Exploring Relationships Between Students' Individual Ways of Reasoning and Normative Ways of Reasoning

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Through the lens of the emergent perspective (Cobb & Yackel, 1996), this study examined the nature and extent of variation in individuals' ways of reasoning from ways of reasoning that were accepted by a classroom community. This was done by interviewing seven undergraduate students after they had participated in classroom discussions. In contrast to other studies that have examined this relationship, the individuals' ways of reasoning were qualitatively different from the accepted ways of reasoning. This suggests that even if students actively participate in classroom discourse where students' ideas are considered, debated, and refined, they may not meet the major conceptual goals of the unit. As such, I argue that the relationship between the nature of social interactions students participate in and their subsequent reasoning needs further study, if educators are going to successfully support student learning.

Keywords: Classroom mathematical practices; Emergent perspective; Individual variation

Explorando relaciones entre formas individuales de razonamiento de los estudiantes y formas de razonamiento normativas

A través de la lente de la perspectiva emergente (Cobb y Yackel, 1996), este estudio examinó la naturaleza y el grado de variación en las formas de razonar de los individuos a partir de formas de razonar que fueron aceptadas los integrantes del aula. Se entrevistó a siete estudiantes universitarios después de haber participado en debates en el aula. A diferencia de otros estudios que han examinado esta relación, las formas de razonamiento de los individuos fueron cualitativamente diferentes a las formas de razonamiento aceptadas. Esto sugiere que incluso si los estudiantes participan activamente en el discurso de la clase donde se consideran, debaten y refinan las ideas de los estudiantes, es posible que no desarrollen los objetivos conceptuales principales de la unidad. Como tal, defiendo que la relación entre la naturaleza de las interacciones

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sociales en las que participan los estudiantes y su razonamiento posterior necesita más estudio, si los educadores van a apoyar con éxito el aprendizaje de los estudiantes.

Términos clave: Perspectiva emergente; Prácticas matemáticas en el aula; Variación individual

Over twenty years ago mathematics education research took what Lerman called a "social turn" (2000, p. 19). This meant that researchers began to consider the social nature of knowing more seriously. Researchers began to conceive of knowledge as inseparable from the social context in which that knowledge was developed, explore the semiotic and cultural mediation of thought, and investigate learning as enculturation into practice (Brown et al., 1989; Wenger, 1998; Wertsch, 1991). This is not to say that social interactions were ignored before this time. For example, Piaget acknowledged the contributions of the social world to individuals' construction of knowledge (Cole & Wertsch, 1996). However, after the social turn, mathematics educators often expanded the unit of analysis beyond the individual to explore how social interactions supported mathematical development both in the individual and in the classroom collective.

This expanded conception of learning and learning processes provided important insights that have shaped educators' views of productive teaching. Studies conducted from this perspective argued convincingly that productive social interactions, particularly participating in mathematical discourse in which students' ideas are negotiated and refined, can be beneficial for students (e.g., Empson, 2003; Hufferd-Ackles, et al., 2004). These insights have been formally adopted in that they have influenced standards documents. For example, the National Council of Teachers of Mathematics says that, "effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments" (2014, p. 10). However, as Lerman pointed out, this expanded conception of learning also brought challenges to researchers. "A major challenge for theories from the social turn is to account for individual cognition and difference, and to incorporate the substantial body of research on mathematical cognition, as products of social activity" (Lerman, 2000, p. 27).

Understanding the relationship between the inherent individual variation in students' ways of reasoning as they actively make sense of new mathematics and the productivity of engaging in social interactions in terms of developing powerful mathematical ways of reasoning would be important for teachers in designing classroom experiences and for researchers as they analyze classroom interactions. Yet this relationship remains unclear. On the one hand, educators know that students do not always learn what they hope they would. For nearly 50 years, researchers have been sharing cases of students who reason in seemingly idiosyncratic ways (e.g., Erlwanger, 1973), making sense of the activities they were given in ways that were presumably inconsistent with what their teacher would have hoped. On the other hand, productive social interactions would likely constrain, at least to some degree, the nature of individual variation. Mathematical discourse where a variety of ways of reasoning are compared, connected, and refined has the potential to help students advance their personal conceptions and problematize for the individual mathematically unproductive personal idiosyncrasies. Yet, despite the progress researchers have made in understanding social interactions in the classroom as a result of the social turn, it is unclear the extent to which this constrains individual variation. To contribute to this work, I address the following research question.

What is the nature and extent of individual variation from accepted meanings and ways of reasoning in a classroom community where students analyze and compare various ways of reasoning?

THEORETICAL PERSPECTIVE

The *emergent perspective* (Cobb & Yackel, 1996) is one way to coordinate social aspects of the classroom microculture with psychological features of the individuals. It does this by combining aspects of symbolic interactionism (Bauersfeld et al., 1988) and constructivism (von Glasersfeld, 1984, 1992). In this approach, the social and individual planes have equal weight, in contrast to theories in which the individual plane has primacy (and the social nature of knowing is downplayed) or the social plane has primacy (and the interpretive nature of knowing is downplayed). The emergent perspective has been utilized by mathematics educators around the world to inform a variety of research efforts (Hershkowitz & Jaworski, 2012; Hershkowitz & Schwarz, 1999; Kazemi & Stipek, 2001; Rasmussen et al., 2015; Roy, 2008; Stephan & Rasmussen, 2002; Voigt, 1995; Wawro, 2011).

The emergent perspective outlines three social aspects of the classroom—social norms, socio-mathematical norms, and classroom mathematical practices—and their individual psychological correlates (see Table 1). Social norms are accepted and expected ways of participating in the classroom (e.g., students freely engaging with other students' ideas without necessarily needing teacher mediating). Similarly, socio-mathematical norms are expected ways of participating that are specific to how students engage with the mathematics (e.g., ways of giving valid mathematical arguments).

In this study I focus on the relationship between the third social aspect, classroom mathematical practices, with its individual psychological correlate, mathematical conceptions and activity. Classroom mathematical practices, or simply emergent practices or math practices, are mathematical ways of reasoning and operating that become taken-as-shared. This use of the word "practice" is different from other common uses in mathematics education. In particular, this does not refer to disciplinary practices, or standard activities that people engage in as they do mathematics. Rather, emergent practices are ways of reasoning that arise in the classroom community and eventually become accepted as valid in that micro-community. Acceptance means that classroom participants act as if other participants are familiar with and understand the way of operating. Researchers use the phrase *taken-as-shared* rather than shared when describing these ways of reasoning to emphasize that they are not claiming that all, or even the majority of, students reason in identical ways. Rather, they only claim the mathematical practice is treated as if it is understood and accepted in the community. In this way, the researchers can identify a phenomenon that exists at the classroom level, the acceptance of a particular idea, but not make assumptions about what any individual participant understands. The individual correlate of classroom mathematical practices is students' own mathematical conceptions and activity. This includes students' individual ways of reasoning about a topic, which is the focus on this study.

Table 1
The emergent perspective's interpretive framework (Cobb & Yackel, 1996)

Social Perspective	Individual Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity
Socio-mathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

According to Cobb and Yackel (1996) the relationship between individuals' ways of reasoning and mathematical practices is indirect and reflexive. This means that individuals' ideas give rise to classroom mathematical practices as individuals share and negotiate ways of reasoning. Then, as ways of reasoning become accepted in the community, they influence, but do not determine, students' further reasoning.

Investigating Coordination of Social and Individual Aspects

While most of the research conducted by those who have worked from the emergent perspective has focused on investigating the social constructs of social norms, socio-mathematical norms, and classroom math practices (e.g., Kazemi & Stipek, 2001; Pang, 2000; Yackel, 2001), a few studies have elaborated the relationship between emergent mathematical practices and individuals' ways of reasoning (Cobb, 1999; Rasmussen et al., 2015; Stephan et al., 2003; Tabach et al., 2014). This includes tracking the movement of ideas from small group to whole class and vice versa (Tabach et al., 2014) and expanding the conceptualizations of collective progress (Rasmussen et al., 2015). However, these studies have tended to elucidate the process of establishing mathematical practices in the classroom, rather than investigating students' resulting individual ways of reasoning from participation in those practices.

Two studies have helped researchers understand individuals' interpretations of math practices. Shortly after the emergent perspective was put forth, Cobb (1999) demonstrated how students could participate differently in emergent practices. He showed examples of students who had difficulty in understanding other students' explanations because of the way they were conceiving of the mathematics. Similarly, Stephan et al. (2003) documented some qualitative differences in students' individual ways of reasoning from established math practices. They explored two first grade students', Nancy's and Meagan's, participation in the development of collective mathematical practices around measurement. In the researchers' description, there were brief periods of time when Meagan was reasoning in a way that was not consistent with an established mathematical practice[1], but she was always able to eventually reorganize her knowledge to be consistent with the practice through continued participation in the classroom community.

These studies begin to show how the interpretive nature of knowledge can affect how individual students intellectually engage with emergent practices. However, they generally suggest that students reorganize their knowledge to be consistent with established practices through continued participation in the discourse. If this were always the case, educators could safely assume that they do not need to worry much about individuals' interpretations of classroom events. However, this study will demonstrate that this is not the case. Here I describe an instance where mathematically significant variation in individuals' ways of reasoning from accepted ways of reasoning continued beyond the end of the unit, despite students' active participation in the negotiation of those accepted ways of reasoning.

METHODS

Setting

The research question requires documenting individual variation from ways of reasoning that have been negotiated, refined, and ultimately accepted in the classroom community. To investigate this variation, I compared the ways of reasoning that had been accepted in a classroom community with the individual ways of reasoning of seven students who participated in that class. The class was a capstone course for mathematics majors interested in teaching at the secondary level at a large university in the southwestern part of the United States. There were 26 undergraduates enrolled in the course. The course was taught by an experienced

[1] This can most clearly be seen in the development of the first math practice they describe.

mathematics education researcher familiar with research on orchestrating whole class discussions (e.g., Stein et al., 2008), which seemed to influence her instruction.

The course provided a productive setting to draw participants from because of the way the mathematical discourse was orchestrated. Typically, the teacher would pose a problem that required authentic problem solving. Students would then discuss the problem in small groups, with 3-4 students per group. During this small group discussion, the teacher would visit the groups, listen to their discussion, and ask questions. After students had time to consider the problem in their groups, she would purposefully select several students to share their ways of reasoning. In these whole class discussions, these ways of reasoning would be considered, debated, and refined. This afforded a setting where students actively participated in the negotiation of emergent mathematical practices.

The Mathematical Practice

The research question focuses on examining the nature and extent of individual variation from an accepted math practice in the classroom community. The math practice chosen was developed in the last unit of the course, which focused on developing meanings for exponents and logarithms. Over the 3-week unit (consisting of 7.5 hours of instruction across 6 sessions), the students developed an exponential number line, which was used to cultivate these meanings. The number line was developed over several days during which the students worked on a task to represent the history of the earth on a single timeline (a task adapted from Confrey, 1991). During this time, several ways of representing the timeline were considered, with a wide range of students actively contributing to the negotiation. The timeline they eventually developed was an exponentially scaled number line. On an exponentially scaled number line, any pair of numbers that differ by a particular factor are the same distance apart on a line. This contrasts with a linearly scaled number line where any pair of numbers with a particular arithmetic difference are the same distance apart.

While the idea of placing powers of ten (e.g., 10^1, 10^2, 10^3, etc.) at equal intervals emerged fairly early in the students' development of a timeline, negotiating how to subdivide the segments created by the powers of ten took a substantial amount of time. At first, students wanted to subdivide these segments linearly. However, they eventually rejected this idea. Instead, they accepted two mathematically equivalent ways of reasoning about this subdivision. In the first way of reasoning (Normative Way of Reasoning 1 or NWR 1), students recognized a linear pattern in the exponents and continued that pattern. In the second way of reasoning (Normative Way of Reasoning 2 or NWR 2), students extrapolated from the multiplicative relationships between the powers of 10, reasoning that since powers of 10 were the same distance apart, any same-sized segments should represent a consistent multiplicative increase. Both ways of reasoning yield the same answer, but they are cognitively distinct. NWR 1 relies on an arithmetic pattern in the exponents, while NWR 2 relies on a geometric pattern in the values on the line. Together, these two normative ways of reasoning form an emergent mathematical practice, which I termed Subdividing the Segments. Both ways of reasoning are considered normative because they met one of three criteria outlined in the documenting collective activity method (Cole et al., 2012; Rasmussen & Stephan, 2008). Details about this method and evidence for the emergence of all the mathematical practices that were developed during this unit can be found elsewhere (Gruver, 2018).

I focused on variation from this math practice for several reasons. First, by coordinating the two ways of reasoning, students coordinate arithmetic growth (in the exponents) and geometric growth (in the value on the timeline). Coordination of arithmetic and geometric growth is central to understanding exponential relationships. Second, this coordination was central to developing meanings for fractional exponents in the class. For example, students reasoned that the halfway point between 10^0 and 10^1 should be $10^{1/2}$ so that the linear pattern in the exponents holds. However, they also reasoned the point should be $\sqrt{10}$. Since the segment between 10^0 and 10^1 represents an increase by a factor of 10 and this segment has been divided into two same-sized subsections, each subsection should represent multiplication by a factor that when multiplied by itself (multiplication over the two subsections) is the same as multiplying by 10 (multiplication over the whole segment). That factor is $\sqrt{10}$. These two ways of reasoning together provide an explanation for why $10^{1/2}=\sqrt{10}$. Third, this practice resulted from a shift from linear ways of reasoning to exponential ways of reasoning. This shift can be difficult for students (Alagic & Palenz, 2006; Berezovski, 2004; De Bock et al., 2002).

Data Collection

Within a week after instruction had ended, I conducted individual clinical interviews (Ginsburg, 1997) with seven students from the course. The students were selected based on their willingness to be interviewed. In the interview, students were asked to solve the task shown in Figure 1. This provided opportunities to see how the students reasoned about subdividing an exponential number line and how this was similar to or different from the accepted ways of reasoning in the class community. The interviews were videotaped so that the students' work was clearly visible.

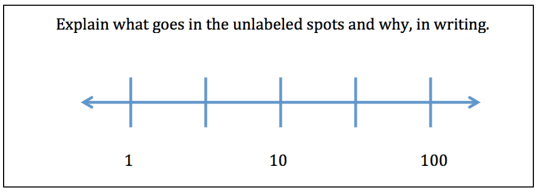

Figure 1. The interview task

Data Analysis

In analyzing the interviews, I first created a descriptive, non-inferential narrative of the reduced data set (Miles & Huberman, 1994). I then engaged in open coding from grounded theory (Strauss & Corbin, 1990). This involved first breaking up the data into smaller episodes, where a student expressed an idea or made use of a strategy. I then grouped similar episodes to form a category. As I inferred these categories, I made use of the constant comparison method (Glaser & Strauss, 1967; Strauss, 1987; Strauss & Corbin, 1990, 1994), which is the comparison of different pieces of data to create and refine categories. As I began to establish categories, I compared the episodes in the category to other episodes in the interviews, both within and between subjects. The purpose of these comparisons is to bring into greater relief the similarities and differences in categories. This was an iterative process. This means that as categories were refined, episodes that had been coded earlier were revisited in light of the new categories (Strauss & Corbin, 1990).

RESULTS

Students reasoned in one of three ways in the interview task shown in Figure 1. The three of ways of reasoning were (a) multiplicative reasoning coordinated with reasoning linearly with the exponents, (b) reasoning linearly with the exponents, and (c) elements of reasoning linearly (see Table 2). The first way of reasoning, multiplicative reasoning coordinated with reasoning linearly with the exponents, is characterized by students recognizing the fact that there was a multiplicative relationship between the subsections generated by subdividing a segment. At a minimum, this means students would reference the fact that $\left(\sqrt{10}\right)\left(\sqrt{10}\right)=10$ and somehow connect that fact to their reasoning about the subsections. The students in this category also reasoned linearly with the exponents, meaning they used the linear pattern in the exponents to determine placements, but this linear reasoning was accompanied by talk of multiplicative patterns. Three students reasoned in this way.

The second way of reasoning, reasoning linearly with the exponents, was characterized by students talking about halving the exponent of 10^1 to find that the midpoints should represent an increase of .5 in the exponent. This differs from the first category in that the linear pattern used in these explanations was not connected with multiplying by $\sqrt{10}$. It is important to note as students reasoned in this way, they may have said the word "factor." Simply uttering this word did not automatically mean that they were reasoning multiplicatively. At times students would call 10^1 a factor, but still reasoned solely about the exponent—dividing it in a linear way. For a way of reasoning to be placed in the first category, students needed to go beyond simply calling something a factor and explain that the factor is being multiplied by something. Two students employed reasoning linearly with the exponents. Finally, elements of linear reasoning means that at some point the student claimed the midpoint was five, presumably because five is half of ten and the midpoint is halfway. Two students made this claim. During the interview one of the students changed his answer as he began reasoning linearly with the exponents, but the other student did not change her answer during the interview. To illustrate the nature and extent of variation from normative ways of reasoning these individual ways of reasoning represent, I will describe the reasoning of one student from each category in detail.

Table 2
The three categories for individuals' ways of reasoning

Code	Characterization
Multiplicative Reasoning Coordinated with Reasoning Linearly with the Exponents	Recognizing the subsections are associated with multiplication by the square root of ten in addition to using the linear pattern in the exponents to determine placements.
Reasoning Linearly with the Exponents	Finding the midpoints by dividing increases in the exponent by two.
Elements of Linear Reasoning	Determining the first midpoint was five by taking half of ten.

The first student's reasoning we consider is Tanya's. Tanya's reasoning was categorized as Multiplicative Reasoning Coordinated with Reasoning Linearly with the Exponents. This means that although she determined the value of the midpoint by reasoning linearly with the exponents, she was also aware that the subsections represented an increase by a factor of $\sqrt{10}$. Tanya began the interview by claiming the first spot should be labeled $10^{.5}$. She said, "Since this is increasing by a factor of 10^1, then half of it would be $10^{1/2}$." She then labeled 1 as 10^0, 10 as 10^1, and 100 as 10^2, as well as marking in a brace over the segment from 1 to 10, which she labeled "$\times\ 10^1$" and a brace over the subsection from 1 to the midpoint, which she labeled "$\times\ 10^{1/2}$" (see Figure 2).

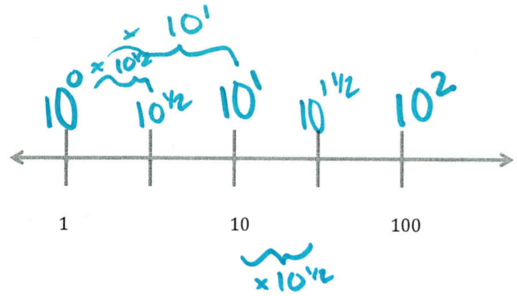

Figure 2. Tanya labeled the multiplicative factors

She then wrote her explanation.

> From 10^0 to 10^1 we increase by a factor of 10^1 ($10^0 \cdot 10^1 = 10^1$). We cut this increment of 10^1 in half, so we half the exponent of 10^1 as well to get $10^{1/2}$. Check by $10^{1/2} \cdot 10^{1/2} = 10^{2/2} = 10^1$. Multiply the previous term by $10^{1/2}$ to obtain the next tick mark, from 10^0 we get the next by $10^0 \cdot 10^{1/2} = 10^{1/2}$, then $10^{1/2} \cdot 10^{1/2} = 10^1$.

When the interviewer asked what she meant by, “We cut this increment of 10^1 in half,” she responded with the following.

Tanya: Since this whole [traces over the segment from 1 to 10], … w[as] 10^1 [points to tick marks labeled 1 and 10 simultaneously] and we only wanted to do half the distance [points to tick marks labeled 1 and the midpoint of 1 and 10], we don’t halve 10, because that just doesn’t make sense. So, we halve the exponent, so instead of moving by a factor of 10^1, we’re moving by a factor of $10^{1/2}$. So, we’re halving the exponent.

Interviewer: How do you know to halve the exponent?

Tanya: When we were first trying to figure it out, it didn’t really make sense to … halve the ten. … We would multiply 10^0 times 10^1 to get 10^1 and we only want to go half the way and so we wouldn't multiply by half of 10, we wouldn’t multiply it by 5, so we would halve the exponent.

While Tanya calculated the midpoint based on linear patterns, multiplication also came up several times in Tanya’s argument. First, she immediately marked in the multiplicative factors of “$\times$ 10^1” and “$\times$ $10^{1/2}$” (see Figure 2). Importantly, these labels included the multiplication symbol “$\times$,” suggesting she saw them as factors. Multiplication was also present in her written explanation. She wrote, “Multiply the previous term by “$10^{1/2}$ to obtain the next tick mark, from 10^0 we get the next by $10^0 \cdot 10^{1/2} = 10^{1/2}$, then $10^{1/2} \cdot 10^{1/2} = 10^1$.” Finally, multiplication was present as she responded to the interviewer’s question, “How do you know to halve the exponent?” In response, she explained that her group in class first tried to halve the ten, but that was inconsistent with the macro-level multiplication. She said, “We wouldn't multiply by half of 10, we wouldn’t multiply it by 5, so we would halve the exponent.” Notice that implicit in her comment is the assumption that the relationship should be multiplicative—what needs to be decided is whether the multiplication should be by 5 or by $10^{.5}$. In summary, even though Tanya talked about halving the exponent in her explanation, this was often coordinated with a recognition of the multiplicative nature of the subsections.

The next student's reasoning we consider is Farah's. Farah's reasoning was categorized as reasoning linearly with the exponents, the second category of individual ways of reasoning. This category describes reasoning that relied solely on the linear pattern in the exponents and did not include any mention of geometric growth. Farah began the task by labeling the spot $10^{.5}$ and relabeling 1 as 10^0 and 10 as 10^1 (see Figure 3) and then wrote the following explanation.

Farah: Because this is an exponential line each label must be representable in exponential form. Each labeled tick mark represents 10^0 , 10^1 , 10^2 respectively. ... The halfway mark is… half of the exponent of the larger endpoint. 1/2 of 1 = 1/2 [therefore] $10^{.5}$.

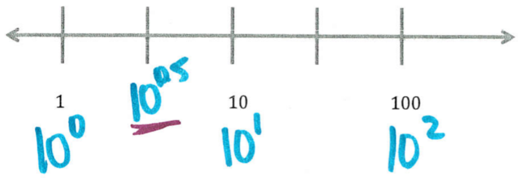

Figure 3. Farah's labels

Here Farah is clear that she was operating on the exponents. She wrote, "The halfway mark is… half of the exponent of the larger endpoint. 1/2 of 1 = 1/2." She did not mention anything about multiplication, despite follow up questions from the interviewer. Furthermore, in her explanation she focused on the form the numbers were written in, which may suggest a focus on the exponents.

From the third category, elements of linear reasoning, we consider Lacey's explanation. This category of reasoning contains explanations where students consider subdividing the segments linearly. Lacey began the interview by saying, "I think it'd [the first unlabeled spot would] be five" and labeled the first unlabeled spot 5. She then wrote her justification. As she wrote her justification, she labeled the second spot 50 (see Figure 4).

Lacey: 5 gets in first spot because it looks like 1/2 distance between 1 and 10. 50 in second spot because 50 is 1/2 of 100.

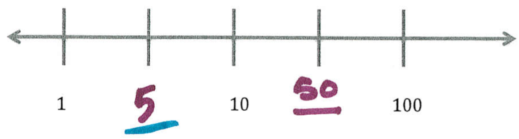

Figure 4. Lacey's labels

After she wrote this, she checked that 5 lay between its surrounding tick marks, 1 and 10. She then said, "This one's going to be fifty because it's half of a hundred and it's still between ten and a hundred." Lacey's way of reasoning differed from the students' ways of reasoning in the other categories in that she reasoned linearly on the values. In particular, she halved 10 to get 5 and halved 100 to get 50. This halving resulted in a number that when added to itself would give the endpoint (5+5) instead of a number that when multiplied by itself would give the endpoint ($\sqrt{10}\cdot\sqrt{10}$). This linear reasoning is similar to the reasoning in the second category in that both require halving, except here it was applied to the actual value of the endpoint (10) instead of the exponent (1) in 10^1, the factor by which the values increased. This difference is crucial as these different ways of reasoning lead to different values.

DISCUSSION

The research question requires investigation into the nature and extent of variation in individuals' ways of reasoning from accepted ways of reasoning. One could argue that the extent of variation was limited. All three ways of reasoning were rooted in ways of reasoning that were discussed in class. The first category of individual reasoning consists of both reasoning linearly with the exponents and coordinating this linear pattern with multiplication by a factor of ($\sqrt{10}$). This strongly parallels the math practice, which also consisted of two coordinated ways of reasoning, reasoning linearly with exponents (NWR 1) and reasoning that preserves multiplicative patterns (NWR 2). One could argue that the second category of individual reasoning also represents a small extent of variation from the math practice in that it consists of a way of reasoning that is essentially the same as NWR 1. The third category of individual way of reasoning represents a larger deviation from the math practice than the other two. Using this way of reasoning, students arrived, at least initially, at a different answer for the values that should lie at the two halfway points. However, despite this variation from the normative ways of reasoning, this individual way of reasoning also bore some similarities to the practice in that it mirrored the development of the practice in the classroom community. Just as the interviewed students considered subdividing linearly, the students in class also considered subdividing linearly. It was only after significant discussion that they decided to reject that idea in favor of preserving the multiplicative structure that existed between the powers of ten, namely that same-sized segments should represent an increase by the same factor. These relationships between the individual ways of reasoning and the ways of reasoning that were expressed in class are summarized in Table 3.

Table 3
The relationship between individual ways of reasoning and the emergent math practice

Individual Ways of Reasoning	Relationship to the Math Practice
Category 1: Multiplicative Reasoning Coordinated with Reasoning Linearly with the Exponents	Similar to the math practice in that it also consists of two coordinated ways of reasoning similar to the NWRs.
Category 2: Reasoning Linearly with the Exponents	Consists of a way of reasoning that is similar to NWR 1 (Subdividing Segments by Reasoning Linearly About Exponents), a constituent piece of the math practice.
Category 3: Elements of Reasoning Linearly Among the Values	Similar to how students in class considered subdividing before establishing the math practice.

Even though one could argue that the extent of variation of individual ways of reasoning from the math practice was constrained in that the ways of reasoning were not idiosyncratic, the nature of the variation was mathematically significant. As the emergent practice was negotiated in class, the idea that each of the smaller same-sized subsections of a subdivided segment should represent an increase by the same factor was central. This idea allowed students to reason about the value of numbers raised to a fractional exponent, such as $10^{.5}$. The number $10^{.5}$ is the halfway point between 10^0 and 10^1 because of the linear pattern in the exponents, but that halfway point should also be $\sqrt{10}$ because the halfway point subdivides a segment that represents an increase by a factor of 10. This means that each of the smaller segments should represent an increase by a number that when multiplied by itself is 10. Fundamental to this way of reasoning is the coordination of geometric growth in the values on the line and arithmetic growth in their exponents–a defining characteristic of exponential relationships. However, in contrast to this type of reasoning, students using category 2 or category 3 ways of reasoning showed no evidence of coordination whatsoever. For them, the timeline was likely a linear timeline, whether of the exponents or the values. This conception would not offer the same power when reasoning about exponential relationships generally or fractional exponents specifically.

Understanding the nature and extent of individual variation from the negotiated math practice in the case presented here can give insights to the relationship between the interpretive nature of knowledge and participating in the negotiation of ideas. In this case, participation in the negotiation of emergent practices did seem to constrain the extent of individual variation from the negotiated practice. All the ways of reasoning expressed in the interviews were rooted in and consistent with ways of reasoning that were discussed in class as the math practice was being negotiated. However, this constraint of extent of individual variation did not mean that all students met the conceptual goals of the unit. In fact, presumably because of this similarity between what was being discussed in class and individuals' personal ways of reasoning, some individuals did not perceive a need to adjust their ways of thinking. This led to four of the seven students interviewed showing no evidence of meeting a major conceptual goal of the unit.

This case illustrates the complex nature of individual learning as students participate in mathematically rich classroom discourse. This study suggests that even if students participate in conceptually focused discussions where a variety of students' ways of reasoning are considered, debated, and refined, the participating students may not meet the central learning goals of instruction. As they participate in the negotiation of accepted ways of reasoning, they may interpret these ways of reasoning as consistent with their own, even if there are significant conceptual differences.

This means that NCTM's vision of effective teaching as teaching that "facilitates discourse among students to build shared understanding of mathematical ideas" is not straightforward. However, when and why participating in mathematically rich discussions is insufficient to help students meet conceptual goals is still poorly understood. This suggests that the challenge Lerman (2000) identified is still relevant today. Teachers need a better understanding of the nature of individual learning processes that occur in social environments if they are to design activities and orchestrate discussions that foster conceptual learning for a range of students. Future research could elaborate the conditions under which subtle variation in individual ways of reasoning, especially those that might be hard for a teacher to detect in the moment, could pose challenges to students developing powerful ways of reasoning by engaging in class discussions. Future research could also elaborate types of activities or ways of orchestrating discourse that could help problematize for students' ways of reasoning that may be similar to, but qualitatively different from, what is being developed at the whole class level. These future insights could help teachers create learning experiences that allow a wider range of students to develop powerful mathematical conceptions.

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