. *Let e be an idempotent element in C^*. Then the following statements are equivalent.*

(a) *e is a split idempotent in C^*.*

(b) *eCe is a subcoalgebra of C.*

Proof. Let us denote $f = 1 - e$. By Proposition 2.1.2, for any subspace $V \subseteq C$, V is a subcoalgebra of C if and only if $V^{\perp}$ is a twosided ideal of C^*. Then we proceed as follows in order to compute the orthogonal of eCe.

$$\begin{aligned} (eCe)^{\perp} &= (eC \cap Ce)^{\perp} \\ &= (eC)^{\perp} + (Ce)^{\perp} \\ &= C^*f + fC^* \\ &= eC^*f + fC^*f + fC^*e + fC^*f \\ &= eC^*f + fC^*e + fC^*f \\ &= H_e \end{aligned}$$

Thus eCe is a subcoalgebra of C if and only if H_e is a twosided ideal of C^* if and only if e is a split idempotent in C^*. □

Let us give a description of the split idempotents. Suppose that C is a pointed coalgebra, that is, C is an admissible subcoalgebra of a path coalgebra. We recall from Theorem 3.5.2 and Proposition 3.5.4 that the left (right) may be described as follows:

Proposition 4.3.5. *Let C be an admissible subcoalgebra of a path coalgebra KQ and e be an idempotent element in $(KQ)^*$ associated to a subset $X \subseteq Q_0$. Then:*

(a) *e is left semicentral if and only if there is no arrow $y \to x$ in Q such that $y \notin X$ and $x \in X$.*

(b) *e is right semicentral if and only if there is no arrow $x \to y$ in Q such that $y \notin X$ and $x \in X$.*

We want to give a geometric description of split idempotents in similar a way. In order to do this, we start giving an approach by means of path coalgebras.

Lemma 4.3.6. *Let Q be a quiver and $e \in (KQ)^*$ be the idempotent element associated to a subset of vertices X. Then e is split in $(KQ)^*$ if and only if $I_p \subseteq X$ for any path p in $e(KQ)e$, i.e., there is no cell of length greater than one.*

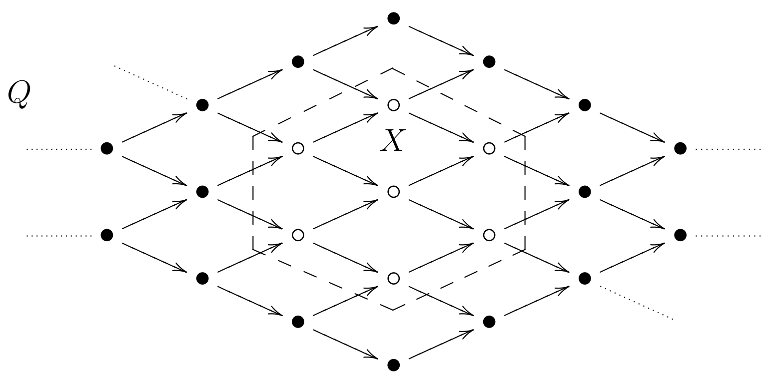
Proof. We have that $e(KQ)e$ is a subcoalgebra of KQ if and only if $\Delta(p) \in e(KQ)e \otimes e(KQ)e$ for any path p in $e(KQ)e$.

Let $p = \alpha_n \cdots \alpha_1 \in e(KQ)e$, $\Delta(p) \in e(KQ)e \otimes e(KQ)e$ if and only if

$$\sum_{j=2}^{n} \alpha_n \cdots \alpha_j \otimes \alpha_{j-1} \cdots \alpha_1 \in e(KQ)e \otimes e(KQ)e.$$

Since all summands are linearly independent, this happens if and only if $s(\alpha_i) \in X$ for all $i = 2, \ldots n$. That is, if and only if $I_p \subseteq X$. □

Therefore the subset of vertices X is a convex set in the quiver Q. In the following picture we show an example of a set X of vertices associated to a split idempotent, denoted by the white points.



The former proof can be easily generalized to pointed coalgebras. Recall that we denote by $\mathcal{Q}$ the set of all paths in Q.

Lemma 4.3.7. *Let Q be a quiver and C be an admissible subcoalgebra of KQ. Let $e \in C^*$ be the idempotent element associated to a subset of vertices X. Then e is split in C^* if and only if $I_p \subseteq X$ for any path p in $\mathrm{PSupp}(eCe)$.*

The reader might wonder if it is possible a generalization of the previous lemma to any coalgebra considering the Ext-quiver, i.e., if an idempotent e associated to a set of simple comodules $\mathcal{K}$ is split in C^* if and only if, for each path in the Ext-quiver $S_1 \to \cdots \to S_{n-1} \to S_n$, if S_1, $S_n \in \mathcal{K}$ then $S_i \in \mathcal{K}$ for all $i = 2, \ldots, n-1$. The answer is negative.

Example 4.3.8. *Let Q be the quiver*

$$\overset{1}{\circ} \xrightarrow{\alpha} \overset{2}{\circ} \xrightarrow{\beta} \overset{3}{\circ}$$

and C be the admissible subcoalgebra of KQ generated by $\{1, 2, 3, \alpha, \beta\}$. Then the quiver Γ_C is

$$S_1 \longrightarrow S_2 \longrightarrow S_3.$$

But $e \equiv \{1, 3\}$ is a split idempotent because $eCe = S_1 \oplus S_3$ is a subcoalgebra of C.

Following the idea of the Section 3 of Chapter 3 we hope that the following conjecture holds.

Conjecture 4.3.9. *The following are equivalent:*

(a) *The idempotent e is split in C^*.*

(b) *For each non-zero morphism between indecomposable injective right C-comodules $f : E_i \to E_j$ such that f is the composition of $g : E_i \to E_K$ and $h : E_k \to E_j$ for some indecomposable injective right C-comodule E_k, if $i, j \in I_e$ then $k \in I_e$.*

Let us finish the section with an open problem for further development of representation theory of coalgebras.

Problem 4.3.10. *Let C be a coalgebra and $e \in C^*$ be an idempotent element. If eCe is of wild comodule type then C is of wild comodule type.*

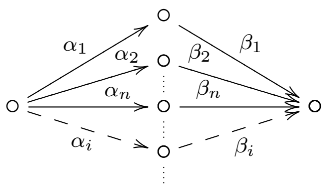
Obviously, Problem 4.2.12 and Problem 4.3.10 are equivalent if the *tame-wild dichotomy* for coalgebras, conjectured by Simson in [Sim05], is true.

4.4 A Gabriel's theorem for coalgebras

We have seen in Chapter 2 that there are examples of admissible subcoalgebras which are not path coalgebras of a quiver with relations (in fact, we have obtained a criterion to know when this occurs). Therefore we cannot describe all pointed coalgebras following that method. Nevertheless, the main aim of the representation theory of coalgebras is to classify all coalgebras by means of their category of comodules, and we know now that wild coalgebras are really difficult to describe in this way. Then we should discard them and reformulate the problem as the following statement: *every basic non-wild coalgebra, over an algebraically closed field, is isomorphic to the path coalgebra of a quiver with relations.* In particular, this implies that *every basic tame coalgebra, over an algebraically closed field, is isomorphic to the path coalgebra of a quiver with relations.* Moreover, if this result holds, this would reduce the proof of the tame-wild dichotomy to path coalgebras of quivers with relations. In this section we will use some results of localization obtained in the present and the previous chapters in order to solve the above-mentioned problem for acyclic quivers.

Firstly, we check that the coalgebra of Example 2.5.5 is of wild comodule type.

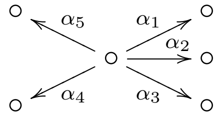
Example 4.4.1. *Consider the coalgebra of Example 2.5.5, that is, let Q be the quiver*



$\gamma_i = \beta_i\alpha_i$ *for all* $i \in \mathbb{N}$

and let H be the admissible subcoalgebra of KQ generated by the set $\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i\in\mathbb{N}}$. It is proved in Example 2.5.5 that H is not the path coalgebra of a quiver with relations. Nevertheless, H is of wild comodule type, as it contains the path coalgebra of the quiver

$\Gamma \equiv$



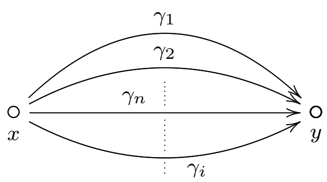
Since $K\Gamma$ is a finite dimensional coalgebra, we have an algebra isomorphism $(K\Gamma)^* \cong K\Gamma$ and there exists an equivalence between the *categories $\mathcal{M}^f_{K\Gamma}$ and $\mathcal{M}_f^{K\Gamma}$. But it is well known that $K\Gamma$ is a wild algebra and hence $K\Gamma$ is a wild coalgebra. By Corollary 4.1.3, this proves that H is wild.*

Theorem 4.4.2. *Let Q be an acyclic quiver and C be an admissible subcoalgebra of KQ which is not the path coalgebra of a quiver with relations. Then C is of wild comodule type.*

Proof. By Corollary 2.5.11, since C is not the path coalgebra of a quiver with relations, there exist an infinite number of paths $\{\gamma_i\}_{n\in\mathbb{N}}$ in Q between two vertices x and y such that:

• None of them is in C.

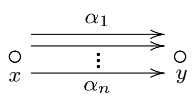
• C contains a set $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$ such that $\Sigma_n = \gamma_n + \sum_{j>n} a_j^n \gamma_j$, where $a_j^n \in K$ for all $j, n \in \mathbb{N}$.



Consider $\mathrm{PSupp}(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ and Γ the finite subquiver of Q formed by the paths γ_i for $i = 1, \ldots, t$.

Then $D = K\Gamma \cap C$ is a finite dimensional subcoalgebra of C (and an admissible subcoalgebra of $K\Gamma$) which contains the elements Σ_1, Σ_2 and Σ_3. It is enough to prove that D is wild.

Consider the idempotent element $e \in D^*$ such that $e(x) = e(y) = 1$ and zero otherwise, i.e., its associated subset of vertices is $X = \{x, y\}$. Then, by Lemma 3.6.3, each Σ_i corresponds to an arrow from x to y in the quiver Γ^e, that is, Γ^e contains the subquiver $\circ \Rrightarrow \circ$ and then $\dim_K \mathrm{Ext}^1_{eDe}(S_x, S_y) \geq 3$. Thus $eDe = K\Gamma^e$ is wild by [Sim05, Corollary 5.5]. Note also that the quiver Γ^e is of the form



an then the simple right eDe-comodule S_x is injective.

Let us prove that the localizing subcategory $\mathcal{T}_e$ of $\mathcal{M}^D$ is perfect colocalizing.

Since Γ is finite and without cycles then $\dim_K K\mathcal{T}ail^\Gamma_X(x)$ is finite and $\dim_K K\mathcal{T}ail^\Gamma_X(y) = 0$ so, by Proposition 3.6.9, the subcategory $\mathcal{T}_e$ is colocalizing.

Let now g be an element in $eC(1-e)$. Then g is a linear combination of tails starting at x and then $\rho_{eC(1-e)}(g) = g \otimes x$ (see the proof of Proposition 3.6.9). Therefore $< g > \cong S_x$ as right eCe-comodules. Suppose that $m = \dim_K eC(1-e)$. Hence $eC = eCe \oplus eC(1-e) = eCe \oplus S_x^m$ and eC is an injective right eCe-comododule. Thus the colocalization is perfect and, by Corollary 4.3.1, D is wild. □

Corollary 4.4.3 (Acyclic Gabriel's theorem for coalgebras)**.** *Let Q be an acyclic quiver. Then any tame admissible subcoalgebra of KQ is the path coalgebra of a quiver with relations.*

Proof. It follows from Theorem 4.4.2 and the weak tame-wild dichotomy. □

Remark. From the proof of Theorem 4.4.2, we have that if Q is acyclic then a basis of a tame admissible subcoalgebra C cannot contain three linearly independent linear combinations of paths with common source and common sink. Then $C = KQ_0 \bigoplus \bigoplus_{x \neq y \in Q_0} C_{xy}$ with $\dim_K C_{xy} \leq 2$ for all $x, y \in Q_0$. Nevertheless, this fact does not imply that the quiver is intervally finite (the number of paths between two vertices is finite). It is enough to consider the quiver

$$\begin{tikzcd}
\cdot \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & {} \arrow[d, dotted] \cdots \\
\cdot & \cdot \arrow[l] & \cdot \arrow[l] & \cdot \arrow[l] & {} \arrow[l] \cdots
\end{tikzcd}$$

and the admissible coalgebra $C = C(Q, \Omega)$, where

$$\Omega = KQ_2 \oplus KQ_3 \oplus \cdots \oplus KQ_n \oplus \cdots .$$

C is a string coalgebra and then it is tame (see [Sim05, Section 6]).

Chapter 5

Hereditary Coalgebras

This final chapter is devoted to the presentation of examples related to the topics considered in the previous chapters. To that end we use some classes of coalgebras whose existence are motivated by the analogous classes in the category of finite dimensional algebras. The main example for us will be the hereditary coalgebras. This is a well-known kind of coalgebras which has been studied with satisfactory results in many papers, see [Chi02], [JLMS06], [JMNR06] and [NTZ96]. The case of a pointed hereditary coalgebra, that is, a path coalgebra of a quiver, is studied extensively. In particular, as a consequence of the results of Chapter 3, we describe the localization of a path coalgebra by means of the cells and tails of its Gabriel quiver. Lastly, we introduce a class of coalgebras close to be hereditary: locally hereditary coalgebras. That kind of coalgebras can be defined by the property that every non-zero morphism between indecomposable injective comodules is surjective, and thus, they are a generalization of the hereditary case.

5.1 Hereditary coalgebras

A coalgebra C is said to be *right hereditary* if, for each subcomodule N of an injective right C-comodule E, the quotient E/N is an injective right C-comodule.

We collect here some known characterizations of a right hereditary coalgebra, see [Chi02], [JLMS06] and [NTZ96].

Theorem 5.1.1. *Let C be a coalgebra. The following conditions are equivalent:*

(a) *C is right hereditary.*

(b) *The global dimension of C is less or equal than one.*

(c) *The injective dimension of any simple right C-comodule is less or equal than one.*

(d) *C/N is an injective right C-comodule for each right coideal N.*

(e) *C/S is an injective right C-comodule for any simple right C-comodule S.*

(f) *C is left hereditary.*

If the coradical C_0 of C is coseparable, these conditions are also equivalent to

(g) *C is formally smooth.*

(h) *The global dimension of the enveloping coalgebra $C \otimes C^{cop}$ is less or equal than 1.*

(i) $\operatorname{Coker} \Delta$ *is an injective (C, C)-bicomodule, where Δ is the comultiplication of C.*

(j) *C is isomorphic to the tensor coalgebra $T_{C_0}(N)$, where N is the injective (C, C)-bicomodule $\frac{C_0 \wedge C_0}{C_0}$.*

Furthermore, if C is pointed then these conditions are equivalent to

(k) *C is isomorphic to the path coalgebra KQ of a quiver Q.*

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$ can be found in [NTZ96]. $(a) \Leftrightarrow (g) \Leftrightarrow (h) \Leftrightarrow (i) \Leftrightarrow (j)$ is proved in [JLMS06]. Finally, $(a) \Leftrightarrow (k)$ appears in [Chi02]. $\square$

Corollary 5.1.2. *The notion of hereditary coalgebra is left-right symmetric.*

Let C be a coalgebra. A right C-comodule M is called *colocal* if $\operatorname{Soc}(M)$ is a simple right C-comodule. Let us now give more characterizations of a hereditary coalgebra.

Proposition 5.1.3. *Let C be a coalgebra. The following conditions are equivalent:*

(a) C/D is an injective right C-comodule for any colocal right coideal D of C.

(b) Every quotient of an indecomposable injective right C-comodule is injective.

(c) Every quotient of an injective right C-comodule by a colocal subcomodule is injective.

(d) Every quotient of an indecomposable injective right C-comodule by its socle is injective.

(e) C/S is an injective right C-comodule for any simple right C-comodule S.

(f) C is right hereditary.

Proof. $(a) \Rightarrow (b)$. Let E_i be an indecomposable injective right C-comodule and N be a subcomodule of E_i. Since $\mathrm{Soc}\, N \subseteq \mathrm{Soc}\, E_i = S_i$ then N is a colocal right coideal of C. Therefore $C/N \cong E_i/N \oplus (\oplus_{k \neq i} E_k)$ is injective and thus so is E_i/N.

$(b) \Rightarrow (c)$. Let E be an injective right C-comodule and N be a colocal subcomodule of E. Then $\mathrm{Soc}\, N = S_i$ and N have the same injective envelope, say E_i, and there exists a monomorphism $f : E_i \to E$. Now, the exact sequence $E_i \xrightarrow{f} E \xrightarrow{p} E/E_i = E'$ splits so $E = E_i \oplus E'$ and E' is injective. Thus $E/N \cong E_i/N \oplus E'$ is injective.

$(c) \Rightarrow (d)$. Trivial.

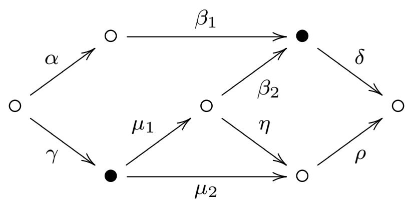
$(d) \Rightarrow (e)$. Let S_i be a simple C-comodule and E_i be its injective envelope, that is, E_i is an indecomposable injective right C-comodule and $\mathrm{Soc}\, E_i = S_i$. Then $C/S_i \cong E_i/\mathrm{Soc}\, E_i \oplus (\oplus_{j \neq i} E_j)$ and thus C/S_i is a direct sum of injective right C-comodules.

$(e) \Rightarrow (f)$. It is proved in Proposition 5.1.1.

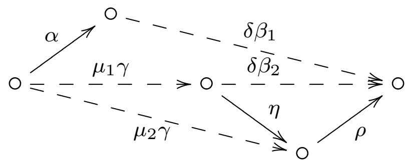
$(f) \Rightarrow (a)$. Trivial $\square$

Let us now suppose that either the field K is algebraically closed or C is a pointed coalgebra. Then, by Corollary 1.5.5 and Theorem 5.1.1, we may assume that C is the path coalgebra of a quiver Q. We recall from Chapter 3 that, for any idempotent element in $(KQ)^*$ associated to a subset of vertices $X \subset Q_0$, the localized coalgebra $e(KQ)e$ is the path coalgebra of the quiver $Q^e = (X, \mathcal{C}ell_X^Q)$.

Example 5.1.4. *Let KQ the path coalgebra of the quiver Q given by*

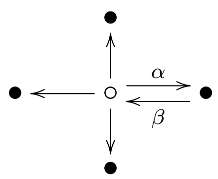


and X be the subset of vertices formed by the white points. Then, the set of cells is $\{\alpha, \eta, \rho, \delta\beta_1, \delta\beta_2, \mu_1\gamma, \mu_2\gamma\}$. Therefore the quiver Q^e is the following:



where the dashed arrows are the cells of length greater than one.

Example 5.1.5. *Let KQ be the path coalgebra of the quiver Q*



and X be the set of vertices formed by the only white point. Then the set of cells is $\{\beta\alpha\}$, that is, the quiver Q^e is

and $e(KQ)e \cong K[\beta\alpha]$.

We apply this idea in order to obtain a description of the quotient functor $T : \mathcal{M}^{KQ} \to \mathcal{M}^{KQ^e}$. Recall that the category of right KQ-comodules is equivalent to the category $\mathrm{Rep}_K^{lnlf}(Q)$ of locally nilpotent representation of finite length of the quiver Q.

Proposition 5.1.6. *Let KQ be a path coalgebra and $e \in (KQ)^*$ be the idempotent element associated to a subset of vertices X. Then, the functor $T : \mathrm{Rep}^{lnlf}(Q) \longrightarrow \mathrm{Rep}^{lnlf}(Q^e)$ maps the representation $(V_x, \varphi_\alpha)_{x \in Q_0, \alpha \in Q_1}$ of Q into the representation $(\bar{V}_y, \bar{\varphi}_\beta)_{y \in X, \beta \in \mathcal{C}ell_X^Q}$ of Q^e given by:*

- $\bar{V}_x = V_x$ *for every* $x \in X$.
- $\bar{\varphi}_\beta = \varphi_{\alpha_n} \cdots \varphi_{\alpha_1}$ *for each* $\beta \in \mathcal{C}ell_X^Q$ *such that* $\beta = \alpha_n \alpha_{n-1} \cdots \alpha_1$ *in* Q.

In Chapter 3 we proved that, for a subset of vertices X of a quiver Q, the localizing subcategory $\mathcal{T}_X$ is is colocalizing if and and if $\mathcal{T}ail_X^Q(x)$ is a finite set for each $x \in X$. Let us now prove that, under this conditions, the colocalizing subcategories are also perfect colocalizing.

Theorem 5.1.7. *Let* Q *be a quiver and* e *be the idempotent element in* $(KQ)^*$ *associated to a subset* $X \subseteq Q_0$. *Then*

$$e(KQ) \cong \bigoplus_{x \in X} \overline{E}_x^{\mathrm{Card}(\mathcal{T}ail_X^Q(x))+1}$$

as right KQ^e*-comodules, where* $\{\overline{E}_x\}_{x \in X}$ *is a complete set of pairwise non-isomorphic indecomposable injective right* KQ^e*-comodules.*

Proof. The right KQ^e-comodule $e(KQ)$ may be decomposed as $e(KQ) = e(KQ)e \oplus e(KQ)(1-e)$. Since there are isomorphisms $e(KQ)e \cong KQ^e \cong \bigoplus_{x \in X} \overline{E}_x$, it is enough to prove that

$$e(KQ)(1-e) \cong \bigoplus_{x \in X} E_x^{\mathrm{Card}(\mathcal{T}ail_X^Q(x))}$$

as right KQ^e-comodules.

Let us assume that, for each $x \in X$, we have

$$\mathcal{T}ail_X^Q(x) = \{\tau_x^i\}_{i \in J_x}.$$

The K-vector space $e(KQ)(1-e)$ is generated by the set of all paths starting at vertices in X and ending at vertices which do not belong to X. Then, for any path $p \in e(KQ)(1-e)$, there exists a unique tail decomposition $p = \tau_x^i p_n \cdots p_1$ for some $\tau_x^i \in \mathcal{T}ail_X^Q(x)$, where $x = t(p_n) \in X$.

We consider the linear map

$$f : e(KQ)(1-e) \longrightarrow \bigoplus_{x \in X} (\bigoplus_{i \in J_x} \overline{E}_{x,i})$$

defined by $f(\tau_x^i p_n \cdots p_1) = p_n \cdots p_1 \in \overline{E}_{x,i}$ for all $p = \tau_x^i p_n \cdots p_1 \in e(KQ)$. Clearly f is well defined and it is a $e(KQ)e$-comodule map. Since $\overline{E}_x$ is generated by the set of all paths in Q^e which end at x, f is bijective. $\square$

Corollary 5.1.8. *Let Q be a quiver and e be the idempotent element in $(KQ)^*$ associated to a subset $X \subseteq Q_0$. Then $e(KQ)$ is an injective right KQ^e-comodule.*

Theorem 5.1.9. *Let Q be a quiver and e be the idempotent element in $(KQ)^*$ associated to a subset $X \subseteq Q_0$. The following conditions are equivalent:*

(a) *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^{KQ}$ is colocalizing.*

(b) *The localizing subcategory $\mathcal{T}_X$ of $\mathcal{M}^{KQ}$ is perfect colocalizing.*

(c) *$\mathcal{T}ail_X^Q(x)$ is a finite set for all $x \in X$. That is, roughly speaking, there are at most a finite number of paths starting at the same point whose only vertex in X is the first one.*

We remark that since any path coalgebra is hereditary, the equivalence between (a) and (b) in the previous Theorem can be obtained from [NT96]. This is not true for any pointed coalgebra.

Example 5.1.10. *Let us consider the quiver Q and the coalgebra C defined on Example 3.6.6, and the subset of vertices $X = \{x_1, x_2, x_3\}$. Then, eCe is the path coalgebra of the quiver*

$$Q^e \equiv \underset{x_2}{\circ} \xleftarrow{\alpha_1} \underset{x_1}{\circ} \xrightarrow{\alpha_3} \underset{x_3}{\circ}$$

and then, the indecomposable injective right eCe-comodules are $E_1 = K < x_1 >$, $E_2 = K < x_2, \alpha_1 >$ and $E_3 = K < x_3, \alpha_3 >$. If $eC(1-e) = K < \alpha_2, \alpha_4, \alpha_2\alpha_1 + \alpha_4\alpha_3 >$ were injective then it would be a sum of indecomposable injective right eCe-comodules. Since $eC(1-e)$ has dimension 3, thus it would be isomorphic to $E_1 \oplus E_1 \oplus E_1$, or $E_1 \oplus E_2$ or $E_1 \oplus E_3$. A straightforward calculation proves that it is not possible.

By [Gab72, Chapter III, Proposition 7], any localizing subcategory of a category of comodules over a path coalgebra is perfect localizing. Then, from the above results and the results obtained in Chapter 4, we have the following:

Proposition 5.1.11. *Let Q be a quiver and e be an idempotent element of $(KQ)^*$ such that $\mathcal{T}_e$ is a colocalizing subcategory of $\mathcal{M}^{KQ}$. If KQ^e is wild then KQ is wild.*

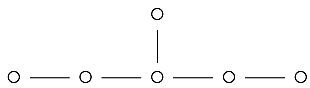
Proposition 5.1.12. *Let Q be a quiver and e be an idempotent element of $(KQ)^*$ such that the section functor $S : \mathcal{M}^{KQ^e} \to \mathcal{M}^{KQ}$ preserves finite dimensional comodules. If KQ^e is wild then KQ is wild.*

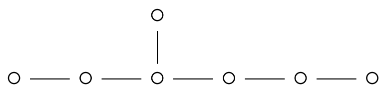
Following [Sim01], we finish the section giving a complete description of all tame path coalgebras. First we recall some special kinds of graphs.

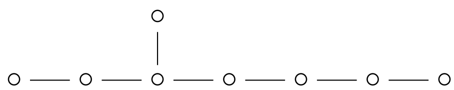
Dynkin diagrams

$\mathbb{A}_n$:  n vértices, $n \geq 1$

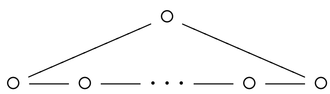
$\mathbb{D}_n$:  n vértices, $n \geq 4$

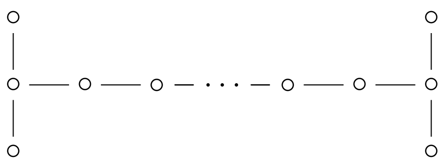
$\mathbb{E}_6$: 

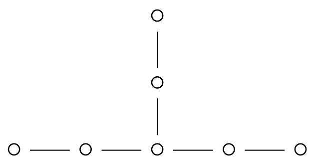
$\mathbb{E}_7$: 

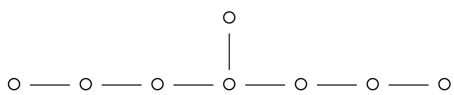
$\mathbb{E}_8$: 

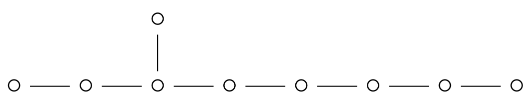
Euclidean graphs

$\tilde{\mathbb{A}}_n$:  $n+1$ vértices, $n \geq 3$

$\tilde{\mathbb{D}}_n$:  $n+1$ vértices, $n \geq 4$

$\tilde{\mathbb{E}}_6$: 

$\tilde{\mathbb{E}}_7$: 

$\tilde{\mathbb{E}}_8$: 

Infinite locally Dynkin diagrams

$$\mathbb{A}_\infty: \quad \circ - \circ - \circ - \cdots - \circ - \circ - \circ - \cdots$$

$$_\infty\mathbb{A}_\infty: \quad \cdots - \circ - \circ - \circ - \cdots - \circ - \circ - \circ - \cdots$$

$$\begin{array}{rl} & \phantom{\circ -{}} \circ \\ & \phantom{\circ -{}} | \\ \mathbb{D}_\infty: & \circ - \circ - \circ - \cdots - \circ - \circ - \circ - \cdots \end{array}$$

Theorem 5.1.13. *[Sim01] Let Q be a quiver and KQ be the path coalgebra of Q. The following conditions are equivalent:*

(a) *KQ is of tame comodule type.*

(b) *KQ is domestic of tame comodule type.*

(c) *The underlying graph of Q is a Dynkin diagram, or a Euclidean graph or a infinite locally Dynkin diagram.*

(d) *KQ is not of wild comodule type.*

Therefore the tame-wild dichotomy holds for this kind of coalgebras.

5.2 Locally hereditary coalgebras

In this section we introduce locally hereditary (or L-hereditary) coalgebras. These compose a class of coalgebras which contains all hereditary coalgebras and whose non-hereditary objects share properties with them.

Locally hereditary algebras were introduced by Simson in the Representation Theory seminar of the Nikołaja Kopenika University of Toruń in 1977. In [Les04], Leszczyński gave description of all tame locally hereditary algebras of finite dimension. An algebra A is said to be right *L-hereditary*, or right *locally hereditary*, if any local right ideal of A is projective.

Theorem 5.2.1. *[Les04] Let A be a finite dimensional algebra. The following conditions are equivalent:*

(a) *A is right L-hereditary.*

(b) *Any local submodule of a projective right A-module is projective.*

(c) *Any nonzero morphism between indecomposable projective right A-modules is a monomorphism.*

(d) *A is left L-hereditary.*

(e) *A is isomorphic to*

$$A' = \begin{pmatrix} F_1 & & & \\ {}_2M_1 & F_2 & 0 & \\ \cdots & \cdots & & \\ {}_nM_1 & {}_nM_2 & \cdots & F_n \end{pmatrix},$$

where $(F_i, {}_iM_j)$ is a K-species such that there exist F_i-F_j- homomorphisms $c_{ijk} : {}_iM_j \otimes_{F_j} {}_jM_k \to {}_iM_k$ verifying that $c_{ijk}(x \otimes y) = 0$ if and only if $x = 0$ or $y = 0$ for all $i, j, k = 1, \ldots, n$.

Let us turn to the case of coalgebras.

Definition 5.2.2. *A coalgebra C is said to be right* L-hereditary *or right* locally hereditary *if, for any right coideal N such that C/N is colocal, the quotient C/N is injective.*

The following lemma shows that there are many examples of locally hereditary coalgebras.

Lemma 5.2.3. *Any hereditary coalgebra is right and left locally hereditary.*

Let us give different characterizations of a locally hereditary coalgebra in the same way that Theorem 5.2.1.

Proposition 5.2.4. *Let C be a coalgebra. The following conditions are equivalent:*

(a) *For any subcomodule N of a injective right C-comodule E such that E/N is colocal, the quotient E/N is an injective right C-comodule.*

(b) *For any nonzero morphism $f : E \to F$, where E and F are right C-comodules such that E is injective and F is colocal, $\operatorname{Im} f$ is an injective right C-comodule.*

(c) *Every nonzero morphism between indecomposable injective right C-comodules is surjective.*

(d) *C is right locally hereditary.*

(e) *For any subcomodule N of an indecomposable injective right C-comodule E_i such that E_i/N is colocal, the quotient E_i/N is an injective right C-comodule.*

Proof. $(a) \Rightarrow (b)$. We have $E/\mathrm{Ker}\, f \cong \mathrm{Im}\, f$ and $\mathrm{Soc}\,(\mathrm{Im}\, f) \subseteq \mathrm{Soc}\, F$. Therefore $\mathrm{Soc}\,(\mathrm{Im}\, f)$ is simple and, by hypothesis, $\mathrm{Im}\, f$ is injective.

$(b) \Rightarrow (c)$. Let $f : E_i \to E_j$ be a nonzero morphism between indecomposable injective right C-comodules. Since $\mathrm{Soc}\, E_j$ is simple, $\mathrm{Im}\, f$ is injective. Therefore, the short exact sequence

$$0 \longrightarrow \mathrm{Im}\, f \longrightarrow E_j \longrightarrow E_j/\mathrm{Im}\, f \longrightarrow 0$$

splits and $E_j = \mathrm{Im}\, f \oplus E_j/\mathrm{Im}\, f$. Since E_j is indecomposable and f is nonzero, we deduce $E_j/\mathrm{Im}\, f = 0$.

$(c) \Rightarrow (d)$. Let N be a right coideal such that $\mathrm{Soc}\,(C/N) = S$ is simple. Let E be the injective envelope of S. Then $f : C/N \to E$ is also the injective envelope of C/N. Therefore, there exists an index $k \in I_C$ such that the composition $E_k \xhookrightarrow{i} C = \oplus_{i \in I_C} E_i \xrightarrow{p} C/N$ is nonzero, where i is the inclusion and p is the projection. Then fpi is surjective and so is f.

$(d) \Rightarrow (e)$. Let $N \leq E_i$ such that $\mathrm{Soc}\,(E_i/N)$ is simple. Let us consider the right coideal $N' = N \oplus (\oplus_{j \neq i} E_j)$. Then $C/N' \cong E_i/N$ has simple socle and thus E_i/N is injective.

$(c) \Rightarrow (a)$. It is similar to the proof of $(c) \Rightarrow (d)$.

$(e) \Rightarrow (c)$. Let $f : E_i \to E_j$ be a nonzero morphism. We have $\mathrm{Im}\, f \cong E_i/\mathrm{Ker}\, f$. Since $\mathrm{Im}\, f$ has simple socle, by hypothesis, it is injective and the result follows as in $(b) \Rightarrow (c)$. □

Problem 5.2.5. *Is the notion of locally hereditary left-right symmetric?*

Let us show that locally hereditary coalgebras are a generalization of finite dimensional locally hereditary algebras.

Lemma 5.2.6. *If R is a finite dimensional right locally hereditary algebra then R^* is a right locally hereditary coalgebra. Conversely, if C is a right locally hereditary coalgebra of finite dimension, C^* is a locally hereditary algebra.*

Proof. Let $f : E_i \to E_j$ be a nonzero morphism between indecomposable injective right R^*-comodules. The dual morphism $f^* : E_j^* \to E_i^*$ is a non-zero morphism between indecomposable projective right R-modules. By hypothesis, f^* is a monomorphism and then f is surjective. The converse result is similar. □

The following example shows that there exist locally hereditary coalgebras which are not hereditary coalgebras.

Example 5.2.7. *Let Q be the quiver*

$$\begin{tikzcd}
 & \overset{x_2}{\circ} \arrow[dr, "\alpha_2"] & \\
x_1\ \circ \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_3"'] & & \circ\ x_4 \\
 & \underset{x_3}{\circ} \arrow[ur, "\alpha_4"'] &
\end{tikzcd}$$

and C be the coalgebra generated by the set

$$\{x_1, x_2, x_3, x_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2\alpha_1 + \alpha_4\alpha_3\}.$$

The indecomposable injective right C-comodules are $E_1 =< x_1 >$, $E_2 =< x_2, \alpha_1 >$, $E_3 =< x_3, \alpha_3 >$ and $E_4 =< x_4, \alpha_2, \alpha_4, \alpha_2\alpha_1 + \alpha_4\alpha_3 >$. We consider the subcomodule $A =< x_4 >$ of E_4 and therefore $E_4/A =< \overline{\alpha_2}, \overline{\alpha_4}, \overline{\alpha_2\alpha_1 + \alpha_4\alpha_3} >$. It is easy to see that $\mathrm{Soc}\,(E_4/A) = S_2 \oplus S_3$ and then the injective envelope $E(E_4/A) = E_2 \oplus E_3 \neq E_4/A$. Thus E_4/A is not injective and C is not hereditary. Nevertheless, a straightforward calculation proves that C is locally hereditary. We sum it up in the following table:

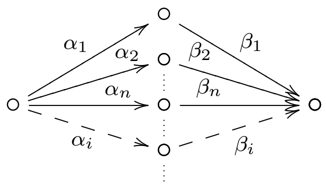
	subcomodules with colocal quotient	*quotient*
E_1	$\emptyset$	—
E_2	$< x_2 >$	E_1
E_3	$< x_3 >$	E_1
E_4	$< 4, \alpha_2 >$, $< 4, \alpha_4 >$, $< 4, \alpha_2, \alpha_4 >$	E_3, E_2, E_1

Example 5.2.8. *The last example may be extended to an infinite dimensional coalgebra. Let Q be the quiver*

$$\begin{tikzcd}
 & \circ \arrow[dr, "\alpha_2"] & & & & \\
\circ \arrow[ur, "\alpha_1"] \arrow[dr, "\beta_1"'] & & \circ \arrow[r, "\gamma_1"] & \circ \arrow[r, "\gamma_2"] & \circ \arrow[r, "\gamma_3"] & \circ \cdots \\
 & \circ \arrow[ur, "\beta_2"'] & & & &
\end{tikzcd}$$

and C be the subcoalgebra of KQ generated by the set of vertices, the set of arrows and $\{\gamma_n \cdots \gamma_1(\alpha_2\alpha_1 + \beta_2\beta_1)\}_{n\geq 0}$ $\{\gamma_n \cdots \gamma_1\alpha_2\}_{n\geq 1}$, $\{\gamma_n \cdots \gamma_1\beta_2\}_{n\geq 1}$ and $\{\gamma_i \cdots \gamma_j\}_{i>j\geq 1}$. Proceeding as above, we may prove that C is a locally hereditary coalgebra. On the other hand, since C is not a path coalgebra, C is not hereditary (see [JLMS06]).

Example 5.2.9. *Consider the quiver Q*



$\gamma_i = \beta_i\alpha_i$ *for all* $i \in \mathbb{N}$

and let H be the admissible subcoalgebra of KQ generated by the set $\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i\geq 1}$. Then H is a locally hereditary non-hereditary coalgebra.

From the above example, one may deduce that not every locally hereditary coalgebra is the path coalgebra of a quiver with relations.

Let C be a coalgebra and $e \in C^*$ be an idempotent element. Then we can consider the functors associated to the localization

$$\mathcal{M}^C \underset{S=-\square_{eCe}Ce}{\overset{T=e(-)=-\square_C eC}{\rightleftarrows}} \mathcal{M}^{eCe}\ .$$

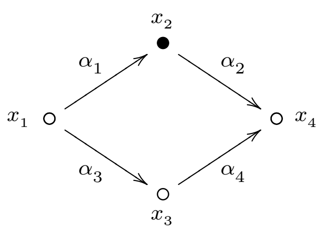
Let us prove that the localization process preserves locally hereditary coalgebras.

Theorem 5.2.10. *If C is a right locally hereditary coalgebra then eCe is a right locally hereditary coalgebra.*

Proof. Let $f : \overline{E}_i \to \overline{E}_j$ be a nonzero morphism between indecomposable injective right eCe-comodules. Then $S(f) : E_i \to E_j$ is a nonzero morphism between indecomposable injective right C-comodules. By hypothesis, $S(f)$ is surjective and since T is exact, $TS(f) = f$ is surjective. $\square$

Unlike it happens with hereditary coalgebras, not every colocalizing subcategory of the category of right comodules over a locally hereditary coalgebra is perfect colocalizing.

Example 5.2.11. *Let Q be the quiver*



and C be the admissible subcoalgebra generated by $\alpha_2\alpha_1 + \alpha_4\alpha_3$. Let us consider $X = \{x_1, x_3, x_4\}$. Then eCe is the path coalgebra of the quiver

$$Q^e \equiv \underset{x_2}{\circ} \xleftarrow{\alpha_1} \underset{x_1}{\circ} \xrightarrow{\alpha_3} \underset{x_3}{\circ}$$

and then, the indecomposable injective right eCe-comodules are $E_1 = K < x_1 >$, $E_2 = K < x_2, \alpha_1 >$ and $E_3 = K < x_3, \alpha_3 >$. If $eC(1-e) = K < \alpha_2, \alpha_4, \alpha_2\alpha_1 + \alpha_4\alpha_3 >$ were injective then it would be a sum of indecomposable injective right eCe-comodules. Since $eC(1-e)$ has dimension 3, it would be isomorphic to $E_1 \oplus E_1 \oplus E_1$ or $E_1 \oplus E_2$ or $E_1 \oplus E_3$. A straightforward calculation proves that it is not possible.

Let us assume that C is a right pure semisimple or a left semiperfect coalgebra.

Lemma 5.2.12. *Let C be a locally hereditary coalgebra and E_i be an indecomposable injective right C-comodule. Then $F_i = \mathrm{End}_C(E_i)$ is a division K-algebra.*

Proof. Let f be a non-zero element of F_i. Then f is surjective. Now, since E_i is finite dimensional, f is bijective. □

Lemma 5.2.13. *Let C be a locally hereditary coalgebra and, E_i and E_j be two non-isomorphic indecomposable injective right C-comodules. If $\mathrm{Hom}_C(E_i, E_j) \neq 0$ then $\mathrm{Hom}_C(E_j, E_i) = 0$*

Proof. Let f and g be two nonzero morphisms (thus surjective) in $\mathrm{Hom}_C(E_i, E_j)$ and $\mathrm{Hom}_C(E_j, E_i)$, respectively. The composition $gf \in \mathrm{End}_C(E_i)$ is a nonzero morphism so it is bijective and therefore so is f. Thus $E_i \cong E_j$ and we get a contradiction. □

For any two non-isomorphic indecomposable injective right C-comodules E_i and E_j, we may consider the set $\mathrm{Rad}^2_C(E_i, E_j)$ formed by all morphisms $f \in \mathrm{Hom}_C(E_i, E_j)$ such that f decomposes as $f = gh$, where $h \in \mathrm{Hom}_C(E_i, E_k)$ and $g \in \mathrm{Hom}_C(E_k, E_j)$ are not isomorphisms, for some indecomposable injective E_k. Therefore, to any pure semisimple locally hereditary coalgebra, there is associated a K-species $(F_i, {}_iM_j)_{i,j \in I_C}$, defined by ${}_iM_j = \mathrm{Hom}_C(E_i, E_j)/\mathrm{Rad}^2_C(E_i, E_j)$ for any $i, j \in I_C$, such that if ${}_iM_j \neq 0$ then ${}_jM_i = 0$. Moreover, we may consider the F_i-F_j-homomorphisms $c_{ijk} : {}_iM_j \otimes_{F_j} {}_jM_k \to {}_iM_k$ defined by the composition of morphisms, and then, it is verified that $c_{ijk}(x \otimes y) = 0$ if and only if $x = 0$ or $y = 0$ for all $i, j, k \in I_C$.

5.3 Other examples

Following [Sim05], a *string coalgebra* is a path coalgebra $C = C(Q, \Omega)$ of a quiver with relations (Q, Ω) which satisfies the following properties:

(a) Each vertex of Q is the source of at most two arrows and the sink of at most two arrows.

(b) The ideal Ω is generated by a set of paths.

(c) Given an arrow $i \xrightarrow{\beta} j$ in Q, there is at most one arrow $j \xrightarrow{\alpha} k$ in Q and at most one arrow $l \xrightarrow{\gamma} i$ in Q such that $\alpha\beta \in C$ and $\beta\gamma \in C$.

In [Sim05], it is proved that every string coalgebra is of tame comodule type. Let us show that the localization process preserves string coalgebras.

Theorem 5.3.1. *Let $C = C(Q, \Omega)$ be a string coalgebra and $e \in C^*$ be an idempotent element. Then, the localized coalgebra eCe is the string coalgebra $C(Q^e, \Omega^e)$, where $\Omega^e = e\Omega e \cap KQ^e$.*

Proof. Since Ω is generated by paths, $KQ = C \oplus \Omega$ as K-vector space. Then $KQ_e \cong e(KQ)e = eCe \oplus e\Omega e$ and therefore $KQ^e = KQ_e \cap KQ^e \cong eCe \oplus (e\Omega e \cap KQ^e)$. It is easy to see that Ω^e is generated by paths in Q^e of length greater than one.

Let us suppose that there is a vertex $i \in (Q^e)_0$ which is the source of three different arrows $\alpha, \beta, \gamma \in (Q^e)_1$. Then there exist three different paths $p_\alpha = \alpha_n \cdots \alpha_1, p_\beta = \beta_m \cdots \beta_1, p_\gamma = \gamma_r \cdots \gamma_1 \in \mathcal{C}ell_X^Q \cap C$ such that their cellular decompositions are α, β and γ, respectively. We have that α_1, β_1 and γ_1 are three arrows in Q starting at i and, since C is string, at least two of them are the same. For instance, suppose that $\alpha_1 = \beta_1$. Furthermore, $p_\alpha \neq p_\beta$ so there exists an integer s such that $\alpha_s \cdots \alpha_1 = \beta_s \cdots \beta_1$ and $\alpha_{s+1}\alpha_s \cdots \alpha_1 \neq \beta_{s+1}\beta_s \cdots \beta_1$.

$$\begin{tikzcd}
& & & & & \circ \arrow[r, squiggly] & \circ \\
& \circ \arrow[r, "\alpha_1=\beta_2"] & \cdots & \circ \arrow[r, "\alpha_s=\beta_s"] & \circ \arrow[ur, "\alpha_{s+1}"] \arrow[dr, "\beta_{s+1}"'] & & \\
\circ \arrow[ur, "\alpha_1=\beta_1"] \arrow[dr, "\gamma_1"'] & & & & & \circ \arrow[r, squiggly] & \circ \\
& \circ \arrow[rrr, squiggly] & & & \circ & &
\end{tikzcd}$$

By the condition (c) in the definition of string coalgebra, $\beta_{s+1}\beta_s \notin C$ or $\alpha_{s+1}\alpha_s \notin C$ and then $p_\alpha \notin C$ or $p_\beta \notin C$. We get a contradiction. We may proceed analogously if there are three different arrows ending at i.

Let $j \xrightarrow{\alpha} i$, $i \xrightarrow{\beta} k$ and $i \xrightarrow{\gamma} l$ be three arrows in Q^e such that $\beta\alpha \in eCe$ and $\gamma\alpha \in eCe$. As above, there exist three different paths $p_\alpha = \alpha_n \cdots \alpha_1, p_\beta = \beta_m \cdots \beta_1, p_\gamma = \gamma_r \cdots \gamma_1 \in \mathcal{C}ell_X^Q \cap C$ such that their cellular decompositions are α, β and γ, respectively. Since $p_\beta \neq p_\gamma$, there exists an integer s such that $\gamma_s \cdots \gamma_1 = \beta_s \cdots \beta_1$ and $\gamma_{s+1}\gamma_s \cdots \gamma_1 \neq \beta_{s+1}\beta_s \cdots \beta_1$.

$$\begin{tikzcd}[row sep=small]
 & & & & & & \circ \arrow[r, squiggly] & \circ \\
\circ \arrow[r, squiggly] & \circ \arrow[r, "\alpha_n"] & \circ \arrow[r, "\beta_1=\gamma_1"] & \cdots \quad \circ \arrow[r, "\beta_s=\gamma_s"] & \circ \arrow[urr, "\beta_{s+1}"] \arrow[drr, "\gamma_{s+1}"'] & & & \\
 & & & & & & \circ \arrow[r, squiggly] & \circ
\end{tikzcd}$$

If $s \geq 1$ then $\beta_{s+1}\beta_s \notin C$ or $\gamma_{s+1}\gamma_s \notin C$ and then $p_\beta \notin C$ or $p_\gamma \notin C$. This is a contradiction, so $s = 0$. But in that case, since C is string, $\beta_1\alpha_n \notin C$ or $\gamma_1\alpha_n \notin C$ and then $\beta\alpha \notin eCe$ or $\gamma\alpha \notin C$. The dual case is similar and the proof follows. $\square$

Definition 5.3.2. *A coalgebra is said to be* gentle *if it is a string coalgebra $C(Q, \Omega)$ which satisfies the following extra statement:*

(d) Given an arrow $i \xrightarrow{\beta} j$ in Q, there is at most one arrow $j \xrightarrow{\alpha} k$ in Q an at most one arrow $l \xrightarrow{\gamma} i$ in Q such that $\alpha\beta \notin C$ and $\beta\gamma \notin C$

Unlike it happens with string coalgebras, the localized coalgebra of a gentle coalgebra does not have to be gentle.

Example 5.3.3. *Let us consider the quiver*

$$\begin{tikzcd}[row sep=small]
 & & \circ \arrow[r, "\beta_2"] & \circ \\
\circ \arrow[r, "\alpha"] & \circ \arrow[ur, "\beta_1"] \arrow[dr, "\gamma"'] & & \\
 & & \circ &
\end{tikzcd}$$

and C the gentle coalgebra generated by all arrows and all vertices, and the paths $\beta_2\beta_1$ and $\beta_1\alpha$. Let e be the idempotent element associated to the subset of vertices $X = Q_0 \backslash s(\beta_2)$. Then eCe is the admissible subcoalgebra of the path coalgebra of the quiver

$$\begin{tikzcd}[row sep=small]
 & & \circ \\
\circ \arrow[r, "\alpha"] & \circ \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & \\
 & & \circ
\end{tikzcd}$$

generated by the set of vertices and the set of arrows. Obviously eCe is not a gentle coalgebra.

Following [CGT04], every right C-comodule M has a filtration

$$0 \subset \mathrm{Soc}\,(M) \subset \mathrm{Soc}\,^2(M) \subset \cdots \subset M$$

called the *Loewy series*, where, for $n > 1$, $\mathrm{Soc}\,^n(M)$ is the unique submodule of M satisfying that $\mathrm{Soc}\,^{n-1}(M) \subset \mathrm{Soc}\,^n(M)$ and

$$\mathrm{Soc}\,(M/\mathrm{Soc}\,^{n-1}(M)) = \mathrm{Soc}\,^n(M)/\mathrm{Soc}\,^{n-1}(M).$$

A right comodule is called *uniserial* if its Loewy series is a composition series. Furthermore, C is said to be *right (left) serial* if its indecomposable injective right (left) comodules are uniserial. We shall call it serial if it is left and right serial.

Let us suppose that C is a serial coalgebra. $\mathrm{Soc}\,^2(E_i)/\mathrm{Soc}\,(E_i) = \mathrm{Soc}\,(E_i/S_i)$ is a simple right comodule for all $i \in I_C$, and then, S_i has a unique predecessor in the Ext-quiver Γ_C for all $i \in I_C$. Furthermore, consider the left version of this property, then each vertex of the left Ext-quiver of C has a unique predecessor, that is, each vertex of the right Ext-quiver Γ_C has a unique successor. A straightforward calculation proves the following result:

Proposition 5.3.4. *Let C be an indecomposable serial coalgebra. Then Γ_C is one of the following quivers:*

(a) $\quad {}_\infty\mathbb{A}_\infty :\quad \cdots\cdots \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \cdots\cdots$

(b) $\quad \mathbb{A}_\infty :\quad \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \cdots\cdots$

(c) $\quad {}_\infty\mathbb{A} :\quad \cdots\cdots \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ$

(d) $\quad \mathbb{A}_n :\quad \circ \longrightarrow \circ \longrightarrow \circ - \cdots - \circ \longrightarrow \circ \longrightarrow \circ \qquad n \geq 1$

(e) $\quad \widetilde{\mathbb{A}}_n :$

$$\begin{tikzcd} & & \circ \arrow[dll] & & \\ \circ \arrow[dr] & & & & \circ \arrow[ull] \\ & \circ \arrow[r] & \circ \arrow[r, dotted, no head] & \circ \arrow[ur] & \end{tikzcd} \qquad n \geq 1$$

Corollary 5.3.5. *Any serial coalgebra over an algebraically closed field is of tame comodule type.*

Proof. By the former proposition, the Ext-quiver must be one of the above list and it is easy to see that the Gabriel quiver of C must also be one of them. The path coalgebra of any of this quiver is of tame comodule type, then any subcoalgebra is also of tame comodule type. $\square$

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