Group Decision-Making with Incomplete Fuzzy Linguistic Preference Relations

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March 27, 2007

Abstract

The aim of this paper is to propose a procedure to estimate missing preference values when dealing with incomplete fuzzy linguistic preference relations assessed using a 2–tuple fuzzy linguistic approach. This procedure attempts to estimate the missing information in an individual incomplete fuzzy linguistic preference relation using only the preference values provided by the respective expert. It is guided by the additive consistency property in order to maintain experts' consistency levels. Additionally, we present a selection process of alternatives in group decision making with incomplete fuzzy linguistic preference relations and analyze the use of our estimation procedure in the decision process.

Keywords: fuzzy linguistic preference relations, group decision making, selection process, incomplete information, consistency.

1   Introduction

Group Decision-Making (GDM) consists of finding the best alternative(s) from a feasible set. To do this, experts have to express their preferences by means of a set of evaluations over a set of alternatives. In this paper we assume that experts use preference relations [2, 21, 22, 23]. According to the nature of the information expressed for every pair of alternatives there exist many different representation formats of preference relations: *fuzzy preference relations* [3, 20, 21], *fuzzy linguistic preference relations* [9, 10, 11, 12, 18, 24], *multiplicative preference relations* [4, 13, 22], *intuitionistic preference relations* [30] and *interval–valued preference relations* [1, 14, 25].

Since each expert has his/her own experience concerning the problem being studied they could have some difficulties in giving all their preferences. This may be due to an expert not possessing a precise or sufficient level of knowledge of the problem, or because that expert is unable to discriminate the degree to which some options are better than others. In such situations, experts are forced to provide *incomplete preference relations* [19]. Therefore, it should be of great importance to provide the experts with tools that allow them to deal with this lack of knowledge in their opinions.

In order to maintain experts' consistency levels many authors have proposed estimation procedures of preferences based on consistency criteria [19, 26, 27, 28, 29]. For fuzzy preference relations, procedures to estimate missing values were proposed in [19, 26] based on *Tanino's additive consistency property* [23]. For ordinal fuzzy linguistic preference relations, some procedures to estimate missing values were proposed in [28, 29] based on *Saaty's consistency property* [22]. However, Saaty's consistency property is defined for multiplicative preference relations and therefore it is not applicable to fuzzy linguistic preference relations. It is well known that the fuzzy translation of Saaty's consistency property coincides with Tanino's additive consistency property [17, 19]. Therefore, it would be desirable to design estimation procedures for fuzzy linguistic preference relations based on the additive consistency property. In [27], a first approach for the case of ordinal fuzzy linguistic preference relations based on the additive consistency property was proposed. However, it fails to use all the estimation possibilities that can be derived from the additive consistency property.

The aim of this paper is to present a complete procedure to estimate missing information in the case of incomplete fuzzy linguistic preference relations. It is based on the linguistic extension of Tanino's consistency principle and makes use of all the estimation possibilities that derive from it. We assume fuzzy linguistic preference relations assessed on a 2–tuple fuzzy linguistic modelling [15] because it provides some advantages with respect to the ordinal fuzzy linguistic modelling [16]. We design a selection process for GDM problems with incomplete fuzzy linguistic preference relations following the choice scheme proposed in [5], i.e., *aggregation* followed by *exploitation*. In this selection procedure we include a new step devoted to complete the fuzzy linguistic preference relations. We also analyse and discuss the use of this estimation procedure of 2-tuple linguistic missing values.

In order to do this, the paper is set out as follows. In Section 2.1, we present the preliminaries, that is, the concepts of incomplete 2–tuple fuzzy linguistic preference relation and linguistic additive consistency property. Section 3 introduces the complete estimation procedure of missing values for incomplete 2–tuple fuzzy linguistic preference relations and an illustrative example. In Section 4, the selection process with incomplete 2–tuple

fuzzy linguistic preference relations is designed and illustrated with an example. In Section 5 we discuss the use of our estimation procedure in the decision process. Finally, in Section 6 we draw our conclusions.

2   Preliminaries

In this section we present the concepts of incomplete 2–tuple fuzzy linguistic preference relation and linguistic additive consistency property.

2.1   Incomplete 2–Tuple Fuzzy Linguistic Preference Relations

There may be situations where it could be very difficult for the experts to provide precise numerical preferences, and therefore linguistic assessments could be used instead [6, 7, 11, 24, 32]. In this paper, we use the 2–tuple fuzzy linguistic model [15] to represent experts' preferences. Many advantages of this representation model to manage linguistic information models were given in [16].

The 2–tuple fuzzy linguistic model takes as a basis the symbolic representation model [7, 9, 8] and, in addition, it defines the concept of symbolic translation to represent the linguistic information by means of a pair of values called linguistic 2–tuple, (s, α), where s is a linguistic term and α is a numeric value representing the symbolic translation.

Definition 1. *Let $\beta \in [0, g]$ be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set $S = \{s_0, s_1, \ldots, s_{g-1}, s_g\}$, i.e., the result of a symbolic aggregation operation. Let $i = round(\beta)$ and $\alpha = \beta - i$ be two values, such that, $i \in [0, g]$ and $\alpha \in [-0.5, 0.5)$, then α is called a symbolic translation.*

This model defines a set of transformation functions to manage the linguistic information expressed by linguistic 2–tuples.

Definition 2. *Let S be a linguistic term set and $\beta \in [0, g]$ a value supporting the result of a symbolic aggregation operation, then the 2–tuple that expresses the equivalent information to β is obtained with the following function:*

$$\begin{aligned} \Delta : \quad & [0, g] \longrightarrow S \times [-0.5, 0.5) \\ & \Delta(\beta) = (s_i, \alpha) \\ & i = round(\beta) \\ & \alpha = \beta - i \end{aligned}$$

where "round" is the usual round operation, s_i has the closest index label to "β" and "α" is the value of the symbolic translation.

There exists a function, Δ^{-1}, such that given a 2–tuple it returns its equivalent numerical value $\beta \in [0, g] \subset \mathbb{R}$:

$$\begin{aligned} \Delta^{-1} : \quad & S \times [-0.5, 0.5) \longrightarrow [0, g] \\ & \Delta^{-1}(s_i, \alpha) = i + \alpha = \beta \end{aligned}$$

A linguistic term can be seen as a linguistic 2–tuple by adding to it the value 0 as symbolic translation, $s_i \in S \equiv (s_i, 0)$, and therefore, this linguistic model can be used to represent linguistic preference relations:

Definition 3. *A 2–tuple linguistic preference relation P on a set of alternatives X is a set of 2–tuples on the product set $X \times X$, i.e., it is characterized by a membership function*

$$\mu_P : X \times X \longrightarrow S \times [-0.5, 0.5)$$

When cardinality of X is small, the preference relation may be conveniently represented by a $n \times n$ matrix $P = (p_{ij})$, being $p_{ij} = \mu_P(x_i, x_j)$ $\forall i, j \in \{1, \ldots, n\}$ and $p_{ij} \in S \times [-0.5, 0.5)$.

As aforementioned, missing information is a problem that needs to be addressed because it is not always possible for the experts to provide all the possible preference assessments on the set of alternatives. A missing value in a linguistic preference relation is not equivalent to a lack of preference of one alternative over another. A missing value can be the result of the incapacity of an expert to quantify the degree of preference of one alternative over another. It must be clear then that when an expert is not able to express the particular value p_{ij}, because he/she does not have a clear idea of how better alternative x_i is over alternative x_j, this does not mean that he/she prefers both options with the same intensity.

In order to model these situations, in the following definitions we express the concept of an incomplete 2–tuple fuzzy linguistic preference relation:

Definition 4. *A function $f : X \times Y$ is* partial *when not every element in the set X necessarily maps to an element in the set Y. When every element from the set X maps to one element of the set Y then we have a* total *function.*

Definition 5. *A 2–tuple fuzzy linguistic preference relation P on a set of alternatives X with a* partial *membership function is an incomplete 2–tuple fuzzy linguistic preference relation.*

Obviously, a 2–tuple fuzzy linguistic preference relation is complete when its membership function is a total one. Clearly, definition 3 includes both definitions of complete and incomplete 2–tuple fuzzy linguistic preference relations. However, as there is no risk of confusion between a complete and incomplete 2–tuple fuzzy linguistic preference relation, in this paper we refer to the first type as simply 2–tuple fuzzy linguistic preference relation.

2.2 Linguistic Additive Consistency

The previous definition of a 2–tuple fuzzy linguistic preference relation does not imply any kind of consistency property. In fact, preference values of a preference relation can be contradictory. Obviously, an inconsistent source of information is not as useful as a consistent one, and thus, it would be quite important to be able to *measure* the consistency of the information provided by experts for a particular problem.

Consistency is usually characterized by *transitivity*. Transitivity seems like a reasonable criterion of coherence for an individual's preferences: if x is preferred to y and y is preferred to z, common sense suggests that x should be preferred to z. Many properties have been suggested to model transitivity , amongst which we can cite [17]: *triangle condition, weak transitivity, max–min transitivity, max-max transitivity, restricted max–min transitivity, restricted max–max transitivity, additive transitivity.*

As shown in [17], additive transitivity for fuzzy preference relations can be seen as the parallel concept of

Saaty's consistency property for multiplicative preference relations [22]. The mathematical formulation of the additive transitivity was given by Tanino in [23]:

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5) \; \forall i, j, k \in \{1, \ldots, n\} \tag{1}$$

This kind of transitivity has the following interpretation: suppose we want to establish a ranking between three alternatives x_i, x_j and x_k, and that the information available about these alternatives suggests that we are in an indifference situation, i.e. $x_i \sim x_j \sim x_k$. When giving preferences this situation would be represented by $p_{ij} = p_{jk} = p_{ik} = 0.5$. Suppose now that we have a piece of information that says $x_i \prec x_j$, i.e. $p_{ij} < 0.5$. This means that p_{jk} or p_{ik} have to change, otherwise there would be a contradiction, because we would have $x_i \prec x_j \sim x_k \sim x_i$. If we suppose that $p_{jk} = 0.5$ then we have the situation: x_j is preferred to x_i and there is no difference in preferring x_j to x_k. We must then conclude that x_k has to be preferred to x_i. Furthermore, as $x_j \sim x_k$ then $p_{ij} = p_{ik}$, and so $(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ij} - 0.5) = (p_{ik} - 0.5)$. We have the same conclusion if $p_{ik} = 0.5$. In the case of $p_{jk} < 0.5$, then we have that x_k is preferred to x_j and this to x_i, so x_k should be preferred to x_i. On the other hand, the value p_{ik} has to be equal to or lower than p_{ij}, being equal only in the case of $p_{jk} = 0.5$ as we have already shown. Interpreting the value $p_{ji} - 0.5$ as the intensity of preference of alternative x_j over x_i, then it seems reasonable to suppose that the intensity of preference of x_i over x_k should be equal to the sum of the intensities of preferences when using an intermediate alternative x_j, that is, $p_{ik} - 0.5 = (p_{ij} - 0.5) + (p_{jk} - 0.5)$. The same reasoning can be applied in the case of $p_{jk} > 0.5$. Additive transitivity implies additive reciprocity. Indeed, because $p_{ii} = 0.5 \; \forall i$, if we make $k = i$ in (1) then we have: $p_{ij} + p_{ji} = 1 \;\; \forall i, j \in \{1, \ldots, n\}$.

Using the transformation functions Δ and Δ^{-1} we define the linguistic additive transitivity property for 2–tuple fuzzy linguistic preference relations as follows:

$$\Delta[(\Delta^{-1}(p_{ij}) - \Delta^{-1}(s_{g/2}, 0)) + (\Delta^{-1}(p_{jk}) - \Delta^{-1}(s_{g/2}, 0))] = \Delta[(\Delta^{-1}(p_{ik}) - \Delta^{-1}(s_{g/2}, 0)] \;\; \forall i, j, k \in \{1, \ldots, n\}. \tag{2}$$

As in the case of additive transitivity, the linguistic additive transitivity implies linguistic additive reciprocity. Indeed, because $p_{ii} = (s_{g/2}, 0) \; \forall i$, if we make $k = i$ in (2) then we have: $\Delta(\Delta^{-1}(p_{ij}) + \Delta^{-1}(p_{ji})) = (s_g, 0) \; \forall i, j \in \{1, \ldots, n\}$.

Expression (2) can be rewritten as:

$$p_{ik} = \Delta(\Delta^{-1}(p_{ij}) + \Delta^{-1}(p_{jk}) - \Delta^{-1}(s_{g/2}, 0)) \;\; \forall i, j, k \in \{1, \ldots, n\} \tag{3}$$

A 2–tuple fuzzy linguistic preference relation will be considered *"additive consistent"* when for every three options in the problem $x_i, x_j, x_k \in X$ their associated linguistic preference degrees p_{ij}, p_{jk}, p_{ik} fulfil (3). An additive consistent 2–tuple fuzzy linguistic preference relation will be referred as consistent throughout the paper, as this is the only transitivity property we are considering.

3 Estimating Missing Values for Incomplete 2–Tuple Fuzzy Linguistic Preference Relations

In this section we present a consistency based procedure to estimate the missing values of a 2–tuple fuzzy linguistic preference relations.

3.1 Estimating Linguistic Values Based on the Linguistic Additive Consistency

Expression (3) can be used to obtain an estimated value of a preference degree using other preference degrees in a fuzzy linguistic preference relation. In [27] an equivalent expression was used to estimate missing values in ordinal fuzzy linguistic preference relations. However, two other possible ways to estimate missing values can be derived from expression (2). Thus, a linguistic preference value p_{ik} ($i \neq k$) can be estimated using an intermediate alternative x_j in three different ways:

1. From $p_{ik} = \Delta(\Delta^{-1}(p_{ij}) + \Delta^{-1}(p_{jk}) - \Delta^{-1}(s_{g/2}, 0))$ we obtain the estimate

$$(cp_{ik})^{j1} = \Delta(\Delta^{-1}(p_{ij}) + \Delta^{-1}(p_{jk}) - \Delta^{-1}(s_{g/2}, 0)) \tag{4}$$

2. From $p_{jk} = \Delta(\Delta^{-1}(p_{ji}) + \Delta^{-1}(p_{ik}) - \Delta^{-1}(s_{g/2}, 0))$ we obtain the estimate

$$(cp_{ik})^{j2} = \Delta(\Delta^{-1}(p_{jk}) - \Delta^{-1}(p_{ji}) + \Delta^{-1}(s_{g/2}, 0)) \tag{5}$$

3. From $p_{ij} = \Delta(\Delta^{-1}(p_{ik}) + \Delta^{-1}(p_{kj}) - \Delta^{-1}(s_{g/2}, 0))$ we obtain the estimate

$$(cp_{ik})^{j3} = \Delta(\Delta^{-1}(p_{ij}) - \Delta^{-1}(p_{kj}) + \Delta^{-1}(s_{g/2}, 0)) \tag{6}$$

3.2 An Estimation Procedure of Missing Values in 2–Tuple Fuzzy Linguistic Preference Relations

To manage incomplete 2–tuple fuzzy linguistic preference relations, we need to introduce the following sets [19]:

$$\begin{aligned} A &= \{(i,j) \mid i,j \in \{1,\ldots,n\} \ \wedge \ i \neq j\} \\ MV &= \{(i,j) \in A \mid p_{ij} \ is \ unknown\} \\ EV &= A \setminus MV \end{aligned} \tag{7}$$

MV is the set of pairs of alternatives whose preference degrees are unknown or missing; EV is the set of pairs of alternatives whose preference degrees are given by the expert. We do not take into account the preference value of one alternative over itself as this is always assumed to be equal to $(s_{g/2}, 0)$.

Expressions (4), (5) and (6) are used to define an iterative estimation procedure of missing values in an incomplete 2–tuple fuzzy linguistic preference relation according to the following two steps: (A) the elements that can be estimated in each iteration of the procedure are established, and (B) the the particular expression that will be used to estimate a particular missing value is produced.

A) Elements to be estimated in each iteration of the procedure

The subset of missing values MV that can be estimated in step h of our procedure is denoted by EMV_h (*estimated missing values*) and defined as follows:

$$EMV_h = \left\{ (i,k) \in MV \setminus \bigcup_{l=0}^{h-1} EMV_l \mid i \neq k \wedge \exists j \in \{H_{ik}^{h1} \cup H_{ik}^{h2} \cup H_{ik}^{h3}\} \right\} \tag{8}$$

with

$$H_{ik}^{h1} = \left\{ j \mid (i,j), (j,k) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \tag{9}$$

$$H_{ik}^{h2} = \left\{ j \mid (j,i),(j,k) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \qquad (10)$$

$$H_{ik}^{h3} = \left\{ j \mid (i,j),(k,j) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \qquad (11)$$

and $EMV_0 = \emptyset$ (by definition). When $EMV_{maxIter} = \emptyset$ with $maxIter > 0$ the procedure will stop as there will not be any more missing values to be estimated. Furthermore, if $\bigcup_{l=0}^{maxIter} EMV_l = MV$ then all missing values are estimated, and consequently, the procedure is said to be successful in the completion of the incomplete 2–tuple fuzzy linguistic preference relation.

B) Expression to estimate a particular missing value

In iteration h, to estimate a particular value p_{ik} with $(i,k) \in EMV_h$, the application of the following function is proposed:

---

function estimate_p(i,k)

1) $cp_{ik}^1 = (s_0, 0),\ cp_{ik}^2 = (s_0, 0),\ cp_{ik}^3 = (s_0, 0),\ \mathcal{K} = 0$

2) $cp_{ik}^1 = \Delta \left(\dfrac{\sum\limits_{j \in H_{ik}^{h1}} \Delta^{-1}(cp_{ik}^{j1})}{\# H_{ik}^{h1}} \right),\ \mathcal{K}++\ if\ H_{ik}^{h1} \neq 0.$

3) $cp_{ik}^2 = \Delta \left(\dfrac{\sum\limits_{j \in H_{ik}^{h2}} \Delta^{-1}(cp_{ik}^{j2})}{\# H_{ik}^{h2}} \right),\ \mathcal{K}++\ if\ H_{ik}^{h2} \neq 0.$

4) $cp_{ik}^3 = \Delta \left(\dfrac{\sum\limits_{j \in H_{ik}^{h3}} \Delta^{-1}(cp_{ik}^{j3})}{\# H_{ik}^{h3}} \right),\ \mathcal{K}++\ if\ H_{ik}^{h3} \neq 0.$

5) Calculate $cp_{ik} = \Delta \left(\dfrac{1}{\mathcal{K}} \left(\Delta^{-1}(cp_{ik}^1) + \Delta^{-1}(cp_{ik}^2) + \Delta^{-1}(cp_{ik}^3) \right) \right)$

end function

The function $estimate\_p(i,k)$ computes the final estimated value of missing value, cp_{ik}, as the average of all estimated values that can be calculated using all possible intermediate alternatives x_j and using the three possible expressions (4)–(6).

Summarizing, the *estimation procedure pseudo–code* of missing values for incomplete 2–tuple fuzzy linguistic preference relations is as follows:

```
        ITERATIVE ESTIMATION PROCEDURE

0.  EMV_0 = ∅
1.  h = 1
2.  while EMV_h ≠ ∅ {
3.     for every (i, k) ∈ EMV_h {
4.           estimate_p(i,k)
5.     }
6.     h + +
7.  }
```

3.3 A Sufficient Condition to Estimate all Missing Values

In the following proposition we provide a sufficient condition that guarantees the success of the above estimation procedure for 2–tuple fuzzy linguistic preference relations.

Proposition 1. *An incomplete 2–tuple fuzzy linguistic preference relation can be completed if a set of $n-1$ non–leading diagonal preference values, where each one of the alternatives is compared at least once, is known.*

Proof: Proof by induction on the number of alternatives will be used:

1. Basis: For $n = 3$, we suppose that two linguistic preference degrees involving the three alternatives are known. These degrees can be provided in three different ways:

   (a) p_{ij} and p_{jk} $(i \neq j \neq k)$ are given.

   In this first case, all the possible combinations of the two 2–tuple linguistic preference values are: $\{p_{12}, p_{23}\}$, $\{p_{13}, p_{32}\}$, $\{p_{21}, p_{13}\}$, $\{p_{23}, p_{31}\}$, $\{p_{31}, p_{12}\}$ and $\{p_{32}, p_{21}\}$. In any of these cases, we can find the remaining 2–tuple linguistic preference degrees of the relation $\{p_{ik}, p_{kj}, p_{ji}, p_{ki}\}$ as follows:

   $$p_{ik} = \Delta(\Delta^{-1}(p_{ij}) + \Delta^{-1}(p_{jk}) - \Delta^{-1}(s_{g/2}, 0)) \quad ; \quad p_{kj} = \Delta(\Delta^{-1}(p_{ik}) - \Delta^{-1}(p_{ij}) + \Delta^{-1}(s_{g/2}, 0))$$
   $$p_{ji} = \Delta(\Delta^{-1}(p_{jk}) - \Delta^{-1}(p_{ik}) + \Delta^{-1}(s_{g/2}, 0)) \quad ; \quad p_{ki} = \Delta(\Delta^{-1}(p_{kj}) - \Delta^{-1}(p_{ij}) + \Delta^{-1}(s_{g/2}, 0))$$

   (b) p_{ji} and p_{jk} $(i \neq j \neq k)$ are given.

   In this second case, all the possible combinations of the two 2–tuple linguistic preference values are: $\{p_{21}, p_{23}\}$, $\{p_{31}, p_{32}\}$ and $\{p_{12}, p_{13}\}$. In any of these cases, we can find the remaining 2–tuple linguistic preference degrees of the relation $\{p_{ik}, p_{ki}, p_{kj}, p_{ij}\}$ as follows:

   $$p_{ik} = \Delta(\Delta^{-1}(p_{jk}) - \Delta^{-1}(p_{ji}) + \Delta^{-1}(s_{g/2}, 0)) \quad ; \quad p_{ki} = \Delta(\Delta^{-1}(p_{ji}) - \Delta^{-1}(p_{jk}) + \Delta^{-1}(s_{g/2}, 0))$$
   $$p_{kj} = \Delta(\Delta^{-1}(p_{ki}) - \Delta^{-1}(p_{ji}) + \Delta^{-1}(s_{g/2}, 0)) \quad ; \quad p_{ij} = \Delta(\Delta^{-1}(p_{kj}) - \Delta^{-1}(p_{ki}) + \Delta^{-1}(s_{g/2}, 0))$$

   (c) p_{ij} and p_{kj} $(i \neq j \neq k)$ are given.

   In this third case, all the possible combinations of the two 2–tuple linguistic preference values are: $\{p_{12}, p_{32}\}$, $\{p_{13}, p_{23}\}$ and $\{p_{21}, p_{31}\}$. In any of these cases, we can find the remaining 2–tuple linguistic preference degrees of the relation $\{p_{ik}, p_{ki}, p_{ji}, p_{jk}\}$ as follows:

   $$p_{ik} = \Delta(\Delta^{-1}(p_{ij}) - \Delta^{-1}(p_{kj}) + \Delta^{-1}(s_{g/2}, 0)) \quad ; \quad p_{ki} = \Delta(\Delta^{-1}(p_{kj}) - \Delta^{-1}(p_{ij}) + \Delta^{-1}(s_{g/2}, 0))$$
   $$p_{ji} = \Delta(\Delta^{-1}(p_{ki}) - \Delta^{-1}(p_{kj}) + \Delta^{-1}(s_{g/2}, 0)) \quad ; \quad p_{jk} = \Delta(\Delta^{-1}(p_{ik}) - \Delta^{-1}(p_{ij}) + \Delta^{-1}(s_{g/2}, 0))$$

2. **Induction hypothesis**: Let us assume that the proposition is true for $n = q - 1$.

3. **Induction step**: Let us suppose that the expert provides only $(q-1)$ 2–tuple linguistic preference degrees where each one of the q alternatives is compared at least once.

   In this case, we can select a set of $(q-2)$ 2–tuple linguistic preference degrees where $(q-1)$ different alternatives are involved. Without loss of generality, we can assume that these $(q-1)$ alternatives are $x_1, x_2, \ldots, x_{q-1}$, and therefore the remaining 2–tuple linguistic preference degree involving the alternative x_q could be p_{qi} $(i \in \{1, \ldots, q-1\})$ or p_{iq} $(i \in \{1, \ldots, q-1\})$.

   By the induction hypothesis we can estimate all the 2–tuple linguistic preference values of the 2–tuple linguistic preference relation of order $(q-1) \times (q-1)$ associated with the set of alternatives $\{x_1, x_2, \ldots, x_{q-1}\}$. Therefore, we have estimates for the following set of 2–tuple linguistic preference degrees

   $$\{p_{ij}, \ i, j = 1, \ldots, q-1, \ i \neq j\}.$$

   If the 2–tuple linguistic value we know is p_{qi}, $i \in \{1, \ldots, q-1\}$ then we can estimate $\{p_{qj}, \ j = 1, \ldots, q-1, \ i \neq j\}$ and $\{p_{jq}, \ j = 1, \ldots, q-1\}$ using $p_{qj} = \Delta(\Delta^{-1}(p_{qi}) + \Delta^{-1}(p_{ij}) - \Delta^{-1}(s_{g/2}, 0))$, $\forall j$, and $p_{jq} = \Delta(\Delta^{-1}(p_{ji}) - \Delta^{-1}(p_{qi}) + \Delta^{-1}(s_{g/2}, 0))$, $\forall j$, respectively.

   If the 2–tuple linguistic value we know is p_{iq}, $i \in \{1, \ldots, q-1\}$ then $\{p_{qj}, \ j = 1, \ldots, q-1\}$ and $\{p_{jq}, \ j = 1, \ldots, q-1, \ i \neq j\}$, are estimated by means of $p_{qj} = \Delta(\Delta^{-1}(p_{ij}) - \Delta^{-1}(p_{iq}) + \Delta^{-1}(s_{g/2}, 0))$, $\forall j$, and $p_{jq} = \Delta(\Delta^{-1}(p_{ji}) + \Delta^{-1}(p_{iq}) - \Delta^{-1}(s_{g/2}, 0))$, $\forall j$, respectively.

3.4 Example

For the sake of simplicity we will assume a low number of alternatives. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of four alternatives and $S = \{N, MW, W, E, B, MB, T\}$ the set of linguistic labels used to provide preferences, with the following meaning:

$$N = Null \quad MW = Much\ Worse \quad W = Worse \quad E = Equally\ Preferred$$
$$B = Better \quad MB = Much\ Better \quad T = Total$$

Suppose the following incomplete fuzzy linguistic preference relation provided by an expert:

$$P = \begin{pmatrix} - & x & W & x \\ x & - & x & MW \\ B & x & - & E \\ x & MB & E & - \end{pmatrix}$$

Note that the expert did not provide any α values, which is a common practice when expressing preferences with linguistic terms. In these cases, we set $\alpha = 0$.

$$P = \begin{pmatrix} - & x & (W,0) & x \\ x & - & x & (MW,0) \\ (B,0) & x & - & (E,0) \\ x & (MB,0) & (E,0) & - \end{pmatrix}$$

Then, the estimation procedure is applied as follows:

Iteration 1: The set of elements that can be estimated is: $EMV_1 = \{(1,4),(2,3),(3,2),(4,1)\}$

- To estimate p_{14} the procedure is as follows:

  $H_{14}^{11} = \{1\} \Rightarrow cp_{14}^1 = \Delta(\Delta^{-1}(cp_{14}^{31})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{13})+\Delta^{-1}(p_{34})-g/2))) = \Delta(\Delta^{-1}(\Delta(2+3-3))) = \Delta(\Delta^{-1}(\Delta(2))) = (W,0)$

  $H_{14}^{12} = \{1\} \Rightarrow cp_{14}^2 = \Delta(\Delta^{-1}(cp_{14}^{32})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{34})-\Delta^{-1}(p_{31})+g/2))) = \Delta(\Delta^{-1}(\Delta(3-4+3))) = \Delta(\Delta^{-1}(\Delta(2))) = (W,0)$

  $H_{14}^{13} = \{1\} \Rightarrow cp_{14}^3 = \Delta(\Delta^{-1}(cp_{14}^{33})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{13})-\Delta^{-1}(p_{43})+g/2))) = \Delta(\Delta^{-1}(\Delta(2-3+3))) = \Delta(\Delta^{-1}(\Delta(2))) = (W,0)$

  $\mathcal{K} = 3 \Rightarrow cp_{14} = \Delta\left(\dfrac{\Delta^{-1}(cp_{14}^1)+\Delta^{-1}(cp_{14}^2)+\Delta^{-1}(cp_{14}^3)}{3}\right) = \Delta\left(\dfrac{2+2+2}{3}\right) = (W,0)$

- To estimate p_{23} the procedure is as follows:

  $H_{23}^{11} = \{1\} \Rightarrow cp_{23}^1 = \Delta(\Delta^{-1}(cp_{23}^{41})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{24})+\Delta^{-1}(p_{43})-g/2))) = \Delta(\Delta^{-1}(\Delta(1+3-3))) = \Delta(\Delta^{-1}(\Delta(1))) = (MW,0)$

  $H_{23}^{12} = \{1\} \Rightarrow cp_{23}^2 = \Delta(\Delta^{-1}(cp_{23}^{42})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{43})-\Delta^{-1}(p_{42})+g/2))) = \Delta(\Delta^{-1}(\Delta(3-5+3))) = \Delta(\Delta^{-1}(\Delta(1))) = (MW,0)$

  $H_{23}^{13} = \{1\} \Rightarrow cp_{23}^3 = \Delta(\Delta^{-1}(cp_{23}^{43})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{24})-\Delta^{-1}(p_{34})+g/2))) = \Delta(\Delta^{-1}(\Delta(1-3+3))) = \Delta(\Delta^{-1}(\Delta(1))) = (MW,0)$

  $\mathcal{K} = 3 \Rightarrow cp_{23} = \Delta\left(\dfrac{\Delta^{-1}(cp_{23}^1)+\Delta^{-1}(cp_{23}^2)+\Delta^{-1}(cp_{23}^3)}{3}\right) = \Delta\left(\dfrac{1+1+1}{3}\right) = (MW,0)$

- To estimate p_{32} the procedure is as follows:

  $H_{32}^{11} = \{1\} \Rightarrow cp_{32}^1 = \Delta(\Delta^{-1}(cp_{32}^{41})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{34})+\Delta^{-1}(p_{42})-g/2))) = \Delta(\Delta^{-1}(\Delta(3+5-3))) = \Delta(\Delta^{-1}(\Delta(5))) = (MB,0)$

  $H_{32}^{12} = \{1\} \Rightarrow cp_{32}^2 = \Delta(\Delta^{-1}(cp_{32}^{42})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{42})-\Delta^{-1}(p_{43})+g/2))) = \Delta(\Delta^{-1}(\Delta(5-3+3))) = \Delta(\Delta^{-1}(\Delta(5))) = (MB,0)$

  $H_{32}^{13} = \{1\} \Rightarrow cp_{32}^3 = \Delta(\Delta^{-1}(cp_{32}^{43})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{34})-\Delta^{-1}(p_{24})+g/2))) = \Delta(\Delta^{-1}(\Delta(3-1+3))) = \Delta(\Delta^{-1}(\Delta(5))) = (MB,0)$

  $\mathcal{K} = 3 \Rightarrow cp_{32} = \Delta\left(\dfrac{\Delta^{-1}(cp_{32}^1)+\Delta^{-1}(cp_{32}^2)+\Delta^{-1}(cp_{32}^3)}{3}\right) = \Delta\left(\dfrac{5+5+5}{3}\right) = (MB,0)$

- To estimate p_{41} the procedure is as follows:

  $H_{41}^{11} = \{1\} \Rightarrow cp_{41}^1 = \Delta(\Delta^{-1}(cp_{41}^{31})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{43})+\Delta^{-1}(p_{31})-g/2))) = \Delta(\Delta^{-1}(\Delta(3+4-3))) = \Delta(\Delta^{-1}(\Delta(4))) = (B,0)$

  $H_{41}^{12} = \{1\} \Rightarrow cp_{41}^2 = \Delta(\Delta^{-1}(cp_{41}^{32})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{31})-\Delta^{-1}(p_{34})+g/2))) = \Delta(\Delta^{-1}(\Delta(4-3+3))) = \Delta(\Delta^{-1}(\Delta(4))) = (B,0)$

  $H_{41}^{13} = \{1\} \Rightarrow cp_{41}^3 = \Delta(\Delta^{-1}(cp_{41}^{33})) = \Delta(\Delta^{-1}(\Delta(\Delta^{-1}(p_{43})-\Delta^{-1}(p_{13})+g/2))) = \Delta(\Delta^{-1}(\Delta(3-2+3))) = \Delta(\Delta^{-1}(\Delta(4))) = (B,0)$

  $\mathcal{K} = 3 \Rightarrow cp_{41} = \Delta\left(\dfrac{\Delta^{-1}(cp_{14}^1)+\Delta^{-1}(cp_{14}^2)+\Delta^{-1}(cp_{14}^3)}{3}\right) = \Delta\left(\dfrac{4+4+4}{3}\right) = (B,0)$

After these elements have been estimated, we have:

$$P = \begin{pmatrix} - & x & (W,0) & (W,0) \\ x & - & (MW,0) & (MW,0) \\ (B,0) & (MB,0) & - & (E,0) \\ (B,0) & (MB,0) & (E,0) & - \end{pmatrix}$$

Iteration 2: The set of elements that can be estimated is: $EMV_2 = \{(1,2),(2,1)\}$

- To estimate p_{12} the procedure is as follows:

$$H_{12}^{21} = \{2\} \Rightarrow cp_{12}^1 = \Delta(\Delta^{-1}\left(\frac{cp_{12}^{31} + cp_{12}^{41}}{2}\right)) = (B,0)$$

$$H_{12}^{22} = \{2\} \Rightarrow cp_{12}^2 = \Delta(\Delta^{-1}\left(\frac{cp_{12}^{32} + cp_{12}^{42}}{2}\right)) = (B,0)$$

$$H_{12}^{23} = \{2\} \Rightarrow cp_{12}^3 = \Delta(\Delta^{-1}\left(\frac{cp_{12}^{33} + cp_{12}^{43}}{2}\right)) = (B,0)$$

$$\mathcal{K} = 3 \Rightarrow cp_{12} = \Delta\left(\frac{\Delta^{-1}(cp_{12}^1) + \Delta^{-1}(cp_{12}^2) + \Delta^{-1}(cp_{12}^3)}{3}\right) = \Delta\left(\frac{4+4+4}{3}\right) = (B,0)$$

- To estimate p_{21} the procedure is as follows:

$$H_{21}^{21} = \{2\} \Rightarrow cp_{21}^1 = \Delta(\Delta^{-1}\left(\frac{cp_{21}^{31} + cp_{21}^{41}}{2}\right)) = (W,0)$$

$$H_{21}^{22} = \{2\} \Rightarrow cp_{21}^2 = \Delta(\Delta^{-1}\left(\frac{cp_{21}^{32} + cp_{21}^{42}}{2}\right)) = (W,0)$$

$$H_{21}^{23} = \{2\} \Rightarrow cp_{21}^3 = \Delta(\Delta^{-1}\left(\frac{cp_{21}^{33} + cp_{21}^{43}}{2}\right)) = (W,0)$$

$$\mathcal{K} = 3 \Rightarrow cp_{21} = \Delta\left(\frac{\Delta^{-1}(cp_{21}^1) + \Delta^{-1}(cp_{21}^2) + \Delta^{-1}(cp_{21}^3)}{3}\right) = \Delta\left(\frac{2+2+2}{3}\right) = (W,0)$$

After these elements have been estimated, we have the following complete 2–tuple fuzzy linguistic preference relation:

$$P = \begin{pmatrix} - & (B,0) & (W,0) & (W,0) \\ (W,0) & - & (MW,0) & (MW,0) \\ (B,0) & (MB,0) & - & (E,0) \\ (B,0) & (MB,0) & (E,0) & - \end{pmatrix}$$

4 A Selection Process for GDM with Incomplete 2–Tuple Fuzzy Linguistic Preference Relations

The aim of the selection process in GDM is to choose the best alternatives according to the opinions given by the experts. A classical selection process consists of two different phases: *aggregation* and *exploitation* (figure 1). Assuming preference relations to represent the experts' opinions, the former defines a collective preference relation indicating the global preference between every ordered pair of alternatives, while the latter transforms the global information about the alternatives into a global ranking of them to identity the best alternatives or the solution set of alternatives.
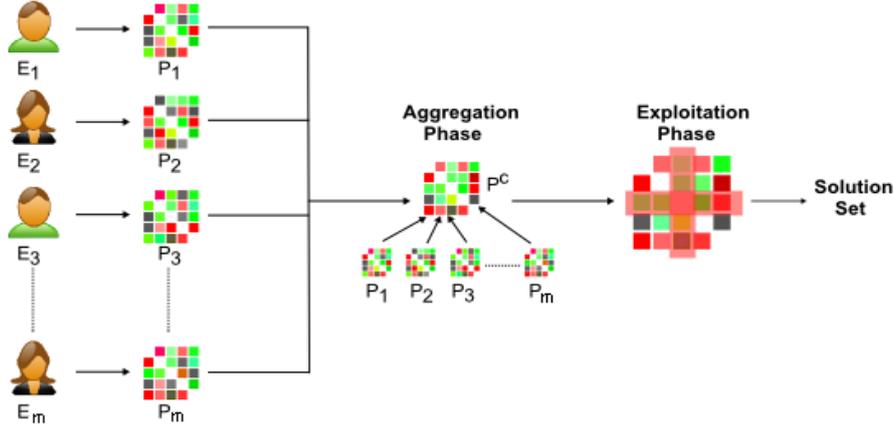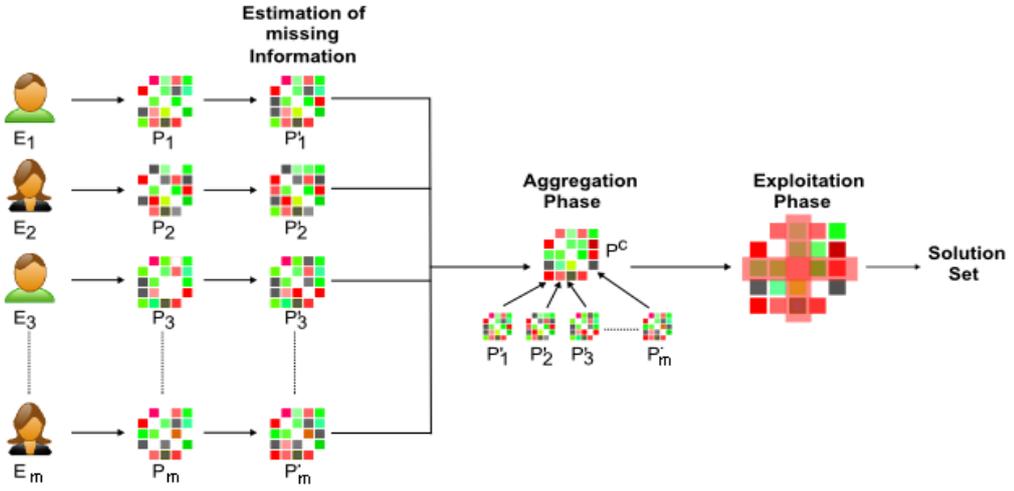
Figure 1: Classical Selection Process

Figure 2: Presented Selection Process

When we deal with GDM situations with incomplete preference relations, there exist cases in which the above classical selection procedure could not be applied satisfactorily. For example, we could find that some preference degrees of the collective preference relation cannot be computed in the aggregation phase and consequently, the ordering of some alternatives cannot be computed in the exploitation phase. To overcome this problem, we present a selection process for GDM with incomplete 2–tuple fuzzy linguistic preference relations that requires three phases (see figure 2): (1) *estimation phase of missing values*, (2) *aggregation phase* and (3) *exploitation phase*.

(1) Estimation of missing information

In this phase, each incomplete 2–tuple fuzzy linguistic preference relation is completed following the estimation procedure of missing values previously presented in section 3.

(2) Aggregation phase: The Collective 2–tuple Linguistic Preference Relation

Once all the missing values in every incomplete 2–tuple fuzzy linguistic preference relation have been estimated, we have a set of m individual 2–tuple fuzzy linguistic preference relations $\{P^1, \ldots, P^m\}$. From this set a collective 2–tuple fuzzy linguistic preference relation $P^c = (p_{ik}^c)$ must be obtained by means of an aggregation procedure. In this case, each value $p_{ik}^c \in S \times [-0.5, 0.5)$ will represent the preference of alternative x_i over alternative x_k according to the majority of the most consistent experts' opinions. To obtain P^c we define the following 2–tuple linguistic OWA operator:

Definition 6: A 2–tuple linguistic OWA operator of dimension n is a function $\phi : (S \times [-0.5, 0.5))^n \longrightarrow S \times [-0.5, 0.5)$, that has a weighting vector associated with it, $W = (w_1, \ldots, w_n)$, with $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$, and it is defined according to the following expression:

$$\phi_W(p_1, \ldots, p_n) = \Delta(\sum_{i=1}^{n} w_i \cdot \Delta^{-1}(p_{\sigma(i)})), \; p_i \in S \times [-0.5, 0.5), \tag{12}$$

being $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ a permutation defined on 2–tuple linguistic values, such that $p_{\sigma(i)} \geq p_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, i.e., $p_{\sigma(i)}$ is the i-highest 2–tuple linguistic value in the set $\{p_1, \ldots, p_n\}$; and being the comparison of two 2–tuple linguistic values (s_k, α_1) and (s_l, α_2) defined as [15]:

- if $k < l$ then (s_k, α_1) is smaller than (s_l, α_2)

- if $k = l$ then

    1. if $\alpha_1 = \alpha_2$ then (s_k, α_1), (s_l, α_2) represent the same information
    2. if $\alpha_1 < \alpha_2$ then (s_k, α_1) is smaller than (s_l, α_2)
    3. if $\alpha_1 > \alpha_2$ then (s_k, α_1) is bigger than (s_l, α_2)

A natural question in the definition of OWA operators is how to obtain W. In [31] it was defined an expression to obtain W that allows to represent the concept of fuzzy majority [20] by means of a fuzzy linguistic non-decreasing quantifier Q [33]:

$$w_i = Q(i/n) - Q((i-1)/n), \quad i = 1, \ldots, n. \tag{13}$$

Therefore, the collective 2–tuple fuzzy linguistic preference relation could be obtained as follows:

$$p_{ik}^c = \phi_Q(p_{ik}^1, \ldots, p_{ik}^m) \tag{14}$$

where Q is the fuzzy quantifier used to implement the fuzzy majority concept.

(3) Exploitation: Choosing the Solution Set

In order to select the solution set of alternatives from the collective 2–tuple fuzzy linguistic preference relation we define two quantifier guided choice degrees of alternatives [11], a dominance and a non-dominance degree.

1. $QGDD_i$: The quantifier guided dominance degree quantifies the dominance that one alternative has over all the others in a fuzzy majority sense and is defined as follows:

$$QGDD_i = \phi_Q(p_{i1}^c, p_{i2}^c, \ldots, p_{i(i-1)}^c, p_{i(i+1)}^c, \ldots, p_{in}^c) \tag{15}$$

This measure allows us to define the set of non-dominated alternatives with maximum linguistic dominance degree:

$$X^{QGDD} = \{x_i \in X \mid QGDD_i = sup_{x_j \in X} QGDD_j\} \tag{16}$$

To calculate $sup_{x_j \in X} QGDD_j$ the above 2–tuple linguistic comparison operator is used.

2. $QGNDD_i$: The quantifier guided non-dominance degree gives the degree in which each alternative is not dominated by a fuzzy majority of the remaining alternatives. Its expression being:

$$QGNDD_i = \phi_Q(Neg(p_{1i}^s), Neg(p_{2i}^s), \ldots, Neg(p_{(i-1)i}^s), Neg(p_{(i+1)i}^s), \ldots, Neg(p_{ni}^s)) \tag{17}$$

where

$$p_{ij}^s = \begin{cases} (s_0, 0) & \text{if } p_{ij} < p_{ji} \\ \Delta(\Delta^{-1}(p_{ij}) - \Delta^{-1}(p_{ji})) & \text{if } p_{ij} \geq p_{ji} \end{cases}$$

represents the degree in which x_i is strictly dominated by x_j, and Neg is the negation operator for 2–tuple linguistic information defined as [15] $Neg(p_{ki}^s) = \Delta(g - \Delta^{-1}(p_{ki}^s))$. The set of of non-dominated alternatives with maximum linguistic non-dominance degree is

$$X^{QGNDD} = \{x_i \in X \mid QGNDD_i = sup_{x_j \in X} QGNDD_j\} \tag{18}$$

As aforementioned, to calculate $sup_{x_j \in X} QGNDD_j$ the above 2–tuple comparison operator is used.

4.1 Example of Application

Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of four alternatives and $S = \{N, MW, W, E, B, MB, T\}$ the same set of linguistic labels used in the previous example. Suppose three experts $\{e_1, e_2, e_3\}$ provide the following incomplete 2–tuple fuzzy linguistic preference relations:

$$P_1 = \begin{pmatrix} - & x & (W,0) & x \\ x & - & x & (MW,0) \\ (B,0) & x & - & (E,0) \\ x & (MB,0) & (E,0) & - \end{pmatrix} ; P_2 = \begin{pmatrix} - & (MW,0) & (W,0) & (W,0) \\ (MB,0) & - & (MB,0) & (MB,0) \\ (E,0) & (MW,0) & - & (W,0) \\ (E,0) & (MW,0) & (E,0) & - \end{pmatrix}$$

$$P_3 = \begin{pmatrix} - & (MW,0) & x & x \\ (B,0) & - & (MB,0) & (MB,0) \\ (W,0) & x & - & (W,0) \\ (W,0) & (MW,0) & (B,0) & - \end{pmatrix}$$

(1) Estimation of missing information

First, we use the estimation procedure presented in Section 3 to obtain the following complete 2–tuple fuzzy linguistic preference relations

$$P_1' = \begin{pmatrix} - & (B,0) & (W,0) & (W,0) \\ (W,0) & - & (MW,0) & (MW,0) \\ (B,0) & (MB,0) & - & (E,0) \\ (B,0) & (MB,0) & (E,0) & - \end{pmatrix} ; P_2' = \begin{pmatrix} - & (MW,0) & (W,0) & (W,0) \\ (MB,0) & - & (MB,0) & (MB,0) \\ (E,0) & (MW,0) & - & (W,0) \\ (E,0) & (MW,0) & (E,0) & - \end{pmatrix}$$

$$P_3' = \begin{pmatrix} - & (MW,0) & (B,0) & (E,0) \\ (B,0) & - & (MB,0) & (MB,0) \\ (W,0) & (N,0) & - & (W,0) \\ (W,0) & (MW,0) & (B,0) & - \end{pmatrix}$$

(2) Aggregation phase

Once the 2–tuple fuzzy linguistic preference relations are completed we aggregate them by means of the 2–tuple linguistic OWA operator. We make use of the linguistic quantifier *most of* defined as $Q(r) = r^{1/2}$, which applying (13), generates a weighting vector of three values to obtain each collective 2–tuple linguistic preference value p_{ik}^c. As an example, the collective 2–tuple linguistic preference value p_{12}^c is obtained as follows

- $p_{12}^1 = (B,0), \ p_{12}^2 = (MW,0), \ p_{12}^3 = (MW,0) \Rightarrow \sigma(1) = 1, \ \sigma(2) = 2, \ \sigma(3) = 3$

- $Q(0) = 0, \ Q(1/3) = 0.58, \ Q(2/3) = 0.82, \ Q(1) = 1 \Rightarrow (w_1, w_2, w_3) = (0.58, 0.24, 0.18$

- $p_{12}^c = \Delta(w_1 \cdot \Delta^{-1}(p_{12}^1) + w_2 \cdot \Delta^{-1}(p_{12}^2) + w_3 \cdot \Delta^{-1}(p_{12}^3)) = \Delta(0.58 \cdot 4 + 0.24 \cdot 1 + 0.18 \cdot 1) = \Delta(2.74) = (E, -0.26)$

The collective 2–tuple fuzzy linguistic preference relation obtained is:

$$P^c = \begin{pmatrix} - & (E,-0.26) & (E,0.16) & (E,-0.42) \\ (B,0.22) & - & (B,0.28) & (B,0.28) \\ (E,0.40) & (E,0.14) & - & (E,-0.42) \\ (E,0.40) & (E,0.32) & (B,-0.42) & - \end{pmatrix}$$

(3) Exploitation phase

Using again the same fuzzy quantifier *most of* and the corresponding weighting vector $W = (0.58, 0.24, 0.18)$, the following quantifier guided dominance degree is obtained

$$(QGDD_1, QGDD_2, QGDD_3, QGDD_4) = \{(E, -0.05), (B, 0.27), (E, 0.19), (E, 0.49)\}$$

To calculate the quantifier guided non-dominance degree we first obtain the matrix P^s:

$$P^s = \begin{pmatrix} - & (N,0) & (N,0) & (N,0) \\ (W,0.48) & - & (W,0.14) & (W,-0.04) \\ (N,0.24) & (N,0) & - & (N,0) \\ (N,0.82) & (N,0) & (W,0) & - \end{pmatrix}$$

The quantifier guided non-dominance degrees are

$$(QGNDD_1, QGNDD_2, QGNDD_3, QGNDD_4) = \{(MB, 0.4), (T, 0), (T, -0.45), (T, -0.18)\}$$

In both cases the maximal sets are $X^{QGDD} = \{x_2\}$ and $X^{QGNDD} = \{x_2\}$ and the solution is the alternative $\{x_2\}$.

5 Discussion

In this section, some important aspects of the use of the estimation procedure within the decision process presented in this paper are analysed. To do so, we compare our model with others models and show the advantages of its use in decision making processes.

1. *Comparison with Xu's model [27].* Proposition 1 establishes the minimum condition of our estimation procedure to solve all possibilities of incomplete information when dealing with individual linguistic preference relations. However, Xu's model [27] does not satisfy that proposition. Then it could not solve all possibilities of incomplete information because it does not use all estimation possibilities that can be derived from Tanino's consistency property. It makes use only of the equation 4, and do not take into account the other two equations 5, 6. Thus, if we have the following incomplete 2–tuple fuzzy linguistic preference relation

$$P = \begin{pmatrix} - & x & (W,0) & x \\ x & - & (E,0) & x \\ x & x & - & x \\ x & x & (MB,0) & - \end{pmatrix}$$

our procedure obtains the following 2-tuple fuzzy linguistic preference relation

$$P' = \begin{pmatrix} - & (W,0) & (W,0) & (N,0) \\ (B,0) & - & (E,0) & (MW,0) \\ (B,0) & (E,0) & - & (MW,0) \\ (T,0) & (MB,0) & (MB,0) & - \end{pmatrix}$$

while Xu's model would not be able to obtain any missing value because there is not any intermediate alternative x_j for which equation 4 can be applied, therefore, the complete 2–tuple fuzzy linguistic preference relation could not be calculated.

2. *On the choice degrees in the selection process.* Given an incomplete fuzzy linguistic preference relation, the selection process could not be carried out because the choice degrees could not be obtained. For example, in the following incomplete fuzzy linguistic preference relation

$$P = \begin{pmatrix} - & W & x & x \\ x & - & x & x \\ x & MW & - & x \\ E & MW & x & - \end{pmatrix}$$

we cannot obtain neither QGDD nor QGNDD for all alternatives. On the one hand, the quantifier guided dominance degree cannot be obtained because there are no values in the second row. On the other hand, matrix P^s is not possible to be calculated because there are missing values on the linguistic preference, and therefore neither the quantifier guided non-dominance degree can be calculated. However, if our procedure is used, we are able to obtain both QGDD and QGNDD.

3. *In the selection process.* In many cases of incomplete information situations if our procedure is not used, a classical selection process could not successfully applied. Indeed, if experts provided incomplete fuzzy

linguistic preference relations, the aggregation phase might be not be possible to be carried out to obtain an incomplete collective 2–tuple linguistic preference and therefore, as we have aforementioned, the choice degrees could not be applied. For example, if four experts provide the following incomplete fuzzy linguistic preference relations

$$P^1 = \begin{pmatrix} - & x & (W,0) & x \\ x & - & (E,0) & x \\ x & x & - & x \\ x & x & (MB,0) & - \end{pmatrix} ; P^2 = \begin{pmatrix} - & x & (B,0) & x \\ x & - & (B,0) & x \\ x & x & - & x \\ x & x & (MW,0) & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & x & (MW,0) & x \\ x & - & (T,0) & x \\ x & x & - & x \\ x & x & (B,0) & - \end{pmatrix} ; P^4 = \begin{pmatrix} - & x & (E,0) & x \\ x & - & (E,0) & x \\ x & x & - & x \\ x & x & (N,0) & - \end{pmatrix}$$

then only the collective values p_{13}^c, p_{23}^c and p_{43}^c could be obtained and therefore, the selection process could not be applied satisfactorily.

6 Conclusions

In this paper we have proposed a complete procedure to estimate missing values in incomplete 2–tuple fuzzy linguistic preference relations which is based on the additive consistency property. This procedure is able to be applied in situations in which other consistency based linguistic approaches are not. Additionally, we have shown its application in a selection procedure of alternatives based on different linguistic choice degrees and have analysed some of its advantages with regards to previous procedures.

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