

# The Frobenius number in the set of numerical semigroups with fixed multiplicity and genus

Aureliano M. Robles-Pérez<sup>\*†</sup> and José Carlos Rosales<sup>\*‡</sup>

## Abstract

We compute all possible numbers that are the Frobenius number of a numerical semigroup when multiplicity and genus are fixed. Moreover, we construct explicitly numerical semigroups in each case.

**Keywords:** Numerical semigroups, Frobenius number, multiplicity, genus.

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## 1 Introduction

Let  $\mathbb{Z}$  and  $\mathbb{N}$  be the set of integers and the set of nonnegative integers respectively. A *numerical semigroup* is a subset  $S$  of  $\mathbb{N}$  that is closed under addition,  $0 \in S$  and  $\mathbb{N} \setminus S$  is finite.

If  $A \subseteq \mathbb{N}$  is a nonempty subset of  $\mathbb{N}$ , then we denote by  $\langle A \rangle$  the submonoid of  $(\mathbb{N}, +)$  generated by  $A$ , that is,

$$\langle A \rangle = \{\lambda_1 a_1 + \dots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, a_1, \dots, a_n \in A, \text{ and } \lambda_1, \dots, \lambda_n \in \mathbb{N}\}.$$

It is well known (see for instance [24]) that  $\langle A \rangle$  is a numerical semigroup if and only if  $\gcd(A) = 1$ , where  $\gcd$  means *greatest common divisor*.

If  $S$  is a numerical semigroup and  $S = \langle A \rangle$ , then we say that  $A$  is a *system of generators* of  $S$ . In addition, if  $S \neq \langle X \rangle$  for all  $X \subsetneq A$ , then we say that  $A$  is a *minimal system of generators* of  $S$ . In [24] it is proved that every numerical semigroup admits a unique minimal system of generators. Moreover, such a minimal system of generators is finite. In this way, the cardinality of the minimal system of generators of  $S$  is called the *embedding dimension* of  $S$  and denoted by  $e(S)$ .

Since  $\mathbb{N} \setminus S$  is finite, we can study three notable invariants of  $S$ . Namely, the maximum integer that does not belong to  $S$ , which is called the *Frobenius number* of  $S$  and denoted by  $F(S)$ ; the least positive integer that belongs to  $S$ , which is called the *multiplicity* of  $S$  and denoted by  $m(S)$ ; and the cardinality of the set  $\mathbb{N} \setminus S$ , which is called the *genus* of  $S$  and denoted by  $g(S)$ .

The Frobenius problem (see [18]) consists in finding formulas for the Frobenius number and the genus of a numerical semigroup in terms of its minimal system of generators. This problem was solved by Sylvester and Curran Sharp (see [28], [29], [30]) when the embedding dimension is equal to two ( $e(S)=2$ ). At present, the Frobenius problem is open for the case  $e(S) \geq 3$ . Indeed, Curtis showed in [8] that it is impossible to find a polynomial formula (that is, a finite set of polynomials) that computes the Frobenius number if the embedding dimension equal to three. Moreover, using Turing reductions, Ramírez-Alfonsín proved in [17] that this problem is NP-hard for arbitrary embedding dimension. Despite of this, there exists a long list of papers about this topic.

With Selmer's formulas (see [26]), the computation of an Apéry set (see [2]) solves the Frobenius problem. Let  $S$  be a numerical semigroup and  $s \in S \setminus \{0\}$ . The *Apéry set of  $s$  in  $S$*  is  $\text{Ap}(S, s) = \{x \in S \mid x - s \notin S\}$ . It is known that this set is finite and, in addition, we have the expressions  $F(S) = \max(\text{Ap}(S, s)) - s$  and  $g(S) = \frac{1}{s} \left( \sum_{w \in \text{Ap}(S, s)} w \right) - \frac{s-1}{2}$ .

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<sup>†</sup>Departamento de Matemática Aplicada, Universidad de Granada, 18071-Granada, Spain.  
E-mail: [arobles@ugr.es](mailto:arobles@ugr.es)

<sup>‡</sup>Departamento de Álgebra, Universidad de Granada, 18071-Granada, Spain.  
E-mail: [jrosales@ugr.es](mailto:jrosales@ugr.es)

In some cases a recursive approach can be used. Let  $\{a_1, \dots, a_n\}$  be the minimal system of generators of a numerical semigroup  $S$ , with  $\gcd(\{a_1, \dots, a_{n-1}\}) = d$ . If we denote by  $S' = \langle \frac{a_1}{d}, \dots, \frac{a_{n-1}}{d}, a_n \rangle$ , then it is well known (see [7], [13], [20]) that  $F(S) = d \cdot F(S') + a_n(d-1)$  and  $g(S) = d \cdot g(S') + \frac{1}{2}(a_n - 1)(d-1)$ .

The ideas of the above paragraphs (together with “L-shaped” regions of lattice points, Euclidean algorithms, continued fractions expansions of certain quotients and Hilbert series) are used, among others, by Brauer and Shockley ([6], [7]), Davison ([9]), Djawadi and Hofmeister ([10]), Greenberg ([12]), Johnson ([13]), Kannan ([14]), Rodseth and Ramírez Alfonsín ([20], [21], [19]), Selmer and Beyer ([26], [27]), in order to solve the Frobenius problem. In most cases, the authors give algorithms and/or lower and upper bounds. Moreover, in some particular settings, exact formulas are provided (see [23]). For more details about these and others works in this line, we refer the reader to the book by Ramírez Alfonsín ([18]) and, in addition, an expository paper by Owens ([16]).

Geometrical tools (in particular, geometric interpretations of continued fractions) were suggested by Arnold (see [3], [4]) for studying the Frobenius problem. Some authors, as Aicardi (see [1]) and Ustinov (see [31]) have used this approach.

Also, Gröbner bases are being used to obtain efficient algorithms to solve the Frobenius problem. Following this via, we should cite the papers by Roune (see [25]), Márquez-Campos, Ojeda, and Tornero (see [15]), and the references therein.

The reader not familiar with the study of numerical semigroups can think that some of the used notation (embedding dimension, genus, multiplicity, ...) is surprising. However, in the literature there are many works devoted to the study of the analytically irreducible one dimensional local domains via their value semigroups, which turn out to be numerical semigroups. All the invariants that are mentioned in this paper have their interpretation in this context, and this is the reason of their names. In this sense, [5] is a very adequate reference in order to translate results between the numerical semigroups theory and the rings theory.

We will denote by  $\mathfrak{L}(m, g)$  the set of all numerical semigroups with multiplicity  $m$  and genus  $g$ . Let us observe that, if  $S$  is a numerical semigroup and  $1 \in S$ , then  $S = \mathbb{N}$ . Therefore, if  $\mathfrak{L}(1, g) \neq \emptyset$ , then  $g = 0$  and  $\mathfrak{L}(1, 0) = \{\mathbb{N}\}$ . From now on we will suppose that  $m$  is an integer greater than or equal to two. We will begin our work showing that  $\mathfrak{L}(m, g) \neq \emptyset$  if and only if  $m \leq g + 1$ . From that point on, the main purpose of this paper is to study the set  $\{F(S) \mid S \in \mathfrak{L}(m, g)\}$ .

As usual, if  $x$  is a rational number, we set  $\lceil x \rceil = \min\{z \in \mathbb{Z} \mid x \leq z\}$ . In Section 2 we will show that, if  $S \in \mathfrak{L}(m, g)$ , then  $F(S) \geq \left\lceil \frac{mg}{m-1} \right\rceil - 1$ . Moreover, we will see that there exists a unique numerical semigroup that achieves such bound.

Let us observe that, if  $m = g + 1$ , then  $\mathfrak{L}(m, g) = \{\{0, g + 1, \rightarrow\}\}$  (where the symbol  $\rightarrow$  means that every integer greater than  $g + 1$  belongs to the set). Thus, from now on, we will suppose that  $m \leq g$ . In Section 3 we will show that, if  $S \in \mathfrak{L}(m, g)$ , then

- $F(S) \leq 2g - 1$  if  $m \nmid (2g - 1)$  (where, if  $a, b$  are positive integers,  $a \nmid b$  means that  $b$  is not divisible by  $a$ ).
- $F(S) \leq 2g - 2$  if  $m \mid (2g - 1)$  (where, if  $a, b$  are positive integers,  $a \mid b$  means that  $a$  divides  $b$ ).

In addition, we will see that both of the bounds are achieved, but not necessarily in a unique numerical semigroup.

Finally, in Section 4 we will show that

$$\{F(S) \mid S \in \mathfrak{L}(m, g)\} = \left\{ F \in \mathbb{N} \mid m \nmid F \text{ and } \left\lceil \frac{mg}{m-1} \right\rceil - 1 \leq F \leq 2g - 1 \right\}.$$

Before go the next section, we want to notice that, in order to facilitate the searching of some results and facts, together with the original references, we have included the corresponding reference in [24].

## 2 A lower bound for the Frobenius number

Let us observe that, if  $S$  is a numerical semigroup, then  $m(S) \in \mathbb{N} \setminus \{0\}$  and  $g(S) \in \mathbb{N}$ .

**Proposition 2.1.** *Let  $(m, g) \in (\mathbb{N} \setminus \{0\}) \times \mathbb{N}$ . Then  $\mathfrak{L}(m, g) \neq \emptyset$  if and only if  $(m, g) = (1, 0)$  or  $2 \leq m \leq g + 1$ .*

*Proof. (Necessity).* If  $m = 1$  and  $S \in \mathfrak{L}(1, g)$ , then it is obvious that  $S = \mathbb{N}$  and, thereby,  $g = 0$ . Therefore, we can suppose that  $2 \leq m$ . Now, if  $S \in \mathfrak{L}(m, g)$ , then  $\{1, \dots, m - 1\} \cap S = \emptyset$  and, consequently,  $g \geq m - 1$ .

*(Sufficiency).* Since  $\mathbb{N} \in \mathfrak{L}(1, 0)$ , we have that  $\mathfrak{L}(1, 0) \neq \emptyset$ . So, let us suppose that  $2 \leq m \leq g + 1$ . Let  $q, r$  be two nonnegative integers such that  $g = q(m - 1) + r$ , with  $0 \leq r \leq m - 2$  (that is,  $q$  and  $r$  are the

quotient and the remainder of the division of  $g$  by  $m - 1$ ). Let  $S = \{0, m, 2m, \dots, qm\} \cup \{qm + r + 1, \rightarrow\}$ . It is clear that  $S$  is a numerical semigroup with multiplicity  $m$  and that  $g(S) = q(m - 1) + r = g$ . Therefore,  $S \in \mathfrak{L}(m, g)$ , that is,  $\mathfrak{L}(m, g) \neq \emptyset$ .  $\square$

Our next aim will be to prove Proposition 2.3. For that we need to introduce some concepts and results.

Following the notation introduced in [22], we say that  $x \in \mathbb{Z} \setminus S$  is a *pseudo-Frobenius number* of  $S$  if  $x + s \in S$  for all  $s \in S \setminus \{0\}$ . The cardinality of the set consisting of all pseudo-Frobenius numbers of  $S$  is an important invariant of  $S$  which is called the *type* of  $S$  (see [5]) and is denoted by  $t(S)$ .

If  $S$  is a numerical semigroup, then we will denote by  $n(S)$  the cardinality of the set  $\{s \in S \mid s < F(S)\}$ . It is clear that  $g(S) + n(S) = F(S) + 1$ . On the other hand, we have the following consequence of [11, Corollary 2, Theorem 20] (in alternation, see [24, Corollary 2.23, Proposition 2.26]).

**Lemma 2.2.** *If  $S$  is a numerical semigroup, then  $g(S) \leq t(S)n(S)$  and  $t(S) \leq m(S) - 1$ . Consequently,  $g(S) \leq (m(S) - 1)n(S)$ .*

Now we are ready to show the announced result.

**Proposition 2.3.** *If  $S$  is a numerical semigroup with  $m(S) \geq 2$ , then*

$$\left\lceil \frac{m(S)g(S)}{m(S) - 1} \right\rceil - 1 \leq F(S).$$

*Proof.* By Lemma 2.2, we know that  $g(S) \leq (m(S) - 1)n(S)$ . Therefore,

$$m(S)g(S) \leq (m(S) - 1)(n(S) + g(S)) = (m(S) - 1)(F(S) + 1),$$

and then the inequality follows easily.  $\square$

In the next result we will see that it is possible to reach the bound of the previous proposition.

**Proposition 2.4.** *Let  $m, g$  be integers such that  $2 \leq m \leq g + 1$ . Then there exists  $S \in \mathfrak{L}(m, g)$  such that  $F(S) = \left\lceil \frac{mg}{m-1} \right\rceil - 1$ .*

*Proof.* Let us have  $q, r \in \mathbb{N}$  such that  $g = q(m - 1) + r$ , with  $0 \leq r \leq m - 2$ . Beginning with proof of Proposition 2.1, we know that  $S = \{0, m, \dots, qm\} \cup \{qm + r + 1, \rightarrow\} \in \mathfrak{L}(m, g)$ . In order to finish this proof, let us see that  $F(S) + 1 = \left\lceil \frac{mg}{m-1} \right\rceil$ . For that, since  $\left\lceil \frac{mg}{m-1} \right\rceil \leq F(S) + 1$  by Proposition 2.3, will be enough to show that  $F(S) + 1 - \frac{mg}{m-1} < 1$ . Let us observe that

$$F(S) = \begin{cases} qm - 1, & \text{if } r = 0, \\ qm + r, & \text{if } r \neq 0, \end{cases}$$

and  $g(S) = (m - 1)q + r$ . Thus, we distinguish two cases.

1. If  $r = 0$ , then  $F(S) + 1 - \frac{mg}{m-1} = qm - \frac{mq(m-1)}{m-1} = 0 < 1$ .
2. If  $r \neq 0$ , then  $F(S) + 1 - \frac{mg}{m-1} = qm + r + 1 - \frac{m(q(m-1)+r)}{m-1} = \frac{m-1-r}{m-1} < 1$ .

$\square$

We finish this section by showing that the semigroup of  $\mathfrak{L}(m, g)$  which reaches the bound of Proposition 2.3 is unique.

**Proposition 2.5.** *Let  $m, g$  be integers such that  $2 \leq m \leq g + 1$ . If  $S \in \mathfrak{L}(m, g)$  and  $F(S) = \left\lceil \frac{mg}{m-1} \right\rceil - 1$ , then  $S = \{0, m, \dots, qm\} \cup \{qm + r + 1, \rightarrow\}$ , where  $q, r$  are the unique non-negative integers such that  $g = q(m - 1) + r$  with  $0 \leq r \leq m - 2$ .*

*Proof.* Let  $T = \{0, m, \dots, qm\} \cup \{qm + r + 1, \rightarrow\}$ . From proof of Proposition 2.4, we know that  $T$  is a numerical semigroup with genus equal to  $g$  and such that  $F(T) = \left\lceil \frac{mg}{m-1} \right\rceil - 1$ . On the other hand, it is clear that, if  $S \in \mathfrak{L}(m, g)$  and  $F(S) + 1 = \left\lceil \frac{mg}{m-1} \right\rceil$ , then  $T \subseteq S$ . Since  $g(T) = g = g(S)$ , we conclude that  $S = T$ .  $\square$

### 3 An upper bound for the Frobenius number

Now we want to give the upper bound of the set  $\{F(S) \mid S \in \mathfrak{L}(m, g)\}$ . In contrast with the given lower bound in the previous section, we will show that the upper bound “almost” does not depend of  $m$ .

The following result is [24, Lemma 2.14].

**Lemma 3.1.** *If  $S$  is a numerical semigroup, then  $F(S) \leq 2g(S) - 1$ .*

**Proposition 3.2.** *Let  $m, g$  be integers such that  $2 \leq m \leq g + 1$ . If  $S \in \mathfrak{L}(m, g)$ , then*

1.  $F(S) = g$  if  $m = g + 1$ .
2.  $F(S) \leq 2g - 1$  if  $m \leq g$  and  $m \nmid (2g - 1)$ .
3.  $F(S) \leq 2g - 2$  if  $m \leq g$  and  $m \mid (2g - 1)$ .

*Proof.* 1. Evident.

2. It is an immediate consequence of Lemma 3.1.

3. By Lemma 3.1, it is sufficient to show that  $F(S) \neq 2g - 1$ . And this is clear because, if  $F(S) = 2g - 1$ , then  $2g - 1 \notin S$ , in contradiction with the fact that  $m \in S$  and  $m \mid (2g - 1)$ . □

Our next goal will be show that the bounds of the previous proposition are reached. For that, we introduce several concepts and results.

A numerical semigroup is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it. Irreducible numerical semigroups with odd (even, respectively) Frobenius number are called *symmetric* (*pseudo-symmetric*, respectively). From [11, Lemma 1, Lemma 3], we have the following result (alternatively see [24, Corollary 4.5]).

**Lemma 3.3.** *Let  $S$  be a numerical semigroup.*

1.  $S$  is symmetric if and only if  $g(S) = \frac{F(S)+1}{2}$ .
2.  $S$  is pseudo-symmetric if and only if  $g(S) = \frac{F(S)+2}{2}$ .

By [24, Theorem 4.2, Lemma 4.3, Proposition 4.4], we have the next lemma.

**Lemma 3.4.** *Let  $\mathfrak{L}(F)$  be the set of all numerical semigroups with Frobenius number equal to  $F$ . If  $S \in \mathfrak{L}(F)$ , then*

1.  $S$  is irreducible if and only if  $S$  is maximal (with respect the set inclusion) in  $\mathfrak{L}(F)$ .
2. If  $h = \max \{x \in \mathbb{N} \setminus S \mid F - x \notin S \text{ and } x \neq \frac{F}{2}\}$ , then  $S \cup \{h\} \in \mathfrak{L}(F)$ .
3.  $S$  is irreducible if and only if  $\{x \in \mathbb{N} \setminus S \mid F - x \notin S \text{ and } x \neq \frac{F}{2}\} = \emptyset$ .
4.  $S \cup \{x \in \mathbb{N} \setminus S \mid F - x \notin S \text{ and } x > \frac{F}{2}\}$  is an irreducible numerical semigroup with Frobenius number equal to  $F$ .

*Remark 3.5.* The first assert of the above lemma is a direct consequence of [11, Proposition 4].

**Proposition 3.6.** *Let  $m, g$  be integers such that  $2 \leq m \leq g$ .*

1. If  $m \nmid (2g - 1)$ , then there exists  $S \in \mathfrak{L}(m, g)$  such that  $F(S) = 2g - 1$ .
2. If  $m \mid (2g - 1)$ , then there exists  $S \in \mathfrak{L}(m, g)$  such that  $F(S) = 2g - 2$ .

*Proof.* 1. It is clear that  $\bar{S} = \langle m \rangle \cup \{2g, \rightarrow\}$  is a numerical semigroup with  $m(\bar{S}) = m$  and  $F(\bar{S}) = 2g - 1$ . By Item 4 of Lemma 3.4, we have that  $S = \bar{S} \cup \left\{x \in \mathbb{N} \setminus \bar{S} \mid F(\bar{S}) - x \notin \bar{S} \text{ and } x > \frac{F(\bar{S})}{2}\right\}$  is a symmetric numerical semigroup with  $F(S) = 2g - 1$ . Let us observe that, if  $x$  is an integer greater than  $\frac{F(\bar{S})}{2}$ , then  $x \geq \lceil \frac{2g-1}{2} \rceil = g \geq m$ . Thereby,  $m(S) = m$ . Moreover, by Lemma 3.3, we have that  $g(S) = \frac{F(S)+1}{2} = g$ . Therefore,  $S \in \mathfrak{L}(m, g)$  and  $F(S) = 2g - 1$ .

2. It is clear that  $\bar{S} = \langle m \rangle \cup \{2g - 1, \rightarrow\}$  is a numerical semigroup with  $m(\bar{S}) = m$  and  $F(\bar{S}) = 2g - 2$  (let us observe that, if  $m \mid (2g - 1)$ , then  $m \nmid (2g - 2)$ ). Following the same reasoning as in the previous item, the proof is complete.  $\square$

We will finish this section with several examples in which the numerical semigroups, that reach the given bounds by Items 2 and 3 of Proposition 3.2, are not unique.

**Example 3.7.** It is easy to check that  $\langle 5, 6, 9 \rangle, \langle 5, 7, 9, 11 \rangle \in \mathfrak{L}(5, 7)$ . Moreover,  $5 \nmid (2 \times 7 - 1)$  and  $F(\langle 5, 6, 9 \rangle) = F(\langle 5, 7, 9, 11 \rangle) = 2 \times 7 - 1 = 13$ .

**Example 3.8.** It is clear that  $\langle 7, 8, 25 \rangle, \langle 7, 11, 19, 24 \rangle \in \mathfrak{L}(7, 18)$ . Moreover,  $7 \mid (2 \times 18 - 1)$  and  $F(\langle 7, 8, 25 \rangle) = F(\langle 7, 11, 19, 24 \rangle) = 2 \times 18 - 2 = 34$ .

## 4 Variability of the Frobenius number

As a consequence of Propositions 2.3 and 3.2, we know that, if  $S \in \mathfrak{L}(m, g)$ , then  $\left\lceil \frac{mg}{m-1} \right\rceil - 1 \leq F(S) \leq 2g - 1$ . Moreover, let us observe that, if  $S \in \mathfrak{L}(m, g)$ , then  $m \nmid F(S)$ . In this section, our aim will be to prove that

$$\{F(S) \mid S \in \mathfrak{L}(m, g)\} = \left\{ F \in \mathbb{N} \mid m \nmid F \text{ and } \left\lceil \frac{mg}{m-1} \right\rceil - 1 \leq F \leq 2g - 1 \right\}.$$

**Lemma 4.1.** *Let  $S, \bar{S}$  be numerical semigroups such that  $S \subseteq \bar{S}$ . Let us suppose that  $\bar{S} \setminus S = \{x_1, \dots, x_r\}$  with  $x_1 > \dots > x_r$ . Then  $S \cup \{x_1, \dots, x_i\}$  is a numerical semigroup for each  $i \in \{1, \dots, r\}$ .*

*Proof.* It is enough to observe that the sum of two elements of  $\{x_1, \dots, x_i\}$ , or the sum of an element of  $S$  and an element of  $\{x_1, \dots, x_i\}$ , is another element of  $\bar{S}$  that is greater than or equal to  $x_i$ . Therefore, this another element belongs to  $S \cup \{x_1, \dots, x_i\}$ .  $\square$

If  $x$  is a rational number, then we will denote  $\lfloor x \rfloor = \max\{z \in \mathbb{Z} \mid z \leq x\}$ . On the other hand, if  $a, b$  are positive integers, then we will denote  $a \bmod b = a - \lfloor \frac{a}{b} \rfloor b$ . Let us observe that  $a = \lfloor \frac{a}{b} \rfloor b + a \bmod b$ , and that  $a \bmod b \in \{0, \dots, b - 1\}$ .

**Lemma 4.2.** *Let  $m, F, g$  be positive integers. Then  $F - \lfloor \frac{F}{m} \rfloor \geq g$  if and only if  $F + 1 \geq \left\lceil \frac{mg}{m-1} \right\rceil$ .*

*Proof. (Necessity).* If  $F - \lfloor \frac{F}{m} \rfloor \geq g$ , then  $Fm - \lfloor \frac{F}{m} \rfloor m \geq gm$ . Thus,  $Fm - (F - F \bmod m) \geq gm$ , and, thereby,  $F \geq \frac{mg}{m-1} - \frac{F \bmod m}{m-1} \geq \frac{mg}{m-1} - 1$ . Therefore,  $F + 1 \geq \left\lceil \frac{mg}{m-1} \right\rceil$ .

*(Sufficiency).* If  $F + 1 \geq \left\lceil \frac{mg}{m-1} \right\rceil$ , then  $F + 1 \geq \frac{mg}{m-1}$ , that is,  $Fm - F + m - 1 \geq mg$ . Consequently,  $F - \lfloor \frac{F}{m} \rfloor + \frac{m-1-F \bmod m}{m} \geq g$ . Since  $F - \lfloor \frac{F}{m} \rfloor$  and  $g$  are integers, and  $0 \leq \frac{m-1-F \bmod m}{m} < 1$ , we conclude that  $F - \lfloor \frac{F}{m} \rfloor \geq g$ .  $\square$

Now, we are prepared to prove the result that we announced at the beginning of this section.

**Theorem 4.3.** *Let  $m, g, F$  positive integers such that  $2 \leq m \leq g$  and  $m \nmid F$ . Then there exists  $S \in \mathfrak{L}(m, g)$  such that  $F(S) = F$  if and only if  $\left\lceil \frac{mg}{m-1} \right\rceil - 1 \leq F \leq 2g - 1$ .*

*Proof. (Necessity).* It is an immediate consequence of Propositions 2.3 and 3.2.

*(Sufficiency).* Depending on  $m < \frac{F}{2}$  or  $m > \frac{F}{2}$ , we distinguish two cases.

- Let us suppose that  $m < \frac{F}{2}$ . Let  $S = \{0, m, \dots, \lfloor \frac{F}{m} \rfloor m\} \cup \{F + 1, \rightarrow\}$ . It is clear that  $S$  is a numerical semigroup with multiplicity  $m$  and Frobenius number  $F$ . Moreover,  $g(S) = F - \lfloor \frac{F}{m} \rfloor$ . Since  $F + 1 \geq \left\lceil \frac{mg}{m-1} \right\rceil$ , by Lemma 4.2, then we have that  $g(S) \geq g$ . From Item 4 of Lemma 3.4, there exist integers  $x_1, x_2, \dots, x_r$  such that  $x_1 > x_2 > \dots > x_r > \frac{F}{2} > m$  and  $\bar{S} = S \cup \{x_1, \dots, x_r\}$  is an irreducible numerical semigroup with Frobenius number  $F$ . In addition, it is clear that  $m(\bar{S}) = m$  and, by Lemma 3.3, that  $g(\bar{S}) = \left\lceil \frac{F+1}{2} \right\rceil$ . Now, let us observe that  $g(S) \geq g \geq \left\lceil \frac{F+1}{2} \right\rceil = g(\bar{S})$ . Applying Lemma 4.1, we have that  $S' = S \cup \{x_1, \dots, x_{g(S)-g}\}$  is a numerical semigroup. Moreover, we have easily that  $m(S') = m$  and  $F(S') = F$ . Finally,  $g(S') = g(S) - (g(S) - g) = g$ .

2. Let us suppose that  $m > \frac{F}{2}$ . The set  $\{m, \dots, F - 1\}$  has cardinality  $F - m$ . Since  $m \leq g$ , then  $F - m \geq F - g$ . Therefore, we can take a subset  $A \subseteq \{m, \dots, F - 1\}$  such that  $m \in A$  and the cardinality of  $A$  is equal to  $F - g$ . Being that all the elements of  $A$  are greater than  $\frac{F}{2}$ , then it is easy to deduce that  $S = A \cup \{F + 1, \rightarrow\} \cup \{0\} \in \mathcal{L}(m, g)$  and  $F(S) = F$ .

□

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