EXECUTABLE FUNCTIONS OF THE REPRESENTATIONS IN LEARNING THE ALGEBRAIC CONCEPTS

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This study aimed to examine the role of multiple representations in learning algebraic concepts for high school students. Using the semi-experimental research method for teaching of numerical, symbolic, and graphical representations, and traditional teaching, 83 female students were selected from the tenth grade of a high school in Tehran. We concluded that there is a significant difference between the mean scores of mathematics in the control and experimental groups. Using the method based on different representations helped the students to become creative and provide similar Algebra examples; thereby analysis power will be increased.

Keywords: Algebra; Graphical representation; Learning; Numerical representation; Symbolic representation

Funciones ejecutables de las representaciones en el aprendizaje de los conceptos algebraicos

Este estudio tiene como objetivo examinar el papel de las representaciones múltiples en el aprendizaje de los conceptos algebraicos en estudiantes de educación secundaria. Se desarrolló una investigación semi-experimental para la enseñanza de representaciones numéricas, simbólicas y gráficas y la enseñanza tradicional, en este estudio participaron 83 estudiantes femeninas del décimo grado de una escuela secundaria en Teherán. Se concluyó que hay una diferencia significativa entre los puntajes promedio de matemáticas en el grupo control y los grupos experimentales. El uso del método basado en diferentes representaciones ayudó a las estudiantes a ser creativas y proporcionar ejemplos de álgebra similares; por lo tanto, la capacidad de análisis aumentará.

Términos clave: Álgebra; Aprendizaje; Representación gráfica; Representación numérica; Representación simbólica

Daryaee, A., Shahvarani, A., Tehranian, A., Lotfi, F. H., & Rostamy-Malkhalifeh, M. (2018). Executable Functions of the Representations in Learning the Algebraic Concepts. *PNA 13*(1), 1-18.

Gouya and Sereshti (2006) consider mathematics to be of a dual nature and believe that while mathematics is extremely abstract, it is also extremely tangible, and this duality poses a serious challenge to the field. This tangible nature comprises many things, including the order of nature and the legitimacy of the various phenomena. Moreover, children experience mathematics, sense the concepts of greater than, less than, and equal to, get familiar with various classifications, and gradually learn the characteristics of each phenomenon since childhood, with their own efforts to understand the world around them. Then they live with mathematics in their various types of games; approximate, estimate, compare, match, count, and do dozens and dozens of other activities that all are mathematical nature. But as soon as children enter the formal education system, usually all of these apparent and hidden acquired mathematical skills are ignored and students are treated as a *tabula rasa,* or blank slate. Consequently, a child who has played and lived with different mathematical concepts is often confused when facing the formal form of those concepts and encounters problems in his/her mathematical learning.

One of the methods through which we can make relationships between the children's informal experiences and knowledge and their formal mathematical knowledge in mathematics education is the use of representations for mathematical concepts and ideas. Representations embody or introduce an idea—or a concept—in different ways (Kilpatrick & Swafford, 2001). For example, in high school mathematics, the exponential function $y = f(x) = 2^x$ can be represented in symbolic, numerical, graphical, and geometric forms, as can be seen in Figure 1 (Mackie, 2002). Regarding the selection and use of different representations, research has shown that those students who are able to use different representational approaches, such as symbolic, graphical, numerical, etc., more flexibly are more successful in mathematics because the ability to move between different representations of mental objects is one of the signs of deep and conceptual learning (Gouya & Sereshti, 2006).

Symbolic representation | Numerical representation | Graphical representation | Geometric representation

$y = f(x) = 2^x$

x	$y = 2^x$
0	1
1	2
2	4
3	8

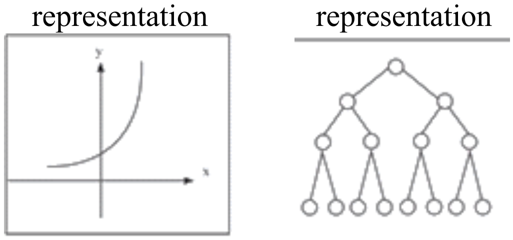

Figure 1. Types of representations for the exponential function

One of the five process standards referred to in the document of the National Council of Mathematical Teachers (NCTM) is the representation standard (NCTM, 2000). The process standards in this document stand for the processes for acquiring the skills and the way of applying the knowledge related to mathematical content. In fact, it can be interpreted that if the content standards are the body of mathematics, then the process standards are the spirit of mathematics. In this document, the representation standard is stated as:

> *Educational programs at all levels of education—K-12—should enable all the students to create and use the representations for organizing, recording and transferring the mathematical ideas; select, apply, and interpret the mathematical representations in solving the problems; use the representations for modeling and interpreting the mathematical, social and physical phenomena"* (NCTM, 2000, p. 360)

According to this council, it is important to expose students to using multiple representations during mathematics instruction. Students need to understand that different representations of mathematical ideas are an important part of their learning process and their mathematical activity. It is very important for students to be encouraged to represent mathematical ideas in a variety of ways that is meaningful to them. It is also important for students to learn that conventional forms of representation have made mathematics learning easier and that students are able to easily communicate with others about mathematical ideas. The NCTM standards also stated that representations can help the students to organize their thoughts. Using representations helps students to have mathematical ideas for thinking in a more tangible and accessible way.

High school students should use representations mostly to solve problems or describe, explain, and develop a mathematical idea. When students study a new mathematical topic, they encounter a lot of new representations for the mathematical concepts. They will need to rigorously make relationship between representations, while a large part of students' mathematical abilities will be created by observing and working on topics from different perspectives. According to Ozel, Yetkiner, and Capraro (2008), the ability to represent a concept in a variety of ways brings a deep understanding of that concept in mind. In the research findings of mathematics education, there are strong reasons why students can understand mathematical concepts through experiencing the different representations and making relationships between different representations. Tall (1991, in Gouya & Sereshti, 2006), proposes four steps for the mathematical learning process as follows: using one type of representation, using more than one type of representation in parallel, making connection and relationship between different representations, and combining the representations and the flexible moves between them. They have quoted from Tall (1991) that recognizing the relationship between equivalent representations and recognizing their common properties leads to the formation of the single concept of objects and mathematical processes. Similarly, Gouya and Sereshti (2006) referring to Dreyfus (1990) state that the idea of using multiple representations of a concept must be in a way that the different aspects of the concept be emphasized and the students be helped to conceptually connect the corresponding aspects in equivalent representations.

Duval (2006) claimed that the representational systems are essential for making relationship and working with mathematical topics and concepts, and argued that mathematical activities clearly involve simultaneous use of at least two representations or changes from one representation to another. Brunner et al. (1997) found that function problems can be taught to students through the use of representations and by the translations between representations. Adu-Gyamfi (2002) provided a review of the literature of representations and concluded that the students who have experience of being taught by the use of multiple representations have a deeper understanding of mathematical concepts because the human mind inherently is an information processing system, and it is very important to choose a representation in an algebraic problem. As a result, the proper illustration of the problem affects the students' performance, and therefore provides strategies to make the problem-solving process effective, which leads to the students' understanding of the algebraic concepts. Therefore, the recognition and use of the representations extremely assists the processing of information in the right direction for leading to the solution, and considering this issue—the proper selection and use of representation—the boundary between successful and unsuccessful solvers is in solving the algebraic problems. In regard to the definitions of three representations; symbolic, numerical, and graphical, researchers are going to teach Algebra regard to cited representations, then, the results of teaching with each representation, indicated separately as sub goals from the main goal. Therefore, the main question/goal of this research addresses the following question: Does using multiple representations impact the learning of Algebraic concepts?

Representations in Algebra

The growing collection of studies in the field of cognitive psychology, cognitive sciences, and mathematics education consider the role of using representations important in mathematical learning (DiSessa, Hammer, & Sherin, 1991; Kaput, 1994; Post, Behr, & Lesh, 1988; Tishman & Perkins, 1997). The students' need for the use of representations is widely accepted. A review of the traditional mathematical curricula and teaching methodology shows that its focus is only on the skills of manipulating symbols and rote learning (Brunner et al., 1997; Moseley & Brenner, 1997). Even basic mathematical concepts are generally presented to the students in abstract forms (Pape & Tchoshanov, 2001). Teaching further focuses on procedural skills, such as solving linear equations, or finding solution sets for a certain unequal system. Some mathematics teachers avoid using representations in algebra classes. However, algebra has used various representation systems to express ideas and processes and has become one of the foundations of mathematics in the school (Herscovics, 1989; Lubinski & Otto, 2002). This branch of mathematics deals with the symbolization of common numerical relations, mathematical structures, and operations on those structures (Kieran, 1989;

Smith, 2004). This does not begin in formal academic years, this kind of thinking appears early in education, spreading over the years, and continues throughout life. The importance of algebra is also supported by the NCTM (2000) standards. According to NCTM:

> *For algebraic thinking; one must be able to understand the patterns, relationships, and functions; present and analyze mathematical conditions and structures using algebraic symbols; use mathematical models to represent and understand the quantitative relationships, and analyze the changes in different fields. Each of these components evolves with the growth and maturity of the students.* (NCTM, 2000, p. 64)

Algebraic reasoning involves the representation, generalization, and compilation of patterns and the order in all aspects of mathematics (Van de Walle, 2001). That is why the branch of algebra is very important in all mathematical areas and students need to deepen their algebraic thinking skills in order to succeed in mathematics and in their own lives. Since algebra is one of the subjects that is less tangible for students than in other subjects, NCTM considers algebra challenging in school mathematics (NCTM, 2000). Due to its difficulty, algebra creates serious obstacles in the process of effective and meaningful learning in mathematics (NCTM, 2000). Students who attend algebra classes often face difficulty in understanding and working with its variables and concepts (Kieran & Chalouh, 1992). However, conceptualizing variables and manipulating them are the key features of algebra education.

One of the ways of making the process of learning algebra meaningful and effective for guiding school students is to use representations. In other words, the use of representations, by expressing algebraic concepts in various forms, including verbal, numerical, and graphical forms has an inevitable contribution to the meaningful learning of algebra (Brunner et al., 1997; Ozgun-Koca, 2001). For example, in order to understand an algebraic variable and to easily work with it, students must be engaged in the use of multiple representations, including the numerical and graphical representations of that variable, so that they can understand that the two different modes of representation show a similar mathematical concept. Many discussions and suggestions have been made in relation to teaching algebra (Davies, 1988; Koedinger & Nathan, 2000; McGregor & Price, 1999; Wagner, 1983; Wagner & Kieran, 1999; Yerushalmy & Gilead, 1997). Most mathematics teachers complain that students consider algebra only as a process of manipulating symbols and obtaining correct results (Blanton & Kaput, 2003; Kaput, 1986; Moseley & Brenner; 1997; Pirie & Martin, 1997; van Dyke & Craine, 1997). They claimed that the students in algebra classes tend to only use the symbolic equation in representing the concept of algebra; they mostly avoid the other types of representations. McCoy et al. (1996) acknowledged that:

> *The traditional algebra curricula where the students learn to simplify algebraic expressions by manipulating the symbols, and solve equations with its low relation to the real world is no more enough. There is a need to develop real-world algebraic models for the students, using exercises related to multiple representations.* (p. 42)

Knowing this problem in the algebra classes, it is considered that students need to learn the types of representations and their interpretation and teachers need to introduce the concepts of representations to the students. After reviewing the Lesh Multiple Representational Translations Model (LMRTM), and the Janvier's Representational Translations Model (JRTM), McCoy et al. (1996) combined these two representational models for the better the conceptual understanding in algebra classes. After deep study of the Lesh model, it can be seen that his definition of representation has similarities with the definition proposed by Janvier (1987a). Lesh defined representation as the external—and thus visible—visualization of the students' conceptualization of internal representations (Lesh, Post, & Behr, 1987). According to Lesh (1979) and Janvier (1987b), a conceptual understanding relies on the students' experience in presenting content in each of the representational modes. Explaining Lesh's view, Cramer & Bezuk (1991) reported that mathematics understanding can be defined as the ability to represent a mathematical idea in multiple ways and to establish relationships between different modes of representation.

Focusing on high school language-based learning-disabled (LD) students, Sauriol (2013) evaluated the effectiveness of a new Algebra 1 course via three studies. As an introduction to algebra, the priority of this new course was to teach the relationship between functions and graphs. Based on the data obtained from the first study, it was found that compared to the control group, the experimental group had significantly better results in their post-test. The second study which had been conducted two years later, similarly showed significantly the better result for the experimental group. However, the third study did not show a significant difference between the two groups, although the experimental group performed somewhat better than control group. They both showed a high performance on their posttest. These results provide support for this point of view, according to which using graphical representations of functions as an introduction to the course of Algebra, instead of focusing on alpha-numeric representations, is considered more appropriate as it allows long lasting learning to the LD students.

According to Rau and Matthews (2017), teachers frequently depend on visual representations to make complex mathematics concepts tangible for the students. Multiple representations are usually used in teaching, for it is not possible to show all aspects of a mathematics concept by using only one type of representation (Rau and Matthews, 2017). In the study by Rau and Matthews (2017), the research-based principles were reviewed to find the effective ways of applying multiple representations in a way that enhanced the students' learning. Fractions, as an illustrative domain, were used by the researchers for discussing the ways through which the visual representation might have an impact on the students' learning according to the conceptual aspects highlighted by the representation for the content that is supposed to be learned. They also proposed ways of helping students to associate between individual representations and to interpret them.

According to Murata and Stewart (2017), the key to help student learning is to use mathematical representations effectively. The study by Murata and Stewart (2017), illustrated some lesson examples in which the mathematical practices are facilitated due to the effective uses of representations while first graders use place value to solve addition problems. Students look at concepts through multiple dimensions through the use of different representations. Since visual representations provide a meaningful space and visual references for the participation of the students, they are considered helpful to involve students in mathematical discussion. Students' discussion of ideas, along with the careful representations of them can effectively facilitate the important aspects of these practices.

METHODS

To begin the research, it is necessary to determine the type of research in fundamental and applied terms. This research is applied. In this study, a semi-experimental method was used. In research that does not allow the researcher to completely control or manipulate the affective—independent—variable or variables, he tries to make his method closer to the empirical method by identifying the variables and expanding the required knowledge. But he can use a method called semi-experimental research method to study and examine the situation. In this study, the teaching of the experimental group was conducted using the representational method, and in the control group, the teaching was conducted in the traditional way. In the experimental group, the researcher mainly valued and stressed the conversions among representational modes.

The study has a main goal, to investigate the teaching based on representations in experimental group versus traditional teaching in control group. We compare the experimental groups as one new intervention versus the traditional group through performance on pretests in experimental and control groups. The pretest was a covariate variable, therefore, we applied semi-experimental research method and the analysis of its results was ANCOVA. Through ANCOVA, we investigated the effect of covariate variable.

Participants

The population of the study included 252 female students in a high school for girls in Tehran's 4th educational district. In tenth grade, students have to learn a lot of Algebra content, therefore, the researchers considered tenth graders. According to the purposive samples, the 10th grade students were considered, the number of which was 86 students. Regarding the number, two groups of experimental and control were randomly selected. The experimental group consisted of 43 students and the control group included 40 students, both of the groups consisted of the students from all high school fields of mathematics, experimental sciences, and humanities. 10th graders have three fields: mathematics, experimental sciences, and humanities with same base of Algebra in their mathematical books. These students have problems in solving algebraic problems.

Instrumentations

In this study, researcher-made mathematical tests—a pretest and a posttest—were used. These tests were designed in accordance with the content of the 10th grade mathematical textbook. Both tests contained ten questions from the algebraic topics in the textbook. The pretest of both control and experimental groups was out of 20 points. The posttest of both control and experimental groups was out of 30 points. The validity of mathematical tests was calculated and estimated according to CVR and CVI indices. The opinions of five skilled experts were recorded about the content of the tests and the content validity of both tests was confirmed according to the criteria of the two aforementioned indicators. To test the reliability of the mathematics test, the Guttman spilt-half coefficient was used. In this test method, the test was administered to a group of 30 students, and then the test was divided into two halves. The best way to split the test into two halves is to put odd numbered questions into one group and put even numbered questions into the other group. The correlation coefficient obtained from the scores of the two halves of the test is the reliability coefficient of each of the two halves of the tests. For this purpose, statistical software was used to examine and measure the Guttman spilt-half coefficient for mathematics tests. Finally, the values of 0.75 and 0.72 were respectively obtained for the reliability of both mathematics tests.

Samples of Teaching Representations

The present study provides a basis for the mathematical learning of algebraic concepts by the 10th grade students for recognizing and using the representations—symbolic, graphical, and numerical—in the teaching of a first-degree equation course, to help the growth and development of the students' algebraic thinking. The following are examples of teaching in numerical, symbolic and graphical methods.

Numerical Representation

Numerical representations, which are comprised of numerical data, are tools that students use to think about a mathematical concept and associate those data to that concept. In this method, the students are asked to get to the answer by relying on their previous acquired knowledge of numbers and the relationships between them and to mentally sort out this information in line with the situation of the problem they are facing. In this method, the students can usually guess the answer by taking very simple steps in a few quick stages.

Example: In 20 years, Mahtab will be five times older than she is today. What is her current age?

x	x+20	5x
0	20	0
1	21	5
2	22	10
3	23	15
4	24	20
5	25	25

Figure 2. A view of teaching in numerical representation method

The equilibrium boundary is where the two columns are supposed to be equal. As can be seen, using numerical calculations, we get to Mahtab's age very simply and through some steps. The interesting point is that after solving a few of such problems, the students can reduce their solving steps and gradually recognize that in facing any situation, they know what number is better to start with to get the answer faster and more accurately. In lesson plans, the teachers first used the problems that only needed integers to solve, followed by using fractional numbers familiar with the concepts of everyday life such as a half, a third,…, or twice, triple, etc.

Symbolic Representation

Symbolic representation refers to the symbolizations students use in written form for thinking about a mathematical concept and associations they have for those symbols to that concept. Symbolic representations include names, symbols, principles, and descriptions. This representation, which is the common representation in students' textbooks, is sometimes referred to as the algebraic solution method. In fact, students discover the problem's unknown parameter or parameters using the given information.

Example: Ali's score on the theoretical exam of computer lesson was 3 times higher than Arash's score. Ali and Arash won high positions in a blogging contest. The teacher added 2 points to Ali's score and 6 points to Arash's scores as their practical scores and so their scores became equal. Calculate the initial scores of Ali and Arash.

$$x + 6 = 3x + 2 \Rightarrow x = 2$$

In this method, we can easily use the help of the symbols, and like the example below, we can even ask the students to write the first-order equation: Bahram wants to find three consecutive numbers, the total of which is 81; write a first-order equation by which we can find this number.

$$x + (x + 1) + (x + 2) = 81 \Rightarrow 3x + 3 = 81$$
$$3x = 78 \Rightarrow x = \frac{78}{3} = 26$$

Graphical Representation

Graphical representation refers to the images that the students intuitively use to think about a mathematical concept or how they associate that image to a concept. Drawing is a phenomenon that all students are familiar with from very young ages. Gradually, the more advanced their imagination becomes, the more creative their drawings become.

Example: The perimeter of an equilateral triangle is equal to the environment of a square. If the length of one side of the triangle is 2/4, what is the length of each side of the square?

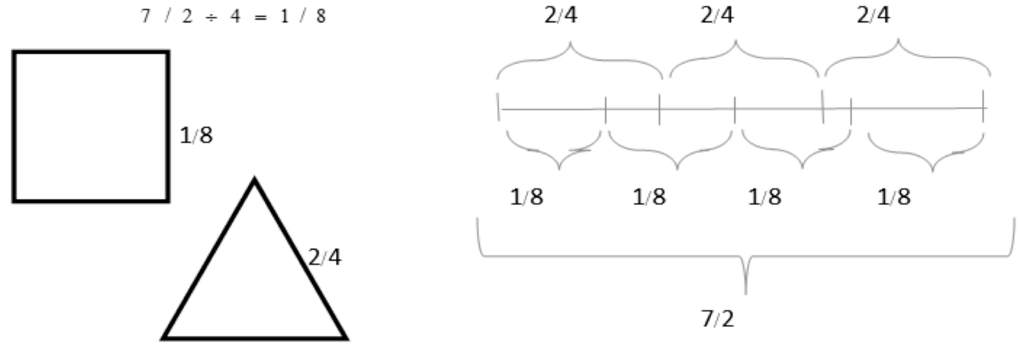

Figure 3. A view of teaching in graphical representation method

FINDINGS

After data collection and after using the new and traditional intervention in the control and experimental groups, in this section, we present the descriptive statistics including central indicators and dispersion in Table 1 for each of the variables.

Table 1
Descriptive Statistics

Test type	Groups	N	Std	Mean
Pretest	Control	40	37.4	14.48
	Experimental	43	45.3	12.82
Posttest	Control	40	4.37	22.20
	Experimental	43	2.35	27.74

As shown Table 1, the means of pretests between the two groups of control and experimental before and after the intervention are somewhat equal and are not significantly different. Moreover, the distribution of posttest and pretest scores in the two groups of control and experimental before and after the intervention was higher in the control group compared to the experimental group. One of the assumptions of Analysis of variance—ANOVA—test is the homogeneity analysis of error variances. For this purpose, the Levene test was used and its result is shown in Table 2. If the p-value is greater than 0.050, the homogeneous assumption is confirmed at a significance level of 0.050.

Table 2
The Results of the Levene Test for Homogeneity of Variances

P-value	df2	df1	F-statistic
0.74	81	1	1.50

According to the results of Table 2, it was found that the P-value is greater than 0.050, therefore it is not significant for any of the variables and it can be claimed that the homogeneity condition of the error variances was met. The assumption of the normal distribution of data—scores—is another assumption of covariance analysis. To test this assumption, K-S and SH-W tests were used and the results are shown in Table 3.

Table 3
The Results of K-S and SH-W Tests to Check the Normal Distribution of Data

		SH-W test		K-S test	
		P-value	Test statistic	P-value	Test statistic
Pretest	Control	07.0	0.93	0.08	0.12
	Experimental	11.0	0.95	0.20	0.09
Posttest	Control	0.25	0.96	0.20	0.08
	Experimental	0.55	0.83	0.14	0.20

According to the results of Table 3, since the p-values are larger than 0.050, the condition of normalization of data is established in the examination of both K-S and SH-W tests, so the distribution of scores in the research variables is normal. Regarding the subject of the study, the population of this research included all 10th grade female students in all three high school fields of mathematics, experimental sciences, and humanities from the fourth district of Tehran during the academic year 2016-2017. In this research, the available sampling method was used which included 83 students. Therefore, randomization of samples is also considered as one of the assumptions of variance analysis. The assumption of homogeneity slope of regression is another assumption of variance analysis. To test this assumption, the interaction effect test was used and the result is shown in Table 4.

Table 4
Results of Homogeneity Slope of Regression

Variable	Source of changes	SST	df	Mean-square	F	P-value
Math scores	Pretest group	846.56	2	423.281	43.97	0.08
	Error	770.00	80	9.62		
	Total	53792.00	83			

According to the results of Table 4, for P-values which show the impact of interaction of the independent group * the pretest Covariate, for the variables are greater than 0.050, the statistics magnitude is 43.970 and the test power is 0.08. Therefore, the hypothesis of Homogeneity slope of regression is accepted. Finally, the final results are shown in Table 5.

Table 5
Results of the Test of Between-Subject Effects

Variable source	Dependent variables	F	Mean Square	ETA	P-value
Group	Post-test	177.93	1051.83	0.69	0.000

According to the results of Table 5, the results of Test of Between-Subject Effects with regard to the pre-test scores as auxiliary variables showed that using representations lead to a significant difference between the control and experimental groups—$p < 0.05$. Therefore, the representation-based teaching—$p < 0.05$ and $F = 177/93$—with $ETA = 0.69$, was effective on the 10^{th} grade students. Therefore, the null hypothesis is rejected and the research hypothesis is accepted and with 95% confidence we conclude that there is a significant difference between the mean of mathematical scores in the two groups of control and experimental.

Conclusion

Representation appears as an initial solution to mathematical activities. Therefore, representation as an introduction to formal mathematical activities is important. Regarding the role of mathematics in all areas of science and business, paying attention to its teaching methods is very important and in this regard, methods that make students interested in mathematics and make it easy to teach and learn should be used. Therefore, as an influential factor in the development of mathematics, representation should be addressed by mathematics teachers in the areas of teaching and its expression and its domain should be specified.

In this study, it has been shown that the use of representations had a positive effect on learning algebraic concepts. The review of the results of this study in the showed that using multiple representation teaching method increases students' ability to solve problems on algebraic concepts, and the students who had been trained by representational method compared to students who had trained by non-representational methods solved mathematical problems better and had better performance. Using representation-based teaching methods and using the tools and providing numerical, symbolic, and graphical instruction make the students more capable of developing the abstract thinking which is the ultimate goal of mathematical learning. In this case, students will have more tools and strategies to address and solve a problem. By looking at the examples, shapes, examining simple and specific modes, and using past experiences, students will be able to understand the formulas and problems. Using multiple representation methods helps students to become creative and provide different examples and applications; thereby their modeling and analysis power will be increased.

The effectiveness of representations-based teaching on the students' performance in Algebra might be due to different reasons. Multiple representations-based teaching methods provide different skills for students, including the importance of translational skills in algebra problem solving, visualization in algebraic objects, and associations between algebraic ideas. This type of teaching helps the students to understand algebraic concepts meaningfully and to avoid rote memorization of these concepts. Through the representations-based type of teaching, students conceptualize algebraic objects and enhance their conceptual understanding of algebra. The findings of this study are in line with the results of earlier studies that provided evidence for the effectiveness of representation-based teaching in involving learners in meaningful learning of algebra. In the traditional algebra teaching method, the conversions among representations may happen only when the students are asked to draw a graph. In such classes, in an example on linear equations, instead of representing the given equation in a table, only identifying two points where the line passes through was enough, of which a graph could be drawn, using this information. Nevertheless, the conversions among different representational modes are highlighted by the representations-based teaching method. Hence, there would be an opportunity for the students to notice that there are several ways to represent one mathematical concept and these ways can be complementary for understanding the concept. The same task in which the students are asked to draw a graph of a given linear equation is taken by the students in a way that they analyze the equation through graphs, tables, plain language, and daily life situations. In this regard, the goal of the task is not to just draw the graph of an equation; it is instead a way of interpreting the current mathematical situation.

In order to improve their mathematical reasoning, the students are required to learn the conventional modes of representations. To establishing a conceptual relationship between conventional modes and these representational modes, idiosyncratic representations which belong to the individual and are specific to certain problems, must be valued in mathematics classes. Moreover, the findings of this study, consistent with the results of previous studies, show that the use of representations provides opportunities to make conversions among representational modes. Findings indicated that after conducting the representations-based teaching method, making connections between the different representational modes became easier for the students. For the most part, the conversions from the symbolic mode of representation to the graphical or tabular modes of representations were helpful for the students to notice that the end point of algebra tasks was not limited only to the symbolic mode of representation. In general, students noticed that equations or the algebraic concepts were in fact the last achieving point, and the application of all other representational modes was only for the purpose of reaching this representational mode. Another result obtained from the experimental group was that students became better problem solvers due to the use of representations-based teaching method, because they were engaged in solving the thematic activities that each of them had, using representations to deal with this problematic situation. For example, they made a conversion from daily life situation. In the literature, it was claimed that for students to become better problem solvers, they need to make conversions among representations and use multiple representations in a flexible way.

The students in experimental group expressed comfortable feeling towards encountering different modes of representations. This study provides support for this point of view which claims that establishing connections between and within the representational modes enhances mathematical understanding. Generally, representations of mathematical concepts are considered as the means for the meaningful learning, and powerful communicating and lead to conceptual understanding. Therefore, according to the above, suggestions for further research are presented:

- Implementing and reviewing the use of representations by developing the creativity of the students in using the representations without teacher's guidance.
- Examining the extent of learning the concepts being taught by the use of representation after a longer period of time.
- Using representation for other lessons and in other grades and levels of study and assessing and evaluating the impacts of this type of teaching.
- Implementing the use of representation, along with the auditory and motor representations in mathematical classes, and examining the results of the students' mathematical performance after teaching based on this method.

- Holding specialized training courses in the field of teaching by the animation-based software method for teaching algebraic concepts, and evaluating and comparing the results with the traditional teaching methods.
- Proposing and implementing a representation-assisted teaching plan by producing electronic contents for mathematics and geometry.
- Implementing and studying the use of representations in order to examine the performance differences of the students based on the factor of gender.

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Received: February, 2018. Accepted: July, 2018.
doi: 10.30827/pna.v13i1.6903

ISSN: 1887-3987