Using a Model to Describe Students' Inductive Reasoning in Problem Solving[1]

María C. Cañadas, Encarnación Castro and Enrique Castro

Department of Didactics of Mathematics, Faculty of Education, University of Granada, Granada

Spain

María C. Cañadas: Departamento de Didáctica de la Matemática, Universidad de Granada. Facultad de Educación. Campus de la Cartuja, s/n. 18071 Granada. Spain. Email: mconsu@ugr.es

[1] This study has been developed within a Spanish national project of Research, Development and Innovation, identified by the code SEJ2006-09056, financed by the Spanish Ministry of Sciences and Technology and FEDER funds.

Abstract

Introduction. We present some aspects of a wider investigation (Cañadas, 2007), whose main objective is to describe and characterize inductive reasoning used by Spanish students in years 9 and 10 when they work on problems that involved linear and quadratic sequences.

Method. We produced a test composed of six problems with different characteristics related to sequences and gave it to 359 Secondary students to work on. The problems could be solved using inductive reasoning. We used an inductive reasoning model made up of seven steps (Cañadas and Castro, 2007) in order to analyze students' responses.

Results. We present some results related to: (a) frequencies of the different steps performed by students, (b) relationships between the frequencies of steps depending on the characteristics of the problems, and (c) the study of the (in)dependence relationships among different steps of the model of inductive reasoning.

Discussion. We can conclude that the inductive reasoning model was useful to describe students' performance. In this paper, we emphasize that the model is not linear. For example, in some problems students reach the generalization step without passing through the previous steps. To describe how students reach more advanced steps without the previous ones, and to analyze whether accessing the intermediate steps could have been helpful for them, are tasks for future research.

Keywords. generalization, inductive reasoning, problem solving, Secondary students, sequences.

Received: 11/29/08 *Initial Acceptance:* 11/29/08 *Final Acceptance:* 01/23/09

Resumen

Introducción. Presentamos algunos aspectos de una investigación más amplia (Cañadas, 2007), cuyo principal objetivo es describir y caracterizar el razonamiento inductive utilizado por estudiantes españoles de 3º y 4º de Educación Secundaria Obligatoria cuando resuelven problemas que involucran sucesiones lineales y cuadráticas.

Método. Propusimos un cuestionario de seis problemas de diferentes características relacionados con sucesiones a 359 estudiantes de Secundaria. Estos problemas podían ser resueltos mediante el razonamiento inductivo. Utilizamos un modelo de razonamiento inductivo compuesto por siete pasos (Cañadas and Castro, 2007) para analizar las respuestas de los estudiantes.

Resultados. Mostramos algunos resultados relacionados son: (a) las frecuencias de los pasos que emplean los estudiantes, (b) las relaciones entre las frecuencias de los pasos según las características del problema y (c) el estudio de las relaciones de (in)dependencia enre los diferentes estados del modelo de razonamiento inductivo.

Discusión. Podemos concluir que el modelo de razonamiento inductive fue útil para describir la actuación de los estudiantes. En este artículo ponemos de manifiesto que este modelo no es linear. Como ejemplo, los estudiantes lograron la generalización sin haber realizado algunos pasos previos en algunos problemas. Describir cómo los estudiantes llegan a pasos más avanzados sin haber realizado los considerados previos y analizar si la realización de los pasos intermedios puede ser útil para los estudiantes son tareas pendientes.

Palabras clave: estudiantes de secundaria, generalización, razonamiento inductivo, resolución de problemas, sucesiones.

Recibido: 29/11/08 *Aceptación inicial:* 29/11/08 *Aceptación final:* 23/01/09

Introduction

Different kinds of reasoning arise from diverse disciplines related to mathematics education such as philosophy, psychology and mathematics: Inductive reasoning, deductive reasoning, abductive reasoning, plausible reasoning, and transformational reasoning are some of them (Harel & Sowder, 1998; Lithner, 2000; Peirce, 1918; Simon, 1996). We consider the general distinction between inductive and the deductive reasoning from the philosophical tradition and from different disciplines and their diverse contexts where this distinction persists. Although some authors today highlight the difficulties in separating these two in practice (Ibañes, 2001; Marrades, & Gutiérrez, 2000; Stenning, & Monaghan, 2005), we make an effort to focus our research on the inductive reasoning process.

From a general viewpoint, we refer to inductive reasoning as a process that starts with particular cases and allows us to obtain more information than that presented by those particular cases (Neubert & Binko, 1992). We can say that inductive reasoning produces a generalization from the initial cases. This is the same sense that Pólya (1967) gave to *induction*. We use this term in a different way than is usually employed in *mathematical induction* or *complete induction*, which is a formal method of proof, based more on deductive than on inductive reasoning. Induction and mathematical induction are not unconnected concepts because some processes of inductive reasoning can conclude with mathematical induction, but this does not always occur.

In this paper we describe some key aspects of a research study (Cañadas, 2007) focused on inductive reasoning. Our main research objective was to *describe and characterize inductive reasoning used by Spanish students in years 9 and 10 when they work on problems that involved linear and quadratic sequences*.

One theoretical contribution of our research was a model comprising seven steps to analyze inductive reasoning, as described by Cañadas and Castro (2007). This model emerged from a pilot study, where we used ideas from Pólya (1967) and Reid (2002), related to the inductive reasoning.

This paper is presented in three main parts. First, we present some general aspects of the theoretical and methodological frameworks of Cañadas (2007). Second, we show some results of students' use of inductive reasoning related to: (a) a general description based on how frequently students performed each step, (b) significant differences in performing these steps, depending on problem characteristics, and (c) the (in)dependence analysis among the steps included in the inductive reasoning model. We finish with a discussion of the results.

Inductive Reasoning Model

One of our research objectives was to produce a systematic way to explore students' inductive reasoning in the context of problem solving. We followed Pólya's idea (1967) about the induction process, considering four steps in a first approximation of a model to describe inductive reasoning:

- Observation of particular cases,
- conjecture formulation based on previous particular cases,
- generalization, and
- conjecture verification with new particular cases.

Reid (2002) used these steps in the context of empirical induction from a finite number of discrete cases, and proposed a reformulation containing five, more detailed states: (a) Work on particular cases, (b) pattern observation, (c) conjecture formulation for the general case (with doubt), (d) generalization, and (e) use of generalization for proving. The main contribution of this proposal is related to conjecture formulation. Reid established a first conjecture formulation for the general case based on particular cases and on the pattern (hypothetically found in a previous step). Since we do not know if the pattern identified is valid for the general case, Reid considered conjecture formulation for the general case with doubt.

We used the previous steps in our pilot study with the main research objective of describing Secondary students' inductive reasoning when they were solving problems that could be solved using this type of reasoning. Through these students' performances, we identified seven steps that allowed us to describe, in a detailed way, students' inductive reasoning from a finite number of discrete cases (Cañadas & Castro, 2007):

- Work on particular cases,
- organization of particular cases,
- search and prediction of pattern,

- conjecture formulation,
- justification (conjecture validation based on particular cases),
- generalization, and
- justification of the generalization (formal proof).

One contribution of these proposed steps is the consideration of organization of particular cases. In Cañadas (2002), it was evident that this was a helpful step where students searched for and predicted a pattern. Moreover, this seven-step model includes two kinds of justification: (a) justification based on particular cases and (b) justification of the generalization or formal proof. We are aware that the last type of justification involved more deductive than inductive reasoning. This is a possible end for the inductive reasoning process where it constitutes a kind of mathematical reasoning in the sense considered by Reid (2002).

These steps can be thought of as levels from particular cases to the general case and its justification beyond the inductive reasoning process. Moreover, they have been successfully used for other kinds of conjecturing processes (Cañadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2008).

The steps of the model can be useful to analyze students' performances in the inductive reasoning process, but not all steps necessarily occur, and they do not have to occur in the proposed order.

Mathematics Subject Matter: Linear and Quadratic Sequences

Given that we chose linear and quadratic sequences as the specific subject matter, we needed to describe this in order to select adequate problems for the students and to obtain criteria to describe students' work on those problems. We used some ideas from *subject matter analysis* (Gómez, 2007) to present a detailed subject matter description of the linear and quadratic sequences. Through some aspects of this analysis, we obtained useful information about linear and quadratic sequences in order to elaborate a procedure to describe inductive strategies. Particularly, we focused on the elements of the sequences, the representation systems of the elements and the transformations among the representations.

The elements of sequences are the particular and general terms, and the limit. Since our interest was inductive reasoning[2], we selected particular and general terms to work on.

Since sequences are a particular kind of function, we took into account four representation systems for functions, following Janvier (1987): (a) Graphic, (b) numeric, (c) verbal and (d) algebraic. On the one hand, particular terms can be expressed numerically, graphically or verbally. On the other hand, general terms can be expressed algebraically or verbally.

Research Questions

The overall research objective of our investigation was broken down into specific objectives. We present them in terms of research questions, which we try to answer in this paper.

- Is there any regularity in the frequencies of the steps performed by students in the proposed problems?
- Are there any significant differences in students' performances of different steps depending on the characteristics of the problem?
- Attending to students' performances, are there any (in)dependence relationships among different steps of the inductive reasoning model?

Method

Participants

We selected 359 students in Grades 9 and 10, from four Spanish public schools. 94% of them were 14, 15 and 16 years old.

Considering our research objective, we first described the students' background related to inductive reasoning, problem solving and sequences, using four complementary

[2] We consider that inductive reasoning is the process that begins with particular cases and produces a generalization from these cases.

sources: (a) Spanish curriculum, (b) informal interviews with students' teachers, (c) mathematics textbooks used by students, and (d) students' notebooks.

The Spanish curriculum includes reasoning as one of its main objectives. However, it contains only a few actions related to inductive reasoning: (a) to recognize numerical regularities, (b) to find strategies to support students' own argumentations, and (c) to formulate and to prove conjectures (Boletín Oficial del Estado, 2004).

Students had previously studied linear sequences but they had not worked on the quadratic ones; and they had worked on problems using inductive reasoning only occasionally, usually in relation to sequences.

We asked the students to work on a written test composed of problems involving linear and quadratic sequences.

Instrument

We developed a written test with the purpose of analyzing inductive reasoning through students' responses to the problems posed. We asked students to work individually, during their usual hour for mathematics class.

The test had six problems which involved linear and quadratic sequences. Test problems were selected in line with our research objective and using the characteristics gathered from the subject matter description of natural number sequences:

- ♦ The order of the sequence. We selected linear or quadratic sequences for problems, according to our research objective.
- ♦ The representation system used in the statements. To analyze inductive reasoning, we considered statements with particular cases expressed verbally, numerically or graphically, the three possible representation systems for particular cases in natural number sequences.
- ♦ The task proposed. We identified four different tasks related to inductive reasoning and sequences: Continuation, extrapolation, generalization, and particularization. We selected continuation and extrapolation for the first task of each problem because gen-

eralization and particularization were part of our analysis through continuation and extrapolation.

The six test problems involved a second task consisting of justifying their responses, thereby allowing us to complete the description of the inductive reasoning model (Figure 1). We show the different characteristics of the problems in relation to the criteria selected for this research in Table 1.

Table 1. ***Characteristics of the problems***

Problem	Task	Representation Systems	Order of the Sequence *n*
1	Continuation	Verbal	1
2	Continuation	Numeric	2
3	Extrapolation	Graphic	1
4	Extrapolation	Verbal	2
5	Extrapolation	Numeric	1
6	Continuation	Graphic	2

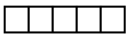
1. A video club rents 50 films a day. The owner observes an increase in renting such that each day they rent three more films than the previous day.

- How many films he will rent in the following five days after his observation?
- Justify your answer.

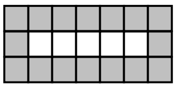
2. We have the following numerical sequence: 3, 7, 13, 21…

- Write down the next four numbers of the sequence.
- Justify your answer.

3. Imagine that you have some white square tiles and some grey square tiles. They are all the same size. You make a row of white tiles.



We surround the row of white tiles with a border of grey tiles.



- How many grey tiles would you need if you had 1320 white tiles and you wanted to surround them in the way you have observed in the drawing?
- Justify your answer.

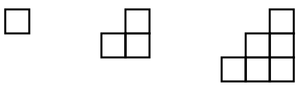
4. We are organizing the first round of a competition. Each team has to play the rest of the participating teams twice (first and second legs). Depending on the number of teams participating, there will be a determined number of matches.

- Calculate the number of matches in the cases that there are 22 and 230 teams.
- Justify your answer.

5. We have the following numerical sequence: 1, 4, 7, 10…

- Write down the number that should be in position 234 of this sequence.
- Justify your answer.

6. Observe the following staircases made of toothpicks, with one, two and three levels. Each square is made of four toothpicks.



- Calculate the number of toothpicks that you need to construct staircases with four, five and six levels.
- Justify your answer.

***Figure 1*. Test Problems**

Data Analysis

In this paper, we focus on some parts of the quantitative data analysis developed in Cañadas (2007). First, we identified the steps performed by each student in his/her responses to each test problem. This information allowed us to:

- Find the frequencies of the steps performed by the students on test problems.
- Analyze the relationship among: (a) steps performed by the students, (b) the representation used in the problem statements, and (c) the order of the sequence involved in problems. We used a logarithmic-linear analysis of three factors: Steps*Representation*Order. This analysis made it possible to identify the effect of each factor in performing the steps, taking into account the possible interaction of any factor pair, and the interaction among all three.
- Analyze the (in)dependence of the different steps identified by the model, through students' performances. We analyzed the dependence or independence relationship of each step in relation to previous steps considered in the model.

Results

Frequencies of Steps

We first identified the steps performed by students on each problem. We present these frequencies in Figure 2.

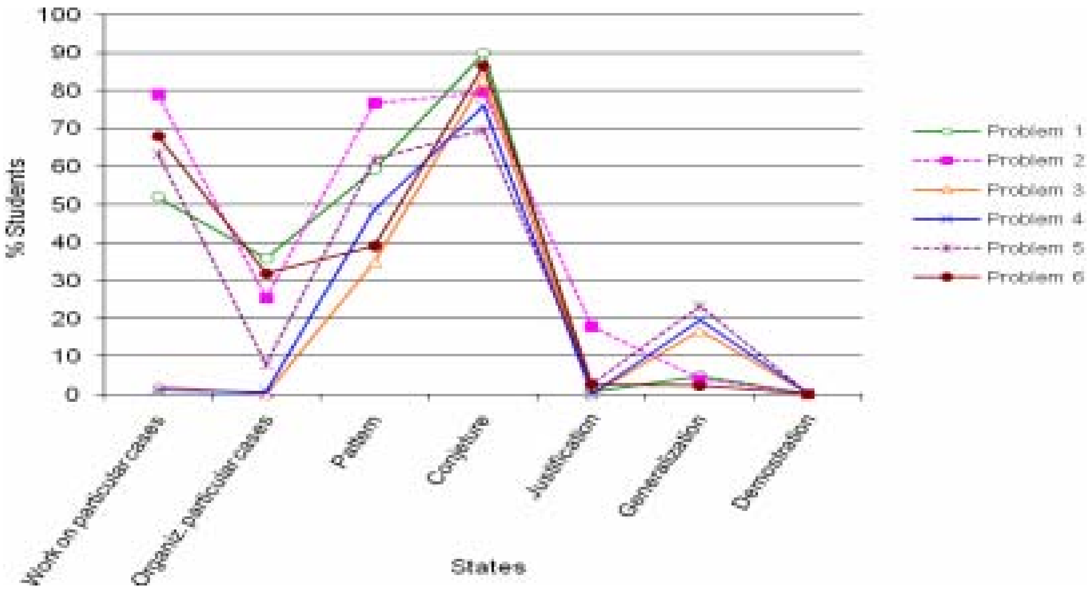


***Figure 2*. Frequency percentages of steps**

The frequencies of steps identified in different problems shows a general tendency, in the sense that frequencies increase and decrease for the same steps on each problem. In general, the steps most frequently used by students were: (a) Work on particular cases, (b) search and prediction of pattern, and (c) conjecture formulation. By contrast, just a few students used particular cases to justify their conjectures and no students demonstrated them.

In spite of this, there are discrepancies among some problems, making us suspect that some characteristics of the problems could be influencing students' performances.

*Logarithmic-Linear Analysis Steps*Representation*Order*

In order to analyze the connection between the representation system used in the problems and the order of the sequences involved, we considered a logarithmic-linear analysis. Given that the residual values are null, the adequate logarithmic-linear model is the saturated one. This kind of analysis includes the three factors and the possible interactions between them.

First, we studied the partial associations of these variables through the chi-square test. We present the results of this analysis in Table 2.

Table 2. *Results of the partial associations*

Effect	Degrees of freedom	Partial chi-square	Prob
Order*Repres	2	175.130	0.0000
Order*States	6	69.441	0.0000
Repres*States	12	255.956	0.0000
Order	1	7.469	0.0063
Repres	2	100.442	0.0000
States	6	4222.179	0.0000

In this table we can observe that all the values of "Prob." are lower than 0.05. This fact allows us to state that all the partial effects are significant. Moreover, looking at the values of the column partial chi-square of Table 2, the effects Representation*States and Order*States associations have high values for associated two-variable effects. This means that these are the significant effects involving two variables. In this paper we analyze these effects, based on the λ parameter and the *z* values.

*Representation*States Association*

We deduce from Table 2 that this is the strongest two-variable associations. Through the logaritmic-linear analysis, we obtain the λ parameter and the *z* values. To sum up these parameters, we present the values of the parameters estimates in relation to the mean in Figure 3.

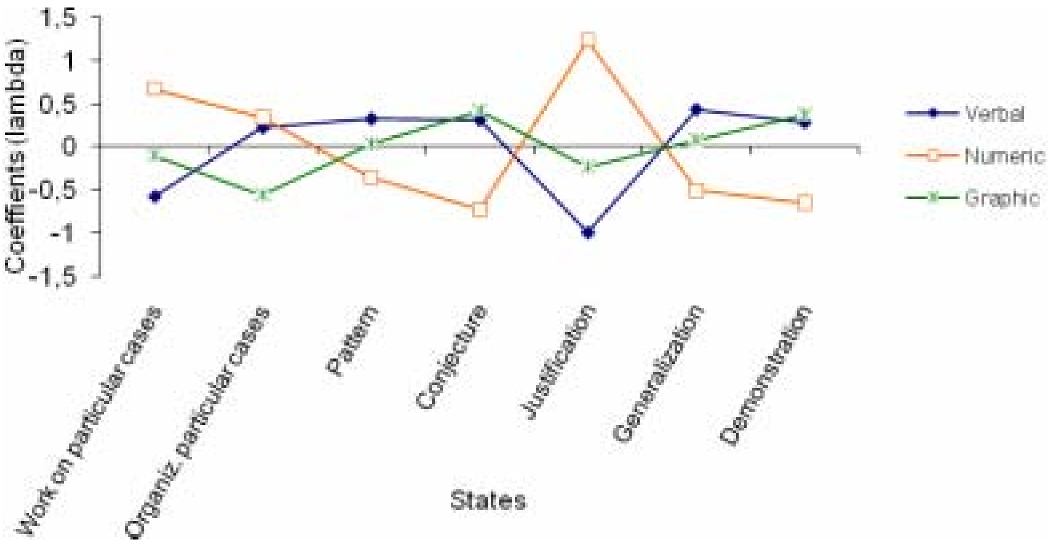


***Figure 3.* Lambda parameters of Representation*States in relation to the mean**

The use of verbal representation is associated with a significant low frequency in work on particular cases (λ = -0.572 and z = -2.719) and in justification (λ = -0.99 and z = -1.95). On the other hand, this representation is associated with a highly significant frequency in pattern recognition (λ = 0.32 and z = 2.05) and generalization (λ = 0.43 and z = 2.36).

The numeric representation is associated with a low frequency in pattern recognition (λ = -0.356 and z = -2.54), in conjectures formulation (λ = -0.726 and z = -5.219), and in generalization (λ = -0.507 and z = -2.983). However, this representation system is associated with frequencies higher that the mean in work on particular cases (λ = 0.322 and z = 2.05), and in conjecture justification (λ = 1.228 and z = 3.597).

Conjecture formulation is the only step associated with graphic representation in a significant way (z = 2.572). As the value of λ shows (0.417), it is higher than the mean.

We summarize the above results in Table 3, indicating a negative association with "-" and a positive association with "+".

Table 3. ***Significant associations Representation*Steps***

Repres	States						
	Work on partic. cases	Organiz. partic. cases	Pattern	Conject.	Justif.	Gen.	Dem.
Verbal	-		+	+	-	+	
Numeric	+		-	-	+	-	
Graphic				+			

*Order*States Association*

Similarly, we present the lambda parameters in Figure 4.

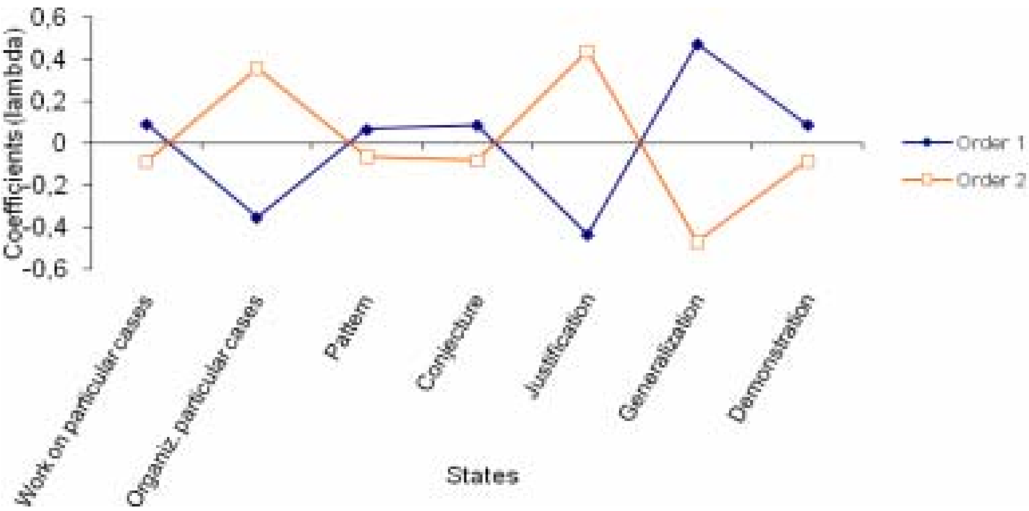


***Figure 4*. Lambda parameters of Representation*States in relation to the mean**

The values of *z* reveal that generalization is the only step that presents significant differences associated with the order of sequences in the problems posed. Observing the λ value, we can conclude that the number of students that generalize in problems that involve linear sequences is higher than the mean; and the number of students who generalize in problems that involve quadratic sequences is lower than the mean.

(In)Dependence Analysis

We analyze the (in)dependence among steps through the chi-square test of statistic independence with a level of significance of 95%. We analyzed the relationship of each step in relation to the previous one(s) because one of our research interests was to determine whether performance of one step helped the student to perform the next step from the inductive reasoning model. This analysis was carried out on each problem independently. We did not include demonstration in the (in)dependence analysis, because there were no students who demonstrated their generalizations.

The first result is that, in the six problems, not all relations showed the same sense of dependence or independence among steps. In order to reach a conclusion from the (in)dependence analysis, we decided that there was evidence of dependence between two steps if more than three problems revealed this characteristic. If not, we considered that there was no evidence of dependence. We summarize these results in Table 4. The grey cells refer to data which were not part of the described analysis.

Table 4. ***(In)Dependence Relationships***

States	Work on particular cases	Organiz. particular cases	Pattern	Conject	Justif	Gen	Dem
Work on particular cases		D	D	I	I	*	
Organiz. particular cases			D	I	*	I	
Pattern				I	I	D	
Conjecture					I	I	
Justif.						I	
Gen.							
Dem.							

D = dependent
I = independent
* The number of dependence and independence relationships are the same in the six problems

Through the dependence analysis described, four steps were dependent on previous ones: (a) organization of particular cases and work on particular cases, (b) pattern and organization of particular cases, (c) pattern and work on particular cases, and (d) generalization and pattern.

Discussion

Even though students were used to working in class on particular cases in tasks related to sequences, this was not evident in the students for all the problems. For three of the problems, less than 10% of the students worked on particular cases (problems 3, 4 and 5; see Figure 1). Similarly, students were accustomed to organizing particular cases in order to reach general terms of sequences in class, yet they tended to not organize particular terms of sequences involved before obtaining a pattern. In every problem, the number of students who worked on particular terms was higher than the number of students who organized particular terms. Analogously, the number of students who identified the patterns was higher than the number of students who organized particular terms.

As for results relating to the kinds of problems, we can conclude that students performed steps more frequently in those problems where particular cases were expressed numerically. One reason for this could be the treatment of sequences in current Spanish Secondary Education, since these students had usually worked on sequences expressed numerically, and they were able to identify the applicability of using certain steps of inductive reasoning that they had previously used in classes.

Frequencies of pattern identification and generalization are higher in problems with graphical statements (problems 3 and 6). In these problems, most of the students reached a generalization without performing previous steps. Most of these cases respond to inadequate patterns. One pending task for research is to determine whether performance of the intermediate steps could have helped students to get the right pattern or to express the generalization.

Taking into account the different order of the sequences involved in the problems, we only identified significant differences in the generalization state, in the sense that the frequency of generalization is significantly higher in problems that involved linear sequences than in problems that involved quadratic sequences. Although students had previously worked on generalization with linear sequences and had studied quadratic sequences, they were not able to generalize in problems that involve quadratic sequences. This could be a consequence of the students' background, since they had worked on generalization activities related to linear sequences.

As we can conclude from Table 4, we did not obtain evidence for dependence relationships for most of the steps of the inductive reasoning model. We therefore affirm that the inductive reasoning model under consideration is not linear.

References

Boletín Oficial del Estado (2004). *Real Decreto 116/2004 de 23 de enero, por el que se desarrolla la ordenación y se establece el currículo de la Educación Secundaria Obligatoria* (BOE, nº 35, pp. 5712-5791). [2004 legislation establishing the currículum for compulsory secondary education.] Madrid: Ministerio de Educación y Ciencia.

Cañadas, M. C. (2007). *Descripción y caracterización del razonamiento inductivo utilizado por estudiantes de educación secundaria al resolver tareas relacionadas con sucesiones lineales y cuadráticas*. [Description and characterization of inductive reasoning used by secondary students in solving tasks related to linear and quadratic sequences.] Doctoral thesis. Granada: Universidad de Granada. (Available at http://cumbia.ath.cx/mcc.htm)

Cañadas, M. C., & Castro, E. (2007). A proposal of categorisation for analysing inductive reasoning. *PNA, 1*(2), 67-78. (Available at ww.pna.es)

Cañadas, M. C., Deulofeu, J., Figueiras, L., Reid, D., & Yevdokimov, A. (2008). Perspectivas teóricas en el proceso de elaboración de conjeturas e implicaciones para la práctica: tipos y pasos. [Theoretical perspectives in the process of forming conjectures, and their implications for practice: types and steps.] *Enseñanza de las Ciencias, 26*(3), 431-444.

Gómez, P. (2007). *Desarrollo del conocimiento didáctico en un plan de formación inicial de profesores de matemáticas de secundaria*. [Developing didactic understanding in a pre-service teacher training plan for Secondary mathematics teachers.] Granada: Universidad de Granada.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In E. Dubinski, A. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate Mathematics Education III* (pp. 234-283). Washington: American Mathematical Society.

Ibañes, M. (2001). *Aspectos cognitivos del aprendizaje de la demostración matemática en alumnos de primer curso de bachillerato.* [Cognitive aspects of learning mathematical proofs in students in fifth year of secondary education.] Doctoral thesis. Valladolid: Universidad de Valladolid.

Lithner, J. (2000). Mathematical reasoning in school tasks. *Educational Studies in Mathematics, 41*(2), 165-190.

Marrades, R., & Gutiérrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 44*, 87-125.

Neubert, G. A., & Binko, J. B. (1992). *Inductive reasoning in the secondary classroom*. Washington: National Education Association.

Peirce (1918). *Tres tipos de razonamiento.* [Three types of reasoning.] Downloaded from http://www.unav.es/gep (20/10/2006).

Pólya, G. (1967). *Le Découverte des Mathématiques*. [The discovery of mathematics.] París: DUNOD.

Reid, D. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education, 33*(1), 5-29.

Simon, M. A. (1996). Beyond inductive and deductive reasoning: The search for a sense of knowing. *Educational Studies in Mathematics, 30*(2), 197-210.

Stenning, K., & Monaghan, P. (2005). Strategies and knowledge representation. In J. P. Leighton, & R. J. Sternberg (Eds.), *The nature of reasoning* (pp. 129-168). Cambridge: Cambridge University Press.