Advances in the study of local and nonlocal partial differential equations

Programa de doctorado en Física y Matemáticas (FisyMat)

Universidad de Granada

TESIS DOCTORAL

Por

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Enero de 2018

Editor: Universidad de Granada. Tesis Doctorales
Autor: Alexis Molino Salas
ISBN: 978-84-9163-751-6
URI: http://hdl.handle.net/10481/49298

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Agradecimientos

En primer lugar quiero agradecer a mis directores David Arcoya y José Carmona su plena dedicación y apoyo, siempre han estado ahí, dispuestos a enseñarme. También quiero agradecer a mi gran amiga Lourdes Moreno, con la que he tenido la suerte de compartir director de tesis y discusiones matemáticas, a Julio Rossi, que ha hecho de mis estancias en Buenos Aires una perfecta combinación de asados y teoremas, a mis otros coautores: José Cañizo, Tommaso Leonori y Sergio Segura por los buenos ratos que hemos pasado delante de una pizarra, a los miembros del departamento de Análisis Matemático y en particular a Salvador Villegas, David Ruiz, José Luis Gámez, Antonio Cañada, Salvador López, Rafael López y Pedro Martínez por hacerme sentir que el grupo es una gran familia, a la gente de Valencia: Mazón, Sergio, Julián y Salvador, por el buen ambiente que se respira cuando se está al lado de ellos.

Por último, y no menos importante, quiero dar mi gratitud a Juan Martínez, Antonio Peralta y Paco Polo por iniciarme en el mundo del Análisis Funcional y porque creyeron en mí, sin ellos no estaría escribiendo esta tesis.

Introduction

In this doctoral thesis we deal with several relevant issues in the theory of local and nonlocal differential equations. The results presented in this manuscript are concentrated in three parts. Each part is divided into chapters. Each chapter corresponds to a paper or a preprint, as follows:

Part (I): **Nonlocal diffusion problems**;

- J.A. Cañizo and A. Molino. Improved Energy Methods for Nonlocal Diffusion Problems, *Discrete and Continuous Dynamical System. Serie A*, **18** no. 3, Art. 17 (2018).
- A. Molino and J.D. Rossi. Nonlocal diffusion problems that approximate a parabolic equation with spatial dependence, *Z. Angew. Math. Phys*, **67** no. 3, Art. 41, 14 pp. (2016).
- A. Molino and J.D. Rossi. Nonlocal approximations to Fokker-Planck equations, to appear in *Funkcialaj Ekvacioj*, (2017).
- T. Leonori, A. Molino and S. Segura de León. Parabolic equations with natural growth approximated by nonlocal equations, *submitted* (2017).

Parte (II): **Elliptic equations with singularity in the quadratic gradient term and Gelfand type problems**;

- J. Carmona, A. Molino and L. Moreno-Mérida. Existence of a continuum of solutions for a quasilinear elliptic singular problem, *J. Math. Anal. Appl.*, **436** no. 2, 1048-1062, (2016).
- J. Carmona, A. Molino and J.D. Rossi. The Gelfand problem for the $1-$homogeneous $p-$laplacian, to appear in *Adv. Nonlinear Anal.* (2017).
- A. Molino. Gelfand type problem for singular quadratic quasilinear equations, *NoDEA. Nonlinear Differential Equations and Applications*, **23** no. 5, Art. 56, 20, (2016).

Parte (III): **Some results in Elliptic Equations modeled by the $p-$laplacian**;

- A. Molino and S. Segura de León. Elliptic equations involving the $1-$Laplacian and a subcritical source term, *submitted* (2017).

- D. Arcoya, A. Molino and L. Moreno-Mérida. Existence and regularizing effect of degenerate lower order terms in elliptic equations beyond the Hardy constant, *submitted* (2017).
- A. Molino and J.D. Rossi. A concave-convex problem with a variable operator, *submitted* (2017).

Thus the memory is divided into ten chapters, each of which contains the results that have been obtained. The chapters are self-contained and can be read independently, except for the incorporation of a complete bibliography at the end of the manuscript. Although each chapter contains its own introduction concerning the problem, it has been considered convenient to present in the following summary all the results obtained in this memory. Finally, it is noted that the methodology, objectives and conclusions of this thesis are included in each chapter.

PART I: Nonlocal diffusion problems

We begin the first part of this introduction with the following nonlocal diffusion differential equation

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} \left(K(x,y)u(y,t) - K(y,x)u(x,t)\right) \mathrm{d}y, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where u_0 is the initial datum and $K : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ is the diffusion kernel which satisfies the following property:

$$\exists\, R, r > 0 \,:\, K(x,y) \geq r \text{ when } |x-y| \leq R. \tag{2}$$

Additional hypotheses about the initial datum u_0 and the kernel K will be added later.

It is interesting to observe that equation (1) is, roughly speaking, the Kolmogorov equation for Markov's process with jumping probability K and density u (Ethier and Kurtz (1986, Chapter 4.2)). This equation has an interesting physics interpretation. More precisely, if the function $u(x,t)$ is thought as the population density of a particular species at position x and at time t (with initial density $u_0(x)$) and the kernel $K(x,y)$ is thought as the probability distribution of jumping from location y to location x, then we have that $\int_{\mathbb{R}^N} K(x,y)u(y,t)\,\mathrm{d}y$ is the rate at which individuals are arriving at position x from all other places, and $-\int_{\mathbb{R}^N} K(y,x)u(x,t)\,\mathrm{d}y$ is the rate at which individuals are leaving the position x to travel to all other places. In this sense, in the absense of external or internal sources, one can easily deduce that the density function $u(x,t)$ satisfies the above equation (1). Furthermore, observe that hypothesis (2) implies that $K(x,x) > 0$ in a neighborhood of x, for all $x \in \mathbb{R}^N$. So that, from the perspective of population dynamics, it means that the probability that some individuals that are in x at time t remain at the same position is positive. Consequently, this kind of nonlocal diffusion equation is relevant in applications, for example, in the study of biological dispersal of species, image processing, particle systems, elasticity and coagulation models, see for instance, Bobaru et al. (2009); Bodnar and Velazquez (2006); Carrillo and Fife (2005); Fife (2003); Fournier and Laurençot (2006) and Hutson et al. (2003). Given an initial datum $u_0(x)$, as a particular application, we also highlight the following unidimensional model proposed by Cortázar et al. (2007)

$$u_t(x,t) = \int_{\mathbb{R}} \tilde{J}\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\,\mathrm{d}y - u(x,t), \quad x \in \mathbb{R}, t > 0, \tag{3}$$

where $\tilde{J}$ is a nonnegative, even and smooth function supported in $[-1,1]$ and whose integral is equal to 1. The function g is a continuous and positive function which accounts for the dispersal distance which depends on the departing point. So that,

g models the heterogeneity of the environment which can affect the distribution of a species through space-dependent dispersal strategies (see also Cortázar et al. (2011); Cortázar et al. (2015) and Cortázar et al. (2016)). Observe that, in this context, if we define

$$K(x,y) = \tilde{J}\left(\frac{x-y}{g(y)}\right)\frac{1}{g(y)},$$

we have $\int_{\mathbb{R}} K(y,x)\,\mathrm{d}y = 1$ and then the equation (3) is a particular case of our above model equation (1). A more general example would be the case in which

$$K(x,y) = J\big(\mathcal{M}(y)(x-y)\big)\det\mathcal{M}(y),$$

where $\mathcal{M}(y)$ is a real $N \times N$ matrix.

Even more, it is worth pointing out that if the diffusion kernel K of (1) is a symmetric function, that is, the probability of jumping from x to y is the same that the probability of jumping from y to x, then we obtain the following diffusion non local problem:

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} K(x,y)(u(y,t)-u(x,t))\,\mathrm{d}y, & x \in \mathbb{R}^N, t>0,\\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{4}$$

This equation has been widely studied during last years as well as its Dirichlet version in bounded domains, that is, the following Dirichlet problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} K(x,y)(u(y,t)-u(x,t))dy, & x \in \Omega,\ t>0,\\ u(x,t) = g(x,t), & x \notin \Omega,\ t>0,\\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5}$$

where g and u_0 are fixed (see Andreu-Vaillo et al. (2010)). We emphasize that in the case in which

$$K(x,y) = \frac{1}{|x-y|^{N+2s}}, \tag{6}$$

the equations (4) and (5) give us a particular and well known kind of problems. Concretely, it appears the fractional laplacian operator and we have the problems

$$u_t(x,t) = -(-\Delta)^s u(x,t),$$

where the integral of the singular kernel is represented by the principal value of the integral (see Valdinoci (2009)).

Asymptotic behavior

In the first Chapter of this manuscript we will deal with the decay rate of the L^p norms of the solutions of (1). We will use some tools known as "energy methods". The aim of these methods is to prove a functional inequality, for a suitable function $\mathcal{F}$, like the following one

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|_p \leq \mathcal{F}\left(\|u(\cdot,t)\|_p\right),$$

where $\|u(\cdot,t)\|_p$ denotes the $L^p(\mathbb{R}^N)$ norm. This kind of inequalities give us ordinary differential inequations. One can deduce the decay rate of the L^p norms solving the above differential inequations. Observe that this strategy (energy method) is very similar to the successful and common one known as *entropy method*. The aim of the entropy method is to compare the time derivative of a Lyapunov functional with the Lyapunov functional itself to obtain a decay rate for solutions (see Jüngel (2016), Arnold et al. (2004); Bakry and Émery (1985); Carrillo et al. (2001); Gross (1975); Otto and Villani (2000); Villani (2002) and Desvillettes and Villani (2004)). There are several advantages to the use of energy methods. Among others, they have the advantage of being quite robust, often being applicable to equations that are not explicitly solvable by Fourier transform methods, and to nonlinear problems. It is convenient to remark that if we take $J \in \mathcal{C}(\mathbb{R}^N,\mathbb{R})$ such that J is nonnegative, radial, $J(0)>0$ and whose integral is equal to 1 and we set $K(x,y)=J(x-y)$, then we obtain a model example which one can solve using Fourier transform methods. Concretely, in this case, we have the following equation

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y,t)\,\mathrm{d}y - u(x,t), & x\in\mathbb{R}^N, t>0,\\ u(x,0)=u_0(x), & x\in\mathbb{R}^N. \end{cases} \tag{7}$$

We highlight that it is equivalent to consider this nonlocal diffusion equation or to consider $u_t(x,t) = J*u - u(x,t)$ since the above integro-differential equation is in convolution form. Hence, in this case, the Fourier transform implies that $\hat{u}_t(\xi,t) = \hat{u}(\xi,t)(\hat{J}(\xi)-1)$ and so that $\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\hat{u}_0(\xi)$. From here, it is possible to obtain the asymptotic behavior (see (Andreu-Vaillo et al., 2010, Chapter 1)).

The use of energy methods for the equation (1) is not new. Indeed, for symmetric kernels, Ignat and Rossi (2008) prove, among others, the following result:

Theorem 1 *Let $N\geq 3$, $K:\mathbb{R}^N\times\mathbb{R}^N\to[0,\infty)$ symmetric satisfying (2) and $u_0\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Then, every solution of (4) satisfies*

$$\|u(\cdot,t)\|_p \leq C\,t^{-\frac{N(p-1)}{2p}},$$

for all $p\in[1,\infty)$ and t big enough, where C is a positive constant which depends on $R,r,N,p,\|u_0\|_1$ and $\|u_0\|_\infty$.

Taking into account this result, the aim of Chapter 1 is to complete it. On the one hand, we want to obtain a result for symmetric and not symmetric kernels. On the other hand, we want to find a result without any restriction about the dimension N. Concretely, we will obtain the following theorem:

Theorem 2 *Let $N \geq 1$. If we assume that $u_\infty : \mathbb{R}^N \to (0,\infty)$ is an equilibrium solution of (1) such that $1/m \leq u_\infty \leq m$, for some $m > 0$, and u is a solution to (1) with initial datum $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, $1 < p < \infty$, then there is a positive constant $C = C(r, R, N, m, p, \|u_0\|_1, \|u_0\|_p)$ such that*

$$\|u(\cdot,t)\|_p \leq C(1+t)^{-\frac{N(p-1)}{2p}},$$

for every $t \geq 0$.

It is convenient to recall that an equilibrium solution is a solution which does not depend on the variable t. Hence, we observe that if we consider a symmetric kernel K, then each positive constant is an equilibrium solution. Therefore, Theorem 2 generalizes and improves the results given by Ignat and Rossi (2008) for symmetric kernels. Even more, we obtain the same decay rate of the nonlocal equation but for every $N \geq 1$ instead of $N \geq 3$.

A direct consequence of Theorem 2 is the asymptotic behavior of the solutions of (3). Indeed, in Cortázar et al. (2007) the authors prove the existence of a positive and bounded equilibrium solution u_∞. Thus, under hypotheses of Theorem 2 one can claim that

$$\|u(\cdot,t)\|_p \leq C(1+t)^{-\frac{p-1}{2p}}, \qquad \textit{for every } t \geq 0.$$

In this first Chapter, we also consider cases in which $K(x,y) = J(x-y)$ where $J : \mathbb{R}^N \to [0,\infty)$ is radial, symmetric and integrable. Therefore, in this context, our problem (1) is

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))\,\mathrm{d}y, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{8}$$

and then, equation (7) is a particular case of this problem. Observe that hypothesis (2) is equivalent to assume that $J(z) \geq r$ when $|z| < R$. Obviously, this condition holds true if J is continuous in a neighborhood of zero and $J(0) > 0$. For this kind of kernels it is possible to obtain a more precise decay of the L^p norms of the solutions. Concretely,

Theorem 3 *Let u be a solution of the equation (8) with initial datum $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. Then, there is a constant $C = C(N,p)$ such that*

$$\|u(\cdot,t)\|_p^p \leq \begin{cases} \|u_0\|_p^p, & 0 \leq t \leq t_0, \\ \left(\|u_0\|_p^{-p\gamma} + C\gamma r R^{N+2} \|u_0\|_1^{-p\gamma}(t-t_0)\right)^{-\frac{1}{\gamma}}, & t \geq t_0, \end{cases}$$

where $\gamma := \frac{2}{N(p-1)}$ *and*

$$t_0 = \max\left\{0,\ \frac{1}{CrR^N}\log\left(R^{\frac{2}{\gamma}}\|u_0\|_1^{-p}\|u_0\|_p^p)\right)\right\}.$$

See also Theorem 1.1.4 for the decay of the L^p norms of the higher derivatives of the solutions. We point out that Theorem 3 implies that, if $t \geq t_0$ for some suitable t_0, then the decay rate of the L^p norms of the solutions of (8) is the same that the decay of the solutions of the heat equation, that is (see Giga et al. (2010)),

$$\|u\|_p^p \leq \left(\|u_0\|_p^{-p\gamma} + C\|u_0\|_1^{-p\gamma}t\right)^{-\frac{1}{\gamma}}, \qquad \text{for every } t \geq 0. \tag{9}$$

As a consequence, roughly speaking, we can say that there is a strong connection between the equation (8) and the heat equation $u_t = \Delta u$. More precisely, it is well known that if we denote by J_ε the rescaled kernel

$$J_\varepsilon(z) := \frac{C(J)}{\varepsilon^{2+N}}J\left(\frac{z}{\varepsilon}\right), \qquad \text{with } C(J)^{-1} = \frac{1}{2}\int_{\mathbb{R}^N} J(z)z_N^2\,\mathrm{d}z, \tag{10}$$

where it is assumed that the second order momentum of J is finite (and thus $C(J)$ is nontrivial), then the solution u^ε of

$$\partial_t u^\varepsilon(x,t) = \int_{\mathbb{R}^N} J_\varepsilon(x-y)((u^\varepsilon(y,t) - u^\varepsilon(x,t))\,\mathrm{d}y, \qquad x \in \mathbb{R}^N,\ t > 0, \tag{11}$$

with initial datum $u_0 \in \mathcal{C}(\mathbb{R}^N)$, converges uniformly in compact subsets of $\mathbb{R}^N \times [0,\infty)$ to the solution v of the heat equation $v_t = \Delta v$ with the same initial datum $v(x,0) = u_0(x)$ (see for instance Andreu-Vaillo et al. (2010) and Rey and Toscani (2013)). As a consequence, if the solutions u^ε tend to the solution of the heat equation, one may wonder if the decay is preserved in the rescaling that leads to the heat equation. That is, we want to know if it is possible to find some ε_0 such that the asymptotic behavior of u^ε is exactly the expression (9) for every $\varepsilon < \varepsilon_0$ and $t \geq 0$. In the following theorem we give the answer to the above question.

Theorem 4 *Let u^ε be a solution of (11) with initial datum $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ and $p \in [2,\infty)$. Then*

$$\|u^\varepsilon(t,\cdot)\|_p^p \leq \left(\|u_0\|_p^{-p\gamma} + C_1\|u_0\|_1^{-p\gamma}(t-t_0)\right)^{-\frac{1}{\gamma}} \qquad \textit{for all } t \geq t_0,$$

where $C_1 = C(N,p)\gamma r R^{N+2}C(J)$ does not depend on ε and

$$t_0 = \max\left\{0,\ \frac{\varepsilon^2}{CrR^NC(J)}\log\left(\varepsilon^{\frac{2}{\gamma}}R^{\frac{2}{\gamma}}\|u_0\|_1^{-p}\|u_0\|_p^p)\right)\right\}.$$

In particular, $t_0 = 0$ for every $\varepsilon < \varepsilon_0 = \|u_0\|_1^{\frac{\gamma p}{2}}/\left(R\|u_0\|_p^{\frac{\gamma p}{2}}\right)$.

Rescaling kernels

In the first chapter we have mentioned that considering the rescaled kernel given by (10), the approximate solutions u^ε of (11) converge uniformly (when $\varepsilon \to 0$) to the solution of the heat equation. In a natural way, one may wonder if it could be possible to consider different rescaled kernels such that their corresponding solutions tend to a solution of a local parabolic equation more general than the heat equation. Chapter 2 and Chapter 3 of this manuscript will deal with this natural question. Concretely, in the first part of Chapter 2, we will show the following result: If$\Omega \subset \mathbb{R}^N$ is a bounded subset, $A(x) = (a_{ij}(x))$ is a $N \times N$ matrix with smooth coefficients in $\overline{\Omega}$, symmetric and positive definite, $g \in L^1\left(\left(\mathbb{R}^N \setminus \Omega\right) \times (0,\infty)\right)$ and $u_0 \in L^1(\Omega)$ denotes the initial datum, then the *smooth* solutions of the following Dirichlet parabolic problem (in divergence form)

$$\begin{cases} v_t(x,t) = \operatorname{div}\left(A(x)\nabla v(x,t)\right), & x \in \Omega, t > 0, \\ v(x,t) = g(x,t), & x \in \partial\Omega, t > 0, \\ v(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{12}$$

can be uniformly approximated by solutions of the nonlocal problem (5), taking a suitable rescaled kernel. We highlight that, to obtain this kind of results it is necessary to assume the existence of *smooth solutions* of (12). However, under suitable regularity hypotheses about g, u_0 and $\partial\Omega$, we can assure that the solutions of (12) belong to $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\Omega} \times [0,T]\right)$ (see, for instance, Lieberman (1996)). More precisely, in the second chapter we prove the following result.

Theorem 5 *Let $v \in \mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\Omega} \times [0,T]\right)$ be a solution of (12), where $0 < \alpha < 1$. For every $\varepsilon > 0$, we consider u^ε solution of*

$$\begin{cases} u_t^\varepsilon(x,t) = \displaystyle\int_{\mathbb{R}^N} K_\varepsilon(x,y)(u^\varepsilon(y,t) - u^\varepsilon(x,t))dy, & x \in \Omega,\ t > 0, \\ u^\varepsilon(x,t) = g(x,t), & x \notin \Omega,\ t > 0, \\ u^\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where

$$K_\varepsilon(x,y) = \frac{C(J)}{\varepsilon^{N+2}} G\left(B^{-1}(x)\frac{x-y}{\varepsilon}\right) G\left(B^{-1}(y)\frac{x-y}{\varepsilon}\right), \tag{13}$$

being $G^2(s) = J(s)$ (with J a smooth nonnegative function, radially symmetric and with compact support), and $B(x) = (b_{ij}(x))$ a $N \times N$ matrix such that

$$\det(B(x))B(x)B^t(x) = A(x). \tag{14}$$

Then

$$\|v - u^\varepsilon\|_{L^\infty(\Omega\times[0,T])} \to 0, \qquad \text{when } \varepsilon \to 0.$$

Consequently, we can claim that the solutions $v \in \mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\Omega}\times[0,T]\right)$ of the Dirichlet problem (12) can be approximate by solutions of a nonlocal parabolic problem using the rescaled kernel given by (13). Note that the existence of the matrix $B^{-1}(x)$ and the matrix factorization (14) is trivial since $A(x)$ is a symmetric and positive definitive matrix. It is worth pointing out that $K_\varepsilon(x,y)$ is also symmetric and this is a very interesting property. Indeed, thanks to this symmetric property, we have the following *integration by parts formula*

$$\iint K(x,y)(u(y)-u(x))\varphi(x)dydx \\ = \frac{-1}{2}\iint K(x,y)(u(y)-u(x))(\varphi(y)-\varphi(x))dydx.$$

We emphasize that this integration by parts formula is very similar to the usual one used for operators in divergence form, i.e.,

$$\int \operatorname{div}(A(x)\nabla v(x))\varphi(x)dx = -\int A(x)\nabla v(x)\nabla\varphi(x)dx.$$

Furthermore, we also highlight the following consequence of the above theorem. If we consider the Dirichlet problem associated to the heat equation, that is to say, the problem (12), where $A(x)$ denotes the identity matrix, then we obtain that the suitable rescaled kernel is

$$K_\varepsilon(x,y) = \frac{C(J)}{\varepsilon^{2+N}} J\left(\frac{x-y}{\varepsilon}\right),$$

which was proved by Cortázar et al. (2009).

Going ahead in the study of these questions, in Theorem 2.1.1 of Chapter 2, it is also obtained an analogous result to the one obtained in Theorem 5 but for more general parabolic equations, i.e., not necessarily parabolic equation in divergence form. More precisely, in Theorem 2.1.1 we consider the following parabolic problem

$$\begin{cases} v_t(x,t) = \displaystyle\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 v(x,t)}{\partial x_i \partial x_j} + \sum_{i}^{N} b_i(x)\frac{\partial v(x,t)}{\partial x_i}, & x\in\Omega, t>0,\\ v(x,t) = g(x,t), & x\in\partial\Omega, t>0,\\ v(x,0) = u_0(x), & x\in\Omega. \end{cases}$$

In this general case, the suitable rescaled kernel is

$$K_\varepsilon(x,y) = \frac{C(x)}{\varepsilon^{N+2}}\, a\Big(x - E(x)(x-y)\Big)\, J\left(L^{-1}(x)\frac{x-y}{\varepsilon}\right), \tag{15}$$

where a is a function given by $a(s) = \sum_i (s_i + M)$, with M a positive and big enough constant to assure that $a(x) \geq \beta > 0$ for some β. The matrix $L(x)$ is the well known

Cholesky's factor for the matrix $A(x)$, i.e., it satisfies $A(x) = L(x)L^t(x)$. The matrix E involves the coefficients $(a_{ij}(x))$ and $b_i(x)$, and $C(x)$ is a normalizing function (see subsection 2.3 for a precise definition). Since, in these new cases, we are working with operators which are not in divergence form, we note that the kernel (15) is not symmetric.

Afterwards, in Chapter 3, we consider a kernel like the following one

$$K(x,y) = J\big(\mathcal{M}(y)(x-y)\big)\det \mathcal{M}(y),$$

where $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative radial function such that

$$J \in \mathcal{C}_c(\mathbb{R}^N) \quad \text{y} \quad \int_{\mathbb{R}^N} J(z)dz = 1\,, \tag{16}$$

being $\mathcal{M}(y)$ a $N \times N$ real matrix with smooth and bounded coefficients such that $\det \mathcal{M}(y) \geq \gamma > 0$. It is convenient to point out that this kind of kernels preserve the mass, that is,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\big(\mathcal{M}(y)(x-y)\big)\det \mathcal{M}(y)\, u(y)dydx = \int_{\mathbb{R}^N} u(x)dx, \quad \forall u \in \mathcal{C}(\mathbb{R}^N).$$

As a consequence, problems like

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J\big(\mathcal{M}(y)(x-y)\big)\det \mathcal{M}(y)u(y,t)dy - u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

can be seen as a particular case of our initial model problem (1). Observe that, in the case in which $\mathcal{M}(y) = g(y)^{-1}\,\mathrm{Id}$, where g is a scalar positive function, then the above equation is

$$u_t(x,t) = \int_{\mathbb{R}^N} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g^N(y)}dy - u(x,t).$$

As we told above, this kind of diffusion kernels was introduced by Cortázar et al. (2007) to model inhomogenous dispersion processes (see Coville (2010) and Cortázar et al. (2015)).

In this third Chapter 3 we will show how a suitable rescaling of this kind of kernels gives us a sequence of approximate solutions which converges to the classical local solution of the Fokker-Planck equation, i.e.,

$$\begin{cases} v_t(x,t) = \displaystyle\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ij}(x)v(x,t)\right), & x \in \mathbb{R}^N, t \in [0,T], \\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{17}$$

where $A(x) = (a_{ij}(x))$ is a $N \times N$ real and positive definite matrix. More precisely, given the rescaled kernel

$$K_\varepsilon(x,y) = \frac{1}{\varepsilon^N} J\Big(B^{-1}(y)\frac{(x-y)}{\varepsilon}\Big)\det B^{-1}(y),$$

where B is such that $BB^t = A$ and J satisfies (16), we prove the following main result.

Theorem 6 *Let $v \in \mathcal{C}^{2+\alpha,1+\alpha/2}\big(\mathbb{R}^N,[0,T]\big)$ be the solution of the classical Fokker-Planck equation (17) with initial datum $v_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. For every $\varepsilon > 0$, let u^ε be the solution of the nonlocal equation*

$$\begin{cases} u_t^\varepsilon = \dfrac{C}{\varepsilon^2}\left\{\displaystyle\int_{\mathbb{R}^N} K_\varepsilon(x,y)u(y,t)dy - u(x,t)\right\}, & x \in \mathbb{R}^N, t \in [0,T], \\ u^\varepsilon(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{18}$$

where $C^{-1} = \frac{1}{2}\int J(z)z_N^2 dz$. Then

$$\sup_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - v(\cdot,t)\|_{L^\infty} \to 0,$$

when $\varepsilon \to 0$.

Observe that, in the particular case $B(y) = g(y)\,\mathrm{Id}$, the equation (18) is

$$u_t^\varepsilon(x,t) = \frac{C}{\varepsilon^2}\left\{\int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} J\Big(\frac{x-y}{\varepsilon g(y)}\Big)\frac{u(y,t)}{g^N(y)}dy - u(x,t)\right\}$$

and so that, its solutions converge to the local differential equation

$$v_t(x,t) = \sum_i (g^2(x)v(x,t))_{x_i x_i}.$$

Consequently, our theorem generalizes the results obtained by Sun et al. (2011).

Approximating the Kardar-Parisi-Zhang equation by nonlocal equations

In Chapter 4 of this manuscript we consider nonlocal problems like the following ones

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)\big(u(y,t) - u(x,t)\big)\,\mathcal{G}\big(x, u(y,t) - u(x,t)\big)\,dy, \tag{19}$$

where J satisfies (16) and $\mathcal{G} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is an auxiliar nonnegative Carathéodory function such that

$$\exists\ \alpha_2 \geq \alpha_1 > 0 : \qquad \alpha_1 \leq \frac{\mathcal{G}(x,s)s - \mathcal{G}(x,\sigma)\sigma}{s-\sigma} \leq \alpha_2, \tag{20}$$

for every s,$\sigma \in \mathbb{R}$ with $s \neq \sigma$ and a.e. $x \in \mathbb{R}^N$. Observe that this implies that $\mathcal{G}$ is a positive and bounded function. Indeed, if we take $\sigma = 0$, one has

$$0 < \alpha_1 \leq \mathcal{G}(x,s) \leq \alpha_2, \qquad \text{for every } s \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N.$$

Moreover, we emphasize that in the particular case $G(x,s) \equiv 1$, we obtain again the diffusion nonlocal equation (7) which is in convolution form.

In this fourth chapter we deal with the Cauchy problem associated to the equation (19), i.e.,

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y;x,t)\; \mathcal{G}(x,u(y;x,t))\,dy & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N\,, \end{cases} \tag{21}$$

where $u(y;x,t) := u(y,t) - u(x,t)$ and $u_0 \in \mathcal{C}(\mathbb{R}^N)$. On the other hand, we deal with the Dirichlet problem associated to the equation (19), i.e.,

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y;x,t)\,\mathcal{G}(x,u(y;x,t))\,dy, & \text{in}\Omega \ \times (0,T), \\ u(x,t) = h(x,t), & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in}\Omega\ , \end{cases} \tag{22}$$

where Ω denotes a bounded domain of $\mathbb{R}^N$ with $N \geq 1$, $T \in \mathbb{R}^+ \cup \{\infty\}$, $h \in L^1\left((\mathbb{R}^N \setminus \Omega) \times (0,T)\right)$ and $u_0 \in L^1(\Omega)$.

Concretely, if u_0 is bounded, then we obtain existence and uniqueness for the Cauchy problem (21). In particular, we prove that there is a solution belonging to $\mathcal{C}\left([0,T);\mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\right)$ (Theorem 4.2.12) which is unique via a comparison principle (Theorem 4.2.14). In a similar way, assuming that $u_0 \in \mathcal{C}(\overline{\Omega})$ and $h \in \mathcal{C}((\mathbb{R}^N \setminus \overline{\Omega}) \times [0,T))$, we prove existence and uniqueness for the Dirichlet problem (22). That is, we show that there is a solution belonging to $\mathcal{C}(\overline{\Omega} \times (0,T))$ (Theorem 4.2.3) which is unique due to another comparison principle (Theorem 4.2.5).

During this fourth chapter, we also study the relation between the nonlocal equation (21) and the well-known deterministic Kardar-Parisi-Zhang equation (KPZ)

$$\begin{cases} u_t - \Delta u = \mu(x)|\nabla u|^2 & \text{in } \mathbb{R}^N \times (0,T)\,, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N\,. \end{cases} \tag{23}$$

We point out that this equation, at least for $\mu(x) = \mu > 0$, was proposed by Kardar et al. (1986) in the physical theory of growth and roughening of surfaces. See also Barabási and Stanley (1995) for others physics applications and the recent and complete work by Wio et al. (2011). It is worth pointing out that the KPZ equation has a natural growth in the gradient, that is to say, this equation has a quadratic growth respect to the gradient. These kind of equations have been widely studied during the last decades since the pioneers works by Ladyzenskaja et al. (1968) and Aronson and Serrin (1967) as well as the results by Boccardo, Murat and Puel in Boccardo et al. (1989). See also the second part of this introduction.

More precisely, Theorem 4.2.15 shows that the Cauchy problem (21), with initial datum $u_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and under the usual rescaled kernel of J

$$J_\varepsilon(z) = \frac{C(J)}{\varepsilon^{2+N}} J\left(\frac{z}{\varepsilon}\right), \tag{24}$$

has a unique solution u^ε (for each $\varepsilon > 0$) which moreover converges uniformly (when $\varepsilon \to 0$) to a classical solution of the KPZ equation (23) with

$$\mu(x) = \frac{2\mathcal{G}_s'(x,0)}{\mathcal{G}(x,0)}. \tag{25}$$

Even more, we prove that every classical solution of the KPZ equation (23) with initial datum $u_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ can be uniformly approximated by solutions of the nonlocal equation (21). Here we consider the rescaled kernels given by (24) and the same initial datum u_0 with the auxiliar function

$$\mathcal{G} \equiv \mathcal{G}_\mu(x,s) = 1 + \frac{\mu(x)\, s}{2(1+\mu(x)^2 s^2)}.$$

Remark that $\mathcal{G}_\mu$ satisfies the hypothesis (20) and moreover (25). We also obtain analogous results for the Dirichlet case (22), see Theorem 4.2.8.

To finish, it is convenient to remark that we also prove two results about the asymptotic behavior of the solutions. On the one hand, for the Dirichlet problem we prove that the solutions converge uniformly to the stationary solution. On the other hand, for the Cauchy problem, we show that the L^2 norm of the solutions has a time decay which depends on $\mathcal{G}$ (absortion or reaction). See Theorem 4.2.16 and Theorem 4.2.17.

PART II: Elliptic equations with singularity in the quadratic gradient term and Gelfand type problems

In the second part of this introduction, we will deal with initial value problems which have two different kind of nonlinearities. On the one hand, we will consider Gelfand nonlinearities (λe^u) and we will study some Gelfand-type problems. On the other hand, we will study some problems whose differential equation has a singular nonlinearity with quadratic growth with respect to the gradient ($|\nabla u|^2/u^\gamma$).

It is convenient to recall that Gelfand-type problems have been widely studied in the literature. In particular, this kind of problems has been extensively applied in some physical models. For instance, for thermal self-ignition problems of a chemically active mixture of gases in a vessel, see Chandrasekhar (1957); Gel'fand (1963); Joseph and Sparrow (1970); Keller and Cohen (1967) and the references therein .

The classical Gelfand problem is the following one

$$\begin{cases} -\Delta u = \lambda\, e^u\,, & \text{in } \Omega, \\ u \geq 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

where$\Omega \subset \mathbb{R}^N$ is an open, bounded subset whose boundary $\partial\Omega$ is smooth, $N \geq 1$ and $\lambda \geq 0$. We remark that in this context, basically, the nonlinear term e^u can be replaced by a regular positive function $f(u)$ which is increasing, convex and moreover $f(0) > 0$.

It is worth pointing out that, roughly speaking, the change of variable $u = \ln(1+v)$ transforms the above semilinear problem in the following quasilinear one

$$\begin{cases} -\Delta v + \dfrac{|\nabla v|^2}{1+v} = \lambda\,(1+v)^2\,, & \text{in } \Omega, \\ v \geq 0\,, & \text{in } \Omega, \\ v = 0\,, & \text{on } \partial\Omega. \end{cases}$$

As a consequence, this example shows how the semilinear problems are strongly connected with the quasilinear ones which have a quadratic growth with respect to the gradient (or equivalently a natural growth). This kind of quasilinear problems with natural growth are very common and they appear in a natural way. Indeed, there are several motivations for these quasilinear equations, some of them coming from Calculus of Variations. For instance, the Euler Lagrange equation associated to the functional

$$I(u) = \frac{1}{2}\int_\Omega a(x,u)|\nabla u|^2 - \int_\Omega f_0(x)u,$$

is, at least formally, the following one

$$-\text{div}\,(a(x,u)\nabla u) + \frac{1}{2}a'_u(x,u)|\nabla u|^2 = f_0(x).$$

We emphasize that if $a(x,u) = 1+|u|^\delta$, with $\delta \in (0,1)$, then the above Euler Lagrange equation not only has a quadratic growth with respect to the gradient but also a singular term. Some applications of these singular equations can be seen in Barenblatt et al. (2000); Berestycki et al. (2001) y Kardar et al. (1986).

Differential operators with natural growth have been thoroughly studied during the last decades since the works by Aronson and Serrin (1967); Ladyzenskaja et al. (1968) and later by Boccardo et al. (1982, 1983). For instance, given $\mu \in L^\infty(\Omega)$ and a continuous function g, we highlight that the existence of solution for the problem

$$\begin{cases} -\Delta u + \mu(x) g(u)\, |\nabla u|^2 = f_0(x) & \text{in} \quad , \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

it has been considered by Bensoussan et al. (1988), Boccardo and Gallouët (1992) and Boccardo et al. (1982).

Singularity in the quadratic term

In Chapter 5 we will consider a singular problem which involves a quasilinear elliptic differential operator with quadratic gradient term. More concretely, our model case will be

$$\begin{cases} -\Delta u + \mu(x) \dfrac{|\nabla u|^2}{u^\gamma} = \lambda u^p + f_0(x) & \text{in} \quad , \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\gamma, p \geq 0$, $\mu \in L^\infty(\Omega)$ is non-negative and $0 \lneqq f_0 \in L^q(\Omega)$ for some $q > N/2$.

A well known case is the above problem when $\lambda = 0$. Indeed, this model with $\lambda = 0$ was introduced by Arcoya and Martínez-Aparicio (2008) when $\gamma = 1$. Afterwards, this one with $\gamma \geq 1$ was extensively studied in Arcoya and Segura de León (2010); Boccardo (2008); Martínez-Aparicio (2009) and Giachetti and Murat (2009). In these above works it is proved the existence of solutions for $\gamma \leq 1$ and the uniqueness for $\gamma < 1$ (see also Arcoya et al. (2017)). It is convenient to recall that, in this problem (with $\lambda = 0$), the case $\gamma > 1$ requieres an additional hypothesis about the nonlinearity f_0. In fact, in Arcoya et al. (2009b), under the additional hypothesis

$$\operatorname{ess\,inf}\{f_0(x) : x \in \omega\} > 0\,, \ \forall \omega \subset\subset \Omega\,,$$

it is proved the existence of a solution if and only if $\gamma < 2$. Even more, we emphasize that if $\gamma \geq 2$, then $\frac{|\nabla u|^2}{u^\gamma} \notin L^1(\Omega)$ for all $u \in W_0^{1,2}(\Omega)$ (see Zhou et al. (2012)), hence there is no solution.

It is worth pointing out that the case $\gamma > 1$ is hardier than the case $\gamma < 1$. Indeed, nowadays the uniqueness of this problem for $\gamma > 1$ is unknown and for $\gamma = 1$ is uncompleted (see Carmona and Leonori (2017)).

Results concerning the above problem for $\lambda \neq 0$ were obtained in Arcoya et al. (2011); Boccardo et al. (2011), being $\mu(x)$ a constant function and $\gamma < 1$. Moreover it is assumed that the parameters γ and p satisfy $\gamma + p < 2$. On the one hand, in

Arcoya et al. (2011), the results were obtained by using topological methods. On the other hand, in Boccardo et al. (2011), the results were proved using some suitable approximate problems and an iterative scheme. Taking into account that the methods used in Arcoya et al. (2011) and Boccardo et al. (2011) can not be applied to the case in which $\mu(x)$ is not a constant or $p < 1 \leq \gamma < 2$, our aim is to complete it in Chapter 5. Even more, in this chapter we will consider more general lower order terms. Indeed we will work with $\mu(x)g(u)|\nabla u|^2$, being g a singular function at zero. We show how the values of λ for which the problem has a solution will be influenced by the singularity of g at zero and moreover by its behavior at infinity. To distinguish the behavior of g at zero and infinity, we take $\gamma \leq \beta$ and we consider the more general model problem

$$\begin{cases} -\Delta u + \mu(x)\dfrac{|\nabla u|^2}{u^\gamma + u^\beta} = \lambda u^p + f_0(x) & \text{in} \quad , \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{Q_λ}$$

Our main result for the case in which $\mu(x)$ is a constant is the following one.

Theorem 7 *Let $\mu(x) = \mu$ be a constant function and we consider $f_0 \in L^q(\Omega)$ with $q > \frac{N}{2}$ satisfying $\operatorname{ess\,inf}\{f_0(x) : x \in \omega\} > 0\,,\ \forall \omega \subset\subset \Omega\,.$ Then*

- *i) If $1 \leq \gamma < 2$ and $0 < p < 1$ the problem (Q_λ) has, at least, a solution for every $\lambda \geq 0$.*
- *ii) If $\gamma < 1 < \beta$ and $1 \leq p$, there are $\lambda_*, \lambda^* > 0$ such that (Q_λ) does not have solutions for $\lambda > \lambda^*$ and has, at least, a solution for $0 \leq \lambda < \lambda_*$.*

Moreover, there is an unbounded continuous, i.e., a connected and closed subset Σ of

$$\{(\lambda, u) \in [0, +\infty) \times C(\overline{\Omega}) : u \textit{ solución de } (Q_\lambda)\}\,,$$

such that there exists a solution u_λ of (Q_λ) with $(\lambda, u_\lambda) \in \Sigma$ for every $\lambda \geq 0$ (item i) or every $0 \leq \lambda < \lambda_*$ (item ii).

Furthermore, the tools used to prove the above theorem also allow us to work with non constant function $\mu(x)$ if we suppose that this function is bounded below and above and the parameter β satisfies $\beta \leq 1$. More precisely, we present the following theorem.

Theorem 8 *Assume that $0 < \gamma \leq \beta \leq 1$, $0 < p < 2 - \beta$, $f_0 \in L^q(\Omega)$ with $q > \frac{N}{2}$ and $m \leq \mu(x) \leq M$, a.e. $x \in \Omega$ (where $M < 2$ in the case $\alpha = \beta = 1$). Then, there exists an unbounded continuum Σ of solutions of (Q_λ), such that there exists u_λ solution of (Q_λ) with $(\lambda, u_\lambda) \in \Sigma$ for every $\lambda \geq 0$.*

We highlight that this theorem not only improves again the results of Arcoya et al. (2011) and Boccardo et al. (2011), but also shows the following property. The hypothesis $p < 2 - \beta$ is a restriction in the behavior of g at infinity, rather than in the singularity at zero. In this sense, working with a more general function $g(s) = 1/(s^\gamma + s^\beta)$, one can observe how the behavior at zero or at infinity of g have a different role in the solutions set.

To prove the above two theorems we use a double approach. Firstly, we take a suitable sequence of approximate problems, as in Boccardo et al. (2011), and we deduce the existence of a continuumΣ_n for the approximate problems using Leray-Schauder degree techniques and Rabinowitz continuation theorem, as in Arcoya et al. (2011). Secondly, we use a topological lemma to obtain a continuum of solutions as the limit of this approximative schemeΣ_n.

Gelfand type problems

In Chapters 6 and 7 we will consider some Gelfand type problems corresponding to different differential operators. Indeed we will consider the 1-homogeneous p-Laplacian and moreover some differential operators having lower order terms with quadratic growth with respect to the gradient.

It is convenient to recall that, if $f(u)$ denotes a regular, positive and convex function with $f(0) > 0$, then the problem

$$\begin{cases} -\Delta u = \lambda\, f(u)\,, & \text{in } \Omega, \\ u \geq 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{G_λ}$$

was studied by Crandall and Rabinowitz (1975) (see also Mignot and Puel (1980) and the references therein). Concretely, if f is superlineal at infinity, that is to say, if $\lim_{s\to+\infty} \frac{f(s)}{s} = \infty$, in Crandall and Rabinowitz (1975) it is proved the following result.

Proposition 0.0.1 *Crandall and Rabinowitz (1975) There exists $\lambda^* > 0$, called the extremal parameter, such that*

- *If $\lambda < \lambda^*$, then (G_λ) admits a minimal bounded solution w_λ.*
- *If $\lambda > \lambda^*$, then (G_λ) has no solution.*

One may wonder if the minimal solution could exist but for Gelfand-type problems corresponding to differential operators which satisfy a comparison principle. In this sense, in Chapter 6, we will prove a comparison principle for the 1-homogeneous p-laplacian which generalizes the well known comparison principles obtained in Barles and Busca (2001); Martínez-Aparicio et al. (2014a). In Chapter 7 we will use the comparison principle contained in Arcoya and Segura de León (2010) (see also Arcoya et al. (2014, 2017)).

Even more, in Crandall and Rabinowitz (1975) it is also proved that the sequence of minimal solutions $\{w_\lambda\}$ of (G_λ) is increasing in λ. Furthermore, the minimal solutions are stable, namely they satisfy the following condition

$$\int_\Omega \left(|\nabla\xi|^2 - \lambda f'(w_\lambda)\xi^2\right) \geq 0, \qquad \forall \xi \in \mathcal{C}_c^\infty(\Omega).$$

We emphasize that this stability condition has an important role to prove some existence and regularity results for the extremal solution. In particular, this condition has been used to achieve optimal results of regularity of extremal solution depending on the dimension N. In this sense, if $f(s) = e^s$ it is obtained regularity results for the extremal solution if $N < 10$. However, if $f(s) = (1+s)^p$ the regularity results are proved for $N < 4 + 2(1-1/p) + 4\sqrt{1-1/p}$ (see Crandall and Rabinowitz (1975)).

We note that it makes sense to extend the above stability condition but for general differential operators with variational structure. Our problem is that in Chapter 6 and 7 we do not have any variational structure. However, in Chapter 7, we will be able to extend the above stability condition and therefore, we will obtain some results concerning to the extremal solution.

Before showing the main results of Chapters 6 and 7, we point out again that Gelfand-type problems constitute one of the most studied fields of semilinear elliptic equations and it has been extensively considered. For instance, see Arcoya et al. (2014); Cabré and Capella (2006); Cabré and Sanchón (2013); Gel'fand (1963) and the references therein.

More concretely, in Chapter 6 we will consider the problem

$$\begin{cases} -\Delta_p^N u = \lambda\, e^u\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda,p}$}$$

where$\Omega \subset \mathbb{R}^N$ is a regular bounded domain, $p \in [2, \infty]$ and the operatorΔ_p^N is the called 1-homogeneous p-laplacian defined, for $p < \infty$, by

$$\Delta_p^N u := \frac{1}{p-1}|\nabla u|^{2-p}\,\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = \frac{1}{p-1}\Delta u + \frac{p-2}{p-1}\Delta_\infty u,$$

and for $p = \infty$ by

$$\Delta_\infty u \equiv \Delta_\infty^N u = \frac{\nabla u}{|\nabla u|}\cdot\left(D^2 u \frac{\nabla u}{|\nabla u|}\right),$$

the 1-homogeneous infinity laplacian

This operator appears when one considers Tug-of-War games with noise, see Manfredi et al. (2012); Peres and Sheffield (2008); Peres et al. (2009), where the Poisson problem is studied. Moreover, the problem with right-hand side λu^q with $0 < q \leq 1$ has been studied in Martínez-Aparicio et al. (2014a) and Martínez-Aparicio et al. (2014b).

Concerning this kind of problems, our first result is the following one.

Theorem 9 *For every $p \in [2, +\infty]$ there is a positive extremal parameter $\lambda^* = \lambda^*(\Omega, N, p)$ such that:*

- *If* $\lambda < \lambda^*$ *the problem* $(P_{\lambda,p})$ *admits a minimal positive solution* w_λ.
- *If* $\lambda > \lambda^*$ *the problem* $(P_{\lambda,p})$ *has no positive solution.*

Moreover, the branch of minimal solutions $\{w_\lambda\}$ *is increasing with* λ.

We highlight that in Chapter 6 we use some arguments from degree theory to study problems whose differential operator is the 1-homogeneous p-laplacian. The use of this tools is not easy due to the lack of regularity. Indeed, to address it we need to use some arguments of Charro et al. (2013) to obtain some compactness results. Using these techniques we will be able to prove the existence of a continuum of solutions either for the parameter λ or the parameter p. In this sense, for every p fixed, we denote by

$$\mathscr{S}_p = \{(\lambda, u) \in [0, \lambda^*(\Omega, N, p)] \times \mathcal{C}(\overline{\Omega}) : u \text{ solution of } (P_{\lambda,p})\},$$

and for every λ fixed, we denote by

$$\mathcal{S}_\lambda = \{(p, u) \in [2, \infty] \times \mathcal{C}(\overline{\Omega}) : u \text{ solution of } (P_{\lambda,p})\}.$$

Theorem 10 *For every fixed* $p \in [2, \infty]$, *there exists an unbounded continuum of solutions* $\mathcal{C} \subset \mathscr{S}_p$ *that emanates from* $\lambda = 0$, *i.e.,* $(0, 0) \in \mathcal{C}$. *Moreover, for every fixed* $\lambda_0 \in (0, \lambda^*)$, *there exists a continuum of solutions* $\mathcal{D} \subset \mathcal{S}_\lambda$, *for all* $\lambda < \lambda_0$, *such that its projection on the axis* p *is* $[2, +\infty]$.

In Chapter 7 we will deal with with some Gelfand-type problems which have a singularity in the gradient term. Concretely, we will consider

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = \lambda\, f(u)\,, & \text{in } \Omega, \\ u > 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

were$\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded and open subset of $\mathbb{R}^N$, $\lambda > 0$ f is a strictly increasing function, derivable in $[0, \infty)$ and such that $f(0) > 0$ and finally g is a nontrivial and positive function that either is continuous in $[0, \infty)$ or it is continuous in $(0, \infty)$ and integrable in a neighborhood of zero. Our model case are $g(s) = \frac{1}{s^\gamma}$ with $\gamma \in (0, 1)$ and $f(s) = e^s$.

We stress that the case g continuous in $[0, +\infty)$ has been studied in Arcoya et al. (2014). Here, the authors showed the existence of a minimal solution in a bounded and maximal interval $(0, \lambda^*)$ and moreover they studied existence and regularity results for extremal solutions. Even more, they characterized minimal solutions as those solutions satisfying a stability condition (see also Brézis and Vázquez (1997) for the semilinear case). Concretely, in Arcoya et al. (2014) a solution is stable, by definition, when it satisfies

$$\int_\Omega |\nabla \phi|^2 \geq \lambda \int_\Omega (f'(u) - g(u) f(u))\phi^2$$

for every $\phi \in W_0^{1,2}(\Omega)$. In Chapter 7 we will say that a solution is stable if it satisfies the above condition given by Arcoya et al. (2014). During this chapter, we will extend the above previous results to the singular framework and moreover we will improve the hypotheses assumed for the continuous case. For instance, among others results, we highlight that the hypothesis $f'(s) - g(s)f(s)$ increasing required by Arcoya et al. (2014), it is necessary only to prove that the stable solutions are minimal.

The results obtained in Chapter 7, apply to the particular case $g(s) = \frac{c}{s^\gamma}$ with $0 < \gamma < 1$, allow us to consider non-convex function $f(s)$. Indeed, if we take $f(s) = e^{\frac{s^{1-\gamma}}{1-\gamma} + (s+\delta)^{1-\gamma}}$ with δ small enough, then $f'(s) - g(s)f(s)$ is a decreasing function. However, there exists $\lambda^* > 0$ such that the problem admits a minimal bounded solution w_λ for every $\lambda < \lambda^*$ but there is no solution for $\lambda > \lambda^*$. Moreover, it is possible to prove the existence of a extremal solution $u^* = \lim\limits_{\lambda \to \lambda^*} w_\lambda$ which is a stable solution (in the above sense) for $\lambda = \lambda^*$. Note that this extremal solution is not, in general, a minimal solution. Even more, if

$$N < \frac{6(1-\gamma) + 2c + 4\sqrt{(c+1-\gamma)(1-\gamma)}}{c+1-\gamma},$$

it is proved that the extremal solution is bounded.

PART III: Some results in Elliptic Equations modeled by the $p-$laplacian

It is considered the following family of elliptic differential equations that involve the $p-$laplacian operator and with Dirichlet conditions at the boundary of a bounded domain$\Omega \subset \mathbb{R}^N$,

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x,u), & \text{in}\Omega\ , \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{26}$$

with $p > 1$ and being the source data $f(x,s)$, with $(x,s) \in \Omega \times \mathbb{R}$, a certain function that we will detail below.

Next we present three classic results related to the previous equation:

- The problem with a subcritical source term: $f(x,s) \leq C_1|s|^q + C_2$, with $0 < q < p^* - 1$ and $p > 1$. There exist at least two nontrival solutions $v \leq 0 \leq w$ (see for instance Dinca et al. (1995)).
- The problem with a Hardy potential: $f(x,s) = \dfrac{\lambda|s|^{p-2}s}{|x|^p}$ and $0 \in \Omega$. There is no solution for $\lambda > ((N-p)/p)^p$, $1 < p < N$ (see García Azorero and Peral Alonso (1998)).
- The concave-convex problem: $f(x,s) = |s|^{r-1}s + \lambda|s|^{q-1}s$, with $0 < q < p-1 < r < p^* - 1$ and $\lambda > 0$. There exists $\lambda^* > 0$ such that there are at least two positive solutions for $\lambda < \lambda^*$ and there is no positive solution for $\lambda > \lambda^*$ (see García Azorero et al. (2000)).

In this third part of the memoir we intend to broaden the study of these problems either by extending the operator or by extending the source data, always without losing the nature of the classical problem. Concretely, in Chapter 8 we study the subcritical problem for the $1-$Laplacian ($p = 1$) in which we prove the existence of two non-trivial solutions for $0 < q < 1^* = 1/(N-1)$ and that are also bounded. Another notable result of the chapter is the proof of a Pohožaev type identity for this kind of operators. The $1-$Laplacian operator was originally treated in Kawohl (1991, 1990), Demengel (1999) y Andreu et al. (2001) leading a huge literature since then. One of the main interests for studying the Dirichlet problem for equations involving the $1-$Laplacian comes from the variational approach to image restoration (we refer to Andreu-Vaillo et al. (2004) for a review on the first variational models in image processing and their connection with the 1–Laplacian, see also the recently work Martín et al. (2017)).

In Chapter 9 it is considered a problem with a Hardy potential for the laplacian operator ($p = 2$). We prove that the presence in the equation of lower order terms $h(x)u(x)^\gamma$ ($h \in L^1_{loc}(\Omega)$, $\gamma > 1$) produces a regularizing effect when obtaining a solution for values of λ greater than the critical $\frac{(N-2)^2}{4}$, even if h vanishes in subsets of Ω.

In addition, this term causes the solutions to be more regular. The Hardy problem for $p = 2$ was treated firstly in Baras and Goldstein (1984). The authors observed that, since $\frac{\lambda}{|x|^2} \in L^r_{loc}(\Omega)$ if and only if $1 \leq r \leq N/2$, the classical theory of uniqueness and regularity could not be applied. They prove that the asymptotic behavior of the solutions depends on the values of λ, determining a critical value $\mathcal{H} = (N-2)^2/4$ also called the Hardy constant. Later, in García Azorero and Peral Alonso (1998) the authors perform a more exhaustive study of the equation for all values of $1 < p < N$ where they effectively reveal that the behavior of the solutions depends on the critical value $\lambda^* = ((N-p)/p)^p$, obtaining solutions for $\lambda < \lambda^*$. Since then a large number of related works have emerged.

In Chapter 10 the concave-convex problem is studied but, instead of making a concave-convex effect to the nonlinearity $f(x,s)$, this effect will be caused to the operator. That is, the operator in consideration is $-\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ being $p(x)$ the constant function 2 in a region of the domain $D_1 \subset \Omega$, and the function constant p (greater than 2) in the remaining region of the domain $D_2 = \Omega \setminus D_1$. Regarding nonlinearity we take $f(x,s) = \lambda|s|^q$ with $1 < q < p-1$. Note that these values of q induce a convex effect in region D_1 and a concave effect in region D_2. The concave-convex problems have received a great interest in the literature of differential equations since the pioneering works of Lions (1982); García Azorero and Peral Alonso (1991); Ambrosetti et al. (1994) and Boccardo et al. (1995). On the other hand, the study of operators $p(x)-$laplacian with $p(x)$ discontinuous have received great attention in recent years in modeling the flow of current in Organic Light-Emitting Diodes (OLEDs) used in the display of portable devices, we refer the works Bulíček et al. (2016); Fischer et al. (2014) and Bulíček et al. (2017). In this chapter we prove the existence of a critical value λ^* such that for $\lambda > \lambda^*$ there is no positive solution, and for $\lambda < \lambda^*$ there is a minimal positive solution. Furthermore, provided that $p < 2N/(N-2)$, there is a second positive solution for almost every $\lambda < \lambda^*$.

The technique to deal with the problems of this third part of this memory is mainly the Calculus of Variations. Observe that the problem (26) has the following functional energy associated $\mathcal{I} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined as follows

$$\mathcal{I}(u) = \int_\Omega \frac{|\nabla u|^p}{p} - \int_\Omega F(x,u)\,,$$

being $F(x,s) = \int_0^s f(x,t)dt$, in the sense that critical points of $\mathcal{I}$ are solutions to problem (26). When dealing with the problems mentioned above, an important step is to replace the space $W_0^{1,p}(\Omega)$ by another more convenient, thus changing the geometry of the functional $\mathcal{I}$ and being more accessible to find their critical points. This will have its advantages as will be seen in the Chapters 9 and 10 turning the functional into coercive, and its disadvantages when the chosen space is not reflexive as is the case of the Chapter 8, not being able to apply well known results such as compactness of Palais-Smale sequences.

Subcritical problem for the $1-$Laplacian

In Chapter 8 we study existence and regularity results of solutions with a Dirichlet problem for an elliptic equation involving the 1–Laplacian operator and a source term, whose model problem is

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) = |u|^{q-1}u, & \text{in}\Omega\ , \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{27}$$

where$\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with Lipschitz boundary and $0 < q < \frac{1}{N-1}$.

The natural energy space to study problems involving the $1-$Laplacian is the space $BV(\Omega)$ of functions of bounded variation, i.e., those L^1-functions such that their distributional gradient is a Radon measure having finite total variation. We point out that $BV(\Omega)$ is a Banach space with norm

$$\|u\|_{BV(\Omega)} = \int_\Omega |Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1},$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$–dimensional Hausdorff measure (we refer for instance Ambrosio et al. (2000)).

Although $BV(\Omega)$ is non reflexive and non separable space. In this way, the $1-$Laplace operator presents an extra difficulty. Another difficulty occurs by defining the quotient $\dfrac{Du}{|Du|}$, being Du just a Radon measure. It can be overcome through the theory of pairings of L^∞–divergence–measure vector fields and the gradient of a BV–function (see Anzellotti (1983)). Using this theory, we may consider a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ such that $\|\mathbf{z}\|_\infty \leq 1$ and $(\mathbf{z}, Du) = |Du|$, so that $\mathbf{z}$ plays the role of ratio $\frac{Du}{|Du|}$. On the other hand, the boundary condition is weaker. Indeed, in general the Dirichlet boundary condition is not achieved in the usual trace form, so that, a very weak formulation must be introduced: $[\mathbf{z}, \nu] \in sign\,(-u)$. Where $[\mathbf{z}, \nu]$ stands for the weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ defined in Anzellotti (1983) as the application $[\mathbf{z}, \nu] : \partial\Omega \to \mathbb{R}$, being ν the outer normal unitary vector of $\partial\Omega$, such that $[\mathbf{z}, \nu] \in L^\infty(\partial\Omega)$ and $\|\,[\mathbf{z}, \nu]\,\|_{L^\infty(\partial\Omega)} \leq \|\mathbf{z}\|_{L^\infty(\Omega;\mathbb{R}^N)}$. Furthermore, this definition coincides with the classical one, that is, $[\mathbf{z}, \nu] = \mathbf{z} \cdot \nu$, for $\mathbf{z} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^N)$.

In this way, we say that $u \in BV(\Omega)$ is a solution of problem (27) if there exists a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ with $\|\mathbf{z}\|_\infty \leq 1$ and such that

(1) $-\,\mathrm{div}\,\mathbf{z} = f(x, u)$ in $\mathcal{D}'(\Omega)$,

(2) $(\mathbf{z}, Du) = |Du|$ as measures on Ω,

(3) $[\mathbf{z}, \nu] \in \mathrm{sign}(-u)$ on $\partial\Omega$.

In order to consider the problem (27) in a variational setting, we establish in Lemma 8.2.6 that solutions are critical points of the functional $\mathcal{I} : BV(\Omega) \to \mathbb{R}$ defined as follows

$$\mathcal{I}(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} - \frac{1}{q+1}\int_\Omega |u|^{q+1}.$$

We recall that one of the approaches to find nontrivial solutions to the Dirichlet problems with p–Laplacian type operator ($p > 1$) and a subcritical term, i.e. with $|u|^{q-1}u$, being $0 < q < p^* - 1$ (where p^* stands for the Sobolev conjugate), is by using the well–known "Mountain Pass Theorem" by Ambrosetti and Rabinowitz Ambrosetti and Rabinowitz (1973). Specifically, first it is proved that the trivial solution is a local minimum of the corresponding energy functional and then, since the functional has a mountain pass geometry, they find other critical points (one positive and another one negative), we refer Dinca et al. (1995). We point out that the proof of the Palais–Smale condition relies on the reflexivity of the energy space $W_0^{1,p}(\Omega)$.

As mentioned above, the space of functions of bounded variation is non reflexive. The strategy is to consider the nontrivial and positive solution, w_p, obtained by "Mountain Pass Theorem" applied in the subcritical problem for the $p-$laplacian with $p > 1$ (similarly reasoning for the negative solution). Then, in certain sense, we take the limit as $p \to 1^+$. However, we carefully have to check that their limit is not the trivial solution. Thus, we prove in Theorem 8.1.1 that there exist at least two nontrivial solutions $v \le 0 \le w$ of problem (27). Moreover, we prove that they are bounded. To this end, it is essential to achieve the existence of a positive constant C independent of p such that

$$\|w_p\|_{W_0^{1,1}(\Omega)} \le C\,,$$

for all $p > 1$ enough small.

Finally, in the last part of this chapter, we state in Proposition 8.4.1 a Pohožaev type identity for solutions belonging to $W^{1,1}(\Omega)$. The important point to note here is, unlike $p-$Laplacian problems, the existence of solutions for supercritical growth. This is confirmed by dealing with explicit examples in the ball.

Regularizing effect of lower order terms in elliptic problems involving a Hardy potential

In Chapter 9 we study the regularizing effect provided by the inclusion of lower order terms in elliptic problems of type (26) whith a Hardy potential. Specifically, we consider equations of type

$$\begin{cases} -\Delta u + h(x)|u|^{p-1}u = \lambda \dfrac{u}{|x|^2} + f(x) & \text{in} \quad , \quad \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

where $p > 1$, $\lambda \in \mathbb{R}$, $0 \leq h \in L^1(\Omega)$ and $f \in L^{\frac{p+1}{p}}(\Omega; hdx)$, i.e., $|f|^{\frac{p+1}{p}} h \in L^1(\Omega)$. Observe that being h integrable, it holds

$$L^m(\Omega; hdx) \subset L^{\frac{p+1}{p}}(\Omega; hdx), \qquad \text{for all } m \geq \frac{p+1}{p}.$$

In the case of not including the regularizing term, that is $h \equiv 0$, it is known that there is a solution for all $f \in W^{-1,2}(\Omega)$ provided that

$$\lambda < \mathcal{H} = \frac{(N-2)^2}{4}, \tag{29}$$

see García Azorero and Peral Alonso (1998). Seen from a variational perspective, the condition (29) implies, due to the Hardy inequality

$$\int_\Omega \frac{|u|^2}{|x|^2} dx \leq \mathcal{H}^{-1} \int_\Omega |\nabla u|^2 ,$$

that the associated energy functional is coercive in $W_0^{1,2}(\Omega)$.

The fact of including the term $h(x)|u|^{p-1}u$ to a problem with a Hardy potential is not new, see Adimurthi et al. (2017); Porzio (2007); Wei and Du (2017) and Wei and Feng (2015). In these last two works the authors study the asymptotic behavior of the solution at 0 for the case $f \equiv 0$ and $h(x) = |x|^\sigma$ with $\sigma > -2$. On the other hand, Porzio (2007) and recentely Adimurthi et al. (2017) treat the case $h(x) \equiv h_0 > 0$ obtaining the following result:

Theorem 11 *Consider $p > 2^* - 1$, $h(x) \equiv h_0 > 0$ and $f \in L^m(\Omega)$ with $\frac{p+1}{p} \leq m < \frac{N}{2}\frac{p-1}{p}$. Then, there exists a solution to problem (28) for all $\lambda \geq 0$. Moreover, the solution belongs to $W_0^{1,2}(\Omega) \cap L^{pm}(\Omega)$.*

We emphasize that the solution provided by the above theorem is obtained as a limit of solutions of a sequence of approximate problems and also the regularity in $L^{pm}(\Omega)$ is tested only for that specific solution.

In this chapter the Theorem 11 is improved in two ways. First, we prove that the solution can be obtained as a minimum of the associated functional, in addition, we obtain regularity for any solution. As a second improvement, we note that we can consider the case $h \in L^1(\Omega)$, not necessarily constant, and it can vanishes in subsets of Ω. For example, we prove the existence and regularity of solution when h vanishes in$\Omega_\delta = \{x \in \Omega : \text{ dist } (x, \partial\Omega) < \delta\}$ for δ enough small. Furthermore, as regards the existence of solutions, it is sufficient that $h \in L^1_{loc}(\Omega)$.

As discussed at the beginning of this Part III, since problem (28) has a variational characterization, the choice of a suitable space of functions will be advantageous when it comes to finding critical points. Indeed, we consider the reflexive space $E = W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega; hdx)$ and the functional energy $\mathcal{I}_\lambda : E \to \mathbb{R}$ defined as follows

$$\mathcal{I}_\lambda(u) = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1} \int_\Omega |u|^{p+1} h - \frac{\lambda}{2} \int_\Omega \frac{u^2}{|x|^2} - \int_\Omega f\, u\, h, \ \forall u \in E\,.$$

Note that every function $f \in L^{\frac{p+1}{p}}(\Omega; hdx)$ has associated a functional φ_f in the dual space E^* given by

$$\langle \varphi_f, g\rangle = \int_\Omega fgh, \ \forall g \in L^{p+1}(\Omega; hdx).$$

We show that $\mathcal{I}_\lambda$ is coercive and bounded from below. By using the Ekeland Variational Principle we also prove that a suitable minimizing sequence of this functional is weakly convergent to a critical point $u \in E$. In this way, in Theorem 9.2.1, we establish the existence of solutions, under the condition of integrability

$$\int_{\Omega\setminus\Omega_\delta} |x|^{\frac{2(p+1)}{1-p}} h(x)^{\frac{2}{1-p}} < \infty\,. \tag{30}$$

Observe that for $h(x) \equiv h_0 > 0$, the above condition is equivalent to $p > 2^* - 1$, imposed in Theorem 11. Moreover, under a condition somewhat stronger than (30): there exists $\bar{s} \in (2, p+1)$ such that

$$\int_{\Omega\setminus\Omega_\delta} |x|^{\frac{2\bar{s}}{2-\bar{s}}} h(x)^{\frac{2\bar{s}}{(p+1)(2-\bar{s})}} < \infty,$$

the functional $\mathcal{I}_\lambda$ is weak lower semicontinuous, then the solution is a minimum of $\mathcal{I}_\lambda$.

Regarding the regularity of the solutions in the Theorem 9.3.1, we establish that every solution to the problem (28) belongs to $W_0^{1,2}(\Omega) \cap L^{pm}(\Omega; h(x)dx)$ provided that:

i) $h \in L^1(\Omega)$ and $h(x) > 0$ for a.e. $x \in \Omega$,

ii) $|x|^{\frac{2pm}{1-p}} h^{1-\frac{pm}{p-1}} \in L^1(\Omega)$,

iii) $f \in L^m(\Omega; h(x)dx)$ with $m \geq \frac{p+1}{p}$.

Once again, the regularizing effect of the term $h(x)|u|^{p-1}u$ is evident since, a priori, the solutions belong to $W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega; h(x)dx)$.

Finally, an interesting case where the previous result is applied is $h(x) \geq \dfrac{\mu}{|x|^\beta}$ with $\mu > 0$ and $0 \leq \beta < N$. Where we obtain that the solutions belong to $W_0^{1,2}(\Omega) \cap L^{pm}(\Omega; h(x)dx)$ with

- $m \in \left[\frac{p+1}{p}, \frac{(N-\beta)(p-1)}{(2-\beta)p}\right)$, if $0 \leq \beta < 2$,
- $m \in \left[\frac{p+1}{p}, \infty\right)$, if $2 \leq \beta < N$.

Thus, in the case $\beta = 0$ (corresponding to h is constant) we obtain the reguarity result of Theorem 11 but for every solution, instead of for a solution obtained as limit of solutions of approximate problems.

Concave-Convex problem with a discontinuous operator

As explained above, in Chapter 10 we study the existence of positive solutions to the following problem

$$\left\{\begin{array}{ll} -\Delta_{p(x)}u = \lambda u^q, & \text{in}\Omega\ , \\ \quad u = 0, & \text{on } \partial\Omega, \end{array}\right. \tag{31}$$

where $\lambda > 0$, $1 < q < p-1$,$\Delta_{\ p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ and $p(x)$ is a discontinuous function given by

$$p(x) = \left\{\begin{array}{ll} 2 & \text{if } x \in D_1, \\ p & \text{if } x \in D_2, \end{array}\right.$$

being $p > 2$ and D_1, D_2 subdomains with smooth boundary and such that

$$\overline{\Omega} = \overline{D_1 \cup D_2}, \qquad D_1 \cap D_2 = \emptyset\,.$$

We call Γ the interface (or surface) inside Ω, $\Gamma = \partial D_1 \cap \Omega = \partial D_2 \cap \Omega$, and we assume that Γ is a smooth surface with finite $(N-1)$ dimensional Hausdorff measure.

To raise the problem (31) variationally, we will decompose the differential equation in two differential equations, one in each subdomain D_i $(i = 1, 2)$. To that end, we must provide a "continuity" of the solution when it crosses from one region to another, in the sense that the trace of u on Γ coincides coming from D_1 and coming from D_2, and also we must provide continuity of the associated fluxes acrossΓ . In this way, we consider solutions to problem (31) as weak solutions to the following problem:

$$\left\{\begin{array}{lll} -\Delta u = \lambda u^q, & & \text{in } D_1, \\ -\Delta_p u = \lambda u^q, & & \text{in } D_2, \\ \dfrac{\partial u}{\partial\eta} = |\nabla u|^{p-2}\dfrac{\partial u}{\partial\eta}, & u|_{D_1} = u|_{D_2}, & \text{on}\Gamma\ , \\ u = 0, & & \text{on } \partial\Omega, \end{array}\right. \tag{32}$$

being η the normal unit vector to Γ pointing outwards D_1. The adequate space to find weak solutions is

$$\mathcal{W}(\Omega) = \left\{ v \in W_0^{1,2}(\Omega) : \int_{D_2} |\nabla v|^p < \infty \right\},$$

which equipped with the norm

$$[v]_{\mathcal{W}(\Omega)} := \|\, \nabla v\, \|_{L^2(D_1)} + \|\, \nabla v\, \|_{L^p(D_2)}\,,$$

is a reflexive and separable Banach space (Lemma 10.2.1). In this way, in Lemma 10.2.5 we prove that positive solutions of (32) are uniquely identified as being positive critical points for the functional

$$\mathcal{F}_\lambda(u) = \int_{D_1} \frac{|\nabla u|^2}{2}\, dx + \int_{D_2} \frac{|\nabla u|^p}{p}\, dx - \lambda \int_\Omega \frac{|u|^{q+1}}{q+1}\, dx.$$

Then, by using the method of sub and supersolution, we prove the existence of $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$ there exists w_λ, minimal and positive solution. For this, a comparison principle and a maximum principle for the problem are needed (see Proposition 10.3.2). Furthermore, w_λ is unique and increasing respect to λ. On the other hand, if $\lambda > \lambda^*$ then there is no positive solution. For the nonexistence we use the fact that solutions to the parabolic problem $u_t = \Delta u + \lambda u^q$ in D_1, with large initial data, blow up in finite time (Theorem 10.1.1).

In Theorem 10.1.2 we establish, under the assumptions $p < 2^*$ and $D_2 \subset\subset \Omega$, the existence of a second solution for almost every $0 < \lambda < \lambda^*$. To prove it we argue in two steps: First, using variational methods and the works of Ambrosetti et al. (1994); Brézis and Nirenberg (1993); García Azorero et al. (2000), we prove that (32) has a solution which is a local minimum of the corresponding energy functional $\mathcal{F}_\lambda$ (Theorem 10.4.6). For this result, since the $p(x)-$laplacian operator with $p(x)$ discontinuous acts differently in D_1 and in D_2, we can only get regularity of solutions at locally Hölder spaces (see Acerbi and Fusco (1994)). Then, to show that there is a local minimum in $\mathcal{W}(\Omega)$, we assume that $D_2 \subset\subset \Omega$ in order to get $\mathcal{C}^1$ regularity close to $\partial\Omega$ and then we show that there is a minimum in the stronger topology $\mathcal{C}^1(F_\delta) \cap \mathcal{C}(\overline{\Omega})$ where

$$F_\delta = \{x \in \Omega \,:\, \text{dist}(x, \partial\,\Omega) < \delta\}.$$

is a small strip around the boundary of Ω. Then, by using a delicate regularity argument, we relax the topology to $\mathcal{W}(\Omega)$. Here we use partially the ideas from Ambrosetti et al. (1994); Brézis and Nirenberg (1993); García Azorero et al. (2000) adapting them to our setting with the introduction of a new original trick while using Stampacchia's approach in Proposition 10.4.5 in order to obtain an $L^\infty-$bound. It is at this point where we use that $p < 2^*$. Note that our space of solutions $\mathcal{W}(\Omega)$ is a subspace of $W_0^{1,2}(\Omega)$ that is larger than $W_0^{1,p}(\Omega)$. Secondly, in order to prove the existence of a second solution, note that the functional $\mathcal{F}_\lambda$ does not have a global minimum. Indeed, let v be a function in $\mathcal{W}(\Omega)$ with compact support in D_1, then, since we have that $q > 1$,

$$F_\lambda(tv) = t^2 \int_{D_1} \frac{|\nabla v|^2}{2}\, dx - t^{q+1}\lambda \int_{D_1} \frac{|v|^{q+1}}{q+1}\, dx \to -\infty \tag{33}$$

as $t \to \infty$. Thus, $\mathcal{F}_\lambda$ has the desired Mountain Pass geometry. However, the main difficulty is to show that Palais-Smale sequences are bounded in $\mathcal{W}(\Omega)$. This question is at present far from being solved and an affirmative answer would allow to find a second solution for all $\lambda \in (0, \lambda^*)$ instead of for almost every $\lambda \in (0, \lambda^*)$. We recall that in previous references involving the search for critical points of Mountain Pass type for semilinear elliptic equations problems like

$$\begin{cases} -\Delta u = f(x,u), & \text{in}\Omega\ , \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

it is usually assumed that the following condition is satisfied

$$\exists\, \kappa > 2 \text{ such that } 0 \le \kappa F(x,s) \le s f(x,s), \quad \forall\, s \ge 0 \text{ and a.e. } x \in \Omega\,. \tag{A-R}$$

This condition was originally introduced in Ambrosetti and Rabinowitz (1973) and it is called Ambrosetti-Rabinowitz type condition. Roughly speaking, the role of (AR) is to ensure that all Palais-Smale sequences at the mountain pass level are bounded. Adapting this result to our variable operator$\Delta\ u\chi_{D_1} + \Delta_p u\chi_{D_2}$ it is not difficult to prove that if $f(x, s)$ satisfies property (AR) for $\kappa > p$, then we have that Palais-Smale sequences are bounded. However, in our setting $f(x, s) = \lambda s^q$ and (AR) is not satisfied for $\kappa > p$ because $q + 1 < p$. Moreover, even conditions weaker than (AR) present in the literature of elliptic equations ensuring the existence of bounded Palais-Smale sequences are not applicable to our problem. To tackle this obstacle, we use some results from the classic works Ambrosetti and Rabinowitz (1973); De Figueiredo (1989); Ghoussoub and Preiss (1989); Jeanjean (1999) again adapting them to our framework. Mainly, relying on a result by Jeanjean (1999) which shows the existence a bounded Palais-Smale sequence at mountain pass level for almost every $0 < \lambda < \lambda^*$.

Chapter 1

Improved energy methods for nonlocal diffusion problems

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DOI:10.3934/dcds.2018057

Abstract

We prove an energy inequality for nonlocal diffusion operators of the following type, and some of its generalisations:

$$Lu(x) := \int_{\mathbb{R}^N} K(x,y)(u(y) - u(x))\,\mathrm{d}y,$$

where L acts on a real function u defined on $\mathbb{R}^N$, and we assume that $K(x,y)$ is uniformly strictly positive in a neighbourhood of $x = y$. The inequality is a nonlocal analogue of the Nash inequality, and plays a similar role in the study of the asymptotic decay of solutions to the nonlocal diffusion equation $\partial_t u = Lu$ as the Nash inequality does for the heat equation. The inequality allows us to give a precise decay rate of the L^p norms of u and its derivatives. As compared to existing decay results in the literature, our proof is perhaps simpler and gives new results in some cases.

1.1 Introduction

In this paper we develop *energy methods* which are useful in the study of some partial differential equations involving nonlocal diffusion terms. We start by the basic example which is the following integro-differential equation in convolution form:

$$\partial_t u(t,x) = \int_{\mathbb{R}^N} J(x-y)\big(u(t,y)-u(t,x)\big)\,\mathrm{d}y, \qquad u(0,x)=u_0(x) \tag{1.1}$$

where $t \geq 0$ is the time variable, $x \in \mathbb{R}^N$ is the space variable, $u = u(t,x) \in \mathbb{R}$ is the unknown, and J is the *diffusion kernel*. Typically one assumes that J is smooth, nonnegative, radially symmetric, and with integral 1; we also mention a variety of models with different assumptions and variations of (1.1) in Section 1.4. Equation (1.1) and its relatives appear as a nonlocal version of the usual diffusion equation $\partial_t u = \Delta u$, and it is known that (1.1) approximates it when J is close to a Delta function (see Theorem 1.1.8 and the remarks before it).

We will apply energy methods to deal with nonlocal problems that not necessarily involve a convolution. That is, problems of the form

$$\partial_t u(t,x) = \int_{\mathbb{R}^N} K(x,y)(u(t,y)-u(t,x))\,\mathrm{d}y, \tag{1.2}$$

where our main hypotheses on K can be summarized as follows: $K(x,y)$ is a nonnegative symmetric function with $\sup_{y\in\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)dx \leq C_K$ and such that K is strictly positive in a neighborhood of the closet set $\{x=y\}$. Furthermore, the symmetry of K can be replaced by integrability conditions (see Subsection 1.4.2). On the other hand, observe that it makes sense to assume that $K(x,x)>0$ since in many models it means that the probability that individuals remain for some time at the point where they are is positive.

As a particular application which motivates our arguments we consider the nonlocal dispersal model proposed by Cortázar et al. (2007) (see also Cortázar et al. (2011); Cortázar et al. (2015); Cortázar et al. (2016)):

$$\partial_t u(t,x) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(t,y)}{g(y)}\,\mathrm{d}y - u(t,x), \qquad \text{in } \mathbb{R}\times[0,\infty), \tag{1.3}$$

with a prescribed initial data $u(x,0)=u_0(x)$ defined on $\mathbb{R}$. Here J is an even, positive, smooth function such that $\int_{\mathbb{R}} J(x)\,\mathrm{d}x = 1$ and supp $J=[-1,1]$, and g is a continuous positive function which accounts for the *dispersal distance* which depends on the departing point. In this model u represents the spatial distribution of a certain species, and g models the heterogeneity of the environment which can affect the distribution of a species through space-dependent dispersal strategies. For this model we are able to give an explicit rate of decay of the L^p norm of solutions, which is to our knowledge a new result (see Theorem 1.4.3).

The driving idea of our methods is that solutions to (1.1) behave in many ways like solutions to the heat equation

$$\partial_t u = \Delta u, \qquad u(0,x) = u_0, \tag{1.4}$$

where as usual the Laplacian Δ acts only on the space variable x (see Theorem 1.1.8 and the comments before it). For more details we refer the reader to Sun et al. (2011) for the Cauchy problem, Cortázar et al. (2009) for Dirichlet boundary conditions (see also Molino and Rossi (2016) in a more general framework) and Cortázar et al. (2008) for Neumann boundary conditions. One important property of (1.4) is the following time decay of solutions: there is a constant $C = C(N,p) > 0$ such that

$$\|u\|_p^p \leq \left(\|u_0\|_p^{-p\gamma} + C\|u_0\|_1^{-p\gamma} t\right)^{-\frac{1}{\gamma}}, \qquad \text{for all } t \geq 0, \tag{1.5}$$

which holds for any $1 < p < +\infty$ and any initial condition $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ nontrivial, and where

$$\gamma := \frac{2}{N(p-1)}.$$

In fact, it still holds for $u_0 \in L^1(\mathbb{R}^N)$ and all $t > 0$ by removing the term $\|u_0\|_p^{-p\gamma}$. Here and below, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space of p-integrable functions on $\mathbb{R}^N$, with associated norm denoted by $\|\cdot\|_p$. There are several ways of showing this decay and regularization property for the heat equation. One of them is noticing that the L^p norms are *Lyapunov functionals* for (1.4): if u solves (1.4) with $u_0 \in L^p(\mathbb{R}^N)$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p = -\frac{4(p-1)}{p}\int_{\mathbb{R}^N} \left|\nabla(u^{\frac{p}{2}})\right|^2. \tag{1.6}$$

One can then compare the right hand side term to $\|u\|_p$ by using the Gagliardo-Nirenberg-Sobolev inequality (which in this particular case is known as the Nash inequality Nash (1958))

$$\|v\|_2 \leq C_N \ \|\nabla v\|_2^\theta \ \|v\|_1^{1-\theta}, \tag{1.7}$$

with

$$\theta := \frac{N}{N+2}.$$

This inequality is valid in any dimension N; in dimensions $N \geq 3$ it can easily be obtained as a consequence of the more familiar Sobolev inequality $\|u\|_{2^*} \leq C\|\nabla u\|_2$, where $2^* := 2N/(N-2)$. By using (1.7) with $v = u^{p/2}$ we obtain for any $p \geq 2$ that

$$\int_{\mathbb{R}^N} \left|\nabla(u^{\frac{p}{2}})\right|^2 \geq C_N^{-\frac{2}{\theta}} \ \|u\|_p^{\frac{p}{\theta}} \ \|u\|_{\frac{p}{2}}^{-\frac{p(1-\theta)}{\theta}} \geq C_N^{-\frac{2}{\theta}} \ \|u\|_p^{p(1+\gamma)} \ \|u\|_1^{-p\gamma}, \tag{1.8}$$

where the last step is obtained through an interpolation of $\|u\|_{p/2}$ between $\|u\|_p$ and $\|u\|_1$. Due to mass conservation for the heat equation we have $\|u\|_1 \leq \|u_0\|_1$ for all times $t \geq 0$ (this inequality is of course an equality for nonnegative, finite-mass solutions). Hence using (1.8) in (1.6) one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p \leq -C\,\|u\|_p^{p(1+\gamma)}\,\|u_0\|_1^{-p\gamma},$$

for some constant $C = C(N,p)$. This is a differential inequality for $\|u\|_p$ that readily gives the decay (1.5).

In the context of diffusion equations, the strategy of using the L^p norm of u and its derivative as a means for studying properties of solutions is known as the *energy method*. It is a close relative of a common and quite successful strategy in kinetic equations and dissipative PDE sometimes known as the *entropy method* (Arnold et al., 2004; Bakry and Émery, 1985; Bonforte et al., 2010; Carrillo et al., 2001; Desvillettes and Villani, 2004; Gross, 1975; Otto and Villani, 2000; Villani, 2002), where one compares the time derivative of a Lyapunov functional with the Lyapunov functional itself via a functional inequality in order to obtain a certain decay rate for solutions. These energy methods have the advantage of being quite robust, often being applicable to equations that are not explicitly solvable by Fourier transform methods, and to nonlinear problems. The question that motivates this paper is whether these ideas can be adapted to equation (1.1) in order to show a decay property similar to (1.5). One important observation is that the same statement cannot be true for solutions of (1.1), since there is no instantaneous L^1 to L^p regularization. In fact, the L^p norms are still a Lyapunov functional for (1.1) (as is well known, any convex function gives a Lyapunov functional for (1.1)): if u is an L^p solution to (1.1) then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p = -\mathcal{D}_p^J(u). \tag{1.9}$$

Here, the L^p *dissipation* $\mathcal{D}_p^J(u)$ is defined for any measurable $u\colon \mathbb{R}^N \to \mathbb{R}$ as

$$\mathcal{D}_p^J(u) := \frac{p}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\big(u(x)-u(y)\big)\big(\phi_{p-1}(u(x))-\phi_{p-1}(u(y))\big)\,\mathrm{d}x\,\mathrm{d}y, \tag{1.10}$$

where for $q > 0$ we denote by ϕ_q the antisymmetric extension of the usual q-th power, that is,

$$\phi_q(s) := |s|^q \operatorname{sgn}(s), \qquad s \in \mathbb{R}.$$

Of course, since ϕ_{p-1} is nondecreasing, the integrand in (1.10) is also nonnegative and always makes sense as a number in $[0,+\infty]$. We point out that for nonnegative u the expression becomes a bit simpler,

$$\mathcal{D}_p^J(u) := \frac{p}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\big(u(x)-u(y)\big)\big(u(x)^{p-1}-u(y)^{p-1}\big)\,\mathrm{d}x\,\mathrm{d}y.$$

Precisely this strategy was discussed in Ignat and Rossi (2009), where it was remarked that no inequality of the following form can hold, for any $q > 2$ and a smooth, nonnegative, compactly supported function J:

$$\mathcal{D}_2^J(u) \geq C\|u\|_q^2.$$

Hence the natural analogue of the usual Sobolev inequality does not hold in the nonlocal case. Similarly, the direct analogue of (1.8) (with $\mathcal{D}_p^J(u)$ on the left hand side) cannot hold, since it would imply an $L^1 - L^p$ regularization effect on (1.1) which is known to fail. In view of this failure, a different strategy was followed there, leading to different inequalities and applications to several linear and nonlinear equations involving nonlocal diffusions. Similar ideas were developed in Brändle and de Pablo (2015) in order to establish decay estimates for fractional diffusions, with modified inequalities used in place of the usual Nash inequality. After the statement of our results we compare them in more detail to those in Brändle and de Pablo (2015); Ignat and Rossi (2009) and other previous works.

Main results. Our purpose is to show a simple inequality that plays the role of (1.8) and provides a means to show precise decay properties of (1.1) and (1.2):

Hypothesis 1 *$J\colon \mathbb{R}^N \to [0,+\infty)$ is a measurable function such that for some $r, R > 0$ we have*

$$J(z) \geq r, \quad \textit{for all } |z| < R\,. \tag{1.11}$$

In particular, this is obviously satisfied if J is continuous in a neighborhood of 0 with $J(0) > 0$.

Theorem 1.1.1 (L^p energy inequality) *Let $J : \mathbb{R}^N \to \mathbb{R}$ be a function satisfying Hypothesis 1. For every $N \geq 1$ and $p \geq 2$, there exists a positive constant $C = C(N,p) > 0$ such that*

$$\mathcal{D}_p^J(u) \geq Cr \min\left\{ R^{N+2} \|u\|_1^{-p\gamma} \, \|u\|_p^{p(1+\gamma)},\ R^N \|u\|_p^p \right\}, \tag{1.12}$$

for all $u \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, where $\gamma := \frac{2}{N(p-1)}$.

This inequality serves as a useful analogue of (1.8) in the nonlocal case, as we will see shortly. If one does not care about the precise dependence of the constant C on J then this can be more simply stated as: there exists a constant $C = C(N,p,J)$ depending only on N, p and J such that

$$\mathcal{D}_p^J(u) \geq C \min\left\{ \|u\|_1^{-p\gamma} \, \|u\|_p^{p(1+\gamma)},\ \|u\|_p^p \right\}. \tag{1.13}$$

The constants in the above inequalities can be estimated explicitly by following the proof. To our knowledge, inequality (1.12) is new. Similar modified Nash inequalities are considered in Carlen et al. (1987); Ignat and Rossi (2009), and especially in Brändle and de Pablo (2015)[Corollary 4.7]. In the latter, (p,q)-inequalities involving the p and q norms of u are given for $p > q > 1$; ours is the limiting case with $q = 1$, not included there. We notice the L^1 case is fundamental for the generalisations we describe later, since mass is a natural conserved quantity in many models.

The inequality in Theorem 1.1.1 immediately allows us to deduce bounds on the asymptotic behaviour of several nonlocal diffusion equations (see Section 1.4). Let us give the argument for equation (1.1), which is the simplest possible model: using (1.9) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p = -\mathcal{D}_p^J(u) \le -Cr \min\left\{R^{N+2}\|u\|_1^{-p\gamma}\,\|u\|_p^{p(1+\gamma)},\ R^N\|u\|_p^p\right\}.$$

Taking into account that $\|u\|_1$ is nonincreasing in time (it is conserved for nonnegative solutions) one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p \le -Cr \min\left\{R^{N+2}\|u_0\|_1^{-p\gamma}\,\|u\|_p^{p(1+\gamma)},\ R^N\|u\|_p^p\right\}.$$

This is a differential inequality for $\|u\|_p$, which can be solved (see Lemma 1.4.1) to yield the following result:

Theorem 1.1.2 *Take a function $J \in L^1(\mathbb{R}^N)$ satisfying Hypothesis 1 and $p \in [2,+\infty)$. Consider the solution u to equation (1.1) with initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. There exists a constant $C = C(N,p)$ (the same as in Theorem 1.1.1) such that*

$$\|u\|_p^p \le \begin{cases} \|u_0\|_p^p & \text{for } 0 \le t \le t_0, \\ \left(\|u_0\|_p^{-p\gamma} + C\gamma r R^{N+2}\|u_0\|_1^{-p\gamma}(t-t_0)\right)^{-\frac{1}{\gamma}} & \text{for } t \ge t_0, \end{cases} \tag{1.14}$$

where $\gamma := \frac{2}{N(p-1)}$ and

$$t_0 = \max\left\{0,\ \frac{1}{CrR^N}\log\left(R^{\frac{2}{\gamma}}\|u_0\|_1^{-p}\|u_0\|_p^p)\right)\right\}.$$

Again, if we are not interested in the precise dependence of the bound on J, $\|u_0\|_1$ and $\|u_0\|_p$ then the following statement is simpler: there exists a constant $C = C(r,R,N,p,\|u_0\|_1,\|u_0\|_p)$ such that

$$\|u\|_p^p \le C(1+t)^{-\frac{N(p-1)}{2}} \qquad \text{for all } t \ge 0. \tag{1.15}$$

This is a direct consequence of the bound (1.14); see Remark 1.4.2. In this sense, Theorem 1.1.1 is a nonlocal analogue of the Gagliardo-Nirenberg-Sobolev (or Nash) inequality: it allows us to give a decay rate of the nonlocal diffusion equation (1.1), and in fact this decay rate approaches that of the heat equation as (1.1) approaches it (see Theorem 1.1.8). Furthermore, due to the interpolation formula $\|u\|_q \le \|u\|_1^s\|u\|_2^{1-s}$ for $q \in [1,2]$ and $s = \frac{2-q}{q}$, (1.15) also holds for $1 \le p \le 2$ and some $C = C(J,N,p,\|u_0\|_1,\|u_0\|_2)$.

We also give inequalities related to higher derivatives of u in Section 1.3, and deduce from them corresponding decay properties of derivatives of u, still at the same asymptotic rate as those for the heat equation. For $k \ge 0$ we define the differential operator D^k acting on a function u as

$$D^k u := -(-\Delta)^{k/2}u.$$

In order to ensure that this expression makes sense we will always assume that $u \in H^k(\mathbb{R}^N)$ (i.e., the classical Sobolev space $W^{k,2}(\mathbb{R}^N)$) when applying D^k. The following result gives an estimate of $\mathcal{D}_2^J(D^k u)$; note that the case $k = 0$ is just the $p = 2$ case of Theorem 1.1.1:

Theorem 1.1.3 *Let $N \geq 1$ be an integer and $J : \mathbb{R}^N \to \mathbb{R}$ be a function satisfying Hypothesis 1. There exists a positive constant $C = C(N)$ such that*

$$\mathcal{D}_2^J(D^k u) \geq Cr \min\left\{ R^{k+N+2} \|u\|_1^{-\frac{4}{N+2k}} \|D^k u\|_2^{2+\frac{4}{N+2k}},\ R^{k+N}\|D^k u\|_2^2 \right\} \tag{1.16}$$

for all $u \in H^k(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.

As a consequence one can obtain a decay of higher derivatives of solutions to (1.1). Notice that the case $k = 0$ of the following result is just Theorem 1.1.2 with $p = 2$:

Theorem 1.1.4 *Take a function J satisfying Hypothesis 1 and a real $k \geq 0$. Consider the solution u to equation (1.1) with initial data $u_0 \in L^1(\mathbb{R}^N) \cap H^k(\mathbb{R}^N)$. There exists a constant $C = C(N,k)$ (the same as in Theorem 1.1.3) such that*

$$\|D^k u\|_2^2 \leq \begin{cases} \|D^k u_0\|_2^2 & \text{for } 0 \leq t \leq t_0, \\ \left(\|D^k u_0\|_2^{-2\gamma} + Cr\gamma R^{k+N+2}\|u_0\|_1^{-2\gamma}(t-t_0)\right)^{-\frac{1}{\gamma}} & \text{for } t \geq t_0, \end{cases}$$

where $\gamma := \frac{2}{N+2k}$ and

$$t_0 = \max\left\{0,\ \frac{1}{CrR^{k+N}} \log\left(R^{\frac{2}{\gamma}}\|u_0\|_1^{-2}\|D^k u_0\|_2^2\right)\right\}.$$

The decay in Theorem 1.1.2 can be interpreted as follows: for large times, the asymptotic decay of the L^p norm of solutions to the nonlocal diffusion equation (1.1) is the same as that of the heat equation. However, there can be an initial time during which a different decay takes place. The threshold between the two is related to the value of the L^p norm of u: if it is large then heuristically (since we are assuming u_0 is integrable) the main contribution to the L^p norm comes from local concentrations of u. Since the smoothing effect of (1.1) is much weaker than that of the heat equation, the rates of decay of the two differ. On the other hand, when $\|u\|_p$ is small, the concentrations of u do not play a major role and the decay of both equations becomes comparable. The inequality (1.12) and the corresponding decay (1.14) are quite precise on the dependence on J and the initial data, giving a direct estimate of the time when the "heat-like" diffusion kicks in: the time t_0 depends logarithmically on the ratio between $\|u_0\|_p$ and $\|u_0\|_1$.

Theorem 1.1.2 as stated is not new; the simplified statement (1.15) can be proved for example by Fourier transform methods (Andreu-Vaillo et al., 2010), and the decay (1.14) can probably be obtained as well. The important advantage of using Theorem 1.1.1 to prove Theorem 1.1.2 is that the method is quite robust under modifications in the linear operator. In Subsection 1.4.2 we prove a result similar to Theorem 1.1.2 which gives decay properties for general nonlocal diffusion equations with a more general kernel $K(x,y)$ instead of $J(x-y)$: consider the equation

$$\partial_t u(t,x) = \int_{\mathbb{R}^N} K(x,y)u(t,y)\,\mathrm{d}y - \sigma(x)u(t,x), \tag{1.17}$$

where $K\colon \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ is a general kernel (not necessarily symmetric) and $\sigma\colon \mathbb{R}^N \to [0,+\infty)$ is a function. Let us keep our discussion at a formal level for the moment and not worry about the problem of existence of solutions to (1.17) or the precise regularity of K and σ. Equation (1.17) is a general form of the *scattering equation* (see for example Michel et al. (2004)), and contains many others as a particular case. The nonlocal diffusion (1.1) is recovered if $K(x,y) = J(x-y)$ and $\sigma(x) = \int_{\mathbb{R}^N} J$ for all x, y. In the case that $\sigma(x) = \int_{\mathbb{R}^N} K(x,y)\,\mathrm{d}y$ the equation can be written as

$$\partial_t u(t,x) = \int_{\mathbb{R}^N} K(x,y)\big(u(t,y) - u(t,x)\big)\,\mathrm{d}y, \tag{1.18}$$

which is a type of nonlocal diffusion equation, where the nonlocality is not given by a convolution. Similarly, if we assume

$$\sigma(x) = \int_{\mathbb{R}^N} K(y,x)\,\mathrm{d}y, \tag{1.19}$$

then equation (1.17) is formally the Kolmogorov forward equation for a Markov jump process with jump rates given by K, where u represents the probability density of the process (Ethier and Kurtz, 1986, Chapter 4.2). Notice that (1.19) is just the statement that the total mass $\int_{\mathbb{R}^N} u(t,x)\,\mathrm{d}x$ is formally conserved in time (as should happen for a probabilistic evolution). In that sense, equation (1.17) contains many evolution equations linked to Markov processes, and has multiple applications. (We give an example linked to a population dispersal in Section 1.4.3.) Equation (1.17) has some properties in common with diffusion processes, but it is important to notice that (1.17) may have finite-mass equilibria (unlike the usual heat equation, whose only finite-mass equilibrium is 0).

Let us state a precise result which is relevant for nonlocal diffusions. For all of them we will assume:

Hypothesis 2 *There exist $r, R > 0$ such that $K(x,y) \geq r$ whenever $|x-y| < R$.*

This is the analogue of Hypothesis 1 in this setting. In order to ensure that L^p solutions of (1.17) exist we will also assume that K is measurable and that for some $C_K > 0$

$$\int_{\mathbb{R}^N} K(x,y)\,\mathrm{d}y \leq C_K, \qquad \int_{\mathbb{R}^N} K(y,x)\,\mathrm{d}y \leq C_K, \qquad \text{for all } x \in \mathbb{R}^N. \tag{1.20}$$

This ensures that the linear operator on the right hand side of (1.17) is bounded in $L^1(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ (and hence, by interpolation, in any $L^p(\mathbb{R}^N)$ with $1 \leq p \leq \infty$).

Theorem 1.1.5 *Take $p \in [2,+\infty)$. Assume that $K\colon \mathbb{R}^N \times \mathbb{R}^N \to [0,+\infty)$ satisfies Hypothesis 2 and (1.20). Consider equation (1.17) with σ given by (1.19), and assume that there exists an equilibrium u_∞ of (1.17) satisfying*

$$\frac{1}{m} \leq u_\infty(x) \leq m, \qquad \textit{for all } x \in \mathbb{R}^N, \tag{1.21}$$

for some $m > 0$. Let u be any solution to equation (1.17) with initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. There exists a constant C depending only on r, R, N, m, p, $\|u_0\|_1$ and $\|u_0\|_p$ such that

$$\|u\|_p^p \leq C(1+t)^{-\frac{N(p-1)}{2}}, \qquad \textit{for all } t \geq 0.$$

In Section 1.4.3 we give an application of these results to a dispersal equation proposed in Cortázar et al. (2007), obtaining an explicit rate of convergence to equilibrium.

Remark 1.1.6 *Condition (1.20) is just included in order to ensure that there are well-defined solutions to (1.17), but it does not play a role in the decay estimates. It can be removed if it can be justified by other means that solutions to (1.17) exist and rigorously satisfy the entropy property (1.9).*

Remark 1.1.7 *In Theorem 1.1.5 one can also give a more precise estimation of the decay and the constants involved, as we did in Theorem 1.1.2. We have preferred in this case to leave the statement in this form for simplicity, but the reader can state the analogue of Theorem 1.1.2 without difficulty.*

We refer to Section 1.4.2 for details on this and a proof of Theorem 1.1.5.

Heat equation scaling. It is worth mentioning that Theorems 1.1.1 and 1.1.2 pass to the limit well when the nonlocal equation (1.1) approximates the heat equation. Let J be a smooth and radially symmetric convolution kernel with $J(0) > 0$, and denote by J_ϵ the rescaling

$$J_\varepsilon(z) := \frac{C(J)}{\varepsilon^{2+N}} J\left(\frac{z}{\varepsilon}\right), \qquad \text{with } C(J)^{-1} = \frac{1}{2}\int_{\mathbb{R}^N} J(z) z_N^2 \, \mathrm{d}z.$$

It is well-known that, u^ε, the solution to the equation

$$\partial_t u^\varepsilon(t,x) = \int_{\mathbb{R}^N} J_\varepsilon(x-y)((u^\varepsilon(t,y) - u^\varepsilon(t,x)) \, \mathrm{d}y, \qquad x \in \mathbb{R}^N,\, t > 0, \tag{1.22}$$

with initial data $u_0 \in \mathcal{C}(\mathbb{R}^N)$ converges to the solution of the heat equation $\partial_t v = \Delta v$ with the same initial data (see for instance Andreu-Vaillo et al. (2010); Rey and Toscani (2013)). Since J satisfies Hypothesis 1 for some $r, R > 0$ one has $J_\varepsilon(z) \geq \frac{rC(J)}{\varepsilon^{2+N}}$, for all $|z| < R\,\varepsilon$ Replacing this in expression (1.14) the ε is cancelled and we obtain the following result:

Theorem 1.1.8 *Assume J satisfies Hypothesis 1. Let u^ε be a solution of (1.22) with initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with $p \in [2, \infty)$. Then it holds*

$$\|u^\varepsilon(t,\cdot)\|_p^p \leq \left(\|u_0\|_p^{-p\gamma} + C_1 \|u_0\|_1^{-p\gamma}(t-t_0)\right)^{-\frac{1}{\gamma}} \qquad \textit{for } t \geq t_0,$$

where $C_1 = C(N,p)\gamma r R^{N+2} C(J)$ *does not depend on* ε *and*

$$t_0 = \max\left\{0,\ \frac{\varepsilon^2}{CrR^N C(J)} \log\left(\varepsilon^{\frac{2}{\gamma}} R^{\frac{2}{\gamma}} \|u_0\|_1^{-p} \|u_0\|_p^p)\right)\right\}.$$

In particular, $t_0 = 0$ *for all* $\varepsilon < \varepsilon_0 = \|u_0\|_1^{\frac{\gamma p}{2}} / \big(R\|u_0\|_p^{\frac{\gamma p}{2}}\big)$.

The interest of the above theorem is that the decay is preserved in the scaling that leads to the heat equation. In addition, for small ε the expression of the decay is exactly of the same form as that of the heat equation, given in (1.5).

Comparison to results in the literature. Several precise results exist already regarding the decay properties of equation (1.1). Let us give a brief review and compare them to our own. Nonlocal diffusions including (1.1) have been studied in Chasseigne et al. (2006), and we refer the reader to the recent book Andreu-Vaillo et al. (2010) for background and an extensive review of the state of the art for equations involving similar nonlocal terms. A similar approximation to the heat equation, with a particular kernel J, was studied in Rey and Toscani (2013), and some nonlocal approximations to Fokker-Planck equations have been recently considered in Mischler and Tristani (2016) and very recently in Toscani (2017).

The observation that solutions to (1.1) decay asymptotically like the heat equation has been present since the first works on the matter, with several analogues of (1.5). The first ones were based on the Fourier transform of (1.1), which is explicitly solvable Chasseigne et al. (2006); Ignat and Rossi (2007, 2008). Energy methods were considered in Ignat and Rossi (2009); results were given on the decay of several models including the linear nonlocal diffusion equation (1.1) and a nonlocal version of the p-Laplacian evolution equation. The method in Ignat and Rossi (2009) is different from ours, and is based on a splitting of the function u into a "smooth" part and a "rough" part. The ideas are somehow reminiscent of ours, since they borrow techniques from Fourier splitting by Schonbek (1980) and there is a parallel with our splitting of the function u in Fourier space. The results from Ignat and Rossi (2009) are in dimensions $N \geq 3$ and K symmetric; on the other hand, they are well-adapted to nonlinear problems like the nonlocal p-Laplacian equation. Our inequality seems to be a simpler argument which works in any dimension, is well-adapted to the linear nonlocal diffusion operator, but does not easily carry over to nonlinear nonlocal operators. It also gives a simple way to track the dependence of the decay on the parameters of the problem, especially the diffusion kernel J.

Inequalities of the type (1.12) were already noticed in Brändle and de Pablo (2015), and used in order to obtain decay and regularisation properties for nonlinear diffusions of the type (1.1) where the function J typically behaves as $|x|^{-N-\alpha}$ as $x \to +\infty$, for some $0 < \alpha \leq 2$. Their proof goes along the lines of Ignat and Rossi (2009). Inequality (1.12) is a limit case of their results, but is not included there for similar reasons as in Ignat and Rossi (2009).

As compared to previous results, we summarise our contributions as follows:

1. Inequality (1.12) seems to be new. Similar ideas were used in Brändle and de Pablo (2015); Ignat and Rossi (2009), but (1.12) is a limiting case not included in these works.

2. Our proof of the inequality (1.12) is straightforward, works in any dimension, and in our opinion simplifies previous arguments for related inequalities. It also leads to a precise estimate of the constants in the inequality, which have in particular the correct scaling when approximating the heat equation (see Theorem 1.1.8).

3. A similar method of proof yields inequalities and decay results involving higher derivatives of the function u; see Section 1.3.

4. The entropy method used allows for an extension to linear mass-conserving equations with general kernels $K(x, y)$ (not necessarily symmetric) instead of $J(x - y)$; see Subsection 1.4.2.

The paper is organised as follows: in Section 1.2 we give the proof of the inequality in Theorem 1.1.1, and in Section 1.3 we prove similar inequalities involving derivatives. Finally, in Section 1.4 we show how these inequalities yield decay properties for several equations involving general kernels $K(x, y)$, in particular proving Theorem 1.1.2 in Subsection 1.4.1.

1.2 Energy inequalities for nonlocal diffusion operators

We are interested in finding useful lower bounds of $\mathcal{D}_p^J(u)$ in terms of L^p norms of u. Since $(|a| - |b|)(|a|^s - |b|^s) \leq (a - b)(\phi_s(a) - \phi_s(b))$ for any $a, b \in \mathbb{R}$ and $s > 1$ (where $\phi_s(a) := |a|^s \operatorname{sgn}(a)$), it is easily seen that

$$\mathcal{D}_p^J(u) \geq \mathcal{D}_p^J(|u|)$$

for any measurable $u \colon \mathbb{R}^N \to \mathbb{R}$. This allows us to work only with nonnegative functions u.

This section is devoted to the proof of Theorem 1.1.1. We first show the case $p = 2$, and then deduce from it the general inequality for $p \geq 2$. The proof of the $p = 2$ case is a modification of a the original proof of the Nash inequality (1.7) appearing in the paper by Nash (1958):

Lemma 1.2.1 *Let I be the normalised characteristic function of the unit ball in $\mathbb{R}^N$,*

$$I(z) := \frac{1}{\omega_N} \quad \text{if } |z| < 1, \qquad I(z) = 0 \quad \text{otherwise,} \tag{1.23}$$

where ω_N is the volume of the unit ball in dimension N. There exists a constant $C = C(N)$ depending only on N such that

$$\mathcal{D}_2^I(u) \geq C \min\left\{ \|u\|_1^{-\frac{4}{N}} \|u\|_2^{2+\frac{4}{N}},\ \|u\|_2^2 \right\}, \tag{1.24}$$

for all $u \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$.

We point out that the constant C can be estimated explicitly by following the calculations in the proof below.

Proof: Along the proof we call $C_1, C_2, \dots$ several constants that depend only on the dimension N. We will use the following property, which holds for some constant $C_1 > 0$:

$$1 - \hat{I}(\xi) \geq \frac{1}{C_1} \min\{1, |\xi|^2\}, \qquad \text{for all } \xi \in \mathbb{R}^N$$

or, in other words,

$$(1 - \hat{I}(\xi))^{-1} \leq C_1 \max\{1, |\xi|^{-2}\}, \qquad \text{for all } \xi \in \mathbb{R}^N. \tag{1.25}$$

Since I has integral one we can write, using that the Fourier transform is an isometry of $L^2(\mathbb{R}^N; \mathbb{C})$,

$$\mathcal{D}_2^I(u) = 2 \langle u, u - I * u \rangle = 2 \big\langle \hat{u}, (1 - \hat{I}) \hat{u} \big\rangle = 2 \int_{\mathbb{R}^N} (1 - \hat{I}) |\hat{u}|^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in the space of L^2 complex functions in $\mathbb{R}^N$. We can break the integral of $\|u\|_2$ in two parts, for any $\delta > 0$:

$$\|u\|_2^2 = \|\hat{u}\|_2^2 = \int_{|\xi| \leq \delta} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi + \int_{|\xi| > \delta} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi. \tag{1.26}$$

These two terms can be estimated as follows: for the first one,

$$\int_{|\xi| \leq \delta} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \leq \|u\|_1^2 \int_{|\xi| \leq \delta} \mathrm{d}\xi \leq \omega_N \delta^N \|u\|_1^2. \tag{1.27}$$

For the second one, using (1.25) and assuming $\delta < 1$ we have

$$\begin{aligned} \int_{|\xi| > \delta} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi &\leq C_1 \int_{|\xi| > \delta} \Big(1 - \hat{I}(\xi)\Big) \max\{1, |\xi|^{-2}\} \, |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \\ &\leq C_1 \int_{|\xi| > \delta} \Big(1 - \hat{I}(\xi)\Big) \max\{1, \delta^{-2}\} \, |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \\ &\leq \frac{C_1}{\delta^2} \int_{|\xi| > \delta} \Big(1 - \hat{I}(\xi)\Big) \, |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \leq \frac{C_1}{\delta^2} \mathcal{D}_2^I(u). \end{aligned} \tag{1.28}$$

Using (1.27) and (1.28) in (1.26) we obtain

$$\|u\|_2^2 \leq \omega_N \delta^N \|u\|_1^2 + \frac{C_1}{\delta^2} \mathcal{D}_2^I(u), \qquad \text{for any } 0 < \delta < 1. \tag{1.29}$$

We would like to optimise this quantity in δ, but *it is only valid for* $0 < \delta < 1$. If we could choose δ freely we would take the one that achieves the best bound in the inequality (1.29), that is,

$$\delta_0 := \left(\frac{2 C_1 \mathcal{D}_2^I(u)}{N \omega_N \|u\|_1^2} \right)^{\frac{1}{N+2}}.$$

Now we discuss two cases:

Case 1. If $\delta_0 < 1$, then replacing δ by δ_0 in (1.29) we have

$$\|u\|_2^2 \le \omega_N^{\frac{2}{N+2}} C_1^{\frac{N}{N+2}} \left(1+\frac{N}{2}\right)\left(\frac{2}{N}\right)^{\frac{N}{N+2}} \|u\|_1^{\frac{4}{N+2}} \mathcal{D}_2^I(u)^{\frac{N}{N+2}}.$$

Equivalently,

$$\mathcal{D}_2^I(u) \ge C_2 \|u\|_1^{-\frac{4}{N}} \|u\|_2^{2+\frac{4}{N}} \tag{1.30}$$

where $C_2 = \omega_N^{-\frac{2}{N}} C_1^{-1} \left(1+\frac{N}{2}\right)^{-\frac{N+2}{N}} \frac{N}{2}$.

Case 2. If $\delta_0 \ge 1$ then this means that

$$N\omega_N \|u\|_1^2 \le 2C_1 \mathcal{D}_2^I(u).$$

In this case, choosing $\delta = 1$ in (1.29) and using the above inequality we get

$$\|u\|_2^2 \le \omega_N \|u\|_1^2 + C_1\, \mathcal{D}_2^I(u) \le \left(1+\frac{2}{N}\right) C_1\, \mathcal{D}_2^I(u),$$

or

$$\mathcal{D}_2^I(u) \ge C_3 \|u\|_2^2 \tag{1.31}$$

with $C_3 := C_1^{-1}\left(1+\frac{2}{N}\right)^{-1}$.

Finally, summarising (1.30) and (1.31) we obtain

$$\mathcal{D}_2^I(u) \ge C_4 \min\left\{ \|u\|_1^{-\frac{4}{N}} \|u\|_2^{2+\frac{4}{N}},\ \|u\|_2^2 \right\} \tag{1.32}$$

with $C_4 := \max\{C_2, C_3\}$. This proves (1.24) with $C = C_4$. $\square$

Notice that $\mathcal{D}_2^J(u)$ satisfies the following scaling property. For $\lambda > 0$ and any function g on $\mathbb{R}^N$ we denote

$$g_\lambda(z) := g(z/\lambda), \qquad z \in \mathbb{R}^N.$$

Then one sees that

$$\mathcal{D}_2^{J_\lambda}(u) = \lambda^{2N} \mathcal{D}_2^J(u_{\frac{1}{\lambda}}). \tag{1.33}$$

This easily gives the following extension of Lemma 1.2.1:

Corollary 1.2.2 (L^2 energy inequality) *Let J satisfy Hypothesis 1. There is some constant $C = C(N)$ that depends only on the dimension N such that*

$$\mathcal{D}_2^J(u) \ge Cr \min\left\{ R^{N+2} \|u\|_1^{-\frac{4}{N}} \|u\|_2^{2+\frac{4}{N}},\ R^N \|u\|_2^2 \right\} \tag{1.34}$$

for all $u \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.

Proof:[Proof of Corollary 1.2.2] Call $I = I(z)$ the normalised characteristic of the unit ball, and define

$$K(z) := \frac{1}{r\,\omega_N} J(Rz), \qquad z \in \mathbb{R}^N.$$

Then

$$K(z) \geq I(z) \qquad \text{for all } z \in \mathbb{R}^N$$

so

$$\mathcal{D}_2^K(u) \geq \mathcal{D}_2^I(u).$$

Since $J = r\omega_N K_R$, due to the scaling (1.33) we have

$$\mathcal{D}_2^J(u) = r\,\omega_N R^{2N} \mathcal{D}_2^K(u_{\frac{1}{R}}) \geq r\,\omega_N R^{2N} \mathcal{D}_2^I(u_{\frac{1}{R}}).$$

Hence we can use Lemma 1.2.1 (writing C_N to denote the constant C in it) to say that

$$\begin{aligned}
\mathcal{D}_2^J(u) &\geq r\omega_N R^{2N} C_N \min\left\{ \|u_{\frac{1}{R}}\|_1^{-\frac{4}{N}} \|u_{\frac{1}{R}}\|_2^{2+\frac{4}{N}},\ \|u_{\frac{1}{R}}\|_2^2 \right\} \\
&= r\omega_N C_N \min\left\{ R^{N+2} \|u\|_1^{-\frac{4}{N}} \|u\|_2^{2+\frac{4}{N}},\ R^N \|u\|_2^2 \right\}.
\end{aligned}$$

This shows the result. □

Corollary 1.2.2 gives the case $p = 2$ of Theorem 1.1.1. In order to obtain the general case for $p \geq 2$ and complete the proof, let us first give a simple inequality in the next lemma:

Lemma 1.2.3 *Let $p > 1$, there exists $c(p) > 0$ such that*

$$(a-b)(a^{p-1} - b^{p-1}) \geq c(p)\,(a^{p/2} - b^{p/2})^2, \qquad \textit{for all } a, b \geq 0. \tag{1.35}$$

Proof: There is no loss of generality in assuming $a > b$. Dividing by a^p (which is not 0) and denoting $\theta = b/a \in [0,1)$, (1.35) is equivalent to showing that

$$F(\theta) := \frac{(1-\theta)(1-\theta^{p-1})}{(1-\theta^{p/2})^2} \geq c(p), \quad \theta \in [0,1).$$

It is a simple matter to check that F is decreasing in $[0,1)$, so one may take $c(p) = \lim_{\theta \to 1} F(\theta) = \frac{4(p-1)}{p^2} > 0$. □

We can now complete the proof of Theorem 1.1.1:

Proof:[Proof of Theorem 1.1.1] As explained at the beginning of Section 1.2, we may assume that u is nonnegative. By using the inequality in Lemma 1.2.3 we obtain

$$\begin{aligned}
\mathcal{D}_p^J(u) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(x)-u(y))(u(x)^{p-1} - u(y)^{p-1})dxdy \\
&\geq c(p) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(x)^{p/2} - u(y)^{p/2})^2 dxdy \\
&= c(p)\mathcal{D}_2^J(u^{p/2}).
\end{aligned}$$

Now, by virtue of Corollary 1.2.2, and calling C_N the constant in it, it follows that

$$\begin{aligned}\mathcal{D}_p^J(u) &\geq c(p)C_N r \min\left\{R^{N+2}\|u^{p/2}\|_1^{-\frac{4}{N}}\,\|u^{p/2}\|_2^{2+\frac{4}{N}},\, R^N\|u^{p/2}\|_2^2\right\}\\ &= c(p)C_N r \min\left\{R^{N+2}\|u\|_{\frac{p}{2}}^{-\frac{2p}{N}}\,\|u\|_p^{p(1+\frac{2}{N})},\, R^N\|u\|_p^p\right\}.\end{aligned}$$

Finally, due to the interpolation formula

$$\|u\|_{\frac{p}{2}} \leq \|u\|_1^{\frac{1}{p-1}}\,\|u\|_p^{\frac{p-2}{p-1}}$$

(note that $p \geq 2$ is used here) we conclude that

$$\mathcal{D}_p^J(u) \geq c(p)C_N\, r \min\left\{R^{N+2}\|u\|_1^{-p\gamma}\,\|u\|_p^{p(1+\gamma)},\, R^N\|u\|_p^p\right\},$$

with $\gamma = \frac{2}{N(p-1)}$. □

1.3 Energy inequalities involving derivatives

We now prove Theorem 1.1.3, an inequality which is useful when studying the decay of derivatives of solutions to nonlocal diffusion equations:

Proof:[Proof of Theorem 1.1.3] The proof is a direct extension of the technique in the proof of Theorem 1.1.1. We follow the same steps. First, we assume that J is the normalised characteristic function of the unit ball in $\mathbb{R}^N$, given by (1.23). Then, closely following Lemma 1.2.1, we claim

$$\mathcal{D}_2^J(D^k u) \geq C_N \min\left\{\|u\|_1^{-\frac{4}{N+2k}}\,\|D^k u\|_2^{2+\frac{4}{N+2k}},\ \|D^k u\|_2^2\right\} \tag{1.36}$$

for some constant $C_N > 0$ depending only on N. As in the proof of Lemma 1.2.1,

$$\mathcal{D}_2^J(D^k u) = 2\int_{\mathbb{R}^N}(1-\hat{J})|\widehat{D^k u}|^2.$$

Now, recalling inequality (1.25) and taking into account that $|\widehat{D^k u}(\xi)|^2 = |\xi|^{2k}|\hat{u}(\xi)|^2 \leq |\xi|^{2k}\|u\|_1^2$ we obtain for $0 < \delta \leq 1$ that

$$\begin{aligned}\|D^k u\|_2^2 = \|\widehat{D^k u}\|_2^2 &= \int_{|\xi|\leq\delta}|\widehat{D^k u}(\xi)|^2\,\mathrm{d}\xi + \int_{|\xi|>\delta}|\widehat{D^k u}(\xi)|^2\,\mathrm{d}\xi\\ &\leq \|u\|_1^2\int_{|\xi|\leq\delta}|\xi|^{2k}\,\mathrm{d}\xi + \frac{C_1}{\delta^2}\int_{|\xi|>\delta}(1-\hat{J}(\xi))|\widehat{D^k u}(\xi)|^2\,\mathrm{d}\xi \qquad (1.37)\\ &\leq \omega_N\delta^{2k+N}\,\|u\|_1^2 + \frac{C_1}{2\delta^2}\,\mathcal{D}_2^J(D^k u).\end{aligned}$$

Choose

$$\delta_0 = \left(\frac{2\, C_1\, \mathcal{D}_2^J(D^k u)}{(N+2k)\, \|u\|_1^2\, \omega_N} \right)^{\frac{1}{N+2k+2}}.$$

We obtain, as in Lemma 1.2.1, two possibilities: if $\delta_0 \le 1$, we get

$$\|D^k u\|_2^2 \le C_2\, \|u\|_1^{2\mu_k}\, \mathcal{D}_2^J(D^k u)^{1-\mu_k} \tag{1.38}$$

with $\mu_k = \frac{2}{N+2+2k}$ and $C_2 = \left(\frac{N}{2}+k\right)^{\mu_k} \left(1 + \frac{2}{N+2k}\right)^{1-\mu_k} C_1 \omega_N^{\mu_k}$. In the other case, $\delta_0 > 1$, we get

$$\|D^k u\|_2^2 \le \frac{2C_1}{(N+2k)\mu_k}\, \mathcal{D}_2^J(D^k u). \tag{1.39}$$

Collecting inequalities (1.38) and (1.39) we have

$$\|D^k u\|_2^2 \le C_N \max\left\{ \|u\|_1^{2\mu_k}\, \mathcal{D}_2^J(D^k u)^{1-\mu_k},\, \mathcal{D}_2^J(D^k u) \right\}$$

where $C_N = \max\left\{C_2, \frac{2C_1}{(N+2k)\mu_k}\right\}$. Reversing the inequality we have thus proved (1.36).

In order to complete the proof we consider any J satisfying Hypothesis 1. We have a scaling property which is an extension of (1.33):

$$\mathcal{D}_2^{J_\lambda}(D^k u) = \lambda^{2N-k} \mathcal{D}_2^J(D^k u_{1/\lambda}), \tag{1.40}$$

for any $\lambda > 0$. Of course, we also have $D^k u_\lambda = \lambda^{-k}(D^k u)_\lambda$, the usual scaling for derivatives. If I denotes the characteristic function of the unit ball on $\mathbb{R}^N$ and we define $K = \frac{1}{r\omega_N} J_{1/R}$ as in the proof of Corollary 1.2.2 then $K \ge I$, and $J = r\omega_N K_R$. Using the scaling property (1.40) and the normalised case (1.36) we see that

$$\begin{aligned}
\mathcal{D}_2^J(D^k u) &= r\omega_N R^{2N-k} \mathcal{D}_2^K(u_{1/R}) \ge r\omega_N R^{2N-k} \mathcal{D}_2^I(u_{1/R}) \\
&\ge r\omega_N R^{2N-k} C_N \min\left\{ \|u_{1/R}\|_1^{-\frac{4}{N+2k}}\, \|D^k u_{1/R}\|_2^{2+\frac{4}{N+2k}},\ \|D^k u_{1/R}\|_2^2 \right\} \\
&= r\omega_N R^{2N-k} C_N \min\left\{ R^{2k-N+2} \|u\|_1^{-\frac{4}{N+2k}}\, \|D^k u\|_2^{2+\frac{4}{N+2k}},\ R^{2k-N} \|D^k u\|_2^2 \right\} \\
&= r\omega_N C_N \min\left\{ R^{k+N+2} \|u\|_1^{-\frac{4}{N+2k}}\, \|D^k u\|_2^{2+\frac{4}{N+2k}},\ R^{k+N} \|D^k u\|_2^2 \right\},
\end{aligned}$$

which shows the result. □

We point out that analogous results can be stated for other differential operators. As an example we consider ∇u. Following the notation of the preceding section we set

$$\mathcal{D}_2^J(\nabla u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)\, |\nabla u(x) - \nabla u(y)|^2\, \mathrm{d}x\, \mathrm{d}y, \tag{1.41}$$

defined for any $u \in H^1(\mathbb{R}^N)$. Reasoning along the same lines as in the previous result one obtains the following result for ∇u (notice that this is not the same as the $k = 1$ case of Theorem 1.1.3, since $D^1 u$ is not equal to ∇u):

Theorem 1.3.1 *Let $N \geq 1$ be an integer and $J : \mathbb{R}^N \to \mathbb{R}$ be a function satisfying Hypothesis 1. There exists a positive constant $C = C(N)$ such that*

$$\mathcal{D}_2^J(\nabla u) \geq Cr \min\left\{ R^{N+3} \|u\|_1^{-\frac{4}{N+2}} \|\nabla u\|_2^{2+\frac{4}{N+2}},\ R^{N+1}\|\nabla u\|_2^2 \right\}, \tag{1.42}$$

for all $u \in H^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.

Proof: If J has integral one we can write, as before,

$$\mathcal{D}_2^J(\nabla u) = 2\langle \nabla u, \nabla u - J * \nabla u\rangle = 2\big\langle \widehat{\nabla u}, (1-\hat{J})\widehat{\nabla u}\big\rangle = 2\int_{\mathbb{R}^N} (1-\hat{J})|\widehat{\nabla u}|^2.$$

Since $|\widehat{\nabla u}(\xi)|^2 = |\xi|^2 |\hat{u}(\xi)|^2 \leq |\xi|^2 \|u\|_1^2$, one can follow the same reasoning as in the $k = 1$ case of Theorem 1.1.3 to obtain the result. □

1.4 Some applications

1.4.1 The linear nonlocal diffusion equation in convolution form

The most direct application of the inequalities in the previous section concerns the long-time behaviour of the linear nonlocal diffusion equation:

$$\partial_t u(t,x) = \int_{\mathbb{R}^N} J(x-y)(u(t,y) - u(t,x))\,\mathrm{d}y, \tag{1.43}$$

where $t \geq 0$ is the time variable, $x \in \mathbb{R}^N$ is the space variable, $u = u(t,x) \in \mathbb{R}$ is the unknown, and J is the *diffusion kernel*. As a straightforward consequence of Theorem 1.1.1 we obtain Theorem 1.1.2, which we prove now.

Proof:[Proof of Theorem 1.1.2] The regularity of the solution u allows us to write the following H-theorem for the L^p norm:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p = -\mathcal{D}_p^J(u). \tag{1.44}$$

Due to Theorem 1.1.1, and taking into account that $\|u(t,\cdot)\|_1 = \|u_0\|_1$ (mass conservation), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p \leq -Cr \min\left\{ R^{N+2}\|u_0\|_1^{-p\gamma} \|u\|_p^{p(1+\gamma)},\ R^N\|u\|_p^p \right\},$$

for some constant $C = C(N,p)$. This is a differential inequality for $\|u\|_p^p$ which allows us to compare it to the solution to the equation

$$X'(t) = -Cr \min\left\{ R^{N+2}\|u_0\|_1^{-p\gamma} X(t)^{(1+\gamma)},\ R^N X(t) \right\}.$$

We can then apply Lemma 1.4.1 with

$$C_1 := CrR^{N+2}\|u_0\|_1^{-p\gamma}, \qquad C_2 := CrR^N,$$

to obtain the result. □

Lemma 1.4.1 *Take $C_1, C_2, \gamma > 0$ and let $X = X(t)$ be a solution on $[0, +\infty)$ to the ordinary differential equation*

$$X'(t) = -\min\left\{C_1 X(t)^{1+\gamma}, C_2 X(t)\right\}. \tag{1.45}$$

with $X(0) > 0$. Then we have

$$X(t) \leq \begin{cases} X(0) & \text{for } t \in [0, t_0], \\ \left(X(0)^{-\gamma} + \gamma C_1 (t - t_0)\right)^{-\frac{1}{\gamma}} & \text{for } t \in (t_0, +\infty) \end{cases} \tag{1.46}$$

where

$$t_0 = \max\left\{0, \ \frac{1}{C_2} \log\left(C_2^{-\frac{1}{\gamma}} C_1^{\frac{1}{\gamma}} X(0)\right)\right\}.$$

Remark 1.4.2 *The solution of the ordinary differential equation in the above lemma is actually explicit (see the proof), and we just aim to give a simple statement that captures the decay of the solution as $t \to +\infty$. One can simplify even further and say that there is a constant $C = C(C_1, C_2, \gamma, X(0))$ such that*

$$X(t) \leq C(1+t)^{-\frac{1}{\gamma}}, \qquad \text{for all } t \geq 0.$$

This is easily deduced from (1.46) with

$$C := \sup_{t \geq 0} \frac{X(t)}{(1+t)^{-\frac{1}{\gamma}}},$$

which is finite since both X and $(1+t)^{-\frac{1}{\gamma}}$ have the same decay as $t \to +\infty$, and obviously depends only on C_1, C_2, γ and $X(0)$.

Proof:[Proof of Lemma 1.4.1] By usual theorems in ordinary differential equations, equation (1.45) has a unique solution on $[0, +\infty)$ with the given initial condition $X(0)$, and this solution is nonnegative on $[0, +\infty)$. The condition that decides which of the two terms achieves the minimum at each time t is whether

$$X(t)^{\gamma} \leq \frac{C_2}{C_1}. \tag{1.47}$$

Since X is nonincreasing, once this condition is satisfied at a certain $t_0 \geq 0$ it will be satisfied for all $t \geq t_0$. With this it is easy to calculate the explicit solution, given by

$$X(t) = \begin{cases} X(0)e^{-C_2 t} & \text{for } t \in [0, t_0], \\ \left(X(t_0)^{-\gamma} + \gamma C_1 (t - t_0)\right)^{-\frac{1}{\gamma}} & \text{for } t \in (t_0, +\infty) \end{cases}$$

where

$$t_0 = \max\left\{0, \ \frac{1}{C_2} \log\left(C_2^{-\frac{1}{\gamma}} C_1^{\frac{1}{\gamma}} X(0)\right)\right\}.$$

One obtains the result by noticing that $X(0)e^{-C_2 t} \leq X(0)$ and $X(t_0) \leq X(0)$. □

Similarly, with the help of the previous lemma the inequalities in Theorem 1.1.3 imply the decay in Theorem 1.1.4:

Proof:[Proof of Theorem 1.1.4] If u satisfies equation (1.1) then $D^k u$ satisfies the same equation, with initial condition $D^k u(0,x) = D^k u_0(x)$. Hence we have, as in (1.9),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|D^k u\|_2^2 = -\mathcal{D}_2^J(D^k u).$$

Using Theorem 1.1.3 we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|D^k u\|_2^2 \leq -Cr \min\left\{R^{k+N+2}\|u\|_1^{-\frac{4}{N+2k}}\|D^k u\|_2^{2+\frac{4}{N+2k}},\ R^{k+N}\|D^k u\|_2^2\right\}.$$

This is again a differential inequality for $\|D^k u\|_2^2$, to which we can apply Lemma 1.4.1 with

$$C_1 = CrR^{k+N+2}\|u_0\|_1^{-\frac{4}{N+2k}}, \qquad C_2 = CrR^{k+N}.$$

This directly gives the result. □

1.4.2 General linear mass-conserving nonlocal equations

In this section we prove Theorem 1.1.5, which concerns equation (1.17), recalled here:

$$\partial_t u(t,x) = \int_{\mathbb{R}^N} K(x,y)u(t,y)\,\mathrm{d}y - \sigma(x)u(t,x), \tag{1.48}$$

where $K\colon \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ is a general kernel (not necessarily symmetric) and $\sigma\colon \mathbb{R}^N \to [0,+\infty)$ is a function. In order to apply our strategy to equation (1.48) we need to have suitable Lyapunov functionals for it. To our knowledge, the most general setting in which one can do this is that of the so-called *general relative entropy* method (Michel et al., 2004, 2005), which we state here in a particular case: assume that (1.19) holds and that

$$\text{There exists a positive equilibrium } u_\infty\colon \mathbb{R}^N \to (0,+\infty) \text{ of (1.48).} \tag{1.49}$$

(That is, a solution u_∞ of (1.48) which does not depend on time t.) Then it is known that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \Phi\left(\frac{u(t,x)}{u_\infty(x)}\right)u_\infty(x)\,\mathrm{d}x \leq 0,$$

whenever Φ is a convex function and u is any solution of (1.48). This fact is well-known in probability theory (see the review by Chafaï (2004)) and is a direct consequence of the general relative entropy method (Michel et al., 2004). The explicit form of its time derivative can be found in Michel et al. (2004):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\Phi\left(f(x)\right)u_\infty(x)\,\mathrm{d}x \\ = -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(x,y)u_\infty(y)\big(\Phi'(f(x))(f(x)-f(y))-\Phi(f(x))+\Phi(f(y))\big)\,\mathrm{d}x\,\mathrm{d}y, \tag{1.50}$$

where we denote $f(t,x) \equiv u(t,x)/u_\infty(x)$, and where the t variable has been omitted for shortness. Notice that the integrand is always nonnegative due to the convexity of Φ. The following particular cases are of interest for us here: for $\Phi(f) = |f|^p$ with $p > 1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p = -\mathcal{E}_p^K(f), \tag{1.51}$$

where the *dissipation* $\mathcal{E}_p^K(f)$ is an operator acting only on the x variable. Its expression is given by the right hand side of (1.50) (with $\Phi(f) = |f|^p$) and is not so simple. But if we additionally assume that

$$K(x,y)u_\infty(y) = K(y,x)u_\infty(x), \qquad \text{for all } x,y \in \mathbb{R}^N, \tag{1.52}$$

then one can check that

$$\mathcal{E}_p^K(f) \\ = p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(x,y)u_\infty(y)\big((f(x))^{p-1}(f(x)-f(y))-(f(x))^p+(f(y))^p\big)\,\mathrm{d}x\,\mathrm{d}y \\ = \frac{p}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big(f(x)-f(y)\big)\left(f(x)^{p-1}-f(y)^{p-1}\right)K(x,y)u_\infty(y)\,\mathrm{d}x\,\mathrm{d}y \tag{1.53}$$

for all nonnegative functions f; note the parallel with (1.9). The last equality in (1.53) is obtained by noticing that the integrals corresponding to $f(x)^p$ and $f(y)^p$ cancel out (easily seen by using (1.52)), and using (1.52) again to symmetrise the remaining integral:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(x,y)u_\infty(y)f(x)^{p-1}(f(x)-f(y))\,\mathrm{d}x\,\mathrm{d}y \\ = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(x,y)u_\infty(y)\left(f(x)^{p-1}-f(y)^{p-1}\right)\big(f(x)-f(y)\big)\,\mathrm{d}x\,\mathrm{d}y.$$

Condition (1.52) is known in probability as the *detailed balance* or *reversibility* condition (it holds for example if $u_\infty \equiv 1$ and K is symmetric). If one works in a setting where (1.51) holds then it may still be possible to use the inequality in Theorem 1.1.1 (or related ones) and deduce some information on the rate of decay of solutions.

Proof:[Proof of Theorem 1.1.5] Condition (1.20) is easily seen to imply that the linear operator given by

$$Lu(x) = \int_{\mathbb{R}^N}K(x,y)u(y)\,\mathrm{d}y - \sigma(x)u(x), \qquad x \in \mathbb{R}^N,$$

is well defined and bounded both in $L^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$. This shows that equation (1.48) with initial condition u_0 has a unique solution in $\mathcal{C}^1([0,+\infty), L^p(\mathbb{R}^N)\cap L^1(\mathbb{R}^N))$ which conserves mass (that is, $\int_{\mathbb{R}^N} u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x$ for all $t \geq 0$), and that it satisfies the entropy property (1.9). It is also seen easily that equation (1.48) preserves sign: if the initial condition is nonnegative (nonpositive) then $u(t,x)$ is nonnegative (nonpositive) for all t, x. As a consequence, it is enough to prove the result when u_0 is nonnegative — the general result is then obtained by linearity from $u_0 = u_0^+ - u_0^-$, with $u_0^+ := \max\{u_0, 0\}$ and to $u_0^- := \max\{-u_0, 0\}$.

For $x, y \in \mathbb{R}^N$ call

$$\tilde{K}(x,y) := r, \quad \text{if } |x-y| \leq R, \qquad \tilde{K}(x,y) := 0 \quad \text{otherwise}$$

and

$$J(x) := r, \quad \text{if } |x| \leq R, \qquad J(x) := 0 \quad \text{otherwise.}$$

Due to Hypothesis 2 and (1.21) we have

$$K(x,y)u_\infty(y) \geq \frac{1}{m}\tilde{K}(x,y).$$

Hence, since $\tilde{K}$ is symmetric, using the same symmetrisation trick as in (1.53),

$$\begin{aligned}
\mathcal{E}_p^K(f) &\geq \mathcal{E}_p^{\tilde{K}}(f) \\
&\geq \frac{p}{2m}\int_{\mathbb{R}^N} \big((f(x))^{p-1}(f(x)-f(y)) - (f(x))^p + (f(y))^p\big)\tilde{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{p}{2m}\int_{\mathbb{R}^N} \big(f(x)^{p-1} - f(y)^{p-1}\big)\big(f(x)-f(y)\big)\tilde{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \mathcal{D}_p^J(f)
\end{aligned}$$

for any nonnegative function f, where $\mathcal{D}_p^J(f)$ is the dissipation in (1.10). Hence for the (nonnegative) solution u, using Theorem 1.1.1 and calling

$$X(t) := \int_{\mathbb{R}^N} f^p u_\infty = \int_{\mathbb{R}^N}\left(\frac{u(t,x)}{u_\infty(x)}\right)^p u_\infty(x)\,\mathrm{d}x$$

we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}X(t) &= -\mathcal{E}_p^K(f) \\
&\leq -\mathcal{D}_p^J(f) \\
&\leq -C\min\{\|f\|_1^{-p\gamma}\|f\|_p^{p(1+\gamma)}, \|f\|_p^p\} \\
&\leq -C_2\min\{\|u_0\|_1^{-p\gamma}X(t)^{1+\gamma}, X(t)\},
\end{aligned}$$

where C_2 also depends on m, and we have used mass conservation and again the bounds in (1.21). Due to the differential inequality in Lemma 1.4.1 we obtain that

$$X(t) \leq C(1+t)^{-\frac{N(p-1)}{2}}, \qquad \text{for all } t \geq 0,$$

for some constant C as stated in the result. We complete the proof by noticing that

$$\|u\|_p^p \le m^{1-p} \int_{\mathbb{R}^N} \left(\frac{u(t,x)}{u_\infty(x)} \right)^p u_\infty(x)\, \mathrm{d}x = m^{1-p} X(t).$$

$\square$

1.4.3 A nonlocal dispersal equation

We consider the following integro-differential equation (the dispersal model that was briefly mentioned in the introduction):

$$\partial_t u(t,x) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)} \right) \frac{u(t,y)}{g(y)}\, \mathrm{d}y - u(t,x), \qquad \text{in } \mathbb{R} \times [0,\infty), \tag{1.54}$$

with a prescribed initial data $u(x,0) = u_0(x)$ defined on $\mathbb{R}$. Here J is an even, positive, smooth function such that $\int_{\mathbb{R}} J(x)\, \mathrm{d}x = 1$ and supp $J = [-1,1]$, and g is a continuous positive function which accounts for the *dispersal distance* which depends on the departing point. In this model u represents the spatial distribution of a certain species, and g models the heterogeneity of the environment which can affect the distribution of a species through space-dependent dispersal strategies. This model was proposed in Cortázar et al. (2007) (see also Cortázar et al. (2011); Cortázar et al. (2015); Cortázar et al. (2016)). It was shown there that if we assume g is bounded above and below then there exists a positive steady state solution of (1.54), that is, a solution of the corresponding stationary problem,

$$u_\infty(x) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)} \right) \frac{u_\infty(y)}{g(y)}\, \mathrm{d}y, \qquad \text{in } \mathbb{R}.$$

Moreover, u_∞ is bounded above and below by positive constants. It was also proved in Cortázar et al. (2007) that any solution u of (1.54) converges to 0 locally as $t \to \infty$. Using the general result in Theorem 1.1.5 we are able to improve this asymptotic behavior obtaining a precise decay rate of the L^p norms of u:

Theorem 1.4.3 *Take $p \in [2,+\infty)$. Let u be a solution of (1.54) with initial data $u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, and assume that*

1. *$J \in L^\infty(\mathbb{R})$ is a bounded, nonnegative function with compact support, satisfying Hypothesis 1,*

2. *and g is a continuous function satisfying*

$$\frac{1}{M} \le g(x) \le M, \qquad \textit{for all } x \in \mathbb{R}$$

and for some $M > 0$.

Then for some constant $C > 0$ depending on J, M, p, $\|u_0\|_1$ and $\|u_0\|_p$,

$$\|u\|_p^p \leq C(1+t)^{-\frac{p-1}{2}}, \qquad \textit{for all } t \geq 0.$$

Proof: Equation (1.54) is of the form (1.48) with $\sigma(x) = 1$ for all $x \in \mathbb{R}$ and

$$K(x,y) = J\left(\frac{x-y}{g(y)}\right)\frac{1}{g(y)}, \qquad \text{for } x, y \in \mathbb{R}.$$

Defining K and σ in this way, (1.19) is satisfied and one can check that this kernel K satisfies Hypothesis 2 and (1.20). By the results in Cortázar et al. (2007) we know that there exists an equilibrium u_∞ satisfying (1.21) (with m depending only on the parameters of the problem), so we are in condition to apply Theorem 1.1.5 and obtain the result. □

Remark 1.4.4 *One can pose equation (1.54) in $\mathbb{R}^N$ instead of $\mathbb{R}$. The only reason in Theorem 1.4.3 why we need the dimension N to be 1 is that we use the results in Cortázar et al. (2007) to ensure there is a positive equilibrium u_∞ which is bounded above and below. Theorem 1.4.3 is still true in dimension N provided the existence of an equilibrium satisfying (1.21) (with the same proof). Such existence of a bounded u_∞ is to our knowledge an open problem in dimension $N > 1$.*

1.4.4 Nonlocal diffusions with a nonlinear source

With very little change in our arguments we can obtain the same decay estimates if we add a nonlinear source to equation (1.48), as long as the nonlinear source "decreases energy". We consider

$$\partial_t u(t,x) = \int_{\mathbb{R}^N} K(x,y)u(t,y)\,\mathrm{d}y - \sigma(x)u(t,x) + f(u(t,x)). \tag{1.55}$$

with K and σ as in Section 1.4.2 and f a locally Lipschitz function satisfying the sign condition

$$f(s)s \leq 0, \qquad \text{for } s \in \mathbb{R}. \tag{1.56}$$

With the same arguments as before we obtain the following:

Theorem 1.4.5 *Take $p \in [2, +\infty)$ and let u be a solution of (1.55) with nonnegative initial data $u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, and assume that K and σ satisfy the conditions of Theorem 1.1.5. Assume that f is a locally Lipschitz function satisfying (1.56). Then for some constant $C > 0$ depending only on K, N, $\|u_0\|_1$ and $\|u_0\|_p$,*

$$\|u\|_p^p \leq C(1+t)^{-\frac{N}{2}}, \qquad \textit{for all } t \geq 0.$$

Proof: The conditions on f, K and σ ensure that there exists a solution of the equation, and that one may differentiate it in time to obtain the usual expression for the time derivative of $\|u\|_p^p$. Dropping the nonpositive term $f(u(t,x))\, u(t,x)\, |u(t,x)|^{p-2}$ we obtain the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p \leq -\,\mathcal{E}_p^K(u),$$

Arguing as in the proof of Theorem 1.1.5 we obtain the asymptotic decay. Observe that the total mass of the solution is nonincreasing, since $f(s) \leq 0$ for $s \geq 0$. $\square$

This equation was treated in Andreu-Vaillo et al. (2010); Ignat and Rossi (2009) where a restriction on the dimension ($N \geq 3$) and K symmetric are required in order to establish the asymptotic behavior.

Chapter 2

Nonlocal diffusion problems that approximate a parabolic equation with spatial dependence

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DOI: 10.1007/s00033-016-0649-8

Abstract

In this paper we show that smooth solutions to the Dirichlet problem for the parabolic equation

$$v_t(x,t) = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 v(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial v(x,t)}{\partial x_i} \qquad x \in \Omega,$$

with $v(x,t) = g(x,t)$, $x \in \partial\Omega$, can be approximated uniformly by solutions of nonlocal problems of the form

$$u_t^\varepsilon(x,t) = \int_{\mathbb{R}^n} K_\varepsilon(x,y)(u^\varepsilon(y,t) - u^\varepsilon(x,t))dy, \; x \in \Omega,$$

with $u^\varepsilon(x,t) = g(x,t)$, $x \notin \Omega$, as $\varepsilon \to 0$, for an appropriate rescaled kernel K_ε. In this way we show that the usual local evolution problems with spatial dependence can be approximated by non-local ones. In the case of an equation in divergence form we can obtain an approximation with symmetric kernels, that is, $K_\varepsilon(x,y) = K_\varepsilon(y,x)$.

2.1 Introduction

Nonlocal diffusion problems of the form

$$u_t(x,t) = \int_{\mathbb{R}^n} K(x,y)(u(y,t) - u(x,t))dy \tag{2.1}$$

and variations of it, have been extensively studied in recent years (see Andreu-Vaillo et al. (2010); Cortázar et al. (2007); Cortazar et al. (2007) and references therein). Here, the kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative, smooth function such that $\int_{\mathbb{R}^N} K(x,y)dx = 1$. A physical interpretation of (2.1) is the following: if $K(x,y)$ is the probability distribution that individuals jump from y to x and $u(x,t)$ is the density at position x at time t, then $\int_{\mathbb{R}^N} K(x,y)u(y,t)dy$ is the rate at which individuals are arriving to position x from all other locations y. Further, with the same reasoning, $\int_{\mathbb{R}^N} K(x,y)u(x,t)dy$ is interpreted as the rate at which they are leaving position x to all other places. Hence, in the absence of external or internal sources, the density $u(x,t)$ satisfies (2.1) (see Andreu-Vaillo et al. (2010); Fife (2003); Hutson et al. (2003)). This kind of nonlocal diffusion equation is relevant in applications, for example, in the study of biological dispersal of species, image processing, particle systems, elasticity and coagulation models, Bobaru et al. (2009); Bodnar and Velazquez (2006); Carrillo and Fife (2005); Fife (2003); Fournier and Laurençot (2006); Hutson et al. (2003).

In this work we consider the following nonlocal diffusion problem: given a bounded domain$\Omega \subset \mathbb{R}^N$, $g \in L^1\left(\left(\mathbb{R}^N \setminus \Omega\right) \times (0,\infty)\right)$ and $u_0 \in L^1(\Omega)$, find $u(x,t)$ such that

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} K(x,y)(u(y,t) - u(x,t))dy, & x \in \Omega,\ t > 0, \\ u(x,t) = g(x,t), & x \notin \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{P_K}$$

where the kernel $K(x,y)$ is a positive function with compact support contained in $\Omega \times B(0,d) \subset \mathbb{R}^N \times \mathbb{R}^N$ with

$$0 < \sup_{y \in B(0,d)} K(x,y) = C(x) \in L^\infty(\Omega). \tag{2.2}$$

As we mentioned before, the integral term in the problem takes into account the individuals arriving or leaving position $x \in \Omega$ from or to other places. In this model, imposing $u(x,t) = g(x,t)$ for $x \notin \Omega$, we are prescribing the values of u outside Ω. In the particular case $g = 0$, we mean that individuals that leave Ω, die (and therefore the density outside Ω is zero).

Existence and uniqueness of solutions of (P_K) is proved in Proposition 2.2.1 using a fixed point argument (see also Appendix A, for an alternative proof). In Proposition 2.2.2 we obtain an appropriate comparison principle.

As a local counterpart to our nonlocal evolution problem, we have the following second order local parabolic differential equation with Dirichlet boundary conditions

$$\begin{cases} v_t(x,t) = \displaystyle\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 v(x,t)}{\partial x_i \partial x_j} + \sum_{i}^{N} b_i(x) \frac{\partial v(x,t)}{\partial x_i}, & x \in \Omega, t > 0, \\ v(x,t) = g(x,t), & x \in \partial\Omega, t > 0, \\ v(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{Q}$$

where the coefficients $a_{ij}(x)$, $b_i(x)$ are smooth in $\overline{\Omega}$ and $(a_{ij}(x))$ is a symmetric positive definite matrix, i.e., $a_{ij} = a_{ji}$ and $\sum_{ij} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2$ for every real vector $\xi = (\xi_1, \ldots, \xi_N) \neq 0$ and for some $\alpha > 0$.

It is important to stress that here we will use that (Q) has smooth solutions. In fact, under regularity assumptions on the boundary data g, the domain Ω and the initial condition u_0, we have that the solutions of (Q) are $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\Omega} \times [0,T]\right)$. For such a regularity result we refer to Lieberman (1996).

Our main goal in this work is to show that the Dirichlet problem for the parabolic equation (Q) can be approximated by nonlocal problems of the form (P_K). More precisely, given $J : \mathbb{R}^N \to \mathbb{R}$ a nonnegative, radial and continuous function with compact support and finite second order momentum, we consider the rescaled kernel

$$K_\varepsilon(x,y) = \frac{C(x)}{\varepsilon^{N+2}}\, a\Big(x - E(x)(x-y)\Big)\, J\left(L^{-1}(x)\frac{x-y}{\varepsilon}\right) \tag{2.3}$$

Here a is given by $a(s) = \sum_i (s_i + M)$, with M large enough to ensure $a(x) \geq \beta > 0$. The matrix $L(x)$ is the Cholesky's factor of $A(x)$, that is, $A(x) = L(x)L^t(x)$, the matrix $E(x)$ is related with the coefficients $(a_{ij}(x))$ and $b_i(x)$ and $C(x)$ is a normalizing function, see Section 2.3 for a precise definition. Then, we prove that u^ε, solutions of rescaled nonlocal problems (P_{K_ε}), approximate uniformly the solution of the corresponding Dirichlet problem for the parabolic equation. We can now formulate our main result.

Theorem 2.1.1 *Let $v \in \mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\Omega} \times [0,T]\right)$ be the solution to (Q). Let, for a given $\varepsilon > 0$, u^ε be the solution to (P_{K_ε}), with initial condition $u_0(x)$ and external datum $g(x,t)$. Then, we have*

$$\|v - u^\varepsilon\|_{L^\infty(\Omega\times[0,T])} \to 0, \qquad \text{as } \varepsilon \to 0.$$

To deal with an equation in divergence form

$$v_t(x,t) = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v(x,t)}{\partial x_j}\right),$$

we can just take

$$b_i(x) := \sum_{j=1}^{N} \frac{\partial a_{ij}(x)}{\partial x_j}$$

and the previous approach works. However, in this case the resulting family of nonlocal approximating problems have non-symmetric kernels. Note that for symmetric kernels, i.e., $K(x, y) = K(y, x)$, one has the desirable property of an "integration by parts formula", that is,

$$\iint K(x,y)(u(y)-u(x))\varphi(x)dydx = -\frac{1}{2}\iint K(x,y)(u(y)-u(x))(\varphi(y)-\varphi(x))dydx.$$

This is similar to the usual integration by parts formula for divergence form operators,

$$\int \operatorname{div}(A(x)\nabla v(x))\varphi(x)dx = -\int A(x)\nabla v(x)\nabla\varphi(x)dx.$$

To obtain a family of symmetric kernels $K_\varepsilon(x, y) = K_\varepsilon(y, x)$ such that the corresponding solutions to the nonlocal problems converge as $\varepsilon \to 0$ to the solution to the Dirichlet problem in divergence form we consider,

$$K_\varepsilon(x,y) = \frac{2}{C(J)\varepsilon^{N+2}} G\left(B^{-1}(x)\frac{x-y}{\varepsilon}\right) G\left(B^{-1}(y)\frac{x-y}{\varepsilon}\right), \tag{2.4}$$

where $G^2(s) = J(s)$ (J is a radially symmetric, compactly supported and smooth kernel), and $B(x) = (b_{ij}(x))$ is a $N \times N$ matrix such that

$$\det(B(x))B(x)B^t(x) = A(x).$$

Note that $B(x)$ is invertible since $A(x)$ is. In this way we obtain a family of non-local symmetric kernels such that the approximation result given in Theorem 2.1.1 holds.

For constant matrices A and $b_i(x) = 0$ in problem (Q), the rescaled kernels (2.3) and (2.4) coincide.

We finish the introduction with a brief description of previous results. When one considers a convolution kernel J (as before, radially symmetric, compactly supported and smooth) and rescale it, that is, for

$$K_\varepsilon(x,y) = \frac{C}{\varepsilon^{N+2}} J\left(\frac{y-x}{\varepsilon}\right) \tag{2.5}$$

one finds in the limit as $\varepsilon \to 0$ solutions to the classical heat equation, $v_t = \Delta v$. This fact was proved in Cortázar et al. (2009) for Dirichlet boundary conditions and in Cortázar et al. (2008) for Newmann boundary conditions. For an evolution problem with the same kernel but with an inhomogeneous term $a(y)$ in front in the whole $\mathbb{R}^N$ we refer to Sun et al. (2011) (see also Cortázar et al. (2007)). In this case the limit equation is given by $v_t = \Delta\,(a(x)v)$. For approximations of models from elasticity (peridynamics) we refer to Bobaru et al. (2009). Concerning nonlinear nonlocal problems (approximating for example the $p-$Laplacian or the porous medium equation) we refer to the book Andreu-Vaillo et al. (2010) and the survey Vázquez (2014). We remark that in the previously mentioned references the case of matrix dependent problems (like the ones included in this paper) was not treated (only scalar coefficients appear).

The rest of this paper is organized as follows: in Section 2, we prove existence and uniqueness for solutions to problem (P_K) using a fixed point theorem (Proposition 2.2.1). In addition, we show a comparison principle (Proposition 2.2.2). In Section 3, using Cholesky's decomposition of the matrix $A(x) = (a_{ij}(x))$, we prove the uniform convergence of u^ε to v, the solution of the local parabolic equation (Theorem 2.1.1). In Section 4 we deal with the divergence form equation proving the convergence result for a symmetric family of kernels. Finally, the Appendix is devoted to give an alternative proof of existence of solutions (Appendix A), additionally, a technical computation using in the proof of Theorem 2.1.1 is postponed to the second part of the Appendix (Appendix B).

2.2 Existence, Uniqueness and Comparison Principle

By a solution of problem (P_K) we mean a function $u \in \mathcal{C}([0,\infty); L^1(\Omega))$ which satisfies

$$u(x,t) = \int_0^t \int_{\mathbb{R}^N} K(x,y)(u(y,s) - u(x,s))dyds + u_0(x), \qquad x \in \Omega,\ t \geq 0,$$

here we understand that $u(y,s) = g(y,s)$ when $y \in \mathbb{R}^N \setminus \Omega$, $s > 0$. Consequently, due to the previous integral expression, we notice that $u \in \mathcal{C}^1([0,\infty); L^1(\Omega))$.

Proposition 2.2.1 *If $u_0 \in L^1(\Omega)$, there exists a unique solution of problem (P_K).*

Proof: Fixed $t_0 > 0$, we set the Banach space $X_{t_0} = \mathcal{C}\left([0,t_0]; L^1(\Omega)\right)$ with norm

$$|||v||| = \max_{0 \leq t \leq t_0} \|v(\cdot,t)\|_{L^1(\Omega)}.$$

Let $\mathcal{T} : X_{t_0} \longrightarrow X_{t_0}$ be the operator defined by

$$\mathcal{T}(v)(x,t) = \int_0^t \int_{\mathbb{R}^N} K(x,y)(v(y,s) - v(x,s))dyds + u_0(x),$$

with $v(x,t) = g(x,t)$ if $x \notin \Omega$.

Note that in the definition of the operator $\mathcal{T}$ we include the fact that we are taking $v(y,s) = g(y,s)$ when $y \notin \Omega$.

In this way, using Fubini's theorem we obtain

$$\begin{aligned}&\|\mathcal{T}(v)(\cdot,t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} \\ &\quad + \int_0^t \left(\int_\Omega \int_{\mathbb{R}^N} K(x,y)|v(y,s)|dydx + \int_\Omega \int_{\mathbb{R}^N} K(x,y)|v(x,s)|dydx \right) ds.\end{aligned}$$

Recalling hypothesis (2.2), let us denote by $C = \|C(x)\|_\infty$. We get

$$\int_{\mathbb{R}^N} K(x,y)|v(y,s)|dy \le C(x)\|v(\cdot,s)\|_{L^1(\Omega)} \le C\|v(\cdot,s)\|_{L^1(\Omega)}$$

and

$$\int_{\mathbb{R}^N} K(x,y)|v(x,s)|dy \le |B(0,d)|C(x)|v(x,s)| \le C|B(0,d)||v(x,s)|.$$

Hence

$$\|\mathcal{T}(v)(\cdot,t)\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)} + \tilde{C}\int_0^t \|v(\cdot,s)\|_{L^1(\Omega)}ds, \tag{2.6}$$

where $\tilde{C} = C(|\Omega| + |B(0,d)|)$. Since $\|v(\cdot,s)\|_{L^1(\Omega)} \le |||v|||$ it follows that

$$\|\mathcal{T}(v)(\cdot,t)\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)} + t\tilde{C}|||v|||,$$

thus operator $\mathcal{T}$ is well defined and

$$|||\mathcal{T}(v)||| \le \|u_0\|_{L^1(\Omega)} + t_0\tilde{C}|||v|||.$$

Now, choosing $t_0 < \tilde{C}^{-1}$, for every $w, z \in X_{t_0}$ we get

$$|||\mathcal{T}(w-z)||| < |||w-z|||.$$

Hence, $\mathcal{T}$ is a contraction on X_{t_0} which maps X_{t_0} into itself, then by the Banach contraction principle there exists a unique $u \in X_{t_0}$ such that $\mathcal{T}(u) = u$, i.e., we get local existence and uniqueness of problem (P_K) for $0 \le t \le t_0$. Moreover, taking the Banach space $X_{2t_0} = \mathcal{C}\left([t_0, 2t_0]; L^1(\Omega)\right)$ with norm $|||v||| = \max_{t_0 \le t \le 2t_0} \|v(\cdot,t)\|_{L^1(\Omega)}$, $\mathcal{T} : X_{2t_0} \longrightarrow X_{2t_0}$ defined by

$$\mathcal{T}(v)(x,t) = \int_{t_0}^t \int_{\mathbb{R}^N} K(x,y)(v(y,s) - v(x,s))dyds + u(x,t_0),$$

and arguing as above, there exists a unique solution in $[t_0, 2t_0]$ and consequently in $[0, 2t_0]$. By an iteration argument, we obtain a unique solution $u \in \mathcal{C}([0,\infty); L^1(\Omega))$ of problem (P_K). $\square$

For an alternative proof we refer the reader to Appendix A.

By a subsolution (respectively supersolution) of problem (P_K) we mean a function $u \in \mathcal{C}^1([0,T]; L^1(\Omega))$ which satisfies the following inequalities

$$\begin{cases} u_t(x,t) \overset{(\ge)}{\le} \displaystyle\int_{\mathbb{R}^N} K(x,y)(u(y,s) - u(x,s))dy, & x \in \Omega,\ t < 0, \\ u(x,t) \overset{(\ge)}{\le} g(x,t), & x \notin \Omega,\ t > 0. \\ u(x,0) \overset{(\ge)}{\le} u_0(x), & x \in \Omega. \end{cases}$$

Clearly, a solution is both a subsolution and a supersolution.

Proposition 2.2.2 *Let $u, v \in \mathcal{C}^1(\overline{\Omega} \times [0,T])$ be a subsolution and supersolution respectively of problem (P_K). Then $u \leq v$.*

Proof: We will denote by $w = v - u$. Obviously $w \in \mathcal{C}^1(\overline{\Omega} \times [0,T])$ and it satisfies

$$\begin{cases} w_t(x,t) \geq \displaystyle\int_{\mathbb{R}^N} K(x,y)(w(y,t) - w(x,t))dy, & x \in \Omega, \ t < 0, \\ w(x,t) \geq 0, & x \notin \Omega, \ t > 0. \\ w(x,0) \geq 0, & x \in \Omega. \end{cases}$$

Now, we assume that $w(x,t)$ is not a nonnegative function, that is, there exists some point $(\tilde{x}, \tilde{t}) \in \Omega \times (0,T]$ such that $w(\tilde{x}, \tilde{t}) < 0$. Then, by the continuity of w, there exists $\varepsilon > 0$ such that $w(\tilde{x}, \tilde{t}) + \varepsilon\tilde{t}$ is also negative. Consider the function $w(x,t) + \varepsilon t \in \mathcal{C}(\overline{\Omega} \times [0,T])$, and let (x_0, t_0) be its minimum, thus

$$w_t(x_0, t_0) + \varepsilon \leq 0.$$

Conversely,

$$w_t(x_0, t_0) + \varepsilon > \int_{\mathbb{R}^N} K(x_0, y)(w(y, t_0) - w(x_0, t_0))dy \geq 0,$$

this leads to a contradiction and we conclude that $w(x,t)$ is a nonnegative function. □

2.3 Proof of Teorem 2.1.1

It is well known that given $A(x) = (a_{ij}(x))$ a symmetric and positive definite matrix there exists a unique lower triangular matrix $L(x) = (l_{ij}(x))$ with real and positive diagonal entries such that

$$A(x) = L(x)L^t(x), \tag{2.7}$$

where $L^t(x)$ denotes the transpose of $L(x)$ which is known as the Cholesky factor and (2.7) is known as the Cholesky factorization, see for instance Householder (1964).

Let $J : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative, radially symmetric, continuous function with $\int_{\mathbb{R}^n} J(z)dz = 1$ and finite second order momentum. Assume also that J is strictly positive in $B(0,r)$ for some $r > 0$ and vanishes in $\mathbb{R}^n \setminus B(0,r)$.

Now we introduce some notations. Given a matrix $A(x) = (a_{ij}(x))$ with $\mathcal{C}^1(\bar{\Omega})$ coefficients we consider:

$$A_i(x) := \sum_{j=1}^N a_{ij}(x),$$

$$W(x) := \begin{pmatrix} b_1(x) & 0 & \dots & 0 \\ 0 & b_2(x) & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & b_N(x) \end{pmatrix}$$

We consider the rescaled kernel

$$K_\varepsilon(x,y) = \frac{C(x)}{\varepsilon^{N+2}}\, a\left(x - E(x)(x-y)\right)\, J\left(L^{-1}(x)\frac{x-y}{\varepsilon}\right). \tag{2.8}$$

Here a is defined as

$$a(s) = \sum_{i=1}^{N} (s_i + M),$$

for some constant $M > 0$ large enough to ensure $a(x) \geq \beta > 0$. The matrix $L(x)$ is given by (2.7) (note that we can take any $N \times N$ matrix $(l_{ij}(x))$, such that $A(x) = L(x)L^t(x)$), the function $C(x)$ is given by

$$C(x) = \frac{2}{C(J)a(x)(\det A(x))^{1/2}}$$

being $C(J) = \int J(z) z_1^2 dz$ and the matrix $E(x)$ by

$$E(x) = \frac{a(x)}{2} W(x) A^{-1}(x).$$

We remark that for this kernel, Proposition 2.2.1 and Proposition 2.2.2 can be used, since J is smooth, $a(x)$ is strictly positive and the coefficients of the envolved matrices are bounded. Therefore, for every $\varepsilon > 0$ we have existence, uniqueness and the comparison principle for the nonlocal problem.

Lemma 2.3.1 *Let u be a $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N \times [0,T]\right)$ function and*

$$\mathcal{L}_\varepsilon(u) := \int_{\mathbb{R}^N} K_\varepsilon(x,y)(u(y,t) - u(x,t))dy.$$

Then

$$\left\| \mathcal{L}_\varepsilon(u) - \left(\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial u(x)}{\partial x_i} \right) \right\|_{L^\infty(\Omega\times[0,T])} \leq \theta(\varepsilon),$$

for some function $\theta(\varepsilon)$ that goes to zero as $\varepsilon \to 0$.

Proof: Under the change variables $y = x - \varepsilon L(x) z$, $\mathcal{L}_\varepsilon(u)$ becomes

$$\frac{C(x)\left(\det A(x)\right)^{1/2}}{\varepsilon^2} \int_{\mathbb{R}^N} a\left(x - \varepsilon D(x) z\right) J(z) (u(x - \varepsilon L(x) z, t) - u(x,t)) dz$$

where $D(x) = \frac{a(x)}{2} W(x) \left(L^t(x)\right)^{-1}$. By a simple Taylor expansion we obtain

$$\begin{aligned}
\mathcal{L}_\varepsilon(u) = \quad & \frac{-C(x)\,(\det A(x))^{1/2}}{\varepsilon} \sum_{i=1}^N \frac{\partial u}{\partial x_i} \sum_{j=1}^N l_{ij}(x) \int_{\mathbb{R}^N} a\left(x - \varepsilon D(x) z\right) J(z) z_j dz \\
& + \frac{1}{2} C(x)\,(\det A(x))^{1/2} \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \sum_{k,m=1}^N l_{ik}(x) l_{jm}(x) \\
& \qquad \times \int_{\mathbb{R}^n} a\left(x - \varepsilon D(x) z\right) J(z) z_k z_m dz + O(\varepsilon^\alpha) \\
= \quad & \mathcal{L}_\varepsilon^1(u) + \mathcal{L}_\varepsilon^2(u) + O(\varepsilon^\alpha).
\end{aligned}$$

For the first expression, $\mathcal{L}_\varepsilon^1(u)$, having in mind the definition of the function $a(s)$ and that J is a radial function, more specifically, we use that $\int J(z) z_j dz = 0$ and $\int J(z) z_m z_j dz = 0$ if $m \neq j$, we get

$$\begin{aligned}
\lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon^1(u) &= C(x)\,(\det A(x))^{1/2} \sum_{i=1}^N \frac{\partial u}{\partial x_i} \sum_{j=1}^N l_{ij}(x) \sum_{k,m=1}^N d_{km}(x) \int_{\mathbb{R}^N} J(z) z_m z_j dz \\
&= C(x)\,(\det A(x))^{1/2}\, C(J) \sum_{i=1}^N \frac{\partial u}{\partial x_i} \sum_{j=1}^N l_{ij}(x) \sum_{k=1}^N d_{jk}^t(x),
\end{aligned}$$

here $d_{jk}^t(x)$ denotes the (j,k)-term of the matrix $D^t(x)$. Finally, since

$$\sum_{j=1}^N l_{ij}(x) \sum_{k=1}^N d_{jk}^t(x) = \sum_{k=1}^N (L(x) D^t(x))_{ik} = \frac{a(x)}{2} b_i(x),$$

it follows that

$$\lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon^1(u) = \sum_{i=1}^N \frac{\partial u(x,t)}{\partial x_i} b_i(x).$$

On the other hand, letting $\varepsilon \to 0$ in $\mathcal{L}_\varepsilon^2(u)$ taking into account the choice of the matrix $L(x)$ we have

$$\lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon^2(u) = \sum_{i,j=1}^N \frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} \sum_{k=1}^N l_{ik}(x) l_{kj}^t(x) = \sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 u(x,t)}{\partial x_i \partial x_j},$$

which concludes the proof. □

Remark 2.3.2 *We want to point out that the use of Cholesky's decomposition is not necessary for the proof. In fact, any matrix $L(x)$ satisfying (2.7) is also allowed. The reason to choose Cholesky's factor is to ensure the uniqueness of the rescaled kernel K_ε defined in (2.8).*

In order to prove our main result, let $\tilde{v}$ be a $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N \times [0,T]\right)$ extension of v, the solution of the parabolic problem (Q). Therefore, $\tilde{v}$ verifies

$$\begin{cases} \tilde{v}_t(x,t) = \Lambda(\tilde{v}(x,t)), & x \in \Omega,\ t \in (0,T], \\ \tilde{v}(x,t) = G(x,t), & x \notin \Omega,\ t \in (0,T], \\ \tilde{v}(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $G(x,t) = g(x,t)$ if $x \in \partial\Omega$ and

$$\Lambda(\tilde{v}(x,t)) = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 \tilde{v}(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial \tilde{v}(x,t)}{\partial x_i}.$$

Moreover, as G is smooth we get

$$G(x,t) = g(x,t) + O(\varepsilon),\ \text{ if } \operatorname{dist}(x,\partial\,\Omega) \le a\varepsilon, \tag{2.9}$$

where $a = r\sqrt{\lambda_{min}}$. Here λ_{min} denotes the $\max_{x\in\overline{\Omega}} \lambda_{min}(A(x)) > 0$. For more details we refer the reader to Appendix B.

Proof:[Proof of Theorem 2.1.1] Set $w^\varepsilon := \tilde{v} - u^\varepsilon$ which satisfies

$$\begin{cases} w_t(x,t) = \Lambda(\tilde{v}) - \mathcal{L}_\varepsilon(\tilde{v}) + \mathcal{L}_\varepsilon(w^\varepsilon), & x \in \Omega,\ t \in (0,T], \\ w^\varepsilon(x,t) = G(x,t) - g(x,t), & x \notin \Omega,\ t \in (0,T], \\ w^\varepsilon(x,0) = 0, & x \in \Omega. \end{cases} \tag{2.10}$$

First, we claim that $\bar{w}(x,t) = K_1\theta(\varepsilon)t + K_2\varepsilon$ is a supersolution with $K_1, K_2 > 0$ sufficiently large but independent of ε. Indeed, taking into account Lemma 2.3.1 and that $\mathcal{L}_\varepsilon(\bar{w}) = 0$ we have

$$\bar{w}_t(x,t) = K_1\theta(\varepsilon) \ge \Lambda(\tilde{v}) - \mathcal{L}_\varepsilon(\tilde{v}) + \mathcal{L}_\varepsilon(\bar{w}).$$

Moreover, $\bar{w}(x,0) > 0$ and by (2.9) we obtain that $\bar{w}(x,t) \ge K_2\varepsilon \ge O(\varepsilon)$, for $t \in (0,T]$ and $x \notin \Omega$ such that $\operatorname{dist}(x,\partial\,\Omega) \le a\varepsilon$, which is our claim. From the comparison result we get

$$\tilde{v} - u^\varepsilon \le \bar{w}(x,t) = K_1\theta(\varepsilon)t + K_2\varepsilon.$$

Similar arguments applied to the case $\underline{w}(x,t) = -\bar{w}(x,t)$ leads us to assert that $\underline{w}(x,t)$ is a subsolution of problem (2.10). We conclude, using again the comparison principle stated in Proposition 2.2.2, that

$$-K_1\theta(\varepsilon)t - K_2\varepsilon \le \tilde{v} - u^\varepsilon \le K_1\theta(\varepsilon)t + K_2\varepsilon,$$

and hence

$$\|v - u^\varepsilon\|_{L^\infty(\Omega\times[0,T])} \le K_1 T\theta(\varepsilon) + K_2\varepsilon \to 0.$$

$\square$

Remark 2.3.3 *It is worth pointing out that the particular case $A(x) = I$ and $b_i(x) = 0$, which corresponds to the heat equation, the rescaled kernel (2.5) considered by Cortázar et al. in Cortázar et al. (2009) is the same K_ε considered here. Moreover, if we take another decomposition of the identity matrix, for example, consider $l_{i,j} = 1$ if $i + j = N + 1$ and 0 otherwise, we can get a different nonlocal approximation by nonlocal diffusion problems of the heat equation.*

2.4 Divergence form operators

In this section, we consider the following rescaled kernel

$$K_\varepsilon(x, y) = \frac{2}{C(J)\varepsilon^{N+2}} G\left(B^{-1}(x)\frac{x-y}{\varepsilon}\right) G\left(B^{-1}(y)\frac{x-y}{\varepsilon}\right), \tag{2.11}$$

where $G^2(s) = J(s)$ and $B(x) = (b_{ij}(x))$ is a $N \times N$ matrix such that

$$\det(B(x))B(x)B^t(x) = A(x).$$

Note that the kernels given in (2.11) are symmetric, that is, they verify

$$K_\varepsilon(x, y) = K_\varepsilon(y, x).$$

For this family of symmetric kernels Proposition 2.2.1 and Proposition 2.2.2 can be used. Therefore, we have that the approximation result stated in Theorem 2.1.1 holds for the divergence form equation

$$v_t(x, t) = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v(x, t)}{\partial x_j}\right).$$

This can be proved exactly as before as soon as one has the following result.

Lemma 2.4.1 *Let u be a $\mathcal{C}^{2+\alpha}\left(\mathbb{R}^N\right)$ function and*

$$\mathcal{L}_\varepsilon(u) := \int_{\mathbb{R}^N} K_\varepsilon(x, y)(u(y) - u(x)dy.$$

Then

$$\left\|\mathcal{L}_\varepsilon(u) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right)\right\|_{L^\infty(\Omega\times[0,T])} \le \theta(\varepsilon),$$

for some function $\theta(\varepsilon)$ that goes to zero as $\varepsilon \to 0$.

Proof: In this proof we will use the following notations for partial derivatives and for the coefficients of the inverse and the adjoint of a matrix,

$$(f(s))'_i = \frac{\partial f(s)}{\partial s_i}, \quad B^{-1}(x) = (b^{-1}_{ij}(x)), \quad B^*(x) = (b^*_{ij}(x)).$$

Using the change of variable $z = \frac{x-y}{\varepsilon}$ and Taylor's expansions we get

$$\mathcal{L}_\varepsilon(u)(x) = F_{1,\varepsilon}(x) + F_{2,\varepsilon}(x) + O(\varepsilon^{2+\alpha})$$

with

$$F_{1,\varepsilon}(x) = \frac{-2}{C(J)\varepsilon} \sum_{i=1}^N \frac{\partial u(x)}{\partial x_i} \int_{\mathbb{R}^N} G(B^{-1}(x-\varepsilon z)z)G(B^{-1}(x)z)z_i\,dz$$

and

$$F_{2,\varepsilon}(x) = \frac{1}{C(J)} \sum_{i,j=1}^N \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \int_{\mathbb{R}^N} G(B^{-1}(x-\varepsilon z)z)G(B^{-1}(x)z)z_i z_j\,dz.$$

Let us first analyze the limit as $\varepsilon \to 0$ of $F_{1,\varepsilon}(x)$. As $\int J(B^{-1}(x)z)z_i dz = 0$ (this follows changing z by $-z$), we can use L'Hopital's rule to obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} F_{1,\varepsilon}(x) &= \frac{2}{C(J)} \sum_{i=1}^N \frac{\partial u(x)}{\partial x_i}\\ &\quad\times \int_{\mathbb{R}^N} \sum_{j=1}^N G_j'(B^{-1}(x)z) \sum_{k,m=1}^N (b_{jk}^{-1})_m'(x) z_k z_m G(B^{-1}(x)z)z_i\,dz.\end{aligned}$$

Now we observe that

$$G_j'(s)G(s) = \frac{1}{2}J_j'(s)$$

and hence

$$\lim_{\varepsilon\to 0} F_{1,\varepsilon}(x) = \frac{1}{C(J)} \sum_{i,j,k,m=1}^N \frac{\partial u(x)}{\partial x_i} (b_{jk}^{-1})_m'(x) \int_{\mathbb{R}^N} J_j'(B^{-1}(x)z) z_k z_m z_i\,dz.$$

Changing variables as $w = B^{-1}(x)z$ we have

$$\begin{aligned}\lim_{\varepsilon\to 0} F_{1,\varepsilon}(x) = \frac{det(B(x))}{C(J)} \sum_{i,j,k,m,p,q,r=1}^N & \frac{\partial u(x)}{\partial x_i} (b_{jk}^{-1})_m'(x) b_{ip}(x) b_{kq}(x) b_{mr}(x)\\ & \times \int_{\mathbb{R}^N} J_j'(w) w_p w_q w_r\,dw.\end{aligned}$$

To continue we have to find the value of the last integral. We have that

$$\int_{\mathbb{R}^N} J_j'(w) w_p w_q w_r\,dw = 0,$$

except for the following cases:

Case 1. $p = q = r = j$. In this case we have

$$\int_{\mathbb{R}^N} J_j'(w)(w_j)^3\,dw = -3\int_{\mathbb{R}^N} J(w)(w_j)^2\,dw = -3C(J).$$

Case 2. ($p = j$ and $q = r \neq j$) or ($q = j$ and $p = r \neq j$) or ($r = j$ and $p = q \neq j$). In any of these cases one index is equal to j and the other two indexes are the same but different from j. Hence, in this case we get

$$\int_{\mathbb{R}^N} J'_j(w) w_j (w_q)^2 \, dw = -\int_{\mathbb{R}^N} J(w)(w_q)^2 \, dw = -C(J).$$

Collecting these cases we obtain

$$\lim_{\varepsilon \to 0} F_{1,\varepsilon}(x) = \sum_{i=1}^{N} \frac{\partial u(x)}{\partial x_i} H_i(x).$$

with

$$\begin{aligned}
H_i(x) = -det(B(x)) \Bigg\{ & \sum_{j,k,m=1}^{N} 3(b_{jk}^{-1})'_m(x) b_{ij}(x) b_{kj}(x) b_{mj}(x) \\
& + \sum_{j,k,m\ p\neq j}^{N} (b_{jk}^{-1})'_m(x) \Big[b_{ij}(x) b_{kp}(x) b_{mp}(x) \Big] \\
& + \sum_{j,k,m,p\neq j}^{N} (b_{jk}^{-1})'_m(x) \Big[b_{ip}(x) b_{kj}(x) b_{mp}(x) \Big] \\
& + \sum_{j,k,m\ p\neq j}^{N} (b_{jk}^{-1})'_m(x) \Big[b_{ip}(x) b_{kp}(x) b_{mj}(x) \Big] \Bigg\} \\
= -det(B(x)) \Bigg\{ & \sum_{j,k,m,p=1}^{N} (b_{jk}^{-1})'_m(x) \Big[b_{ij}(x) b_{kp}(x) b_{mp}(x) \Big] \\
& + \sum_{j,k,m,p=1}^{N} (b_{jk}^{-1})'_m(x) \Big[b_{ip}(x) b_{kj}(x) b_{mp}(x) \Big] \\
& + \sum_{j,k,m,p=1}^{N} (b_{jk}^{-1})'_m(x) \Big[b_{ip}(x) b_{kp}(x) b_{mj}(x) \Big] \Bigg\} = A_1 + A_2 + A_3.
\end{aligned}$$

Let us compute each one of the last three terms A_1, A_2 and A_3. First, using that

$$\sum_{k=1}^{N} b_{ik}^{-1}(x) b_{kj}(x) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

we obtain

$$\sum_{k=1}^{N} (b_{ik}^{-1})'_m(x) b_{kj}(x) = -\sum_{k=1}^{N} b_{ik}^{-1}(x)(b_{kj})'_m(x). \tag{2.12}$$

Using this property, we get

$$\begin{aligned}
A_1 &= -det(B(x)) \sum_{j,k,m,p=1}^{N} (b^{-1}_{jk})'_m(x) \Big[b_{ij}(x) b_{kp}(x) b_{mp}(x) \Big] \\
&= det(B(x)) \sum_{j,k,m,p=1}^{N} \Big[b_{ij}(x) b^{-1}_{jk}(x) (b_{kp})'_m(x) b_{mp}(x) \Big] \\
&= det(B(x)) \sum_{m,p=1}^{N} \Big[(b_{kp})'_m(x) b_{mp}(x) \Big].
\end{aligned}$$

Now, for A_2, using again (2.12) we have

$$\begin{aligned}
A_2 &= -det(B(x)) \sum_{j,k,m,p=1}^{N} (b^{-1}_{jk})'_m(x) \Big[b_{ip}(x) b_{kj}(x) b_{mp}(x) \Big] \\
&= det(B(x)) \sum_{j,k,m,p=1}^{N} \Big[b^{-1}_{jk}(x) (b_{kj})'_m(x) b_{ip}(x) b_{mp}(x) \Big].
\end{aligned}$$

As

$$b^{-1}_{jk}(x) = \frac{1}{det(B(x))} (b^*_{jk}(x))^t = \frac{1}{det(B(x))} b^*_{kj}(x)$$

we get

$$A_2 = \sum_{m,p=1}^{N} b_{ip}(x) b_{mp}(x) \sum_{k,j=1}^{N} b^*_{kj}(x) (b_{kj})'_m(x).$$

Now we use the formula for the derivative of the determinant (see Golberg (1972) for a simple proof),

$$(det(B(x)))'_m = \sum_{k,j=1}^{N} b^*_{kj}(x) (b_{kj})'_m(x),$$

to obtain

$$A_2 = \sum_{m,p=1}^{N} b_{ip}(x) b_{mp}(x) (det(B(x)))'_m.$$

Finally, for A_3, using (2.12) one more time, we have

$$\begin{aligned}
A_3 &= -det(B(x)) \sum_{j,k,m,p=1}^{N} (b^{-1}_{jk})'_m(x) \Big[b_{ip}(x) b_{kp}(x) b_{mj}(x) \Big] \\
&= det(B(x)) \sum_{j,k,m,p=1}^{N} \Big[b_{ip}(x) (b_{kp})'_m(x) b_{mj}(x) b^{-1}_{jk}(x) \Big] \\
&= det(B(x)) \sum_{m,p=1}^{N} \Big[(b_{mp})'_m(x) b_{ip}(x) \Big].
\end{aligned}$$

Hence, collecting these expressions for A_i we obtain

$$H_i(x) = \sum_{j=1}^{N} \Big[det(B(x))(B'_j(x)B^t(x))_{ij} \\ +(det(B(x)))'_j(B(x)B^t(x))_{ij} \\ +det(B(x))(B(x)(B^t)'_j(x))_{ij} \Big] = \sum_{j=1}^{N} \frac{\partial a_{ij}(x)}{\partial x_j}.$$

Therefore, we have obtained

$$\lim_{\varepsilon\to 0} F_{1,\varepsilon}(x) = \sum_{i=1}^{N} \frac{\partial u(x)}{\partial x_i} \sum_{j=1}^{N} \frac{\partial a_{ij}(x)}{\partial x_j}. \tag{2.13}$$

Next, we deal with the limit as $\varepsilon \to 0$ of $F_{2,\varepsilon}(x)$. It holds that

$$\lim_{\varepsilon\to 0} F_{2,\varepsilon}(x) = \frac{1}{C(J)} \sum_{i,j=1}^{N} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \int_{\mathbb{R}^N} G^2(B^{-1}(x)z) z_i z_j \, dz.$$

Changing variables as $w = B^{-1}(x)z$ we get

$$\lim_{\varepsilon\to 0} F_{2,\varepsilon}(x) = \frac{det(B(x))}{C(J)} \sum_{i,j=1}^{N} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \int_{\mathbb{R}^N} J(w) \sum_{k=1}^{N} b_{ik}(x) w_k \sum_{m=1}^{N} b_{jm}(x) w_m \, dw.$$

Now we only have to observe that

$$\int_{\mathbb{R}^N} J(w) w_k w_m \, dw = \begin{cases} C(J) & k = m, \\ 0 & k \neq m, \end{cases}$$

to obtain

$$\lim_{\varepsilon\to 0} F_{2,\varepsilon}(x) = \sum_{i,j=1}^{N} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} det(B(x)) \sum_{k=1}^{N} b_{ik}(x) b_{jk}(x) = \sum_{i,j=1}^{N} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} a_{ij}(x). \tag{2.14}$$

Finally, from (2.13) and (2.14) we conclude that

$$\lim_{\varepsilon\to 0} \mathcal{L}_\varepsilon(u)(x) = \sum_{i,j=1}^{N} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} a_{ij}(x) + \sum_{i=1}^{N} \frac{\partial u(x)}{\partial x_i} \sum_{j=1}^{N} \frac{\partial a_{ij}(x)}{\partial x_j} \\ = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \right)$$

as we wanted to show. □

2.5 Appendix

Appendix A

For any arbitrary $T > 0$ we claim that $\mathcal{T}$ is a contraction on $\mathcal{C}\left([0,T]; L^1(\Omega)\right)$ with norm

$$|||v||| = \max_{0\le t\le T} e^{-Mt}\|v(\cdot,t)\|_{L^1(\Omega)}$$

being M some constant greater than $\tilde{C} = C(|\Omega| + |B(0,d)|)$. Indeed, from (2.6)

$$\|\mathcal{T}(v)(\cdot,t)\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)} + \frac{\tilde{C}}{M}\left(e^{Mt}-1\right)|||v|||,$$

therefore

$$|||T(v)|||\le \max_{0\le t\le T}\left(e^{-Mt}\|u_0\|_{L^1(\Omega)} + \frac{\tilde{C}}{M}\left(1-e^{-Mt}\right)|||v|||\right) \le \|u_0\|_{L^1(\Omega)} + \frac{\tilde{C}}{M}|||v|||,$$

and the claim is proved. The rest of the proof is similar in spirit to the proof of Proposition 2.2.1.

Appendix B

Given $B(x)$, matrix $n\times n$ defined for each $x\in\overline{\Omega}$, we wish to recall that the induced matrix norm to the euclidian matrix norm

$$\|B(x)\|_2 = \sup_{y\neq 0}\frac{\|B(x)y\|_2}{\|y\|_2}$$

is the spectral norm, i.e., $\|B(x)\|_2 = \sqrt{\lambda_{Max}(B^t(x)B(x))}$. Thus

$$\|L^{-1}(x)\|_2 = \sqrt{\lambda_{Max}(A^{-1}(x))} = \left(\lambda_{min}(A(x))\right)^{-1/2},$$

and hence $L^{-1}(x)\dfrac{x-y}{\varepsilon}\in B(0,r)$ if $y\in B(x,\frac{r\varepsilon}{\|L^{-1}(x)\|_2})\subset B(x,a\varepsilon)$.

Ackonwledgement

This work was carried out during a stay by the first author at Universidad of Buenos Aires (Argentina) supported by Secretaría de Estado de Investigación, Desarrollo e Innovación EEBB2014 (Spain).

Chapter 3

Nonlocal approximations to Fokker-Planck equations

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Abstract

We show that solutions to a classical Fokker-Plank equation can be approximated by solutions to nonlocal evolution problems when a rescaling parameter that controls the size of the nonlocality goes to zero.

3.1 Introduction

Nonlocal reaction-diffusion equations of the form

$$u_t(x,t) = \int_{\mathbb{R}^N} K(x,y)u(y,t)dy - u(x,t), \tag{3.1}$$

where $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative smooth kernel (usually assumed to be symmetric, but here this may not be the case) such that $\int K(x,y)dx = 1$, and variations of it, have been recently studied to model diffusion process. If $u(y,t)$ is thought of as a density of a population at location y at time t and $K(x,y)$ as the probability distribution of jumping from y to x, then the rate at which individuals are arriving to x is $\int K(x,y)u(y,t)dy$. On the other hand, the rate at which individuals are leaving location x to travel to other places is $-\int K(y,x)u(x,t)dy = -u(x,t)$. In the absence of external sources this implies that the density satisfies equation (3.1).

New in this work is to consider kernels of the form

$$K(x,y) = J\big(\mathcal{M}(y)(x-y)\big) \det \mathcal{M}(y). \tag{3.2}$$

Here $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative radial function such that

$$J \in \mathcal{C}_c(\mathbb{R}^n), \quad \int_{\mathbb{R}^N} J(z)dz = 1 \;\text{ and }\; \int_{\mathbb{R}^N} J(z)z_N^2 dz = C(J) < \infty \tag{3.3}$$

and $M(y)$ is a $N \times N$ real matrix with smooth and bounded coefficients such that $\det M(y) \geq \gamma > 0$. Note that, for this kind of kernels, we have a mass preserving property, that is,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\big(\mathcal{M}(y)(x-y)\big) \det \mathcal{M}(y)\, u(y)dydx = \int_{\mathbb{R}^N} u(x)dx, \qquad \forall u \in \mathcal{C}(\mathbb{R}^N).$$

Our main goal in this work is to show that solutions to the nonlocal problem (3.1) with kernels of the form (3.2) adequately rescaled approximate solutions to the classical local Fokker-Plank equation.

In more detail, consider the following local diffusion problem

$$\begin{cases} v_t(x,t) = \displaystyle\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(a_{ij}(x)v(x,t)\right), & x \in \mathbb{R}^N, t \in [0,T], \\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.4}$$

where $A(x) = (a_{ij}(x))$ is a real positive-definite matrix.

Throughout the paper, we make the following assumptions on the matrix A: $A(x) = (a_{ij}(x))$ is a real $N \times N$ symmetric and positive-definite matrix with smooth coefficients such that

$$\delta\|\xi\|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \beta\|\xi\|^2, \qquad \forall x,\xi \in \mathbb{R}^N,$$

for some constants $0 < \delta < \beta$ and we will also assume that

$$\max_x \left\{ \sum_{i,j} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j} \right\} < \infty. \tag{3.5}$$

Given $A(x)$, we let $B(x) = (b_{ij}(x))$ be a real $N \times N$ matrix with strictly positive determinant and smooth coefficients satisfying $B(x)B^t(x) = A(x)$, $x \in \mathbb{R}^N$. Note that such decomposition is possible since A is a positive-definite matrix (e.g. using Cholesky factorization).

Now, let us consider the following nonlocal equation

$$\begin{cases} u_t^\varepsilon = \dfrac{C}{\varepsilon^2} \left\{ \displaystyle\int_{\mathbb{R}^N} K_\varepsilon(x,y)u(y,t)dy - u(x,t) \right\}, & x \in \mathbb{R}^N, t \in [0,T], \\ u^\varepsilon(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.6}$$

where $C^{-1} = \frac{1}{2}\int J(z)z_N^2 dz$ is a constant that depends only on J and the kernel K_ε is given by

$$K_\varepsilon(x,y) = \frac{1}{\varepsilon^N} J\Big(B^{-1}(y)\frac{(x-y)}{\varepsilon}\Big) \det B^{-1}(y),$$

with B as above, that is, such that $BB^t = A$ and J satisfying (3.3).

As we have mentioned, our aim is to show that solutions of (3.6) converge uniformly to solutions of (3.4). Our main result reads as follows:

Theorem 3.1.1 *Let v be a classical solution of Fokker-Planck equation (3.4) with initial datum $v_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. For every $\varepsilon > 0$, consider u^ε the solution of the nonlocal equation (3.6). Then,*

$$\sup_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - v(\cdot,t)\|_{L^\infty} \to 0,$$

as $\varepsilon \to 0$.

Now, let us comment briefly on previous results concerning approximations of local PDEs by nonlocal problems.

Kernels of the form (3.2) cover a wide variety of nonlocal diffusion problems treated in the past twenty years. For example, taking the simplest case $M(y) = Id$, equation (3.1) reduces to the following convolution type diffusion problem

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - u(x,t).$$

This model has been treated by several authors in different contexts, see for example Bates et al. (1997); Chasseigne et al. (2006); Cortazar et al. (2007) and the references given therein. In addition, in Cortázar et al. (2009) the authors prove that, under an appropriate rescaling of the kernel, that is, solutions to

$$u_t^\varepsilon(x,t) = \frac{C}{\varepsilon^2}\left\{\int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} J\Big(\frac{x-y}{\varepsilon}\Big)u(y,t)dy - u(x,t)\right\},$$

converge, as $\varepsilon \to 0$, to solutions to the local heat equation, $v_t = \Delta v$.

Another example is the kernel (3.2) with $M(y) = g^{-1}(y)Id$, being g a positive scalar function. In this case (3.1) takes the form

$$u_t(x,t) = \int_{\mathbb{R}^N} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g^N(y)}dy - u(x,t).$$

Note that in this evolution problem the step size, $g(y)$, depends on the position y. Such kind of diffusion kernel was introduced in Cortázar et al. (2007) in order to model a non-homogeneous dispersal process. See also Cortázar et al. (2015) and Coville (2010). For this problem in Sun et al. (2011) the authors prove that under appropriate rescaling of the kernel, i.e. when the problem takes the form

$$u_t^\varepsilon(x,t) = \frac{C}{\varepsilon^2}\left\{\int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} J\Big(\frac{x-y}{\varepsilon g(y)}\Big)\frac{u(y,t)}{g^N(y)}dy - u(x,t)\right\}, \tag{3.7}$$

solutions converge to solutions to the local equation

$$v_t(x,t) = \sum_i (g^2(x)v(x,t))_{x_i x_i}.$$

Closely related to this work is Molino and Rossi (2016) where we find kernels for nonlocal evolution problems that, when appropriately rescaled as above, have solutions that approximate solutions to local problems with spatial dependence in divergence form,

$$v_t(x,t) = \sum_{i,j}(a_{ij}(x)v_{x_j})_{x_i}(x,t)$$

or in non-divergence form,

$$v_t(x,t) = \sum_{i,j} a_{ij}(x) v_{x_i x_j}(x,t).$$

Notations. Given $A(x) = (a_{ij}(x))$ we denote by $a_{ij}^t(x)$ and $a_{ij}^{-1}(x)$ the coefficients of the matrices $A^t(x)$, $A^{-1}(x)$ respectively. Also, for any given function $f : \mathbb{R}^N \to \mathbb{R}$ we denote by $f_i'(s) = \frac{\partial f(s)}{\partial s_i}$ and by $[f]_+(s) = \max\{0, f(s)\}$.

The paper is organized as follows: in Section 3.2 we show existence, uniqueness and a comparison principle for the nonlocal problem and in Section 3.3 we prove the convergence of the solutions as the scaling parameter ε goes to zero.

3.2 Existence, uniqueness and comparison principle

We start this section proving the comparison principle for our problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right) \det M(y)\, u(y,t)dy - u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{P}$$

For this purpose, we first set the notion of sub and supersolution for (P).

Definition 3.2.1 *A function $u \in \mathcal{C}^1\left([0,\infty), \mathcal{C}(\mathbb{R}^N)\right)$ is a subsolution of problem (P) if it satisfies*

$$\begin{cases} u_t(x,t) \le \displaystyle\int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right) \det M(y)\, u(y,t)dy - u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(x,0) \le u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

As usual, a supersolution is defined analogously by replacing " $\le$ " by " $\ge$ ".

Theorem 3.2.2 *[Comparison Principle] Let u, v be a subsolution and supersolution respectively of problem (P). Then $u \le v$.*

Proof: To prove this result we follow closely (Sun et al., 2011, Theorem 2.5). Set $w = u - v$, then

$$\begin{cases} w_t(x,t) \le \displaystyle\int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right) \det M(y) w(y,t)dy - w(x,t) & x \in \mathbb{R}^N, t > 0, \\ w(x,0) \le 0, & x \in \mathbb{R}^N. \end{cases} \tag{3.8}$$

Let us consider the following function

$$s(x,t) = \begin{cases} 1, & \text{if } w(x,t) \ge 0, \\ 0, & \text{if } w(x,t) < 0. \end{cases}$$

Multiplying (3.8) by $s(x,t)$ and taking into account that $w_t(x,t)s(x,t) = \left([w]_+\right)_t(x,t)$ and $w(y,t) \le [w]_+(y,t)$, we obtain, dropping the positive term $w(x,t)s(x,t)$, that

$$\left([w]_+\right)_t(x,t) \le \int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right) \det M(y)\, [w]_+(y,t)dy,$$

integrating in $\mathbb{R}^N$ and by using the mass preserving property, we get

$$\int_{\mathbb{R}^N} \left([w]_+\right)_t(x,t) \le \int_{\mathbb{R}^N} [w]_+(y,t)dy.$$

Finally, integrating in $(0,t)$ and since $[w]_+(x,0)=0$ we can assert, using Fubini's theorem, that

$$h(t) \le \int_0^t h(s)ds, \tag{3.9}$$

where

$$h(t) = \int_{\mathbb{R}^N} [w]_+(x,t)dx.$$

Hence, applying Gronwall's Lemma in (3.9), we conclude that

$$h(t) \le 0.$$

Now, since $[w]_+(x,t) \ge 0$ and by the continuity of $[w]_+$, we get that $[w]_+(x,t)=0$ and, consequently,

$$u(x,t) \le v(x,t)$$

for all $x \in \mathbb{R}^N,\ t>0$. □

Note that the previous proof works locally in time, that is, a supersolution v and a subsolution u defined both for $t \in [0,T]$ verify $u(x,t) \le v(x,t)$ for all $x \in \mathbb{R}^N,\ 0 \le t \le T$.

Definition 3.2.3 *By a solution of the problem (P), we mean a function $u \in \mathcal{C}\left([0,\infty);\mathcal{C}(\mathbb{R}^N)\right)$ that satisfies*

$$u(x,t) = \int_0^t \int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right) \det M(y)\, u(y,s)dyds - \int_0^t u(x,s)ds + u_0(x),$$

for all $x \in \mathbb{R}^N$, $t \in [0,\infty)$. Consequently, due to this integral expression, we can assert that $u \in \mathcal{C}^1\left([0,\infty);\mathcal{C}(\mathbb{R}^N)\right)$.

Now, we prove existence and uniqueness of a solution which is bounded in $\mathbb{R}^N$.

Theorem 3.2.4 *[Existence] For every continuous and bounded initial data u_0 there exists a unique solution $u \in \mathcal{C}\left([0,\infty);\mathcal{C}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\right)$ of problem (P).*

Proof: For $T>0$ we consider the Banach space

$$X = \mathcal{C}\left([0,T];\mathcal{C}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\right),$$

with the norm

$$\|w\| = \max_{0\le t\le T} e^{-k(M+1)t}\|w(\cdot,t)\|_{L^\infty}.$$

Here $M = \max_{x\in\mathbb{R}^N} \det M(x) > 0$ and k is any value greater than one.

Now, let Y be the closed ball of X with radius $k\|u_0\|_\infty$ and centered at the origin. Note that Y is a complete metric space with the induced metric $d(w_1,w_2)=\|w_1-w_2\|$.

In order to establish the existence and uniqueness of solutions of (P) via Banach contraction principle, we define the operator $\mathcal{T}: Y \longrightarrow Y$ by

$$\mathcal{T}(w)(x,t) = \int_0^t \int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right) \det M(y)\, w(y,s)dyds - \int_0^t w(x,s)ds + u_0(x).$$

Let us first prove that this operator is well defined. Clearly $\mathcal{T}(w)$ is belongs to X and satisfies

$$\|\mathcal{T}(w)(\cdot,t)\|_{L^\infty} \le \max_x \left| \int_0^t \int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right) \det M(y)\, w(y,s)dyds \right|$$
$$+ \int_0^t \|w(\cdot,s)\|_{L^\infty} ds + \|u_0\|_{L^\infty} \le (M+1)\int_0^t \|w(\cdot,s)\|_{L^\infty} ds + \|u_0\|_{L^\infty}.$$

Since $\|w\| \le k\|u_0\|_{L^\infty}$, we obtain that, for $0 \le t \le T$,

$$\|\mathcal{T}(w)(\cdot,t)\|_{L^\infty} \le e^{k(M+1)T}\|u_0\|_{L^\infty},$$

therefore, for T small, $\|\mathcal{T}(w)\| \le k\|u_0\|_{L^\infty}$ and $\mathcal{T}(w)$ belongs to Y.

Now, let us show that the operator $\mathcal{T}$ is a contraction. we have

$$d(\mathcal{T}(w_1),\mathcal{T}(w_2)) \le \max_{0\le t\le T} e^{-k(M+1)t}(M+1)\int_0^t \|w_1(\cdot,s)-w_2(\cdot,s)\|_{L^\infty}ds.$$

Arguing as above, we obtain

$$d(\mathcal{T}(w_1),\mathcal{T}(w_2)) \le \max_{0\le t\le T}\frac{1}{k}\|w_1-w_2\|\left(1-e^{-k(M+1)t}\right) \le \frac{1}{k}d(w_1,w_2).$$

Hence, using Banach's Fixed Point Theorem there exists u a fix point of $\mathcal{T}$, that is the unique solution of problem (P) for $t\in[0,T]$ and belongs to Y. Finally, since from the comparison principle we have that

$$-\|u_0\|_{L^\infty}e^{(\max_x\int K(x,y)dy-1)t} \le u(x,t) \le \|u_0\|_{L^\infty}e^{(\max_x\int K(x,y)dy-1)t},$$

we obtain a global solution, $u\in\mathcal{C}\left([0,\infty);\mathcal{C}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\right)$. □

3.3 Approximations of the Fokker-Planck equation by nonlocal problems

In this section we prove our main result, that is, that solutions of the Fokker-Planck equation can be approximated by solutions of the nonlocal problem by rescaling the kernel.

Recall that the general Fokker-Planck equation is given by

$$\begin{cases} v_t(x,t)=\displaystyle\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\left(a_{ij}(x)v(x,t)\right), & x\in\mathbb{R}^N, t\in[0,T],\\ v(x,0)=v_0(x), & x\in\mathbb{R}^N,\end{cases} \tag{$F-P$}$$

We will call a solution to the Cauchy problem for the Fokker-Planck equation ($F-P$) a *classical solution* if $v\in\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N,[0,T]\right)$. Note that the regularity of v is related with smoothness of $a_{ij}(x)$ and the initial datum v_0; see Evans (1998); Ladyzenskaja et al. (1968).

We first need to prove the following technical lemmas.

Lemma 3.3.1 *Let J be a function satisfying hypothesis (3.3). Then, the following properties are satisfied:*

1.

$$\int_{\mathbb{R}^N} J_j'(w)w_pw_qw_r\,dw = \begin{cases} -3C(J), & \text{if } p=q=r=j,\\ -C(J), & \text{if } \begin{cases} p=j \text{ and } r=q\ne j, \text{ or}\\ q=j \text{ and } r=p\ne j, \text{ or}\\ r=j \text{ and } p=q\ne j,\end{cases}\\ 0, & \text{in other case.}\end{cases}$$

2.

$$\int_{\mathbb{R}^N} J_{jj}''(w)w_lw_nw_sw_t\,dw = \begin{cases} 12C(J), & \text{if } l=n=s=t=j,\\ 2C(J), & \begin{cases} \text{if two indexes are equal to } j\\ \text{and the others two are equal}\\ \text{to each other and different to } j.\end{cases}\\ 0, & \text{in other case.}\end{cases}$$

3. For $j \neq p$

$$\int_{\mathbb{R}^N} J''_{jp}(w) w_l w_n w_s w_t dw = \begin{cases} 3C(J), & \begin{cases} \text{if three indexes are equal to } j \\ \text{and the other one are equal} \\ \text{to } p, \text{ or viceversa.} \end{cases} \\ \\ C(J), & \begin{cases} \text{if one index is equal to } j, \text{ another} \\ \text{index is equal to } p, \text{ and the} \\ \text{others two are equal to each} \\ \text{other but different to } j \text{ and } p. \end{cases} \\ \\ 0, & \text{in other case.} \end{cases}$$

Proof: (1) If $p = q = r = j$ and since J has compact support, integrating by parts it follows that

$$\int_{\mathbb{R}^N} J'_j(w) w_j^3 dw = -3 \int_{\mathbb{R}^N} J(w) w_j^2 dw = -3C(J).$$

Similarly, if one of the indexes is equal to j and the others two are equal between them and different from j, integrating by parts respect to the variable j, we obtain

$$\int_{\mathbb{R}^N} J'_j(w) w_j w_s^2 dw = -\int_{\mathbb{R}^N} J(w) w_s^2 dw = -C(J),$$

for $s = p, q, r$. Finally, in the same way we show that is zero occurs in any different case.

(2) For the first case, integrating by parts twice, we get

$$\int_{\mathbb{R}^N} J''_{jj}(w) w_j^4 dw = -4 \int_{\mathbb{R}^N} J'_j(w) w_j^3 dw = 12 \int_{\mathbb{R}^N} J(w) w_j^2 dw = 12C(J).$$

We proceed likewise, if two indexes are equal to j and the other two are equal between them and different from j (there are 6 cases). For example, taking $l = n = j$ and $s = t \neq j$, we obtain integrating by parts twice

$$\int_{\mathbb{R}^N} J''_{jj}(w) w_j^2 w_s^2 dw = -2 \int_{\mathbb{R}^N} J'_j(w) w_j w_s^2 dw = 2 \int_{\mathbb{R}^N} J(w) w_j^2 dw = 2C(J).$$

Finally, the proof in any other case follows similarly and is left to the reader.

(3) We apply the same reasoning, integrating by parts twice, respect to the variable p and j. First, if three indexes are equal to j and the other one is equal to p (there are 8 cases) we get, taking for example $l = n = s = j$ and $t = p$, that

$$\int_{\mathbb{R}^N} J''_{jp}(w) w_j^3 w_p dw = -\int_{\mathbb{R}^N} J'_j(w) w_j^3 dw = 3 \int_{\mathbb{R}^N} J(w) w_j^2 dw = 3C(J).$$

Analogously, if one index is equal to j, another index is equal to p, and the other two are equalbetween them but different from j and p (there are 12 cases) we have, choosing for example $l = j$, $n = p$ and $s = t \neq j, p$, that

$$\int_{\mathbb{R}^N} J''_{jp}(w) w_j w_p w_s^2 dw = -\int_{\mathbb{R}^N} J'_j(w) w_j w_s^2 dw = \int_{\mathbb{R}^N} J(w) w_s^2 dw = C(J).$$

We leave it to the reader to verify that in any other case the integral expression is equal to zero. □

Lemma 3.3.2 *Let $A(x) = (a_{ij}(x))$ be a $N \times N$ non-singular real matrix with smooth coefficients $a_{ij} : \mathbb{R}^N \to \mathbb{R}$, $i, j = 1 \ldots N$, then the following properties are satisfied:*

1.

$$\sum_k \left(a_{ik}^{-1}\right)'_m (x) a_{kj}(x) = -\sum_k a_{ik}^{-1}(x) \left(a_{kj}\right)'_m (x),$$

2.
$$\sum_k (a_{jk}^{-1})''_{mp}(x)a_{kq}(x) = -\sum_k \left\{(a_{jk}^{-1})'_m(x)(a_{kq})'_p(x) + (a_{jk}^{-1})'_p(x)(a_{kq})'_m(x) + a_{jk}^{-1}(x)\,(a_{kq})''_{mp}(x)\right\},$$

3.
$$\sum_{j,k} a_{jk}^{-1}(x)(a_{kj})'_m(x) = \det A^{-1}(x)\,(\det A(x))'_m\,.$$

Proof:
*(*1) It follows by computing the derivate of $\sum_k a_{ik}^{-1}(x)a_{kj}(x) = \delta_{ij}$.
*(*2) It is easy to prove when we compute the derivate of the expression in *(*1).
*(*3) See Golberg (1972) for a simple and original proof. □

Also the following propositions will be needed in the proof of our main theorem. To simplify the notation, in what follows we let
$$J_\varepsilon(s) = \frac{1}{\varepsilon^N} J\Big(\frac{s}{\varepsilon}\Big).$$

Proposition 3.3.3 *Let u be a $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N \times [0,T]\right)$ function and let $\mathcal{L}^1_\varepsilon$ and Λ be the operators given by*
$$\mathcal{L}^1_\varepsilon(u(x,t)) = \frac{C}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon\left(B^{-1}(y)(x-y)\right)\det B^{-1}(y)(u(y,t)-u(x,t))dy,$$
$$\Lambda(u(x,t)) = \sum_{i,j}\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j}a_{ij}(x) + 2\sum_{i,j}\frac{\partial u(x,t)}{\partial x_i}\frac{\partial a_{ij}(x)}{\partial x_j}.$$
Then,
$$\sup_{t\in[0,T]} \|\left(\mathcal{L}^1_\varepsilon - \Lambda\right)(u(\cdot,t))\|_{L^\infty} \to 0 \ \ as\ \ \varepsilon \to 0.$$

Proof: Under the change variables $y = x - \varepsilon z$ and by a simple Taylor expansion we obtain
$$\mathcal{L}^1_\varepsilon(u(x,t)) = \sum_{i,j}\frac{\partial^2 u(x,t)}{\partial x_i\partial x_j}H^1_\varepsilon(x) + \sum_i \frac{\partial u(x,t)}{\partial x_i}H^2_\varepsilon(x) + O(\varepsilon^\alpha),$$
being
$$H^1_\varepsilon(x) = \frac{C}{2}\int_{\mathbb{R}^N} J\left(B^{-1}(x-\varepsilon z)z\right)\det B^{-1}(x-\varepsilon z)z_i z_j dz,$$
and
$$H^2_\varepsilon(x) = -\frac{C}{\varepsilon}\int_{\mathbb{R}^N} J\left(B^{-1}(x-\varepsilon z)z\right)\det B^{-1}(x-\varepsilon z)z_i dz.$$

First, we claim that
$$H^1_\varepsilon(x) \to a_{ij}(x)$$
as $\varepsilon \to 0$. Indeed, changing variables as $\omega = B^{-1}(x)z$ we get
$$\begin{aligned}\lim_{\varepsilon\to 0} H^1_\varepsilon(x) &= \frac{C}{2}\det B^{-1}(x)\int_{\mathbb{R}^N} J\left(B^{-1}(x)z\right) z_i z_j dz \\ &= \frac{C}{2}\sum_{k,m} b_{ik}(x)b_{jm}(x)\int_{\mathbb{R}^N} J(w)w_k w_m dw.\end{aligned}$$

Taking into account that
$$\int_{\mathbb{R}^N} J(w)w_k w_m dw = 0$$
if $k \neq m$ and the value of the constant C, we get that
$$\lim_{\varepsilon\to 0} H^1_\varepsilon(x) = \sum_k b_{ik}(x)b_{jk}(x) = \sum_k b_{ik}(x)b^t_{kj}(x) = a_{ij}(x).$$

Now, we claim that

$$H^2_\varepsilon(x) \to 2\sum_j \frac{\partial a_{ij}(x)}{\partial x_j}$$

as $\varepsilon \to 0$. Indeed, since J is a radial function, it follows that

$$\int_{\mathbb{R}^N} J(B^{-1}(x)z)z_i dz = 0.$$

Therefore, $\lim_{\varepsilon\to 0} H^2_\varepsilon(x) = \frac{0}{0}$ and we can use L'Hopital rule to obtain

$$\lim_{\varepsilon\to 0} H^2_\varepsilon(x) = \lim_{\varepsilon\to 0} -C\int_{\mathbb{R}^N} \left(F^1_\varepsilon(x,z) + F^2_\varepsilon(x,z)\right) dz,$$

where

$$F^1_\varepsilon(x,z) = \frac{\partial}{\partial\varepsilon}\left(J(B^{-1}(x-\varepsilon z)z)\right)\det B^{-1}(x-\varepsilon z)z_i,$$

and

$$F^2_\varepsilon(x,z) = J(B^{-1}(x-\varepsilon z)z)\frac{\partial}{\partial\varepsilon}\left(\det B^{-1}(x-\varepsilon z)\right)z_i.$$

To compute the first part, we note that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\left(J(B^{-1}(x-\varepsilon z)z)\right) &= \sum_j\left\{J'_j(B^{-1}(x-\varepsilon z)z)\frac{\partial}{\partial\varepsilon}\sum_k b^{-1}_{jk}(x-\varepsilon z)z_k\right\}\\ &= \sum_{j,k,m} J'_j(B^{-1}(x-\varepsilon z)z)\left(b^{-1}_{jk}\right)'_m(x-\varepsilon z)z_k(-z_m).\end{aligned} \tag{3.10}$$

In this way we obtain

$$\lim_{\varepsilon\to 0} -C\int_{\mathbb{R}^N} F^1_\varepsilon(x,z)dz = C\det B^{-1}(x)\sum_{j,k,m}\left(b^{-1}_{jk}\right)'_m(x)\int_{\mathbb{R}^N} J'_j(B^{-1}(x)z)z_kz_mz_i dz. \tag{3.11}$$

Now, we change variables as $w = B^{-1}(x)z$ to obtain

$$C\sum_{j,k,m,p,q,r}\left(b^{-1}_{jk}\right)'_m(x)b_{kp}(x)b_{mq}(x)b_{ir}(x)\int_{\mathbb{R}^N} J'_j(w)w_pw_qw_r dw.$$

Using property *(*1) from Lemma 3.3.1 we get

$$\begin{aligned} &= -6\sum_{j,k,m}(b^{-1}_{jk})'_m(x)b_{kj}(x)b_{mj}(x)b_{ij}(x)\\ &\quad -2\sum_{j,k,m,q\neq j}(b^{-1}_{jk})'_m(x)b_{kj}(x)b_{mq}(x)b_{iq}(x)\\ &\quad -2\sum_{j,k,m,p\neq j}(b^{-1}_{jk})'_m(x)b_{kp}(x)b_{mj}(x)b_{ip}(x)\\ &\quad -2\sum_{j,k,m,p\neq j}(b^{-1}_{jk})'_m(x)b_{kp}(x)b_{mp}(x)b_{ij}(x)\\ &= -2\sum_{j,k,m,p}(b^{-1}_{jk})'_m(x)\left[b_{kj}(x)b_{mp}(x)b_{ip}(x)\right.\\ &\qquad +b_{kp}(x)b_{mj}(x)b_{ip}(x)\\ &\qquad \left.+b_{kp}(x)b_{mp}(x)b_{ij}(x)\right],\end{aligned} \tag{3.12}$$

which by property *(*1) from Lemma 3.3.2 turns out to be equal to

$$\begin{aligned} &2\sum_{j,k,m,p} b^{-1}_{jk}(x)(b_{kj})'_m(x)b_{mp}(x)b_{ip}(x)\\ &\quad +2\sum_{k,p}(b_{kp})'_k(x)b_{ip}(x) + 2\sum_{m,p}(b_{ip})'_m(x)b_{mp}(x)\\ &= 2\sum_{j,k,m,p} b^{-1}_{jk}(x)(b_{kj})'_m(x)b_{mp}(x)b_{ip}(x) + 2\sum_j\frac{\partial a_{ij}(x)}{\partial x_j},\end{aligned} \tag{3.13}$$

where in the last equality we have used that $a_{ij}(x) = \sum_p b_{ip}(x)b_{jp}(x)$ and we have replaced the indexes k and p by j.

To conclude the claim, we have to compute the second part and to verify that it is cancelled with the first term of the last part of (3.13). To be more specific, we need to show that

$$\lim_{\varepsilon\to 0} C\int_{\mathbb{R}^N} F_\varepsilon^2(x,z)dz = 2\sum_{j,k,m,p} b_{jk}^{-1}(x)(b_{kj})'_m(x)b_{mp}(x)b_{ip}(x).$$

In fact, by virtue of

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\left(\det B^{-1}(x-\varepsilon z)\right) &= \sum_m \left(\det B^{-1}(x-\varepsilon z)\right)'_m(-z_m)\\ &= \sum_m \det B^{-2}(x-\varepsilon z)\left(\det B(x-\varepsilon z)\right)'_m z_m,\end{aligned} \tag{3.14}$$

we have that

$$\lim_{\varepsilon\to 0} C\int_{\mathbb{R}^N} F_\varepsilon^2(x,z)dz = C\det B^{-2}(x)\sum_m\left(\det B(x)\right)'_m\int_{\mathbb{R}^N} J\left(B^{-1}(x)z\right)z_m z_i dz$$

changing variables again $w = B^{-1}(x)z$

$$= 2\det B^{-1}(x)\sum_{m,p}\left(\det B(x)\right)'_m b_{mp}(x)b_{ip}(x)$$

and finally, using property *(*3) from Lemma 3.3.2 we get

$$= 2\sum_{j,k,m,p} b_{jk}^{-1}(x)(b_{kj})'_m(x)b_{mp}(x)b_{ip}(x)$$

and the proof is finished. □

Proposition 3.3.4 *Let u be a $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N\times[0,T]\right)$ function and let $\mathcal{L}_\varepsilon^2$, Γ be the operators defined as*

$$\mathcal{L}_\varepsilon^2(u(x,t)) = \frac{C}{\varepsilon^2}\left[\int_{\mathbb{R}^N} J_\varepsilon\left(B^{-1}(y)(x-y)\right)\det B^{-1}(y)dy - 1\right]u(x,t),$$

and

$$\Gamma(u(x,t)) = \sum_{i,j}\frac{\partial^2 a_{ij}(x)}{\partial x_i\partial x_j}u(x,t),$$

Then,

$$\sup_{t\in[0,T]}\|\left(\mathcal{L}_\varepsilon^2-\Gamma\right)(u(x,t))\|_{L^\infty}\to 0 \ \ \textit{as} \ \ \varepsilon\to 0.$$

Proof: Under the change variables $y = x-\varepsilon z$ we obtain

$$\mathcal{L}_\varepsilon^2(u(x,t)) = \frac{C}{\varepsilon^2}\left[\int_{\mathbb{R}^N} J\left(B^{-1}(x-\varepsilon z)z\right)\det B^{-1}(x-\varepsilon z)dz - 1\right]u(x,t). \tag{3.15}$$

Note that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} J\left(B^{-1}(x-\varepsilon z)z\right)\det B^{-1}(x-\varepsilon z)dz = \int_{\mathbb{R}^N} J\left(B^{-1}(x)z\right)\det B^{-1}(x)dz = 1.$$

Therefore, using L'Hopital rule in (3.15) we get

$$\lim_{\varepsilon\to 0}\mathcal{L}_\varepsilon^2(u(x,t)) = \frac{C}{2\varepsilon}\int_{\mathbb{R}^N}\left(G_\varepsilon^1(x,z)+G_\varepsilon^2(x,z)\right)dz\,u(x,t), \tag{3.16}$$

where

$$G_\varepsilon^1(x,z) = \frac{\partial}{\partial\varepsilon}\left(J(B^{-1}(x-\varepsilon z)z)\right)\det B^{-1}(x-\varepsilon z),$$

and

$$G_\varepsilon^2(x,z) = J(B^{-1}(x-\varepsilon z)z)\frac{\partial}{\partial\varepsilon}\left(\det B^{-1}(x-\varepsilon z)\right).$$

Now, the proof splits naturally into two parts:
Part 1: To compute

$$\lim_{\varepsilon\to 0}\frac{C}{2\varepsilon}\int_{\mathbb{R}^N} G_\varepsilon^1(x,z)dz.$$

Using equality (3.10), it is equivalent to compute

$$\lim_{\varepsilon\to 0}\frac{C}{2\varepsilon}\sum_{j,k,m}\int_{\mathbb{R}^N} J_j'(B^{-1}(x-\varepsilon z)z)\,(b_{jk}^{-1})_m'(x-\varepsilon z)\,z_k(-z_m)\det B^{-1}(x-\varepsilon z)dz.$$

Taking into account that

$$\int_{\mathbb{R}^N} J_j'(w)w_q w_p dw = 0,$$

a simple computation gives that the above expression is $\frac{0}{0}$ and we can use L'Hopital rule again, to obtain

$$\lim_{\varepsilon\to 0}\frac{C}{2\varepsilon}\int_{\mathbb{R}^N} G_\varepsilon^1(x,z)dz = \lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N}\left(A_\varepsilon^1(x,z)+A_\varepsilon^2(x,z)+A_\varepsilon^3(x,z)\right)dz, \tag{3.17}$$

where

$$A_\varepsilon^1(x,z) = \frac{\partial}{\partial\varepsilon}\left[J_j'(B^{-1}(x-\varepsilon z)z)\right](b_{jk}^{-1})_m'(x-\varepsilon z)\det B^{-1}(x-\varepsilon z)z_k(-z_m)dz,$$

$$A_\varepsilon^2(x,z) = J_j'(B^{-1}(x-\varepsilon z)z)\,\frac{\partial}{\partial\varepsilon}\left[(b_{jk}^{-1})_m'(x-\varepsilon z)\right]\det B^{-1}(x-\varepsilon z)z_k(-z_m)dz$$

and

$$A_\varepsilon^3(x,z) = J_j'(B^{-1}(x-\varepsilon z)z)\,(b_{jk}^{-1})_m'(x-\varepsilon z)\,\frac{\partial}{\partial\varepsilon}\left[\det B^{-1}(x-\varepsilon z)\right]z_k(-z_m)dz.$$

Therefore, the Part 1 will be splitter again into three steps:
Part 1.a: Compute

$$\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N} A_\varepsilon^1(x,z)dz.$$

By an argument similar to (3.10), we have

$$\frac{\partial}{\partial\varepsilon}\left[J_j'(B^{-1}(x-\varepsilon z)z)\right] = \sum_{p,q,r} J_{jp}''(B^{-1}(x-\varepsilon z)z)\,(b_{pq}^{-1})_r'(x-\varepsilon z)\,z_q(-z_r),$$

thus

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N} A_\varepsilon^1(x,z)dz\\ &=\frac{C}{2}\sum_{j,k,m,p,q,r}(b_{pq}^{-1})_r'(x)(b_{jk}^{-1})_m'(x)\det B^{-1}(x)\int_{\mathbb{R}^N} J_{jp}''(B^{-1}(x)z)\,z_q z_r z_k z_m dz.\end{aligned}$$

Now we change variables as $w = B^{-1}(x)z$ to obtain

$$\begin{aligned}\frac{C}{2}\sum_{j,k,m,p,q,r,l,n,s,t}&(b_{pq}^{-1})_r'(x)(b_{jk}^{-1})_m'(x)b_{ql}(x)b_{rn}(x)b_{ks}(x)b_{mt}(x)\\ &\times\int_{\mathbb{R}^N} J_{jp}''(w)w_l w_n w_s w_t dw.\end{aligned}$$

Finally, by properties *(*2) and *(*3) from Lemma 3.3.1, proceeding with similar arguments applied in (3.12) with easy modifications, we obtain that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N} A_\varepsilon^1(x,z)dz = \\
&\sum_{j,k,m,p,q,r,s}(b_{pq}^{-1})'_r(x)(b_{jk}^{-1})'_m(x)\left[b_{qj}(x)b_{rp}(x)b_{ks}(x)b_{ms}(x)\right.\\
&\qquad +b_{qj}(x)b_{rs}(x)b_{kp}(x)b_{ms}(x)+b_{qj}(x)b_{rs}(x)b_{ks}(x)b_{mp}(x)\\
&\qquad +b_{qp}(x)b_{rj}(x)b_{ks}(x)b_{ms}(x)+b_{qs}(x)b_{rj}(x)b_{kp}(x)b_{ms}(x)\\
&\qquad +b_{qs}(x)b_{rj}(x)b_{ks}(x)b_{mp}(x)+b_{qp}(x)b_{rs}(x)b_{kj}(x)b_{ms}(x)\\
&\qquad +b_{qs}(x)b_{rp}(x)b_{kj}(x)b_{ms}(x)+b_{qs}(x)b_{rs}(x)b_{kj}(x)b_{mp}(x)\\
&\qquad +b_{qp}(x)b_{rs}(x)b_{ks}(x)b_{mj}(x)+b_{qs}(x)b_{rp}(x)b_{ks}(x)b_{mj}(x)\\
&\qquad \left.+b_{qs}(x)b_{rs}(x)b_{kp}(x)b_{mj}(x)\right].
\end{aligned}\tag{3.18}$$

Part 1.b: Compute

$$\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N} A_\varepsilon^2(x,z)dz.$$

Since

$$\frac{\partial}{\partial\varepsilon}\left[(b_{jk}^{-1})'_m(x-\varepsilon z)\right]=\sum_p (b_{jk}^{-1})''_{mp}(x-\varepsilon z)(-z_p),$$

it follows, letting $\varepsilon\to 0$ and changing variables $w=B^{-1}(x)z$, that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N} A_\varepsilon^2(x,z)dz &= \frac{C}{2}\sum_{j,k,m,p,q,r,s}(b_{jk}^{-1})''_{mp}(x)b_{pq}(x)b_{kr}(x)b_{ms}(x)\\
&\qquad\times\int_{\mathbb{R}^N} J_j'(w)w_qw_rw_s dw
\end{aligned}$$

which due to property *(*1) from Lemma 3.3.1 and arguing as in (3.12) is equal to

$$-\sum_{j,k,m,p,q}(b_{jk}^{-1})''_{mp}(x)\left[b_{pq}(x)b_{kq}(x)b_{mj}(x)+b_{pj}(x)b_{kq}(x)b_{mq}(x)+b_{pq}(x)b_{kj}(x)b_{mq}(x)\right].$$

Thus, using *(*2) from Lemma 3.3.2, we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N} A_\varepsilon^2(x,z)dz = \sum_{j,k,m,p,q}\left\{(b_{jk}^{-1})'_m(x)(b_{kq})'_p(x)b_{pq}(x)b_{mj}(x)\right.\\
&+(b_{jk}^{-1})'_p(x)(b_{kq})'_m(x)b_{pq}(x)b_{mj}(x)+(b_{kq})''_{mp}(x)b_{mj}(x)b_{jk}^{-1}(x)b_{pq}(x)\\
&+(b_{jk}^{-1})'_m(x)(b_{kq})'_p(x)b_{pj}(x)b_{mq}(x)+(b_{jk}^{-1})'_p(x)(b_{kq})'_m(x)b_{pj}(x)b_{mq}(x)\\
&+(b_{kq})''_{mp}(x)b_{pj}(x)b_{jk}^{-1}(x)b_{mq}(x)+(b_{jk}^{-1})'_m(x)(b_{kj})'_p(x)b_{pq}(x)b_{mq}(x)\\
&+(b_{jk}^{-1})'_p(x)(b_{kj})'_m(x)b_{pq}(x)b_{mq}(x)+(b_{kj})''_{mp}(x)b_{pq}(x)b_{jk}^{-1}(x)b_{mq}(x).
\end{aligned}\tag{3.19}$$

Note that, thanks to *(*1) from Lemma 3.3.2, some terms from expressions (3.18) and (3.19) cancel. In fact, the 12^{th} term of (3.18) verifies

$$\begin{aligned}
&\sum_{j,k,m,p,q,r,s}(b_{pq}^{-1})'_r(x)(b_{jk}^{-1})'_m(x)b_{qs}(x)b_{rs}(x)b_{kp}(x)b_{mj}(x)\\
&\qquad = -\sum_{j,k,m,p,q,r,s}b_{pq}^{-1}(x)(b_{jk}^{-1})'_m(x)(b_{qs})'_r(x)b_{rs}(x)b_{kp}(x)b_{mj}(x)
\end{aligned}$$

and since

$$\sum_p b_{kp}(x) b^{-1}_{pq}(x) = 1$$

if $k = q$ and vanishes if $k \neq q$ we obtain

$$-\sum_{j,k,m,r,s} (b^{-1}_{jk})'_m(x)(b_{ks})'_r(x) b_{rs}(x) b_{mj}(x).$$

Replacing s by q and r by p, this last expression is cancelled by the 1^{st} term of (3.19). We leave it to the reader to verify that, in the same way, the 2^{nd}, 4^{th}, 5^{th} and 7^{th} terms of expression (3.19) are cancelled by the 5^{th}, 3^{rd}, 1^{st} and 2^{nd} terms of expression (3.18) respectively. Hence, from Part 1.b only the 3^{rd}, 6^{th}, 8^{th} and 9^{th} terms remain.

Part 1.c: Compute

$$\lim_{\varepsilon\to 0} \frac{C}{2} \sum_{j,k,m} \int_{\mathbb{R}^N} A^3_\varepsilon(x,z)dz.$$

By equality (3.14) we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{C}{2} \sum_{j,k,m} \int_{\mathbb{R}^N} A^3_\varepsilon(x,z)dz \\ &= -\frac{C}{2} \det B^{-2}(x) \sum_{j,k,m,p} (b^{-1}_{jk})'_m(x)(\det B(x))'_p \int_{\mathbb{R}^N} J'_j(B^{-1}(x)z) z_k z_m z_p dz. \end{aligned} \tag{3.20}$$

Furthermore, thanks to the result obtained from equality (3.11) in (3.13), inside the proof of Proposition 3.3.3 we get

$$\begin{aligned} &C \det B^{-1}(x) \textstyle\sum_{j,k,m} (b^{-1}_{jk})'_m(x) \int_{\mathbb{R}^N} J'_j(B^{-1}(x)z) z_k z_m z_p dz \\ &\quad = 2 \sum_{j,k,m,s} b^{-1}_{jk}(x)(b_{kj})'_m(x) b_{ms}(x) b_{ps}(x) \\ &\quad\quad + 2\sum_{k,j} (b_{jk})'_j(x) b_{pk}(x) + 2\sum_{k,j} (b_{pk})'_j(x) b_{jk}(x). \end{aligned} \tag{3.21}$$

In addition, we have Golberg (1972), that is,

$$\det B^{-1}(x)(\det B(x))'_p = \sum_{q,r} b^{-1}_{qr}(x)(b_{rq})'_p(x). \tag{3.22}$$

Replacing (3.21) and (3.22) in equality (3.20), we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{C}{2} \sum_{j,k,m} \int_{\mathbb{R}^N} A^3_\varepsilon(x,z)dz \\ &= -\sum_{j,k,m,p,q,r,s} b^{-1}_{qr}(x)(b_{rq})'_p(x) b^{-1}_{jk}(x)(b_{kj})'_m(x) b_{ms}(x) b_{ps}(x) \\ &\quad - \sum_{j,k,m,p,q,r} b^{-1}_{qr}(x)(b_{rq})'_p(x) \left\{ (b_{jk})'_j(x) b_{pk}(x) + (b_{pk})'_j(x) b_{jk}(x) \right\}. \end{aligned} \tag{3.23}$$

Note that above expression is cancelled with the 7^{th}, 10^{th} and 4^{th} terms of expression (3.18).

Summarizing, we conclude Part 1 of the proof as follows:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{C}{2\varepsilon}\int_{\mathbb{R}^N} G^1_\varepsilon(x,z)dz = \lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N}\big(A^1_\varepsilon(x,z)+A^2_\varepsilon(x,z)+A^3_\varepsilon(x,z)\big)\,dz\\
&= \sum_{j,k,m,p,q,r,s}(b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x)\,[b_{qs}(x)b_{rj}(x)b_{ks}(x)b_{mp}(x)\\
&\quad + b_{qs}(x)b_{rp}(x)b_{kj}(x)b_{ms}(x) + b_{qs}(x)b_{rs}(x)b_{kj}(x)b_{mp}(x)\\
&\quad + b_{qs}(x)b_{rp}(x)b_{ks}(x)b_{mj}(x)] \qquad\qquad (3.24)\\
&\quad + \sum_{j,k,m,p,q}\big\{(b_{kq})''_{mp}(x)b_{mj}(x)b^{-1}_{jk}(x)b_{pq}(x)\\
&\quad + (b_{kq})''_{mp}(x)b_{pj}(x)b^{-1}_{jk}(x)b_{mq}(x) + (b^{-1}_{jk})'_p(x)(b_{kj})'_m(x)b_{pq}(x)b_{mq}(x)\\
&\quad + (b_{kj})''_{mp}(x)b_{pq}(x)b^{-1}_{jk}(x)b_{mq}(x)\big\}\,.
\end{aligned}$$

Part 2: We have to compute

$$\lim_{\varepsilon\to 0}\frac{C}{2\varepsilon}\int_{\mathbb{R}^N} G^2_\varepsilon(x,z)dz.$$

Which due to relation (3.14), it is equivalent to compute

$$\lim_{\varepsilon\to 0}\frac{C}{2\varepsilon}\sum_p\int_{\mathbb{R}^N} J(B^{-1}(x-\varepsilon z)z)\,\frac{(\det B(x-\varepsilon z))'_p}{\det B^2(x-\varepsilon z)}\,z_p\,dz.$$

Note that since

$$\int_{\mathbb{R}^N} J(B^{-1}(x)z)\,z_p\,dz = 0,$$

letting $\varepsilon\to 0$, we have that the above expression is $\frac{0}{0}$. Consequently, by L'Hopital rule we obtain

$$\lim_{\varepsilon\to 0}\frac{C}{2\varepsilon}\int_{\mathbb{R}^N} G^2_\varepsilon(x,z)dz = \frac{C}{2}\sum_p\int_{\mathbb{R}^N}\big(R^1_\varepsilon(x,z)+R^2_\varepsilon(x,z)+R^3_\varepsilon(x,z)\big)\,dz,$$

where

$$R^1_\varepsilon(x,z) = \frac{\partial}{\partial\varepsilon}\big[J(B^{-1}(x-\varepsilon z)z)\big]\,\frac{(\det B(x-\varepsilon z))'_p}{\det B^2(x-\varepsilon z)}\,z_p,$$

$$R^2_\varepsilon(x,z) = J(B^{-1}(x-\varepsilon z)z)\frac{\partial}{\partial\varepsilon}\left[\frac{(\det B(x-\varepsilon z))'_p}{\det B(x-\varepsilon z)}\right]\det B^{-1}(x-\varepsilon z)z_p,$$

and

$$R^3_\varepsilon(x,z) = J(B^{-1}(x-\varepsilon z)z)\frac{(\det B(x-\varepsilon z))'_p}{\det B(x-\varepsilon z)}\frac{\partial}{\partial\varepsilon}\big[\det B^{-1}(x-\varepsilon z)\big]\,z_p.$$

Therefore, the Part 2 will be divided into three steps:

Part 2.a: Compute

$$\lim_{\varepsilon\to 0}\frac{C}{2}\sum_p\int_{\mathbb{R}^N} R^1_\varepsilon(x,z)dz.$$

By identity (3.10) and letting ε to 0, we get

$$= -\frac{C}{2}\det B^{-2}(x)\sum_{j,k,m,p}(b^{-1}_{jk})'_m(x)(\det B(x))'_p\int_{\mathbb{R}^N} J'_j(B^{-1}(x)z)z_kz_mz_pdz.$$

Which coincides with expression (3.20). Hence,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{C}{2}\sum_p\int_{\mathbb{R}^N} R^1_\varepsilon(x,z)dz\\
&= -\sum_{j,k,m,p,q,r,s} b^{-1}_{qr}(x)(b_{rq})'_p(x)b^{-1}_{jk}(x)(b_{kj})'_m(x)b_{ms}(x)b_{ps}(x) \qquad\qquad (3.25)\\
&- \sum_{j,k,m,p,q,r} b^{-1}_{qr}(x)(b_{rq})'_p(x)\,\big\{(b_{jk})'_j(x)b_{pk}(x)+(b_{pk})'_j(x)b_{jk}(x)\big\}\,.
\end{aligned}$$

Part 2.b: Compute

$$\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{p}\int_{\mathbb{R}^N} R^2_\varepsilon(x,z)dz.$$

If we compute de derivative of (3.22), we obtain that

$$\begin{aligned}&\frac{\partial}{\partial\varepsilon}\left[\frac{(\det B(x-\varepsilon z))'_p}{\det B(x-\varepsilon z)}\right]=\frac{\partial}{\partial\varepsilon}\sum_{j,k} b^{-1}_{jk}(x-\varepsilon z)(b_{kj})'_p(x-\varepsilon z)\\&=-\sum_{j,k,m}\left\{(b^{-1}_{jk})'_m(x-\varepsilon z)z_m(b_{kj})'_p(x-\varepsilon z)+b^{-1}_{jk}(x-\varepsilon z)(b_{kj})''_{pm}(x-\varepsilon z)z_m\right\}.\end{aligned}$$

Therefore, replacing the above expression, letting $\varepsilon\to 0$ and change variables as $w=B^{-1}(x)z$, Part 2.b reads as follows

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{p}\int_{\mathbb{R}^N} R^2_\varepsilon(x,z)dz\\&=-\sum_{j,k,m,p,q}\left\{(b^{-1}_{jk})'_m(x)(b_{kj})'_p(x)b_{pq}(x)b_{mq}(x)\right.\\&\qquad\left.+b^{-1}_{jk}(x)(b_{kj})''_{pm}(x)b_{pq}(x)b_{mq}(x)\right\}.\end{aligned}\tag{3.26}$$

Part 2.c: Compute

$$\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{p}\int_{\mathbb{R}^N} R^3_\varepsilon(x,z)dz.$$

Using again the equalities (3.22) and (3.14), letting $\varepsilon\to 0$ and change variables $w=B^{-1}(x)z$, we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{p}\int_{\mathbb{R}^N} R^3_\varepsilon(x,z)dz\\&=\sum_{j,k,m,p,q,r,s} b^{-1}_{jk}(x)(b_{kj})'_m b^{-1}_{qr}(x)(b_{rq})'_p(x)b_{ms}(x)b_{ps}(x).\end{aligned}\tag{3.27}$$

Note that this expression cancels with the first part of (3.25) from Part 2.a.

Summarizing, we conclude Part 2 of the proof as follows:

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{C}{2\varepsilon}\int_{\mathbb{R}^N} G^2_\varepsilon(x,z)dz=\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{p}\int_{\mathbb{R}^N}\left(R^1_\varepsilon(x,z)+R^2_\varepsilon(x,z)+R^3_\varepsilon(x,z)\right)dz\\&=-\sum_{j,k,m,p,q,r} b^{-1}_{qr}(x)(b_{rq})'_p(x)\left\{(b_{jk})'_j(x)b_{pk}(x)+(b_{pk})'_j(x)b_{jk}(x)\right\}\\&\quad-\sum_{j,k,m,p,q}\left\{(b^{-1}_{jk})'_m(x)(b_{kj})'_p(x)b_{pq}(x)b_{mq}(x)\right.\\&\qquad\left.+b^{-1}_{jk}(x)(b_{kj})''_{pm}(x)b_{pq}(x)b_{mq}(x)\right\}.\end{aligned}\tag{3.28}$$

Finally, taking into account that the first sum of (3.28) is cancelled with the 2^{nd} and 3^{rd} term of (3.24) and the second sum of (3.28) is cancelled with the last two terms of (3.24). We have, adding Part 1 and Part 2 in (3.16), that

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathcal{L}^2_\varepsilon(u(x,t))=\Bigg\{&\sum_{j,k,m,p,q,r,s}(b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x)b_{qs}(x)b_{rj}(x)b_{ks}(x)b_{mp}(x)\\&+\sum_{j,k,m,p,q,r,s}(b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x)b_{qs}(x)b_{rp}(x)b_{ks}(x)b_{mj}(x)\\&+\sum_{j,k,m,p,q}(b_{kq})''_{mp}(x)b_{mj}(x)b^{-1}_{jk}(x)b_{pq}(x)\\&+\sum_{j,k,m,p,q}(b_{kq})''_{mp}(x)b_{pj}(x)b^{-1}_{jk}(x)b_{mq}(x)\Bigg\}u(x,t).\end{aligned}$$

Now, applying property *(*1) from Lemma 3.3.2, each sum satisfies

$$\begin{aligned}\sum_{j,k,m,p,q,r,s} & (b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x)b_{qs}(x)b_{rj}(x)b_{ks}(x)b_{mp}(x) \\ &= \sum_{j,k,m,p,q,r,s} b^{-1}_{pq}(x)b^{-1}_{jk}(x)(b_{qs})'_r(x)b_{rj}(x)(b_{ks})'_m(x)b_{mp}(x) \\ &= \sum_{k,q,s}(b_{qs})'_k(x)(b_{ks})'_q(x) = \sum_{i,j,k}(b_{ik})'_j(x)(b_{jk})'_i(x),\end{aligned} \tag{3.29}$$

replacing, in the last equality, indexes $\{q,k,s\}$ by $\{i,j,k\}$ respectively. We have

$$\begin{aligned}\sum_{j,k,m,p,q,r,s} & (b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x)b_{qs}(x)b_{rp}(x)b_{ks}(x)b_{mj}(x) \\ &= \sum_{j,k,m,p,q,r,s} b^{-1}_{pq}(x)b^{-1}_{jk}(x)(b_{rp})'_r(x)b_{qs}(x)(b_{mj})'_m(x)b_{ks}(x) \\ &= \sum_{m,p,r}(b_{rp})'_r(x)(b_{mp})'_m(x) = \sum_{i,j,k}(b_{ik})'_i(x)(b_{jk})'_j(x),\end{aligned} \tag{3.30}$$

replacing, in the last equality, indexes $\{r,m,p\}$ by $\{i,j,k\}$ respectively.

Now,

$$\begin{aligned}\sum_{j,k,m,p,q}(b_{kq})''_{mp}(x)b_{mj}(x)b^{-1}_{jk}(x)b_{pq}(x) &= \sum_{k,p,q}(b_{kq})''_{kp}(x)b_{pq}(x) \\ &= \sum_{i,j,k}(b_{ik})''_{ij}(x)b_{jk}(x),\end{aligned} \tag{3.31}$$

replacing, in the last equality, indexes $\{k,p,q\}$ by $\{i,j,k\}$ respectively.

Also, we have

$$\begin{aligned}\sum_{j,k,m,p,q}(b_{kq})''_{mp}(x)b_{pj}(x)b^{-1}_{jk}(x)b_{mq}(x) &= \sum_{k,m,q}(b_{kq})''_{mk}(x)b_{mq}(x) \\ &= \sum_{i,j,k}(b_{jk})''_{ij}(x)b_{ik}(x),\end{aligned} \tag{3.32}$$

replacing, in the last equality, indexes $\{m,k,q\}$ by $\{i,j,k\}$ respectively.

Summarizing, from (3.29), (3.30), (3.31) and (3.32), we conclude that

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathcal{L}^2_\varepsilon(u(x,t)) &= \sum_{i,j,k}\left\{(b_{ik})'_j(x)(b_{jk})'_i(x) + (b_{ik})'_i(x)(b_{jk})'_j(x)\right. \\ &\left. +(b_{ik})''_{ij}(x)b_{jk}(x) + (b_{jk})''_{ij}(x)b_{ik}(x)\right\}u(x,t) = \sum_{i,j}\frac{\partial^2 a_{ij}(x)}{\partial x_i\partial x_j}u(x,t),\end{aligned}$$

and the Proposition gets proved. □

Proposition 3.3.5 *Let u be a $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N\times[0,T]\right)$ function and let $\mathcal{L}_\varepsilon$ be the operator defined as*

$$\mathcal{L}_\varepsilon(u(x,t)) = \frac{C}{\varepsilon^2}\left\{\int_{\mathbb{R}^N} J_\varepsilon\left(B^{-1}(y)(x-y)\right)\det B^{-1}(y)u(y,t)dy - u(x,t)\right\}. \tag{3.33}$$

Then,

$$\sup_{t\in[0,T]}\left\|\mathcal{L}_\varepsilon(u(x,t)) - \sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\left(a_{ij}(x)u(x,t)\right)\right\|_{L^\infty} \to 0 \qquad \text{as } \varepsilon\to 0.$$

Proof: Thanks to Propostion 3.3.3 and Proposition 3.3.4 we obtain that

$$\begin{aligned}&\sup_{t\in[0,T]}\left\|\mathcal{L}_\varepsilon(u(x,t)) - \sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\left(a_{ij}(x)u(x,t)\right)\right\|_{L^\infty} \\ &\quad\le \sup_{t\in[0,T]}\left\|\left(\mathcal{L}^1_\varepsilon-\Lambda\right)(u(x,t))\right\|_{L^\infty} + \sup_{t\in[0,T]}\left\|\left(\mathcal{L}^2_\varepsilon-\Gamma\right)(u(x,t))\right\|_{L^\infty}\to 0,\end{aligned}$$

as $\varepsilon \to 0$. □

We are now ready to prove our main result.

Proof:[Proof of Theorem 3.1.1] We will denote by $w^\varepsilon = u^\varepsilon - v$. Note that w^ε satisfies the following equation

$$\begin{cases} w_t^\varepsilon(x,t) = \mathcal{L}_\varepsilon(w^\varepsilon(x,t)) + \tilde{F}(x,t), & x \in \mathbb{R}^N, \ t \in [0,T], \\ w^\varepsilon(x,0) = 0, & x \in \mathbb{R}^N, \end{cases} \tag{3.34}$$

where

$$\tilde{F}(x,t) = \mathcal{L}_\varepsilon(v(x,t)) - \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(a_{ij}(x) v(x,t)\right).$$

In addition, thanks to Proposition 3.3.5, we can assert that there exists a positive function θ such that $|\tilde{F}(x,t)| \le \theta(\varepsilon) \to 0$ as $\varepsilon \to 0$, for every $x \in \mathbb{R}^N$, $t \in [0,T]$.

Next, let us consider

$$\eta(\varepsilon) = \max \left\{ \frac{C}{\varepsilon^2} \left[\int_{\mathbb{R}^N} J_\varepsilon \left(B^{-1}(y)(x-y)\right) \det B^{-1}(y) dy - 1 \right], x \in \mathbb{R}^N \right\},$$

it is easy to check that $\eta(\varepsilon) < \infty$, for every $\varepsilon > 0$. Futhermore, by Proposition 3.3.4 and (3.5) we obtain

$$\eta(\varepsilon) \to \max \left\{ \sum_{i,j} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j}, x \in \mathbb{R}^N \right\} < \infty.$$

In this way, we set the following function

$$\overline{w}(x,t) = \begin{cases} \dfrac{\theta(\varepsilon)}{\eta(\varepsilon)} \left(e^{\eta(\varepsilon)t} - 1\right) + \varepsilon\, e^{\eta(\varepsilon)t}, & \text{if } \eta(\varepsilon) \neq 0, \\ \theta(\varepsilon)t + \varepsilon, & \text{if } \eta(\varepsilon) = 0, \end{cases}$$

for $x \in \mathbb{R}^N$, $t \in [0,T]$. Now, we claim that $\overline{w}$ is a supersolution of (3.34). Indeed, for $\eta(\varepsilon) \neq 0$

$$\overline{w}_t(x,t) = \theta(\varepsilon) e^{\eta(\varepsilon)t} + \varepsilon\, \eta(\varepsilon)\, e^{\eta(\varepsilon)t} = \eta(\varepsilon)\, \overline{w}(x,t) + \theta(\varepsilon)$$

$$\ge \mathcal{L}_\varepsilon^2(\overline{w}(x,t)) + \tilde{F}(x,t) = \mathcal{L}_\varepsilon(\overline{w}(x,t)) + \tilde{F}(x,t),$$

taking into account that $\mathcal{L}_\varepsilon^1(\overline{w}(x,t)) = 0$ in the last equality. We left to the reader to check the case $\eta(\varepsilon) = 0$. Finally, as $\overline{w}(x,0) = \varepsilon$, the claim is proved.

Similar arguments applied to the case $\underline{w}(x,t) = -\overline{w}(x,t)$ leads us to assert that $\underline{w}(x,t)$ is a subsolution of problem (3.34).

We conclude from the comparison principle, Theorem 3.2.2, that

$$\underline{w} \le w^\varepsilon \le \overline{w}$$

and since $\overline{w}(x,t)$, $\underline{w}(x,t) \to 0$ as $\varepsilon \to 0$ our main result gets proved. □

Remark 3.3.6 *One can easily check that, for all test function $\varphi \in \mathcal{C}_c^2(\mathbb{R}^N)$ and $u \in L^1(\mathbb{R}^N) \cap C^{2+\alpha}(\mathbb{R}^N)$, it holds that*

$$\int_{\mathbb{R}^N} \mathcal{L}_\varepsilon u(x)\, \varphi(x) dx = \int_{\mathbb{R}^N} \sum_{j,k} \varphi''_{x_j x_k}(x) \left(B(x) B^t(x)\right)_{(j,k)} u(x)\, dx + 0(\varepsilon^\alpha) \tag{3.35}$$

and hence, integrating by parts twice, we get

$$\int_{\mathbb{R}^N} \mathcal{L}_\varepsilon u(x)\, \varphi(x) dx = \int_{\mathbb{R}^N} \sum_{j,k} \frac{\partial^2}{\partial x_j x_k} (a_{jk}(x) u(x))\, \varphi(x) dx + 0(\varepsilon^\alpha).$$

In fact, for $\varphi \in \mathcal{C}_c^2(\mathbb{R}^N)$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \mathcal{L}_\varepsilon u(x)\,\varphi(x)dx \\
&= \int_{\mathbb{R}^N} \frac{C}{\varepsilon^2}\left\{\int J_\varepsilon(B^{-1}(y)(x-y)) \det B^{-1}(y)u(y)\varphi(x)dy - u(x)\varphi(x)\right\}dx \\
&= \int_{\mathbb{R}^N} \frac{C}{\varepsilon^2}\left(\int J_\varepsilon(B^{-1}(y)(x-y)) \det B^{-1}(y)\varphi(x)dx\right)u(y)dy - \frac{C}{\varepsilon^2}\int_{\mathbb{R}^N} u(x)\varphi(x)dx \\
&= \int_{\mathbb{R}^N} u(y)\frac{C}{\varepsilon^2}\left\{\int J_\varepsilon(B^{-1}(y)y - z)\,\varphi(B(y)z)\,dz - \varphi(y)\right\}dy \\
&= \int_{\mathbb{R}^N} u(y)\frac{C}{\varepsilon^2}\left\{\int J_\varepsilon(B^{-1}(y)y - z)\,\phi(z)\,dz - \phi(B^{-1}(y)y)\right\}dy,
\end{aligned}
$$

with $\phi(z) := \varphi(B(y)z)$. Now we observe that it is well known (see Chasseigne et al. (2006)) that this last expression verifies

$$= \int_{\mathbb{R}^N} u(y)\Delta\phi(B^{-1}(y)y)\,dy + O(\varepsilon^\alpha).$$

Using that

$$\Delta\phi(B^{-1}(x)x) = \sum_{j,k} \varphi''_{x_j x_k}(x)\,(B(x)B^t(x))_{(j,k)}$$

we obtain (3.35).

Remark 3.3.7 *Our results can be interpreted from a stochastic processes viewpoint. In fact, given the stochastic differential equation*

$$d\,\mathbf{X}_t = B(\mathbf{X}_t)\,d\,\mathbf{W}_t,$$

where $\mathbf{X}_t$ is an N-dimensional random variable vector and $\mathbf{W}_t$ is an N-dimensional standard Wiener process. Our main result states that

Solutions of the rescaled nonlocal problem (3.6), $u^\varepsilon(x,t)$, converge uniformly to the probability density, $u(x,t)$, that corresponds to the process $\mathbf{X}_t$.

See Risken (1984) for more details.

Chapter 4

Parabolic equations with natural growth approximated by nonlocal equations

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Abstract

In this paper we study several aspects related with solutions of nonlocal problems whose prototype is

$$\begin{cases} u_t = \displaystyle\int_{\mathbb{R}^N} J(x-y)\big(u(y,t)-u(x,t)\big)\mathcal{G}\big(u(y,t)-u(x,t)\big)dy & \text{in}\,\Omega \quad \times (0,T)\,, \\ u(x,0) = u_0(x) & \text{in} \quad , \end{cases} \qquad \Omega$$

where we take, as the most important instance, $\mathcal{G}(s) \sim 1 + \frac{\mu}{2}\frac{s}{1+\mu^2 s^2}$ with $\mu \in \mathbb{R}$ as well as $u_0 \in L^1(\Omega)$, J is a smooth symmetric function with compact support and Ω is either a bounded smooth subset of $\mathbb{R}^N$, with nonlocal Dirichlet boundary condition, or $\mathbb{R}^N$ itself.

The results deal with existence, uniqueness, comparison principle and asymptotic behavior. Moreover we prove that if the kernel rescales in a suitable way, the unique solution of the above problem converges to a solution of the deterministic Kardar-Parisi-Zhang equation.

4.1 Introduction

This work is concerned with the study the existence, uniqueness, comparison principle and asymptotic behavior for the following nonlinear parabolic equation with nonlocal diffusion,

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)\big(u(y,t)-u(x,t)\big)\mathcal{G}\big(u(y,t)-u(x,t)\big)dy & \text{in}\Omega \quad \times (0,T)\,, \\ u(x,0) = u_0(x) & \text{in} \quad , \end{cases} \tag{4.1}$$

for an appropriate functions J and $\mathcal{G}$ (see below (J) and ($\mathcal{G}$)), and its relationship with the local problem

$$\begin{cases} u_t - \Delta u = \mu|\nabla u|^2 & \text{in}\Omega \quad \times (0,T)\,, \\ u(x,0) = u_0(x) & \text{in} \quad , \quad \Omega \end{cases} \tag{4.2}$$

where

1. Ω is either $\mathbb{R}^N$ itself (Cauchy problem) or a bounded smooth subset of $\mathbb{R}^N$ adding the boundary condition $u(x,t) = h(x,t)$ on $\partial\Omega \times (0,T)$ for h sufficiently smooth (Dirichlet problem);
2. $T > 0$ (possibly infinite) and $\mu \in \mathbb{R}$;
3. u_0 is a smooth enough datum.

4.1.1 Local problem

The equation $u_t - \Delta u = \mu|\nabla u|^2$, at least for $\mu > 0$, is known in the literature as the deterministic Kardar-Parisi-Zhang (KPZ) equation. It was proposed in Kardar et al. (1986) in the physical theory of growth and roughening of surfaces. Further developments on physical applications of the KPZ equation can be found in Barabási and Stanley (1995) (for a survey on more recent aspects we refer to Wio et al. (2011)).

The Kardar–Parisi–Zhang equation has given rise to a rich mathematical theory which has had a spectacular recent progress (see Corwin (2012); Hairer (2013)). From the point of view of Partial Differential Equations, equations having a gradient term with the so-called natural growth have been largely studied in the last decades by many mathematicians: in addition to the classical reference Ladyzenskaja et al. (1968) let us just mention the pioneer paper by Aronson and Serrin Aronson and Serrin (1967) and also the result due to Boccardo, Murat and Puel Boccardo et al. (1989).

4.1.2 Nonlocal problem

Nonlocal evolution equations have been extensively studied to model diffusion processes. The prototype example in this framework is the following one

$$u_t(x,t) = \int_{\mathbb{R}^N} K(x,y)(u(y,t)-u(x,t))dy, \tag{4.3}$$

where the kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative smooth function (not necessarily symmetric) satisfying $\displaystyle\int_{\mathbb{R}^N} K(x,y)dx = 1$ for any $y \in \mathbb{R}^N$ (or variations of it, see for instance Andreu et al. (2008/09)). If $u(y,t)$ is thought of as a density at location y at time t and $K(x,y)$ as the probability distribution of jumping from place y to place x, then the rate at which individuals from any other location go to the place x is given by $\int_{\mathbb{R}^N} K(x,y)u(y,t)dy$. On the other hand, the rate at which individuals leave the location x to travel to all other places is $-\int_{\mathbb{R}^N} K(y,x)u(x,t)dy = -u(x,t)$. In the absence of external sources this implies that the density must satisfy equation (4.3).

We are especially interested in symmetric kernels (we denote them by J) that have compact support; it means that the individuals can jump from a place to other, but they cannot go “too far away". On the contrary, for instance, nonlocal operators that allow "long jumps"correspond to a different choice of kernels. It is the case of the fractional laplacian that involves a kernel that is singular and that does not have compact support (see, for instance Valdinoci (2009) for a survey on this latter class of processes).

In particular, we consider $J : \mathbb{R}^N \to \mathbb{R}$ as a nonnegative radial symmetric function such that

$$J \in \mathcal{C}_c(\mathbb{R}^n), \qquad \int_{\mathbb{R}^N} J(z)\, dz = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} J(z) z_N^2 dz < \infty, \qquad z = (z_1, \ldots, z_N).$$

With this choice of the kernel, equation (4.3) changes into a diffusion equation of convolution type, namely

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - u(x,t), \qquad \text{in}\Omega \times (0,T) \tag{4.4}$$

(see for instance Bates et al. (1997); Chasseigne et al. (2006); Cortazar et al. (2007)).

4.1.3 Background

One of the most important features of nonlocal equations is that can be rescaled to approximate local ones.

In Cortázar et al. (2009) (see also Molino and Rossi (2016) for the same type of result in a more general case) it has been proved that, under an appropriate rescaling kernel, solutions of (4.4) converge uniformly to solutions of heat equation. To be more specific, solutions of

$$u_t^\varepsilon(x,t) = \frac{C}{\varepsilon^2}\left[\int_{\mathbb{R}^N} J_\varepsilon(x-y)u(y,t)dy - u(x,t)\right] \qquad \text{in}\Omega \times (0,T) \tag{4.5}$$

converge uniformly to solutions of

$$v_t = \Delta v \qquad \text{in}\Omega \times (0,T)\,,$$

where $C^{-1} = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^N} J(z) z_N^2\, dz$ and $J_\varepsilon(s) = \dfrac{1}{\varepsilon^N} J(\dfrac{s}{\varepsilon})$.

Let us mention that results in this direction, with the presence of a gradient term of convection type can be found, for instance, in Ignat and Rossi (2007): in such a case the equation is the sum of two terms, one corresponding to the diffusion one, the other to the convection term.

In general, we consider nonlocal problems of the type

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)\big(u(y,t) - u(x,t)\big)\, \mathcal{G}\big(u(y,t) - u(x,t)\big)\, dy, \tag{4.6}$$

where $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ is a suitable continuous function. For instance, if $\mathcal{G} \equiv 1$, then we recover problem (4.4). Let us mention the case $\mathcal{G}(s) = |s|^{p-2}$, with $p \geq 2$ has been treated in Andreu et al. (2008/09) where it is proved that solutions to the rescaled nonlocal problem converge to solutions of the Dirichilet problem for the p–Laplacian evolution equation.

On the contrary, the kind of kernels $\mathcal{G}$ we consider does not have the same structure of the previous ones, since they are bounded and do not satisfy any symmetry assumptions (neither odd nor even).

With this background, it is not surprising that problem (4.2) can be approximated by nonlocal equations. The question is to identify what kind of nonlocal equation approximates, under rescaling, problem (4.2). At first glance, one could think that a good approximation for (4.2) might be a nonlocal equation such as

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)\big(u(y,t) - u(x,t)\big)\, dy + \mu \int_{\mathbb{R}^N} J(x-y)|u(y,t) - u(x,t)|^2\, dy,$$

that is, taking $\mathcal{G}(s) = 1 + \mu s$ in (4.6). We explicitly point out that this is an unbounded function that satisfies $\mathcal{G}(0) = 1$ and $\mathcal{G}'(0) = \mu$ (compare with condition ($\mathcal{G}$) below). Anyway, for our approach the lack of boundedness of $\mathcal{G}$ leads to an obstacle for proving the existence of a solution to (4.6) via a fixed point argument. By the other hand, we recall that one of the main tools to deal with problem (4.2) is the so–called Hopf–Cole change of unknown which is defined by $w(x,t) = e^{\mu u(x,t)}$. This transforms every classical solution to (4.2) into a classical solution to problem

$$\begin{cases} w_t(x,t) = \Delta w(x,t) & \text{in} \quad \Omega\,(0,T)\,, \\ w(x,0) = e^{\mu u_0(x)} & \text{in}\Omega \ \ , \end{cases}$$

for a smooth enough datum u_0. However, the same kind of difficulty are found if one try to reproduce the Hopf–Cole transformation and try to approximate the solution of (4.2) by something of the form

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)\Big(e^{\mu u(y,t)} - e^{\mu u(x,t)}\Big)\, dy\,.$$

4.1.4 Main results

To conclude this introduction we want to state the most relevant results of our work. In order to not enter in technicalities, let us fix a family of kernels $\mathcal{G}_\mu$ that are the easiest (not trivial) example we can consider: for $\mu \in \mathbb{R}$ let

$$\mathcal{G}_\mu(s) = 1 + \frac{\mu s}{2(1+\mu^2 s^2)}\,, \qquad s \in \mathbb{R}\,, \quad \mu \in \mathbb{R},$$

and the corresponding family of nonlocal Dirichlet problems

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)\big(u(y,t)-u(x,t)\big)\,\mathcal{G}_\mu\big(u(y,t)-u(x,t)\big)\,dy & \text{in}\Omega \quad \times (0,T)\,, \\ u(x,0) = u_0(x) & \text{in} \qquad , \\ u(x,t) = h(x,t) & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T)\,. \end{cases} \tag{4.7}$$

with Ω a bounded domain and u_0 and h smooth enough.

After have proved the existence, uniqueness (see Theorem 4.2.3) and a Comparison Principle (see Theorem 4.2.5) for solutions of (4.7), we face the problem of *r*escaled kernels.

The result we prove, in this model case, reads like this.

Let u be the unique smooth solution to (4.2), with suitable initial data u_0 and boundary condition $u(x,t) = h(x,t)$ on $\partial\Omega \times (0,T)$. Then there exists a family of functions $\{u^\varepsilon\}$, $\varepsilon > 0$, such that u^ε solves the approximating nonlocal problem

$$\begin{cases} u_t^\varepsilon(x,t) = \dfrac{C}{\varepsilon^2}\displaystyle\int_{\Omega_{J_\varepsilon}} J_\varepsilon(x-y)\left[\big(u^\varepsilon(y,t)-u^\varepsilon(x,t)\big) + \frac{\mu}{2}\frac{\big(u^\varepsilon(y,t)-u^\varepsilon(x,t)\big)^2}{1+\mu^2\big(u^\varepsilon(y,t)-u^\varepsilon(x,t)\big)^2}\right]dy, \\ \hfill \text{in}\Omega \quad \times (0,T), \\ u^\varepsilon(x,0) = u_0(x), \hfill \text{in}\Omega \quad , \\ u^\varepsilon(x,t) = h(x,t), \hfill \text{in} \quad (\Omega_{J_\varepsilon}\backslash\Omega) \times (0,T), \end{cases}$$

with C a suitable constant, $\Omega_{J_\varepsilon} = \Omega + \operatorname{supp} J_\varepsilon$ and the family $\{u^\varepsilon\}$ satisfies

$$\lim_{\varepsilon\to 0} \quad \sup_{t\in[0,T]} \Big\| u^\varepsilon(x,t) - u(x,t) \Big\|_{L^\infty(\Omega)} = 0\,.$$

The same kind of results (i.e. existence, uniqueness and convergence for a suitable rescaled kernel to a solution of a local problem) are also proved for the corresponding Cauchy problem associated (i.e., $\Omega = \mathbb{R}^N$).

In addition, we deal with the asymptotic behavior of the solutions of problem (4.1). Concretely, we have two kind of results: if Ω is a bounded domain of $\mathbb{R}^N$, we prove that the solutions of (4.7) converge uniformly to the stationary one. On the other hand, if $\Omega = \mathbb{R}^N$, we prove that the L^2-norm of the solution has a suitable decay in time, depending on the nature (absorption or reaction) of the kernel (see for more details Theorems 4.2.16 and 4.2.17, respectively).

Plan of the paper

Section 2 is devoted to show the precise statements of the main results. Preliminaries are contained in Section 3. Section 4 deals with the Dirichlet problem in a bounded domain, while the results concerning the Cauchy problem can be found in Section 5.

4.2 Statement of the results

This section is devoted to the statement of the main results we prove in the present paper.

Let us consider the following equation:

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y;x,t)\,\mathcal{G}(x,u(y;x,t))\,dy, \tag{4.8}$$

where $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative radial symmetric function such that

$$J \in \mathcal{C}_c(\mathbb{R}^n), \quad \int_{\mathbb{R}^N} J(z)\,dz = 1 \quad \text{and} \quad C(J) := \int_{\mathbb{R}^N} J(z)z_N^2 dz, \quad z = (z_1, z_2, \dots, z_N) \tag{J}$$

and where, here and throughout the paper, we denote $u(y;x,t) := u(y,t) - u(x,t)$.

As far as the function $\mathcal{G}$ is concerned, we assume that $\mathcal{G} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a nonnegative Carathéodory function (namely, $\mathcal{G}(\cdot, s)$ is measurable for every $s \in \mathbb{R}$ and $\mathcal{G}(x,\cdot)$ is continuous for almost every $x \in \mathbb{R}^N$) satisfying

$$\exists\ \alpha_2 \geq \alpha_1 > 0 \,:\, \alpha_1 \leq \frac{\mathcal{G}(x,s)s - \mathcal{G}(x,\sigma)\sigma}{s-\sigma} \leq \alpha_2, \ \forall s,\sigma \in \mathbb{R} \ \ s \neq \sigma\,, \text{ and for a.e. } x \in \mathbb{R}^N \tag{$\mathcal{G}$}$$

Let us first point out that the above condition implies that $\mathcal{G}$ is a positive bounded function, since taking $\sigma = 0$ in ($\mathcal{G}$), we get

$$0 < \alpha_1 \leq \mathcal{G}(x,s) \leq \alpha_2, \qquad \text{for any } s \in \mathbb{R} \text{ and for a.e. } x \in \mathbb{R}^N.$$

Moreove observe that the above condition relies to be a sort of uniform ellipticity for the operator, while ($\mathcal{G}$) corresponds to a strong monotonicity.

Further remarks about the condition on $\mathcal{G}$ are addressed to Section 3.

Anyway, let us stress again that, in contrast with all the known results about nonlocal equation of the above type, in our case we do not require any symmetry (neither odd nor even) assumption to $\mathcal{G}$.

The prototype of $\mathcal{G}$ we have in mind (we will come back on this example later) is the following one:

$$\mathcal{G}_\mu(x,s) = 1 + \frac{\mu(x)\,s}{2(1+\mu(x)^2 s^2)}, \quad x \in \Omega, \quad s \in \mathbb{R}\,,$$

where $\mu : \Omega \to \mathbb{R}$ stands for a measurable function.

4.2.1 Dirichlet problem

The first kind of results we want to prove deals with the existence and uniqueness of solutions of a nonlocal Dirichlet boundary value problem. More precisely, consider the following problem in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$.

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y;x,t)\,\mathcal{G}(x,u(y;x,t))\,dy, & \text{in}\,\Omega \times (0,T) \\ u(x,t) = h(x,t), & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in}\,\Omega\,, \end{cases}$$

with $h \in L^1\left((\mathbb{R}^N \setminus \Omega) \times (0,\infty)\right)$ and $u_0 \in L^1(\Omega)$.

Let us first observe that the integral expression vanishes outside of $\Omega_J = \Omega + \operatorname{supp}(J)$. In this way, h is only needed to be fixed, in fact, in $\Omega_J \setminus \Omega$ and we can rewrite the above problem as

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\Omega_J} J(x-y)u(y;x,t)\,\mathcal{G}(x,u(y;x,t))\,dy, & \text{in}\,\Omega \times (0,T), \\ u(x,t) = h(x,t), & \text{in } (\Omega_J \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in}\,\Omega\,, \end{cases} \tag{P}$$

where $T > 0$ may be finite or $+\infty$.

Due to the aim of the paper, we give now two definitions of solution.

Definition 4.2.1 *Assume that J and $\mathcal{G}$ satisfy (J) and ($\mathcal{G}$), respectively.*
For $h(x,t) \in L^1((\Omega_J \setminus \Omega) \times (0,T))$ and $u_0(x) \in L^1(\Omega)$, we define a weak solution of problem (P) a function $u \in \mathcal{C}([0,T); L^1(\Omega))$ such that:

$$u(x,t) = \int_0^t \int_{\Omega_J} J(x-y)u(y;x,\tau)\mathcal{G}(x,u(y;x,\tau))\,dy\,d\tau + u_0(x), \quad \text{for a.e. } x \in \Omega,\ t \in (0,T), \tag{4.9}$$

$$u(y,t) = h(y,t), \quad \text{for a.e. } y \in \Omega_J \setminus \Omega \text{ and } t \in (0,T)$$

$$\lim_{t \to 0^+} \|u(x,t) - u_0(x)\|_{L^1(\Omega)} = 0\,.$$

Moreover, if $h(x,t) \in \mathcal{C}((\Omega_J \setminus \overline{\Omega}) \times (0,T))$ and $u_0(x) \in \mathcal{C}(\overline{\Omega})$, we define a regular solution of problem (P) as a function $u \in \mathcal{C}([0,\infty); \mathcal{C}(\overline{\Omega}))$ such that:

$$u(x,t) = \int_0^t \int_{\Omega_J} J(x-y)u(y;x,\tau)\mathcal{G}(x,u(y;x,\tau))\,dy\,d\tau + u_0(x), \quad \text{for any } x \in \overline{\Omega},\ t \in (0,T),$$

$$u(y,t) = h(y,t) \qquad \text{for any } y \in \Omega_J \setminus \overline{\Omega} \text{ and } t \in (0,T)$$

$$\lim_{t \to 0^+} \|u(x,t) - u_0(x)\|_{\mathcal{C}(\overline{\Omega})} = 0\,.$$

Some more remarks about the meaning of weak and regular solutions are in order to be given.

Remark 4.2.2

i) Observe that, in addition to the different smoothness of the boundary condition and/or the initial datum, the main difference lies on the prescription of data on $\partial\Omega$. Indeed, for weak solutions, h is prescribed in $(\Omega_J \setminus \Omega) \times (0,T)$ and u_0 in Ω, while for regular solutions, h is prescribed in $(\Omega_J \setminus \overline{\Omega}) \times (0,T)$ and u_0 in $\overline{\Omega}$.

ii) As already noticed in Chasseigne et al. (2006) (in a different context) the boundary conditions cannot be meant in a classical way, i.e. it is not true that the solutions of problem (P) pointwise coincide with the prescribed boundary data $h(x,t)$. This is due to the fact that the value at any *point $(x,t) \in \partial\Omega \times (0,T)$ depends both on the values of u inside $\overline{\Omega} \times [0,T]$ and on the boundary datum $h(x,t)$, since*

$$u(x,t) = \int_0^t \int_{\Omega \cap suppJ} J(x-y)u(y;x,\tau)\ \mathcal{G}\big(x, u(y,\tau) - u(x,\tau)\big)\, dy\, d\tau$$
$$+ \int_0^t \int_{\Omega^c \cap suppJ} J(x-y)\big(h(y,\tau) - u(x,\tau)\big)\ \mathcal{G}\big(x, h(y,\tau) - u(x,\tau)\big)\, dy\, d\tau + u_0(x)\,.$$

Consequently, in contrast with the local case, the equation is solved up to the boundary, depending, near $\partial\Omega$, also of the prescribed boundary condition.

iii) Let us stress that the regularity required in the definition of weak solutions is the less restrictive in order to give sense to the formulation and to the boundary and initial conditions. Anyway from (4.9) we deduce that the time derivative $u_t(x,t)$ of u also belongs to $\mathcal{C}((0,\infty); L^1(\Omega))$.

Let us also point out that the weak solutions framework is the more natural one in order to prove the existence of a solution. Indeed we only require an L^1 regularity to prove the existence of a solution.

Finally we want to underline that the nonlocal operator involved in such equation does not have the regularizing effect that is typical of the Laplacian, but leave unchanged the regularity of the initial and boundary data.

Our existence result is the following.

Theorem 4.2.3 *[Existence] Consider problem (P) and suppose that (J) and ($\mathcal{G}$) are in force. Then:*

i) For any $u_0 \in L^1(\Omega)$ and $h \in L^1((\Omega_J \setminus \Omega) \times (0,T))$ there exists a unique weak solution;

ii) For any $u_0 \in \mathcal{C}(\overline{\Omega})$ and $h \in \mathcal{C}((\Omega_J \setminus \overline{\Omega}) \times [0,T))$ there exists a unique regular solution and moreover its time derivative belongs to $\mathcal{C}(\overline{\Omega} \times (0,T))$.

Once we have deduced the existence of a solution, one important tool is to compare two solutions, or, more generally a sub and a supersolution. Here we recall what we mean by those concepts in our setting.

Definition 4.2.4 *A function $u \in \mathcal{C}(\overline{\Omega} \times [0,T])$ is a regular subsolution to problem (P) if it satisfies $u_t \in \mathcal{C}(\overline{\Omega} \times (0,T))$ and*

$$\begin{cases} u_t(x,t) \le \displaystyle\int_{\Omega_J} J(x-y)u(y;x,t)\,\mathcal{G}(x, u(y;x,t))\, dy, & in\ \overline{\Omega} \times (0,T), \\ u(x,t) \le h(x,t), & in\ (\Omega_J \setminus \overline{\Omega}) \times (0,T), \\ u(x,0) \le u_0(x), & in\ \overline{\Omega}, \end{cases} \tag{4.10}$$

with $u_0(x) \in \mathcal{C}(\overline{\Omega})$ and $h(x,t) \in \mathcal{C}((\Omega_J \setminus \overline{\Omega}) \times (0,T))$.
As usual, a regular supersolution is defined analogously by replacing "$\le$" with "$\ge$". Clearly, a regular solution is both a regular subsolution and a regular supersolution.

Next, we state our comparison principle.

Theorem 4.2.5 *[Comparison Principle] Let u an v be a regular subsolution and a regular supersolution of problem (P), respectively, with boundary data $h_1(x,t)$ and $h_2(x,t)$ and initial data $u_0(x)$ and $v_0(x)$, respectively. If $h_1(x,t) \le h_2(x,t)$ in $\Omega_J \setminus \overline{\Omega}$ and $u_0(x) \le v_0(x)$ in $\overline{\Omega}$, then $u \le v$ in $\overline{\Omega} \times [0,T]$.*

Remark 4.2.6 *The existence, uniqueness and comparison principle are also true relaxing the hypotheses on the kernel $J(x-y)$ by considering a more general one of the form $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^+$ with compact support in $\Omega \times B(0,\rho)$, with $\rho > 0$ such that*

$$0 < \sup_{y \in B(0,\rho)} K(x,y) = R(x) \in L^\infty(\Omega).$$

The next result we want to prove relates solutions of local and nonlocal equations. In order to do it, let us fix a Hölder continuous function $\mu : \overline{\Omega} \to \mathbb{R}$ with exponent $\alpha \in (0,1)$, and consider

$$\mathcal{G}_\mu(x,s) = 1 + \frac{\mu(x)\, s}{2(1+\mu(x)^2 s^2)}, \qquad (x,s) \in \overline{\Omega} \times \mathbb{R}. \tag{4.11}$$

The local problem we are interested in is the following

$$\begin{cases} v_t(x,t) = \Delta v(x,t) + \mu(x)|\nabla v(x,t)|^2 & \text{in}\Omega \ \times (0,T), \\ v(x,t) = h_0(x,t) & \text{on } \partial\Omega \times (0,T), \\ v(x,0) = v_0(x) & \text{in} \quad . \qquad\qquad \Omega \end{cases} \tag{4.12}$$

Observe that if for the same $0 < \alpha < 1$ we have that $\partial\Omega \in \mathcal{C}^{2+\alpha}$, $v_0 \in \mathcal{C}^{1+\alpha}(\overline{\Omega})$, $h \in \mathcal{C}^{1+\alpha,1+\alpha/2}(\partial\Omega \times [0,T])$ with v_0 and h compatible (namely, they are globally a $C^{1+\alpha,1+\alpha/2}$ function of the parabolic boundary of the cylinder) and the equation holds up to the boundary, then Theorem 6.1 of Chapter V in Ladyzenskaja et al. (1968) provides a solution $v \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times (0,T])$.
Such a result becomes trivial if we assume $\mu(x) = \mu \in \mathbb{R}$, after the Hopf–Cole transformation, since solutions of the heat equation satisfy the required regularity.

We set here the definition of classical solution.

Definition 4.2.7 *We say that $v \in \mathcal{C}(\overline{\Omega} \times [0,T]) \cap \mathcal{C}^{2+\alpha,1+\alpha/2}(\Omega \times (0,T))$ is a* classical solution *for the Dirichlet problem (4.12) if it satisfies both the equations and the boundary and initial conditions in a pointwise sense.*

Consider now, for any $\varepsilon > 0$ the rescaling nonlocal problem

$$\begin{cases} u_t^\varepsilon(x,t) = \dfrac{C(x)}{\varepsilon^2} \displaystyle\int_{\Omega_{J_\varepsilon}} J_\varepsilon(x-y) u^\varepsilon(y;x,t)\, \mathcal{G}_\mu(x, u^\varepsilon(y;x,t))dy & \text{in } \overline{\Omega} \times (0,T), \\ u^\varepsilon(x,t) = h(x,t) & \text{in } (\ \ _{J_\varepsilon} \setminus \overline{\Omega}) \times (0,T), \\ u^\varepsilon(x,0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{4.13}$$ Ω

where $\mathcal{G}_\mu$ defined in (4.11) and $C(x)$, u_0 and h are suitable measurable functions.

Here we state our converging result.

Theorem 4.2.8 *Let Ω be a $\mathcal{C}^{2+\alpha}$, with $\alpha \in (0,1)$, bounded domain of $\mathbb{R}^N$, $N \geq 1$, and let v be a classical solution of the quasilinear problem (4.12) with $h \in \mathcal{C}^{1+\alpha}(\Omega_{J_\varepsilon} \backslash \Omega \times (0,T])$ such that $h\big|_{\partial\Omega \times (0,T)} = h_0(x,t)$ and $v_0 \in \mathcal{C}^{1+\alpha}(\overline{\Omega})$. Assume that J satisfies (J) and that for a.e. x in Ω, $\mathcal{G}(x,s)$ is a $\mathcal{C}^{1+\alpha}$ function with respect to the s variable such that that ($\mathcal{G}$) holds true. For any $\varepsilon > 0$, let u^ε denote the solution to*

$$\begin{cases} u_t^\varepsilon(x,t) = \dfrac{C(x)}{\varepsilon^2} \displaystyle\int_{\Omega_{J_\varepsilon}} J_\varepsilon(x-y) u^\varepsilon(y;x,t)\, \mathcal{G}(x, u^\varepsilon(y;x,t))dy & \text{in } \overline{\Omega} \times (0,T), \\ u^\varepsilon(x,t) = h(x,t) & \text{in } (\Omega_{J_\varepsilon} \setminus \overline{\Omega}) \times (0,T), \\ u^\varepsilon(x,0) = v_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{4.14}$$

with $C(x)^{-1} = \frac{1}{2}C(J)\mathcal{G}(x,0)$ *and* $\mu(x) = \dfrac{2\mathcal{G}'_s(x,0)}{\mathcal{G}(x,0)}$ *for any a.e.* $x \in \Omega$*. Then we have*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \left\| u^\varepsilon(x,t) - v(x,t) \right\|_{L^\infty(\Omega)} = 0\,.$$

Let us stress that the same kind of result (as well as the existence, uniqueness and Comparison Principle one) can be proved in a more general framework. First of all, we might consider the same equation adding on the right hand side a (smooth enough) function. On the other hand, a more general kernel, that depends also on y could be considered (see Remark 4.4.3 for some more details). We decided to skip these generalizations in order to keep the paper more readable.

The last type of results of this section deals with the asymptotic behavior of the solutions to (P_K). More precisely we prove, as it is usual for parabolic equations, that a solution of problem (P_K) converges, for large times, to a stationary solution of the same problem.

In order to avoid technicalities, we assume that the lateral condition is homogeneous, i.e. $h(x,t) \equiv 0$.

Here we state our result that asserts such a convergence, even if, under some additional hypotheses, we provide results on the rate of convergence (see Remark 4.4.6 for more details).

Theorem 4.2.9 *For every* $0 \le u_0 \in \mathcal{C}_0(\overline{\Omega})$*, the regular solution to problem*

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\Omega_J} J(x-y)u(y;x,t)\ \mathcal{G}(x,u(y;x,t))\,dy & \textit{in } \overline{\Omega}\times(0,+\infty), \\ u(x,t) = 0, & \textit{in } \Omega_J\setminus\overline{\Omega}\times(0,+\infty),\ t>0, \\ u(x,0) = u_0(x) & \textit{in } \overline{\Omega}, \end{cases} \tag{4.15}$$

satisfies

$$\lim_{t\to\infty} u(x,t) = 0 \qquad \textit{uniformly in} \quad \overline{\Omega}\,.$$

Remark 4.2.10 *We want to stress that the hypothesis* $u_0 \ge 0$ *is not, in fact, necessary, but we assume it just to let the proof easier.*

Let us just point out that we have two special cases whose asymptotic behavior is well known in the local setting. If we assume that

$$\exists\beta>0 : \mathcal{G}(x,s)s \le \beta s, \qquad \forall s\in\mathbb{R},\ \text{for a.e. } x\in\mathbb{R}^N\,, \tag{4.16}$$

it corresponds to the *a*bsorption case, i.e. the case in which we have (at least) the same decay estimates as if $\mathcal{G}\equiv 1$. In fact we can deduce (see Remark 4.4.6) that in the absorption case the rate of convergence at 0 is of exponential type. On the other hand, if

$$\exists\beta>0 : \mathcal{G}(x,s)s \ge \beta s, \qquad \forall s\in\mathbb{R},\ \text{for a.e. } x\in\mathbb{R}^N. \tag{4.17}$$

the result is more surprising since it correspond to the *r*eaction case. In this framework it is crucial to deal with smooth solutions, since we exploit, in the proof, the comparison principle.

4.2.2 Cauchy problem

This section deals with the Cauchy Problem related to (4.8), that is

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y;x,t)\ \mathcal{G}(x,u(y;x,t))\,dy & \text{in } \mathbb{R}^N\times(0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N\,, \end{cases} \tag{C}$$

with $\mathcal{G}$ as in ($\mathcal{G}$), J as in (J) and $u_0 \in \mathcal{C}(\mathbb{R}^N)$. First let us give the notion of solution.

Definition 4.2.11 *Given* $u_0 \in \mathcal{C}(\mathbb{R}^N)$ *we define a solution of problem* (C) *as a function* $u \in \mathcal{C}\left([0,T);\mathcal{C}(\mathbb{R}^N)\right)$ *such that it satisfies*

$$u(x,t) = \int_0^t \int_{\mathbb{R}^N} J(x-y)u(y;x,\tau)\mathcal{G}(x,u(y;x,\tau))dyd\tau + u_0(x) \qquad \text{in } \mathbb{R}^N \times (0,T).$$

Consequently, due to the integral expression above, $u \in \mathcal{C}^1\left((0,T);\mathcal{C}(\mathbb{R}^N)\right)$.

The first result we want to present in this framework deals with the existence of a bounded solution.

Theorem 4.2.12 *[Existence] For every continuous and bounded initial data* u_0 *there exists a unique solution* $u \in \mathcal{C}\left([0,T);\mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\right)$ *of problem* (C).

We continue this section proving the comparison principle for our problem. For this purpose, we first set the notion of sub and supersolution.

Definition 4.2.13 *A function* $u \in \mathcal{C}^0\left([0,T),\mathcal{C}(\mathbb{R}^N)\right) \cap \mathcal{C}^1\left((0,T),\mathcal{C}(\mathbb{R}^N)\right)$ *is a subsolution of problem* (C) *if it satisfies*

$$\begin{cases} u_t(x,t) \leq \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y;x,t)\mathcal{G}(x,u(y;x,t))dy, & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) \leq u_0(x), & \text{in } \mathbb{R}^N. \end{cases}$$

As usual, a supersolution is defined analogously by replacing " $\leq$ *" by "* $\geq$ *".*

Next we state the comparison principle in this framework.

Theorem 4.2.14 *[Comparison Principle] Let* u, v *be a subsolution and supersolution respectively of problem* (C) *with initial data* $u_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $v_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$*, respectively, such that* $u_0 \leq v_0$ *in* $\mathbb{R}^N$*. Then* $u \leq v$ *in* $\mathbb{R}^N \times (0,T)$.

Now, we prove that given a classical solution (i.e., $v \in \mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N \times [0,T]\right)$) of the parabolic problem with a quadratic gradient term of the form

$$\begin{cases} v_t(x,t) = \Delta v(x,t) + \mu(x)|\nabla v(x,t)|^2 & \text{in } \mathbb{R}^N \times (0,T) \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{4.18}$$

with $v_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\mu(x) \in \mathcal{C}^\alpha\left(\mathbb{R}^N\right) \cap L^\infty(\mathbb{R}^N)$, it can be approximated by a solution of the nonlocal problem

$$\begin{cases} u_t^\varepsilon = \dfrac{C(x)}{\varepsilon^2} \displaystyle\int_{\mathbb{R}^N} J_\varepsilon(x-y)u^\varepsilon(y;x,t)\,\mathcal{G}(x,u^\varepsilon(y;x,t))dy & \text{in } \mathbb{R}^N \times (0,T), \\ u^\varepsilon(x,0) = v_0(x), & \text{in } \mathbb{R}^N, \end{cases} \tag{4.19}$$

such that $\dfrac{2\mathcal{G}'_s(x,0)}{\mathcal{G}(x,0)} = \mu(x)$. As usual $C(x)^{-1} = \frac{1}{2}C(J)\mathcal{G}(x,0) \neq 0$ and $J_\varepsilon(s) = \frac{1}{\varepsilon^N}J(\frac{s}{\varepsilon})$.

Theorem 4.2.15 *Let* v *be a classical solution of quasilinear differential equation* (4.18)*. Let, for a given* $\varepsilon > 0$, u^ε *be the solution to* (4.19)*, with the same initial datum* $v_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$*. Then, we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \left\| u^\varepsilon(\cdot,t) - v(\cdot,t) \right\|_{L^\infty(\mathbb{R}^N)} = 0\,.$$

Finally, we study the asymptotic behavior of the solutions associated to the Cauchy problem. Our result depends on the nature of $\mathcal{G}$, i.e. if it is of absorption or reaction type.

Summarizing, we obtain the following results:

Theorem 4.2.16 *For $N \geq 1$, let u be a solution of Cauchy problem (C) satisfying (4.16) and positive initial datum $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$. Then there exists a positive constant $C = C(J, N, \beta, q)$ such that*

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^N)} \leq C\|u_0\|_{L^1(\mathbb{R}^N)} t^{-\frac{N}{2}\left(1-\frac{1}{q}\right)}, \ \textit{for any } q \in [1, \infty),$$

for t sufficiently large.

Theorem 4.2.17 *For $N \geq 1$, let u be a solution of of Cauchy problem (C) with $\mathcal{G} \equiv \mathcal{G}_\mu$, $0 \leq \mu \in L^\infty(\mathbb{R}^N)$ and positive initial datum $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ satisfying*

$$\|u_0\|_{L^\infty(\mathbb{R}^N)}\|\mu\|_{L^\infty(\mathbb{R}^N)} < 1\,. \tag{4.20}$$

Then,

$$\|u(\cdot, t)\|^2_{L^2(\mathbb{R}^N)} \leq \tilde{C}\|u_0\|_{L^1(\mathbb{R}^N)} t^{-\frac{N}{2}},$$

for some $\tilde{C} = \tilde{C}(\|\mu\|_{L^\infty(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, N, J) > 0$ and for t sufficiently large.

4.3 Preliminaries

Notation. Throughout this paper, we always use the following notation:
we denote in a short way $u(y; x, t) = u(y, t) - u(x, t)$. Moreover the time variable will always get values between 0 and T, with $T > 0$. As far as the kernel J is concerned, we assume that it is defined as in (J) and such that $\mathcal{G}$ satisfies ($\mathcal{G}$) and $C = 2C(J)^{-1}$, $J_\varepsilon(s) = \frac{1}{\varepsilon^N} J(\frac{s}{\varepsilon})$.

As far as the the function $\mathcal{G}(x, s)$ is concerned, we observe that, for a function $\mathcal{G}$ differentiable with respect to s we have, thanks to the Mean Value Theorem, that

$$\alpha_1 \leq \mathcal{G}'_s(x, s)s + \mathcal{G}(x, s) \leq \alpha_2, \qquad \text{for any } s \in \mathbb{R} \text{ a.e. in } x \in \mathbb{R}^N. \tag{4.21}$$

Moreover, if $\mathcal{G}$ is differentiable with respect to s, condition ($\mathcal{G}$) is equivalent to define $\psi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with

$$0 < \alpha_1 \leq \psi(x, s, \sigma) \leq \alpha_2 \qquad \text{for a.e. x } \in \Omega, \quad \forall s, \sigma \in \mathbb{R},$$

such that

$$\psi(x, s, \sigma) = \begin{cases} \dfrac{\mathcal{G}(x, s)s - \mathcal{G}(x, \sigma)\sigma}{s - \sigma} & \text{if} \quad s \neq \sigma\,, \\ \mathcal{G}'_s(x, s)s + \mathcal{G}(x, s) & \text{if} \quad s = \sigma. \end{cases} \tag{4.22}$$

We also remark that, in particular, condition ($\mathcal{G}$) implies $\mathcal{G}(x, 0) \neq 0$ for any $x \in \mathbb{R}^N$.

Here, we state the following technical result which allow us to see that the function defined in (4.11) satisfies the basic condition ($\mathcal{G}$).

Proposition 4.3.1 *Let p, q and k be real numbers, then the following properties hold true*

$$\frac{3}{4} \leq 1 + \frac{kp}{2(1 + k^2p^2)} \leq \frac{5}{4}\,,$$

$$p\left[1 + \frac{kp}{2(1 + k^2p^2)}\right] - q\left[1 + \frac{kq}{2(1 + k^2q^2)}\right] = (p - q)\left[1 + \frac{k(p + q)}{2(1 + k^2p^2)(1 + k^2q^2)}\right],$$

$$1-\frac{3\sqrt{3}}{16}\leq 1+\frac{k(p+q)}{2(1+k^2p^2)(1+k^2q^2)}\leq 1+\frac{3\sqrt{3}}{16}\,.$$

Moreover, for any measurable function $\mu:\mathbb{R}^N\to\mathbb{R}$, *the function defined by* $\mathcal{G}_\mu(x,s)=1+\dfrac{\mu(x)s}{2(1+\mu(x)^2s^2)}$ *satisfies the following conditions*

(i) $\left(1-\frac{3\sqrt{3}}{16}\right)(s-\sigma)\leq\mathcal{G}_\mu(x,s)s-\mathcal{G}_\mu(x,\sigma)\sigma\leq\left(1+\frac{3\sqrt{3}}{16}\right)(s-\sigma),\quad$ *for* $s>\sigma,\ x\in\mathbb{R}^N$*;*

(ii) if $\mu\geq 0$, *then* $\mathcal{G}_\mu(x,s)s\leq s$, *for any* $(x,s)\in\mathbb{R}^N\times\mathbb{R}$*;*

(iii) if $\mu\leq 0$ *then* $\mathcal{G}_\mu(x,s)s\geq s$, *for any* $(x,s)\in\mathbb{R}^N\times\mathbb{R}$.

Proof: The first two inequalities are straightforward while for the third one we just remark that the function given by

$$f(x,y)=\frac{|x|+|y|}{(1+x^2)(1+y^2)}$$

attains its maximum $\frac{3\sqrt{3}}{8}$ at the point $(\frac{1}{3},\frac{1}{3})$.
Now, (i) is a consequence of the previous inequalities. Conditions (ii) and (iii) follow by the fact that

$$\begin{cases}\frac{3}{4}\leq G_\mu(x,s)\leq 1, & \text{if } (x,s)\in\mathbb{R}^N\times[0,\infty),\\ 1\leq G_\mu(x,s)\leq\frac{5}{4}, & \text{if } (x,s)\in\mathbb{R}^N\times(-\infty,0],\end{cases}$$

for $\mu(x)\leq 0$, and

$$\begin{cases}1\leq G_\mu(x,s)\leq\frac{5}{4}, & \text{if } (x,s)\in\mathbb{R}^N\times[0,\infty),\\ \frac{3}{4}\leq G_\mu(x,s)\leq 1, & \text{if } (x,s)\in\mathbb{R}^N\times(-\infty,0],\end{cases}$$

for $\mu(x)\geq 0$. $\square$

Remark 4.3.2 *Let us stress that in the above result we only assume that* $\mu(x)$ *is measurable, without any hypotheses on its regularity.*

Lemma 4.3.3 *Let* $q\geq 1$, *there exists* $c(q)>0$ *such that*

$$(a-b)(a^{q-1}-b^{q-1})\geq c(q)\,(a^{\frac{q}{2}}-b^{\frac{q}{2}})^2,\qquad \text{for any } a,b\geq 0. \tag{4.23}$$

Proof: Without loss of generality we assume $a>b$. Therefore, (4.23) is equivalent to prove that the function

$$F(\theta)=\frac{(1-\theta)(1-\theta^{q-1})}{(1-\theta^{\frac{q}{2}})^2}\qquad \theta\in[0,1),$$

is bounded below by a $0<c(q)$, being $\theta=b/a$.

The result just follows by computing the derivative of F and noticing that it is decreasing. Hence the minimum of F is achieved at $\theta=1$, and $\lim_{\theta\to1^-}F(\theta)=4\dfrac{q-1}{q^2}$. $\square$

4.4 Proofs of results about Dirichlet Problem

We start by proving the existence result.

Proof:[Proof of Theorem 4.2.3] i) Fixed an arbitrary $T>0$, we set the Banach space $X_T=\mathcal{C}([0,T];L^1(\Omega))$ endowed with norm

$$|||v|||=\max_{0\leq t\leq T}e^{-Mt}\|v(\cdot,t)\|_{L^1(\Omega)}\,, \tag{4.24}$$

for some $M\geq\tilde{C}=\alpha_2\,\|J\|_{L^\infty(\Omega)}\,(|\Omega|+|\mathrm{supp}(J)|)$.

Let $\mathcal{T} : X_T \to X_T$ be the operator defined by

$$\mathcal{T}(v)(x,t) = \int_0^t \int_{\Omega_J} J(x-y)v(y;x,\tau)\mathcal{G}(x, v(y;x,\tau))\, dy\, d\tau + u_0(x),$$

with $v(y,t) = h(y,t)$ for $y \in \Omega_J \setminus \Omega$. Then, we prove the existence and uniqueness of solutions of (P) via the standard Banach contraction principle applied to the operator $\mathcal{T}$. In this way, using Fubini's Theorem and since $\mathcal{G}$ is bounded, we obtain

$$\begin{aligned} \|\mathcal{T}(v(\cdot,t))\|_{L^1(\Omega)} &\le \|u_0\|_{L^1(\Omega)} + \alpha_2 \int_0^t \int_\Omega \int_{\Omega_J} J(x-y)|v(y;x,\tau)|\, dy\, dx\, d\tau \\ \le \|u_0\|_{L^1(\Omega)} + \alpha_2 \int_0^t &\left(\int_\Omega \int_{\Omega_J} J(x-y)|v(y,\tau)|\, dy\, dx + \int_\Omega \int_{\Omega_J} J(x-y)|v(x,\tau)|\, dy\, dx \right) d\tau \\ &\le \|u_0\|_{L^1(\Omega)} + \frac{\tilde{C}_1}{M}\left(e^{Mt}-1\right) |||v||| + \tilde{C}_2\,, \end{aligned} \tag{4.25}$$

where $\tilde{C}_1 = \alpha_2\|J\|_{L^\infty(\mathbb{R}^N)}(|\Omega| + |\mathrm{supp}(J)|)$ and $\tilde{C}_2 = \alpha_2\|J\|_{L^\infty(\mathbb{R}^N)}|\Omega|\, \|h\|_{L^1((\Omega_J\setminus\Omega)\times(0,\infty))}$. Therefore

$$\begin{aligned} |||\mathcal{T}(v)||| &\le \max_{0\le t\le T}\left(e^{-Mt}(\|u_0\|_{L^1(\Omega)} + \tilde{C}_2) + \frac{\tilde{C}_1}{M}\left(1-e^{-Mt}\right)|||v|||\right) \\ &\le \|u_0\|_{L^1(\Omega)} + \tilde{C}_2 + \frac{\tilde{C}_1}{M}|||v|||\,. \end{aligned}$$

Hence, $\mathcal{T}$ maps X_T into itself. Note that all the involved constants do not depend on the value T.

Now, by virtue of ($\mathcal{G}$), we can assert that for every $w, z \in X_T$

$$\begin{aligned} \left|(\mathcal{T}(w)-\mathcal{T}(z))(x,t)\right| &\le \int_0^t \int_{\Omega_J} J(x-y)\left|w(y;x,\tau)\mathcal{G}(x,w(y;x,\tau)) - z(y;x,\tau)\mathcal{G}(x,z(y;x,\tau))\right| dy\, d\tau \\ &\le \alpha_2 \int_0^t \int_{\Omega_J} J(x-y)\, |w(y;x,\tau) - z(y;x,\tau)|\, dy\, d\tau. \end{aligned}$$

Therefore, arguing as in (4.25), we get

$$\|\mathcal{T}(w)-\mathcal{T}(z)\|_{L^1(\Omega)} \le \frac{\tilde{C}_1}{M}\left(e^{Mt}-1\right)|||w-z|||\,.$$

Thus, since $M > \tilde{C}$, we get

$$|||\mathcal{T}(w)-\mathcal{T}(z)||| \le \vartheta |||w-z|||\,,$$

with $0 < \vartheta < 1$. Hence $\mathcal{T}$ is a contraction and by the Banach's Fixed Point Theorem there exists a unique $u \in X_T$ such that $\mathcal{T}(u) = u$, i.e., consequently we get local existence and uniqueness of problem (P) for $0 \le t \le T$. Moreover, since this argument is independent of the value T, we obtain a unique solution $u \in \mathcal{C}([0,\infty); L^1(\Omega))$ of problem (P).

ii) For the second part it is sufficient to change the definition of $|||\cdot|||$ in (4.24), replacing $L^1(\Omega)$ with $\mathcal{C}(\overline{\Omega})$. The regularity of u_t easily follows by using the equation solved by u. □

Next we deal with the proof of the comparison principle.

Proof:[Proof of Theorem 4.2.5] We denote by $w = u - v$. Obviously $w \in \mathcal{C}(\overline{\Omega}\times[0,T])$, $w_t \in \mathcal{C}(\overline{\Omega}\times(0,T))$ and it satisfies

$$\begin{cases} w_t(x,t) \le \displaystyle\int_{\Omega_J} J(x-y)(w(y,t)-w(x,t))\psi(w(y;x,t))\, dy, & \text{in } \overline{\Omega}\times(0,T), \\ w(x,t) \le 0, & \text{in} \Omega_J \setminus \overline{\Omega}\times(0,T), \\ w(x,0) \le 0, & \text{in } \overline{\Omega}, \end{cases}$$

where ψ is the function defined in (4.22).

Assume by contradiction that $w(x,t)$ is positive at some point $(\tilde{x},\tilde{t})$ that, without loss of generality, we can assume that belongs to$\Omega \times (0,T]$. Thus, by the continuity of u and v, there exists a $\delta > 0$ such that $w(\tilde{x},\tilde{t}) - \delta\tilde{t} > 0$. Let us denote by (x_0,t_0) the maximum point of $w(x,t) - \delta t$ which is, by construction, positive. Consequently being u_t continuous in$\Omega \times (0,T)$, we have that

$$w_t(x_0,t_0) - \delta \geq 0\,.$$

On the other hand, plugging it into the equation in (4.10), we get

$$\begin{aligned}
w_t(x_0,t_0) \leq \int_{\Omega_J} J(x_0-y)\big(w(y,t_0)-w(x_0,t_0)\big)\psi\big(w(y,t_0)-w(x_0,t_0)\big)dy \\
= \int_{\Omega} J(x_0-y)\big((w(y,t_0)-\delta t_0)-(w(x_0,t_0)-\delta t_0)\big)\psi\big((w(y,t_0)-\delta t_0)-(w(x_0,t_0)-\delta t_0)\big)dy \\
+ \int_{\Omega_J\setminus\Omega} J(x_0-y)\big((w(y,t_0)-\delta t_0)-(w(x_0,t_0)-\delta t_0)\big)\psi\big((w(y,t_0)-\delta t_0)-(w(x_0,t_0)-\delta t_0)\big)dy\,,
\end{aligned}$$

and the last two integrals are nonpositive. Indeed as far as the first one is concerned, we observe that (x_0,t_0) is a maximum point, while ψ is positive; moreover outside Ω we use that the boundary condition is negative and that $w(x_0,t_0) - \delta t_0 > 0$, as well as the positivity of ψ. Hence we get a contradiction. □

Our goal is now to get a proof of Theorem 4.2.8. Here, we start with a preliminary Lemma.

Lemma 4.4.1 *Let $u \in \mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N \times [0,T]\right)$, $\mathcal{G}(x,s)$ a $\mathcal{C}^{1+\alpha}$ function with respect to variable s such that $\mathcal{G}(x,0) \neq 0$ for a.e. $x \in \Omega$, and let $\mathcal{L}_\varepsilon$ be the following operator*

$$\mathcal{L}_\varepsilon(u(x,t)) = \frac{C(x)}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon(x-y)\,u(y;x,t)\,\mathcal{G}(x,u(y;x,t))\,dy, \tag{4.26}$$

where $\frac{1}{C(x)} = \frac{1}{2}C(J)\ \mathcal{G}(x,0)$. Then, $\exists\, c = c(T) > 0$ such that, $\forall \varepsilon > 0$

$$\sup_{t\in[0,T]} \left\|\mathcal{L}_\varepsilon(u(x,t)) - \Delta u(x,t) - \mu(x)|\nabla u(x,t)|^2\right\|_{L^\infty(\Omega)} \leq c\,\varepsilon^\alpha$$

where $\mu(x) = \dfrac{2\mathcal{G}_s'(x,0)}{\mathcal{G}(x,0)}$.

Remark 4.4.2 *Observe that the integral expression above vanishes outside of $\Omega_{J_\varepsilon} = \Omega + \varepsilon\,\mathrm{supp}(J)$. In this way, h is only needed to be prescribed in $\Omega_{J_\varepsilon} \setminus \overline{\Omega}$. Observe also that, thanks to the hypothesis of Theorem 4.2.8 we use, in the following, that*

$$h(x,t) = h_0(x,t) + O(\varepsilon) \qquad in\ \ \Omega \setminus \Omega_{J_\varepsilon}\,.$$

Proof: In order to compute $\mathcal{L}_\varepsilon(u(x,t))$ we make the change of variables $y = x - \varepsilon z$, and we get

$$\mathcal{L}_\varepsilon(u(x,t)) = \frac{C(x)}{\varepsilon^2}\int_{\mathbb{R}^N} J(z)u(x-\varepsilon z;x,t)\,\mathcal{G}(x,u(x-\varepsilon z;x,t))dz. \tag{4.27}$$

Moreover by Taylor formula we have that $\mathcal{G}(x,\delta) = \mathcal{G}(x,0) + \mathcal{G}_s'(x,0)\delta + O(\delta^{1+\alpha})$, and

$$u(x-\varepsilon z;x,t) = -\varepsilon\sum_i \frac{\partial u(x,t)}{\partial x_i}z_i + \frac{\varepsilon^2}{2}\sum_{i,j}\frac{\partial^2 u(x,t)}{\partial x_i\partial x_j}z_iz_j + O\left(\varepsilon^{2+\alpha}\right),$$

Consequently

$$\mathcal{L}_\varepsilon(u(x,t)) = S_1(x,t) + S_2(x,t) + S_3(x,t) \tag{4.28}$$

being

$$S_1(x,t) = \frac{C(x)\mathcal{G}(x,0)}{\varepsilon^2}\int_{\mathbb{R}^N} J(z)u(x-\varepsilon z;x,t)\,dz,$$

$$S_2(x,t) = \frac{C(x)\mathcal{G}'_s(x,0)}{\varepsilon^2}\int_{\mathbb{R}^N} J(z)u(x-\varepsilon z;x,t)^2\,dz,$$

$$S_3(x,t) = \frac{C(x)}{\varepsilon^2}\int_{\mathbb{R}^N} J(z)u(x-\varepsilon z;x,t)^{2+\alpha}\,dz = O(\varepsilon^\alpha).$$

First, we deal with $S_1(x,t)$ and we obtain

$$\begin{aligned} S_1(x,t) &= -\frac{C}{\varepsilon}\sum_i \frac{\partial u(x,t)}{\partial x_i}\int_{\mathbb{R}^N} J(z)z_i dz + C(J)^{-1}\sum_{i,j}\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j}\int_{\mathbb{R}^N} J(z)z_i z_j + O\left(\varepsilon^\alpha\right) \\ &= \sum_{i,j}\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} + O\left(\varepsilon^\alpha\right), \end{aligned} \tag{4.29}$$

using in the last equality that J is radially symmetric, that is, $\int_{\mathbb{R}^N} J(z)z_i dz = 0$ and

$$\int_{\mathbb{R}^N} J(z)z_i z_j\,dz = 0 \qquad \text{if i} \neq \text{j} .$$

In order to compute $S_2(x,t)$, using the expansion of $u(x-\varepsilon z;x,t)$ up to the first order, we get

$$\begin{aligned} S_2(x,t) &= \frac{C(x)\mathcal{G}'_s(x,0)}{\varepsilon^2}\int_{\mathbb{R}^N} J(z)\left(-\varepsilon\sum_i \frac{\partial u(x,t)}{\partial x_i}z_i + O\left(\varepsilon^{1+\alpha}\right)\right)^2 dz \\ &= C(x)\mathcal{G}'_s(x,0)\sum_{i,j}\frac{\partial u(x,t)}{\partial x_i}\frac{\partial u(x,t)}{\partial x_j}\int_{\mathbb{R}^N} J(z)z_i z_j dz + O\left(\varepsilon^\alpha\right) \\ &= \frac{2\mathcal{G}'_s(x,0)}{\mathcal{G}(x,0)}\sum_i\left(\frac{\partial u(x,t)}{\partial x_i}\right)^2 + O\left(\varepsilon^\alpha\right), \end{aligned} \tag{4.30}$$

using again, in the last equality, that J is radially symmetric. Finally, setting $u(x-\varepsilon z;x,t) = O(\varepsilon)$, we obtain that $S_3(x,t) = O\left(\varepsilon^\alpha\right)$ and gathering together (4.28) with (4.29) and (4.30), we deduce that (4.27) becomes

$$\mathcal{L}_\varepsilon(u(x,t)) = \Delta\, u(x,t) + \mu(x)|\nabla u(x,t)|^2 + O\left(\varepsilon^\alpha\right)$$

concluding the proof. □

Remark 4.4.3 *Arguing as the in the proof of the above Lemma, we can state the following assertion: the operator defined as*

$$\tilde{\mathcal{L}}_\varepsilon(u(x,t)) = \frac{C(x)}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon(x-y)\,u(y;x,t)\,\mathcal{G}(y,u(y;x,t))\,dy,$$

converges uniformly in $[0,T]\times\overline{\Omega}$, *as* $\varepsilon \to 0$, *to the operator*

$$\Delta u(x,t) + \nabla_y \eta(x,0)\nabla u(x,t) + \eta'_s(x,0)|\nabla u(x,t)|^2,$$

being $\eta(x,s) = \log G(x,s)^2$. *Therefore, the role of the variables is not symmetric.*

Remark 4.4.4 *Let us recall that given* $\mu : \mathbb{R}^N \to \mathbb{R}$, *then* $\mathcal{G}_\mu(x,s)$ *defined in* (4.11) *satisfies* $\frac{2\mathcal{G}'_s(x,0)}{\mathcal{G}(x,0)} = \mu(x)$, *for any* $x \in \mathbb{R}^N$.

Now, we prove the main result of this section. That is, classical solutions of (4.12) can be approximated by solutions of problem (4.13) which in a general setting reads as follows,

Proof:[Proof of Theorem 4.2.8] Let $\tilde{v}$ be a $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N \times [0,T]\right)$ extension of v, the solution to (4.12). Denote by $h(x,t) = \tilde{v}(x,t)$ for any $(x,t) \in (\mathbb{R}^N \backslash \Omega) \times (0,T]$. Then h is smooth and $h(x,t) = h_0(x,t)$ if $x \in \partial\Omega$ and we get

$$h(x,t) = h_0(x,t) + O(\varepsilon), \quad \text{for } x \in \Omega_{J_\varepsilon} \setminus \Omega. \tag{4.31}$$

Observe that $\tilde{v}$ verifies

$$\begin{cases} \tilde{v}_t(x,t) = \Delta\tilde{v}(x,t) + \mu(x)|\nabla\tilde{v}(x,t)|^2 & \text{in}\Omega\ , \\ \tilde{v}(x,t) = h(x,t) & \text{in } (\ \ _{J_\varepsilon} \setminus \Omega) \times (0,T), \qquad \Omega \\ \tilde{v}(x,0) = v_0(x) & \text{in} \quad . \qquad\qquad\qquad \Omega \end{cases}$$

Theorem 4.2.3 asserts that, for any given $\varepsilon > 0$, there exists a unique u^ε which is solution to (4.14).

Set $w^\varepsilon := \tilde{v} - u^\varepsilon$, which satisfies

$$\begin{cases} w^\varepsilon_t(x,t) = \Delta\tilde{v}(x,t) + \mu(x)|\nabla\tilde{v}(x,t)|^2 - \mathcal{L}_\varepsilon(u^\varepsilon(x,t)) & \text{in}\Omega\ \times (0,T), \\ w^\varepsilon(x,t) = 0 & \text{in } (\ \Omega_{J_\varepsilon} \setminus \Omega) \times (0,T), \\ w^\varepsilon(x,0) = 0 & \text{in}\Omega\ . \end{cases} \tag{4.32}$$

By using condition (4.22), we set

$$\begin{aligned} \mathcal{M}_\varepsilon(w^\varepsilon(x,t)) &:= \mathcal{L}_\varepsilon(\tilde{v}(x,t)) - \mathcal{L}_\varepsilon(u^\varepsilon(x,t)) \\ &= \frac{C(x)}{\varepsilon^2}\int_{\Omega_{J_\varepsilon}} J_\varepsilon(x-y)\,\psi\left(x,\tilde{v}(y;x,t),u^\varepsilon(y;x,t)\right)\, w^\varepsilon(y;x,t)dy. \\ \Lambda_\varepsilon(\tilde{v}(x,t)) &:= \Delta\tilde{v}(x,t) + \mu(x)|\nabla\tilde{v}(x,t)|^2 - \mathcal{L}_\varepsilon(\tilde{v}(x,t)). \end{aligned}$$

In this way, we replace equation (4.32) by the following

$$\begin{cases} w^\varepsilon_t(x,t) = \Lambda_\varepsilon(\tilde{v}(x,t)) + \mathcal{M}_\varepsilon(w^\varepsilon(x,t)), & \text{in}\Omega\ \times (0,T), \\ w^\varepsilon(x,t) = 0\,, & \text{in } (\Omega_{J_\varepsilon} \setminus \Omega) \times (0,T), \\ w^\varepsilon(x,0) = 0, & \text{in}\Omega\ . \end{cases} \tag{4.33}$$

We begin by proving that for $K_1, K_2 > 0$ sufficiently large, $\overline{w}(x,t) = K_1\varepsilon^\alpha t + K_2\varepsilon$ is a supersolution of (4.33). Indeed, taking into account Lemma 4.4.1 and that $\mathcal{M}_\varepsilon(\overline{w}(x,t)) = 0$, we obtain

$$\overline{w}_t(x,t) = K_1\varepsilon^\alpha \geq \Lambda_\varepsilon(\tilde{v}(x,t)) = \Lambda\ _\varepsilon(\tilde{v}(x,t)) + \mathcal{M}_\varepsilon(\overline{w}(x,t)),$$

for $x \in \Omega$, $t \in (0,T]$. Moreover, $\overline{w}(x,0) > 0$ and by (4.31), we have that $\overline{w}(x,t) \geq K_2\varepsilon \geq O(\varepsilon)$, for $x \in \Omega_{J_\varepsilon} \setminus \Omega$ and $t \in (0,T]$. Consequently, $\overline{w}$ is a supersolution of (4.33).
Now, by the comparison principle stated in Theorem 4.2.5, we get

$$\tilde{v} - u^\varepsilon \leq K_1\varepsilon^\alpha t + K_2\varepsilon. \tag{4.34}$$

By the other hand, similar arguments applied to the case $\underline{w} = -\overline{w}$ leads us to assert that $\underline{w}$ is a subsolution of (4.33) and using again the comparison principle we obtain

$$\tilde{v} - u^\varepsilon \geq -K_1\varepsilon^\alpha t - K_2\varepsilon. \tag{4.35}$$

Hence, by virtue of (4.34) and (4.35)

$$\sup_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - v(\cdot,t)\|_{L^\infty(\Omega)} \leq K_1\varepsilon^\alpha T + K_2\varepsilon,$$

that vanishes as ε goes to 0. □

Here, we deal with the asymptotic behavior of the solution. In order to prove the main result (i.e. Theorem 4.2.9), we start with an intermediate result.

Theorem 4.4.5 *Given $\lambda \neq 0$, consider the problem*

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\Omega_J} J(x-y)\mathcal{G}(x,u(y;x,t))u(y;x,t)\,dy, & x \in \overline{\Omega},\ t>0, \\ u(x,t) = 0, & x \in \Omega_J \setminus \overline{\Omega},\ t>0. \\ u(x,0) = \lambda, & x \in \Omega. \end{cases} \tag{4.36}$$

Then the unique solution to problem (4.36) *satisfies*

$$\lim_{t\to\infty} u(\cdot,t) = 0\,, \quad \textit{uniformly in } \overline{\Omega}\,. \tag{4.37}$$

*Proof:*We assume that $\lambda > 0$, the other case may similarly be proved.

Let $u \in \mathcal{C}(\overline{\Omega}\times[0,\infty))$ be the unique solution to problem (4.36) with $\lambda > 0$. Since $v^1(x,t) = \lambda$ and $v^2(x,t) = 0$ define a supersolution and a subsolution, respectively, it follows from the Comparison Principle that

$$0 \le u(x,t) \le \lambda\,, \quad \text{for every in } \overline{\Omega}\times(0,+\infty)\,. \tag{4.38}$$

Moreover, fixed $\tau > 0$, the function $u^\tau(x,t) = u(x,t+\tau)$ defines a solution with initial datum $u_0^\tau(x) = u(x,\tau)$. Thus, the basic inequality (4.38) implies $0 \le u_0^\tau(x) \le \lambda$. Appealing again to the Comparison Principle, it yields

$$0 \le u(x,t+\tau) \le u(x,t)\,, \quad \text{for every in } \Omega \text{ and for any } \tau > 0\,.$$

Hence, we obtain that our solution is nonincreasing with respect to t. As a consequence, there exists

$$w(x) = \lim_{t\to\infty} u(x,t)\,, \quad \text{for any } x \in \overline{\Omega}\,.$$

We have to prove that $w(x) = 0$ for any $x \in \overline{\Omega}$. Observe that this limit function satisfies

$$w(x) = \int_0^\infty \int_{\Omega_J} J(x-y)\mathcal{G}(x,u(y;x,t))u(y;x,t)\,dy\,dt + \lambda\,, \quad x \in \overline{\Omega}$$

and $w\big|_{\Omega_J\setminus\overline{\Omega}} \equiv 0$.

Fixed any $x \in \Omega$, consider a sequence $\{t_n\}_{n\in\mathbb{N}}$ satisfying $t_n \to \infty$. We deduce that

$$\lim_{n\to\infty} u_t(x,t_n) = \int_{\Omega_J} J(x-y)\mathcal{G}(x,w(y;x))w(y;x)\,dy\,,$$

and so this limit does not depend on the chosen sequence. Thus, there exists $\lim_{t\to\infty} u_t(x,t) = \ell$ and this limit is nonpositive since our solution is nonincreasing in t. (We remark that the limit ℓ depends on the considered point x.) Assume by contradiction that $\ell < 0$. Then there exists $t_0 > 0$ such that

$$u_t(x,t) < \frac{\ell}{2}\,, \quad \text{for any } t \ge t_0\,.$$

It follows that $u(x,t) - u(x,t_0) < \frac{\ell}{2}(t-t_0)$, which implies $u(x,t) < \lambda + \frac{\ell}{2}(t-t_0)$ and this quantity is negative for t large enough. Since this contradicts (4.38), we have $\ell = 0$. Obviously, this argument holds for every $x \in \Omega$, wherewith

$$\lim_{t\to\infty} u_t(x,t) = \int_{\Omega_J} J(x-y)\mathcal{G}(x,w(y;x))w(y;x)\,dy = 0\,, \quad x \text{ in}\Omega\ \,.$$

By continuity, we conclude that

$$\int_{\Omega_J} J(x-y)\mathcal{G}(x,w(y;x))w(y;x)\,dy = 0\,, \quad x \text{ in } \overline{\Omega}\,. \tag{4.39}$$

Recalling that the function w is the limit of a nonincreasing family of continuous functions, we deduce that w is lower semicontinuous in $\overline{\Omega}$. So w attains its maximum in $\overline{\Omega}$; let $x_0 \in \overline{\Omega}$ satisfy $w(x) \le w(x_0)$ for any $x \in \overline{\Omega}$.

Since the function J is radial symmetric, it is positive in an open ball centered at the origin; we denote its radius is r Let n be the integer part of $\operatorname{dist}(x_0, \partial\Omega)/r$. Applying (4.39) it yields

$$\int_{\Omega_J} J(x-y)\mathcal{G}(x, w(y) - w(x_0))(w(y) - w(x_0))\,dy = 0\,.$$

Since the integrand is nonpositive, it vanishes, so that $w(y) = w(x_0)$ for any $y \in \overline{\Omega}$ satisfying $y - x_0 \in \operatorname{supp} J$, that is, for any $y \in \overline{\Omega} \cap B_1(x_0)$. If $n \ge 1$ and so $B_r(x_0) \subset \Omega$, taking y_0 close to the boundary of $B_r(x_0)$ and applying the same argument, we infer that $w(y) = w(x_0)$ for any $y \in \overline{\Omega} \cap B_{2r}(x_0)$. We may follow this procedure n times to find some $x \in \Omega$ such that $w(x) = w(x_0)$ and $\operatorname{dist}(x,\partial\,\Omega) < r$ (this fact can already be attained in the first step if $n = 0$). Then

$$0 = \int_{\overline{\Omega}} J(x-y)\mathcal{G}(x, w(y) - w(x))(w(y) - w(x))\,dy + \int_{\Omega_J\setminus\overline{\Omega}} J(x-y)\mathcal{G}(x, -w(x))(-w(x))\,dy\,.$$

Notice that both integrands are nonpositive, so that both vanish. We deduce from the first integral that w is constant in $\overline{\Omega} \cap B_r(x)$ and from the second one that this constant is equal to 0. Therefore, $w(x_0) = w(x) = 0$ and as a consequence $w(x) = 0$ for any $x \in \overline{\Omega}$.

Recalling that the function $u(x,t)$ is nonincreasing in t and $\lim_{t\to\infty} u(x,t) = 0$ for any $x \in \overline{\Omega}$, we deduce from Dini's Theorem that this convergence is uniform. □

With the help of Theorem 4.4.5, we are ready to prove Theorem 4.2.9.

Proof:[Proof of Theorem 4.2.9] Consider u^1 the solution to (4.15) with initial datum $u_0^1(x) = \|u_0\|_{L^\infty(\Omega)}$, and $u^2 \equiv 0$. On the one hand, it follows from the Comparison Principle that

$$0 \le u(x,t) \le u^1(x,t)\,, \quad \text{for any } x \in \overline{\Omega} \text{ and } t > 0\,.$$

On the other hand, we deduce from Proposition 4.4.5 that

$$\lim_{t\to\infty} u^1(x,t) = 0\,, \quad \text{uniformly in } \overline{\Omega}$$

and thus the result follows. □

Remark 4.4.6 *As already mentioned, if hypothesis* (4.16) *holds true, we have that the decay at* 0 *is of exponential type. Indeed,*

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^2(x,t)dx &= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\mathcal{G}(x, u(y;x,t))\,u(y;x,t)\,u(x,t)\,dy\,dx\\ &= -\beta\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))^2\,dy\,dx.\end{aligned}$$

Now, due to Chasseigne et al. (2006), there exists a pair $(\lambda_1, \phi(x)) \in \mathbb{R}^+ \times \mathcal{C}(\Omega)$ *such that*

$$0 < \lambda_1 = \inf_{u\in L^2(\Omega)\setminus\{0\}} \frac{\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y)-u(x))^2 dy\,dx}{\displaystyle\int_\Omega u(x)^2 dx}$$

and a function $\phi(x)$ *where the infimum is attained. Consequently, we conclude that*

$$\frac{d}{dt}\int_\Omega u^2(x,t)dx \le -2\beta\lambda_1\int_\Omega u(x,t)^2 dx,$$

and integrating over $[0,t]$*, we have that* $\|u(\cdot,t)\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)}\, e^{-\lambda_1\beta\ t}$.

4.5 Proofs of results about Cauchy Problem

As in the previous Section, we start by proving the existence and uniqueness result.

Proof:[Proof of Theorem 4.2.12] For $T > 0$ we consider the Banach space

$$X = \mathcal{C}\left([0,T]; \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\right),$$

endowed with the norm

$$|||w||| = \max_{0 \le t \le T} e^{-kMt} \|w(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}.$$

Here $M = 2\,\alpha_2$ and $k \ge 1$.

Now, let Y be the closed ball of X with radius $k\|u_0\|_{L^\infty(\mathbb{R}^N)}$ and centered at the origin. Note that Y is a complete metric space with the induced metric $d(w_1, w_2) = |||w_1 - w_2|||$.

In order to establish the existence and uniqueness of solutions of (C) via Banach contraction principle, we define the operator $\mathcal{T} : Y \longrightarrow Y$ by

$$\mathcal{T}(w)(x,t) = \int_0^t \int_{\mathbb{R}^N} J(x-y) w(y;x,\tau\,) \mathcal{G}(x, w(y;x,\tau\,)) dy d\tau + u_0(x).$$

Let us first prove that this operator is well defined. Clearly $\mathcal{T}(w)$ is belongs to X and satisfies

$$\begin{aligned} \|\mathcal{T}(w)(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} &\le \alpha_2 \max_{x\in\mathbb{R}^N} \int_0^t \int_{\mathbb{R}^N} J(x-y)|w(y;x,s)| dy ds + \|u_0\|_{L^\infty(\mathbb{R}^N)} \\ &\le 2\alpha_2 \int_0^t \|w(\cdot,s)\|_{L^\infty(\mathbb{R}^N)} ds + \|u_0\|_{L^\infty(\mathbb{R}^N)} \\ &\le 2\alpha_2 |||w||| \textstyle\int_0^t e^{kMs} ds + \|u_0\|_{L^\infty(\mathbb{R}^N)} \le e^{kMt} \|u_0\|_{L^\infty(\mathbb{R}^N)}. \end{aligned} \tag{4.40}$$

Therefore,

$$|||\mathcal{T}(w)||| = \max_{0\le t\le T} e^{-kMt} \|\mathcal{T}(w)(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \|u_0\|_{L^\infty(\mathbb{R}^N)}.$$

Since $k > 1$, we obtain that $|||\mathcal{T}(w)||| \le \; k\|u_0\|_{L^\infty(\mathbb{R}^N)}$ and $\mathcal{T}(w)$ belongs to Y.

Now, let us show that the operator $\mathcal{T}$ is a contraction. By using that $\mathcal{G}$ satisfies ($\mathcal{G}$) and arguing as (4.40), we obtain

$$\begin{aligned} \|\left(\mathcal{T}(w_1) - \mathcal{T}(w_2)\right)(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} &\le \alpha_2 \max_{x\in\mathbb{R}^N} \int_0^t \int_{\mathbb{R}^N} J(x-y)|w_1(y;x,\tau\,) - w_2(y;x,\tau\,)| dy d\tau \\ &\le 2\,\alpha_2 \int_0^t \|w_1(\cdot,\tau) - w_2(\cdot,\tau)\|_{L^\infty(\mathbb{R}^N)} d\tau \le 2\,\alpha_2 |||w_1 - w_2||| \int_0^t e^{kM\tau} d\tau \\ &\le \tfrac{1}{k}\left(e^{kMt} - 1\right) |||w_1 - w_2|||. \end{aligned}$$

Therefore,

$$d(\mathcal{T}(w_1), \mathcal{T}(w_2)) \le \frac{1}{k} |||w_1 - w_2||| \max_{0\le t\le T} \left(1 - e^{-kMt}\right) \le \frac{1}{k} d(w_1, w_2).$$

Since $k > 1$, $\mathcal{T}$ is a contraction. Hence, using Banach's Fixed Point Theorem there exists u a fix point of $\mathcal{T}$, that is the unique solution of problem (C) for $t \in [0,T]$ and belongs to Y. Finally, since T is arbitrary, we obtain a global solution, $u \in \mathcal{C}\left([0,\infty); \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\right)$. □

Now we can prove the Comparison Principle.

Proof:[Proof of Theorem 4.2.14] Set $w = u - v$, then in virtue of (4.22) w satisfies

$$\begin{cases} w_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)\, w(y;x,t)\; \psi(x, u(y;x,t), v(y;x,t)) dy & \text{in } \mathbb{R}^N \times (0,+\infty) \\ w(x,0) \le 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{4.41}$$

where ψ is the function defined in (4.22). Let us consider the following function

$$\varsigma(x,t) = \begin{cases} 1 & \text{if } w(x,t) \geq 0, \\ 0 & \text{if } w(x,t) < 0. \end{cases}$$

Multiplying (4.41) by $\varsigma(x,t)$ and taking into account that $w_t(x,t)\varsigma(x,t) = (w_+)_t\,(x,t)$ and $w(y,t)\varsigma(x,t) \leq w_+(y,t)$, we obtain, dropping the positive term $w(x,t)\varsigma(x,t)$, that

$$\begin{aligned} (w_+)_t\,(x,t) &= \int_{\mathbb{R}^N} J(x-y)\,(w(y,t)\varsigma(x,t) - w(x,t)\varsigma(x,t))\,\psi(x,u(y;x,t),v(y;x,t))dy \\ &\leq \int_{\mathbb{R}^N} J(x-y)\,w_+(y,t)\,\psi(x,u(y;x,t),v(y;x,t))dy \leq \alpha_2 \int_{\mathbb{R}^N} J(x-y)\,w_+(y,t)dy, \end{aligned}$$

integrating in $\mathbb{R}^N$ and by using $\int_{\mathbb{R}^N} J(z)dz = 1$, we get

$$\int_{\mathbb{R}^N} (w_+)_t\,(x,t)dx \leq \alpha_2 \int_{\mathbb{R}^N} w_+(y,t)dy.$$

Finally, integrating in $(0,T]$ and since $w_+(x,0) = 0$ we can assert, using Fubini's theorem, that

$$k(t) \leq \alpha_2 \int_0^t k(\tau)d\tau, \qquad \text{where} \qquad k(t) = \int_{\mathbb{R}^N} w_+(x,t)dx. \tag{4.42}$$

Hence, applying Gronwall's Lemma in (4.42), we conclude that

$$k(t) \leq 0.$$

Now, since $w_+(x,t) \geq 0$ and by the continuity of w_+, we get that $w_+(x,t) = 0$ and, consequently,

$$u(x,t) \leq v(x,t)$$

for any $x \in \mathbb{R}^N,\ t > 0$. □

Note that the previous proof works locally in time, that is, a supersolution v and a subsolution u defined both for $t \in [0,T]$ verify $u(x,t) \leq v(x,t)$ for any $x \in \mathbb{R}^N,\ 0 \leq t < T$.

Proof:[Proof of Theorem 4.2.15] By Theorem 4.2.12, for any $\varepsilon > 0$ there exists u^ε the unique solution of problem (4.19). Set $w^\varepsilon := v - u^\varepsilon$, wich satisfies

$$\begin{cases} w^\varepsilon_t(x,t) = \Delta v(x,t) + \mu(x)|\nabla v(x,t)|^2 - \mathcal{L}_\varepsilon(u^\varepsilon(x,t)), & \text{in } \mathbb{R}^N \times (0,T], \\ w^\varepsilon(x,0) = 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{4.43}$$

being

$$\mathcal{L}_\varepsilon(u^\varepsilon(x,t)) = \frac{C(x)}{\varepsilon^2} \int_{\mathbb{R}^N} J_\varepsilon(x-y)u^\varepsilon(y;x,t)\,\mathcal{G}(x,u^\varepsilon(y;x,t))dy.$$

Now, the proof follows the one of Theorem 4.2.3.
Choosing $\overline{w}(x,t) = K\varepsilon^\alpha t$ and $\underline{w}(x,t) = -\overline{w}(x,t)$. Then for K sufficiently large we have that $\overline{w}$ and $\underline{w}$ are super and subsolution of (4.43) respectively. Therefore, by the principle comparison of Theorem 4.2.14 we obtain $\underline{w} \leq w^\varepsilon \leq \overline{w}$ and the proof is straightforward. □

As far as the asymptotic behavior is concerned, we observe that $\hat{J}(\xi)$, the Fourier transform of J, satisfies

$$\hat{J}(\xi) \leq 1 - C(J)|\xi|^2 + o(|\xi|^2), \quad \text{as } \xi \to 0\,.$$

where the above estimates follows since

$$\frac{1}{2}\partial^2_{\xi_i\xi_i}\hat{J}(0) = \frac{1}{2}\int_{\mathbb{R}^N} J(z)z_N^2 dz = \frac{1}{2}C(J) < \infty,$$

thanks to (3.3).

For the convenience of the reader we repeat the following Lemma that is proved in Cañizo and Molino (2016) including also a sketch of the proof (in order to make this part of the paper self-contained).

Lemma 4.5.1 *Let $u \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and J satisfying hypothesys (J). In addition, consider*

$$D_J(u) = \int_{\mathbb{R}^N} \left(1 - \hat{J}(\xi)\right) |\hat{u}(\xi)|^2 d\xi.$$

Then, $\exists\, \tilde{C} = \tilde{C}(N, J) > 0$ such that

$$\|u\|^2_{L^2(\mathbb{R}^N)} \leq \tilde{C} \max \left\{ \|u\|^{\frac{4}{N+2}}_{L^{12}(\mathbb{R}^N)} D_J(u)^{\frac{N}{N+2}}, D_J(u) \right\},$$

and consequently

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)\,(u(y)-u(x))^2\, dxdy \geq K\, \min \left\{ \|u\|^{-\frac{4}{N}}_{L^1(\mathbb{R}^N)} \|u\|^{2+\frac{4}{N}}_{L^2(\mathbb{R}^N)}, \|u\|^2_{L^2(\mathbb{R}^N)} \right\}. \tag{4.44}$$

Proof: First, we set the following quantities

$$C = \max_{|\xi| \geq 1} \frac{1}{1 - \hat{J}(\xi)} > 0, \qquad \delta_0 = \left(\frac{C\, D_J(u)}{C(N) \|u\|^2_{L^1(\mathbb{R}^N)} C(J)} \right)^{\frac{1}{N+2}},$$

where $C(N) = \frac{N \pi^{N/2}}{2\Gamma\left(\frac{N}{2}+1\right)}$ and Γ denotes the Gamma function. Since $u \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ it follows that $\hat{u} \in L^2(\mathbb{R}^N)$ and consequently we obtain for $0 < \delta \leq 1$ that

$$\|\hat{u}\|^2_{L^2(\mathbb{R}^N)} = \int_{|\xi| \leq \delta} |\hat{u}(\xi)|^2 d\xi + \int_{|\xi| > \delta} |\hat{u}(\xi)|^2 d\xi \leq \|u\|^2_{L^1(\mathbb{R}^N)} \frac{2C(N)}{N} \delta^N + \frac{C}{C(J)\, \delta^2} D_J(u). \tag{4.45}$$

Now, if we assume that $\delta_0 \leq 1$. Replacing δ by δ_0 in (4.45), we have

$$\|\hat{u}\|^2_{L^2(\mathbb{R}^N)} \leq C_1 \|u\|^{\frac{4}{N+2}}_{L^1(\mathbb{R}^N)} D_J(u)^{\frac{N}{N+2}}, \tag{4.46}$$

where $C_1 = \left(\frac{2}{N}+1\right) C(N)^{\frac{2}{N+2}} C^{\frac{N}{N+2}}$. Alternatively, if we assume that $\delta_0 > 1$, i.e.,

$$C(N)\, \|u\|^2_{L^1(\mathbb{R}^N)} < C\, D_J(u),$$

choosing $\delta = 1$ in (4.45) and using the above inequality, we get

$$\|\hat{u}\|^2_{L^2(\mathbb{R}^N)} \leq \|u\|^2_{L^1(\mathbb{R}^N)} \frac{2C(N)}{N} + C\, D_J(u) \leq \left(\frac{2}{N}+1\right) C\, D_J(u). \tag{4.47}$$

Finally, using Plancherel's theorem on $\|\hat{u}\|^2_{L^2(\mathbb{R}^N)}$ and summarizing (4.46) and (4.47), it follows that

$$\|u\|^2_{L^2(\mathbb{R}^N)} \leq \tilde{C} \max \left\{ \|u\|^{\frac{4}{N+2}}_{L^1(\mathbb{R}^N)} D_J(u)^{\frac{N}{N+2}}, D_J(u) \right\}$$

where $\tilde{C} = \max\left\{C_1, \left(\frac{2}{N}+1\right) C\right\}$ and the proof is concluded. Due to the above formula, we can state the following inequality

$$D_J(u) \geq K\, \min \left\{ \|u\|^{-\frac{4}{N}}_{L^1(\mathbb{R}^N)} \|u\|^{2+\frac{4}{N}}_{L^2(\mathbb{R}^N)}, \|u\|^2_{L^2(\mathbb{R}^N)} \right\},$$

being $K = K(N, J)$. Thus, it is easy to check that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)\,(u(y)-u(x))^2\, dxdy = -2 \int_{\mathbb{R}^N} (J * u - u)(x)\, u(x)\, dx,$$

having in mind that Fourier transform preserves inner product we deduce (4.44) $\square$

Next Lemma gives the L^1 boundedness from above or from below of solutions depending on how the function $\mathcal{G}(x,s)s$ behaves. To be more specific we have the following result.

Lemma 4.5.2 *Let u be a solution of Cauchy problem (C) with $0 \le u_0 \in L^1(\mathbb{R}^N)$. Then*

(i) If $\mathcal{G}$ satisfies (4.16), it follows that $t \mapsto \|u(\cdot,t)\|_{L^1(\mathbb{R}^N)}$ is decreasing on $[0,\infty)$, therefore

$$\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)} \le \|u_0\|_{L^1(\mathbb{R}^N)}\,.$$

(ii) If $\mathcal{G}$ satisfies (4.17), it follows that $t \mapsto \|u(\cdot,t)\|_{L^1(\mathbb{R}^N)}$ is increasing on $[0,\infty)$, therefore

$$\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)} \ge \|u_0\|_{L^1(\mathbb{R}^N)}\,.$$

Proof: Since $0 \le u_0$ and Comparison Principle of Proposition 4.2.14 we can assume that $u(x,t) \ge 0$. Furthermore, if $\mathcal{G}(x,s)s \le \beta s$ for any $(x,s) \in \mathbb{R}^N \times \mathbb{R}$, since

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N} u(x,t)dx &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)u(y;x,t)\ \mathcal{G}(x,u(y;x,t))dydx \\ &\le \beta\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dydx = 0,\end{aligned}$$

where the last identity follows since, by Fubini Theorem,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dydx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(x,t)-u(y,t))dxdy\,.$$

Hence $\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)}$ is nonincreasing in time and we state (i). Equivalently, if $\mathcal{G}(x,s)s \ge \beta s$ for any $(x,s) \in \mathbb{R}^N \times \mathbb{R}$, reasoning as above we obtain the opposite inequality and, consequently, $\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)}$ is nondecreasing in time and (ii) is proved. □

Now we can prove the asymptotic behavior of the solution for $\mathcal{G}$ satisfying (4.16),

Theorem 4.5.3 *Let u be a solution of Cauchy problem (C) with $\mathcal{G}$ satisfying (4.16) and positive prescribed data $u_0 \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for $q \ge 2$. Then there exists $C = C(J,N,\beta,q) > 0$ such that*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^N)} \le C\|u_0\|_{L^1(\mathbb{R}^N)}t^{-\frac{N}{2}\left(1-\frac{1}{q}\right)},$$

for any t sufficiently large.

Proof:[Proof of Theorem 4.5.3]

Let $q \ge 2$ and let us multiply the equation in (C) by $u^{q-1}(x,t)$ (observe that $u \ge 0$): thus we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{q}\int_{\mathbb{R}^N} u(x,t)^q dx &= \int_{\mathbb{R}^N} u_t(x,t)u(x,t)^{q-1}dx \\ &\le \beta\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))u(x,t)^{q-1}dxdy \\ &= -\frac{\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))(u(y,t)^{q-1}-u(x,t)^{q-1})dxdy \\ &\le -C(q,\beta)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y,t)^{q/2}-u(x,t)^{q/2})^2dxdy,\end{aligned}$$

where in the last inequality we have used Lemma 4.3.3. Hence by (4.44), we get

$$\frac{d}{dt}\int_{\mathbb{R}^N} u(x,t)^q dx \le -C\ \min\left\{\|u(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R}^N)}^{-\frac{2q}{N}}\|u(\cdot,t)\|_{L^q(\mathbb{R}^N)}^{q\left(1+\frac{2}{N}\right)}, \|u(\cdot,t)\|_{L^q(\mathbb{R}^N)}^q\right\}$$

where $C = C(q,\beta,N,J)$. Now, by interpolation $\|u(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R}^N)} \le \|u(\cdot,t)\|_{L^1(\mathbb{R}^N)}^{\frac{1}{q-1}} \|u(\cdot,t)\|_{L^q(\mathbb{R}^N)}^{\frac{q-2}{q-1}}$ and denoting by $Y(t) = \|u(\cdot,t)\|_{L^q(\mathbb{R}^N)}^q$, we obtain, in virtue of Lemma 4.5.2, the following differential inequality

$$Y'(t) \le -C \min\left\{\|u_0\|_{L^1(\mathbb{R}^N)}^{-q\gamma} Y(t)^{1+\gamma}, Y(t)\right\} \tag{4.48}$$

being $\gamma = \dfrac{2}{N(q-1)}$. Therefore, $Y(t)$ is decreasing. We claim that there exists $t_0 \ge 0$ such that

$$Y(t) \le \|u_0\|_{L^1(\mathbb{R}^N)}^q, \qquad t \ge t_0.$$

Indeed, otherwise, using that $Y(t)$ is decreasing, we would have that $\|u_0\|_{L^1(\mathbb{R}^N)}^q \le Y(t)$ for any $t \ge t_0$. Replacing in (4.48) we obtain

$$Y'(t) \le -C\,Y(t), \qquad t \ge t_0,$$

and integrating on $[t_0,t]$ we get that $Y(t) \le Y(t_0)e^{-C(t-t_0)} \to 0$ as $t \to \infty$ which leads to a contradiction and the claim is proved.

Thus, since

$$Y(t) = Y(t)^{1+\gamma}Y(t)^{-\gamma} \ge Y(t)^{1+\gamma}Y(t_0)^{-\gamma} \ge Y(t)^{1+\gamma}\|u_0\|_{L^1(\mathbb{R}^N)}^{-q\gamma},$$

it follows, by inequality (4.48), that

$$Y'(t) \le -C\,\|u_0\|_{L^1(\mathbb{R}^N)}^{-q\gamma}\,Y(t)^{1+\gamma}, \qquad t \ge t_0.$$

Integrating on $[t_0,t]$ we get

$$Y(t) \le \frac{\|u_0\|_{L^1(\mathbb{R}^N)}^q}{(\gamma\, C)^{1/\gamma}}\,(t-t_0)^{-1/\gamma}.$$

Having in mind that $Y(t) = \|u(\cdot,t)\|_{L^q(\mathbb{R}^N)}^q$ and $\dfrac{-1}{q\,\gamma} = -\dfrac{N}{2}\left(1-\dfrac{1}{q}\right)$ we conclude that, for any time t large enough, $\exists\ C = C(J,N,\beta,q)$, such that

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^N)} \le C\|u_0\|_{L^1(\mathbb{R}^N)}\,t^{-\frac{N}{2}\left(1-\frac{1}{q}\right)}.$$

□

With the help of the above result, we can now prove Theorem 4.2.16.

Proof:[Proof of Theorem 4.2.16] Theorem 4.5.3 covers the case $q \ge 2$, while for $q \in (1,2]$ the interpolation inequality yields to

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^N)} \le \|u(\cdot,t)\|_{L^1(\mathbb{R}^N)}^{\frac{2}{q}-1}\,\|u(\cdot,t)\|_{L^2(\mathbb{R}^N)}^{2\left(1-\frac{1}{q}\right)} \le C\|u_0\|_{L^1(\mathbb{R}^N)}\,t^{-\frac{N}{2}\left(1-\frac{1}{q}\right)},$$

being $C = C(J,N,\beta,q)$ a positive constant. □

In order to obtain a decay estimate of the norm of the solution u, for functions $\mathcal{G}_\mu$ with $\mu(x) \ge 0$, a L^1 boundedness from above of u is required. For this purpose, we must to control de L^∞-norm of initial data u_0 with respect to function μ.

Lemma 4.5.4 *Let u be a solution of of Cauchy problem (C) with $\mathcal{G} \equiv \mathcal{G}_\mu$, $0 \le \mu \in L^\infty(\mathbb{R}^N)$ and positive prescribed data $u_0 \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ satisfying $\|u_0\|_{L^\infty(\mathbb{R}^N)}\|\mu\|_{L^\infty(\mathbb{R}^N)} = \theta < 1$. Then*

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^2(\mathbb{R}^N)}^2 \le -(1-\theta)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))^2dydx. \tag{4.49}$$

If, in addition, $u_0 \in L^1(\mathbb{R}^N)$ then

$$\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)} \le c\|u_0\|_{L^1(\mathbb{R}^N)}, \tag{4.50}$$

with $c = c(\|u_0\|_{L^\infty(\mathbb{R}^N)}, \|\mu\|_{L^\infty(\mathbb{R}^N)}) > 1$.

Proof: Since $u_0 \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$, by Theorem 4.2.12 there exists a unique solution of problem (C) and it satisfies $u \in \mathcal{C}\big([0,\infty); \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\big)$. Moreover, since 0 and $\|u_0\|_{L^\infty(\mathbb{R}^N)}$ are sub and supersolution respectively of problem (C), we get, due the comparison principle Theorem 4.2.14, that

$$0 \le u(x,t) \le \|u_0\|_{L^\infty(\mathbb{R}^N)}, \qquad\qquad (x,t) \in \mathbb{R}^N \times [0,\infty).$$

Let us multiply the equation in (C) by $u(x,t)$ and integrate in $\mathbb{R}^N$, so that

$$\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} = 2\int_{\mathbb{R}^N} u_t(x,t)u(x,t)dx$$

$$= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)u(y;x,t)\ \mathcal{G}_\mu(u(y;x,t))\,u(x,t)\ dydx$$

$$= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)u(y;x,t)\ u(x,t)dydx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\frac{\mu(x)u(y;x,t)^2}{1+\mu^2(x)u(y;x,t)^2}u(x,t)dydx$$

$$\le -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)u(y;x,t)^2dydx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\mu(x)u(y;x,t)^2u(x,t)dydx$$

$$= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)u(y;x,t)^2(1-\mu(x)u(x,t))dydx$$

$$\le -(1-\theta)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))^2dydx,$$

which proves the first part of lemma.

In order to get (4.50), we compute the derivate of L^1-norm of u, and we get

$$\begin{aligned}
\frac{d}{dt}\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\mathcal{G}_\mu\big(u(y,t)-u(x,t)\big)\big(u(y,t)-u(x,t)\big)dydx\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\frac{\mu(x)}{2}\,\frac{\big(u(y,t)-u(x,t)\big)^2}{1+\mu^2(x)\big(u(y,t)-u(x,t)\big)^2}dxdy\\
&\le \frac{\|\mu\|_{L^\infty(\mathbb{R}^N)}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\big(u(y,t)-u(x,t)\big)^2dydx\\
&\le -\frac{\|\mu\|_{L^\infty(\mathbb{R}^N)}}{2}\,\frac{1}{1-\theta}\,\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)},
\end{aligned}$$

where we have used (4.49) in the last inequality. Hence, we obtain the following differential inequality:

$$\exists\ c_1 > 0 : \qquad \frac{d}{dt}\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)} + c_1\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \le 0,$$

being $c_1 = \frac{\|\mu\|_{L^\infty(\mathbb{R}^N)}}{2(1-\theta)} > 0$. Consequently, integrating on $[0,t]$,

$$\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)} + c_1\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \le \|u_0\|_{L^1(\mathbb{R}^N)} + c_1\|u_0\|_{L^\infty(\mathbb{R}^N)}\,\|u_0\|_{L^1(\mathbb{R}^N)},$$

where we have used the interpolation formula, $\|u_0\|^2_{L^2(\mathbb{R}^N)} \le \|u_0\|_{L^\infty(\mathbb{R}^N)}\,\|u_0\|_{L^1(\mathbb{R}^N)}$. Finally we conclude that $\|u(\cdot,t)\|_{L^1(\mathbb{R}^N)} \le c\|u_0\|_{L^1(\mathbb{R}^N)}$, for $c = 1 + c_1\|u_0\|_{L^\infty(\mathbb{R}^N)}$.

$\square$

Proof:[Proof of Theorem 4.2.17] Applying inequality (4.44) in (4.49) from Lemma 4.5.4, it follows

$$\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \le -C_1\min\left\{\|u(\cdot,t)\|^{-\frac{4}{N}}_{L^1(\mathbb{R}^N)}\|u(\cdot,t)\|^{2+\frac{4}{N}}_{L^2(\mathbb{R}^N)}, \|u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)}\right\},$$

where $C_1 = C_1(\|\mu\|_{L^\infty(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, N, J) > 0$. Writing $X(t) = \|u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)}$ and using the boundedness of L^1-norm in inequality (4.50) we have that

$$X'(t) \leq -C_2 \min\left\{ \|u_0\|_{L^1(\mathbb{R}^N)}^{-\frac{4}{N}} X(t)^{1+\frac{2}{N}},\, X(t) \right\},$$

where $C_2 = C_2(\|\mu\|_{L^\infty(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, N, J) > 0$. Thus, arguing as in proof of Theorem 4.5.3, we can assume that there exists $t_0 \geq 0$ such that $X(t) \leq \|u_0\|^2_{L^1(\mathbb{R}^N)}$ for $t \geq t_0$ and therefore,

$$X'(t) \leq -C_2 \|u_0\|_{L^1(\mathbb{R}^N)}^{-\frac{4}{N}} X(t)^{1+\frac{2}{N}}, \qquad t \geq t_0.$$

Finally, integrating on $[t_0, t]$, we obtain the L^2-norm decay estimate for any t sufficiently large. $\square$

Chapter 5

Existence of a continuum of solutions for a quasilinear elliptic singular problem

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DOI: 10.1016/j.jmaa.2015.12.034

Abstract

In this paper we study the existence of positive solution $u \in H_0^1(\Omega)$ for some quasilinear elliptic equations, having lower order terms with quadratic growth in the gradient and singularities, whose model is

$$-\Delta u + \mu(x)\frac{|\nabla u|^2}{u^\gamma + u^\beta} = \lambda u^p + f_0(x),\ x \in \Omega,\ 0 < \gamma \leq \beta, 0 < p.$$

Using topological methods we obtain the existence of an unbounded continuum of solutions. In the case $\mu(x)$ constant we derive the existence of solution for every $\lambda > 0$ if $1 < \gamma < 2$ for any β and $p < 1$. Even more for $\mu \in L^\infty(\Omega)$ we prove this result if $\beta \leq 1$ and $p < 2 - \beta$.

5.1 Introduction

We consider the following boundary value problem

$$\begin{cases} -\Delta u + \mu(x) g(u) \, |\nabla u|^2 = \lambda u^p + f_0(x) & \text{in} \quad , \; \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where Ω is a smooth bounded and open subset of $\mathbb{R}^N$, $N \geq 3$, $p \geq 0$. The functions $\mu \in L^\infty(\Omega)$ and $g \in C^1((0, +\infty))$ are nonnegative; notice that g can become singular at zero. We are assume $0 \lneqq f_0 \in L^q(\Omega)$ for some $q > N/2$.

By a subsolution (respectively, supersolution) of problem (P_λ) we mean a function $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$ with $u > 0$ a.e. $x \in \Omega$, $g(u) \, |\nabla u|^2 \in L^1(\Omega)$ and which satisfies the following inequality:

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega \mu(x) g(u) \, |\nabla u|^2 \varphi \overset{(\geq)}{\leq} \int_\Omega (\lambda u^p + f_0) \varphi \,,$$

for every $0 \leq \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. A solution is a function which is both a subsolution and a supersolution.

Problem (P_λ) involves a quasilinear elliptic differential operator with quadratic gradient terms. This kind of differential operators with natural growth were considered in Boccardo et al. (1982, 1983) and since then different associated boundary value problems have been studied. A well known case is the existence of the solution of (P_0) when g is continuous at $u = 0$ (see for instance Bensoussan et al. (1988), Boccardo and Gallouët (1992) and Boccardo et al. (1982)).

Alternatively, problem (P_0) for functions g with a singularity at zero, has also been extensively studied in Arcoya and Martínez-Aparicio (2008); Arcoya and Segura de León (2010); Boccardo (2008); Martínez-Aparicio (2009). Existence of solutions was discussed in Arcoya et al. (2009b) in the case $\sqrt{g} \in L^1(0,1)$ by imposing the following condition

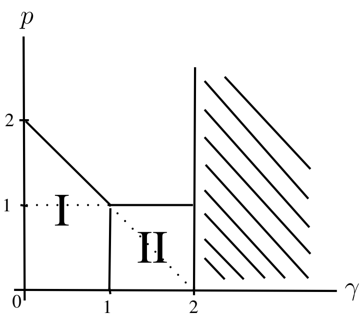
$$\operatorname{ess\,inf}\{f_0(x) : x \in \omega\} > 0 \,, \; \forall \omega \subset\subset \Omega \,. \tag{5.1}$$

Results concerning (P_λ) for $\lambda \neq 0$ were obtained in Arcoya et al. (2011); Boccardo et al. (2011) in the case $g(s) = 1/s^\gamma$ where the model problem is

$$\begin{cases} -\Delta u + \mu(x) \dfrac{|\nabla u|^2}{u^\gamma} = \lambda u^p + f_0(x) & \text{in} \quad , \; \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{R_λ}$$

with $\mu(x)$ as a constant function. More precisely, with $\gamma < 1$ and $\gamma + p < 2$ (region I in Figure 1 below), the existence of a solution for each $\lambda \geq 0$ was proved in Arcoya et al. (2011) by means of topological methods and in Boccardo et al. (2011) by using an approximative scheme.

Figure 1: Existence regions



Notice that if $\gamma \geq 2$ it makes no sense to search solutions of (R_λ). Indeed, as it is proved in Zhou et al. (2012), $\frac{|\nabla u|^2}{u^\gamma} \notin L^1(\Omega)$.

However, the techniques employed in Arcoya et al. (2011); Boccardo et al. (2011) can not be applied in the case $\mu(x)$ not constant or where $p < 1 \leq \gamma < 2$ (region II in Figure 1 above). In this paper, we complete the previous results and we extend them for a more general function g in order to show the following: *"the values of λ for which there exists a solution of (P_λ) depends on the behavior of g at infinity"*. In fact, in contrast with the results when $g \equiv 0$, in some cases we obtain solutions for every positive λ, that is, the gradient term produces a regularizing effect. We deal with (P_λ) for a function g exhibiting a different behavior at zero and at infinity. In particular, we are mainly interested in the case of functions $g(s) = 1/(s^\gamma + s^\beta)$ with $\gamma \leq \beta$. In this way, we consider the model problem

$$\begin{cases} -\Delta u + \mu(x)\dfrac{|\nabla u|^2}{u^\gamma + u^\beta} = \lambda u^p + f_0(x) & \text{in} \quad , \ \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{Q_λ}$$

as a natural extension of the problem (R_λ). Observe that for $\lambda = 0$, as was mentioned above, problem (Q_0) has been extensively studied. Our main goal is to exploit this known case to obtain an unbounded continuum Σ of solutions of (Q_λ), namely, a connected and closed subset of

$$\{(\lambda, u) \in [0, +\infty) \times C(\overline{\Omega}) : \ u \text{ is a solution of } (Q_\lambda)\},$$

for suitable values of p,γ and β, which extend the previous existence results. In particular, beginning with the case $\mu(x)$ constant and $\gamma < 2$, we prove in Theorem 5.1.1 the existence of an unbounded continuum Σ. In Theorem 5.1.2 we deal with non-constant $\mu(x)$ in the case $\beta \leq 1$.

Theorem 5.1.1 *Assume $\mu(x) = \mu$ is constant and that $f_0 \in L^q(\Omega)$ with $q > \frac{N}{2}$ satisfies (5.1). Then:*

i) If $1 \leq \gamma < 2$ and $0 < p < 1$ then problem (Q_λ) admits at least one solution for every $\lambda \geq 0$.

ii) If $\gamma < 1 < \beta$ and $1 \leq p$, then there exists $\lambda_*, \lambda^* > 0$ such that (Q_λ) admits no solution for $\lambda > \lambda^*$ and at least one solution for $0 \leq \lambda < \lambda_*$.

Moreover, there exists an unbounded continuum Σ of solutions of (Q_λ), such that there exists u_λ solution of (Q_λ) with $(\lambda, u_\lambda) \in \Sigma$ for every $\lambda \geq 0$ (item i)) or every $0 \leq \lambda < \lambda_*$ (item ii)).

We would like to stress that in the case of item i), it is not required assumptions on the parameter β. This is because in order to $\frac{|\nabla u|^2}{u^\gamma + u^\beta}$ be an integrable function we only need the natural hypothesis $\gamma < 2$ which is a condition at zero. In other words, the behavior of g at infinity has not a role in the solutions set. Conversely, item ii) shows that no regularizing effect take place since there is no solution for all positive λ.

Moreover, observe that this theorem improve the results of Arcoya et al. (2011); Boccardo et al. (2011) since item *i)* with $\gamma = \beta$ gives us existence results of the problem (R_λ) in the case that (γ, p) belongs to Region II of Figure 1 above.

Furthermore, our techniques also allow us to work with non-constant function $\mu(x)$ when the parameter (γ, p) belongs to the corresponding Region I of the Figure 1 above. In fact, if we suppose that there exist positive constants m, M such that

$$m \leq \mu(x) \leq M, \text{ a.e. } x \in \Omega, \tag{5.2}$$

we prove the following theorem.

Theorem 5.1.2 *Assume that $0 < \gamma \leq \beta \leq 1$, $0 < p < 2-\beta$, $f_0 \in L^q(\Omega)$ with $q > \frac{N}{2}$ and (5.2) where $M < 2$ in the case $\gamma = \beta = 1$ and $M > 0$ otherwise. Then there exists an unbounded continuum Σ of solutions of (Q_λ), such that there exists u_λ solution of (Q_λ) with $(\lambda, u_\lambda) \in \Sigma$ for every $\lambda \geq 0$.*

Note that this theorem with $\gamma = \beta < 1$ improves again the results of Arcoya et al. (2011) since we can consider non-constant function $\mu(x)$. Furthermore, it improves also Boccardo et al. (2011) except regularity of f_0; in this work the authors consider data f_0 belonging to $L^{\frac{2N}{2N-\gamma(N-2)}}(\Omega)$.

In addition, since we deal with $\gamma < \beta$ and the function $g(s) = 1/(s^\gamma + s^\beta)$ behaves at infinity as $1/s^\beta$ do, we also show that the hypothesis $p < 2 - \beta$ is a restriction in the behavior of g at infinity, rather than in the singularity at zero.

We obtain the existence of the continuum in the above two theorems by using a double approach. Initially, for a convenient sequence of approximated problems, we can derive the existence ofΣ_n by means of Leray-Schauder degree techniques and Rabinowitz continuation theorem as in Arcoya et al. (2011). This requires the uniqueness of the solution for the problem (P_0), in order to set the problem as a fixed point problem for a compact operator. This uniqueness result can not be deduced from Arcoya and Segura de León (2010) if μ is not a constant. Conditions to have uniqueness results for (P_0) were obtained in Arcoya et al. (2017). Secondly, we use a topological lemma to obtain a continuum of solutions as the limit of this approximative schemeΣ_n. It is also important to note that condition (5.1) becomes crucial when applying this approach in Theorem 5.1.1.

The rest of the paper is structured as follows, Section 2 presents the main a priori estimates (this is essentially contained in Stampacchia (1966) and Boccardo et al. (2011)). Section 3 provides, for sequences of solutions of (P_λ), compactness properties and continua of solutions. Section 4 provides proofs of the main theorems. Finally the Appendix contains the proof of some a priori estimates and results related to the uniqueness of solution of the problem (P_0).

5.2 Preliminaries

In this section, according the values for p, we obtain L^∞ estimates for solutions of problem (P_λ).

As usual, for every $s \in \mathbb{R}$, we denote by $s^+ = \max\{s, 0\}$, $s^- = s - s^+$, $T_\varepsilon(s) = s \min\{1, \varepsilon /|s|\}$ and $G_\varepsilon(s) = s - T_\varepsilon(s)$.

Next lemma is consequence of the classical Stampacchia method (Stampacchia (1966)). We include the proof in the Appendix, by convenience of the reader, using the Hartman-Stampacchia variant (Hartman and Stampacchia (1966), see also Ladyzhenskaya and Ural'tseva (1968)).

Lemma 5.2.1 *Let Λ be a positive number. Assume that $0 < p < 1$ and $f_0 \in L^q(\Omega)$ with $q > \frac{N}{2}$, then there exists a positive constant $C > 0$ such that, for every $g \geq 0$ and every solution u of (P_λ) with $0 < \lambda < \Lambda$, one has $\|u\|_{L^\infty(\Omega)} \leq C$.*

The next lemma shows that, for a convenient decay of g at infinity, the previous result is true even for some cases where $p \geq 1$.

Lemma 5.2.2 *Let Λ be a positive number. Assume (5.2) and that $f_0 \in L^q(\Omega)$ with $q > \frac{N}{2}$. Let g_0 also be a nonnegative function in $C((0, +\infty))$ sastifying*

$$\liminf_{t \to \infty} t^\beta g_0(t) > 0, \tag{5.3}$$

where $1 \leq p < 2-\beta$. *Then there exists a positive constant* $C > 0$ *such that, for every* $g \geq g_0$ *and every solution* u *of* (P_λ) *with* $0 < \lambda < \Lambda$, *one has* $\|u\|_{L^\infty(\Omega)} \leq C$.

Proof: We follow the arguments of (Boccardo et al., 2011, Theorem 2.1) and we prove that the right hand side of (P_λ) is (uniformly) bounded in $L^r(\Omega)$, for some $r > \frac{N}{2}$. Thus the conclusion follows by the classical Stampacchia boundedness theorem and by the positive sign on the quadratic gradient lower order term.

We claim that there exists a positive constant $C > 0$ and $\sigma \geq pN/2$ such that, for every $g \geq g_0$ and every solution u of (P_λ) with $0 < \lambda < \Lambda$, one has $\|u\|_{L^\sigma(\Omega)} \leq C$. Thus we can take $r = \min\{q, \sigma/p\}$ to complete the proof.

In order to prove the claim we take $\sigma = (2-\beta)s^{**}$ for some s with

$$\max\left\{\frac{Np}{2(2-\beta+p)}, \frac{2N}{2N-\beta(N-2)}\right\} < s < \frac{N}{2}. \tag{5.4}$$

We observe that since $\frac{Np}{2(2-\beta+p)} < s$ we have that $(2-\beta)s^{**} > pN/2$. In addition, (5.4) assures that $\theta = \frac{(2-\beta)s^{**}}{2^*} > 1$ and, for $0 < \delta < 1$, we use $(u+\delta)^{2\theta+\beta-2} - \delta^{2\theta+\beta-2}$ as test function taking into account (Arcoya et al., 2011, Lemma 2.1).

We obtain, dropping negative terms,

$$\begin{aligned}&\int_\Omega |\nabla u|^2 (u+\delta)^{2\theta+\beta-3}\Big[(2\theta+\beta-2) + m(u+\delta)g(u)\Big] \\ &\quad \leq M\delta^{2\theta+\beta-2}\int_\Omega g(u)|\nabla u|^2 + \int_\Omega [\Lambda u^p + f_0](u+\delta)^{2\theta+\beta-2}.\end{aligned} \tag{5.5}$$

Using (5.3) we deduce the existence of a positive constant $C > 0$ such that

$$\frac{1+tg_0(t)}{(t+1)^{1-\beta}} \geq C, \, \forall t \geq 0.$$

Hence, since $g \geq g_0$ and $\delta < 1$, we have the inequality

$$1 + tg(t) \geq C(t+\delta)^{1-\beta}, \forall t \geq 0.$$

Therefore, from (5.5) we obtain, using also Sobolev inequality,

$$\begin{aligned}C\mathcal{S}\left(\int_\Omega \left[(u+\delta)^\theta - \delta^\theta\right]^{2^*}\right)^{\frac{2}{2^*}} &\leq C\int_\Omega |\nabla u|^2 (u+\delta)^{2\theta-2} \\ &\leq M\delta^{2\theta+\beta-2}\int_\Omega g(u)|\nabla u|^2 + \int_\Omega [\Lambda u^p + f_0](u+\delta)^{2\theta+\beta-2},\end{aligned} \tag{5.6}$$

where $\mathcal{S}$ is the Sobolev embedding constant. Letting δ tend to zero, we get

$$C\mathcal{S}\left(\int_\Omega u^{2^*\theta}\right)^{\frac{2}{2^*}} \leq C\int_\Omega |\nabla u|^2 u^{2\theta-2} \leq \Lambda\int_\Omega u^{2\theta+\beta+p-2} + \int_\Omega f_0 u^{2\theta+\beta-2}. \tag{5.7}$$

Thanks to the choice of θ, we have $2^*\theta = (2\theta+\beta-2)s' = (2-\beta)s^{**}$. Thus, using Hölder inequality, and recalling that $s^{**}(2-\beta) \geq 2^* > 2 > 2-\beta > p$, we deduce

$$\begin{aligned}\left(\int_\Omega u^{(2-\beta)s^{**}}\right)^{\frac{2}{2^*}} &\leq C\left(\int_\Omega u^{(2-\beta)s^{**}}\right)^{\frac{2\theta+\beta+p-2}{(2-\beta)s^{**}}} \\ &+ C\,\|f_0\|_{L^s(\Omega)}\left(\int_\Omega u^{(2-\theta)s^{**}}\right)^{\frac{1}{s'}}.\end{aligned} \tag{5.8}$$

Now we point out that $\frac{2}{2^*} > \frac{1}{s'}$, since $s < \frac{N}{2}$, and that $\frac{2}{2^*} > \frac{2\theta+\beta+p-2}{(2-\beta)s^{**}}$, since $2-\beta > p$. Therefore, from (5.8) it follows the claim which allows to finish the proof. □

5.3 Global continua of solutions

Let $\mathcal{M}$ be the solution set for (P_λ), namely

$$\mathcal{M} = \{(\lambda, u) \in [0, +\infty) \times C(\overline{\Omega}) : u \text{ is a solution of } (P_\lambda)\}.$$

Continua of solutions in $\mathcal{M}$ are obtained in this section by using degree computations and Rabinowitz continuation theorem. In this way, we set (P_λ) as a fixed point problem for a compact operator.

Next result gives sufficient conditions to assure that solutions of (P_λ) are uniformly bounded from below by a positive constant in compact subsets. In fact, we can consider lower order terms of the form $h(u)|\nabla u|^2$ with

$$\begin{aligned} h \in C((0, +\infty)) \text{ is a nonnegative function, nonincreasing} \\ \text{in a neighborhood of zero with } \sqrt{h} \in L^1(0, 1), \end{aligned} \tag{5.9}$$

and data f_0 satisfying

(F) If $e^{-\int_1^s h(t)dt} \in L^1(0, 1)$ then f_0 is nonnegative and nontrivial. In other case f_0 satisfies (5.1).

Lemma 5.3.1 *Assume that h verifies (5.9) and $f_0 \in L^1(\Omega)$ satisfies (F). Then for each $\omega \subset\subset \Omega$ there exists a positive constant c_ω such that $z(x) \geq c_\omega > 0$ a.e. $x \in \omega$, for every $0 < z \in H_0^1(\Omega) \cap C(\Omega)$ supersolution of*

$$-\Delta z + h(z)|\nabla z|^2 = f_0 \quad \text{in } \Omega.$$

Proof: On the one hand, if f_0 satisfies (5.1) the proof can be found in (Arcoya et al., 2009b, Proposition 2.3). On the other hand, if $e^{-\int_1^s h(t)dt} \in L^1(0, 1)$, then f_0 is a general nonnegative and nontrivial function and we split the proof into two cases: when $h \in L^1(0, 1)$ we conclude by (Arcoya et al., 2011, Proposition 2.4), while if $h \notin L^1(0, 1)$ we follow the arguments of (Martínez-Aparicio, 2009, Theorem 3.1). □

Remark 5.3.2 *We notice that if we assume $h(s) = \frac{C}{s^\gamma}$ then $e^{-\int_1^s h(t)dt} \in L^1(0, 1)$ if and only if $\gamma < 1$ or if $\gamma = 1$ and $C < 1$.*

The following lemma ensures the compactness properties required later to deal with our topological approach.

Lemma 5.3.3 *Assume that $0 \lneqq f_0 \in L^q(\Omega)$ with $q > \frac{N}{2}$ and $\mu \in L^\infty(\Omega)$. Let assume that $0 < u_n \in H_0^1(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$\begin{cases} -\Delta u_n + \mu(x) g_n(u_n)|\nabla u_n|^2 = \lambda_n w_n^p + f_0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.10}$$

with $0 \leq \lambda_n$ bounded in $\mathbb{R}$, $0 \leq w_n$ bounded in $C(\overline{\Omega})$ and $0 \leq g_n$ a sequence of functions in $C((0, +\infty))$. Then, up to a subsequence, u_n is strongly convergent in $C(\overline{\Omega})$ to $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$. If, in addition, $\lambda_n \to \lambda$, $w_n \to w$ in $C(\overline{\Omega})$, $g_n(s) \to g(s)$ uniformly in $C([a, b])$ for every $0 < a < b < \infty$, $g_n(s) \leq h(s)$ for some h verifying (5.9) and f_0 satisfies (F), then u is a solution of problem

$$\begin{cases} -\Delta u + \mu(x) g(u)|\nabla u|^2 = \lambda w^p + f_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

Moreover, if the problem (5.11) admits a unique solution then the whole sequence u_n converges strongly to u in $C(\overline{\Omega})$.

Proof: Since the sequence $f_n := \lambda_n w_n^p + f_0$ is bounded in $L^q(\Omega)$ for some $q > N/2$, we can deduce, as in the proof of Lemma 5.2.1, or by using the Stampacchia technique in Stampacchia (1966) that $\|u_n\|_{L^\infty(\Omega)} \leq c_\infty$ for some positive constant c_∞. In addition, applying (Ladyzhenskaya and Ural'tseva, 1968, Theorem 6.1) we deduce that the sequence u_n is bounded in $C^{0,\alpha}(\overline{\Omega})$. Consequently, Ascolí-Arzelá Theorem assures that u_n possesses a subsequence converging in $C(\overline{\Omega})$. This conclude the first part of the lemma.

In order to prove the second part we observe that, since the sequence u_n is bounded in $H_0^1(\Omega)$ (arguing again as in the proof of Lemma 5.2.1, Step I) we can assume that u_n converges weakly to u in $H_0^1(\Omega)$. Now we prove that u is solution of problem (5.11), i.e. $u > 0$, $g(u)|\nabla u|^2 \in L^1(\Omega)$ and satisfies,

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega \mu(x) g(u)\, |\nabla u|^2 \varphi = \int_\Omega (\lambda w^p + f_0)\varphi, \tag{5.12}$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

By Lemma 5.3.1 given $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $u_n(x) \geq c_\omega$ a.e. $x \in \omega$ for every $n \in \mathbb{N}$. In particular, using that u_n converges strongly to u in $C(\overline{\Omega})$, we deduce $u > 0$ in Ω. Even more, the strong convergence of g_n to g in $C([c_\omega, c_\infty])$ assures that $g_n(u_n) \to g(u)$ a.e. in Ω .

Next, by the first part of the proof of Theorem 3.1 in Boccardo (2008) we have that $\mu(x)g(u)|\nabla u|^2 \in L^1(\Omega)$. We include the proof by convenience of the reader. Indeed, taking $\varphi = \frac{T_\epsilon(u_n)}{\epsilon}$ as test function in (5.10) and dropping the positive term coming from the principal part we get

$$\int_\Omega \mu(x) g_n(u_n)|\nabla u_n|^2 \frac{T_\epsilon(u_n)}{\epsilon} \leq \int_\Omega (\lambda_n w_n^p + f_0)\frac{T_\epsilon(u_n)}{\epsilon}.$$

Since $\displaystyle\int_\Omega (\lambda_n w_n^p + f_0) \leq C$, we obtain

$$\int_\Omega \mu(x) g_n(u_n)|\nabla u_n|^2 \frac{T_\epsilon(u_n)}{\epsilon} \leq C.$$

The limit as $\epsilon \to 0$ implies, using that $\lim\limits_{\epsilon\to 0} \frac{T_\epsilon(u_n)}{\epsilon} = 1$,

$$\int_\Omega \mu(x) g_n(u_n)|\nabla u_n|^2 \leq C\,.$$

Furthermore, the results of (Boccardo and Murat, 1992, Theorem 2.1) imply that (up to a subsequence) $\nabla u_n \to \nabla u$ strongly in $(L^q(\Omega))^N$ $(1 < q < 2)$, particularly, it converges almost everywhere inΩ . Then, the last inequality gives us after applying the Fatou lemma that

$$\int_\Omega \mu(x) g(u)|\nabla u|^2 \leq C\,,$$

which proves our claim.

To finish, following closely Boccardo (2008), we prove that u satisfies the equation (5.12). Since $\varphi = \varphi^+ + \varphi^-$, it is enough to prove (5.12) for every nonegative function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Furthermore, by density, it is sufficient to prove it when $0 \leq \varphi \in H_0^1(\Omega) \cap C_c(\Omega)$.

We divide the proof into two steps.

Step I. The function u satisfies

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega \mu(x) g(u)|\nabla u|^2 \phi \leq \int_\Omega \lambda w^p\, \phi + \int_\Omega f_0 \phi,$$

for all $0 \leq \phi \in H_0^1(\Omega) \cap C_c(\Omega)$. Indeed, since $\mu(x) g_n(u_n)|\nabla u_n|^2 \geq 0$, $g_n(u_n) \to g(u)$ a.e. $x \in \Omega$, ∇u_n converges weakly in $(L^2(\Omega))^N$ and a.e. $x \in \Omega$ to ∇u and w_n^p converges to w^p strongly in $L^2(\Omega)$, then we obtain the result taking a function $0 \leq \phi \in H_0^1(\Omega) \cap C_c(\Omega)$ as a test function in (5.10) and applying Fatou lemma.

Step II. The function u satisfies

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega \mu(x) g(u)|\nabla u|^2 \phi \geq \int_\Omega \lambda w^p \phi + \int_\Omega f_0 \phi,$$

for all $0 \leq \phi \in H_0^1(\Omega) \cap C_c(\Omega)$. We fix $0 \leq \phi \in H_0^1(\Omega) \cap C_c(\Omega)$ and we define the function

$$H(t) = \int_1^t M h\,(s)\, ds,$$

where $M = \|\mu\|_{L^\infty(\Omega)}$. Let call $\omega = \text{supp}\ \phi$ and observe, thanks to Lemma 5.3.1 there exists a positive constants c_ω such that $c_\omega \leq u_n$ in ω for every $n \in \mathbb{N}$. Moreover, the boundedness in $L^\infty(\Omega)$ of the sequence $\{u_n\}$ implies $u_n \leq c_\infty$. Therefore, for n big enough

$$|H(u) - H(u_n)| \leq M \int_{c_w}^{c_\infty} h(s) ds \leq M(c_\infty - c_\omega) \max_{s \in [c_\omega, c_\infty]} h(s) < \infty,$$

a.e. $x \in \omega$. In addition, one can similarly deduce, that

$$|H(u) - H(u_n)| \leq M|u - u_n| \max_{s \in [c_\omega, c_\infty]} h(s), \text{ a.e. } x \in \omega.$$

In particular, there exists a positive constant C_ϕ (depending only on ϕ) such that

$$e^{H(u)-H(u_n)} \phi \leq C_\phi.$$

Even more,

$$\nabla \left(e^{H(u)-H(u_n)} \phi \right) = \\ e^{H(u)-H(u_n)} \left(M\phi h(u) \nabla u - M \phi h(u_n) \nabla u_n + \nabla \phi \right) \in L^2(\Omega).$$

Thus, taking $\varphi = e^{H(u)-H(u_n)}\phi$ as a test function in (5.10), we get

$$\begin{aligned} \int_\Omega \nabla u_n \nabla \phi\, e^{H(u)-H(u_n)} + M \int_\Omega h(u) \nabla u \nabla u_n e^{H(u)-H(u_n)} \phi \\ - \int_\Omega \left(\lambda_n w_n^p + f_0\right) e^{H(u)-H(u_n)} \phi \\ = \int_\Omega \left(M h(u_n) - \mu(x) g_n(u_n)\right) |\nabla u_n|^2 e^{H(u)-H(u_n)}\, \phi\,. \end{aligned} \tag{5.13}$$

Next, we want to pass to the limit in the above expression. Observe that, since ∇u_n converges weakly in $(L^2(\Omega))^N$, we have

$$\int_\Omega \nabla u_n \nabla \phi\, e^{H(u)-H(u_n)} \longrightarrow \int_\Omega \nabla u \nabla \phi.$$

In addition, since the function $\phi h(u)$ and the sequence $\lambda_n w_n^p$ are bounded, we obtain using the Lebesgue Theorem

$$\int_\Omega h(u) \nabla u \nabla u_n e^{H(u)-H(u_n)}\, \phi \longrightarrow \int_\Omega h(u) |\nabla u|^2 \phi,$$

and

$$\int_\Omega \left(\lambda_n w_n^p + f_0\right) e^{H(u)-H(u_n)}\, \phi \longrightarrow \int_\Omega \left(\lambda w^p + f_0\right)\, \phi\,.$$

To finish, since $Mh(u_n) - \mu(x) g_n(u_n) \geq 0$, we deduce the inequality desired applying the Fatou Lemma in the right hand side of (5.13).

Summarizing Step I and Step II we conclude the proof. □

As can be observed, uniqueness of solution for (P_0) plays a fundamental role. In order to use the uniqueness result in (Arcoya et al., 2017, Theorem 1.1) we have to assume that the function g satisfies in addition that for every $\nu > 0$ there exists $\theta_\nu \geq 0$ and a nonnegative function $\tilde{g} \in C^1((0, +\infty))$ with $e^{-\int_1^s \tilde{g}(t) dt} \in L^1(0,1)$ such that for every $0 < s < \nu$ and for a.e. $x \in \Omega$

$$\begin{aligned} \theta_\nu \left[(\mu(x) g'(s) - \tilde{g}'(s)) + \tilde{g}(s)(\mu(x) g(s) - \tilde{g}(s)) \right] \\ \geq \left(\mu(x) g(s) - \tilde{g}(s) \right)^2. \end{aligned} \tag{5.14}$$

Remark 5.3.4 *In the case $\mu(x) = \mu$ for some positive constant μ, we can use the uniqueness result for problem (P_0) in Arcoya and Segura de León (2010) for functions $g \in L^1(0,1)$. Observe that, in that case, condition (5.14) is also trivially satisfied with $\tilde{g}(s) = \mu g(s)$. On the other hand, for a non-constant function $\mu(x)$, it is proved in Arcoya et al. (2017) that condition (5.14) is also satisfied in the case $g(s) = 1/s^\gamma$ with $\gamma < 1$. Moreover, in the case $g(s) = 1/s$, assuming in addition that $M < 1$, we can choose $\tilde{g}(s) = c/s$ for some $M < c < 1$ and we have that (5.14) is satisfied with $\theta_\nu \geq \frac{c}{1-c}$. Others particular cases that it will be used in the proof of Theorem 5.1.2 can be found in the Appendix.*

Finally, next result ensures existence of an unbounded, connected and closed subset of $\mathcal{M}$.

Theorem 5.3.5 *Assume (5.2), g satisfies (5.14), $g(s) \leq h(s)$ for some function h verifying (5.9) and $f_0 \in L^q(\Omega)$ with $q > N/2$ satisfies (F). Then there exists an unbounded continuum $\Sigma \subset \mathcal{M}$ such that $(0, u_0) \in \Sigma$, where u_0 is the unique solution of (P_0).*

Proof: Firstly, we observe the problem (P_0) admits a unique solution $0 < u \in H_0^1(\Omega) \cap C(\overline{\Omega})$. Indeed, the existence is due to Boccardo (2008) and Martínez-Aparicio (2009) if $0 \lneqq f_0$ and due to (Arcoya et al., 2009b, Theorem 1.1) if f_0 satisfies (5.1). Alternatively, the uniqueness is deduced using (Arcoya et al., 2017, Theorem 1.1).

Hence, we can define $K : [0,1] \times \mathbb{R} \times C(\overline{\Omega}) \to C(\overline{\Omega})$ by setting $K(t,\lambda, w)$ as the unique solution $0 < u \in C(\overline{\Omega})$ of the problem

$$\begin{cases} -\Delta u + t\mu(x)g(u)\,|\nabla u|^2 = \lambda^+(w^+)^p + f_0 & \text{in}\Omega\,, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.15}$$

for every $\lambda \in \mathbb{R}$, $t \in [0,1]$ and $w \in C(\overline{\Omega})$. With this notation problem (P_λ) can be rewritten as a fixed point problem, namely,

$$u = K_\lambda^1(u),$$

with $K_\lambda^t(u) = K(t,\lambda, u)$. Moreover, since g satisfies (5.14) Lemma 5.3.3 assures that K is compact and we can use Leray-Schauder degree to study (P_λ).

The result follows, as in Arcoya et al. (2011), from the Rabinowitz's Theorem (Rabinowitz, 1971, Theorem 3.2). We only have to compute the index of the solution u_0 and show that it is different from zero. Let us denote $u_t = K(1-t, 0, 0)$ i.e., u_t is the unique positive solution in $H_0^1(\Omega) \cap C(\overline{\Omega})$ of the problem

$$\begin{cases} -\Delta u + (1-t)\mu(x)g(u)\,|\nabla u|^2 = f_0 & \text{in}\Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Define the homotopy $J : [0,1] \times C(\overline{\Omega}) \to C(\overline{\Omega})$ by $J(t,w) = u_t$ for every $t \in [0,1]$ and $w \in C(\overline{\Omega})$. Observe that $J(t,w) = K(1-t,0,0)$ and thus, using Lemma 5.3.3, we have that J is compact. Moreover, $J(0,w) = u_0$ and $J(1,w) = (-\Delta)^{-1}(f_0(x))$. Therefore

$$i(K_0^1, u_0) = i(J(0,\cdot), u_0) = i(J(1,\cdot), u_1) = i((-\Delta)^{-1}(f_0(x)), u_1) = 1.$$

Consequently, since $i(K_0^1, u_0) = 1$, we conclude the proof by using Rabinowitz's theorem. □

5.4 Proof of the main results

In order to prove Theorem 5.1.1 and Theorem 5.1.2 we recall, for the convenience of the reader, the following definition and topological result (see Whyburn (1958)):

Definition 5.4.1 *Let $\{S_n\} \subset X$ be any infinite collection of point sets, not necessarily different. The set of all points x of our space X such that every neighborhood of x contains points of infinitely many sets of $\{S_n\}$ is called the* superior limit. *The set of all points y such that every neighborhood of y contains points of all but a finite number of the sets of $\{S_n\}$ is called the* inferior limit.

From the definiton, we have at once for any system $\{S_n\}$

$$\liminf S_n \subset \limsup S_n$$

Lemma 5.4.2 (Whyburn (1958)) *Let X be a metric space. If $\{S_n\}$ is a sequence of connected subsets of X such that $\bigcup S_n$ is relatively compact and $\liminf S_n$ is not empty, then the $\limsup S_n$ is connected.*

The trick, in the proof of Theorem 5.1.1 and Theorem 5.1.2 is to use Lemma 5.4.2 where S_n is a continuum of solutions of the following approximated problems

$$\begin{cases} -\Delta u + \mu(x)\,\dfrac{|\nabla u|^2}{(u+\frac{1}{n})^{\gamma} + (u+\frac{1}{n})^{\beta}} = \lambda u^p + f_0(x)\,, & \text{in}\Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{$Q_{n,\lambda}$}$$

for $n \in \mathbb{N}$ and $\gamma \leq \beta$.

Proof:[Proof of Theorem 5.1.1] First we deal with item i). We consider, for $n \in \mathbb{N}$, the approximated problems ($Q_{n,\lambda}$) and the idea is to use Theorem 5.3.5 with $h(s) = \frac{1}{s^\gamma + s^\beta}$ and f_0 satisfying (5.1). We observe that, under the assumption μ constant, the function $g_n(s) = \frac{1}{(s+1/n)^\gamma + (s+1/n)^\beta}$ satisfies (5.14) without restrictions in γ and β (recall Remark 5.3.4). Now by Theorem 5.3.5, there exists a continuumΣ_n in $[0,+\infty)\times C(\overline{\Omega})$ of positive solutions of ($Q_{n,\lambda}$) such that $(0,u_n)\in\Sigma_n$ with u_n solution of ($Q_{n,0}$). One can observe that by Lemma 5.2.1, one has $\mathrm{Proj}_{[0,\infty)}\Sigma_n = [0,\infty)$.

For obtaining the existence of an unbounded continuum Σ of solutions of (Q_λ) we apply the result of Lemma 5.4.2. Indeed, for every$\Lambda > 0$ we take $S_{n,\Lambda}$ the connected component of$\Sigma_n \cap \big([0,\Lambda]\times C(\overline{\Omega})\big)$ such that $(0,u_n)\in S_{n,\Lambda}$. Since Σ_n is unbounded and $\mathrm{Proj}_{[0,\infty)}\Sigma_n = [0,\infty)$, we deduce that $\mathrm{Proj}_{[0,\Lambda]}S_{n,\Lambda} = [0,\Lambda]$. Moreover, Lemma 5.3.3 with $\lambda_n = 0$ assures that, up to (not relabeled) subsequences, u_n converges strongly to u solution of (Q_0), which implies $(0,u)\in\liminf S_{n,\Lambda}$. Even more, given a sequence $(\lambda_m,u_m)\in\bigcup_{k\in\mathbb{N}} S_{k,\Lambda}$ we have that, for some $k_m\in\mathbb{N}$

$$\begin{cases} -\Delta u_m + \mu(x) g_{k_m}(u_m)|\nabla u_m|^2 = \lambda_m u_m^p + f_0(x) & \text{in} \quad \Omega \\ u_m = 0 & \text{on } \partial\Omega, \end{cases}$$

with $0\leq\lambda_m<\Lambda$ and $\|u_m\|_{L^\infty(\Omega)}\leq c_\Lambda$. As we can suppose that $k_m\to\infty$, then the first part of Lemma 5.3.3, with $w_n = u_m$, assures that (λ_m,u_m) admits a strongly convergent subsequence. In particular we deduce that $\bigcup_{k\in\mathbb{N}} S_{k,\Lambda}$ is relatively compact. We notice that if the sequence k_m is bounded then, up to a sequence, (λ_m,u_m) converges in $\bigcup_{k\in\mathbb{N}} S_{k,\Lambda}$. Now we can use Lemma 5.4.2 to deduce that $\Gamma_\Lambda = \limsup S_{n,\Lambda}$ is a continuum which, using the second part of Lemma 5.3.3, is contained in $\mathcal{M}$. In fact, since for every $n\in\mathbb{N}$ there exists $(\Lambda,u_n)\in S_{n,\Lambda}$, then we have that $\mathrm{Proj}_{[0,\Lambda]}\Gamma_\Lambda = [0,\Lambda]$. Furthermore, by construction,Γ_Λ is increasing in Λ and we can take $\Sigma = \bigcup_{n\in\mathbb{N}}\Gamma_n$. Observe that since $(0,u)\in\Gamma_n$ for every $n\in\mathbb{N}$ then$\Sigma\subset\mathcal{M}$ is a connected set in $[0,+\infty)\times C(\overline{\Omega})$. Moreover, $\mathrm{Proj}_{[0,\infty)}\Sigma = \bigcup_{n\in\mathbb{N}}[0,n] = [0,\infty)$.

Now we deal with the proof in the case of item *ii*). In this case, since $\mu(x)$ is constant and $\gamma<1$, we have that $g(s) = \frac{1}{s^\gamma+s^\beta}$ verifies (5.14) and (5.9). Thus, the unbounded continuum Σ of solutions of (Q_λ) is obtained from Theorem 5.3.5. In addition, the projection of Σ to the λ-axis has to be bounded, since we can use (Arcoya et al., 2011, Theorem 5.1) to deduce the existence of λ^*. Observe that $g\in L^1(0,+\infty)$ and

$$\begin{aligned} \lim_{s\to\infty}\frac{s^p}{\int_0^s e^{\int_r^s g(t)dt}dr} &= \lim_{s\to\infty}\frac{e^{\int_1^s\left(\frac{p}{t}-g(t)\right)dt}}{\int_0^s e^{-\int_1^r g(t)dt}dr} = \lim_{s\to\infty}\left(\frac{p}{s}-g(s)\right)s^p \\ &= \lim_{s\to\infty} s^{p-1}\left(p-\frac{s^{1-\beta}}{s^{\gamma-\beta}+1}\right) = \begin{cases} 1, & p=1, \\ \infty, & p>1. \end{cases} \end{aligned}$$

Therefore g verifies condition (1.6) in Arcoya et al. (2011). □

Proof:[Proof of Theorem 5.1.2] We observe that, for every $n \in \mathbb{N}$ fixed, the function $g_n(s) = \frac{1}{(s+1/n)^\gamma+(s+1/n)^\beta}$ satisfies (5.14) for $\beta \leq 1$ and general $\mu(x)$ (see *Cases 1-3* of Appendix). Thus by Theorem 5.3.5 there exists a continuum$\Sigma_{\ n}$ in $[0,+\infty) \times C(\overline{\Omega})$ of positive solutions of $(Q_{n,\lambda})$ such that $(0, u_n) \in \Sigma_n$ with u_n solution of $(Q_{n,0})$. We claim that $\mathrm{Proj}_{[0,\infty)}\Sigma_n = [0,\infty)$. Indeed, this is a consequence of the bound on the norm, for λ in bounded sets, of the solutions of $(Q_{n,\lambda})$. More precisely, this bound is obtained by means of Lemma 5.2.1, for $p < 1$ and Lemma 5.2.2 with $g_0(s) = \frac{1}{(s+1)^\gamma+(s+1)^\beta}$ for $p \geq 1$.

The existence of the unbounded continuum Σ with $\mathrm{Proj}_{[0,\infty)}\Sigma = [0,\infty)$ is deduced now arguing as in the proof of Theorem 5.1.1, observe that Lemma 5.3.3 with $\lambda_n = 0$ assures that, passing to subsequence, u_n converges strongly to u solution of (Q_0). To conclude, we note by Remark 5.3.4 the need to consider $M < 2$ in the case $\gamma = \beta = 1$. $\square$

Remark 5.4.3 *Thanks to* Case 4 *of Appendix it is worth stressing that the previous theorem could be extended to $\gamma = 1 < \beta$ if $M \leq 1$.*

Remark 5.4.4 *A simplest proof of Theorem 5.1.2 can be obtained in the particular case $\gamma = \beta \leq 1$. Indeed, the function $g(s) = 1/s^\gamma$ with $\gamma < 1$ satisfies condition (5.14) and this condition is also satisfied in the case $\gamma = 1$ if, in addition, we assume that $M < 1$ (see Remark 5.3.4). Consequently applying directly Theorem 5.3.5 for $\gamma < 1$ and Remark 5.3.2 for $\gamma = 1$ we can deduce the existence of an unbounded continuum Σ of solutions of (R_λ). Moreover, using Lemma 5.2.1 in the case $p < 1$ or Lemma 5.2.2, with $\beta = \gamma$ and $g_0(s) = 1/s^\gamma$, in the case $p > 1$, we can assure that $Proj_{[0,\infty)}\Sigma = [0,\infty)$, concluding the claim.*

Appendix

We devote this appendix to include the proof of Lemma 5.2.1 as well as the proof of (5.14) in some particular cases.

Proof:[Proof of Lemma 5.2.1] We choose suitable test functions taking into account (Arcoya et al., 2011, Lemma 2.1). We divide the proof into two steps:

STEP I. There exists a positive constant C such that, for every $g \geq 0$ and every solution u of (P_λ) with $0 < \lambda < \Lambda$, one has $\|u\|_{H_0^1(\Omega)} \leq C$.

Indeed, take $\varphi = u$ as a test function to obtain, dropping the positive term given by the lower order term, that

$$\int_\Omega |\nabla u|^2 \leq \int_\Omega \lambda u^{p+1} + \int_\Omega f_0\, u\,.$$

Since $p+1 < 2$, we can use Hölder and Sobolev inequalities in the right hand side to conclude

$$\int_\Omega |\nabla u|^2 \leq c\left(\left(\int_\Omega |\nabla u|^2\right)^{\frac{p+1}{2}} + \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}\right),$$

for some positive constant c depending only onΛ ,Ω, f_0 and p. This inequality give us Step I with C the unique positive solution of the equation $s^2 = c(s^{p+1} + s)$.

STEP II. There exists $C > 0$ such that, for every $g \geq 0$ and every solution u of (P_λ) with $0 < \lambda < \Lambda$, one has $\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^1(\Omega)}$.

Given $k > 1$, we take $\varphi = G_k(u)$ as a test function in (P_λ). Hence, dropping the positive lower order term and using Hölder's inequality in the right hand side, we have

$$\int_\Omega |\nabla G_k(u)|^2 \leq \int_{A_k} (\lambda + f_0)u^2 \leq \|\lambda + f_0\|_{L^q(\Omega)} \left(\int_{A_k} u^{2q'}\right)^{\frac{1}{q'}}, \tag{5.16}$$

where $A_k = \{x \in \Omega\ :\ u(x) > k\}$. Throughout the proof, C denotes different positive constants depending only onΛ, f_0, p and Ω.

Firstly, we estimate the right hand side of (5.16) using Hölder and Sobolev inequalities and the fact that $u = T_k(u) + G_k(u)$. Thus,

$$\begin{aligned}\left(\int_{A_k} u^{2q'}\right)^{\frac{1}{q'}} &\le C\left(k^{2q'}|A_k| + \int_{A_k} G_k(u)^{2q'}\right)^{\frac{1}{q'}} \\ &\le C\left(\int_{A_k} G_k(u)^{2q'}\right)^{\frac{1}{q'}} + C\,k^2|A_k|^{\frac{1}{q'}} \\ &\le C\left(\int_{\Omega} G_k(u)^{2^*}\right)^{\frac{2}{2^*}} |A_k|^{\frac{1}{q'}-\frac{2}{2^*}} + C\,k^2|A_k|^{\frac{1}{q'}} \\ &\le C\,|A_k|^{\frac{1}{q'}-\frac{2}{2^*}} \int_{\Omega} |\nabla G_k(u)|^2 + C\,k^2|A_k|^{\frac{1}{q'}}\,.\end{aligned}$$

Consequently, from (5.16) we have,

$$\int_{\Omega} |\nabla G_k(u)|^2 \le C\,\|\lambda + f_0\|_{L^q(\Omega)}\left(|A_k|^{\frac{1}{q'}-\frac{2}{2^*}} \int_{\Omega} |\nabla G_k(u)|^2 + k^2|A_k|^{\frac{1}{q'}}\right).$$

Using Step I we have that $k|A_k| \le \|u\|_{L^1(\Omega)} \le C$ and, since $\frac{1}{q'} - \frac{2}{2^*} > 0$, we can choose k big enough such that

$$\int_{\Omega} |\nabla G_k(u)|^2 \le C\,\|\lambda + f_0\|_{L^q(\Omega)}\,k^2|A_k|^{\frac{1}{q'}}\,.$$

Using Hölder and Sobolev inequalities and the above inequality we conclude

$$\int_{A_k} G_k(u) \le C\,k\,|A_k|^{1+\frac{1}{2q'}-\frac{1}{2^*}}\,,$$

which gives us the result applying (Hartman and Stampacchia, 1966, Lemma 7.2) (see also (Ladyzhenskaya and Ural'tseva, 1968, Lemma 5.1, pag 71)).

Summarizing Step I and Step II, we conclude the proof. □

Now we prove (5.14) for $g(s) = \frac{1}{(s+\frac{1}{n})^{\gamma}+(s+\frac{1}{n})^{\beta}}$ and $0 < \gamma \le \beta \le 1$ or $M < 1 = \gamma < \beta$.

Proof:[Proof of (5.14)] For every $\nu > 0$, we take $\tilde{g}(s) = h(s)g(s)$ for a convenient function $h \in C^1([0,+\infty))$, such that, for some $\theta_\nu \ge 0$

$$\begin{aligned}\theta_\nu\left(\left(\mu(x)\frac{g'(s)}{g^2(s)} - h(s)\frac{g'(s)}{g^2(s)} - \frac{h'(s)}{g(s)}\right) + h(s)(\mu(x) - h(s))\right) \\ \ge (\mu(x) - h(s))^2, \quad \forall s < \nu\,.\end{aligned}$$

Observe that this inequality is trivially satisfied if $h(s) = \mu(x)$ and $h'(s) \le 0$ while, in other case, it is equivalent to prove that the function

$$\sigma(x,s) \equiv \frac{(\mu(x) - h(s))\left(h(s) + \frac{g'(s)}{g^2(s)}\right) - \frac{h'(s)}{g(s)}}{(\mu(x) - h(s))^2}$$

is bounded from below by a positive constant. We point out that

$$\frac{g'(s)}{g^2(s)} = -\gamma\left(s + \frac{1}{n}\right)^{\gamma-1} - \beta\left(s + \frac{1}{n}\right)^{\beta-1}.$$

Now we choose the function $h(s)$ based on the different values of γ and β.
Case 1. $\gamma \le \beta < 1$.

In this case we take $h(s) = -g'(s)/g^2(s) = \gamma(s+\frac{1}{n})^{\gamma-1} + \beta(s+\frac{1}{n})^{\beta-1}$. Thus

$$h'(s) = \gamma(\gamma-1)\left(s + \frac{1}{n}\right)^{\gamma-2} + \beta(\beta-1)\left(s + \frac{1}{n}\right)^{\beta-2} < 0.$$

In particular, we have that $\sigma(x,s)$ is given by

$$\frac{\left(\gamma(1-\gamma)\left(s+\frac{1}{n}\right)^{\gamma-2}+\beta(1-\beta)\left(s+\frac{1}{n}\right)^{\beta-2}\right)\left(\left(s+\frac{1}{n}\right)^{\gamma}+\left(s+\frac{1}{n}\right)^{\beta}\right)}{\left(\mu(x)-\gamma\left(s+\frac{1}{n}\right)^{\gamma-1}-\beta\left(s+\frac{1}{n}\right)^{\beta-1}\right)^2}$$
$$=\frac{\left(\gamma(1-\gamma)\left(s+\frac{1}{n}\right)^{\gamma-\beta}+\beta(1-\beta)\right)\left(\left(s+\frac{1}{n}\right)^{\gamma-\beta}+1\right)}{\left(\mu(x)\left(s+\frac{1}{n}\right)^{1-\beta}-\gamma\left(s+\frac{1}{n}\right)^{\gamma-\beta}-\beta\right)^2}$$

We conclude by taking into account that this function (which may take infinite values) only vanishes for $s\to+\infty$.
Case 2. $\gamma<\beta=1$.
In this case we take again $h(s)=-g'(s)/g^2(s)=\gamma(s+\frac{1}{n})^{\gamma-1}+1$. Thus

$$h'(s)=\gamma(\gamma-1)\left(s+\frac{1}{n}\right)^{\gamma-2}<0.$$

In particular, we have

$$\sigma(x,s)=\frac{\left(\gamma(1-\gamma)\left(s+\frac{1}{n}\right)^{\gamma-2}\right)\left(\left(s+\frac{1}{n}\right)^{\gamma}+\left(s+\frac{1}{n}\right)\right)}{\left(\mu(x)-\gamma\left(s+\frac{1}{n}\right)^{\gamma-1}-1\right)^2}.$$

We conclude, as before, by taking into account that this function only vanishes for $s\to+\infty$.
Case 3. $\gamma=\beta=1$.
In this case we can choose $h(s)=2+\frac{1}{1+3ns}$ and we have

$$\sigma(x,s)=\frac{(\mu(x)-2-\frac{1}{1+3ns})\frac{1}{1+3ns}+\frac{6n(s+1/n)}{(1+3ns)^2}}{\left(\mu(x)-2-\frac{1}{1+3ns}\right)^2}$$
$$>\frac{\frac{-3-6ns}{(1+3ns)^2}+\frac{6n(s+1/n)}{(1+3ns)^2}}{\left(\mu(x)-2-\frac{1}{1+3ns}\right)^2}=\frac{3}{((\mu(x)-2)(1+3ns)-1)^2}$$

We conclude again using that this function only vanishes for $s\to+\infty$.
Case 4. $M\le 1=\gamma<\beta$.
In this case we can choose $h(s)=1$ and, since $\frac{g'(s)}{g^2(s)}=-1-\beta(s+\frac{1}{n})^{\beta-1}$, we have

$$\sigma(x,s)=\frac{1-1-\beta(s+\frac{1}{n})^{\beta-1}}{\mu(x)-1}=\frac{\beta(s+\frac{1}{n})^{\beta-1}}{1-\mu(x)}\ge\frac{\beta}{n^{\beta-1}(1-\mu(x))}.$$

$\square$

Chapter 6

The Gelfand problem for the 1-homogeneous p-laplacian

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DOI: 10.1515/anona-2016-0233

Abstract

In this paper we study the existence of viscosity solutions to the Gelfand problem for the 1-homogeneous p-laplacian in a bounded domain $\Omega \subset \mathbb{R}^N$, that is, we deal with $-\frac{1}{p-1}|\nabla u|^{2-p}\,\mathrm{d}iv\left(|\nabla u|^{p-2}\nabla u\right) = \lambda\, e^u$ in Ω with $u = 0$ on $\partial\Omega$. For this problem we show that, for $p \in [2, \infty]$, there exists a positive critical value $\lambda^* = \lambda^*(\Omega, N, p)$ such that:

- If $\lambda < \lambda^*$, the problem admits a minimal positive solution w_λ.
- If $\lambda > \lambda^*$, the problem admits no solution.

Moreover, the branch of minimal solutions $\{w_\lambda\}$ is increasing with λ.

In addition, using degree theory, for fixed p we show that there exists an unbounded continuum of solutions that emanates from the trivial solution, $u = 0$ with $\lambda = 0$ and for a small fixed λ we also obtain a continuum of solutions with $p \in [2, \infty]$.

6.1 Introduction

This paper deals with the Gelfand problem corresponding to the 1-homogeneous p-Laplacian,

$$\begin{cases} -\Delta_p^N u = \lambda\, e^u\,, & \text{in}\Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda,p}$}$$

where$\Omega \subset \mathbb{R}^N$ is a regular bounded domain, $p \in [2, \infty]$ and the operatorΔ_p^N is the called 1-homogeneous p-laplacian (also called the normalized $p-$Laplacian) defined, for $p < \infty$, by

$$\Delta_p^N u := \frac{1}{p-1} |\nabla u|^{2-p}\, div\left(|\nabla u|^{p-2} \nabla u\right) = \alpha \Delta u + \beta \Delta_\infty u, \tag{6.1}$$

being $\alpha = 1/(p-1)$, $\beta = (p-2)/(p-1)$ and for $p = \infty$

$$\Delta_\infty u \equiv \Delta_\infty^N u = \frac{\nabla u}{|\nabla u|} \cdot \left(D^2 u \frac{\nabla u}{|\nabla u|}\right)$$

the 1-homogeneous infinity laplacian. This kind of elliptic operators for $2 \le p < \infty$ have 1 and $1/(p-1)$ as ellipticity constants, hence there is a lack of uniform ellipticity when we let $p \to \infty$. Therefore, the theory of uniformly elliptic operators can not be applied. Moreover, we remark the lack of variational structure and differentiability of this operator, in contrast to what happens with the classical p-laplacian. This fact implies that the theory concerning "stable solutions" can not be applied to our problem.

Note that the 1-homogeneous p-laplacian is a convex combination of laplacian and infinity laplacian operators since $\alpha + \beta = 1$. Moreover, $\alpha = 1$, $\beta = 0$ if $p = 2$ and $\alpha \to 0$, $\beta \to 1$ as $p \to \infty$. This operator appears when one considers Tug-of-War games with noise, see Manfredi et al. (2012); Peres and Sheffield (2008); Peres et al. (2009), where the Poisson problem is studied. Moreover, the sublinear problem and the eigenvalue problem associated to the 1-homogeneous p-Laplacian, namely, the problem with right-hand side λu^q for $0 < q \le 1$, has been studied in Martínez-Aparicio et al. (2014a) and Martínez-Aparicio et al. (2014b). In view of these two references it seems natural to deal with the superlinear case (that for this operator is challenging due to the fact that there is no variational structure and no Sobolev spaces framework).

Concerning the Gelfand problem, since it is a classical problem, there is a large number of references. We quote: Arcoya et al. (2014); Cabré and Capella (2006); Cabré and Sanchón (2013); Gel'fand (1963); Molino (2016) and references therein for the Laplacian and Ros-Oton (2014) for the fractional Laplacian.

Our first result for this problem reads as follows:

Theorem 6.1.1 *For every fixed $p \in [2, +\infty]$ there exists a positive extremal parameter $\lambda^* = \lambda^*(\Omega, N, p)$ such that:*

- *If $\lambda < \lambda^*$, problem ($P_{\lambda,p}$) admits a minimal positive solution w_λ.*
- *If $\lambda > \lambda^*$, problem ($P_{\lambda,p}$) has no positive solution.*

Moreover, the branch of minimal solutions $\{w_\lambda\}$ is increasing with λ. Even more, in the case of a ball, $\Omega = B_r$, the minimal solution is radial.

One of our main tools for the proof of this result is a comparison principle (that we prove here) adapted to the particular structure of the 1-homogeneous p-laplacian (see Theorem 6.3.3). This result generalizes previous ones in Barles and Busca (2001); Martínez-Aparicio et al. (2014a). We believe that this comparison principle is of independent interest.

Using arguments from degree theory we can obtain the following result concerning solutions that are not necessarily the minimal one. Remark that we even obtain a continuum of solutions for a fixed

p using λ as parameter or for fixed λ small taking p as parameter. More precisely, fixed p we denote by $\mathscr{S}_p$ to the solution set, i.e.

$$\mathscr{S}_p = \{(\lambda, u) \in [0, \lambda^*(\Omega, N, p)] \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (P_{\lambda,p})\}.$$

Analogously, fixed λ we denote by $\mathcal{S}_\lambda$ to the solution set

$$\mathcal{S}_\lambda = \{(p, u) \in [2, \infty] \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (P_{\lambda,p})\}.$$

Theorem 6.1.2 *For every fixed $p \in [2, \infty]$, there exists an unbounded continuum of solutions $\mathcal{C} \subset \mathscr{S}_p$ that emanates from $\lambda = 0, u = 0$, i.e. $(0, 0) \in \mathcal{C}$. Moreover, for every fixed $\lambda < \lambda_0 = \min\{\lambda^*(\Omega, N, 2), (2d^2e)^{-1}\}$, where d is the diameter of Ω, there exists a continuum of solutions $\mathcal{D} \subset \mathcal{S}_\lambda$, with $Proj_{[2,+\infty]}\mathcal{D} = [2, +\infty]$ and $\|u\|_\infty \le 1, \forall (p, u) \in \mathcal{D}$.*

We remark that, as a consequence of the previous theorem there is a lower bound for the extremal parameter found in Theorem 6.1.1, $0 < \lambda_0 \le \lambda^*(\Omega, N, p)$ for every $p \in [2, +\infty]$.

The use of degree theory is new for this kind of operators. Here we perform homotopies both in the parameters λ and p. The deformation in p is needed in order to start the argument with the trivial solution $u = 0$ for the problem with $p = 2$ and $\lambda = 0$, $\Delta u = 0$, that is known to have degree 1. Note that, due to the non smoothness of the operator, there is a nontrivial difficulty in the computation of the degree of the trivial solution to$\Delta_p^N u = 0$. Also note that the necessary compactness is nontrivial, we rely here in results from Charro et al. (2013).

Remark 6.1.3 *Our results can be generalized to handle the equation*

$$-\Delta_p^N u = \lambda f(u),$$

with a general continuous nonlinearity f that verifies

$$f(0) > 0, \qquad f(s) \text{ is increasing} \qquad \text{and} \qquad \frac{f(s)}{s} \ge k > 0.$$

To simplify the exposition we just write the details for $f(s) = e^s$ and we make a comment at the end of the paper on how to deal with this general case.

The rest of the paper is organized as follows: in Section 6.2 we collect some preliminaries and state the definition of a viscosity solution to our equation, in Section 6.3 we prove our comparison result, and finally in Sections 6.4 and 6.5 we prove our main results concerning the Gelfand problem.

6.2 Preliminaries

In this section we introduce the notion of viscosity solution for problem $(P_{\lambda,p})$. Actually we give the definition for a more general family of nonlinearities and we consider the following boundary value problem:

$$\begin{cases} -\Delta_p^N u = \lambda\, f(x, u)\,, & \text{in}\Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Since the normalized infinity laplacian,$\Delta_\infty u = \frac{\nabla u}{|\nabla u|} \cdot \left(D^2 u \frac{\nabla u}{|\nabla u|}\right)$ is not well defined at the points where $|\nabla u(x)| = 0$, we have to use the semicontinuous envelopes of the operator

$$(\xi, X) \mapsto \frac{\xi}{|\xi|} \cdot \left(X \frac{\xi}{|\xi|}\right), \qquad \xi \in \mathbb{R}^N,\ X \in \mathbb{S}_N$$

in order to define viscosity solutions for problem (6.2) (see Chen et al. (1991); Crandall et al. (1992)). To this end, we denote the largest and the smallest eigenvalue for $A \in \mathbb{S}_N$ by $M(A)$ and $m(A)$, respectively. That is,

$$M(A) = \max_{|\eta|=1} \eta \cdot (A\eta)\,, \qquad m(A) = \min_{|\eta|=1} \eta \cdot (A\eta)\,.$$

Let us denote by $USC(\omega)$ the set of upper semicontinuous functions $u : \omega \subset \mathbb{R}^N \to \mathbb{R}$, and we denote by $LSC(\omega)$ the set of lower semicontinuous functions.

Definition 6.2.1

1. *$\underline{u} \in USC(\Omega)$ is a viscosity subsolution of the equation $-\Delta_p^N u = \lambda\, f(x,u)$ if whenever $x_0 \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ such that $\varphi(x_0) = \underline{u}(x_0)$ and $\varphi - \underline{u} > 0$ in $\Omega \setminus \{x_0\}$, then*

$$\begin{cases} -\Delta_p^N \varphi(x_0) \le \lambda\, f(x_0, \varphi(x_0))\,, & \text{if } \nabla\varphi(x_0) \neq 0, \\ -\alpha\Delta\varphi(x_0) - \beta M\left(D^2\varphi(x_0)\right) \le \lambda\, f(x_0,\varphi(x_0))\,, & \text{if } \nabla\varphi(x_0) = 0. \end{cases} \tag{6.3}$$

   If, in addition, $\underline{u} \in USC(\overline{\Omega})$ and $\underline{u} \le 0$ on $\partial\Omega$ we say that $\underline{u}$ is a subsolution of (6.2).

2. *$\overline{u} \in LSC(\Omega)$ is a viscosity supersolution of the equation $-\Delta_p^N u = \lambda\, f(x,u)$ if whenever $x_0 \in \Omega$ and $\psi \in \mathcal{C}^2(\Omega)$ such that $\psi(x_0) = \overline{u}(x_0)$ and $\overline{u} - \psi > 0$ in $\Omega \setminus \{x_0\}$, then*

$$\begin{cases} -\Delta_p^N \psi(x_0) \ge \lambda\, f(x_0, \psi(x_0))\,, & \text{if } \nabla\psi(x_0) \neq 0, \\ -\alpha\Delta\psi(x_0) - \beta m\left(D^2\psi(x_0)\right) \ge \lambda\, f(x_0,\psi(x_0))\,, & \text{if } \nabla\psi(x_0) = 0. \end{cases} \tag{6.4}$$

   If, in addition, $\overline{u} \in LSC(\overline{\Omega})$ and $\overline{u} \ge 0$ on $\partial\Omega$ we say that $\overline{u}$ is a supersolution of (6.2).

3. *A continuous function $u : \overline{\Omega} \to \mathbb{R}$ is a viscosity solution of (6.2) if it is both, a viscosity supersolution and a viscosity subsolution.*

In what follows, φ stands for test functions whose graph touches the graph of u from above, and ψ denotes test functions whose graph touches the graph of u from below. Notice that the inequalities $\varphi - \underline{u} > 0$ and $\overline{u} - \psi > 0$ have to be satisfied in a neighborhood of $\{x_0\}$ and not necessarily in the whole$\Omega\ \setminus \{x_0\}$.

Remark 6.2.2 *Let u be a classical subsolution of (6.2), that is, $u \in \mathcal{C}^2(\overline{\Omega})$, $u \le 0$ on $\partial\Omega$ and for every $x \in \Omega$ satisfies*

$$\begin{cases} -\Delta_p^N u(x) \le \lambda\, f(x, u(x))\,, & \text{if } \nabla u(x) \neq 0, \\ -\alpha\Delta u(x) - \beta M\left(D^2 u(x)\right) \le \lambda\, f(x,u(x))\,, & \text{if } \nabla u(x) = 0. \end{cases}$$

Then u is a viscosity subsolution. Indeed, let $\varphi \in \mathcal{C}^2(\Omega)$ be such that $\varphi(x_0) = u(x_0)$ and $\varphi - u > 0$ in $\Omega \setminus \{x_0\}$, then $\nabla(\varphi - u)(x_0) = 0$ and $D^2(\varphi - u)(x_0)$ is a positive definite $N \times N$ matrix. Therefore,

$$\eta \cdot \left(D^2\varphi(x_0)\eta\right) \ge \eta \cdot \left(D^2 u(x_0)\right)\eta, \eta \quad \in \mathbb{R}^N,$$

and $tr\left(D^2\varphi(x_0)\right) \ge tr\left(D^2 u(x_0)\right)$ (i.e. $\Delta\varphi(x_0) \ge \Delta u(x_0)$). Hence, if $\nabla u(x_0) \neq 0$, we obtain

$$-\alpha\Delta\varphi(x_0) - \beta\Delta_\infty^N \varphi(x_0) \le -\alpha\Delta u(x_0) - \beta\Delta_\infty^N u(x_0) \le \lambda f(x_0,\varphi(x_0)).$$

Finally, using that $M(D^2\varphi(x_0)) \ge M(D^2 u(x_0))$ for $\nabla u(x_0) = 0$, it follows that u is a viscosity subsolution. We can proceed analogously with the supersolution case. Thus, classical solutions of (6.2) are solutions in the viscosity sense.

Let us observe that $\underline{u} \in USC(\Omega)$ is a viscosity subsolution of $-\Delta_p^N u = \lambda f(x,u)$ if

$$\begin{cases} -\alpha\,\mathrm{tr}(X) - \beta\dfrac{\eta}{|\eta|}\cdot\left(X\dfrac{\eta}{|\eta|}\right) \le \lambda f(x_0,\varphi(x_0)), & \text{if } \eta \neq 0, \\ -\alpha\,\mathrm{tr}(X) - \beta M(X) \le \lambda f(x_0,\varphi(x_0)), & \text{if } \eta = 0. \end{cases} \tag{6.5}$$

whenever $x_0 \in \Omega$ and $(\eta, X) = \big(\nabla\varphi(x_0), D^2\varphi(x_0)\big) \in \mathbb{R}^N \times \mathbb{S}_N$ for some $\varphi \in \mathcal{C}^2(\Omega)$ such that $\varphi(x_0) = \underline{u}(x_0)$ and $\varphi - \underline{u} > 0$ in $\Omega \setminus \{x_0\}$. Thus, as in Crandall et al. (1992), we can characterize viscosity sub and supersolutions using the concept of upper and lower semijets in the sense of the following definition.

Definition 6.2.3 *For $u \in USC(\mathcal{O})$ and $x_0 \in \mathcal{O}$ we define the upper semijet*

$$J^{2,+}_{\mathcal{O}}u(x_0) = \big\{\big(\nabla\varphi(x_0), D^2\varphi(x_0)\big) : \varphi \in \mathcal{C}^2(\mathcal{O}),\ \varphi(x_0) = u(x_0) \text{ and } \varphi - u \text{ has a local minimum at } x_0\big\}.$$

Analogously, for $u \in LSC(\mathcal{O})$ and $x_0 \in \mathcal{O}$, we define the lower semijet

$$J^{2,-}_{\mathcal{O}}u(x_0) = \big\{\big(\nabla\psi(x_0), D^2\psi(x_0)\big) : \psi \in \mathcal{C}^2(\mathcal{O}),\ \psi(x_0) = u(x_0) \text{ and } \psi - u \text{ has a local maximum at } x_0\big\}.$$

Finally, we introduce the sets $\overline{J}^{2,+}_{\mathcal{O}}u(x_0)$, $\overline{J}^{2,-}_{\mathcal{O}}u(x_0)$ as follows: $(p, X) \in \overline{J}^{2,+}_{\mathcal{O}}u(x_0)$ if there exist $x_n \in B_r(x_0)$ and $(p_n, X_n) \in J^{2,+}_{\mathcal{O}}u(x_n)$, such that $u(x_n) \to u(x_0)$ and $(x_n, p_n, X_n) \to (x_0, p, X)$ as $n \to \infty$. An analogous statement holds for $\overline{J}^{2,-}_{\mathcal{O}}u(x_0)$.

Remark 6.2.4 *It is clear that $\underline{u} \in USC(\Omega)$ is a viscosity subsolution of $-\Delta_p^N u = \lambda f(x,u)$ if (6.5) is verified for every $(\eta, X) \in J^{2,+}_{\Omega}\underline{u}(x_0)$. Moreover, if $\underline{u}$ is a subsolution then (6.5) is verified for every $(\eta, X) \in \overline{J}^{2,+}_{\Omega}\underline{u}(x_0)$. The analogous statement holds for supersolutions.*

Remark 6.2.5 *In Imbert et al. (2016) a parabolic equation of the form*

$$u_t = |\nabla u|^{\gamma}(\Delta u + (p-2)\Delta_\infty^N u)$$

was studied using viscosity solutions. The definition of viscosity solutions given there (inspired in Ohnuma and Sato (1997)) differs from ours. In fact, in Imbert et al. (2016) the authors restrict the class of test functions in order to give sense to the equation when the gradient vanishes (note that this parabolic problem can be singular or degenerate according to the value of γ). In our definition we do not restrict the test functions but we give a meaning to $\Delta_\infty^N u$ in terms of the largest and the smallest eigenvalue of D^2u at points where the gradient vanishes. With our definition we can prove a comparison principle in the next section.

6.3 Comparison principle and Uniqueness

In this section, we start giving sufficient conditions on f to prove a comparison principle and hence obtain uniqueness for (6.2).

Definition 6.3.1 *Given a positive function $h \in C^1(0, +\infty)$ such that $h \in L^1(0,1)$ and $h'(s)/h^2(s)$ is nondecreasing, we say that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the h-decreasing condition if for every $x \in \Omega$*

$$h(s)f(x,s) \text{ is decreasing respect to } s. \tag{6.6}$$

Remark 6.3.2 *Observe that if $f(x,s)=f_0(x)>0$, that is, f does not depend on s, then f satisfies the h-decreasing condition for $h(s)=\frac{1}{s^q}$ for any $0<q<1$. In addition, when $f(x,s)=f_0(x)s^q>0$ for some $0\le q<1$ then f satisfies the h-decreasing condition for $h(s)=\frac{1}{s^{q+\varepsilon}}$ for any $0<\varepsilon<1-q$. Moreover, taking h a decreasing function, we obtain that any function $0<f\in\mathcal{C}^1(\Omega\times\mathbb{R})$ nonincreasing with respect to s also satisfies the h-decreasing condition (since $h'(s)f(x,s)+h(s)f'_s(x,s)<0$ in this case).*

Theorem 6.3.3 *Assume that $0<f\in\mathcal{C}(\Omega\times\mathbb{R})$ satisfies the h-decreasing condition. Let $\underline{u},\overline{u}\in C(\overline{\Omega})$ be respectively a sub and a supersolution of $-\Delta_p^N u=f(x,u)$ such that $\overline{u}>0$ in Ω and $\underline{u}\le\overline{u}$ on $\partial\Omega$. Then $\underline{u}\le\overline{u}$ in $\overline{\Omega}$.*

Proof: We argue by contradiction following closely the ideas in Crandall et al. (1992). Suppose that $\Omega^+=\{x\in\overline{\Omega}\,:\,\underline{u}(x)>\overline{u}(x)\}$ is non empty. Let

$$H(s)=\int_0^s h(t)dt,$$

for $s\ge 0$. By hypothesis $\underline{u}\le\overline{u}$ on $\partial\Omega$ and using that $\underline{u},\overline{u}\in C(\overline{\Omega})$ we have that there exists $\hat{x}\in\Omega^+$ with

$$H(\underline{u}(\hat{x}))-H(\overline{u}(\hat{x}))=\sup_{x\in\Omega^+}H(\underline{u}(x))-H(\overline{u}(x))>0.$$

SinceΩ^+ is an open set we can take $\hat{\Omega}$, an open neighborhood of $\hat{x}$, such that $\overline{\hat{\Omega}}\subset\Omega^+$. Now, let $\underline{w}$ and $\overline{w}$ be the positive functions defined for $x\in\hat{\Omega}$ by

$$\underline{w}(x)=H(\underline{u}(x))\qquad\text{and}\qquad\overline{w}(x)=H\left(\overline{u}(x)\right).$$

Clearly $\underline{w},\overline{w}\in C(\overline{\hat{\Omega}})$ and

$$\underline{w}(x)>\overline{w}(x)>0,\qquad x\in\hat{\Omega}. \tag{6.7}$$

Now, we claim that $\underline{w},\overline{w}$ are a sub and a supersolution (in the viscosity sense) of the equation

$$-\Delta_p^N w+\frac{h'(H^{-1}(w))}{h^2(H^{-1}(w))}|\nabla w|^2=h(H^{-1}(w))f(x,H^{-1}(w)),\qquad\text{in }\hat{\Omega}. \tag{Q}$$

Indeed, we proceed to show that $\underline{w}$ is subsolution (the fact that $\overline{w}$ is a supersolution can be proved in the same way). For every $x_0\in\hat{\Omega}$ we take $\varphi\in\mathcal{C}^2(\hat{\Omega})$ with $\varphi(x_0)=\underline{w}(x_0)$ and $\varphi(x)>\underline{w}(x)$ for every $x\in\hat{\Omega}\setminus\{x_0\}$. If $\nabla\varphi(x_0)\neq 0$ and we take $\tilde{\varphi}=H^{-1}(\varphi)$, then it is easy to check that

$$\begin{aligned}
&-\Delta_p^N\varphi(x_0)+\frac{h'(H^{-1}(\varphi(x_0)))}{h^2(H^{-1}(\varphi(x_0)))}|\nabla\varphi(x_0)|^2\\
&\quad=-\alpha\Delta\varphi(x_0)-\beta\Delta_\infty\varphi(x_0)+h'(\tilde{\varphi}(x_0))|\nabla\tilde{\varphi}(x_0)|^2\\
&\quad=-\alpha\Delta\tilde{\varphi}(x_0)h(\tilde{\varphi}(x_0))-\alpha h'(\tilde{\varphi}(x_0))|\nabla\tilde{\varphi}(x_0)|^2-\beta\Delta_\infty\tilde{\varphi}(x_0)h(\tilde{\varphi}(x_0))\\
&\qquad-\beta h'(\tilde{\varphi}(x_0))|\nabla\tilde{\varphi}(x_0)|^2+h'(\tilde{\varphi}(x_0))|\nabla\tilde{\varphi}(x_0)|^2\\
&\quad=-\Delta_p^N\tilde{\varphi}(x_0)h(\tilde{\varphi}(x_0)).
\end{aligned}$$

Now, taking into account that $\tilde{\varphi}(x_0)=\underline{u}(x_0)$ and $(\tilde{\varphi}-\underline{u})(x)>0$ in $\hat{\Omega}\setminus\{x_0\}$, it follows that $\tilde{\varphi}$ is a test function touching from above u at x_0. Thus, since $\underline{u}$ is subsolution of $-\Delta_p^N u=f(x,u)$ we get

$$-\Delta_p^N\tilde{\varphi}(x_0)\le f(x_0,H^{-1}(\tilde{\varphi}(x_0))).$$

Consequently

$$-\Delta_p^N\varphi(x_0)+\frac{h'(H^{-1}(\varphi(x_0)))}{h^2(H^{-1}(\varphi(x_0)))}|\nabla\varphi(x_0)|^2\le h(H^{-1}(\varphi(x_0)))f(x_0,H^{-1}(\varphi(x_0))).$$

In case $\nabla\varphi(x_0)=0$, since $\nabla\tilde\varphi(x_0)=0$ and $D^2\varphi(x_0)=h(\tilde\varphi(x_0))D^2\tilde\varphi(x_0)$, we have

$$\begin{aligned}-\alpha\Delta\varphi(x_0)-\beta M\left(D^2\varphi(x_0)\right)&=-\alpha\Delta\tilde\varphi(x_0)h(\tilde\varphi(x_0))-\beta M\left(D^2\tilde\varphi(x_0)\right)h(\tilde\varphi(x_0))\\&\le h(H^{-1}(\varphi(x_0)))f(x_0,H^{-1}(\varphi(x_0))).\end{aligned}$$

Therefore, we conclude that $\underline{w}$ is a subsolution of problem (Q), which was our claim.

Now, consider the sequence of functions

$$\Psi_n(x,y)=\underline{w}(x)-\overline{w}(y)-\frac{n}{4}|x-y|^4,\ \ (x,y)\in\overline{\hat\Omega}\times\overline{\hat\Omega},\quad n\in\mathbb{N}.$$

For every $n\in\mathbb{N}$, let $(x_n,y_n)\in\overline{\hat\Omega}\times\overline{\hat\Omega}$ be such that

$$\Psi_n(x_n,y_n)=\sup_{\overline{\hat\Omega}\times\overline{\hat\Omega}}\Psi_n(x,y),$$

we note that$\Psi_n(x_n,y_n)$ is finite since $\underline{w}-\overline{w}$ is continuous and $\overline{\hat\Omega}$ is compact. Moreover$\Psi_n(x_n,y_n)\ge\Psi(x,x)=\underline{w}(x)-\overline{w}(x)>0$. Furthermore, we can assume that $x_n,y_n\to\hat x,\hat y$, $\underline{w}(x_n)\to\underline{w}(\hat x)$ and $\overline{w}(y_n)\to\overline{w}(\hat y)$ as $n\to\infty$ and that $\hat x=\hat y$ (see (Crandall et al., 1992, Lemma 3.1 and Proposition 3.7)). Next, by (Crandall et al., 1992, Theorem 3.2), there exist $X_n,Y_n\in\mathbb{S}_N$ satisfying

(i) $X_n\le Y_n$,

(ii) $(\eta_n,X_n)\in\overline{J}^{2,+}_{\hat\Omega}(\underline{w}(x_n))$, $(\eta_n,Y_n)\in\overline{J}^{2,-}_{\hat\Omega}(\overline{w}(y_n))$,

(iii) $X_n\le 0\le Y_n$, for $x_n=y_n$,

where $\eta_n=n|x_n-y_n|^2(x_n-y_n)$.

Hence, if $x_n\neq y_n$, having in mind that $\underline{w}$ and $\overline{w}$ are sub and supersolution of (Q) and using Remark 6.2.4, we obtain that

$$\begin{aligned}&h(H^{-1}(\overline{w}(y_n)))f(y_n,H^{-1}(\overline{w}(y_n)))\\&\quad\le-\alpha\,\mathrm{tr}(Y_n)-\beta\,\frac{\eta_n}{|\eta_n|}\cdot\left(Y_n\frac{\eta_n}{|\eta_n|}\right)+\frac{h'(H^{-1}(\overline{w}(y_n)))}{h^2(H^{-1}(\overline{w}(y_n)))}|\eta_n|^2\\&\quad\le-\alpha\,\mathrm{tr}(X_n)-\beta\,\frac{\eta_n}{|\eta_n|}\cdot\left(X_n\frac{\eta_n}{|\eta_n|}\right)+\frac{h'(H^{-1}(\underline{w}(x_n)))}{h^2(H^{-1}(\underline{w}(x_n)))}|\eta_n|^2\\&\qquad+\left(\frac{h'(H^{-1}(\overline{w}(y_n)))}{h^2(H^{-1}(\overline{w}(y_n)))}-\frac{h'(H^{-1}(\underline{w}(x_n)))}{h^2(H^{-1}(\underline{w}(x_n)))}\right)|\eta_n|^2\\&\quad\le h(H^{-1}(\underline{w}(x_n)))f(x_n,H^{-1}(\underline{w}(x_n)))\\&\qquad+\left(\frac{h'(H^{-1}(\overline{w}(y_n)))}{h^2(H^{-1}(\overline{w}(y_n)))}-\frac{h'(H^{-1}(\underline{w}(x_n)))}{h^2(H^{-1}(\underline{w}(x_n)))}\right)|\eta_n|^2,\end{aligned}$$

letting $n\to\infty$ and by continuity of $\overline{w},\underline{w}$, f, h, h' and using that h'/h^2 is nondecreasing, we get

$$h(H^{-1}(\overline{w}(\hat x)))f(\hat x,H^{-1}(\overline{w}(\hat x)))\le h(H^{-1}(\underline{w}(\hat x)))f(\hat x,H^{-1}(\underline{w}(\hat x))).$$

This is a contradiction with (6.7) since it implies, using (6.6) that

$$h(H^{-1}(\overline{w}(\hat x)))f(\hat x,H^{-1}(\overline{w}(\hat x)))>h(H^{-1}(\underline{w}(\hat x)))f(\hat x,H^{-1}(\underline{w}(\hat x))).$$

If $x_n=y_n$ for $n\ge n_0$, then $\eta_n=0$ and by (iii) we have

$$\begin{aligned}h(H^{-1}(\overline{w}(y_n)))f(y_n,H^{-1}(\overline{w}(y_n)))&\le-\alpha\,\mathrm{tr}(Y_n)-\beta\,m(Y_n)\\&\le-\alpha\,\mathrm{tr}(X_n)-\beta\,M(X_n)\\&\le h(H^{-1}(\underline{w}(x_n)))f(x_n,H^{-1}(\underline{w}(x_n))),\end{aligned}$$

and, arguing as above, it leads to the contradiction. □

Let us extract easy consequences of this comparison principle.

Proposition 6.3.4 *[Uniqueness] Assume that $0 < f \in C(\Omega \times \mathbb{R})$ satisfies the h-decreasing condition. Then, there exists at most one positive viscosity solution of*

$$\begin{cases} -\Delta_p^N u(x) = f(x,u)\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Proof: Suppose that there exist $u_1, u_2 \geq 0$ two solutions of (P). Using twice Theorem 6.3.3 we obtain that $u_1 \leq u_2$ and $u_2 \leq u_1$, and we conclude that $u_1 = u_2$. □

The next result improves Martínez-Aparicio et al. (2014a) where a starshaped condition on the domain Ω was required.

Corollary 6.3.5 *As a particular case, we can assert that there exists a unique positive solution of*

$$\begin{cases} -\Delta_p^N u(x) = \lambda u^q\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

for every $\lambda > 0$ and $0 < q < 1$. Moreover, for $\lambda = 0$, the problem admits as unique solution, $u = 0$.

Proof: For $\lambda > 0$, the uniqueness is due to Proposition 6.3.4 and existence to (Martínez-Aparicio et al., 2014a, Theorem 3.1) (which can be extended to the case $p = \infty$ using the same iterative procedure as in (Martínez-Aparicio et al., 2014a, Theorem 3.1)). For $\lambda = 0$, we observe that u is a solution of $-\Delta_p^N u = 0$ if and only if $-\Delta_p u = 0$ in the viscosity sense, (this holds since it is enough to test the equation $-\Delta_p u = 0$ with test functions with $\nabla\varphi \neq 0$, see Juutinen et al. (2001)). Thus, the trivial solution $u = 0$ is the unique solution when $\lambda = 0$. □

6.4 Existence of Minimal Solutions for the Gelfand problem

The first result of this section shows how one can pass to the limit in a sequence of viscosity solutions of a sequence of problems to obtain a viscosity solution of the limit problem.

Lemma 6.4.1 *Let u_n, $f_n \in \mathcal{C}(\Omega)$ and $p_n \in [2,\infty]$ be three sequences satisfying*

$$-\Delta_{p_n}^N u_n = f_n, \tag{6.8}$$

in the viscosity sense, such that $f_n \to f$, $u_n \to u$ uniformly for every $\omega \subset\subset \Omega$ and $p_n \to p \in [2,\infty]$. Then, u is a viscosity solution to the problem

$$-\Delta_p^N u = f. \tag{6.9}$$

Proof: First, we prove that u is a subsolution. For every $x_0 \in \Omega$ we take $\varphi \in \mathcal{C}^2(\Omega)$ such that $\varphi(x_0) = u(x_0)$ and $\varphi - u > 0$ in $\Omega \setminus \{x_0\}$. Fix $\delta > 0$ such that $\overline{B_\delta(x_0)} \subset \Omega$, and for every $n \in \mathbb{N}$, we consider x_n as the strict minimum point (not necessarily unique) of $\varphi - u_n$ in $\overline{B_\delta(x_0)}$, i.e.,

$$(\varphi - u_n)(x_n) \leq (\varphi - u_n)(x), \qquad \text{for all } x \in \overline{B_\delta(x_0)}.$$

Up to a subsequence, we can assume that $x_n \to x^* \in \overline{B_\delta(x_0)}$. Using that u_n is continuous and that the sequence u_n uniformly converges to u we deduce that $u_n(x_n) \to u(x^*)$. We obtain, taking limits in the above inequality, that

$$(\varphi - u)(x^*) \leq (\varphi - u)(x), \qquad \text{for all } x \in \overline{B_\delta(x_0)},$$

and we can assert that $x^* = x_0$. We set

$$\varphi_n(x) = \varphi(x) + u_n(x_n) - \varphi(x_n) + \|x - x_n\|^4, \qquad x \in \overline{B_\delta(x_0)}.$$

It is easy to check that φ_n satisfies:

$$\varphi_n(x_n) = u_n(x_n), \qquad \nabla\varphi_n(x_n) = \nabla\varphi(x_n), \qquad D^2\varphi_n(x_n) = D^2\varphi(x_n)$$

and

$$(\varphi_n - u_n)(x) > 0$$

in a neighborhood of x_n. Thus, using that u_n is a subsolution of (6.8), taking φ_n as test function we obtain that

1. If $\nabla\varphi_n(x_n) \neq 0$ then $-\alpha_n\Delta\varphi_n(x_n) - \beta_n\Delta_\infty\varphi_n(x_n) \leq f_n(x_n)$ and thus
$$-\alpha_n\Delta\varphi(x_n) - \beta_n\Delta_\infty\varphi(x_n) \leq f_n(x_n). \tag{6.10}$$
2. If $\nabla\varphi_n(x_n) = 0$ then $-\alpha_n\Delta\varphi_n(x_n) - \beta_n M(D^2\varphi_n(x_n)) \leq f_n(x_n)$ and thus
$$-\alpha_n\Delta\varphi(x_n) - \beta_n M(D^2\varphi(x_n)) \leq f_n(x_n), \tag{6.11}$$

where $\alpha_n = \dfrac{1}{p_n-1}$, $\beta_n = \dfrac{p_n-2}{p_n-1}$ if $p_n < +\infty$ and $\alpha_n = 0, \beta_n = 1$ if $p_n = \infty$.

Now, denoting $\alpha = \dfrac{1}{p-1}$, $\beta = \dfrac{p-2}{p-1}$ if $p < +\infty$ and $\alpha = 0, \beta = 1$ in other case, we distinguish three different cases:

Case i): $\nabla\varphi(x_0) \neq 0$. In this case, we can suppose that, up to a subsequence, $\nabla\varphi_n(x_n) \neq 0$ for $n \geq n_0$ and, taking into account that $\varphi \in \mathcal{C}^2$ and the continuity and uniform convergence of f_n, we can pass to the limit in (6.10) as $n \to \infty$ to obtain

$$-\alpha\Delta\varphi(x_0) - \beta\Delta_\infty\varphi(x_0) \leq f(x_0).$$

Case ii): $\nabla\varphi(x_0) = 0$ and, up to a subsequence, $\nabla\varphi_n(x_n) \neq 0$ for $n \geq n_0$. In this case, since

$$\Delta_\infty\varphi(x_n) \leq M(D^2\varphi(x_n))$$

replacing in (6.10) we get (6.11) and taking limits we obtain the desired inequality

$$-\alpha\Delta\varphi(x_0) - \beta M(D^2\varphi(x_0)) \leq f(x_0). \tag{6.12}$$

Case iii): $\nabla\varphi(x_0) = \nabla\varphi_n(x_n) = 0$, for $n \geq n_0$ we obtain (6.12) directly from (6.11).

On the other hand, to prove that u is a supersolution, we argue in a similar way. To be more specific, for every $x_0 \in \Omega$ we take the test function $\psi \in \mathcal{C}^2(\Omega)$, satisfying $u - \psi$ has a strict minimum at x_0 with $\psi(x_0) = u(x_0)$. Now, taking x_n, the strict minimum of $u_n - \psi$ in $\overline{B_\delta(x_0)} \subset \Omega$, we set $\psi_n(x) = \psi(x) + u_n(x_n) - \psi(x_n) - \|x - x_n\|^4$ as the test function in (6.8) touching the graph of u_n from below in x_n. The rest of the proof runs as before. □

Now we can prove the existence of minimal solutions of $(P_{\lambda,p})$ for λ small and nonexistence of solutions for λ large, that is, we prove Theorem 6.1.1.

Proof:[Proof of Theorem 6.1.1] Let $z \in \mathcal{C}^2([0,1])$ be a classical solution to the problem

$$\begin{cases} -z''(r) - \alpha(N-1)\dfrac{z'(r)}{r} = \lambda\, e^{z(r)}\,, & r \text{ in } (0,1), \\ z(1) = 0, \;\; z'(0) = 0, \end{cases} \tag{6.13}$$

with

$$\alpha = \frac{1}{p-1} \quad \text{if } p < +\infty \text{ and } \alpha = 0 \text{ in other case.}$$

Then $u(x) := z(|x|)$ is a solution to the problem

$$\begin{cases} -\Delta_p^N u = \lambda\, e^u\,, & \text{in } B_1, \\ u > 0, & \text{in } B_1, \\ u = 0\,, & \text{on } \partial B_1, \end{cases} \tag{6.14}$$

in the sense of Definition 6.2.1-*(iii)* (see also Remark 6.2.2). Due to Joseph and Lundgren (1972/73), it is well known that there exists a positive number $\tilde\lambda(B_1)$, depending only on p, N, such that problem (6.13) has no solution for $\lambda > \tilde\lambda(B_1)$. Moreover, for every $0 \le \lambda < \tilde\lambda(B_1)$ there exists a classical solution $z \in \mathcal{C}^2([0,1])$ (see also Jacobsen and Schmitt (2002) for a complete description of the multiplicity of solutions). Observe that for any classical solution $z \in \mathcal{C}^2([0,1])$, $\lambda \ge 0$, of (6.13) it holds that $\lambda \le \tilde\lambda(B_1)$ (we refer again to Jacobsen and Schmitt (2002) for a complete description of the multiplicity of solutions).

Note also that the relationship between classical solutions of (6.13) and viscosity radial solutions of (6.14) is bidirectional. Given $u \in \mathcal{C}(\bar B_1)$ solution of (6.14) radially symmetric and decreasing then $z(r) = u(|x|)$ for some $x \in \Omega$ with $|x| = r$ satisfies (6.13) in the weak sense (which is equivalent to be a classical solution in this case).

Taking into account Remark 6.2.2, u is also a solution to our problem in the viscosity sense.

Now, for any fixed $R > 0$, we can rescale the problem and consider

$$v(r) := z(r/R).$$

It is easy to check that we arrive to the ODE

$$\begin{cases} -v''(r) - \alpha(N-1)\dfrac{v'(r)}{r} = \dfrac{\lambda}{R^2}\, e^{v(r)}\,, & \text{in } (0,R), \\ v(R) = 0, \;\; v'(0) = 0. \end{cases} \tag{6.15}$$

Summarizing, we have that there exists a positive value

$$\tilde\lambda(B_R) = \frac{\tilde\lambda(B_1)}{R^2} > 0,$$

which is decreasing with respect to R, such that problem ($P_{\lambda,p}$) admits at least a solution for every $\lambda < \tilde\lambda(B_R)$ in the ball of radius R, $\Omega = B_R$.

Let now Ω be a bounded domain and $R_1 > 0$ given by

$$R_1 = \min\{R > 0 \,:\, \Omega \subset B_R\}. \tag{6.16}$$

Notice that if u_{R_1} is a solution in B_{R_1} for some$\Lambda \;<\; \tilde\lambda(B_{R_1})$ then it is a supersolution in Ω for $\lambda \le \Lambda < \tilde\lambda(B_{R_1})$. We claim that there exists a solution of problem ($P_{\lambda,p}$) with $\lambda = \Lambda$. Indeed, to prove this fact we use a standard monotone iteration argument: let $w_0 = 0$ and for every $n \ge 1$ we define the recurrent sequence $\{w_n\}$ by

$$(Q_n) \qquad \begin{cases} -\Delta_p^N w_n = \lambda\, e^{w_{n-1}}\,, & \text{in}\Omega, \\ w_n > 0\,, & \text{in}\Omega. \\ w_n = 0\,, & \text{on } \partial\Omega, \end{cases}$$

The sequence $\{w_n\} \in \mathcal{C}(\bar\Omega)$ is well defined by Manfredi et al. (2012); Peres and Sheffield (2008), see also Lu and Wang (2008). Note that we are solving a problem of the form $-\Delta_p^N w_n = f$ inΩ , with $f > 0$ and $w_n = 0$ on $\partial\Omega$ as boundary condition. Then existence is a consequence of a limit procedure involving game theory (in this problem the right hand side, f, enters into the problem as a running payoff and the boundary condition $w_n = 0$ as the final payoff). The existence of such solution can be also proved directly using Perron's method thanks to our general comparison principle.

Moreover, the sequence $\{w_n\}$ is increasing with n. Indeed, taking in account that $0 < w_1$ we obtain $\lambda e^{w_0} \le \lambda e^{w_1}$ and using the comparison principle in Theorem 6.3.3 it follows that $w_1 \le w_2$. By an inductive argument, we get $0 < w_1 \le w_2 \le \cdots \le w_n$, for all $n \ge 1$. From the fact that u_{R_1} is a supersolution of problem ($P_{\lambda,p}$), with a similar inductive argument, we prove that $w_n \le u_{R_1}$ for every $n \in \mathbb{N}$.

Since $u_{R_1} \in L^\infty(\Omega)$, the sequence $\{w_n(x)\}$ is increasing and bounded by $u_{R_1}(x)$, therefore, exists

$$w_\lambda(x) := \lim_{n\to\infty} w_n(x).$$

In addition, thanks to the subtle Krylov-Safonov $\mathcal{C}^{0,\alpha}$-estimates of w_n for every $p \in [2, \infty]$ (here we refer to Caffarelli and Cabré (1995); Charro et al. (2013)), we obtain that $w_n \to w_\lambda$ uniformly. Taking $f_n = \lambda e^{w_{n-1}}$ and $p_n = p$ in Lemma 6.4.1 we get that w_λ is a solution of problem $(P_{\lambda,p})$.

To prove that the obtained solution w_λ is minimal, let v_λ be a solution of problem $(P_{\lambda,p})$, by a similar argument using the comparison principle and induction in n we have $w_n \le v_\lambda$ for all $n \in \mathbb{N}$. As $w_\lambda(x) = \lim_{n\to\infty} w_n(x)$ (we use again comparison here), we obtain $w_\lambda \le v_\lambda$.

We have thus proved that for every $\lambda < \tilde{\lambda}(B_{R_1})$ there exists w_λ, minimal solution of problem $(P_{\lambda,p})$. In particular

$$0 < \tilde{\lambda}(B_{R_1}) \le \lambda^*(\Omega, N, p) = \sup\{\lambda > 0 \,:\, \exists \text{ a minimal solution of } (P_{\lambda,p})\} \le \infty.$$

Now to ensure that $\lambda^*(\Omega, N, p) < \infty$ let

$$R_2 = \max\{R > 0 \,:\, B_R \subset \Omega\},$$

we remark that without loss of generality we can assume that $0 \in \Omega$. In that way, taking w_λ, the minimal solution in Ω, as a supersolution in B_{R_2} and applying the above argument again, withΩ replaced by B_{R_2}, we obtain that $\lambda^*(\Omega, N, p) \le \lambda^*(B_{R_2}, N, p)$.

Note that in the case $\Omega =\ B_r$ we can perform the previous argument starting with $w_0 = 0$ and obtain that the minimal solution is radial. In fact, by uniqueness, in this case w_n is radial for every n. Remark that in this case the unique minimal solution leads to a solution of the ODE (6.15) and thus $\lambda^*(B_{R_2}, N, p) \le \tilde{\lambda}(B_{R_2})$. □

Remark 6.4.2 *The arguments used in the previous proof shows that the extremal parameter verifies*

$$\begin{aligned}\lambda^*(\Omega, N, p) \quad &= \sup\{\lambda > 0 \,:\, \textit{there exists a minimal solution of } (P_{\lambda,p})\}\\ &= \sup\{\lambda > 0 \,:\, \textit{there exists a solution of } (P_{\lambda,p})\}\\ &= \sup\{\lambda > 0 \,:\, \textit{there exists a nonnegative supersolution of } (P_{\lambda,p})\}.\end{aligned}$$

Also note that

$$\lambda^*(\Omega_1, N, p) \le \lambda^*(\Omega_2, N, p) \qquad \textit{when } \Omega_2 \subset \Omega_1,$$

and that the extremal value for a ball, $\Omega = B_R$, is the one that corresponds to the existence of a radial solution, we refer to Jacobsen and Schmitt (2002) and Joseph and Lundgren (1972/73) for the analysis of the resulting ODE.

In addition, we note that, if we have a solution to our problem, it holds

$$-\Delta_p^N u = \lambda e^u \ge \lambda u.$$

Therefore we must have $\lambda \le \lambda_{1,p}(\Omega)$, where $\lambda_{1,p}(\Omega)$ is the first eigenvalue of the operator $-\Delta_p^N$ with Dirichlet boundary conditions. We conclude that

$$\lambda^*(\Omega, N, p) \le \lambda_{1,p}(\Omega).$$

6.5 Unbounded Continua of Solutions

For the reader's convenience, we recall the following general results from the theory of global continua of solutions using degree theory which will be essential for our analysis. For the proofs we refer to Ambrosetti and Arcoya (2011), Schmitt (1995) and Leray and Schauder (1934).

Theorem 6.5.1 *[Continuation Theorem of Leray-Schauder] Let X be a real Banach space, $\mathcal{O}$ an open bounded subset of X and assume that $T : \mathbb{R} \times X \to X$ is completely continuous (i.e., relatively compact and continuous). Furthermore, assume that for $\lambda = \lambda_0$ we have that $u \ne T(\lambda_0, u)$ for every $u \in \partial\mathcal{O}$ and deg$(I - T(\lambda_0, \cdot), \mathcal{O}, 0) \ne 0$. Let*

$$\Sigma = \{(\lambda, u) \in [\lambda_0, \infty) \times X \,:\, u = T(\lambda, u)\}.$$

Then there exists a maximal connected and closed $\mathcal{C} \subset \Sigma$. Even more, the following statements are valid:

1. $\mathcal{C} \cap \{\lambda_0\} \times \mathcal{O} \neq \emptyset$.
2. *Either* $\mathcal{C}$ *is unbounded or else* $\mathcal{C} \cap \{\lambda_0\} \times X \setminus \overline{\mathcal{O}} \neq \emptyset$.

Theorem 6.5.2 *[Homotopy property] Let* X *be a real Banach space,* $\mathcal{O}$ *an open subset of* X *and let* $T \in \mathcal{C}([0,1] \times \overline{\mathcal{O}}, X)$ *be completely continuous in* $[0,1] \times \overline{\mathcal{O}}$*. If* $b : [0,1] \to X$ *is continuous and* $b(t) \neq u - T(t,u)$ *in* $[0,1] \times \partial\mathcal{O}$*, then deg*$(I - T, \mathcal{O}, b(t))$ *remains constant* $\forall\, t \in [0,1]$.

Theorem 6.5.3 *[Classical Leray-Schauder's theorem] Let* X *be a real Banach space,* $\mathcal{O} \subset X$ *an open and bounded subset of* X *and* $\Phi : [a,b] \times \overline{\mathcal{O}} \to X$ *given by* $\Phi(t,u) = u - T(t,u)$ *being* T *completely continuous. We also assume that*

$$\Phi(t,u) \neq u, \qquad \forall\, (t,u) \in [a,b] \times \partial\mathcal{O}.$$

Then, if deg $(\Phi(a,\cdot), \mathcal{O}, 0) \neq 0$*, it holds that*

1. *The equation* $\Phi(t,u) = 0$ *with* $u \in X$ *has a solution in* $\mathcal{O}$ *for every* $a \leq t \leq b$.
2. *There exists a closed and connected set,* $\Sigma_{a,b} \subset \{(t,u) \in [a,b] \times X \,:\, u = T(t,u)\}$*, that intersects* $t = a$ *and* $t = b$.

Let us consider the operator

$$K : [0,1] \times \mathbb{R} \times \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega})$$

by defining, for every $t \in [0,1]$, $\lambda \in \mathbb{R}$ and $w \in \mathcal{C}(\overline{\Omega})$, $u := K(t,\lambda, w)$ as the unique solution in $\mathcal{C}(\overline{\Omega})$ of the problem

$$\begin{cases} -\Delta^N_{p(t)} u = \lambda^+ \, e^{w^+} \,, & \text{in}\Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

where

$$p(t) = \frac{t-2}{t-1}.$$

That is, $-\Delta^N_{p(t)} u = -(1-t)\,\Delta u - t\,\Delta_\infty u$. Notice that every $p(t) \in [2,\infty]$ is labeled by a unique $t \in [0,1]$ (and conversely), thus K is well defined.

Now, we prove that K is completely continuous, which allows us to apply the Leray-Schauder degree techniques (see Leray and Schauder (1934)), in order to study the existence of "continua of solutions" of $(P_{\lambda,p})$, i.e., connected and closed subsets in the solution set

$$\mathscr{S}_p = \left\{ (\lambda, u) \in [0,\infty) \times \mathcal{C}(\overline{\Omega}) \,:\, K\left(\frac{p-2}{p-1}, \lambda, u\right) = u \right\},$$

for every fixed $p \in [2,+\infty]$ or, if we fixed λ instead, in

$$\mathcal{S}_\lambda = \left\{ (p, u) \in [2,\infty] \times \mathcal{C}(\overline{\Omega}) \,:\, K\left(\frac{p-2}{p-1}, \lambda, u\right) = u \right\},$$

Lemma 6.5.4 *Let assume that* $u_n \in \mathcal{C}(\overline{\Omega})$ *satisfies*

$$\begin{cases} -\Delta^N_{p(t_n)} u_n = \lambda_n \, e^{w_n} \,, & \text{in } \Omega, \\ u_n = 0\,, & \text{on } \partial\Omega, \end{cases}$$

with $t_n \in [0,1]$ *and* $0 \leq \lambda_n$*,* w_n *bounded in* $\mathbb{R} \times \mathcal{C}(\overline{\Omega})$*. Then, up to a subsequence,* u_n *is strongly convergent to* $u \in \mathcal{C}(\overline{\Omega})$*. If, in addition,* $\lambda_n \to \lambda$*,* $t_n \to t$ *and* w_n *converges in* $\mathcal{C}(\overline{\Omega})$ *to* w*, then* u *is solution of problem*

$$\begin{cases} -\Delta^N_{p(t)} u = \lambda \, e^{w} \,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega. \end{cases}$$

Proof: If $\lambda_n = 0$ then $u_n = 0$ is the unique solution (Corollary 6.3.5) and the proof is immediate. In other case, since the sequence $0 < \lambda_n e^{w_n} \leq C$ for some positive constant, u_n is a subsolution of problem

$$\begin{cases} -\Delta^N_{p(t_n)} v = C\,, & \text{in}\Omega, \\ v = 0\,, & \text{on } \partial\Omega. \end{cases}$$

It is well-known, by the theory of uniformly elliptic fully nonlinear equations, that for every fixed $n \in \mathbb{N}$, $u_n \in \mathcal{C}^{0,\nu(n)}(\overline{\Omega})$, whenever $2 \leq p(t_n) \leq M$ for some M sufficiently large (for instance, greater than the dimension N), being $0 < \nu(n) < 1$ (Caffarelli and Cabré (1995); Gilbarg and Trudinger (1983)). We stress that this Hölder estimates depend on the ratio between the ellipticity constants, which in this case is $p(t_n) - 1$ and, consequently, it blows-up as $p(t_n) \to \infty$. However, for $p(t_n) \in [M, \infty]$, it is shown in (Charro et al., 2013, Theorem 7) that $u_n \in \mathcal{C}^{0,\rho(n)}(\overline{\Omega})$ being $\rho(n) = \frac{p(t_n)-N}{p(t_n)-1}$.

Thus, we can assert that the sequence $u_n \in \mathcal{C}^{0,\gamma}(\overline{\Omega})$ where $\gamma = \min\{\nu(n), \rho(n) : n \in \mathbb{N}\}$. Hence, Ascolí-Arzelá Theorem gives that u_n possesses a subsequence converging in $\mathcal{C}(\overline{\Omega})$ which conclude the first part of the lemma. Finally, the second part is a direct consequence of the uniqueness of solutions Proposition 6.3.4 and Lemma 6.4.1. □

The following is the main result in this section, Theorem 6.1.2.

Proof:[Proof of Theorem 6.1.2] Fixed $R > 0$, let $\mathcal{O}_R$ be the open ball of radius R of $\mathcal{C}(\overline{\Omega})$ and we fix some λ_R with

$$0 < \lambda_R < \frac{R}{2d^2 e^R},$$

where d is the diameter ofΩ.

By Lemma 6.5.4, we obtain that $K \in \mathcal{C}([0,1] \times [0, \lambda_R] \times \overline{\mathcal{O}_R}, \mathcal{C}(\overline{\Omega}))$ and $K(t,\lambda, \cdot)$ is completely continuous for every $(t,\lambda) \in [0,1] \times [0, \lambda_R]$. Now, in order to apply twice Theorem 6.5.2 for the parameters (t,λ) with $b(t,\lambda) \equiv 0 \in \mathcal{C}(\overline{\Omega})$, we must check an *priori bound* of the solutions of the equation $u = K(t,\lambda, u)$. That is, $u \neq K(t,\lambda, u)$ in $[0,1] \times [0, \lambda_R] \times \partial\mathcal{O}_R$. In fact, we argue by contradiction, suppose that $\|u\|_\infty = R$ and there exist $t \in [0,1]$ and $\lambda \in [0, \lambda_R]$ such that u satisfies the equation

$$-\Delta^N_{p(t)} u = \lambda e^u \qquad \text{in}\Omega \quad ,$$

hence u is a subsolution of problem

$$-\Delta^N_{p(t)} v = \lambda e^R \qquad \text{in}\Omega \quad .$$

On the other hand, a simple computation of (Charro et al., 2013, Theorem 1 and Theorem 3) shows that if $v \in \mathcal{C}(\overline{\Omega})$ is a nonnegative subsolution of the Poisson problem

$$-\Delta^N_p v = f(x) \qquad \text{in} \quad ,\Omega$$

with $0 \leq f \in \mathcal{C}(\overline{\Omega})$ and $p \in [2, \infty]$, then $\|v\|_\infty \leq 2d^2 \|f\|_\infty$. Applying this last result we get the following contradiction

$$R = \|u\|_\infty \leq 2d^2 \lambda e^R \leq 2d^2 \lambda_R e^R < R.$$

In this way, due to the Homotopy Property, we obtain

$$\deg(I - K(t,\lambda, \cdot), \mathcal{O}_R, 0) = const \qquad \forall\,(t,\lambda) \in [0,1] \times [0, \lambda_R].$$

Moreover, since

$$K(0, \lambda, w) = (-\Delta)^{-1}\left(\lambda e^{w^+}\right)$$

is the inverse of the laplacian operator, and it is well known that

$$\deg(I - K(0, 0, \cdot), \mathcal{O}_R, 0) = 1,$$

we get

$$1 = \deg(I - K(0, 0, \cdot), \mathcal{O}_R, 0) = \deg(I - K(t,\lambda, \cdot), \mathcal{O}_R, 0).$$

To conclude the proof, we apply the Continuation Theorem of Leray-Schauder (Theorem 6.5.1) with $T(\lambda,u)=K(t,\lambda,u)$, for every fixed $t\in[0,1]$, which is completely continuous (Lemma 6.5.4). Therefore, by using that $\deg(I-T(0,\cdot),\mathcal{O}_R,0)=1\neq 0$, we can assert that there exists a maximal connected subset $\mathcal{C}$ of $\mathscr{S}_p$ that contains $(0,0)$. Furthermore, since 0 is the unique solution for $\lambda=0$, $\mathcal{C}$ is not bounded. Finally, since for every λ such that there is a solution of ($P_{\lambda,p}$) we can construct a minimal solution, we can state that $\mathcal{C}\subset[0,\lambda^*]\times\mathcal{C}(\overline{\Omega})$.

With the same arguments, using Theorem 6.5.3 with $T(t,u)=K(t,\lambda,u)$ and $[a,b]=[0,1]$, for every fixed $\lambda\in(0,\lambda_0=\min\{\lambda^*(\Omega,N,2),\frac{1}{2d^2e}\})$, we can obtain the existence of a continuum of solutions moving $p\in[2,\infty]$. More precisely, since $\deg(I-K(0,\lambda,\cdot),\mathcal{O}_1,0)=1$ we can apply Theorem 6.5.3 obtaining the existence of a continuum$\Sigma_{0,1}\subset\{(t,u)\in[0,1]\times\mathcal{O}_1 : u=T(t,u)\}$ such that $Proj_{[0,1]}\Sigma_{0,1}=[0,1]$. Note that the upper bound for λ is used to ensure an a priori bound. Thus, we finish the proof by taking

$$\mathcal{D}=\left\{\left(\frac{t-2}{t-1},u\right)\in[2,+\infty]\times\mathcal{O}_1 : (t,u)\in\Sigma_{0,1}\right\}.$$

□

Remark 6.5.5 *Now we briefly comment on possible extensions for more general nonlinearities. Note that we can also deal with the equation*

$$-\Delta_p^N u=\lambda f(u),$$

with a general continuous nonlinearity f that verifies $f(0)>0$, increasing and $\frac{f(s)}{s}\geq k>0$. In fact, we only need to show existence and nonexistence of radial solutions (the rest of the arguments can be extended without much difficulties). Hence we arrive to the problem

$$\begin{cases} -z''(r)-\theta\dfrac{z'(r)}{r}=\lambda f(z(r)), & r\in(0,1),\\ z(r)>0, & r\in(0,1),\\ z(1)=z'(0)=0, \end{cases}\tag{6.17}$$

where $\theta=\frac{N-1}{p-1}\in[0,\infty)$ due to the fact that $p\in[2,\infty]$. Multiplying by r^θ and integrating twice we obtain

$$\begin{aligned} z(r) &= \lambda\int_r^1\frac{1}{\tau^\theta}\int_0^\tau s^\theta f(z(s))dsd\tau\\ &\geq \lambda\int_r^1\frac{1}{\tau^\theta}\int_0^r s^\theta f(z(s))dsd\tau\\ &\geq \lambda\int_r^1\frac{1}{\tau^\theta}\int_0^r s^\theta f(z(r))dsd\tau. \end{aligned}$$

Therefore, for every $r\in(0,1)$ it must hold

$$\frac{1}{k}\geq\frac{z(r)}{f(z(r))}\geq\lambda\int_r^1\int_0^r\left(\frac{s}{\tau}\right)^\theta dsd\tau:=\lambda\,F_\theta(r).$$

As $F_\theta(r)$ is positive in $(0,1)$ and is bounded above we concluded that $\lambda\leq\frac{1}{c(\theta)k}$. Hence there is no solutions for λ greater than a constant that depends only on p and N.

To look for existence of solutions for small λ, we can use degree theory for the operator $T:[0,\infty)\times C([0,1])\to C([0,1])$ given by

$$T(\lambda,u)=\lambda\int_r^1\frac{1}{\tau^\theta}\int_0^\tau s^\theta f(u(s))dsd\tau.$$

Since f is assumed to be continuous it is easy to check that T is completely continuous. Now, as $T(0,u)=0$ for every $u\in C([0,1])$ using Leray-Schauder's theorem we obtain the existence of a continuum of solutions $\mathcal{C}\subset[0,\infty)\times C([0,1])$ that is unbounded with $(0,0)\in\mathcal{C}$. In particular, there exist solutions for values of λ close to 0.

Chapter 7

Gelfand type problem for singular quadratic quasilinear equations

A. Molino, *NoDEA. Nonlinear Differential Equations and Applications*, **23** no. 5 (2016), Art. 56, 20.
DOI: 10.1007/s00030-016-0409-7

Abstract

In this paper, we study the existence of positive solutions for the quasilinear elliptic singular problem

$$\begin{cases} -\Delta u + c\,\frac{|\nabla u|^2}{u^\gamma} = \lambda\, f(u)\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

where $c, \lambda > 0$, $\gamma \in (0,1)$, f is strictly increasing and derivable in $[0,\infty)$ with $f(0) > 0$. We show that there exists $\lambda^* > 0$ such that $(0, \lambda^*]$ is the maximal set of values such there exists solution. In addition, we prove that for $\lambda < \lambda^*$ there exists minimal and bounded solutions. Moreover, we give sufficient conditions for existence and regularity of solutions for $\lambda = \lambda^*$.

7.1 Introduction

Gelfand-type problems constitute one of the most studied fields of semilinear elliptic equations and it has been considered since the very earliest stages of development of the theory of Partial Differential Equations. There are several reasons for this interest, foremost among them are the wide applications to physical models (we refer to Chandrasekhar (1957); Gel'fand (1963); Joseph and Sparrow (1970); Keller and Cohen (1967) and references therein) and the open problems relating to the existence and boundedness of solutions which still remain unsolved. We recall that a Gelfand-type problem aims to study the following semilinear elliptic equation

$$\begin{cases} -\Delta u = \lambda\, f(u)\,, & \text{in } \Omega, \\ u \geq 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{G_λ}$$

where Ω is a smooth bounded, open subset of $\mathbb{R}^N$ $(N \geq 1)$, $\lambda \geq 0$ and the nonlinearity term satisfies

$$f \text{ is } \mathcal{C}^1[0,\infty), \text{ positive, increasing and convex such that } f(0) > 0. \tag{F}$$

Typical examples for f are the power-like $(1+u)^p$ with $p > 1$ and the exponential e^u. If a solution u of (G_λ) belongs to $L^\infty(\Omega)$ it is said that it is *regular* and *minimal* if $u \leq v$ being v any other solution of (G_λ).

M.G. Crandall and P.H. Rabinowitz in Crandall and Rabinowitz (1975) (see also F. Mignot and J.P. Puel Mignot and Puel (1980)) proved, under the hypothesis f is superlinear at infinity (i.e. $\frac{f(s)}{s} \to \infty$), the following result

Proposition 7.1.1 *[Crandall-Rabinowitz, 1973] Crandall and Rabinowitz (1975) There exists a positive number λ^* called the extremal parameter such that*

- *If $\lambda < \lambda^*$ the problem (G_λ) admits a minimal bounded solution w_λ.*
- *If $\lambda > \lambda^*$ the problem (G_λ) admits no solution.*

Even more, they showed that the sequence of minimal solutions $\{w_\lambda\}$ of (G_λ) is increasing. Furthermore, the minimal solutions are stable, namely they satisfy the following condition

$$\int_\Omega \left(|\nabla \xi|^2 - \lambda f'(w_\lambda)\xi^2\right) \geq 0, \qquad \forall \xi \in \mathcal{C}_c^\infty(\Omega).$$

An important role is played by the stability condition in order to prove the existence and regularity of $u^* := \lim_{\lambda \to \lambda^*} w_\lambda$, called *extremal solution*. In particular, it has been used to achieve optimal results of regularity of extremal solution depending on the dimension N. Special mention should be made of the exponential case $f(s) = e^s$, obtaining regularity for $N < 10$ as well as the power-like $f(s) = (1+s)^p$ for $N < 4 + 2(1-1/p) + 4\sqrt{1-1/p}$ (see Crandall and Rabinowitz (1975)).

In Brézis and Vázquez (1997) H. Brezis and J.L. Vázquez proved that u^* is a weak solution of (G_{λ^*}). But, as far as regularity of u^* is concerned , for general nonlinearities f satisfying (F), a few results are obtained. More specifically, assuming the superlinearity of f, G. Nedev proved the boundedness of extremal solutions for dimension $N \leq 3$ (Nedev (2000)) and S. Villegas in Villegas (2013) for $N = 4$. See also X. Cabré et al. in Cabré and Capella (2006); Cabré and Sanchón (2013) for convex domains Ω.

On the other hand, quasilinear Dirichlet problems having lower order terms with quadratic growth with respect to the gradient whose simplest model is the following boundary value problem

$$\begin{cases} -\Delta u + H(x,u)\,|\nabla u|^2 = f_0(x)\,, & \text{in}\Omega, \\ u > 0\,, & \text{in}\Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{Q}$$

have also been extensively studied. A simple motivation relies in the fact that they arise naturally in Calculus of Variations. For example, the Euler-Lagrange equation of the functional

$$I(u) = \frac{1}{2}\int_\Omega a(x,u)|\nabla u|^2 - \int_\Omega f_0(x)u,$$

is formally

$$-\mathrm{div}\,(a(x,u)\nabla u) + \frac{1}{2}a'_u(x,u)|\nabla u|^2 = f_0(x),$$

wich contains a quadratic gradient term.

In the 1980s, L. Boccardo, F. Murat and J.P. Puel discussed, among other important aspects, the case $H(x,s) = g(s)$ continuous in $[0,\infty)$, giving a huge literature since then (see Boccardo et al. (1982, 1983) and references therein). It can be observed in the previous example of Calculus of Variations that if we consider functions with unbounded derivative in zero, for instance $a(x,u) = 1 + |u|^\delta$ with $\delta \in (0,1)$, it shows that the Euler-Lagrange equation associated should have a singularity in the quadratic term. In recent years, the case $H(x,s)$ with a singularity at $s = 0$ has been studied by D. Arcoya et al. (Arcoya and Segura de León (2010); Arcoya et al. (2009a,b, 2010)) and some applications are described by this kind of equations, see for instance Barenblatt et al. (2000); Berestycki et al. (2001); Kardar et al. (1986).

The goal of this work is to bring together the two areas above, that is, a Gelfand-type problem with a singularity in the gradient term. To be more specifically, we propose to study the existence and regularity of positive solutions for the following problem

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = \lambda\, f(u)\,, & \text{in}\Omega, \\ u > 0\,, & \text{in}\Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

were Ω is a smooth bounded and open subset of $\mathbb{R}^N$ $(N \geq 3)$, $\lambda > 0$, f strictly increasing, derivable in $[0,\infty)$ with $f(0) > 0$ and respect to g a nontrivial and positive function that either is continuous in $[0,\infty)$ or it is continuous in $(0,\infty)$, decreasing and integrable in a neighborhood of zero. Typical example is $g(s) = \frac{1}{s^\gamma}$ with $\gamma \in (0,1)$.

Most recently in Arcoya et al. (2014) D. Arcoya et al. solved problem (P_λ) in the case g continuous in $[0,\infty)$. Consequently, in the just mentioned paper the authors proved analogous results to that of semilinear elliptic problem (G_λ). They established that the maximal set of λ for which the problem (P_λ) has at least one solution is a closed interval $[0,\lambda^*]$, with $\lambda^* > 0$, and there exists a minimal regular solution for every $\lambda \in [0,\lambda^*)$ (compare Proposition 7.1.1). They also proved, under suitable conditions, that for $\lambda = \lambda^*$ there exists a minimal regular solution. Even more, they characterized minimal solutions as those solutions satisfying a stability condition. Motivated by this paper, our intention in the current work is to address this matter and provide statements that apply to the quasilinear problem having a singularity in the quadratic gradient term. To make our discussion more precise, under suitable hypotheses (see below hypotheses (H1)-(H4)) we prove in Theorem 7.2.9 a similar version of Crandall-Rabinowitz result (Proposition 7.1.1) for problem (P_λ). Moreover, assuming that

$$\lim_{s\to\infty} \frac{s(f'(s) - g(s)f(s))}{f(s)} = \alpha \in (1,\infty]$$

then u^* is a stable solution of (P_{λ^*}) (Theorem 7.3.6 and Corollary 7.3.7). We suggest that the reader refers to Brézis and Vázquez (1997) and compare this condition with $\lim_{s\to\infty} \frac{sf'(s)}{f(s)} = \alpha \in (1,\infty]$. We recall, following the definition introduced by D. Arcoya et al. Arcoya et al. (2014), that a stable solution in the literature of elliptic equations with quadratic growth in the gradient is a positive solution satisfying

$$\int_\Omega |\nabla\phi|^2 \geq \lambda \int_\Omega (f'(u) - g(u)f(u))\phi^2$$

for every $\phi \in \mathcal{W}(\Omega)$. Stability condition plays an important role in the process to determine when the extremal solutions are regular, we give sufficient conditions in Theorem 7.4.1. Finally, under the extra condition $f'(s) - g(s)f(s)$ is strictly increasing, we prove that stable solutions are minimal (Theorem

7.3.8). We would like to point out that, unlike the work of D. Arcoya et al., we use this extra condition exclusively for this last result.

The rest of this paper proceeds as follows: in Section 2, it is shown the existence of bounded minimal solutions for (P_λ) up to a given value λ^*. In addition, we prove that sequence of minimal solutions is increasing respect to λ. In Section 3, we deal with the stability and the issue of the circumstances under which u^* is a stable solution. Also, we establish the relation between minimal and stable solution. Finally, in Section 4 we proceed with the study of regularity of extremal solution and some examples are stated.

Notation. We denote by $|\Omega|$ the Lebesgue measure of$\Omega \subset \mathbb{R}^N$ and by 2^* the critical Sobolev exponent $2N/(N-2)$, $N > 2$. For every $s \in \mathbb{R}$ we consider $s^+ = \max\{s, 0\}$, $s^- = \min\{s, 0\}$ and the functions $G(s) = \int_0^s g(t)dt$, $\psi(s) = \int_0^s e^{-G(t)}dt$.

7.2 Existence of bounded minimal solutions

This section is devoted to the study of solutions of problem (P_λ). As in the semilinear case, it is expected that there exists an interval of values of λ such that there is at least one solution. Even more, we prove that there exists a parameter $\lambda^* > 0$ such that the problem has a minimal solution u_λ which is bounded if $0 < \lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$.

We recall that a function $0 < u \in \mathcal{W}(\Omega)$ is a (weak) solution of (P_λ) if $g(u)|\nabla u|^2$, $f(u) \in L^1(\Omega)$ and it satisfies

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega g(u)\, |\nabla u|^2\, \phi = \int_\Omega \lambda\, f(u)\, \phi, \tag{7.1}$$

for all test function $\phi \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$. As usual, supersolution (respectively subsolution) is defined analogously by replacing the equality "=" by the inequality "$\geq$", (resp. $\leq$), for positive test function.

We are interested in the case of functions g which are singular at zero, as a model case $g(s) = \dfrac{1}{s^\gamma}$, $\gamma \in (0, 1)$. In this way, the function g will be required to satisfy the following hypotheses

$$\lim_{s\to\infty} \sup g(s) < \infty \tag{H1}$$

$$f'(s) - g(s)f(s) > 0 \text{ and non-singular } (s \geq 0) \tag{H2}$$

$$e^{-G(s)} \in L^1(1, \infty) \tag{H3}$$

$$\forall C > 0,\ \exists \tilde{C} > 0 : g(Cs) \leq \tilde{C} g(s), \quad \forall s < 1 \tag{H4}$$

Remark 7.2.1 *We want to point out that the hypothesis (H2), which involves function f, in particular it implies that the function $f(s)e^{-G(s)}$ is increasing for $s \geq 0$. Moreover, model case satisfies hypotheses (H1), (H3), (H4) and (H2) taking for instance functions of kind $f(s) = h(s)e^{\frac{s^{1-\gamma}}{1-\gamma}}$, with $h(s)$ increasing and $h(0) > 0$, which also implies that $f(s)$ is concave in a neighborhood of zero. Another interesting case is $g(s) = \frac{1}{log(1+s^\gamma)}$ with $\gamma \in (0, 1)$. Additionally, we would like to highlight that functions $g(s) = c$ $(c > 0)$ are also considered.*

One of the main keys to study problems with singularities in the quadratic gradient term is to treat with test functions with compact support. For this reason it is appropriate to enunciate the following result, which ensures that solutions have a convenient estimate from below in compact sets.

Proposition 7.2.2 *For every compactly contained open subset $\omega \subset \Omega$ (i.e., $\omega \subset\subset \Omega$) there exists a constant $c_\omega > 0$ such that $u(x) \geq c_\omega$ a. e. $x \in \omega$ for every $u \in \mathcal{W}(\Omega)$ supersolution of problem (P_λ).*

Proof: To prove it we follow closely (Arcoya et al., 2011, Proposition 2.4). By the fact that $\lambda f(s) \geq \lambda f(0) \neq 0$ for every $s \geq 0$ then every supersolution $u \in \mathcal{W}(\Omega)$ of (P_λ) is a supersolution of problem

$$\begin{cases} -\Delta w + g(w)\,|\nabla w|^2 = \lambda\, f(0)\,, & \text{in}\Omega, \\ w > 0\,, & \text{in}\Omega, \\ w = 0\,, & \text{on } \partial\Omega. \end{cases} \tag{P_0}$$

The problem (P_0) has a solution w_0 in $\mathcal{W}(\Omega) \cap \mathcal{C}(\Omega)$ (see (Boccardo, 2008, Theorem 3.1)), in particular, since w_0 is continuous, it follows that for every compactly contained subset $\omega \subset \Omega$ there exists $\min_{\overline{\omega}} w_0 = c_\omega > 0$. Now by comparison principle due to (Arcoya and Segura de León, 2010, Theorem 2.7) we obtain that $u(x) \geq w_0(x) \geq c_\omega$ a.e. $x \in \omega$. $\square$

Lemma 7.2.3 *If g satisfies (H1), (H2) and (H3), then there exists $\bar{\lambda}$ such that (P_λ) admits no solution for $\lambda > \bar{\lambda}$.*

Proof: Let $u \in \mathcal{W}(\Omega)$ be a solution of (P_λ) and let ϕ_1 be the positive eigenfunction associated to λ_1, the first positive eigenvalue of the Laplacian operator $-\Delta$ with zero Dirichlet boundary conditions. We take $\varphi_n = e^{-G(u)}\tilde{\phi}_n$, $n \in \mathbb{N}$, where $0 \leq \tilde{\phi}_n \in \mathcal{C}_c^\infty(\Omega)$ such that $\tilde{\phi}_n \to \phi_1$ in $\mathcal{W}(\Omega)$. Since $\varphi_n \in L^\infty(\Omega)$ and $|\nabla\varphi_n| \leq e^{-G(u)}g(u)\tilde{\phi}_n|\nabla u| + e^{-G(u)}|\nabla\tilde{\phi}_n| \in L^2(\Omega)$ (by Proposition 7.2.2 and hypothesis (H1)), the function φ_n belongs to $\mathcal{W}(\Omega) \cap L^\infty(\Omega)$ and we can take it as test function in (7.1) to have

$$\int_\Omega e^{-G(u)}\nabla u \nabla\tilde{\phi}_n \geq \lambda \int_\Omega f(u)e^{-G(u)}\tilde{\phi}_n,$$

taking limits as n tends to ∞, we get

$$\int_\Omega e^{-G(u)}\nabla u \nabla\phi_1 = \lambda \int_\Omega f(u)e^{-G(u)}\phi_1.$$

On the one hand, let ψ be given by $\psi(s) = \int_0^s e^{-G(t)}dt$, then $e^{-G(u)}\nabla u = \nabla\psi(u)$ and $\psi(u) \in \mathcal{W}(\Omega)$ since $\psi(s)$ is a Lipschitz function, and on the other hand by hypothesis (H2) $f(s)e^{-G(s)} \geq f(0)$, we obtain

$$\int_\Omega \nabla\phi_1\nabla\psi(u) \geq \lambda f(0)\int_\Omega \phi_1.$$

Taking into account $\psi(s) \leq c_1$ by hypothesis (H3) and integrability of g near to zero,

$$\int_\Omega \nabla\phi_1\nabla\psi(u) \geq \frac{\lambda f(0)}{c_1}\int_\Omega \phi_1\psi(u).$$

Lastly, using that ϕ_1 is the eingefunction associated to λ_1, we conclude the proof taking $\bar{\lambda} \geq \dfrac{\lambda_1 c_1}{f(0)}$.

$\square$

Remark 7.2.4 *Even more, there exists $\bar{\lambda}$ such that (P_λ) admits no supersolution for $\lambda > \bar{\lambda}$. Indeed, the proof is similar starting with u a supersolution in place of a solution of (P_λ).*

We will consider $\mathcal{I}$ the set of values of $\lambda > 0$ such that there exists a solution of (P_λ). By the previous lemma $\mathcal{I} \subset (0, \bar{\lambda}]$. In order to prove the main result of this section let $\Phi(s)$ be a positive function given by

$$\Phi(s) = \psi^{-1}\left(\frac{\lambda}{\mu}\psi(s)\right), \quad 0 < \lambda < \mu. \tag{7.2}$$

We give some properties of function $\Phi(s)$.

Lemma 7.2.5 *Let $\Phi(s)$ be a positive function defined by (7.2). Then, following properties are satisfied:*

1. $0 \leq \Phi(s) \leq s$.
2. *If (H3) is satisfied then Φ is bounded.*
3. $0 < \Phi'(s) \leq \dfrac{\lambda}{\mu}$.
4. $\Phi''(s) = \Phi'(s)\left[g\left(\Phi(s)\right)\Phi'(s) - g(s)\right]$.

Proof:

1. Clearly $\Phi(s) \geq 0$. On the other hand, since $\dfrac{\lambda}{\mu}\psi(s) \leq \psi(s)$ and ψ^{-1} is increasing then
$$\Phi(s) = \psi^{-1}\left(\frac{\lambda}{\mu}\psi(s)\right) \leq \psi^{-1}\left(\psi(s)\right) = s.$$
2. Since $\psi(\infty) < \infty$ and $\dfrac{\lambda}{\mu} < 1$ we get the result.
3. An easy computation shows that
$$\Phi'(s) = \frac{\lambda}{\mu}\frac{e^{-G(s)}}{e^{-G(\Phi(s))}} = \frac{\lambda}{\mu}e^{G(\Phi(s))-G(s)} \leq \frac{\lambda}{\mu}.$$
using in the last inequality that G is increasing and $\Phi(s) \leq s$. Consequently, Φ is strictly increasing.
4. We may now compute the second derivative to conclude that
$$\Phi''(s) = \left(\frac{\lambda}{\mu}e^{G(\Phi(s))-G(s)}\right)' = \frac{\lambda}{\mu}e^{G(\Phi(s))-G(s)}\left(g(\Phi(s))\Phi'(s) - g(s)\right).$$

□

Proposition 7.2.6 *If g satisfies hypothesis (H1)-(H4) and u is a solution of (P_μ) $(\mu > 0)$ then, for every fixed $\lambda < \mu$, $\Phi(u)$ is a bounded supersolution of (P_λ).*

Proof: $\psi(s)$ is well-defined since g is continuous in $(0,\infty)$ and integrable near to zero. Furthermore, by hypothesis (H3) it is bounded, therefore $\Phi(u)$ is bounded using property (1) from Lemma 7.2.5. By the other hand, taking into account

$$|\nabla\Phi(u)| = \Phi'(u)|\nabla u| \leq \frac{\lambda}{\mu}|\nabla u| \in L^2(\Omega),$$

and $\Phi(u) = 0$ on $\partial\Omega$, it therefore follows that $\Phi(u) \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$. Moreover, we claim that functions $f(\Phi(u))$ and $g(\Phi(u))|\nabla\Phi(u)|^2$ are in $L^1(\Omega)$. Indeed, since f is continuous and Φ is bounded we deduce that $f(\Phi(u)) \in L^1(\Omega)$. Now we prove that $g(\Phi(u))|\nabla\Phi(u)|^2 \in L^1(\Omega)$, to this end, we define the subset of$\Omega_{\,\varepsilon}$ as $\{x \in \Omega :\ u(x) < \varepsilon\}$ where $0 < \varepsilon < 1$ is such that $g(s)$ is decreasing in $(0,\varepsilon)$. On one side, if $u \geq \varepsilon$ then$\Phi(\ u) \geq \Phi(\varepsilon)$ since Φ is increasing, in addition of $\Phi(u)$ is bounded and g is continuous gives $g(\Phi(u)) \leq C$ a.e. $x \in \Omega \setminus \Omega_\varepsilon$ and from the fact that $\Phi(u) \in \mathcal{W}(\Omega)$ we obtain that $g(\Phi(u))|\nabla\Phi(u)|^2 \in L^1(\Omega \setminus \Omega_\varepsilon)$.

On the other side, again by property (1) from Lemma 7.2.5 we obtain $0 < \Phi(s) \leq \varepsilon$, $s \in (0,\varepsilon)$ and since

$$\lim_{s\to 0^+}\frac{\Phi(s)}{s} = \lim_{s\to 0^+}\Phi'(s) = \frac{\lambda}{\mu},$$

let $C_\varepsilon > 0$ be the infimum of $\frac{\Phi(s)}{s}$ for $s \in (0,\varepsilon)$, namely,$\Phi(\ s) \geq C_\varepsilon s \ \forall s \in (0,\varepsilon)$. Now, by the fact that $g(s)$ is decreasing in $(0,\varepsilon)$ and $\Phi(s), C_\varepsilon s \in (0,\varepsilon)$ then $g(\Phi(s)) \leq g(sC_\varepsilon)$ in $(0,\varepsilon)$. Taking also into account the hypothesis (H4) there exists $\tilde{C}_\varepsilon > 0$ such that $g(sC_\varepsilon) \leq \tilde{C}_\varepsilon g(s)$ and

$$g(\Phi(u))|\nabla\Phi(u)|^2 \leq \tilde{C}_\varepsilon \left(\frac{\lambda}{\mu}\right)^2 g(u)|\nabla u|^2 \in L^1(\Omega_\varepsilon),$$

proving the claim. As a result, up to now $\Phi(u) \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$ and $f(\Phi(u)), g(\Phi(u))|\nabla\Phi(u)|^2 \in L^1(\Omega)$. To conclude the proof we verify that $\Phi(u)$ is a supersolution of (P_λ), i.e.,

$$\int_\Omega \nabla\Phi(u)\nabla\varphi + \int_\Omega g(\Phi(u))\,|\nabla\Phi(u)|^2\,\varphi \geq \int_\Omega \lambda\, f(\Phi(u))\,\varphi,$$

for all $0 \leq \varphi \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$. For every fixed $0 \leq \varphi \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$ let $\{\varphi_n\}_{n\in\mathbb{N}}$ be positive functions in $\mathcal{C}_c^\infty(\Omega)$ such that $\varphi_n \to \varphi$ in $\mathcal{W}(\Omega)$. Then $\phi_n = \Phi'(u)\varphi_n \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$, indeed, since $\Phi'(u) \leq \frac{\lambda}{\mu}$ then $\phi_n \in L^\infty(\Omega)$ and by property (4) from Lemma 7.2.5

$$|\nabla\phi_n|^2 \leq \left(\frac{\lambda}{\mu}\right)^2 \left(|\nabla\varphi_n|^2 + \left(g(\Phi(u))|\nabla\Phi(u)| + g(u)|\nabla u|\right)^2 \varphi_n^2\right),$$

and the fact that $u(x) \geq c_{\omega_n}$ for a. e. $x \in \omega_n$, where $\omega_n = \operatorname{supp}\varphi_n$, in addition to hypothesis (H1) we obtain that $g(u), g(\Phi(u)) \in L^\infty(\omega_n)$ and $|\nabla\phi_n|^2 \in L^1(\Omega)$.

Therefore, taking ϕ_n as a test function in problem (P_μ)

$$\int_\Omega \nabla u\left(\Phi''(u)\nabla u\,\varphi_n + \Phi'(u)\nabla\varphi_n\right) + \int_\Omega g(u)|\nabla u|^2\Phi'(u)\varphi_n =$$
$$\mu\int_\Omega f(u)\Phi'(u)\varphi_n \geq \lambda\int_\Omega f(\Phi(u))\varphi_n,$$

using in the last inequality that $\mu f(u)\Phi'(u) = \lambda f(\Phi(u))\dfrac{e^{-G(u)}f(u)}{e^{-G(\Phi(u))}f(\Phi(u))}$ and hypothesis (H2).

Lastly, adding and subtracting $|\nabla\Phi(u)|^2 g(\Phi(u))\varphi_n$ together with the fact that the term $\frac{\Phi''(u)}{\Phi'(u)} + g(u) - \Phi'(u)g(\Phi(u))$ is equal to zero, we have for all $n \in \mathbb{N}$

$$\int_\Omega \nabla\Phi(u)\nabla\varphi_n + \int_\Omega g(\Phi(u))\,|\nabla\Phi(u)|^2\,\varphi_n \geq \int_\Omega \lambda\, f(\Phi(u))\,\varphi_n,$$

since $|\nabla\Phi(u)|^2$, $g(\Phi(u))\,|\nabla\Phi(u)|^2$, $f(\Phi(u)) \in L^1(\Omega)$ and $\varphi_n \to \varphi$ in $\mathcal{W}(\Omega)$, we take the limit when n tends to ∞ and we conclude the proof. □

Remark 7.2.7 *Contrary to others works on this topic, this supersolution depends on the quadratic gradient term $g(s)$, and not on the nonlinearity term $f(s)$ (compare Arcoya et al. (2014) and Brézis et al. (1996)). This allows us to deal with functions f less restrictive, for instance, in Arcoya et al. (2014) the authors impose $f'(s) - g(s)f(s)$ is an increasing function, conversely this condition is not required in this section, in fact no-convex functions such as $f(s) = e^{G(s)}e^{(s+\delta)^\delta}$ with δ small enough are allowed, being $f'(s) - g(s)f(s)$ decreasing near to zero.*

This result will prove to be extremely useful in the following theorem which ensures that set $\mathcal{I}$ is an interval.

Theorem 7.2.8 *Assume that g satisfies hypotheses (H1)-(H4) and fix $\mu \in \mathcal{I}$, then for every $\lambda \in (0,\mu)$ there exists a bounded minimal solution of (P_λ).*

Proof: First we prove that there exists a bounded solution. To prove it we use a standard monotone iteration argument: let w_0 the bounded solution of problem (P_0) in the proof of Proposition

7.2.2, we point out that w_0 is unique due to (Arcoya and Segura de León, 2010, Theorem 2.9). For every $n \geq 1$ we define the recurrent sequence $\{w_n\}$ by

$$\begin{cases} -\Delta w_n + g(w_n)\,|\nabla w_n|^2 = \lambda\, f(w_{n-1})\,, & \text{in}\Omega, \\ w_n > 0\,, & \text{in}\Omega, \\ w_n = 0\,, & \text{on } \partial\Omega. \end{cases} \tag{P_n}$$

The sequence $\{w_n\}$ is well defined by Boccardo (2008) and Arcoya and Segura de León (2010), even more, the sequence is increasing, to check that it suffices to prove that $w_0 \leq w_1$. Indeed, taking in account that $0 < w_0$ and f is increasing we obtain $\lambda f(0) \leq \lambda f(w_0)$ and by comparison principle, which is due to Arcoya and Segura de León (2010), it follows that $w_0 \leq w_1$ and by induction argument $0 < w_0 \leq w_1 \leq \cdots \leq w_n$, for all $n \geq 1$. By the fact that $\Phi(u)$, defined by (7.2), is a supersolution of problem (P_0), with a similar argument we prove that $w_n \leq \Phi(u)$ for every $n \in \mathbb{N}$.

Since $\Phi(u) \in L^\infty(\Omega)$, the sequence $\{w_n(x)\}$ is increasing and bounded by $\Phi(u)(x)$ for a. e. $x \in \Omega$. Let $w_\lambda(x)$ be the limit almost every where in Ω (i. e., $w_\lambda(x) := \lim_{n\to\infty} w_n(x)$ a. e. $x \in \Omega$). We claim that $w_\lambda \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$. Indeed, clearly $w_\lambda \in L^\infty(\Omega)$ since $w_\lambda \leq \Phi(u) \in L^\infty(\Omega)$. Moreover, as $w_n \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$ we can take it as a test function in problem (P_n)

$$\int_\Omega |\nabla w_n|^2 + \int_\Omega g(w_n)|\nabla w_n|^2 w_n = \lambda \int_\Omega f(w_{n-1})w_n,$$

dropping the positive term $g(w_n)|\nabla w_n|^2 w_n$, since $w_{n-1} \leq w_n \leq \Phi(u)$ and f is increasing it follows that

$$\int_\Omega |\nabla w_n|^2 \leq \lambda \int_\Omega f(\Phi(u))\Phi(u) \leq \lambda f(\|\Phi(u)\|_\infty)\|\Phi(u)\|_\infty|\Omega|.$$

That is, $\{w_n\}$ is uniformly bounded in $\mathcal{W}(\Omega)$ and, up to a subsequence, there exists $\tilde{w}$ such that w_n converges weakly to $\tilde{w}$ in $\mathcal{W}(\Omega)$ and $w_n(x) \to \tilde{w}(x)$ a. e. $x \in \Omega$, by the unicity of the limit $w_\lambda = \tilde{w} \in \mathcal{W}(\Omega)$ and we conclude the claim.

We now verify that w_λ is solution of (P_λ). In order to prove it we define the operator $K : \mathcal{W}(\Omega) \to \mathcal{W}(\Omega)$ by $K[v]$ as the unique solution of problem

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = v^+ + \lambda\, f(0)\,, & \text{in}\Omega, \\ u > 0\,, & \text{in}\Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

K is well defined (see Boccardo (2008) and Arcoya and Segura de León (2010)), even more, due to (Arcoya et al., 2011, Proposition 2.5) K is a compact operator. We remark that with this notation w_n is solution of (P_n) if and only if $w_n = K[\lambda(f(w_{n-1}) - f(0))]$. Now taking limits and considering that w_n converges weakly to w_λ in $\mathcal{W}(\Omega)$ we obtain that $w_\lambda = K[\lambda(f(w_\lambda) - f(0))]$, that is, w_λ is a solution of (P_λ).

Our next claim is that the interval $\mathcal{I}$ is not empty. Indeed, we proceed to show that there exists $\tilde{\lambda} \in \mathcal{I}$. In order to get this, we fix $k > 0$ and we consider $\tilde{u} \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$, $\|\tilde{u}\|_\infty \leq \tilde{c}$, the unique solution of problem

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = k\,, & \text{in}\Omega, \\ u > 0\,, & \text{in}\Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

we take $\tilde{\lambda} \in (0, \delta)$, where $0 < \delta \leq \dfrac{k}{f(\tilde{c})}$, to obtain for all $\varphi \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$

$$\int_\Omega \nabla\tilde{u}\nabla\varphi + \int_\Omega g(\tilde{u})|\nabla\tilde{u}|^2\varphi = \int_\Omega k\varphi \geq \int_\Omega \delta f(\tilde{c})\varphi \geq \tilde{\lambda}\int_\Omega f(\tilde{u})\varphi,$$

that $\tilde{u}$ is a bounded supersolution of $(P_{\tilde{\lambda}})$. We now apply the standard monotone iteration argument again, with the bounded supersolution $\Phi(u)$ replaced by $\tilde{u}$, to obtain $u_{\tilde{\lambda}}$ a bounded solution of problem $(P_{\tilde{\lambda}})$ and finally that $\mathcal{I} \neq \emptyset$.

Note that we have actually proved that if $\mu \in \mathcal{I}$ then $(0,\mu] \subset \mathcal{I}$, even more, for every $\lambda \in (0,\mu)$ there exists a bounded solution of (P_λ). The proof is completed by showing that solutions w_λ are minimal, indeed, let v_λ be a solution of problem (P_λ), by a similar argument of comparison principle and by induction in n we have $w_n \leq v_\lambda$ for all $n \in \mathbb{N}$ as $w_\lambda(x) := \lim_{n\to\infty} w_n(x)$ a. e. $x \in \Omega$ thus $w_\lambda \leq v_\lambda$.

□

Theorem 7.2.8 and Lemma 7.2.3 may be summarized by formulating our main result of this section

Theorem 7.2.9 *Assume that g satisfies hypotheses (H1)-(H4). Then there exists $\lambda^* \in (0,\bar{\lambda}]$ such that there is a bounded minimal solution of (P_λ) for every $\lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$.*

Remark 7.2.10 *We note that if $\lambda_1 \leq \lambda_2 < \lambda^*$, taking w_{λ_2} as a supersolution of problem (P_{λ_1}) and arguing as the proof of Theorem 7.2.8 we obtain $w_{\lambda_1} \leq w_{\lambda_2}$. That is, the family of functions $\{w_\lambda\}_{\lambda\in\mathcal{I}}$ are increasing.*

Remark 7.2.11 *It is worth pointing out that for every fixed arbitrary $\mu \in \mathcal{I}$ sufficiently small and u a solution of (P_μ), it follows that $\Phi(u) = \psi^{-1}\left(\frac{\lambda}{\mu}\psi(u)\right)$ tends to zero as $\lambda \to 0$. Hence, for every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that $w_\nu(x) < \varepsilon$ for every $0 < \nu < \eta$.*

7.3 Stability and extremal solutions

As we have stated at the Remark 7.2.10, the mapping $\lambda \to u_\lambda$ is increasing in $(0,\lambda^*)$, a.e. $x \in \Omega$. This allows one define $u^* := \lim_{\lambda\to\lambda^*} u_\lambda$ and we call u^* the extremal solution of problem (P_λ). In Brézis et al. (1996) and Arcoya et al. (2014) the authors proved that u^* is a weak solution for the semilinear and quasilinear problem, respectively. In order to prove the same effect for the singular quadratic quasilinear case we give a property of the minimal solutions, its stability.

Definition 7.3.1 *Let u be a solution of (P_λ), we say that u is stable if $f'(u) - g(u)f(u) \in L^1_{loc}(\Omega)$ and*

$$\int_\Omega |\nabla\phi|^2 \geq \lambda \int_\Omega (f'(u) - g(u)f(u))\phi^2 \tag{7.3}$$

holds for every $\phi \in \mathcal{C}_c^\infty(\Omega)$.

Since $f'(u) - g(u)f(u) > 0$ it follows that, by a standard approximation argument and Fatou Lemma, one can take $\phi \in \mathcal{W}(\Omega)$ in the above definition.

The following result may be proved in much the same way as (Arcoya et al., 2014, Lemma 3.7).

Lemma 7.3.2 *Minimal bounded solutions of (P_λ) are stable.*

Our next goal is to prove that stability condition (7.3) (and under extra condition) allows us to ensure that minimal bounded solutions are uniformly bounded in $\mathcal{W}(\Omega)$. For that purpose we give the following technnical lemma.

Lemma 7.3.3 *Let f and g be two positive continuous functions in $(0,\infty)$ with f increasing and sastifying the condition*

$$\lim_{s\to\infty} \frac{s(f'(s) - g(s)f(s))}{f(s)} > 0.$$

Then, for every positive $\delta < \alpha$, there exits a positive constant $C(\delta)$ (depending only on δ) such that $f(s)s \leq \frac{1}{\delta}s^2(f'(s) - g(s)f(s)) + C(\delta)$ for all $s \geq 0$.

Proof: By definition of limit: for all $\varepsilon > 0$ there exists $s_0(\varepsilon)$ depends to ε such that

$$\left|\frac{s(f'(s) - g(s)f(s))}{f(s)} - \alpha\right| < \varepsilon \ , \ \forall s \geq s_0(\varepsilon),$$

choosing $\varepsilon = \alpha - \delta$ and multiplying by s we obtain that there exists $s_0(\delta)$ such that

$$s^2(f'(s) - g(s)f(s)) \geq \delta s f(s), \ \ \forall s \geq s_0(\delta),$$

By the other hand, since f is increasing, $f(s)s < f(s_0(\delta))s_0(\delta)$ for all $s < s_0(\delta)$. Hence taking $C(\delta) = \frac{f(s_0(\delta))s_0(\delta)}{\delta}$ we conclude the proof. $\square$

Proposition 7.3.4 *Let $\{w_\lambda\}$ be a sequence of minimal bounded solutions of problem (P_λ) such that f and g satisfy the condition*

$$\lim_{s\to\infty} \frac{s(f'(s) - g(s)f(s))}{f(s)} = \alpha \in (1, \infty]. \tag{7.4}$$

Then, the sequence is uniformly bounded in $\mathcal{W}(\Omega)$.

Proof: Let w_λ be the minimal bounded solution of (P_λ) taken as a test function in (7.1) and dropped the positive term $g(w_\lambda)|\nabla w_\lambda|^2 w_\lambda$ we obtain

$$\int_\Omega |\nabla w_\lambda|^2 \leq \lambda \int_\Omega f(w_\lambda)w_\lambda.$$

In addition, by Lemma 7.3.3

$$\int_\Omega |\nabla w_\lambda|^2 \leq \frac{\lambda}{\delta}\int_\Omega \big(f'(w_\lambda) - g(w_\lambda)f(w_\lambda)\big)\, w_\lambda^2 + C_1,$$

with $C_1 = \lambda^* C(\delta)|\Omega|$.

While on the other hand, by Lemma 7.3.2 w_λ satisfies the stability condition, hence choosing $\phi = w_\lambda$ in (7.3)

$$\int_\Omega |\nabla w_\lambda|^2 \geq \lambda \int_\Omega \big(f'(w_\lambda) - g(w_\lambda)f(w_\lambda)\big)\, w_\lambda^2.$$

Finally, by combining the last two inequalities and taking $\delta > 1$ the proposition follows. $\square$

Remark 7.3.5 *We note that above proof also involves the boundedness of $\int_\Omega f(w_\lambda)w_\lambda$ for $\lambda \in (0, \lambda^*)$.*

The remainder of this section will be devoted to the proof of our main result, namely the extremal solution u^* is a solution of problem (P_{λ^*}).

Theorem 7.3.6 *Under the hypotheses (H1)-(H4) and condition (7.4), $w_\lambda(x)$ converges to $u^*(x)$ a. e. $x \in \Omega$, a solution of (P_{λ^*}).*

Proof: Thanks to Proposition 7.3.4 there exists $C_1 > 0$ independent of λ such that $\|w_\lambda\|_{\mathcal{W}(\Omega)} \leq C_1$ for all $\lambda \in (0, \lambda^*)$. Therefore, up to a subsequence, w_λ converges to u^* weakly in $H_0^1(\Omega)$ ($w_\lambda \rightharpoonup u^*$), strongly in $L^s(\Omega)$ $(1 \leq s < 2^*)$ and almost everywhere in Ω,

$$w_\lambda(x) \longrightarrow u^*(x), \qquad \text{a.e. } x \in \Omega. \tag{7.5}$$

It should be noted that, as $w_\lambda(x)$ is increasing, the whole sequence converges almost everywhere to $u^*(x) > 0$.

Now we prove that u^* is a solution of (P_{λ^*}), i. e. $g(u^*)|\nabla u^*|^2$, $f(u^*) \in L^1(\Omega)$ and satisfies (7.1). First we claim that $f(w_\lambda)$ is uniformly bounded in $L^1(\Omega)$, indeed fixed $\rho > 0$ then $f(s) \leq f(\rho) + \frac{1}{\rho} f(s)s$ for every $s \geq 0$, thus

$$\int_\Omega f(w_\lambda) \leq f(\rho)|\Omega| + \frac{1}{\rho}\int_\Omega f(w_\lambda)w_\lambda$$

and by Remark 7.3.5, the last expression is bounded, proving the claim. Therefore the boundedness of $f(w_\lambda)$ in $L^1(\Omega)$ combined with the fact that $f(w_\lambda)$ is increasing, the monotone convergence theorem implies that $f(u^*) \in L^1(\Omega)$.

Concerning the term $g(u^*)\,|\nabla u^*|^2$, taking $\varphi = \frac{T_\varepsilon(w_\lambda)}{\varepsilon}$ as test function in (7.1), where $T_\varepsilon(s) := \min\{s,\varepsilon\}$, thereby $\frac{T_\varepsilon(w_\lambda)}{\varepsilon} \leq 1$ and $\nabla T_\varepsilon(w_\lambda) = \nabla w_\lambda \cdot \chi_{\{w_\lambda \leq \varepsilon\}}$, we get

$$\int_\Omega |\nabla w_\lambda|^2 \cdot \chi_{\{w_\lambda \leq \varepsilon\}} + \int_\Omega g(w_\lambda)|\nabla w_\lambda|^2 \frac{T_\varepsilon(w_\lambda)}{\varepsilon} \leq \lambda^* \int_\Omega f(w_\lambda).$$

Dropping the positive term $|\nabla w_\lambda|^2 \cdot \chi_{\{w_\lambda \leq \varepsilon\}}$ and taking into account the boundedness of $f(w_\lambda)$ in $L^1(\Omega)$ we obtain that there exists a positive constant C_2 such that

$$\int_\Omega g(w_\lambda)|\nabla w_\lambda|^2 \frac{T_\varepsilon(w_\lambda)}{\varepsilon} \leq C_2.$$

Taking the limit as $\varepsilon \to 0$ and having in mind that $\lim\limits_{\varepsilon \to 0} \frac{T_\varepsilon(w_\lambda)}{\varepsilon} = 1$, we get from the Lebesgue dominated convergence theorem

$$\int_\Omega g(w_\lambda)|\nabla w_\lambda|^2 \leq C_2,$$

for every $\lambda \in (0, \lambda^*)$. Now, the result of (Boccardo and Murat, 1992, Theorem 2.1) yields that (up to a subsequence) $\nabla w_\lambda \to \nabla u^*$ converges strongly in $(L^q(\Omega))^N$ $(1 < q < 2)$, particularly it converges almost everywhere in Ω. Then we have, by Fatou lemma, $g(u^*)|\nabla u^*|^2 \in L^1(\Omega)$.

To close, following closely Boccardo (2008), we proceed to show that u^* satisfies the equation (7.1). Since $\phi = \phi^+ + \phi^-$, it is enough to prove it for every nonegative function $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Furthermore, by density, it is sufficient to prove it when $0 \leq \phi \in H_0^1(\Omega) \cap C_c(\Omega)$. First we claim that u^* is a subsolution. Indeed, from

$$\int_\Omega g(w_\lambda)\,|\nabla w_\lambda|^2\,\phi = \lambda \int_\Omega f(w_\lambda)\phi - \int_\Omega \nabla w_\lambda \nabla \phi,$$

we apply the Fatou lemma on the left side. In regards to the right-hand side, since w_λ converges weakly to u^* in $\mathcal{W}(\Omega)$ and the boundedness of $f(w_\lambda)$ in $L^1(\Omega)$ we take limits and the claim is proved.

On the other hand, our next claim is that u^* is a supersolution. Choosing $\varphi = e^{G(T_k(u^*)) - G(w_\lambda)}\,\phi$ as a test function we obtain

$$\int_\Omega e^{G(T_k(u^*)) - G(w_\lambda)} \nabla w_\lambda \nabla \phi + \int_\Omega e^{G(T_k(u^*)) - G(w_\lambda)} g(T_k(u^*)) \nabla T_k(u^*) \nabla w_\lambda \phi$$
$$= \lambda \int_\Omega f(w_\lambda) e^{G(T_k(u^*)) - G(w_\lambda)}\,\phi.$$

Since w_λ converges weakly to u^* and by the strong convergence of $e^{G(T_k(u^*)) - G(w_\lambda)}$ to $e^{G(T_k(u^*)) - G(u^*)}$, hence taking limits as λ tends to λ^* and again by Fatou lemma on the right side it follows that

$$\int_\Omega e^{G(T_k(u^*)) - G(u^*)} \nabla u^* \nabla \phi + \int_\Omega e^{G(T_k(u^*)) - G(u^*)} g(T_k(u^*)) \nabla T_k(u^*) \nabla u^* \phi$$
$$\geq \lambda^* \int_\Omega f(u^*) e^{G(T_k(u^*)) - G(u^*)}\,\phi.$$

Finally, since ϕ has compact support, there exists a positive constant such that $u^* \geq w_\lambda \geq C_\phi$, that is, $g(u^*)$ is bounded in $supp\phi$. We pass to the limit as $k \to \infty$ and by dominated convergence theorem we obtain the desired converse inequality for compact support functions. Using density argument we finish the proof.

$\square$

Corollary 7.3.7 *Under the hypotheses of Theorem 7.3.6 the extremal solution u^* is stable.*

Proof: Since w_λ is stable, it follows that

$$\int_\Omega |\nabla\phi|^2 \geq \lambda \int_\Omega (f'(w_\lambda) - g(w_\lambda)f(w_\lambda))\phi^2,$$

letting $\lambda \to \lambda^*$ and by Fatou lemma imply that u^* satisfies condition (7.3). Theorem 7.3.6 now shows that u^* is stable. $\square$

We have been working under the assumption that $f'(s) - g(s)f(s)$ is not necessarily increasing. In the remainder of this section we assume $f'(s) - g(s)f(s)$ to be increasing.

Theorem 7.3.8 *Assume the hypotheses (H1)-(H4) hold and $f'(s) - g(s)f(s)$ is strictly increasing. Then every stable solution of problem (P_λ) is minimal.*

Proof: Let u be a stable solution of (P_λ) and suppose, contrary to our claim, that there exists $v \in \mathcal{W}(\Omega)$ a solution of (P_λ) and $\mathcal{O} \subset \Omega$ $(|\mathcal{O}| \neq 0)$ such that $v < u$ in $\mathcal{O}$.

On the one hand, choosing $e^{-G(u)}\phi$ $(\phi \in \mathcal{C}_c^\infty)$ as a test function in the equation (7.1) satisfied by u

$$\int_\Omega e^{-G(u)}\nabla u \nabla\phi = \lambda \int_\Omega f(u)e^{-G(u)}\phi, \tag{7.6}$$

and by a standard approximation argument the above equation is satisfied for every $\phi \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$.

Analogously, choosing $e^{-G(v)}\phi$ on the equation which is satisfied by v,

$$\int_\Omega e^{-G(v)}\nabla v \nabla\phi = \lambda \int_\Omega f(v)e^{-G(v)}\phi, \tag{7.7}$$

for every $\phi \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$. Now, subtracting (7.7) from (7.6) and writing $\psi(s)$ instead of $\int_0^s e^{-G(t)}dt$, this gives

$$\int_\Omega \nabla(\psi(u) - \psi(v))\nabla\phi = \lambda \int_\Omega \left(f(u)e^{-G(u)} - f(v)e^{-G(v)}\right)\phi.$$

Taking $\phi = (\psi(u) - \psi(v))^+$ in the above equation, which is zero in$\Omega \setminus \mathcal{O}$, since ψ is increasing and $v < u$ in $\mathcal{O}$. We have

$$\int_\mathcal{O} |\nabla(\psi(u) - \psi(v))|^2 = \lambda \int_\mathcal{O} \left(f(u)e^{-G(u)} - f(v)e^{-G(v)}\right)(\psi(u) - \psi(v)). \tag{7.8}$$

On the other hand, taking $\phi = (\psi(u) - \psi(v))^+$ on the stability condition (7.3) satisfied by u, it gives

$$\int_\mathcal{O} |\nabla(\psi(u) - \psi(v))^+|^2 \geq \lambda \int_\mathcal{O} (f'(u) - g(u)f(u))\left[(\psi(u) - \psi(v))^+\right]^2. \tag{7.9}$$

Now combining (7.8) with (7.9) yields

$$\int_\mathcal{O} \left[(f'(u) - g(u)f(u))z - \left(f(u)e^{-G(u)} - f(v)e^{-G(v)}\right)\right] z \leq 0, \tag{7.10}$$

here and subsequently, z denotes $\psi(u) - \psi(v)$. Note that $z > 0$ in $\mathcal{O}$. Our claim is that $(f'(u) - g(u)f(u))z - (f(u)e^{-G(u)} - f(v)e^{-G(v)}) > 0$, which leads to a contradiction with (7.10), therefore

$z \leq 0$ and concluding that $u \leq v$ in $\mathcal{O}$. To prove the claim it is sufficient to show that $f'(u) - g(u)f(u) - \frac{f(u)e^{-G(u)} - f(v)e^{-G(v)}}{z}$ is positive. Thus, since by the Mean Value Theorem there exists $\tilde{u} \in [v, u]$, a. e. $x \in \mathcal{O}$, such that

$$\frac{f(u)e^{-G(u)} - f(v)e^{-G(v)}}{z} = \frac{f'(\tilde{u})e^{-G(\tilde{u})} - g(\tilde{u})f(\tilde{u})e^{-G(\tilde{u})}}{e^{-G(\tilde{u})}}$$

$$= f'(\tilde{u}) - g(\tilde{u})f(\tilde{u}),$$

hence, with the fact that $f'(s) - g(s)f(s)$ is strictly increasing and $\tilde{u} \leq u$ a. e. in $\mathcal{O}$, the claim is proved and the theorem follows. □

Corollary 7.3.9 *Under the hypotheses of Theorem 7.3.6. If in addition, $f'(s) - g(s)f(s)$ is strictly increasing. Then the extremal solution u^* is stable and minimal.*

Proof: Clearly, by Corollary 7.3.7 the extremal solution u^* given by Theorem 7.3.6 is stable and consequently, applying Theorem 7.3.8 we complete the proof. □

Corollary 7.3.10 *Under the assumptions of Theorem 7.3.8. If u is an stable and singular solution of (P_λ) then $\lambda = \lambda^*$.*

Proof: By Theorem 7.3.8 u is the minimal solution of (P_λ) and Theorem 7.2.8 assures that u is bounded for $\lambda \in (0, \lambda^*)$ which implies, since u is singular, that $\lambda = \lambda^*$. □

7.4 Regularity of extremal solutions

The *extremal solution* u^* may be bounded or singular. In Brézis and Vázquez (1997) H. Brezis and J.L. Vázquez raised the question of determining the regularity of u^* depending on the dimension N, this problem led to the study of the regularity theory of stable solutions which many authors are interested (Cabré and Capella (2006); Nedev (2000); Villegas (2013)). In this section, we will obtain, under suitable conditions depending on the dimension N, the regularity of extremal solutions for the quasilinear case with singularity in the quadratic gradient term.

In what follows, we write the nonlinearity term of (P_λ) as $e^{G(s)}h(s)$ instead of $f(s)$, where $h(0) > 0$ and h is a derivable function in $[0, \infty)$. We note that with this notation hypothesis (H2) is equivalent to impose $h(s)$ is increasing. In this way, we replace problem (P_λ) by the following

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = \lambda\, e^{G(u)} h(u)\,, & \text{in}\Omega, \\ u > 0\,, & \text{in}\Omega, \\ u = 0\,, & \text{on } \partial\Omega. \end{cases} \tag{Q_λ}$$

We can now formulate our main result of this section.

Theorem 7.4.1 *Under hypotheses (H1)-(H4) and*

$$\lim_{s \to \infty} \frac{sh'(s)}{h(s)} > 1 \tag{7.11}$$

The extremal solution of (Q_λ) given in Theorem 7.3.6 is bounded whenever

$$N < \frac{4 + 2(\tilde{\mu} + \tilde{\alpha}) + 4\sqrt{\tilde{\mu} + \tilde{\alpha}}}{1 + \tilde{\alpha}}, \tag{7.12}$$

$\tilde{\alpha}$ and $\tilde{\mu}$ being the following parameters

$$\tilde{\alpha} := \lim_{s \to \infty} \frac{g(s)h(s)}{h'(s)}, \quad \tilde{\mu} := \lim_{s \to \infty} \frac{h''(s)h(s)}{(h'(s))^2}. \tag{7.13}$$

Remark 7.4.2 *Comparing the above theorem with (Arcoya et al., 2014, Theorem 4.7) we obtain similar results replacing $\tilde{\alpha}$ and $\tilde{\mu}$ by*

$$\alpha = \frac{\tilde{\alpha}}{1+\tilde{\alpha}}, \qquad \mu = \frac{\tilde{\alpha}+\tilde{\mu}}{\tilde{\alpha}+1}.$$

However, in addition to the singularity of function g, some hypotheses of (Arcoya et al., 2014, Theorem 4.7) such as $\alpha < 1$, $\frac{1}{f} \in L^1$, $\left|\frac{f'(s)}{f^2(s)}\right| \leq c_2(1+\sqrt{g(s)})$ or $f'(s)-g(s)f(s)$ is increasing, are not necessary. We wish to emphasize that last hypothesis allow us to deal with functions $f(s)$ no-convex.

Proof: Due to Stampacchia Lemma ((Stampacchia, 1966, Lemma 5.1)), we have to show that $e^{G(u^*)}h(u^*) \in L^\beta(\Omega)$ with $\beta > N/2$.

By (7.12) we fix

$$\beta \in \left(\frac{N}{2}, \frac{2+(\tilde{\mu}+\tilde{\alpha})+2\sqrt{\tilde{\mu}+\tilde{\alpha}}}{1+\tilde{\alpha}}\right), \tag{7.14}$$

and let us consider the following positive differentiable function

$$\phi(s) = \sqrt{\frac{h(s)^\beta \left(e^{G(s)}\right)^{\beta-1}}{h'(s)}}, \quad s \geq R,$$

such that $\phi(0) = 0$ and $\phi \in \mathcal{C}^1[0, R]$. For $\lambda < \lambda^*$ let u_λ be the bounded minimal solution of (Q_λ) given by Theorem 7.2.8 which, under the assumptions of Theorem 7.3.6 with condition (7.4) replaced by condition (7.11), converges to $u^*(x)$ a. e. $x \in \Omega$. In addition to Lemma 7.3.2, u_λ satisfies the stability condition, in this way, taking $\phi(u_\lambda)$ in (7.3) (clearly $\phi(u_\lambda) \in \mathcal{W}(\Omega)$ since u_λ is bounded) we obtain

$$\begin{aligned}\int_\Omega (\phi'(u_\lambda))^2|\nabla u_\lambda|^2 \geq \lambda \int_{\Omega_R} e^{G(u_\lambda)}h'(u_\lambda)\phi^2(u_\lambda) + \lambda \int_\Omega e^{\beta G(u_\lambda)}h^\beta(u_\lambda) \\ -\lambda \int_{\Omega_R} e^{\beta G(u_\lambda)}h^\beta(u_\lambda)\end{aligned} \tag{7.15}$$

where$\Omega_R = \{x \in \Omega : u_\lambda(x) < R\}$. Computing, we have

$$\phi'(s) = \frac{\phi(s)}{2}\left(\beta\frac{h'(s)}{h(s)} + (\beta-1)g(s) - \frac{h''(s)}{h'(s)}\right). \tag{7.16}$$

While on the other hand, we define

$$\zeta(s) := e^{-G(s)} \int_0^s (\phi'(t))^2 e^{G(t)} dt,$$

since u_λ is bounded if follows that $\zeta(u_\lambda) \in L^\infty(\Omega)$, and applying L'Hôpital rule we obtain

$$\lim_{s\to 0} \frac{\int_0^s (\phi'(t))^2 e^{G(t)} dt}{s} = (\phi'(0))^2 < \infty$$

since $\phi \in \mathcal{C}^1[0, R]$. Thus

$$\zeta'(0) = \lim_{s\to 0} \frac{\zeta(s)}{s}$$

and therefore $\zeta'(u_\lambda) \in L^\infty(\Omega)$ and $\zeta(u_\lambda) \in \mathcal{W}(\Omega)$. Furthermore, using (7.13) and L'Hôpital rule we get

$$\begin{aligned}\lim_{s\to\infty} \frac{\zeta(s)}{(e^{G(s)}h(s))^{\beta-1}} &= \lim_{s\to\infty} \frac{\int_0^s (\phi'(t))^2 e^{G(t)} dt}{e^{\beta G(s)}h(s)^{\beta-1}} \\ &= \lim_{s\to\infty} \frac{(\phi'(s))^2 e^{(1-\beta)G(s)}}{h(s)^{\beta-2}(\beta g(s)h(s) + (\beta-1)h'(s))}\end{aligned}$$

$$= \lim_{s\to\infty} \frac{h^2(s)\left(\beta\dfrac{h'(s)}{h(s)} + (\beta-1)g(s) - \dfrac{h''(s)}{h'(s)}\right)^2}{4h'(s)\left(\beta g(s)h(s) + (\beta-1)h'(s)\right)} = \frac{(\beta+(\beta-1)\tilde{\alpha}-\tilde{\mu})^2}{4(\tilde{\alpha}\beta+\beta-1)},$$

which is less than 1 due to (7.14). Thereby, there exist $\gamma < 1$ and $K > 0$ such that

$$\zeta(s) \le \gamma\left(e^{G(s)}h(s)\right)^{\beta-1} + K, \quad s \ge R.$$

In this way, choosing $\zeta(u_\lambda)$ as a test function in (7.1) we obtain

$$\int_\Omega (\phi'(u_\lambda))^2|\nabla u_\lambda|^2 = \lambda\int_\Omega e^{G(u_\lambda)}h(u_\lambda)\zeta(u_\lambda)$$

$$\le \gamma\lambda\int_\Omega e^{\beta G(u_\lambda)}h^\beta(u_\lambda) + K\lambda\int_\Omega e^{G(u_\lambda)}h(u_\lambda).$$

Combining this last inequality with (7.15) (and dropping the positive term $e^{G(u_\lambda)}h'(u_\lambda)\phi^2(u_\lambda)$) we can assert that

$$(1-\gamma)\lambda\int_\Omega e^{\beta G(u_\lambda)}h^\beta(u_\lambda) \le \lambda\int_{\Omega_R} e^{\beta G(u_\lambda)}h^\beta(u_\lambda) + K\lambda\int_\Omega e^{G(u_\lambda)}h(u_\lambda),$$

and taking into account that h is increasing (hypothesis (H2)) together with the Lebesgue dominated convergence theorem we deduce that

$$\int_\Omega e^{\beta G(u_\lambda)}h^\beta(u_\lambda) \le \frac{f(R)^\beta|\Omega|}{\lambda^*(1-\gamma)} + \frac{K}{1-\gamma}\int_\Omega e^{G(u^*)}h(u^*)$$

and $e^{G(u^*)}h(u^*) \in L^1(\Omega)$ since u^* is a solution of (P_{λ^*}) (Theorem 7.3.6). Finally we conclude, from the Fatou Lemma applied on the left-hand side of the above inequality, that $e^{G(u^*)}h(u^*) \in L^\beta(\Omega)$ with $\beta > N/2$ which is the desired conclusion. $\square$

We now give few examples, according to the different types of function g.

Example 1 *Let us consider the problem*

$$\begin{cases} -\Delta u + c\,|\nabla u|^2 = \lambda\, e^u\,, & \text{in } \Omega,\\ u > 0\,, & \text{in } \Omega,\\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \qquad (P_\lambda)$$

with $c < 1$. By Theorem 7.2.9, since $g(s) = c$ satisfies hypotheses (H1)-(H4), there exists $\lambda^* > 0$ such that there is a bounded minimal solution for every $\lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$. Moreover, there exists u^* solution for $\lambda = \lambda^*$ (Theorem 7.3.6) and it is stable and minimal (Corollary 7.3.9). Furthermore, since $\tilde{\alpha} = \frac{c}{1-c}$ and $\tilde{\mu} = 1$, it follows from Theorem 7.4.1 that u^* is bounded provided that

$$N < 4(1-c) + 2 + 4\sqrt{1-c}.$$

We remark that letting $c \to 0$ we obtain the regularity of extremal solution for the well known semilinear elliptic equation (G_λ) in the exponential case, i.e., $N < 10$.

Example 2 *In the singularity case $g(s) = \frac{c}{s^\gamma}$ with $0 < \gamma < 1$, a relevant example would be the case $f(s)$ no-convex. Thus, if we take as $h(s) = e^{(s+\delta)^{1-\gamma}}$ with δ small enough then $f'(s) - g(s)f(s)$ is not increasing (see Remark 7.2.7). Therefore, Theorem 7.2.9 ensures that there exist $\lambda^* > 0$ and bounded minimal solutions for $\lambda < \lambda^*$, and no solutions for $\lambda > \lambda^*$. Even more, since condition 7.4 is satisfied, u^* is a stable solution for $\lambda = \lambda^*$ (Theorem 7.3.6* and *Corollary 7.3.7) and not necessarily minimal. In addition, since $\tilde{\alpha} = \frac{c}{1-\gamma}$ and $\tilde{\mu} = 1$, due to Theorem 7.4.1 we obtain for*

$$N < \frac{6(1-\gamma) + 2c + 4\sqrt{(c+1-\gamma)(1-\gamma)}}{c+1-\gamma},$$

the regularity of the extremal solution. We would like to stress that letting $c \to 0$ we have $N < 10$.

Chapter 8

Elliptic equations involving the 1–Laplacian and a subcritical source term

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Abstract

In this paper we deal with a Dirichlet problem for an elliptic equation involving the 1–Laplacian operator and a source term. We prove that, when the growth of the source is subcritical, there exist two bounded nontrivial solutions to our problem. Moreover, a Pohožaev type identity is proved, which holds even when the growth is supercritical. We also show explicit examples of our results.

8.1 Introduction

This paper is concerned to the following Dirichlet problem for the 1–Laplacian operator and a subcritical source term, whose model problem is

$$\begin{cases} -\text{div}\left(\dfrac{Du}{|Du|}\right) = |u|^{q-1}u, & \text{in}\Omega\ , \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.1}$$

where$\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an open bounded set with Lipschitz boundary and $0 < q < \frac{1}{N-1}$. Our aim is to obtain nontrivial solutions (in the sense of Definition 8.2.1) and study their properties.

We point out that similar problems have many applications and have been studied for a long time. Indeed, the study of steady states of reaction–diffusion equations have systematically been studied since the late 1970s (see Fife (1979) and Ni (2011) for a more recent survey). More precisely, Dirichlet problems with p–Laplacian type operator $(p > 1)$ having a term with a subcritical growth, that is:

$$\begin{cases} -\Delta_p u = |u|^{q-1}u, & \text{in}\Omega\ , \\ \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{8.2}$$

with $0 < q < p^* - 1$ (where p^* stands for the Sobolev conjugate), have extensively been considered in the theory of Partial Differential Equations by using different approaches (for a background we refer to Ambrosetti and Arcoya (2011) and Dinca et al. (2001)). For instance in Dinca et al. (1995) the authors, by using the well–known "Mountain Pass Theorem" by Ambrosetti and Rabinowitz Ambrosetti and Rabinowitz (1973), firstly proved that the trivial solution is a local minimum of the corresponding energy functional and then, since the functional has a mountain pass geometry, they find other critical points (one positive and another one negative), which obviously are solutions to problem (8.2) . We point out that the proof of the Palais–Smale condition relies on the reflexivity of the energy space $W_0^{1,p}(\Omega)$. Moreover, the restriction $q < p^* - 1$ ensures that the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, being this fact essential for the approach used in Dinca et al. (1995).

The 1–Laplace operator appearing in (8.1) introduces some extra difficulties and special features. We recall that in recent years there have been many works devoted to this operator (we refer to the pioneering works Andreu et al. (2001); Demengel (1999); Kawohl (1991, 1990) and the related papers Andreu et al. (2002); Andreu-Vaillo et al. (2002); Bellettini et al. (2002); Cicalese and Trombetti (2003); Demengel (2002a,b)). One of the main interests for studying the Dirichlet problem for equations involving the 1–Laplacian comes from the variational approach to image restoration (we refer to Andreu-Vaillo et al. (2004) for a review on the first variational models in image processing and their connection with the 1–Laplacian). This has led to a great amount of papers dealing with problems that involve the 1–Laplacian operator. In spite of this situation, up to our knowledge, this is the first attempt to analyze problem (8.1).

The natural energy space to study problems involving the 1–Laplacian is the space $BV(\Omega)$ of functions of bounded variation, i.e., those L^1–functions such that their distributional gradient is a Radon measure having finite total variation. In order to deal with the 1–Laplacian operator, a first difficulty occurs by defining the quotient $\dfrac{Du}{|Du|}$, being Du just a Radon measure. It can be overcome through the theory of pairings of L^∞–divergence–measure vector fields and the gradient of a BV–function (see Anzellotti (1983)). Using this theory, we may consider a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ such that $\|\mathbf{z}\|_\infty \leq 1$ and $(\mathbf{z}, Du) = |Du|$, so that $\mathbf{z}$ plays the role of the above ratio. In general, the Dirichlet boundary condition is not achieved in the usual trace form, so that a very weak formulation must be introduced: $[\mathbf{z}, \nu] \in sign\,(-u)$, where $[\mathbf{z}, \nu]$ stands for the weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$.

We point out that the space $BV(\Omega)$ is not reflexive, so that we cannot follow the arguments of Dinca et al. (1995). Instead, we apply the results in Dinca et al. (1995) for problem (8.2) getting nontrivial solutions w_p and then we let p goes to 1. Hence, one of our biggest concerns will be that constants appearing in the proof do not depend on p. The other major difficulty we have to overcome is to check that the limit function $w = \lim_{p\to 1} w_p$ is not trivial.

8.1.1 Assumptions and main result

Let us state our problem and assumptions more precisely. We consider the general problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) = f(x,u), & \text{in}\Omega\ , \\ \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Here, the source term $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following hypotheses

(i) There exists $\alpha > 0$ such that

$$\limsup_{s\to 0} \frac{|f(x,s)|}{|s|^\alpha} < \infty, \qquad \text{uniformly in } x \in \Omega.$$

(ii) There exist $q \in \left(0, \frac{1}{N-1}\right)$ and $C > 0$ such that

$$|f(x,s)| \le C\left(1 + |s|^q\right), \qquad x \in \Omega, s \in \mathbb{R}.$$

(iii) There exist $\kappa > 1$ and $s_0 > 0$ such that

$$0 < \kappa\, F(x,s) \le s f(x,s), \qquad x \in \Omega, |s| \ge s_0,$$

where $F(x,s) = \int_0^s f(x,t)dt$. We deal with solutions of problem (P) in the sense of Definition 8.2.1 (see next section). Our main result is stated as follows:

Theorem 8.1.1 *Under the above assumptions, there exist at least two nontrivial solutions $v, w \in BV(\Omega) \cap L^\infty(\Omega)$ of problem (P). Moreover, $v \le 0 \le w$ a.e. $x \in \Omega$.*

The proof of existence considers approximating p–Laplacian problems and then the limit as $p \to 1^+$ of their nontrivial solutions w_p is taken. To this end, it is essential to achieve the existence of a positive constant $\tilde{C}$ independent of p such that

$$\|w_p\|_{W_0^{1,1}(\Omega)} \le \tilde{C}\,, \tag{8.3}$$

so that they are uniformly bounded in $W_0^{1,1}(\Omega)$. However, we carefully have to check that their limit is not the trivial solution.

As far as the regularity of solutions is concerned, we further prove that they are bounded. To prove the boundedness of the solutions a crucial point is the estimate (8.3). We would like to highlight that the usual Stampacchia truncation method with $p-$Laplacian problem does not work here since the problem becomes superlineal when p tends to 1 (i.e. $p - 1 < q$).

Finally, in Proposition 8.4.1 we state a Pohožaev type identity for solutions belonging to $W^{1,1}(\Omega)$. The important point to note here is, unlike $p-$Laplacian problems, the existence of solutions for any growth conditions of the source term. This is confirmed by dealing with explicit examples in the ball.

This paper is organized as follows: in the next section on Preliminaries we introduce the space of functions of bounded variation and we give some definitions and properties of Anzellotti's theory. In addition, we raise the problem (P) in a variational framework. Section 3 is devoted to the proof of existence and regularity of nontrivial solutions. To finish, in Section 4 a Pohožaev type identity is obtained. For the sake of completeness, we include there some examples.

8.2 Preliminaries

Throughout this paper, the symbol $\mathcal{H}^{N-1}(E)$ stands for the $(N-1)$–dimensional Hausdorff measure of a set $E \subset \mathbb{R}^N$ and $|E|$ for its Lebesgue measure. Moreover,$\Omega \subset \mathbb{R}^N$ denotes an open bounded set with Lipschitz boundary. Thus, an outward normal unit vector $\nu(x)$ is defined for $\mathcal{H}^{N-1}$–almost every $x \in \partial\Omega$.

We will denote by $W_0^{1,q}(\Omega)$ the usual Sobolev space, of measurable functions having weak gradient in $L^q(\Omega;\mathbb{R}^N)$ and zero trace on $\partial\Omega$. Finally, if $1 \le p < N$, we will denote by $p^* = Np/(N-p)$ its Sobolev conjugate exponent. Furthermore, $BV(\Omega)$ will denote the space of functions of bounded variation:

$$BV(\Omega) = \left\{u \in L^1(\Omega) \,:\, Du \text{ is a bounded Radon measure } \right\}$$

where $Du : \Omega \to \mathbb{R}^N$ denotes the distributional gradient of u. In what follows, we denote the distributional gradient by ∇u if it belongs to $L^1(\Omega;\mathbb{R}^N)$. We recall that the space $BV(\Omega)$ with norm

$$\|u\|_{BV(\Omega)} = \int_\Omega |Du| + \int_\Omega |u|$$

is a Banach space which is non reflexive and non separable.

On the other hand, the notion of a trace on the boundary can be extended to functions $u \in BV(\Omega)$, so that we may write $u\big|_{\partial\Omega}$, through a bounded operator $BV(\Omega) \hookrightarrow L^1(\partial\Omega)$, which is also onto. As a consequence, an equivalent norm on $BV(\Omega)$ can be defined (see Ambrosio et al. (2000)):

$$\|u\| = \int_\Omega |Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1},$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$–dimensional Hausdorff measure. We will often use this norm in what follows. In addition, the following continuous embeddings hold

$$BV(\Omega) \hookrightarrow L^m(\Omega)\,, \quad \text{for every } 1 \le m \le \frac{N}{N-1}\,,$$

which are compact for $1 \le m < \frac{N}{N-1}$.

Moreover, we will use some functionals which are lower semicontinuous with respect to the L^1–convergence. Besides the BV–norm, we also apply the lower semicontinuity of the functional given by

$$u \mapsto \int_\Omega \varphi\,|Du|,$$

where φ is a nonnegative smooth function. For further properties of functions of bounded variations, we refer to Ambrosio et al. (2000)

Since our concept of solution lies on the Anzellotti theory, we next introduce it. Consider $X_N(\Omega) = \left\{\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N) \,:\, \mathrm{d}\,\mathbf{z} \in L^N(\Omega)\right\}$. For $\mathbf{z} \in X_N(\Omega)$ and $u \in BV(\Omega)$ we denote by $(\mathbf{z}, Du) : \mathcal{C}_c^\infty(\Omega) \to \mathbb{R}$ the distribution introduced by Anzellotti (Anzellotti (1983)):

$$\langle(\mathbf{z}, Du), \varphi\rangle = -\int_\Omega u\,\varphi\, \mathrm{d}\,\mathbf{z} - \int_\Omega u\,\mathbf{z}\,\nabla\varphi, \quad \forall\,\varphi \in \mathcal{C}_c^\infty(\Omega)\,. \tag{8.4}$$

Moreover, in Anzellotti (1983) (see also (Andreu-Vaillo et al., 2004, Corollary C.7, C.16)) it is proved that $(\mathbf{z}, Du)$ is a Radon measure with finite total variation and for every Borel B set with $B \subseteq U \subseteq \Omega$ (U open) it holds

$$\left|\int_B (\mathbf{z}, Du)\right| \le \int_B |(\mathbf{z}, Du)| \le \|\mathbf{z}\|_{L^\infty(U)} \int_B |Du|\,. \tag{8.5}$$

We recall the notion of weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ defined in Anzellotti (1983) as the application $[\mathbf{z}, \nu] : \partial\Omega \to \mathbb{R}$, being ν the outer normal unitary vector of $\partial\Omega$, such that $[\mathbf{z}, \nu] \in$

$L^\infty(\partial\Omega)$ and $\| [\mathbf{z},\nu] \|_{L^\infty(\partial\Omega)} \le \|\mathbf{z}\|_{L^\infty(\Omega;\mathbb{R}^N)}$. Furthermore, this definition coincides with the classical one, that is,

$$[\mathbf{z},\nu] = \mathbf{z}\cdot\nu, \quad \text{for } \mathbf{z}\in\mathcal{C}^1(\overline{\Omega}_\delta;\mathbb{R}^N)\,, \tag{8.6}$$

where$\Omega_{\,\delta} = \{x\in\Omega \,:\, dist(x,\partial\,\Omega)<\delta\}$, for some $\delta>0$ sufficiently small. In Anzellotti (1983) a Green formula involving the measure $(\mathbf{z},Du)$ and the weak trace $[\mathbf{z},\nu]$ is established, namely:

$$\int_\Omega (\mathbf{z},Du) + \int_\Omega u \;\mathrm{d}\,\mathbf{z} = \int_{\partial\Omega} u\,[\mathbf{z},\nu]\,d\mathcal{H}^{N-1} \tag{8.7}$$

being $\mathbf{z}\in X_N(\Omega)$ and $u\in BV(\Omega)$.

Next, we give the definition of solution to our problem

Definition 8.2.1 *We say that $u\in BV(\Omega)$ is a solution of problem (P) if there exists a vector field $\boldsymbol{z}\in L^\infty(\Omega;\mathbb{R}^N)$ with $\|\boldsymbol{z}\|_\infty\le 1$ and such that*

(1) $-\,div\,\boldsymbol{z} = f(x,u)$ in $\mathcal{D}'(\Omega)$,

(2) $(\boldsymbol{z},Du) = |Du|$ as measures on Ω,

(3) $[\boldsymbol{z},\nu]\in sign(-u)$ on $\partial\Omega$.

Remark 8.2.2 *We remark that our solution belongs to $BV(\Omega)\subset L^{\frac{N}{N-1}}(\Omega)$. Thus condition (ii) satisfied by function f leads to*

$$|f(x,u(x))| \le C\,(1+|u(x)|^q) \in L^{\frac{N}{q(N-1)}}(\Omega)$$

for certain $1<q<\frac{1}{N-1}$, wherewith $f(\cdot,u)\in L^N(\Omega)$. It follows from (1) in the above definition that $\mathrm{d}\boldsymbol{z}\in L^N(\Omega)$, so that the Anzellotti theory is available.

Remark 8.2.3 *In principle, condition (1) in Definition 8.2.1 only allows us to take test functions in the space $\mathcal{C}_c^\infty(\Omega)$. We explicitly point out that, as a consequence of the Anzellotti theory, we may choose any $w\in BV(\Omega)$ as a test function. Then, Green's formula (8.7) implies*

$$\int_\Omega (\boldsymbol{z},Dw) - \int_\Omega f(x,u)w = \int_{\partial\Omega} w[\boldsymbol{z},\nu]\,d\mathcal{H}^{N-1}\,.$$

Observe that the vector field $\mathbf{z}$ need not be unique. For instance, we may choose $\mathbf{z}=(1,0,\cdots,0)$ or $\mathbf{z}=(0,1,\cdots,0)$ to check that $u\equiv 0$ is solution of (8.1).

In order to introduce a variational setting of problem (P) we recall the notion of subdifferential of a convex operator.

Definition 8.2.4 *Let $H : BV(\Omega)\to\mathbb{R}$ be a convex operator. For every $u\in BV(\Omega)$ we denote by $\partial H(u)$, the subdifferential of H in u, as the set*

$$\big\{\xi\in BV(\Omega)' \,:\, H(u)+\xi(v-u)\le H(v),\ \textit{for all } v\in BV(\Omega)\big\}$$

Remark 8.2.5 *Using this definition it is easy to check that u_0 is a global minimum of H if and only if $0\in\partial H(u_0)$.*

Lemma 8.2.6 *Given $u\in BV(\Omega)$ and $\boldsymbol{z}\in L^\infty(\Omega;\mathbb{R}^N)$ with $\|\boldsymbol{z}\|_\infty\le 1$, $\mathrm{d}\boldsymbol{z}\in L^N(\Omega)$, $(\boldsymbol{z},Du)=|Du|$ and $[\boldsymbol{z},\nu]\in sign(-u)$ on $\partial\Omega$. Let $\xi_u : BV(\Omega)\to\mathbb{R}$ be a linear map defined as*

$$\xi_u(v) := -\int_\Omega v\;\mathrm{d}\boldsymbol{z}.$$

Then, $\xi_u\in\partial\|u\|$.

Proof: Observe that $\xi_u \in BV(\Omega)'$ as a consequence of the Anzellotti theory. Indeed, Green's formula (8.7) and $\|\mathbf{z}\|_\infty \le 1$ imply

$$|\xi_u(v)| \le \left|\int_\Omega (\mathbf{z}, Dv)\right| + \left|\int_{\partial\Omega} v\,[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}\right| \le \int_\Omega |Dv| + \int_{\partial\Omega} |v|\,d\mathcal{H}^{N-1}\,,$$

for every $v \in BV(\Omega)$. So $\xi_u \in BV(\Omega)'$ and $\|\xi_u\| \le 1$.

On the other hand, for every $v \in BV(\Omega)$ we obtain

$$\begin{aligned}
\xi_u\,(v-u) &= \int_\Omega -\,\mathrm{d}\,\mathbf{z}\,(v-u)\\
&= \int_\Omega (\mathbf{z}, D(v-u)) - \int_{\partial\Omega} (v-u)\;[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}\\
&= \int_\Omega (\mathbf{z}, Dv) - \int_\Omega |Du| - \int_{\partial\Omega} (v\;[\mathbf{z},\nu] + |u|)d\mathcal{H}^{N-1}\\
&\le \|\mathbf{z}\|_\infty \int_\Omega |Dv| - \int_\Omega |Du| + \|\mathbf{z}\|_\infty \int_{\partial\Omega} |v|d\mathcal{H}^{N-1} - \int_{\partial\Omega} |u|d\mathcal{H}^{N-1}\\
&\le \|v\| - \|u\|.
\end{aligned}$$

□

Let $J : BV(\Omega) \to \mathbb{R}$ be defined as

$$J(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1} - \int_\Omega F(x,u).$$

We will say that $u_0 \in BV(\Omega)$ is a critical point of functional J if there exists $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ with $\|\mathbf{z}\|_\infty \le 1$ such that

$$-\int_\Omega w\;\mathrm{d}\mathbf{z} = \int_\Omega f(x,u_0)w, \qquad \text{for all } w \in BV(\Omega),$$
$$(\mathbf{z}, Du_0) = |Du_0| \;\text{ in } \Omega \quad \text{ and } \quad [\mathbf{z},\nu] \in \operatorname{sign}(-u_0) \;\text{ on } \partial\Omega.$$

In virtue of Lemma 8.2.6, the functional given by $\xi(w) = -\int_\Omega w\;\mathrm{d}\mathbf{z}$ belongs to $\partial\|u_0\|$. We point out that critical points of J coincide with solutions of problem (P).

8.3 Proof of Theorem 1

8.3.1 Existence of non trivial solutions

We shall prove that (P) has a nontrivial solution $w \ge 0$. A similar argument shows that there exists a nontrivial solution $v \le 0$.

Let $\tilde{p} = \min\{1+\alpha, \kappa, q+1\}$. For each $1 < p < \tilde{p}$, consider the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x,u), & \text{in}\Omega\;, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.8}$$

By our hypotheses and the choice of $\tilde{p}$, the following assertions are true for every $p \in (1,\tilde{p})$:

(a) $|f(x,s)| \le C(1+|s|^q)$ with $0 < q < p^* - 1$,

(b) $\lim_{s\to 0}\sup \dfrac{f(x,s)}{|s|^{p-2}s} = 0$, uniformly with $x \in \Omega$,

(c) $0 < \kappa\, F(x,s) \le sf(x,s)$ for $x \in \Omega$, $|s| \ge s_0$ and $\kappa > p$.

Then, it is well–know that problem (8.8) has nontrivial solutions $v_p \le 0 \le w_p$ (see e.g. Dinca et al. (2001)). These solutions are obtained using the "Mountain Pass Theorem" by Ambrosetti and Rabinowitz (Ambrosetti and Rabinowitz (1973)) for the two following functionals $J_p^{\pm} : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J_p^{\pm}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega F_{\pm}(x,u),$$

where $F_{\pm}(x,s) = \int_0^s f_{\pm}(x,t)\mathrm{d}t$, being $f_{\pm} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$f_+(x,s) = \begin{cases} 0 & \text{if } s \le 0, \\ f(x,s) & \text{if } s > 0. \end{cases} \qquad f_-(x,s) = \begin{cases} f(x,s) & \text{if } s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Concretely, for the nonnegative solution w_p it is used J_p^+ (while J_p^- is used for the nonpositive one v_p). Now consider the functional

$$I_p(u) = J_p^+(u) + \frac{p-1}{p}\,|\Omega|\,.$$

Since, by Young's inequality

$$\int_\Omega |\nabla u|^{p_1} \le \frac{p_1}{p_2}\int_\Omega |\nabla u|^{p_2} + \frac{p_2 - p_1}{p_2}|\Omega|, \quad 1 \le p_1 \le p_2,$$

it follows that I_p is nondecreasing with respect to p. On the other hand, we fix $0 < \phi \in \mathcal{C}_c^\infty(\Omega)$ and since $I_p(t\phi) \to -\infty$ as $t \to \infty$, it yields $e = T\phi$ (for some $T > 0$) such that $I_{\tilde p}(e) < 0$. Then, by monotonicity, we obtain

$$I_p(e) < 0, \qquad \text{for all } p \in (1, \tilde p).$$

Moreover, due to the fact that critical points of J_p^+ are uniquely determined by critical points of I_p, it follows that $u \equiv 0$ is a local minimum of I_p and $w_p \ge 0$ is a nontrivial critical point of I_p which can be obtained invoking to the Mountain Pass Theorem. That is, it satisfies

$$I_p(w_p) = \inf_{\gamma \in \Gamma_p} \max_{t \in [0,1]} I_p(\gamma(t)),$$

where

$$\Gamma_p = \left\{\gamma \in \mathcal{C}\left([0,1], W_0^{1,p}(\Omega)\right) \,:\, \gamma(0) = 0, \gamma(1) = e\right\}.$$

Next we claim that the sequence $\{I_p(w_p)\}_{1<p<\tilde p}$ is increasing. Indeed, let $1 < p_1 < p_2 < \tilde p$ and thanks to the monotony of I_p and the fact that$\Gamma_{p_2} \subset \Gamma_{p_1}$ (because $W_0^{1,p_2}(\Omega) \subset W_0^{1,p_1}(\Omega)$), it holds

$$\begin{aligned} I_{p_1}(w_{p_1}) &= \inf_{\gamma \in \Gamma_{p_1}} \max_{t \in [0,1]} I_{p_1}(\gamma(t)) \\ &\le \inf_{\gamma \in \Gamma_{p_2}} \max_{t \in [0,1]} I_{p_1}(\gamma(t)) \\ &\le \inf_{\gamma \in \Gamma_{p_2}} \max_{t \in [0,1]} I_{p_2}(\gamma(t)) \\ &= I_{p_2}(w_{p_2}) \end{aligned}$$

and the claim is proved. Thus, for a fixed $p_0 \in (1, \tilde p)$ we get $I_p(w_p) \le I_{p_0}(w_{p_0})$ for all $p \in (1, p_0)$ and hence

$$\frac{1}{p}\int_\Omega |\nabla w_p|^p - \int_\Omega F(x, w_p) \le C, \qquad \text{for all } p \in (1, p_0), \tag{8.9}$$

with $C = C(p_0) > 0$ independent of p. Observe that we write $F(x,w_p)$ instead $F_+(x,w_p)$ because $w_p \ge 0$ (an analogous remark holds for $f_+(x,w_p)$).

We denote$\Omega_p = \{x \in \Omega : w_p(x) \le s_0\}$, for any $p \in (1, p_0)$. Then, by condition (a) and the definition of $F(x,s)$, we obtain

$$\int_{\Omega_p} F(x, w_p) \le C s_0\left(1 + s_0^q\right)|\Omega| = C_1, \tag{8.10}$$

where C_1 is independent of p. Also, by condition (c) and since w_p is a solution, it holds

$$\int_{\Omega\setminus\Omega_p} F(x,w_p) \leq \frac{1}{\kappa}\int_\Omega w_p f(x,w_p) = \frac{1}{\kappa}\int_\Omega |\nabla w_p|^p. \tag{8.11}$$

Substituting (8.10) and (8.11) into (8.9), we get

$$\left(\frac{1}{p_0}-\frac{1}{\kappa}\right)\int_\Omega |\nabla w_p|^p \leq \left(\frac{1}{p}-\frac{1}{\kappa}\right)\int_\Omega |\nabla w_p|^p \leq C + C_1\,.$$

Then, since $\kappa > p_0$, we conclude that

$$\int_\Omega |\nabla w_p|^p \leq \tilde{C}, \qquad \forall\, p \in (1,p_0), \tag{8.12}$$

for some positive constant $\tilde{C} = \tilde{C}(p_0)$, independent of p.

This last inequality (8.12) allows us to establish the following statements (see (Andreu et al., 2001, Proposition 3), and also (Mercaldo et al., 2013, Theorem 3.3)): there exists a bounded vector field $\mathbf{z} \in L^\infty(\Omega\colon\ \mathbb{R}^N)$ with $\|\mathbf{z}\|_\infty \leq 1$ such that

$$|\nabla w_p|^{p-2}\nabla w_p \rightharpoonup \mathbf{z}, \ \text{ weakly in } L^r(\Omega;\mathbb{R}^N), \ \text{ for all } 1 \leq r < \infty, \tag{8.13}$$

as $p \to 1^+$. In particular,

$$\int_\Omega |\nabla w_p|^{p-2}\nabla w_p \cdot \nabla\varphi \to \int_\Omega \mathbf{z}\cdot\nabla\varphi, \ \text{ for all } \varphi \in \mathcal{C}_c^1(\Omega). \tag{8.14}$$

On the other hand, (8.12) and Young's inequality imply

$$\|w_p\| \leq \int_{\partial\Omega} |w_p|\, d\mathcal{H}^{N-1} + \frac{1}{p}\int_\Omega |\nabla w_p|^p + \frac{p-1}{p}|\Omega| \leq \tilde{C} + |\Omega|\,,$$

so that $\{w_p\}_{p>1}$ is bounded in $BV(\Omega)$. It follows that there exists $w \in BV(\Omega)$ such that, up to a subsequence (no relabeled),

(A) $w_p \to w$, in $L^m(\Omega)$, for $1 \leq m < \frac{N}{N-1}$.

(B) $w_p(x) \to w(x)$, almost everywhere $x \in \Omega$.

(C) $\exists\, g \in L^m(\Omega)$ $(1 \leq m < \frac{N}{N-1})$ such that $|w_p(x)| \leq g(x)$.

Observe that $w \geq 0$ because $w_p \geq 0$ for all $p > 1$. Then, thanks to (B) and the fact that $f(x,s)$ is a Carathéodory function, we obtain

$$f(x,w_p(x)) \to f(x,w(x)), \qquad \text{a.e. } x \in \Omega.$$

Moreover, we deduce from (C) that

$$|f(x,w_p(x))| \leq C(1+|w_p(x)|^q) \leq C(1+g(x)^q) \in L^N(\Omega).$$

Consequently, by the Dominated Convergence Theorem,

$$\int_\Omega f(x,w_p)\varphi \to \int_\Omega f(x,w)\varphi, \qquad \text{for all } \varphi \in \mathcal{C}_c^1(\Omega). \tag{8.15}$$

Expressions (8.14) and (8.15) imply that

$$-\operatorname{d}\mathbf{z} = f(x,w) \text{ in } \mathcal{D}'(\Omega). \tag{8.16}$$

In order to prove that $(\mathbf{z},Dw) = |Dw|$, we note that it is enough to show $\langle(\mathbf{z},Dw),\varphi\rangle = \langle|Dw|,\varphi\rangle$ for all $0 \leq \varphi \in \mathcal{C}_c^1(\Omega)$. Since $\|\mathbf{z}\|_\infty \leq 1$ and (8.5) holds, we just prove the inequality $\langle(\mathbf{z},Dw),\varphi\rangle \geq \langle|Dw|,\varphi\rangle$. Due to the definition of $(\mathbf{z},Dw)$, we must check that:

$$-\int_\Omega w \operatorname{d}\mathbf{z}\,\varphi - \int_\Omega w\,\mathbf{z}\cdot\nabla\varphi \geq \int_\Omega |Dw|\,\varphi, \ \text{ for all } 0 \leq \varphi \in \mathcal{C}_c^1(\Omega). \tag{8.17}$$

To this end, taking $0 \le w_p\,\varphi \in W_0^{1,p}(\Omega)$ as a test function in problem (8.8), we get

$$\int_\Omega |\nabla w_p|^p \varphi + \int_\Omega w_p |\nabla w_p|^{p-2} \nabla w_p \cdot \nabla \varphi = \int_\Omega f(x, w_p) w_p\, \varphi. \tag{8.18}$$

We estimate the first integral term in (8.18) using Young's inequality:

$$\int_\Omega \varphi |\nabla w_p| \le \frac{1}{p} \int_\Omega \varphi |\nabla w_p|^p + \frac{p-1}{p} \int_\Omega \varphi\,.$$

Now, from the lower semicontinuity of the involved functional, we obtain

$$\begin{aligned}\liminf_{p\to 1^+} \int_\Omega \varphi |\nabla w_p|^p &\ge \liminf_{p\to 1^+} \int_\Omega \varphi |\nabla w_p| \\ &= \int_\Omega \varphi |Dw|\,.\end{aligned}$$

On the other hand, by (A) and (8.13)

$$\int_\Omega w_p |\nabla w_p|^{p-2} \nabla w_p \cdot \nabla \varphi \to \int_\Omega w\, \mathbf{z} \cdot \nabla \varphi, \qquad \text{as } p \to 1^+.$$

The right hand side of (8.18) is analyzed as follows. We deduce from

$$|f(x, w_p) w_p\, \varphi| \le MC |w_p| (1 + |w_p|^q) \le C_1 g(x)(1 + g(x)^q) \in L^1(\Omega)$$

and the pointwise convergence, that

$$\int_\Omega f(x, w_p) w_p\, \varphi \to \int_\Omega f(x, w) w\, \varphi = -\int_\Omega \mathrm{d}\,\mathbf{z}\, w\, \varphi\,.$$

Then, letting $p \to 1^+$ in (8.18), we obtain the required inequality (8.17) to conclude that

$$(\mathbf{z}, Dw) = |Dw|. \tag{8.19}$$

Next, we will show that $[\mathbf{z}, \nu] \in \operatorname{sign}(-w)$ on $\partial\Omega$. It is easy to check that this fact is equivalent to show

$$\int_{\partial\Omega} (|w| + w\,[\mathbf{z}, \nu])\, d\mathcal{H}^{N-1} = 0, \tag{8.20}$$

because $|\,[\mathbf{z}, \nu]\,| \le \|\mathbf{z}\|_\infty \le 1$. Since $-w\,[\mathbf{z}, \nu] \le \|\mathbf{z}\|_\infty |w| \le |w|$ and so

$$\int_{\partial\Omega} (|w| + w\,[\mathbf{z}, \nu])\, d\mathcal{H}^{N-1} \ge 0\,,$$

it remains to prove the reverse inequality. To do this, we take $w_p - \varphi$, with $\varphi \in \mathcal{C}_c^1(\Omega)$, as a test function in (8.8), to obtain

$$\int_\Omega |\nabla w_p|^p = \int_\Omega |\nabla w_p|^{p-2} \nabla w_p \cdot \nabla \varphi + \int_\Omega f(x, w_p)(w_p - \varphi). \tag{8.21}$$

Hence, using Young's inequality, we get

$$\begin{aligned}p \int_\Omega |\nabla w_p| &\le \int_\Omega |\nabla w_p|^p + (p-1)|\Omega| \\ &= \int_\Omega |\nabla w_p|^{p-2} \nabla w_p \cdot \nabla \varphi + \int_\Omega f(x, w_p)(w_p - \varphi) + (p-1)|\Omega|.\end{aligned}$$

Now, having in mind (8.13), the weak lower semicontinuity of the total variation and from the previous arguments, we can pass to the limit as $p \to 1^+$, to have

$$\begin{aligned}\int_\Omega |Dw| + \int_{\partial\Omega} |w| d\mathcal{H}^{N-1} &\le \int_\Omega \mathbf{z} \cdot \nabla\varphi - \int_\Omega f(x,w)\varphi + \int_\Omega f(x,w)w \\ &= \int_\Omega f(x,w)w, \end{aligned} \tag{8.22}$$

due to (8.16). Furthermore, by (8.16), (8.7) and (8.19), we get

$$\begin{aligned}\int_\Omega f(x,w)w &= -\int_\Omega w \,\mathrm{d}\,\mathbf{z} \\ &= -\int_{\partial\Omega} w\,[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_\Omega (\mathbf{z}, Dw) \\ &= -\int_{\partial\Omega} w\,[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_\Omega |Dw|\,. \end{aligned}$$

Replacing this equality in (8.22) gives the desired equality in (8.20) and we conclude that

$$[\mathbf{z},\nu] \in \operatorname{sign}(-w) \text{ on } \partial\Omega. \tag{8.23}$$

Then, (8.16), (8.19) and (8.23) lead to conclude that w is a nonnegative solution of problem (P) in the sense of Definition 8.2.1.

In order to check that w is nontrivial, by hypothesis (i), $f(x,0) = 0$ and there exists $\delta > 0$, small enough, such that $|f(x,s)| \le K_1|s|^\alpha$ for all $|s| \in (0,\delta)$ and for some $K_1 > 0$. Observe that hypothesis (ii) implies $\alpha < q < \frac{1}{N-1}$. Moreover, by definition of $F_+(x,s)$ it follows

$$F_+(x,s) = \int_0^s f_+(x,t)dt \le \int_0^s |f(x,s)| \le \frac{K_1}{1+\alpha}|s|^{1+\alpha},$$

for $|s| \in (0,\delta)$. Let $\rho \in (0,\delta)$ to be determined. Then, for $u \in BV(\Omega)$ with $\|u\| = \rho$, it holds

$$\begin{aligned} J(u) &= \|u\| - \int_\Omega F_+(x,u) \\ &\ge \|u\| - \frac{K_1}{1+\alpha}\int_\Omega |u|^{1+\alpha} \\ &\ge \|u\| - K_2\|u\|^{1+\alpha} \\ &= \rho(1 - K_2\rho^\alpha). \end{aligned}$$

We define ρ, so small, such that $1 - K_2\rho^\alpha \ge \frac{1}{2}$, so that

$$J(u) \ge \frac{\rho}{2}, \qquad \text{for } \|u\| = \rho > 0.$$

Observing that $J(e) < 0$, we deduce that $\|e\| > \rho$. Since, by Young's inequality, we get that $I_p(u) \ge J(u)$ for all $u \in W_0^{1,p}$, it follows that

$$I_p(w_p) = \inf_{\gamma\in\Gamma_p} \max_{t\in[0,1]} I_p(\gamma(t)) \ge \frac{\rho}{2}. \tag{8.24}$$

On the other hand, we have

$$\begin{aligned} \lim_{p\to 1^+} \frac{1}{p}\int_\Omega |\nabla w_p|^p &= \lim_{p\to 1^+} \frac{1}{p}\int_\Omega f(x,w_p)w_p \\ &= \int_\Omega f(x,w)w \\ &= \int_\Omega (\mathbf{z}, Dw) - \int_{\partial\Omega} w[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} \\ &= \int_\Omega |Dw| + \int_{\partial\Omega} |w|\, d\mathcal{H}^{N-1}, \end{aligned}$$

where in the last equality we have used that w is a solution of (P). In addition, it is easy to check that

$$\lim_{p\to 1^+}\int_\Omega F(x,w_p)=\int_\Omega F(x,w).$$

By using these last two equalities, we can assert that

$$\lim_{p\to 1^+} I_p(w_p)=J(w). \tag{8.25}$$

Summarizing (8.24) and (8.25) we conclude that $J(w)\geq \frac{\varrho}{2}$ and then w is nontrivial, because $J(0)=0$.

With regard to the existence of a nontrivial solution $v\leq 0$ of problem (P), we use the same reasoning applied to the functional

$$\tilde{I}_p(u)=\frac{1}{p}\int_\Omega|\nabla u|^p-\int_\Omega F_-(x,u)+\frac{p-1}{p}\,|\Omega|\,,$$

getting that $v_p\to v$ as $p\to 1^+$. Where v_p is the nonpositive solution of $p-$Laplacian problem (8.8).

8.3.2 Boundedness of the solutions

In this subsection, we will write $\mathcal{S}_1$ to denote the best constant of the Sobolev embedding $W_0^{1,1}(\Omega)\hookrightarrow L^{\frac{N}{N-1}}(\Omega)$. Moreover, for every $k\geq 0$ and $0\leq w_p\in W_0^{1,p}(\Omega)$ solution of (8.8) defined in the proof of Theorem 8.1.1, we set

$$A_k(w_p)=A_{k,p}=\{x\in\Omega\,:\,|w_p(x)|>k\}\,.$$

Lemma 8.3.1 *For every $\varepsilon>0$ there exists $k_0>0$ (which does not depend on p) such that*

$$\int_{A_{k,p}}(1+w_p^q)^N<\varepsilon$$

for every $k\geq k_0$ and for all $p>1$ small enough.

Proof: Using Hölder's inequality twice, Sobolev's inequality and taking into account that

$$|A_{k,p}|\leq\frac{1}{k^{\frac{N}{N-1}}}\int_{A_{k,p}}w_p^{\frac{N}{N-1}},$$

we obtain

$$\begin{aligned}
\int_{A_{k,p}}(1+w_p^q)^N &\leq 2^{N-1}\left(|A_{k,p}|+\int_{A_{k,p}}w_p^{qN}\right)\\
&\leq 2^{N-1}\left(|A_{k,p}|+\left(\int_{A_{k,p}}w_p^{\frac{N}{N-1}}\right)^{q(N-1)}|A_{k,p}|^{1-q(N-1)}\right)\\
&\leq\frac{2^{N-1}(1+k^{qN})}{k^{\frac{N}{N-1}}}\int_\Omega w_p^{\frac{N}{N-1}}\\
&\leq\frac{2^{N-1}(1+k^{qN})}{k^{\frac{N}{N-1}}}\mathcal{S}_1^{\frac{N}{N-1}}\left(\int_\Omega|\nabla w_p|\right)^{\frac{N}{N-1}}\\
&\leq\frac{2^{N-1}(1+k^{qN})}{k^{\frac{N}{N-1}}}\mathcal{S}_1^{\frac{N}{N-1}}\left(\int_\Omega|\nabla w_p|^p\right)^{\frac{N}{p(N-1)}}|\Omega|^{\frac{p-1}{p}\frac{N}{N-1}},
\end{aligned}$$

now, having in mind inequality (8.12) which asserts the existence of a positive constant $\tilde{C}$, which does not depend on p, satisfying

$$\left(\int_\Omega |\nabla w_p|^p\right)^{\frac{1}{p}} \le \tilde{C}^{1/p} < 1 + \tilde{C}$$

and since $|\Omega|^{\frac{p-1}{p}} < 1 + |\Omega|$, it follows that there exists a positive constant $C = C(N, q, \mathcal{S}_1, |\Omega|)$ such that

$$\int_{A_{k,p}} (1 + w_p^q)^N < \frac{C(1 + k^{qN})}{k^{\frac{N}{N-1}}} \to 0$$

as $k \to \infty$, because $q < \frac{1}{N-1}$. □

Remark 8.3.2 *By a similar argument we can state the existence of a $k_0 > 0$ (which does not depend on p) such that*

$$\int_{A_{k,p}} (1 + |v_p|^q)^N < \varepsilon$$

for every $k \ge k_0$ and for all $p > 1$ sufficiently small. Where $0 \ge v_p \in W_0^{1,p}(\Omega)$ is the negative solution of (8.8) and $A_{k,p} = A_k(v_p)$.

Now, we are ready to prove the boundedness of the solutions v and w of problem (P). *Proof:*[Proof of Boundedness] We prove the boundedness of the positive solution w. The proof for the negative one is similar in spirit.

For every $k > 0$, we define the auxiliary function $G_k : \mathbb{R} \to \mathbb{R}$ as usual

$$G_k(s) = \begin{cases} s - k, & s > k, \\ 0, & |s| \le k, \\ s + k, & s < -k. \end{cases}$$

Then, choosing $G_k(w_p)$ as a test function in (8.8), we get

$$\int_\Omega |\nabla G_k(w_p)|^p = \int_\Omega f(x, w_p) G_k(w_p). \tag{8.26}$$

Now, computing and using (8.26), Sobolev's embedding, and the Young and Hölder inequalities, we have

$$\begin{aligned}
\left(\int_\Omega G_k(w_p)^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}} &\le \mathcal{S}_1 \int_\Omega |\nabla G_k(w_p)| \\
&\le \frac{\mathcal{S}_1}{p} \int_\Omega |\nabla G_k(w_p)|^p + \frac{\mathcal{S}_1(p-1)}{p}|\Omega| \\
&\le \mathcal{S}_1 \int_\Omega |f(x, w_p| G_k(w_p) + \frac{\mathcal{S}_1(p-1)}{p}|\Omega| \\
&\le C\mathcal{S}_1 \int_{A_k} (1 + w_p^q) G_k(w_p) + \frac{\mathcal{S}_1(p-1)}{p}|\Omega| \\
&\le C\mathcal{S}_1 \left(\int_{A_{k,p}} (1 + w_p^q)^N\right)^{\frac{1}{N}} \left(\int_\Omega G_k(w_p)^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}} \\
&\qquad + \frac{\mathcal{S}_1(p-1)}{p}|\Omega|.
\end{aligned}$$

By Lemma 8.3.1, there exists $\tilde{k}_0 > 0$ (which does not depend on p) such that

$$\int_{A_{k,p}} (1 + w_p^q)^N < \frac{1}{(2C\mathcal{S}_1)^N}, \text{ for all } k \ge \tilde{k}_0,$$

and for all $p > 1$ sufficiently small. Consequently, we obtain

$$\int_\Omega G_k(w_p)^{\frac{N}{N-1}} \le \left(\frac{2\mathcal{S}_1(p-1)|\Omega|}{p} \right)^{\frac{N}{N-1}}.$$

Since $w_p(x) \to w(x)$ a.e. $x \in \Omega$, by Fatou lemma, we can pass to the limit on $p \to 1$, to conclude that

$$\int_\Omega (w(x) - k)^{\frac{N}{N-1}} = 0, \text{ for every } k \ge \tilde{k}_0.$$

Thus, $\|w\|_\infty \le \tilde{k}_0$. □

8.4 A Pohožaev type identity and explicit examples

In this section we provide a Pohožaev type identity for elliptic problems involving the 1–Laplacian operator

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) = f(u), & \text{in}\Omega\ , \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.27}$$

From now on, for any function g evaluated on $\partial\Omega$, we write $\int_{\partial\Omega} g$ instead of $\int_{\partial\Omega} g\, d\mathcal{H}^{N-1}$ when no confusion can arise.

Proposition 8.4.1 *[Pohožaev type identity for the* $1-$*Laplacian] Let* $u \in W^{1,1}(\Omega)$ *be a solution of problem* (8.27) *in the sense of Definition 8.2.1 with* $\boldsymbol{z} \in \mathcal{C}^1(\overline{\Omega}_\delta)$ *(for some* $\delta > 0$ *sufficiently small) and assume that* $x \cdot \nabla u \in W^{1,1}(\Omega)$*. Then,* u *satisfies the identity*

$$\begin{aligned} &(N-1)\int_\Omega u f(u) - N \int_\Omega F(u) + \int_{\partial\Omega} F(u) x \cdot \nu \\ &= \int_{\partial\Omega} |\nabla u| x \cdot \nu - \int_{\partial\Omega} (x \cdot \nabla u)\,(\boldsymbol{z} \cdot \nu) + (N-1)\int_{\partial\Omega} |u|\,. \end{aligned} \tag{8.28}$$

Proof:

By our assumption $x \cdot \nabla u \in W^{1,1}(\Omega)$, we have

$$\nabla\,(x \cdot \nabla u) = \nabla u + D^2 u \cdot x,$$

where $\left(D^2 u \cdot x\right)_j = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} x_i$ $(j = 1, \dots, N)$ belong to $L^1(\Omega)$. Moreover, by Stampacchia's Theorem, $\nabla\,(x \cdot \nabla u) = 0$ a.e. in the set $\{x \cdot \nabla u = 0\}$ which implies

$$\left(D^2 u \cdot x\right)_j = 0, \quad \text{a.e. in } \{|\nabla u| = 0\}.$$

Hence, integrating by parts and taking into account (8.6), we obtain

$$\begin{aligned} \int_\Omega \mathrm{d}\,\mathbf{z}\,(x \cdot \nabla u) &= \int_{\partial\Omega} (x \cdot \nabla u)\,(\mathbf{z} \cdot \nu) - \int_\Omega \mathbf{z} \cdot \nabla(x \cdot \nabla u) \\ &= \int_{\partial\Omega} (x \cdot \nabla u)\,(\mathbf{z} \cdot \nu) - \int_\Omega |\nabla u| - \int_\Omega \left(D^2 u \cdot x\right) \cdot \mathbf{z}\,, \end{aligned} \tag{8.29}$$

On the other hand, we also get

$$\begin{aligned} N \int_\Omega |\nabla u| &= \int_{\partial\Omega} |\nabla u| x \cdot \nu - \int_\Omega x \cdot \nabla(|\nabla u|) \\ &= \int_{\partial\Omega} |\nabla u| x \cdot \nu - \int_\Omega \left(D^2 u \cdot x\right) \cdot \mathbf{z}\,, \end{aligned} \tag{8.30}$$

where in the last integral term we replace $\frac{\nabla u}{|\nabla u|}$ by $\mathbf{z}$ since we can assume that $|\nabla u| > 0$. Then, combining (8.29) and (8.30), we obtain

$$\int_\Omega \mathrm{d}\,\mathbf{z}\,(x\cdot\nabla u) = \int_{\partial\Omega}(x\cdot\nabla u)\,(\mathbf{z}\cdot\nu) + (N-1)\int_\Omega|\nabla u| - \int_{\partial\Omega}|\nabla u|x\cdot\nu. \tag{8.31}$$

Since u is a solution, we can choose $x\cdot\nabla u \in W^{1,1}(\Omega)$ as a test function and by using integration by parts we get

$$\begin{aligned}\int_\Omega \mathrm{d}\,\mathbf{z}\,(x\cdot\nabla u) &= -\int_\Omega f(u)(x\cdot\nabla u)\\ &= -\sum_i\int_\Omega x_i\frac{\partial F(u)}{\partial x_i}\\ &= -\int_{\partial\Omega}F(u)x\cdot\nu + N\int_\Omega F(u).\end{aligned}$$

Also, taking u as a test function we have

$$\int_\Omega|\nabla u| = \int_\Omega uf(u) + \int_{\partial\Omega}u\,(\mathbf{z}\cdot\nu)\ .$$

Replacing the above two equalities in (8.31) and remembering that $u\,(\mathbf{z}\cdot\nu) = -|u|$, it yields the equality (8.28). Finally, we point out that in case $|\nabla u| = 0$ in the whole Ω, we obtain the identity

$$(N-1)\int_\Omega uf(u) - N\int_\Omega F(u) + \int_{\partial\Omega}F(u)x\cdot\nu = (N-1)\int_{\partial\Omega}|u|$$

□

Corollary 8.4.2 *In case $\Omega = B_R$ (the ball of radius $R > 0$). Under the hypotheses of Proposition 8.4.1, solutions of (8.27) must satisfy the inequality*

$$(N-1)\int_{B_R}uf(u) - N\int_{B_R}F(u) + R\int_{\partial B_R}F(u) \ge (N-1)\int_{\partial B_R}|u|.$$

Proof: Since $x\cdot\nu = R$ and by (8.5) , it follows that

$$\begin{aligned}&\int_{\partial B_R}|\nabla u|x\cdot\nu - \int_{\partial B_R}(x\cdot\nabla u)\,(\mathbf{z}\cdot\nu)\\ &\qquad\ge R\int_{\partial B_R}|\nabla u| - \|\mathbf{z}\|_\infty\int_{\partial B_R}(x\cdot\nabla u)\\ &\qquad\qquad\ge R\,(1-\|\mathbf{z}\|_\infty)\int_{\partial B_R}|\nabla u| \ge 0.\end{aligned}$$

Substituting into (8.28), we obtain the desired inequality. □

The following result, first obtained by F. Demengel in (Demengel, 1999, Section 4), is now a consequence of Proposition 8.4.1.

Corollary 8.4.3 *Besides the hypotheses of Proposition 8.4.1, assume that $u_{|\partial\Omega} \equiv 0$. Then*

$$(N-1)\int_\Omega uf(u) = N\int_\Omega F(u).$$

In particular, for $f(s) = |s|^{q-1}s$ it follows $q = \frac{1}{N-1}$.

It is worth noting that in the Pohožaev inequalities, there is no restriction on the possible values of q. We give some explicit examples about radial solutions of problem (P) in the ball $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. We point out that they also satisfy the Pohožaev identity (8.28).

Example 3 *For $f(s) = |s|^{q-1}s$, with $q > 0$*

$$u(x) \equiv \left(\frac{N}{R}\right)^{1/q}, \quad \boldsymbol{z}(x) = -\frac{x}{R},$$

defines a positive constant solution, while a negative solution is defined by

$$u(x) \equiv -\left(\frac{N}{R}\right)^{1/q}, \quad \boldsymbol{z}(x) = \frac{x}{R}.$$

Furthermore thanks to Proposition 8.4.1, for a general continuous and increasing function f, constant solutions of (8.27) in B_R must satisfy

$$u \equiv f^{-1}\left(\frac{N}{R}\right).$$

In the next examples, we assume a supercritical growth, so that in the supercritical case, two positive (and two negative) solutions are obtained. A further remark is in order. We have considered the Anzellotti theory of pairing gradients of BV–functions and bounded vector fields whose divergence is an L^N–function. It should be remarked that analogous results hold for bounded vector fields whose divergence is a function belonging to the Marcinkiewicz space $L^{N,\infty}(\Omega)$. This fact is a consequence of the continuous embedding of $BV(\Omega) \hookrightarrow L^{\frac{N}{N-1},1}(\Omega)$, where $L^{\frac{N}{N-1},1}(\Omega)$ denotes the Lorentz space (see Alvino (1977)). Hence, the Radon measure $(\mathbf{z}, Du)$ is well–defined for the vector field $\mathbf{z}(x) = \frac{x}{|x|}$, whose distributional divergence is given by $\mathrm{d}\mathbf{z}(x) = \frac{N-1}{|x|}$ and belongs to $L^{N,\infty}(B_R)$, and for any $u \in BV(B_R)$.

Example 4

1. *For $f(s) = s_+^q$ with $q > \frac{1}{N-1}$*

$$u(x) = \left(\frac{N-1}{|x|}\right)^{1/q}, \quad \boldsymbol{z}(x) = -\frac{x}{|x|},$$

is a positive solution in $W^{1,1}(B_R)$.

2. *For $f(s) = \left(\left(\frac{N-1}{R}\right)^{1/q} + s\right)_+^q$ with $q > \frac{1}{N-1}$*

$$u(x) = \left(\frac{N-1}{|x|}\right)^{1/q} - \left(\frac{N-1}{R}\right)^{1/q}, \quad \boldsymbol{z}(x) = -\frac{x}{|x|},$$

is a positive solution belongs to $W_0^{1,1}(B_R)$.

Chapter 9

Existence and regularizing effect of degenerate lower order terms in elliptic equations beyond the Hardy constant

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Abstract

In this paper we study the regularizing effect of lower order terms in elliptic problems involving a Hardy potential. Concretely, our model problem is

$$-\Delta u + h(x)|u|^{p-1}u = \lambda \frac{u}{|x|^2} + f(x) \quad \text{in} \quad \Omega$$

with Dirichlet conditions on $\partial\Omega$, where $p > 1$ and $f \in L^m(\Omega; hdx)$ with $m \geq \frac{p+1}{p}$. We prove that there is a solution of the above problem even for $\lambda \geq \mathcal{H} = \frac{(N-2)^2}{4}$ and $0 \leq h \in L^1(\Omega)$ which could be vanished in a subset of Ω. Moreover, we show that all the solutions are in $L^{pm}(\Omega; hdx)$. These results improve and generalize the case $h(x) \equiv h_0$ treated in Porzio (2007) and recently in Adimurthi et al. (2017).

9.1 Introduction

For a bounded domain$\Omega \subset \mathbb{R}^N$ $(N > 2)$ with smooth boundary $\partial\Omega$ and $0 \in \Omega$, we consider the following problem

$$\begin{cases} -\Delta u + h(x)|u|^{p-1}u = \lambda \dfrac{u}{|x|^2} + f(x) & \text{in} \quad , \quad \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.1}$$

being $\lambda > 0$, $p > 1$, $0 \le h \in L^1_{loc}(\Omega)$ and $f \in L_h^{\frac{p+1}{p}}(\Omega)$, (i.e. $|f|^{\frac{p+1}{p}} h \in L^1(\Omega)$).

If $h \equiv 0$, it is proved in García Azorero and Peral Alonso (1998) the existence of a solution for every $f \in W^{-1,2}(\Omega)$ when $\lambda < \mathcal{H} = \frac{(N-2)^2}{4}$ ($\mathcal{H}$ is called the Hardy constant). From this pioneering paper the case $h \equiv 0$ has been studied by many authors. More recently, it is proved in Adimurthi et al. (2017); Porzio (2007) that if $h(x) \equiv h_0 > 0$, then the lower order term $h_0|u|^{p-1}u$ has a regularizing effect: Consider $f \in L^m(\Omega)$, then there exists a solution belonging to $W_0^{1,2}(\Omega) \cap L^{pm}(\Omega)$ for every $\lambda \ge 0$ provided that $\frac{p+1}{p} \le m < \frac{N}{2}\frac{p-1}{p}$. The solution is obtained as limit of solutions of a sequence of suitable approximate problems. In particular the $L^{pm}(\Omega)$-regularity of the solution is only obtained for this specific solution obtained by approximation. We remark explicitly that the assumption that $h(x)$ is uniformly away from zero is essential in these papers.

Our first goal is to deal with the existence of solutions for $\lambda \ge \mathcal{H}$ and terms h which can vanish in a subset of Ω. Indeed, in Section 2 we handle functions $h(x)$ that can be zero in a neighbourhood $\Omega_\delta = \{x \in \Omega : \text{ dist } (x,\partial\Omega) < \delta\}$ of $\partial\Omega$. First we prove in Theorem 9.2.1-a) that if

$$\int_{\Omega\setminus\Omega_\delta} |x|^{\frac{2(p+1)}{1-p}} h(x)^{\frac{2}{1-p}} < \infty\,, \tag{9.2}$$

then there exists a solution u of (9.1) for every $\lambda \le \Lambda(\delta)$, where $\Lambda(\delta) \to \infty$ as $\delta \to 0$. Observe that in the particular case that $h(x) \equiv a > 0$, the above condition is satisfied provided that $p > 2^* - 1$. Hence, our result contains also the existence result of Adimurthi et al. (2017); Porzio (2007) when $m = \frac{p+1}{p}$ (see Corollary 9.2.3). The case that h is zero inΩ_δ is also considered in Corollary 9.2.5.

For the proof of Theorem 9.2.1-a) we take advantage of the variational nature of (9.1) by finding its solution as a critical point of the associated Euler $\mathcal{C}^1$-functional I_λ (see (9.4) below). Indeed, we show that I_λ is coercive and bounded from below. By using the Variational Principle of Ekeland we also prove that a suitable minimizing sequence of this functional is weakly convergent to a critical point $u \in W_0^{1,2}(\Omega) \cap L_h^{p+1}(\Omega)$ of I_λ, i.e., a solution of (9.1).

In addition, in Theorem 9.2.1-b) we also prove that if we strengthen the condition (9.2) by assuming that there exists $\bar{s} \in (2, p+1)$ such that

$$\int_{\Omega\setminus\Omega_\delta} |x|^{\frac{2\bar{s}}{2-\bar{s}}} h(x)^{\frac{2\bar{s}}{(p+1)(2-\bar{s})}} < \infty, \tag{9.3}$$

then I_λ is weakly lower semicontinuous (see Remark 9.2.2-iv) for a comparison with the result of (García Azorero and Peral Alonso, 1998, Theorem 3.4)) and thus u is a minimum of the functional I_λ. We also use this additional variational characterization of this found solution to obtain the existence of a non-zero solution of the problem (9.1) when $f \equiv 0$ (see Corollary 9.2.6) and improve the corresponding existence results of Wei and Du (2017); Wei and Feng (2015) (see Remark 9.2.7).

We devote the section 3 to study the regularity of every solution of (9.1). Specifically we prove in Theorem 9.3.1 that if $f \in L_h^m(\Omega)$ with $m \ge \frac{p+1}{p}$ and $|x|^{\frac{2pm}{1-p}} h^{1-\frac{pm}{p-1}} \in L^1(\Omega)$, then every solution u of (9.1) verifies $u \in L_h^{pm}(\Omega)$ improving the previously mentioned regularity result of Adimurthi et al. (2017); Porzio (2007) for solutions which are only obtained as limit of solutions of approximate problems (see Remark 9.3.4-ii)).

9.2 Coercivity and existence of solutions

For $0 \le h \in L^1_{loc}(\Omega)$ let $L_h^{p+1}(\Omega)$ be the linear space of all measurable functions in Ω such that $|f|^{p+1}h \in L^1(\Omega)$. It can be equiped with the seminorm

$$|u|_{L_h^{p+1}(\Omega)} = \left(\int_\Omega |u|^{p+1}\, h\right)^{\frac{1}{p+1}}, \ \forall u \in L_h^{p+1}(\Omega),$$

which is a norm in the particular case that $h(x) > 0$ a.e. $x \in \Omega$.

We consider the reflexive space

$$E = W_0^{1,2}(\Omega) \cap L_h^{p+1}(\Omega)$$

endowed with the norm

$$\|u\|_E = \|\nabla u\|_{L^2(\Omega)} + |u|_{L_h^{p+1}(\Omega)}.$$

Observe that every function $f \in L_h^{\frac{p+1}{p}}(\Omega)$ has associated a functional φ_f in the dual space E^* (of E) given by

$$\langle \varphi_f, g\rangle = \int_\Omega fgh, \ \forall g \in L_h^{p+1}(\Omega).$$

Hence, we understand that a solution of (9.1) is just a critical point of the $\mathcal{C}^1$-functional I_λ defined in E by setting

$$I_\lambda(u) = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1}\int_\Omega |u|^{p+1}\, h - \frac{\lambda}{2}\int_\Omega \frac{u^2}{|x|^2} - \int_\Omega f\, u\, h, \ \forall u \in E; \tag{9.4}$$

i.e. a function $u \in E$ satisfying

$$\int_\Omega \nabla u \nabla v + \int_\Omega |u|^{p-1} u\, v\, h - \lambda \int_\Omega \frac{u}{|x|^2} v - \int_\Omega f(x)\, v\, h = 0, \ \forall v \in E.$$

On the other hand, for every $\delta \ge 0$, we define the set

$$\Omega_\delta = \{x \in \Omega : \ \text{dist}\ (x, \partial\Omega) < \delta\}.$$

Observe that$\Omega_0 = \emptyset$ and that clearly there exists $\delta_0 > 0$ such that for every $\delta \in [0, \delta_0]$ the boundary $\partial\Omega_\delta$ ofΩ_δ is smooth and $0 \notin \bar\Omega_\delta$, where $\bar\Omega_\delta$ denotes the clousure ofΩ_δ. We point out that in the sequel the positive constant δ will be always assumed to be smaller than δ_0.

Our first goal is to study the existence of solutions for the problem (9.1) with functions h that can vanish inΩ_δ. Concretely, we are going to prove the following existence theorem.

Theorem 9.2.1 *Assume that $p > 1$, $f \in L_h^{\frac{p+1}{p}}(\Omega)$ and that there exists $\delta \ge 0$ such that $\partial\Omega_\delta$ is smooth, $0 \notin \bar\Omega_\delta$ and $h > 0$ a.e. in $\Omega \setminus \Omega_\delta$.*

a) If condition (9.2) holds true, then there exists $\Lambda(\delta)$ such that (9.1) has a solution $u \in E$ for every $\lambda \le \Lambda(\delta)$. In addition, $\Lambda(\delta) \to \infty$ as $\delta \to 0$.

b) If, in addition, there exists $\bar s \in (2, p+1)$ such that condition (9.3) holds true, then u is a minimum of functional I_λ given by (9.4).

Remarks 9.2.2

i) As it has been previously observed, every function $f \in L_h^{\frac{p+1}{p}}(\Omega)$ can be considered as an element of the dual space E^* of E. We will see in the proof that for the above existence result the hypothesis $f \in L_h^{\frac{p+1}{p}}(\Omega)$ can be relaxed to $f \in E^*$.

ii) Observe that condition (9.2) is equivalent to $\frac{1}{|x|h^{\frac{1}{p+1}}} \in L^{\frac{2(p+1)}{p-1}}(\Omega)$, while condition (9.3) means that $\frac{1}{|x|h^{\frac{1}{p+1}}} \in L^{\frac{2\bar{s}}{\bar{s}-2}}(\Omega)$. Observe that if $2 < \bar{s} < p+1$, then $2 < \frac{2(p+1)}{p-1} < \frac{2\bar{s}}{\bar{s}-2}$ and it follows that (9.3) implies (9.2).

iii) Moreover, (9.3) is clearly satisfied in the case in which $h(x)$ is a Hardy potential term of order $p+1$ on the left hand of equation (9.1), i.e. $h(x) = 1/|x|^{p+1}$. Indeed, in this context condition, (9.3) holds true due to the boundedness of the domain Ω.

iv) In the case $h \equiv 0$, the part b) of the above theorem has to be compared with the result of (García Azorero and Peral Alonso, 1998, Theorem 3.4) where the authors proved the existence of a minimum of the functional by using an argument that do not require the weak lower semicontinuity of the functional I_λ leaving this semicontinuity as an open problem. As for us, we prove that the hypothesis (9.3) implies that I_λ is w.l.s.c.

Proof: a) By (9.2), using the Hölder inequality with exponent $\dfrac{p+1}{2}$, we obtain for every $u \in E$

$$\int_\Omega \frac{u^2}{|x|^2} = \int_{\Omega_\delta} \frac{u^2}{|x|^2} + \int_{\Omega\setminus\Omega_\delta} \frac{u^2}{|x|^2} = \int_{\Omega_\delta} \frac{u^2}{|x|^2} + \int_{\Omega\setminus\Omega_\delta} \frac{u^2 h(x)^{\frac{2}{p+1}}}{h(x)^{\frac{2}{p+1}}|x|^2}$$
$$\leq \frac{1}{\rho(\delta)^2}\int_{\Omega_\delta} u^2 + C_1\left(\int_{\Omega\setminus\Omega_\delta} |u|^{p+1}\,h\right)^{\frac{2}{p+1}},$$

where $\rho(\delta) := \text{dist}(0, \Omega_\delta) > 0$.

Moreover, since $u = 0$ in $\partial\Omega$ and $\partial\Omega \subset \partial\Omega_\delta$ we can use a Poincaré inequality inΩ_δ (see e.g. Maz'ya (2011), (Ziemer, 1989, Section 4.6) see also (Adams, 1998, Section 8)) to assert that

$$\int_{\Omega_\delta} u^2 \leq C(\delta)\int_{\Omega_\delta} |\nabla u|^2$$

with the positive constant $C(\delta)$ satisfying

$$C(\delta) = C_2\sqrt{\frac{|\Omega_\delta|}{C_{1,2}(\partial\Omega)}} \to 0, \quad \text{as } \delta \to 0, \tag{9.5}$$

where $C_{1,2}(\partial\Omega)$ denotes the capacity of $\partial\Omega$.

Hence, the functional I_λ given by (9.4) satisfies for every $u \in E$ that

$$I_\lambda(u) \geq \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1}\int_\Omega |u|^{p+1}\,h - \frac{\lambda C(\delta)}{\rho(\delta)^2}\int_{\Omega_\delta}\frac{|\nabla u|^2}{2}$$
$$- \frac{\lambda\, C_1}{2}\left(\int_{\Omega\setminus\Omega_\delta} |u|^{p+1}\,h\right)^{\frac{2}{p+1}} - \int_\Omega f\,u\,h$$
$$\geq \left(1 - \frac{\lambda C(\delta)}{\rho(\delta)^2}\right)\int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1}\int_\Omega |u|^{p+1}\,h - \frac{\lambda\,C_1}{2}\left(\int_\Omega |u|^{p+1}\,h\right)^{\frac{2}{p+1}}$$
$$- \|f\|_{E^*}\|u\|_E.$$

Thus, since $\frac{2}{p+1} < 1$, we obtain that I_λ is coercive and bounded from below provided that

$$\lambda \leq \Lambda(\delta) := \frac{\rho(\delta)^2}{C(\delta)}.$$

As a consequence, by the *Variational Principle of Ekeland* Ekeland (1974), there is a bounded minimizing sequence $\{u_n\} \subset E$ such that

$$I_\lambda(u_n) \to \inf_E I_\lambda \tag{9.6}$$

and $I'_\lambda(u_n) \to 0$ in E^*, i.e., there exists a sequence of positive numbers $\{\varepsilon_n\}$ converging to zero such that

$$\left| \int_\Omega \nabla u_n \nabla v + \int_\Omega |u_n|^{p-1} u_n \, v \, h - \lambda \int_\Omega \frac{u_n}{|x|^2} v - \int_\Omega f(x) \, v \, h \right| \le \varepsilon_n \|v\|_E, \ \forall v \in E. \tag{9.7}$$

We are going to pass to the limit in this inequality as n tends to infinity. The boundedness of $\{u_n\}$ in E implies that, up to a subsequence, we have the weak convergence of u_n in E to some $u \in E$. In particular, up to a subsequence, we can assume that

(A) $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$,

(B) $u_n h^{\frac{1}{p+1}} \rightharpoonup u h^{\frac{1}{p+1}}$ in $L^{p+1}(\Omega)$,

(C) $u_n \to u$ in $L^q(\Omega)$ $(1 \le q < 2^*)$,

(D) $u_n(x) \to u(x)$ a.e. in Ω ,

(E) $\exists \, g \in L^q(\Omega)$ $(1 \le q < 2^*)$ such that $|u_n(x)| \le g(x)$.

Obviously, by (A),

$$\lim_{n\to\infty} \int_\Omega \nabla u_n \nabla v = \int_\Omega \nabla u \nabla v, \ \forall v \in W_0^{1,2}(\Omega)$$

and by (B) the sequence $|u_n|^{p-1}u_n$ is bounded in $L_h^{p+1}(\Omega)$ and due to almost every convergence (D), it follows that $|u_n|^{p-1}u_n \rightharpoonup |u|^{p-1}u$ in $L^{p+1}(\Omega; hdx)$. Hence, by (E), Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p-1} u_n \, v \, h = \int_\Omega |u|^{p-1} u \, v \, h, \quad \forall v \in L^{p+1}(\Omega).$$

In order to get the convergence of the term with Hardy potential, i.e., $\int_\Omega \frac{u_n}{|x|^2} v$, we point out that for each $v \in W_0^{1,2}(\Omega)$ the operator $T_v : W_0^{1,2}(\Omega) \to \mathbb{R}$ defined as

$$T_v(u) = \int_\Omega \frac{u}{|x|^2} \, v, \ \forall v \in W_0^{1,2}(\Omega)$$

is linear and continuous since (by using Hölder and Hardy inequalities)

$$|T_v(u)| \le \left(\int_\Omega \left(\frac{u}{|x|} \right)^2 \right)^{1/2} \left(\int_\Omega \left(\frac{v}{|x|} \right)^2 \right)^{1/2} \le \mathcal{H} \, \|u\|_{W_0^{1,2}(\Omega)} \|v\|_{W_0^{1,2}(\Omega)}$$

for every $v \in W_0^{1,2}(\Omega)$, ($\mathcal{H}$ is the Hardy constant).

In particular, since T_v has finite range, it is also compact and hence $T_v(u_n)$ strongly converges to $T_v(u)$, i.e.

$$\lim_{n\to\infty} \int_\Omega \frac{u_n(x)}{|x|^2} v(x) = \int_\Omega \frac{u(x)}{|x|^2} v(x).$$

In conclusion, taking limits in (9.7) we obtain that $u \in E$ is a solution of problem (9.1) for $\lambda < \Lambda(\delta)$.

In addition, since $\rho(\delta) \to \operatorname{dist}(0, \partial\Omega) > 0$ as $\delta \to 0$, then (9.5) implies that $\Lambda(\delta) \to \infty$ as $\delta \to 0$.

b) As it has been seen in the proof of the part a), for every $\lambda \le \Lambda(\delta)$ the functional I_λ is bounded from below and coercive. Thus, in order to deduce that I_λ attains its minimum, it suffices to show that it is weak lower semicontinuous. Assume hence that $\{u_n\}$ is a sequence weakly convergent in E. As before, up to a subsequence, we can assume that $\{u_n\}$ verifies the convergences (A)-(E). In addition, we note that the boundedness of $u_n h^{\frac{1}{p+1}}$ in $L^{p+1}(\Omega)$ and the a.e. convergence (D) of u_n imply the strong convergence of $u_n h^{\frac{1}{p+1}}$ in $L^s(\Omega)$ for every $1 \le s < p+1$. As a consequence, there exists $G \in L^s(\Omega)$ such that (again up to a subsequence) $|u_n(x) h^{\frac{1}{p+1}}(x)| \le G(x)$, for all $n \in \mathbb{N}$.

We claim that

$$\lim_{n\to\infty} \int_\Omega \frac{u_n(x)^2}{|x|^2} = \int_\Omega \frac{u(x)^2}{|x|^2}. \tag{9.8}$$

Indeed, if we consider the function $g \in L^2(\Omega)$ given in (E) with $q = 2$ which satisfies that $|u_n(x)| \leq g(x)$ for every $n \in \mathbb{N}$ and almost everywhere for $x \in \Omega$ then

$$\frac{u_n^2(x)}{|x|^2} \leq H(x) \text{ a.e. } x \in \Omega,$$

where the function H is defined in Ωas

$$H(x) = \begin{cases} \dfrac{g^2(x)}{|x|^2}, & \text{if } x \in \bar{\Omega}_\delta, \\ \\ \dfrac{G^2(x)}{|x|^2 h(x)^{\frac{2}{p+1}}}, & \text{if } x \in \Omega \setminus \bar{\Omega}_\delta. \end{cases}$$

By (D) we also have the convergence of $\frac{u_n(x)^2}{|x|^2}$ to $\frac{u(x)^2}{|x|^2}$ for almost every $x \in \Omega$. Therefore, by the dominated convergence theorem, the claim will be proved if we show that $H \in L^1(\Omega)$. For this purpose, observe that taking into account that $0 \notin \bar{\Omega}_\delta$, we deduce that $\frac{g^2(x)}{|x|^2} \in L^1(\bar{\Omega}_\delta)$, i.e., $H \in L^1(\bar{\Omega}_\delta)$. To prove the integrability in$\Omega \setminus \bar{\Omega}_\delta$, we use the Hölder inequality with exponent $\frac{s}{2} > 1$ to obtain

$$\int_{\Omega\setminus\Omega_\delta} \frac{G^2(x)}{|x|^2 h(x)^{\frac{2}{p+1}}} \leq \left(\int_{\Omega\setminus\Omega_\delta} \frac{1}{|x|^{\frac{2s}{s-2}} h(x)^{\frac{2s}{(s-2)(p+1)}}} \right)^{\frac{s-2}{s}} \left(\int_{\Omega\setminus\Omega_\delta} G(x)^s \right)^{\frac{2}{s}}.$$

The last two integral terms are finite due to hypothesis (9.3) and that $G \in L^s(\Omega)$. Consequently, we also have $H \in L^1(\Omega \setminus \bar{\Omega}_\delta)$ and the claim is proved.

By the other hand, the result of (Boccardo and Murat, 1992, Theorem 2.1) implies that (up to a subsequence) $\nabla u_n \to \nabla u$ strongly in $(L^q(\Omega))^N$ $(1 < q < 2)$ and in particular (up to a subsequence) it converges almost everywhere in Ω. Then, applying the Fatou lemma we have

$$\liminf_{n\to\infty} \left(\int_\Omega \frac{|\nabla u_n|^2}{2} + \frac{1}{p+1} \int_\Omega |u_n|^{p+1} h \right) \geq \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1} \int_\Omega |u|^{p+1} h \tag{9.9}$$

Summarizing (9.8) and (9.9) we obtain

$$\liminf_{n\to\infty} I_\lambda(u_n) \geq I_\lambda(u),$$

i.e. the functional I_λ is w.l.s.c. and the proof is concluded. □

If we take $\delta = 0$, then $\Omega_\delta = \emptyset$ and by observing that $\int_\Omega |x|^{\frac{2(p+1)}{1-p}} < \infty$ provided that $p > 2^* - 1$, we derive from Theorem 9.2.1 the following consequence for the case that h is a positive constant in all Ω.

Corollary 9.2.3 *Assume $p > 2^* - 1$, $f \in L^{\frac{p+1}{p}}(\Omega)$ and $h(x) \equiv h_0 > 0$ in Ω. There exists $u \in E$, solution of problem (9.1) for every $\lambda \in \mathbb{R}$.*

Remark 9.2.4 In particular, we recover the existence result of Adimurthi et al. (2017); Porzio (2007): there exists a solution in $E = W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)$.

A simple case in which h vanishes inΩ_δ is the following one.

Corollary 9.2.5 *Let $p > 2^* - 1$, $0 < \delta \leq \delta_0$, $f \in L^{\frac{p+1}{p}}(\Omega \setminus \Omega_\delta)$ and $h \equiv h_0 \chi_{\Omega\setminus\Omega_\delta}$ for some $h_0 > 0$. Then, there is a solution of (9.1) in E for $\lambda \leq \Lambda(\delta)$.*

If $\mathcal{H} < \lambda$ then it is possible to choose $w \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega |\nabla w|^2 - \lambda \int_\Omega \frac{w^2}{|x|^2} < 0.$$

and since $p > 1$, we deduce in the case $f \equiv 0$ that $\inf_E I_\lambda \leq I_\lambda(tw) < 0 = I_\lambda(0)$ provided that t is close to zero. This allows to conclude this section by showing a simple consequence of the additional information that the solution u given in Theorem 9.2.1 is a minimum of I_λ.

Corollary 9.2.6 *If $p > 1$, the function h satisfies (9.3) with $h > 0$ a.e. in $\Omega \setminus \Omega_\delta$ and $\mathcal{H} < \lambda \leq \Lambda(\delta)$, then the problem*

$$\begin{cases} -\Delta u + h(x)|u|^{p-1}u = \lambda \dfrac{u}{|x|^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.10}$$

has at least one nonzero solution.

Remark 9.2.7 As usual by considering instead of I_λ the functional J_λ given by

$$J_\lambda(u) = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1}\int_\Omega |u|^{p+1}\, h - \frac{\lambda}{2}\int_\Omega \frac{(u^+)^2}{|x|^2},\ u \in E,$$

it is possible to deduce the existence of a positive solution of the problem (9.10). Therefore we improve the corresponding existence result of Wei and Feng (2015) where it is required additionally that h is a continuous and positive function in $\bar{\Omega}$ and of Wei and Du (2017), where the case $h(x) = 1/|x|^\beta$ with $\beta > 2$ is studied. (Observe that in both cases considered in those papers, $\Lambda(\delta) = \infty$ in the above corollary).

9.3 Regularity of the solutions

In this section, for the reader's convenience we assume that $h \in L^1(\Omega)$. In this case, by Hölder inequality, it is easy to verify that $L_h^r(\Omega) \subset L_h^s(\Omega)$ for every $r \geq s \geq 1$. Next, we give a sufficient condition on the function h for which if we strength the condition $f \in L_h^{\frac{p+1}{p}}(\Omega)$ by assuming that $f \in L_h^m(\Omega)$ with $m \geq \frac{p+1}{p}$, then the solution (given by Theorem 9.2.1) $u \in W_0^{1,2}(\Omega) \cap L_h^{p+1}(\Omega)$ of (9.1) is more regular: it belongs also to $L_h^{pm}(\Omega)$.

Theorem 9.3.1 *Assume that $h \in L^1(\Omega)$ with $h(x) > 0$ a.e. in Ω and that there exists $m \geq \frac{p+1}{p}$ such that*

i) $f \in L_h^m(\Omega)$,

ii) $|x|^{\frac{2pm}{1-p}} h^{1-\frac{pm}{p-1}} \in L^1(\Omega)$.

If u is a solution of (9.1), then $u \in L_h^{pm}(\Omega)$.

Remark 9.3.2 If instead of assuming that $h \in L^1(\Omega)$ we only assume that $h \in L^1_{loc}(\Omega)$, then the above hypothesis i) should be replaced by $f \in L_h^{\frac{p+1}{p}}(\Omega) \cap L_h^m(\Omega)$.

Proof: For every $k > 0$, we define the auxiliary function $T_k : \mathbb{R} \to \mathbb{R}$ as usual

$$T_k(s) = \begin{cases} k, & s > k, \\ s, & |s| \leq k, \\ -k, & s < -k. \end{cases}$$

Let $u \in E$ be a solution of (9.1). Since $m \geq (p+1)/p$, we have $\gamma := pm - 1 - p > 0$ and we can choose $|T_k(u)|^\gamma T_k(u)$ as a test function in problem (9.1) to obtain, by dropping the positive term coming from the principal part, that

$$\int_\Omega h|u|^p|T_k(u)|^{\gamma+1} \leq \lambda \int_\Omega \frac{|u||T_k(u)|^{\gamma+1}}{|x|^2} + \int_\Omega f|T_k(u)|^{\gamma+1}h. \tag{9.11}$$

Next, we estimate each term of the above inequality. In order to do it, we define

$$F_k(u) := |u|^{p-\delta}|T_k(u)|^{1+\gamma+\delta}h\,,$$

where

$$\delta = \frac{(1+\gamma)(p-1)}{\gamma+2} = \frac{p(m-1)(m-1)}{pm-p+1} \in (0, p-1).$$

Using that $|T_k(s)| \leq |s|$ for all $s \in \mathbb{R}$, we deduce that

$$|u|^p|T_k(u)|^{\gamma+1}h = F_k(u)|T_k(u)|^{-\delta}/|u|^{-\delta} \geq F_k(u)$$

and thus

$$\int_\Omega h|u|^p|T_k(u)|^{\gamma+1} \geq \int_\Omega F_k(u), \tag{9.12}$$

On the other hand, using Hölder inequality with exponent $p-\delta > 1$ and that $1+\delta+\gamma = (1+\gamma)(p-\delta)$, we get

$$\begin{aligned}\lambda \int_\Omega \frac{|u||T_k(u)|^{\gamma+1}}{|x|^2} &= \lambda \left(\int_\Omega |x|^{\frac{2pm}{1-p}} h^{1-\frac{pm}{p-1}}\right)^{\frac{1}{(p-\delta)'}} \left(\int_\Omega F_k(u)\right)^{\frac{1}{p-\delta}} \\ &\leq C_1 \left(\int_\Omega F_k(u)\right)^{\frac{1}{p-\delta}}, \end{aligned}\tag{9.13}$$

where the last inequality is a consequence of hypothesis ii).

In addition, using Hölder with exponent m and taking into account that

$$\frac{(\gamma+1)m}{m-1} = pm = \gamma + 1 + p$$

we obtain by i)

$$\begin{aligned}\int_\Omega f|T_k(u)|^{\gamma+1}\,h &= \int_\Omega f\,h^{\frac{1}{m}}\,|T_k(u)|^{\gamma+1}\,h^{\frac{m-1}{m}} \\ &\leq \left(\int_\Omega |f|^m\,h\right)^{\frac{1}{m}} \left(\int_\Omega |T_k(u)|^{\frac{(1+\gamma)m}{m-1}}\,h\right)^{\frac{m-1}{m}} \leq C_2 \left(\int_\Omega F_k(u)\right)^{\frac{m-1}{m}}. \end{aligned}\tag{9.14}$$

In conclusion, substituting (9.12), (9.13) and (9.14) into (9.11), we deduce that

$$\int_\Omega F_k(u) \leq C_1 \left(\int_\Omega F_k(u)\right)^{\frac{1}{p-\delta}} + C_2 \left(\int_\Omega F_k(u)\right)^{\frac{m-1}{m}}. \tag{9.15}$$

Since $\frac{1}{p-\delta}$ and $\frac{m-1}{m}$ are less than 1, (9.15) implies the existence of $k_0 > 0$ and $C_3 > 0$ (independent of k and u) such that

$$\int_\Omega |u|^{p-\delta}|T_k(u)|^{1+\gamma+\delta}h = \int_\Omega F_k(u) \leq C_3, \qquad \text{for all } k \geq k_0.$$

Fatou's lemma when k tends to ∞ and the fact that $\gamma + 1 + p = pm$ implies that

$$\int_\Omega |u|^{pm}h(x)dx = \int_\Omega |u|^{p+1+\gamma}h \leq C_3$$

as we desired.

□

A particular interesting case is when the function h can be compared with a Hardy potential of different order.

Corollary 9.3.3 *Assume that $f \in L^m_h(\Omega)$ for $m \geq \frac{p+1}{p}$, and that there exist $\mu > 0$ and $\beta \geq 0$ such that the function $h \in L^1(\Omega)$ satisfies*

$$h(x) \geq \frac{\mu}{|x|^\beta}, \text{ a.e. } x \in \Omega.$$

If u is a solution of (9.1), then $u \in L^{pm}\left(\Omega; \frac{dx}{|x|^\beta}\right)$ for every

$$m \in \begin{cases} \left[\frac{p+1}{p}, \frac{(N-\beta)(p-1)}{(2-\beta)p}\right), & \text{if } \beta \in [0,2), \\ \left[\frac{p+1}{p}, \infty\right), & \text{if } \beta \geq 2. \end{cases}$$

Remarks 9.3.4

i) The integrability of h implies that necessarily $\beta < N$.

ii) Observe that if $\beta \in [0, 2)$, then the interval $[\frac{p+1}{p}, \frac{(N-\beta)(p-1)}{(2-\beta)p}]$ of the possibles values of m is not empty (i.e., $\frac{p+1}{p} < \frac{(N-\beta)(p-1)}{(2-\beta)p}$) if and only if h satisfies condition (9.2).

iii) We note that in the particular case $\beta = 0$ the regularity result is proved in Adimurthi et al. (2017) only for a solution obtained as limit of solutions of a sequence of suitable approximate problems, but not for every solution as in the previous result.

Chapter 10

A concave-convex problem with a variable operator

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Abstract

We study the following elliptic problem $-A(u) = \lambda u^q$ with Dirichlet boundary conditions, where $A(u)(x) = \Delta u(x)\chi_{D_1}(x) + \Delta_p u(x)\chi_{D_2}(x)$ is the Laplacian in one part of the domain, D_1, and the $p-$Laplacian (with $p > 2$) in the rest of the domain, D_2. We show that this problem exhibits a concave-convex nature for $1 < q < p-1$. In fact, we prove that there exists a positive value λ^* such that the problem has no positive solution for $\lambda > \lambda^*$ and a minimal positive solution for $0 < \lambda < \lambda^*$. If in addition we assume that p is subcritical, that is, $p < 2N/(N-2)$ then there are at least two positive solutions for almost every $0 < \lambda < \lambda^*$, the first one (that exists for all $0 < \lambda < \lambda^*$) is obtained minimizing a suitable functional and the second one (that is proven to exist for almost every $0 < \lambda < \lambda^*$) comes from an appropriate (and delicate) mountain pass argument.

10.1 Introduction

Given a smooth bounded domain Ω we split it into two smooth subdomains

$$\overline{\Omega} = \overline{D_1 \cup D_2}, \qquad D_1 \cap D_2 = \emptyset$$

(we assume that both D_1 and D_2 are Lipschitz). We call Γ the interface inside Ω,

$$\Gamma = \partial D_1 \cap \Omega = \partial D_2 \cap \Omega,$$

and we assume that Γ is a smooth surface with finite $(N-1)$ dimensional Hausdorff measure.

For a fixed $p > 2$ we consider the operator which acts as the Laplacian in the region D_1 and as the p-Laplacian in the region D_2. To be more precise, we consider equations of the form

$$-\Delta u = f(u), \text{ in } D_1 \qquad \text{and} \qquad -\Delta_p u = f(u), \text{ in } D_2,$$

with a Dirichlet boundary condition, $u = 0$ on $\partial\Omega$, a suitable continuity condition on Γ and a power nonlinearity f.

Note that this problem can also be rewritten involving a variable exponent operator, a $p(x)$-Laplacian, with a discontinuous exponent $p(x)$. That is, we deal with

$$\begin{cases} -\Delta_{p(x)} u = f(u), & \text{in}\Omega\ , \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases}$$

where$\Delta_{\ p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ and the variable discontinuous exponent $p(x)$ is given by

$$p(x) = \begin{cases} 2 & \text{if } x \in D_1, \\ p > 2 & \text{if } x \in D_2. \end{cases} \tag{10.1}$$

With regard to equations involving $p(x)$-Laplacian terms, with a general $p(x)$ (not necessarily discontinuous) we refer the reader to the recent book Diening et al. (2011) for background and an extensive review of recent results. In addition, problems that involve the $p(x)$-Laplacian with a discontinuous variable exponent, which is assumed to be constant in disjoint pieces of the domain Ω, are recently used to model organic semiconductors (i.e., carbon-based materials conducting an electrical current). In these models $p(x)$ describes a jump function that characterizes Ohmic and non-Ohmic contacts of the device material, see Bulíček et al. (2016) and Bulíček et al. (2017). In fact, let us consider the Organic Light-Emitting Diodes (OLEDs) which are constituted by thin-film heterostructures made up by organic molecules or polymers. Each functional layer has its own current-voltage characteristics and hence, the current-flow equation is of $p(x)$-Laplacian type. Since the exponent $p(x)$ describes non-Ohmic behavior of materials, it changes abruptly in passing from one to another. For example, in electrodes the parameter $p(x)$ is typically 2 (Ohmic) while in organic materials $p(x)$ takes larger values, e.g. $p(x) = 9$ (Fischer et al. (2014)).

This work is devoted to the study of this kind of operators with a power nonlinearity on the right hand side that has a concave-convex nature with respect to the variable operator$\Delta_{\ p(x)}$. That is, convex (superlinear) for the Laplacian and concave (sublinear) for the p-Laplacian. Concretely, we look for existence and multiplicity of positive weak solutions for the following problem

$$\begin{cases} -\Delta u = \lambda u^q, & & \text{in } D_1, \\ -\Delta_p u = \lambda u^q, & & \text{in } D_2, \\ \dfrac{\partial u}{\partial \eta} = |\nabla u|^{p-2}\dfrac{\partial u}{\partial \eta}, & u|_{D_1} = u|_{D_2}, & \text{on}\Gamma\ , \\ u = 0, & & \text{on } \partial\Omega, \end{cases} \tag{10.2}$$

in the following function space

$$\mathcal{W}(\Omega) = \left\{ v \in W_0^{1,2}(\Omega) : \int_{D_2} |\nabla v|^p < \infty \right\}.$$

Here

$$\lambda > 0, \qquad 2 < q + 1 < p,$$

and η is the normal unit vector to Γ pointing outwards D_1. This space $\mathcal{W}(\Omega)$ is a reflexive and separable Banach space equipped with the norm

$$[v]_{\mathcal{W}(\Omega)} := \| \nabla v \|_{L^2(D_1)} + \| \nabla v \|_{L^p(D_2)} \tag{10.3}$$

(see Lemma 10.2.1 for a detailed proof). We refer to the Preliminaries section in order to justify the definition of this convenient space.

Observe that in (10.2) we have continuity of the solution, in the sense that the trace of u onΓ coincides coming from D_1 and coming from D_2, and also we have continuity of the associated fluxes across Γ. In addition, note that the exponent q is a superlinear exponent (convex) for the problem in D_1 and a $p-$sublinear one (concave) for the problem in D_2. Therefore this problem has both a concave part and a convex one (but acting in different regions).

It is fairly easy to see that problem (10.2) has a variational structure. Indeed, if we consider the functional $F : \mathcal{W}(\Omega) \to \mathbb{R}$

$$F_\lambda(u) = \int_{D_1} \frac{|\nabla u|^2}{2}\, dx + \int_{D_2} \frac{|\nabla u|^p}{p}\, dx - \lambda \int_\Omega \frac{|u|^{q+1}}{q+1}\, dx, \tag{10.4}$$

as we will see in Lemma 10.2.5, positive solutions of (10.2) are uniquely identified as being positive critical points for this functional.

From a pure mathematical perspective concave–convex problems have received some interest in the literature in recent times, including several kinds of boundary conditions and generalizations to other operators such as the p–Laplacian or fully nonlinear uniformly elliptic operators. The subject goes back to the pioneering works Boccardo et al. (1995), García Azorero and Peral Alonso (1991), García Azorero and Peral Alonso (1994) and Lions (1982). However, Ambrosetti et al. (1994) is regarded as a first detailed analysis of the main properties of such type of problems, especially its bifurcation diagrams (see also Lions (1982), Section 1.1). We also quote Ambrosetti et al. (1996) and García Azorero et al. (2000) that deal with Dirichlet conditions and the p–Laplacian operator; Charro et al. (2009), dedicated to fully nonlinear uniformly elliptic operators with Dirichlet boundary conditions; García-Azorero et al. (2004), dealing with flux–type nonlinear boundary conditions and source nonlinearities and García-Melián et al. (2012) handling concave–convex terms of absorption nature. Of course, this list is far from being complete and is only a sample of the previous research on the topic.

In this framework we have the following results:

Theorem 10.1.1 *There exists $\lambda^* > 0$ such that:*

1. *For $0 < \lambda < \lambda^*$ there exists w_λ a minimal positive solution. Moreover, this minimal solution, w_λ, is unique and increasing with respect to λ.*

2. *For $\lambda > \lambda^*$ there is no positive solution.*

The proof is based on the method of sub and supersolution. For this, a comparison principle and a maximum principle for this problem are needed. For the nonexistence of solutions for λ large we use the fact that solutions to the parabolic problem $u_t = \Delta u + \lambda u^q$ in D_1, with large initial data, blow up in finite time. Theorem 10.1.1 is proved in Section 3.

Our next result shows that this problem has a second solution for almost every $0 < \lambda < \lambda^*$ when p is subcritical, in our case that is, $p < 2^*$. Here $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ when $N = 1, 2$. Note that we also have that q is subcritical since $1 < q < p - 1 < 2^* - 1$.

Theorem 10.1.2 *Assume, in addition, $p < 2^*$ and $D_2 \subset\subset \Omega$. Then, there exists a second positive solution v_λ for almost every $0 < \lambda < \lambda^*$.*

To prove the existence of a second solution we argue in two steps: First, using variational methods, we prove that (10.2) has a solution which is a local minimum of the corresponding energy functional (Theorem 10.4.6). This fact is subtle and we run into new difficulties. To be more precise, as the operator acts differently in D_1 and in D_2, we can only get regularity of solutions at locally Hölder spaces (we refer the seminal paper Acerbi and Fusco (1994)). Then, to show that there is a local minimum in $\mathcal{W}(\Omega)$, we assume that $D_2 \subset\subset \Omega$ in order to get $\mathcal{C}^1$ regularity close to $\partial\Omega$ and then we show that there is a minimum in the stronger topology $\mathcal{C}^1(F_\delta) \cap \mathcal{C}(\overline{\Omega})$ where F_δ is a small strip around the boundary of Ω. Then, by using a delicate regularity argument, we relax the topology to $\mathcal{W}(\Omega)$. Here we use partially the ideas from Ambrosetti et al. (1994); Brézis and Nirenberg (1993); García Azorero et al. (2000) adapting them to our setting with the introduction of a new original trick while using Stampacchia's approach in Proposition 10.4.5 in order to obtain an $L^\infty-$bound. It is at this point where we use that $p < 2^*$. Note that our space of solutions $\mathcal{W}(\Omega)$ is a subspace of $W_0^{1,2}(\Omega)$ that is larger than $W_0^{1,p}(\Omega)$.

Next, in order to prove the existence of a second positive solution, the crucial fact is to try to apply a Mountain Pass argument. The main difficulty here is to show that Palais-Smale sequences are bounded in $\mathcal{W}(\Omega)$. This question is at present far from being solved and an affirmative answer would allow to find a second solution for all $\lambda \in (0, \lambda^*)$ instead of for almost every $\lambda \in (0, \lambda^*)$. Let us discuss some difficulties: Initially, we point out that the usual trick combining $F_\lambda(u_n) \to c$ with $F'_\lambda(u_n)u_n = o(\|u_n\|)$ does not work here. In addition, we would like to comment that in previous references involving the search for critical points of Mountain Pass type for problems like

$$\begin{cases} -\Delta u = f(x,u), & \text{in}\Omega\ , \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases}$$

it is usually assumed that

$$\exists\, \kappa > 2 \ \text{ such that } \ \forall\, s \geq 0 \text{ and a.e. } x \in \Omega \Rightarrow 0 \leq \kappa F(x,s) \leq s f(x,s), \tag{AR}$$

where $F(x,s) = \int_0^s f(x,t)dt$. This condition was originally introduced in Ambrosetti and Rabinowitz (1973) and it is called Ambrosetti-Rabinowitz type condition. Roughly speaking, the role of (AR) is to ensure that all Palais-Smale sequences at the mountain pass level are bounded. Adapting this result to our variable operator$\Delta\ u\chi_{D_1} + \Delta_p u\chi_{D_2}$ it is not difficult to prove that if $f(x,s)$ satisfies property (AR) for $\kappa > p$, then we have that Palais-Smale sequences are bounded (see Appendix). However, in our setting $f(x,s) = \lambda s^q$ and (AR) is not satisfied for $\kappa > p$ because $q+1 < p$. Moreover, even conditions weaker than (AR) present in the literature of elliptic equations ensuring the existence of bounded Palais-Smale sequences are not applicable to our problem. To tackle this obstacle, we use some results from the classic works Ambrosetti and Rabinowitz (1973); De Figueiredo (1989); Ghoussoub and Preiss (1989); Jeanjean (1999) again adapting them to our framework. Mainly, relying on a result by Jeanjean Jeanjean (1999) which shows the existence a bounded Palais-Smale sequence at mountain pass level for almost every $0 < \lambda < \lambda^*$. We remark that once we have a bounded Palais-Smale sequence we are able to prove that there is a subsequence that converges strongly in $\mathcal{W}(\Omega)$.

Finally, we note that with the same ideas used here we can obtain similar results for the following problem

$$\begin{cases} -\Delta u = \lambda u^{q_1}\chi_{D_1} + \lambda u^{q_2}\chi_{D_2}, & \text{in}\Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $q_1 < 1 < q_2$. See García-Melián et al. (2016) for similar results for the same problem with $\lambda u^{q(x)}$, with a continuous exponent $q(x)$.

Also remark that when we take $D_1 = D_2 = \Omega$, that is, for the problem

$$\begin{cases} -\Delta u - \Delta_p u = \lambda u^q, & \text{in}\Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $1 < q < p-1$ one has existence of a minimal positive solution for large λ, $\lambda > \tilde{\lambda}$ and nonexistence for small λ, $\lambda < \tilde{\lambda}$. This result (that can be obtained just constructing adequate sub and supersolution) has to be contrasted with ours for (10.2) where we have existence for small λ and nonexistence for large λ.

The rest of this paper is organized as follows: in the Preliminaries, Section 10.2, we give some definitions and motivate the use of the space $\mathcal{W}(\Omega)$. In Section 10.3 we deal with the proof of Theorem 10.1.1. Finally, in Section 10.4 we prove the existence of a second solution provided $p < 2^*$. For completeness, in the Appendix we include a proof that shows that Palais-Smale sequences are bounded when we assume (AR) with $\kappa > p$.

10.2 Preliminaries

In this section we motivate the use of the space $\mathcal{W}(\Omega)$ to define weak solutions for our problem and also we collect some results that will be used throughout this work.

In order to justify the definition of space $\mathcal{W}(\Omega)$, let us give a briefly description about $W_0^{1,p(x)}$ spaces with $p(x)$ defined in (10.1). Following Diening et al. (2011) we define the Banach space

$$L^{p(x)}(\Omega) = \left\{v : \Omega \to \mathbb{R} \text{ mesurable } : \|v\|_{L^2(D_1)} + \|v\|_{L^p(D_2)} < \infty\right\}.$$

equipped with the Luxemburg norm

$$\|v\|_{L^{p(x)}(\Omega)} = \inf_{\tau>0}\left\{\int_{D_1}\left(\frac{u}{\tau}\right)^2 + \int_{D_2}\left(\frac{u}{\tau}\right)^p \leq 1\right\}.$$

The space $L^{p(x)}(\Omega)$ is a reflexive and separable Banach space. Accordingly, we set the Sobolev space

$$W^{1,p(x)}(\Omega) = \left\{v : \Omega \to \mathbb{R} \text{ mesurable } : v, |\nabla v| \in L^{p(x)}(\Omega)\right\}$$

and we have that $W^{1,p(x)}(\Omega)$ is a reflexive and separable Banach space with the norm

$$\|v\|_{W^{1,p(x)}(\Omega)} = \|v\|_{L^{p(x)}(\Omega)} + \|\nabla v\|_{L^{p(x)}(\Omega)}.$$

Moreover, since $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(x)}(\Omega)$ ((Fan et al., 2006, Theorem 2.4 and 2.7)). Then, $W_0^{1,p(x)}(\Omega)$ is well-defined as the closure of $C_c^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and it satisfies

$$W_0^{1,p}(\Omega) \subset W_0^{1,p(x)}(\Omega) \subset W_0^{1,2}(\Omega).$$

However, we can not use Poincaré's inequality in $\widehat{W}$ since, in general, it does not hold for discontinuous exponents, see (Diening et al., 2011, Sec. 8.2). Thus, we deal with a different Sobolev space that will be appropriate for our problem. Concretely, we define the Sobolev space $\mathcal{W}(\Omega)$

$$\mathcal{W}(\Omega) = \left\{v \in W_0^{1,2}(\Omega) : \int_{D_2} |\nabla v|^p < \infty\right\},$$

equipped with the following norm

$$\|v\|_{\mathcal{W}(\Omega)} = \|v\|_{W_0^{1,2}(\Omega)} + \|\nabla v\|_{L^p(D_2)}.$$

The space $\mathcal{W}(\Omega)$ is a separable and reflexive Banach space, since it is a closed subspace of $W_0^{1,2}(\Omega)$. The following result asserts that, by using Poincaré inequality, we can use the norm $[\,\cdot\,]_{\mathcal{W}(\Omega)}$ defined in (10.3) which only depends on the gradient terms.

Lemma 10.2.1 *$\big(\mathcal{W}(\Omega), [\,\cdot\,]_{\mathcal{W}(\Omega)}\big)$ is a reflexive and separable Banach space.*

Proof: Since $\big(\mathcal{W}(\Omega), \|\cdot\|_{\mathcal{W}(\Omega)}\big)$ is a reflexive and separable Banach space, it is sufficient to show that the norms $[\,\cdot\,]_{\mathcal{W}(\Omega)}$ and $\|\cdot\|_{\mathcal{W}(\Omega)}$ are equivalent. For this purpose we use the fact that functions in the classical Sobolev space $W_0^{1,2}(\Omega)$ satisfies the Poincaré inequality and also that the continuous embedding of variable Lebesgue spaces to obtain for arbitrary $v \in \mathcal{W}(\Omega)$,

$$\begin{aligned}
\|v\|_{\mathcal{W}(\Omega)} &= \|v\|_{W^{1,2}(\Omega)} + \|\nabla v\|_{L^p(D_2)} \\
&\le c_1\|\nabla v\|_{L^2(D_1)} + c_1\|\nabla v\|_{L^2(D_2)} + \|\nabla v\|_{L^p(D_2)} \\
&\le c_1\|\nabla v\|_{L^2(D_1)} + c_2\|\nabla v\|_{L^p(D_2)} \\
&\le c_3\left(\|\nabla v\|_{L^2(D_1)} + \|\nabla v\|_{L^p(D_2)}\right).
\end{aligned}$$

and

$$\begin{aligned}
\|v\|_{\mathcal{W}(\Omega)} &\ge \|\nabla v\|_{L^2(\Omega)} + \|\nabla v\|_{L^p(D_2)} \\
&\ge \|\nabla v\|_{L^2(D_1)} + \|\nabla v\|_{L^p(D_2)}.
\end{aligned}$$

In these estimates, positive constants are denoted by c_i, $i \ge 1$. □

Remark 10.2.2 *It is worth pointing out that the* $\|\cdot\|_{L^p(D_2)}$*-norm is controlled by the* $[\,\cdot\,]_{\mathcal{W}(\Omega)}$*-norm (in particular, if* $[u]_{\mathcal{W}(\Omega)} < \infty \Rightarrow \|u\|_{L^p(D_2)} < \infty$*). Moreover, there exists* $C > 0$ *such that* $\|u\|_{L^p(D_2)} \le C\left(\|\nabla u\|_{L^p(D_2)} + \|u\|_{L^2(D_2)}\right)$*. To see this fact, arguing by contradiction, suppose that for every* $n \in \mathbb{N}$ *there exists* u_n *such that*

$$\|u_n\|_{L^p(D_2)} > n\left(\|\nabla u_n\|_{L^p(D_2)} + \|u_n\|_{L^2(D_2)}\right) \tag{10.5}$$

which is equivalent to write the above expression as

$$1 > n\left(\|\nabla v_n\|_{L^p(D_2)} + \|v_n\|_{L^2(D_2)}\right).$$

being

$$v_n = \frac{u_n}{\|u_n\|_{L^p(D_2)}}.$$

Since $\|\nabla v_n\|_{L^p(D_2)} < \frac{1}{n}$ *and* $\|v_n\|_{L^p(D_2)} = 1$ *it follows that the sequence* $\{v_n\}$ *is bounded in* $W^{1,p}(D_2)$ *and hence, up to a subsequence,* v_n *converges weakly to* $w \in W^{1,p}(D_2)$*. Consequently,* $v_n \to w$ *in* $L^r(D_2)$ *for every* $r \in [2, p^*)$*. Taking* $r = p$*, and the fact* $\|v_n\|_{L^p(D_2)} = 1$ *implies* $\|w\|_{L^p(D_2)} = 1$*. However, taking* $r = 2$ *from (10.5) we have* $\|u_n\|_{L^2(D_2)} < \frac{1}{n}$ *and then we get that* $\|w\|_{L^2(D_2)} = 0$ *leading to a contradiction.*

Remark 10.2.3 *Let* $\mathcal{W}(\Omega)'$ *be the dual space of* $\mathcal{W}(\Omega)$*. We have that for every fixed* $w \in \mathcal{W}(\Omega)$ *the functional* $\hat{w} : \mathcal{W}(\Omega) \to \mathbb{R}$ *defined as*

$$\hat{w}(v) := \int_{D_1} \nabla w \nabla v + \int_{D_2} |\nabla w|^{p-2}\nabla w \nabla v + \int_\Omega wv, \qquad v \in \mathcal{W}(\Omega)$$

belongs to $\mathcal{W}(\Omega)'$*.*

Since we are considering positive solutions to the following $p(x)$-laplacian equation

$$\begin{cases} -\Delta_{p(x)} u = \lambda u^q, & \text{in}\Omega\ , \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $p(x)$ defined in (10.1), a natural idea of what is a positive weak solution is a positive function that vanishes on $\partial\Omega$ (in an appropriate trace sense) and such that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \varphi = \int_{D_1} \nabla u \nabla \varphi + \int_{D_2} |\nabla u|^{p-2}\nabla u \nabla \varphi = \lambda \int_\Omega u^q\, \varphi$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

Hence, let us state the definition of weak positive solutions to our problem as follows:

Definition 10.2.4 *Let $u \in \mathcal{W}(\Omega)$ be a positive function, it is said that u is a weak positive solution of* (10.2) *if it satisfies*

$$\int_{D_1} \nabla u \nabla \varphi + \int_{D_2} |\nabla u|^{p-2} \nabla u \nabla \varphi = \lambda \int_{\Omega} u^q \varphi \tag{10.6}$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

Note that (10.6) is formally equivalent to the following conditions:

$$\int_{D_1} \nabla u \nabla \varphi = \lambda \int_{D_1} u^q \varphi + \int_{\Gamma} \frac{\partial u}{\partial \eta} \varphi,$$

$$\int_{D_2} |\nabla u|^{p-2} \nabla u \nabla \varphi = \lambda \int_{D_2} u^q \varphi - \int_{\Gamma} |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} \varphi,$$

and

$$\int_{\Gamma} \frac{\partial u}{\partial \eta} \varphi = \int_{\Gamma} |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} \varphi.$$

In the next lemma we prove that we can study critical points of functional (10.4) instead of solutions of equation (10.2).

Lemma 10.2.5 *Solutions of* (10.2) *are characterized by positive critical points of functional in* (10.4)

Proof: From Definition 10.2.4, weak solutions satisfy

$$\int_{D_1} \nabla u \nabla \varphi + \int_{D_2} |\nabla u|^{p-2} \nabla u \nabla \varphi = \lambda \int_{\Omega} u^q \varphi$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Therefore, weak solutions are positive critical points of the functional (10.4). Conversely, if $u \in \mathcal{W}(\Omega)$ is a critical point, we obtain in particular that

$$\int_{D_1} \nabla u \nabla \phi = \lambda \int_{D_1} |u|^{q-1} u \, \phi, \quad \forall \phi \in \mathcal{C}_c^\infty(D_1).$$

Thus, u is a weak solution of the laplacian problem: $-\Delta u = \lambda |u|^{q-1} u$ in D_1. Hence, multiplying by test functions $\varphi \in \mathcal{C}_c^\infty(\Omega)$, integrating by parts and taking into account that $\Gamma = \partial D_1 \cap \Omega$, we obtain

$$\int_{D_1} \nabla u \nabla \varphi = \lambda \int_{D_1} |u|^{q-1} u \, \varphi + \int_{\Gamma} \frac{\partial u}{\partial \eta} \varphi, \tag{10.7}$$

being η the normal unit vector to Γ pointing outwards D_1. Analogously, choosing test functions belongs to $\mathcal{C}_c^\infty(D_2)$, we get that critical points are weak solutions to the p-laplacian problem: $-\Delta_p u = \lambda |u|^{q-1} u$ in D_2. The same arguments used above applied to this case give

$$\int_{D_2} |\nabla u|^{p-2} \nabla u \nabla \varphi = \lambda \int_{D_2} |u|^{q-1} u \, \varphi - \int_{\Gamma} |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} \varphi. \tag{10.8}$$

Finally, since equalities (10.7) and (10.8) hold together, the fact that u is a critical point imply that $\int_\Gamma \frac{\partial u}{\partial \eta} \varphi = \int_\Gamma |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} \varphi$. Therefore, it follows that positive critical points of functional F_λ are weak solutions to our problem. □

Finally, let us introduce the concept of sub and supersolution.

Definition 10.2.6 *By a subsolution (respectively, supersolution) to the problem* (10.2) *we mean a function $u \in \mathcal{W}(\Omega)$ that satisfies the following inequality:*

$$\int_{D_1} \nabla u \nabla \varphi + \int_{D_2} |\nabla u|^{p-2} \nabla u \nabla \varphi \leq (\geq) \, \lambda \int_{\Omega} |u|^{q-1} u \, \varphi \, ,$$

for every $0 \leq \varphi \in C_c^\infty(\Omega)$.

Note that a solution is just a function which is both a subsolution and a supersolution.

10.3 Existence and Non-Existence of Solutions

This section deals with existence and non existence of solutions. Initially, note that the functional F does not have a global minimum (and therefore the direct method of calculus of variations is not applicable). Indeed, let v be a function in $\mathcal{W}(\Omega)$ with compact support in D_1, then, since we have that $q > 1$,

$$F_\lambda(tv) = t^2 \int_{D_1} \frac{|\nabla v|^2}{2}\, dx - t^{q+1}\lambda \int_{D_1} \frac{|v|^{q+1}}{q+1}\, dx \to -\infty \tag{10.9}$$

as $t \to \infty$.

Hence, we use sub and supersolution techniques in order to get existence of solutions to problem (10.2). Our first step is to prove existence, uniqueness and a comparison principle for the problem

$$\begin{cases} -\Delta u = f, & & \text{in } D_1, \\ -\Delta_p u = f, & & \text{in } D_2, \\ \dfrac{\partial u}{\partial \eta} = |\nabla u|^{p-2}\dfrac{\partial u}{\partial \eta} & u|_{D_1} = u|_{D_2}, & \text{on} \Gamma\ , \\ u = 0, & & \text{on } \partial\Omega. \end{cases} \tag{10.10}$$

Here solutions, sub and supersolutions are understood as in Definitions 10.2.4 and 10.2.6 with λu^q replaced by f.

Proposition 10.3.1 *For every $f \in L^2(\Omega)$, the problem (10.10) has a unique weak solution in $u \in \mathcal{W}(\Omega)$.*

Proof: It is sufficient to prove that the functional

$$I(u) := \int_{D_1} \frac{|\nabla u|^2}{2}\, dx + \int_{D_2} \frac{|\nabla u|^p}{p}\, dx - \int_\Omega f\, u\, dx,$$

has a unique critical point in $\mathcal{W}(\Omega)$. First, observe that is straightforward that this functional is weakly lower semi continuous in $\mathcal{W}(\Omega)$. Moreover, there exists $0 < C = C(N, p, \|f\|_{L^2(\Omega)}, |\Omega|)$ such that

$$I(u) \geq C\left(\|\nabla u\|^2_{L^2(D_1)} - \|\nabla u\|_{L^2(D_1)} + \|\nabla u\|^p_{L^p(D_2)} - \|\nabla u\|_{L^p(D_2)}\right).$$

Thus, the functional is coercive (i.e., $I(u) \to \infty$ as $[u]_{\mathcal{W}(\Omega)} \to \infty$) and since $\mathcal{W}(\Omega)$ is a reflexive Banach space there exists $u^* \in \mathcal{W}(\Omega)$ such that

$$I(u^*) = \min\{I(u)\ :\ u \in \mathcal{W}(\Omega)\}.$$

The uniqueness is due to the strict convexity of I. Indeed, by using the inequality $|\xi|^r \geq |\xi_0|^r + r|\xi_0|^{r-2}\xi_0(\xi - \xi_0)$, for $\xi,\xi_0 \in \mathbb{R}^N$ and $r = 2, p$ (which is strict if $\xi \neq \xi_0$) it follows that $I(w) > I(v) + I'(v)(w - v)$ for $v \neq w \in \mathcal{W}(\Omega)$. $\square$

Proposition 10.3.2 *Let $u_1, u_2 \in \mathcal{W}(\Omega)$ be sub and supersolution respectively of (10.10). Then $u_1 \leq u_2$ a.e. in Ω.*

Proof: From the definition of sub and supersolution we get, for every test function $0 \leq \varphi \in \mathcal{C}_c^\infty(\Omega)$,

$$\int_{D_1} \nabla u_1 \nabla\varphi + \int_{D_2} |\nabla u_1|^{p-2}\nabla u_1 \nabla\varphi \leq \int_\Omega f\,\varphi\ , \tag{10.11}$$

$$\int_{D_1} \nabla u_2 \nabla\varphi + \int_{D_2} |\nabla u_2|^{p-2}\nabla u_2 \nabla\varphi \geq \int_\Omega f\,\varphi\ . \tag{10.12}$$

Note that since $\mathcal{W}(\Omega) \subset W_0^{1,2}(\Omega) = \overline{\mathcal{C}_c^\infty(\Omega)}^{W^{1,2}}$, by density we can choose test functions in $\mathcal{W}(\Omega)$. In this way, consider the test function

$$\varphi = (u_1 - u_2)^+ := \max\{u_1 - u_2, 0\}$$

in the above inequalities and subtract (10.12) from (10.11) to obtain

$$\int_{\{x \in D_1 : u_1 > u_2\}} |\nabla(u_1 - u_2)|^2 \\ + \int_{\{x \in D_2 : u_1 > u_2\}} \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)(\nabla u_1 - \nabla u_2) \le 0.$$

Finally, taking into account the well-known inequality

$$\left(|\xi|^{r-2}\xi - |\xi_0|^{r-2}\xi_0\right)(\xi - \xi_0) \ge c(r)|\xi - \xi_0|^r, \quad \xi, \xi_0 \in \mathbb{R}^N, \tag{10.13}$$

for $r = 2, p$, we conclude that $(u_1 - u_2)^+ \equiv 0$ finishing the proof. □

As a direct consequence, there exists $u \ge 0$ the unique weak solution of (10.10) for every $0 \le f \in L^2(\Omega)$. The next result shows that in fact the solution is strictly positive when f is nontrivial.

Proposition 10.3.3 *For every nontrivial $0 \le f \in L^2(\Omega)$, every supersolution of (10.10) is strictly positive in Ω.*

Proof: Let $u \ge 0$ in Ω be a supersolution (or a solution) to (10.10). There is no loss of generality in assuming that $f_{|D_2} \ne 0$ (the argument when $f_{|D_1} \ne 0$ is completely analogous). Consider $0 < v \in W_0^{1,p}(D_2)$ the solution to the problem

$$\begin{cases} -\Delta_p v = f, & \text{in } D_2, \\ v = 0, & \text{on } \partial D_2. \end{cases} \tag{10.14}$$

Since $u \ge 0$, it follows that $u \ge 0$ on Γ and hence u is a supersolution to (10.14). From the comparison principle we obtain that $u \ge v > 0$ in D_2. Furthermore, if $u(x_0) = 0$ for some $x_0 \in \Gamma$, by Hopf's lemma we have, in addition, that

$$\frac{\partial u(x_0)}{\partial \eta} = |\nabla u(x_0)|^{p-2}\frac{\partial u(x_0)}{\partial \eta} < 0$$

which means that x_0 is not a minimum of u and this contradicts the fact that $u(x_0) = 0$. Therefore, $u > 0$ on Γ. Finally, to show the that u is positive in the region D_1, consider $w \in W^{1,2}(D_1)$ the solution to the following problem

$$\begin{cases} -\Delta w = 0, & \text{in } D_1, \\ w = u, & \text{on } \partial D_1. \end{cases} \tag{10.15}$$

Since $u > 0$ on $\Gamma \subset \partial D_1$, the strong maximum principle applied in problem (10.15) shows that $w > 0$ in D_1. Taking into account that u is a supersolution to problem (10.15), we conclude from the comparison principle that $u \ge w > 0$ in D_1. □

Corollary 10.3.4 *Let $u \in \mathcal{W}(\Omega)$ be a nonnegative solution to problem (10.2). Then either $u(x) = 0$ a.e. x in Ω or $u(x) > 0$ a.e. $x \in \Omega$.*

The method of proof of Proposition 10.3.3 can be applied to solutions that are nonnegative and nontrivial on the boundary. To be more precisely, we state the following proposition whose proof is almost the same as the previous one and is therefore omitted.

Proposition 10.3.5 *Let $0 \le f \in L^2(\Omega)$ (maybe trivial) and u solution of (10.10) with boundary conditions $0 \lneqq u$ on $\partial\Omega$. Then $u > 0$ in Ω.*

Now, we are ready to prove one of the main goals of this section.

Proposition 10.3.6 *There exists a minimal bounded and positive solution of problem (10.2) for every $0 < \lambda \le \tilde{\lambda}$, being $\tilde{\lambda}$ sufficiently small.*

Proof: First, we find a supersolution of (10.2) for λ small. By Proposition 10.3.1, let $\overline{u} \in \mathcal{W}(\Omega)$ be the unique positive solution to the problem

$$\begin{cases} -\Delta w = 1, & & \text{in } D_1, \\ -\Delta_p w = 1, & & \text{in } D_2, \\ \dfrac{\partial w}{\partial \eta} = |\nabla w|^{p-2} \dfrac{\partial w}{\partial \eta}, & w|_{D_1} = w|_{D_2}, & \text{on} \Gamma\ , \\ w = 0, & & \text{on } \partial\Omega. \end{cases}$$

Classical regularity for p-laplacian operators states that there exist $C_1, C_2 > 0$ such that $\|\overline{u}\|_{L^\infty(D_1)} \le C_1$ and $\|\overline{u}\|_{L^\infty(D_2)} \le C_2$. Furthermore, setting $\tilde{\lambda} = \frac{1}{(C_1+C_1)^q}$, we get

$$\int_{D_1} \nabla \overline{u}\, \nabla \varphi + \int_{D_2} |\nabla \overline{u}|^{p-2}\, \nabla \overline{u}\, \nabla \varphi = \int_\Omega \varphi = \tilde{\lambda} \int_\Omega (C_1 + C_2)^q\, \varphi \ge \lambda \int_\Omega \overline{u}^q\, \varphi,$$

for all $\lambda \le \tilde{\lambda}$ and $0 \le \varphi \in \mathcal{C}_c^\infty(\Omega)$. Therefore, $\overline{u}$ is a supersolution of (10.2) for $\lambda \le \tilde{\lambda}$. Note that this argument shows the existence of a bounded supersolution only for λ small.

Next, to get a subsolution, take $v \in W_0^{1,p}(D_2)$ the positive solution to

$$\begin{cases} -\Delta_p v = \lambda v^q, & \text{in } D_2, \\ v = 0, & \text{on } \partial D_2. \end{cases} \tag{10.16}$$

Note that there is a unique v for every $\lambda > 0$ due to the fact that $q < p - 1$. Then we define

$$\underline{u}(x) = \begin{cases} v(x) & x \in D_2, \\ 0 & x \in \overline{D}_1. \end{cases} \tag{10.17}$$

Clearly, $\underline{u}$ belongs to $\mathcal{W}(\Omega)$. Moreover, due to Hopf's Lemma Sakaguchi (1987), we get that $|\nabla \underline{u}|^{p-2} \frac{\partial \underline{u}}{\partial \eta} < 0$ on Γ (recal that η is the normal unit vector to Γ pointing outwards D_1), then

$$\int_{D_2} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi = \lambda \int_{D_2} \underline{u}^q\, \varphi + \int_\Gamma |\nabla \underline{u}|^{p-2} \frac{\partial \underline{u}}{\partial \eta}\, \varphi \le \lambda \int_{D_2} \underline{u}^q\, \varphi,$$

for every $\lambda > 0$ and $0 \le \varphi \in \mathcal{C}_c^\infty(\Omega)$. Thus, $\underline{u}$ is the required subsolution of (10.2) without any restriction on $\lambda > 0$. We stress that, thanks to Hopf's Lemma, the above inequality is strict for tests functions that verify $\varphi > 0$ on Γ. Thus, $\underline{u}$ is not a solution.

Clearly, $0 = \underline{u}(x) \le \overline{u}(x)$ for $x \in \overline{D}_1$. In addition, since $\underline{u}$, $\overline{u}$ are a solution and a supersolution respectively of problem (10.16) for $\lambda \le \tilde{\lambda}$, it follows by the comparison principle for $p-$sublinear terms in p-laplacian operators that $\underline{u} \le \overline{u}$ a.e. in D_2. Finally, since $\underline{u} = \overline{u} = 0$ on $\partial\Omega$, we can state that

$$\underline{u} \le \overline{u}, \qquad \text{a.e. in } \overline{\Omega}.$$

To conclude, we use the standard monotone iteration argument in order to find a solution for our problem. For every $n \ge 1$ we define the recurrent sequence $\{w_n\}$ by

$$\begin{cases} -\Delta w_n = \lambda\, w_{n-1}^q, & & \text{in } D_1, \\ -\Delta_p w_n = \lambda\, w_{n-1}^q, & & \text{in } D_2, \\ \dfrac{\partial w_n}{\partial \eta} = |\nabla w_n|^{p-2} \dfrac{\partial w_n}{\partial \eta}, & w_n|_{D_1} = w_n|_{D_2}, & \text{on} \Gamma\ , \\ w_n = 0, & & \text{on } \partial\Omega, \end{cases} \tag{10.18}$$

where $w_0 = \underline{u}$. The sequence $\{w_n\}$ is well defined by Proposition 10.3.1. Moreover, the sequence is increasing. To check this property it suffices to prove that $w_0 \le w_1$ (and then proceed by induction). Indeed, taking into account that w_0 is a subsolution of problem (10.18) for $n = 1$, we obtain by comparison principle Proposition 10.3.2 that $w_0 \le w_1$. Hence, by an inductive argument: $w_0 \le w_1 \le \cdots \le w_n$, for all $n \ge 1$. By the fact that $\overline{u}$ is a supersolution of problem (10.18) for $n = 1$, with a similar argument we prove that $w_n \le \overline{u}$ for every $n \in \mathbb{N}$. Since $\overline{u} \in L^\infty(\Omega)$, the sequence $\{w_n(x)\}$ is increasing and bounded by $\overline{u}(x)$ for a.e. $x \in \Omega$. Let $w_\lambda(x)$ be the limit almost everywhere in Ω (i.e., $w_\lambda(x) := \lim_{n\to\infty} w_n(x)$ a.e. $x \in \Omega$) which is bounded since $\overline{u}$ is bounded. We claim that $w_\lambda \in \mathcal{W}(\Omega)$. Indeed, since $w_n \in \mathcal{W}(\Omega)$ we can take it as a test function in equation (10.18) to obtain

$$\int_{D_1} |\nabla w_n|^2 + \int_{D_2} |\nabla w_n|^p = \lambda \int_\Omega w_{n-1}^q w_n \le \lambda \int_\Omega \overline{u}^{\,q+1} \le \lambda \|\overline{u}\|_{L^\infty(\Omega)}^{q+1} |\Omega|.$$

That is, $\{w_n\}$ is uniformly bounded in the norm of $\mathcal{W}(\Omega)$ and since this space is reflexive, up to a subsequence, w_n converges weakly to $\tilde{w} \in \mathcal{W}(\Omega)$. Furthermore, $w_n(x) \to \tilde{w}(x)$ a.e. $x \in \Omega$. Finally, by the uniqueness of the limit $w_\lambda = \tilde{w} \in \mathcal{W}(\Omega)$ and we conclude the claim.

To finish the proof, we verify that w_λ is a weak solution of (10.2). To this end, fix $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and observe that from (10.18) we get

$$\int_{D_1} \nabla w_n \nabla \varphi + \int_{D_2} |\nabla w_n|^{p-2} \nabla w_n \nabla \varphi = \lambda \int_\Omega w_{n-1}^q\, \varphi.$$

Now, let $n \to \infty$ to obtain

$$\int_{D_1} \nabla w_\lambda \nabla \varphi + \int_{D_2} |\nabla w_\lambda|^{p-2} \nabla w_\lambda \nabla \varphi = \lambda \int_\Omega w_\lambda^q\, \varphi,$$

as desired. We note that w_λ is positive by Corollary 10.3.4 and minimal by construction. In fact, let $\tilde{w}_\lambda$ be another solution of problem (10.2), by a similar argument using the comparison principle and induction in n we obtain $w_n \le \tilde{w}_\lambda$ for all $n \in \mathbb{N}$, thus $w_\lambda(x) = \lim_{n\to\infty} w_n(x) \le \tilde{w}_\lambda(x)$ a.e. $x \in \Omega$. □

Now we are ready to proceed with the proof of Theorem 10.1.1.

Proof:[Proof of Theorem 10.1.1] First, we observe that if there exists $\hat{u} \in \mathcal{W}(\Omega)$, a solution to problem (10.2) for some $\hat{\lambda} > 0$, then there exists w_λ a minimal solution for every $\lambda \in (0, \hat{\lambda})$. Indeed, for a fixed $0 < \lambda < \hat{\lambda}$, we take $\hat{u}$ as a supersolution and $\underline{u}$ from (10.17) as a subsolution of problem (10.2). Recall that we have showed existence of this subsolution for any value of $\lambda > 0$. Arguing as in the proof of Proposition 10.3.6, it holds that the sequence $\underline{u} < w_1 \le w_2 \le \cdots \le w_n \le \cdots \le \hat{u}$ is uniformly bounded in $\mathcal{W}(\Omega)$ and, by our previous argument, there exists w_λ, the minimal solution. In this way we set

$$\lambda^* = \sup\{0 \le \lambda : \text{exists a solution to problem (10.2)}\}.$$

By Propositon 10.3.6 it follows that $\lambda^* > 0$. Thus, for every $0 < \lambda < \lambda^*$ there exists w_λ a minimal positive solution.

Next, in order to prove that $\lambda^* < \infty$, we take again $v \in W_0^{1,p}(D_2)$ the unique positive solution to (10.16) and let us observe that

$$v(x) = \lambda^\gamma\, v_1(x), \qquad \text{in } D_2,$$

with $\gamma = \frac{1}{p-1-q} > 0$ and v_1 the unique solution to

$$\begin{cases} -\Delta_p v_1 = (v_1)^q, & \text{in } D_2, \\ v_1 = 0, & \text{on } \partial D_2. \end{cases}$$

Now, fix a ball $B \subset\subset D_2$. Since $v_1 \ge c > 0$ in B, it holds that

$$v(x) \ge c\lambda^\gamma, \qquad x \in B.$$

That is, v is uniformly large in B for λ large.

Now, let us consider z the solution to

$$\begin{cases} -\Delta z = 0, & \text{in } D_1, \\ -\Delta_p z = 0, & \text{in } D_2, \setminus B \\ \dfrac{\partial z}{\partial \eta} = |\nabla z|^{p-2}\dfrac{\partial z}{\partial \eta}, \qquad z|_{D_1} = z|_{D_2}, & \text{on} \Gamma \ , \\ z = 0, & \text{on } \partial\Omega, \\ z = c\lambda^{\gamma}, & \text{on } \partial B. \end{cases} \tag{10.19}$$

Such solution can be obtained as the minimum from the following coercive functional

$$H(u) = \int_{D_1} \frac{|\nabla u|^2}{2}\, dx + \int_{D_2 \setminus B} \frac{|\nabla u|^p}{p}\, dx$$

in the set $\mathcal{A} = \{u \in \tilde{\mathcal{W}}(\Omega \setminus B) \,:\, u_{|\partial B} \equiv c\lambda^{\gamma}\}$ being $\tilde{\mathcal{W}}(\Omega \setminus B)$ the Banach space defined as

$$\tilde{\mathcal{W}}(\Omega \setminus B) = \left\{u \in W^{1,2}(\Omega \setminus B) \cap W^{1,p}(D_2 \setminus B) \,:\, u_{|\partial\Omega} \equiv 0\right\}.$$

We note that such minimum is attained because $\mathcal{A}$ is a nonempty convex and weakly close subset of $\tilde{\mathcal{W}}(\Omega \setminus B)$.

Now fix a different ball $B_2 \subset\subset D_1$. We claim that z is uniformly large in B_2 when λ is large. Indeed, z should be large on Γ and therefore large in B_2.

In order to prove the nonexistence of solutions to (10.2) for λ large. Assume, arguing by contradiction, that there is a solution u for λ large. By a comparison argument, we have that

$$u \geq v, \qquad \text{in } D_2.$$

Hence u is a supersolution of problem (10.19) in $\tilde{\mathcal{W}}(\Omega \setminus B)$ and due to Proposition 10.3.5 in the space $\tilde{\mathcal{W}}(\Omega \setminus B)$, it holds by comparison principle

$$u \geq z \qquad \text{in } B_2.$$

This gives a contradiction, since the solution to the parabolic problem

$$\begin{cases} w_t - \Delta w = \lambda w^q, & \text{in } B_2 \times (0,T), \\ w = 0, & \text{on } \partial B_2 \times (0,T), \\ w_0 = z, & \text{in } B_2, \end{cases} \tag{10.20}$$

blows up in finite time (due to the fact that z is uniformly large in the ball B_2, see for instance Ball (1977)) and also must satisfy

$$w(x,t) \leq u(x),$$

since u is a supersolution to the parabolic problem (10.20).

Finally, we note that if $\lambda_1 \leq \lambda_2 < \lambda^*$, taking w_{λ_2} as a supersolution of problem (10.2) for $\lambda = \lambda_1$ and arguing as the proof of Proposition 10.3.6 we obtain $w_{\lambda_1} \leq w_{\lambda_2}$. That is, the family of functions $\{w_\lambda\}_{0<\lambda<\lambda^*}$ is increasing with λ. □

10.4 Multiplicity of solutions

In this section we show that problem (10.2) has at least two positive different solutions provided $p < 2^*$ if $N \geq 3$ (with no restriction on p for $N = 1, 2$) and $D_2 \subset\subset \Omega$. Concretely, we prove that (10.2) has a first solution which corresponds to the global minimum of an appropriated functional and then a second solution is found by means of Mountain Pass theory.

Since our objective is to find positive solutions of our problem, we observe that they correspond to critical points of the following functional

$$G_\lambda(u) = \int_{D_1} \frac{|\nabla u|^2}{2}\, dx + \int_{D_2} \frac{|\nabla u|^p}{p}\, dx - \lambda \int_\Omega \frac{u_+^{q+1}}{q+1}\, dx,$$

where $u_+ = \max\{u, 0\}$. We will write it simply G instead G_λ when no confusion can arise. Of course, $F(u) = G(u)$ whenever $u \geq 0$ and then, positive critical points of G correspond to positive solutions of (10.2).

In general, for a $p(x)$ discontinuous, the $\mathcal{C}^1(\Omega)$-regularity of minimizers of G are not satisfied, in fact, one can find some counter-examples in Zhikov (1997). However, as it mentioned in (Harjulehto et al., 2010, Theorem 9.15) which refers to Fan and Zhao (2006), for our class of discontinuous exponents one can arrive at locally Hölder continuity (see also Acerbi and Fusco (1994)). Therefore, due to lack of $\mathcal{C}^1$-results in whole Ω, we impose that $D_2 \subset\subset \Omega$ in order to get regularity close to $\partial\Omega$. Concretely, as we will see later, we need that local minimizers of functional G belongs to $\mathcal{C}^1(F_\delta) \cap \mathcal{C}(\overline{\Omega})$ where F_δ is a small strip around the boundary,

$$F_\delta = \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\,\Omega) < \delta\} \tag{10.21}$$

being δ enough small to ensure that $F_{3\delta} \subset D_1$ and ∂F_δ is smooth.

Following partially the ideas in Ambrosetti et al. (1994), we begin by showing the next result.

Lemma 10.4.1 *For every $\lambda \in (0, \lambda^*)$ there exists a local minimum of G in the $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(F_\delta)$-topology.*

Proof: Fixed $0 < \lambda < \lambda^*$, we take $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 < \lambda < \lambda_2 < \lambda^*$ and let us denote by u_1 and u_2 their respective minimal solutions for λ_1 and λ_2 obtained in Theorem 10.1.1. Since the minimal solutions are increasing, we have $u_1 \leq u_2$. Even more, since $\lambda_1 < \lambda_2$ it follows by the Strong Maximum Principle applied in each region D_i, $i = 1, 2$ (see for instance Damascelli (1998); Guedda and Véron (1989)) and the Hopf Maximum Principle that

$$u_1 < u_2, \qquad \text{in}\Omega\ ,$$

$$\frac{\partial u_2}{\partial \nu} < \frac{\partial u_1}{\partial \nu} < 0, \quad \text{on } \partial\Omega,$$

being ν the outer unit normal on $\partial\Omega$.

Consider,

$$h(x, s) = \begin{cases} u_2^q(x)\,, & s \geq u_2(x), \\ s^q\,, & u_1(x) < s < u_2(x), \\ u_1^q(x)\,, & s \leq u_1(x), \end{cases}$$

and the truncated functional

$$\tilde{G}(u) = \int_{D_1} \frac{|\nabla u(x)|^2}{2} + \int_{D_2} \frac{|\nabla u(x)|^p}{p} - \lambda \int_\Omega H(x, u)$$

where $u \in \mathcal{W}(\Omega)$ and $H(x, u) = \int_0^u h(x, s) ds$. Clearly, $\tilde{G}$ is coercive and weakly lower semicontinuous (because $q < \frac{N+2}{N-2}$) . Hence, there exists its global minimum at some $\tilde{u} \in \mathcal{W}(\Omega)$ and for every $0 \leq \varphi \in \mathcal{C}_c^\infty(\Omega)$ it holds

$$\begin{aligned}\int_{D_1} \nabla\tilde{u}(x)\nabla\varphi(x) + \int_{D_2} |\nabla\tilde{u}(x)|^{p-2}\nabla\tilde{u}(x)\nabla\varphi(x) &= \lambda \int_\Omega h(x, \tilde{u})\varphi(x) \\ &> \lambda_1 \int_\Omega u_1^q(x)\varphi(x).\end{aligned}$$

That is, $\tilde{u}$ is a supersolution of (10.10) with $f = \lambda_1 u_1^q$ and since u_1 is a solution it follows by the comparison principle from Proposition 10.3.2 that $u_1 \leq \tilde{u}$. We proceed analogously to obtain that

$\tilde{u} \le u_2$. Moreover, using again the Strong Maximum Principle and the Hopf Maximum Principle we obtain that

$$0 < u_1 < \tilde{u} < u_2, \quad \text{in}\Omega\ , \tag{10.22}$$

and

$$\frac{\partial u_2}{\partial \nu} < \frac{\partial \tilde{u}}{\partial \nu} < \frac{\partial u_1}{\partial \nu} < 0, \quad \text{on } \partial\Omega. \tag{10.23}$$

Next, we claim that $\tilde{u} \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(F_\delta)$. Indeed, let $K = \Omega \setminus F_{\delta/2}$ be a compact set. Since $\tilde{u}$ is a local minimizer and u_1, u_2 are bounded then $\tilde{G}$ is in the framework of the work Fan and Zhao (2006). It follows a higher integrability of the gradient of $\tilde{u}$ which implies locally Hölder continuity, hence $\tilde{u} \in \mathcal{C}^\alpha(K)$. Moreover, $\tilde{u}$ satisfies the equation

$$\begin{cases} -\Delta \tilde{u} = \lambda \tilde{u}^q, & \text{in } F_\delta, \\ \tilde{u} = 0, & \text{on } \partial\Omega, \end{cases}$$

and $\tilde{u}$ is continuous on $\partial F_\delta \cap \Omega$. Then, the well-known classical regularity for the laplacian operator (see Gilbarg and Trudinger (1983)) implies that $\tilde{u} \in \mathcal{C}^1(F_\delta) \cap \mathcal{C}(\overline{F}_\delta)$ and the claim is proved.

Finally, in virtue of inequalities (10.22) and (10.23), there exists $\varepsilon > 0$ sufficiently small such that $u_1 < v < u_2$ in Ω for all $v \in B_\varepsilon(\tilde{u})$ the ball of center $\tilde{u}$ and radius ε in the topology of $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(F_\delta)$. Therefore,

$$G(v) = \tilde{G}(v) \ge \tilde{G}(\tilde{u}) = G(\tilde{u}), \qquad \text{for all } v \in B_\varepsilon(\tilde{u}).$$

Equivalently, $\tilde{u}$ is a local minimum of G in $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(F_\delta)$-topology. □

Remark 10.4.2 *Concerning the regularity of local minimizers of functional $\tilde{G}$ in the proof of above lemma, the same reasoning applied to the functional G states that local minimizers of G also belong to $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(F_\delta)$.*

Our first goal is to show that there exists a local minimum of G in $\mathcal{W}(\Omega)$. In fact, we will prove that $\tilde{u}$, the local minimum in $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(F_\delta)$-topology of the proof of Lemma 10.4.1, is the desired local minimizer. To prove it, we argue by contradiction following closely the ideas of (De Figueiredo, 1987, Lemma 1) (see also Brézis and Nirenberg (1993)). Thus, we suppose that there exists $\varepsilon_0 > 0$ such that

$$G(v_\varepsilon) := \min\{G(u) \,:\, u \in V_\varepsilon(\tilde{u})\} < G(\tilde{u}), \quad \text{for all } \varepsilon < \varepsilon_0, \tag{10.24}$$

where $V_\varepsilon(\tilde{u})$ is the closed set

$$V_\varepsilon(\tilde{u}) = \left\{ u \in \mathcal{W}(\Omega) \,:\, \int_{D_1} \frac{|\nabla(u-\tilde{u})|^2}{2} + \int_{D_2} \frac{|\nabla(u-\tilde{u})|^p}{p} \le \varepsilon \right\}.$$

Note that such minimum is attained as G is weakly lower semicontinuous and $V_\varepsilon(\tilde{u})$ is weakly compact in the reflexive space $\mathcal{W}(\Omega)$. Moreover, $v_\varepsilon \to \tilde{u}$ as $\varepsilon \to 0$ in norm in $\mathcal{W}(\Omega)$.

The strategy is to prove that $v_\varepsilon \to \tilde{u}$ in $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(F_\delta)$-topology contradicting the fact that $\tilde{u}$ is a local minimum in $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(F_\delta)$-topology by the above lemma.

For that purpose, we note that the corresponding Euler equation for v_ε contains a nonpositive Lagrange multiplier $\mu_\varepsilon \le 0$. Namely, v_ε must be satisfy the following:

$$\begin{aligned} &\int_{D_1} \nabla u \nabla \varphi + \int_{D_2} |\nabla u|^{p-2} \nabla u \nabla \varphi - \int_\Omega g(u)\varphi \\ &\qquad = \mu_\varepsilon \left[\int_{D_1} \nabla(u-\tilde{u}) \nabla \varphi + \int_{D_2} |\nabla(u-\tilde{u})|^{p-2} \nabla(u-\tilde{u}) \nabla \varphi \right], \end{aligned} \tag{10.25}$$

for all $\varphi \in \mathcal{W}(\Omega)$, being $g(u) = \lambda u_+^q$.

Our first step is to prove that v_ε are uniformly L^∞-bounded by a constant independent of ε.

Lemma 10.4.3 *Given* $0 \leq \varepsilon < \varepsilon_0 < 1$*, there exists* $M > 0$ *such that* v_ε *defined by* (10.24) *satisfies*

$$\|v_\varepsilon\|_{L^\infty(\Omega)} \leq M,$$

for all $\varepsilon \in [0, \varepsilon_0)$.

Proof: We adapt the techniques applied in García Azorero et al. (2000) by using the classical lemma due to Stampacchia Stampacchia (1966). First, since

$$\int_{D_1} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \nabla \phi + \int_{D_2} \nabla \tilde{u} \phi = \lambda \int_\Omega \tilde{u}^q \phi, \quad \forall\, \phi \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(F_\delta),$$

and a density argument, the above equality holds for test functions belonging to $\mathcal{W}(\Omega)$. Hence, we write equation (10.25), which satisfies v_ε, as follows

$$\int_{D_1} \nabla(u - \tilde{u}) \nabla \varphi + \int_{D_2} (|\nabla u|^{p-2} \nabla u - |\nabla \tilde{u}|^{p-2} \nabla \tilde{u}) \nabla \varphi - \int_\Omega (g(u) - g(\tilde{u}))\varphi$$
$$= \mu_\varepsilon \left[\int_{D_1} \nabla(u - \tilde{u}) \nabla \varphi + \int_{D_2} |\nabla(u - \tilde{u})|^{p-2} \nabla(u - \tilde{u}) \nabla \varphi \right],$$

for all $\varphi \in \mathcal{W}(\Omega)$. We consider now for every $k \in \mathbb{R}^+$ the function $T_k : \mathbb{R} \to \mathbb{R}$ given by

$$T_k(s) = \begin{cases} s + k, & s \leq -k, \\ 0, & -k < s \leq k, \\ s - k, & s > k. \end{cases}$$

Thus, taking

$$\varphi = T_k(u - \tilde{u})$$

as test function in the previous equation we get

$$\int_{D_1 \cap \Omega_k} \nabla(u - \tilde{u}) \nabla T_k(u - \tilde{u}) + \int_{D_2 \cap \Omega_k} (|\nabla u|^{p-2} \nabla u - |\nabla \tilde{u}|^{p-2} \nabla \tilde{u}) \nabla T_k(u - \tilde{u})$$
$$= \int_\Omega (g(u) - g(\tilde{u})) T_k(u - \tilde{u}) + \mu_\varepsilon \left[\int_{D_1 \cap \Omega_k} |\nabla(u - \tilde{u})|^2 + \int_{D_2 \cap \Omega_k} |\nabla(u - \tilde{u})|^p \right],$$

where$\Omega_k \equiv \{x \in \Omega \, : \, |u(x) - \tilde{u}(x)| > k\}$.

Hence, dropping the negative term

$$\mu_\varepsilon \left[\int_{D_1} |\nabla(u - \tilde{u})|^2 + \int_{D_2} |\nabla(u - \tilde{u})|^p \right]$$

and using the inequality (10.13), we arrive to

$$\begin{aligned} &\int_{D_1 \cap \Omega_k} |\nabla T_k(u - \tilde{u})|^2 + c(p) \int_{D_2 \cap \Omega_k} |\nabla T_k(u - \tilde{u})|^p \\ &\qquad \leq \int_\Omega (g(u) - g(\tilde{u})) T_k(u - \tilde{u}). \end{aligned} \tag{10.26}$$

We can also assume that $\|u - \tilde{u}\|_{L^r(\Omega)} \leq R$ independent of ε. Note that due $u \in V_\varepsilon(\tilde{u})$ then r is at least equal to 2^*. Therefore, since $|T_k(s)| \leq |s|$ and applying Hölder inequality for this $r \geq 2^*$, the right hand side can be estimated as follows

$$\begin{aligned} \int_\Omega (g(u) - g(\tilde{u})) T_k(u - \tilde{u}) &\leq \int_{\Omega_k} |g(u) - g(\tilde{u})| |T_k(u - \tilde{u})| \\ &\leq \lambda \int_{\Omega_k} (|u|^q + |\tilde{u}|^q) |T_k(u - \tilde{u})| \\ &\leq \lambda \left(\int_{\Omega_k} (|u|^q + |\tilde{u}|^q)^{\frac{r}{q}} \right)^{\frac{q}{r}} \left(\int_{\Omega_k} |T_k(u - \tilde{u})|^{2^*} \right)^{\frac{1}{2^*}} |\Omega_k|^{1 - \frac{q}{r} - \frac{1}{2^*}} \\ &\leq C_1 \left(\int_{\Omega_k} |T_k(u - \tilde{u})|^{2^*} \right)^{\frac{1}{2^*}} |\Omega_k|^{1 - \frac{q}{r} - \frac{1}{2^*}}, \end{aligned} \tag{10.27}$$

for some positive constant $C_1(\lambda, q, N, R, \|\tilde{u}\|_{L^r(\Omega)})$. For the reader's convenience, we will explain the last inequality in more detail, we have

$$\begin{aligned}
\lambda\left(\int_{\Omega_k}(|u|^q+|\tilde{u}|^q)^{\frac{r}{q}}\right)^{\frac{q}{r}} &\leq c_1(\lambda)\left(\int_{\Omega}|u|^r+\int_{\Omega}|\tilde{u}|^r\right)^{\frac{q}{r}}\\
&\leq c_2(\lambda, q, N, \|u\|_{L^r(\Omega)}, \|\tilde{u}\|_{L^r(\Omega)})\\
&\leq c_3(\lambda, q, N, R, \|\tilde{u}\|_{L^r(\Omega)}).
\end{aligned}$$

Replacing inequality (10.27) in (10.26) we have that

$$\begin{aligned}
\int_{D_1\cap\Omega_k}&|\nabla T_k(u-\tilde{u})|^2+c(p)\int_{D_2\cap\Omega_k}|\nabla T_k(u-\tilde{u})|^p\\
&\leq C_1\left(\int_{\Omega_k}|T_k(u-\tilde{u})|^{2^*}\right)^{\frac{1}{2^*}}|\Omega_k|^{1-\frac{q}{r}-\frac{1}{2^*}}\,.
\end{aligned}\tag{10.28}$$

Concerning to the left hand side, we use the inequality

$$a+b^c\geq 2^{-c}(a+b)^c, \qquad 0\leq a,b\leq 1\leq c,$$

to obtain

$$\begin{aligned}
\int_{D_1\cap\Omega_k}&|\nabla T_k(u-\tilde{u})|^2+c(p)\int_{D_2\cap\Omega_k}|\nabla T_k(u-\tilde{u})|^p\\
&\geq C_2\left(\int_{D_1\cap\Omega_k}|\nabla T_k(u-\tilde{u})|^2+\left(\int_{D_2\cap\Omega_k}|\nabla T_k(u-\tilde{u})|^2\right)^{\frac{p}{2}}\right)\\
&\geq C_3\left(\int_{\Omega_k}|\nabla T_k(u-\tilde{u})|^2\right)^{\frac{p}{2}}\\
&\geq C_4\left(\int_{\Omega_k}|T_k(u-\tilde{u})|^{2^*}\right)^{\frac{p}{2^*}}.
\end{aligned}\tag{10.29}$$

Going back to (10.28), we get

$$\left(\int_{\Omega_k}|T_k(u-\tilde{u})|^{2^*}\right)^{\frac{p-1}{2^*}}\leq C_5\,|\Omega_k|^{1-\frac{q}{r}-\frac{1}{2^*}}\,.\tag{10.30}$$

On the other hand, it is easy to check that $h-k\leq|T_k(s)|$, for $s\geq h\geq k$. Therefore, $h-k\leq|T_k(u-\tilde{u})|$, for $x\in\Omega_h$ and $h\geq k$. Hence, we obtain the inequality

$$|\Omega_h|(h-k)^{2^*}\leq\int_{\Omega_h}|T_k(u-\tilde{u})|^{2^*}\leq\int_{\Omega_k}|T_k(u-\tilde{u})|^{2^*}\tag{10.31}$$

and combining with (10.30) we have that

$$|\Omega_h|\leq\frac{C_6}{(h-k)^{2^*}}|\Omega_k|^{\beta}, \qquad \text{for } h>k.$$

being $\beta=\left(1-\frac{q}{r}-\frac{1}{2^*}\right)\frac{2^*}{p-1}$. Therefore we can apply Stampacchia Lemma Stampacchia (1966), to deduce that

(i) if $u-\tilde{u}\in L^r(\Omega)$ with $r>\dfrac{2^*q}{2^*-p}$, then $u-\tilde{u}\in L^\infty(\Omega)$ and

$$\|u-\tilde{u}\|_{L^\infty(\Omega)}\leq c\,C_6^{1/2^*},$$

for some specific $c>0$,

(ii) if $u-\tilde{u}\in L^r(\Omega)$ with $r=\dfrac{2^*q}{2^*-p}$, then $u-\tilde{u}\in L^s(\Omega)$ for $s\in[1,\infty)$,

(iii) if $u-\tilde{u}\in L^r(\Omega)$ with $r<\dfrac{2^*q}{2^*-p}$, then $u-\tilde{u}\in L^s(\Omega)$ for $s=\dfrac{2^*}{1-\beta}-\rho$ and $\rho>0$ arbitrary small.

Since $u\in L^{2^*}(\Omega)$ we can argue as above for $r=2^*$. Thus, if $2^*>\frac{2^*q}{2^*-p}$ we conclude by item (i) that $u-\tilde{u}\in L^\infty(\Omega)$ and, in virtue of the regularity of $\tilde{u}$, we get that $\|u\|_{L^\infty(\Omega)}\le M$. In the case $2^*=\frac{2^*q}{2^*-p}$ we use item (ii) to choose $s>\frac{2^*q}{2^*-p}$ and after repeating the argument we lie under the conditions of item (i) and conclude again the desired bound. Finally, in the case $2^*<\frac{2^*q}{2^*-p}$, by using item (iii) we can take

$$r_1=\frac{2^*(p-1)}{p-2^*+q}-\rho_1>2^*.$$

As before, if $r_1\ge\frac{2^*q}{2^*-p}$ we conclude easily. In other cases we take

$$r_2=\frac{2^*(p-1)r_1}{(p-2^*)r_1+2^*q}-\rho_2.$$

We claim that arguing by iteration, there exists $k_0\in\mathbb{N}$ such that $r_k>\frac{2^*q}{2^*-p}$ for $k\ge k_0$, i.e, we can conclude after a finite number of steps. Indeed, in other cases, we have that the sequence $\{r_k\}$ is bounded and it satisfies the recurrence

$$\begin{cases} r_{k+1}=\dfrac{2^*(p-1)r_k}{(p-2^*)r_k+2^*q}-\rho_{k+1}, \\ r_0=2^*. \end{cases} \tag{10.32}$$

Where $\rho_{k+1}\to 0$. Moreover, it is easy to check that the sequence is increasing and therefore it is convergent and the limit r_∞ satisfies

$$r_\infty=\frac{2^*(p-1)r_\infty}{(p-2^*)r_\infty+2^*q},$$

namely,

$$r_\infty=\frac{2^*(p-1-q)}{p-2^*}<0,$$

which is a contradiction, proving the claim. Note that here we use the condition $p<2^*$. □

Remark 10.4.4 *Note that the hypothesis $p<2^*$ is necessary in order to apply Stampacchia's idea in the proof of the previous lemma.*

Proposition 10.4.5 *Let v_ε defined in (10.24). Then $v_\varepsilon\to\tilde{u}$ in $\mathcal{C}(\overline{\Omega})\cap\mathcal{C}^1(F_\delta)$-topology for $\delta>0$ sufficiently small.*

Proof: Due to the construction of F_δ in (10.21), we have that v_ε satisfies

$$\begin{cases} -(1-\mu_\varepsilon)\Delta v_\varepsilon=\lambda v_\varepsilon^q, & \text{in } F_{2\delta}, \\ v_\varepsilon=0, & \text{on } \partial\Omega. \end{cases}$$

Moreover, by using Lemma 10.4.3 it follows that v_ε is bounded on $\partial F_{2\delta}\cap\Omega$. Then by interior regularity, one may bootstrap the bound $\|v_\varepsilon\|_{W^{1,2}(F_\delta)}\le M$ to arrive to $\|v_\varepsilon\|_{\mathcal{C}^{1,\alpha}(F_\delta)}\le M$ independent of ε. Thus, since $v_\varepsilon\to\tilde{u}$ in $\mathcal{W}(\Omega)$ it follows by Arzelà-Ascoli that $v_\varepsilon\to\tilde{u}$ in $\mathcal{C}^1(F_\delta)$. This concludes the first part of the proof.

In order to prove that $v_\varepsilon \to \tilde{u}$ uniformly in $\mathcal{C}(\overline{\Omega})$ we adapt part of the method of Stampacchia used in the proof of Lemma 10.4.3 to get an estimate. Concretely, let $\kappa \in \mathbb{N}$ such that r_κ, the κ-term of the sequence (10.32), satisfies $r_\kappa > \frac{2^* q}{2^*-p}$. We adapted (10.27) replacing by r_κ in the following form

$$\begin{aligned}\int_\Omega (g(v_\varepsilon) - g(\tilde{u}))T_k(v_\varepsilon - \tilde{u}) &\le \lambda \int_{\Omega_k} (|v_\varepsilon|^q + |\tilde{u}|^q)|T_k(v_\varepsilon - \tilde{u})| \\ &\le \lambda \left(\int_{\Omega_k} (|v_\varepsilon|^q + |\tilde{u}|^q)^{\frac{r_\kappa}{q}} \right)^{\frac{q}{r_\kappa}} \left(\int_{\Omega_k} |T_k(v_\varepsilon - \tilde{u})|^{2^*} \right)^{\frac{1}{2^*}} |\Omega_k|^{1-\frac{q}{r_\kappa}-\frac{1}{2^*}} \\ &\le C \left(\int_{\Omega_k} |T_k(v_\varepsilon - \tilde{u})|^{2^*} \right)^{\frac{1}{2^*}} |\Omega_k|^{1-\frac{q}{r_\kappa}-\frac{1}{2^*}},\end{aligned}$$

here $C = C(\lambda, q, \kappa, N, \|\tilde{u}\|_{L^{r_\kappa}(\Omega)})$. Let us consider $0 < \tau < 1/2^*$ sufficiently small, that we will specify later, and we write the last expression as follows

$$\begin{aligned}&\int_\Omega (g(v_\varepsilon) - g(\tilde{u}))T_k(v_\varepsilon - \tilde{u}) \\ &\quad\le C \left(\int_{\Omega_k} |T_k(v_\varepsilon - \tilde{u})|^{2^*} \right)^{\tau} \left(\int_{\Omega_k} |T_k(v_\varepsilon - \tilde{u})|^{2^*} \right)^{\frac{1}{2^*}-\tau} |\Omega_k|^{1-\frac{q}{r_\kappa}-\frac{1}{2^*}} \\ &\quad\le C \left(\int_{\Omega} |v_\varepsilon - \tilde{u}|^{2^*} \right)^{\tau} \left(\int_{\Omega_k} |T_k(v_\varepsilon - \tilde{u})|^{2^*} \right)^{\frac{1}{2^*}-\tau} |\Omega_k|^{1-\frac{q}{r_\kappa}-\frac{1}{2^*}}.\end{aligned}$$

Therefore, using this inequality in (10.26) and having in mind (10.29), it holds that

$$\left(\int_{\Omega_k} |T_k(v_\varepsilon - \tilde{u}|^{2^*} \right)^{\frac{p-1}{2^*}+\tau} \le C\,\theta(\varepsilon)\; |\Omega_k|^{1-\frac{q}{r_\kappa}-\frac{1}{2^*}},$$

here $\theta(\varepsilon) = \left(\int_\Omega |v_\varepsilon - \tilde{u}|^{2^*} \right)^{\tau}$ (note that $\theta(\varepsilon) \to 0$ since $v_\varepsilon \to \tilde{u}$ in $\mathcal{W}(\Omega)$). Thus, by using inequality (10.31), we get

$$|\Omega_h| \le \frac{\tilde{C}\hat{\theta}(\varepsilon)}{(h-k)^{2^*}} |\Omega_k|^{\hat{\beta}}, \qquad h > k.$$

Where $\hat{\theta}(\varepsilon) = \theta(\varepsilon)^{\frac{2^*}{p-1+\tau 2^*}}$ and

$$\hat{\beta} = \frac{1 - \frac{q}{r_\kappa} - \frac{1}{2^*}}{\frac{p-1}{2^*} + \tau}.$$

Then, choosing τ such that $\hat{\beta} > 1$ (note that it is possible due to the choice of r_κ) it is straightforward by item (i) from Stampacchia Lemma that

$$\|v_\varepsilon - \tilde{u}\|_{L^\infty(\Omega)} \le c\,\hat{\theta}(\varepsilon)^{\frac{1}{2^*}} \to 0, \qquad \text{as } \varepsilon \to 0,$$

which completes the proof. □

Summarizing, we have proved the following result:

Theorem 10.4.6 *For every $\lambda \in (0, \lambda^*)$, there exists, $\tilde{u}_\lambda$, a positive local minimum of G_λ in $\mathcal{W}(\Omega)$.*

The last goal is to obtain a second positive solution of problem (10.2). Taking into account (10.9), one may expect that G_λ possesses a mountain-pass geometry and, by using results by Ghoussoub-Preiss (Ghoussoub and Preiss (1989)) and Jeanjean (Jeanjean (1999)) in the spirit of the celebrated Mountain Pass theorem due to Ambrosetti and Rabinowitz (Ambrosetti and Rabinowitz (1973)), to find a critical point different from the minimum. To make sure that this critical point is nontrivial we consider, for every fixed $\lambda \in (0, \lambda^*)$, the truncated functional $\widehat{G}_\lambda : \mathcal{W}(\Omega) \to \mathbb{R}$ as follows:

$$\widehat{G}_\lambda(u) = \int_{D_1} \frac{|\nabla u(x)|^2}{2} + \int_{D_2} \frac{|\nabla u(x)|^p}{p} - \lambda \int_\Omega \widehat{H}(x, u), \tag{10.33}$$

as usual $\widehat{H}(x,s) = \int_0^s \widehat{h}(x,t)dt$, being in this case

$$\widehat{h}(x,t) = \begin{cases} t^q\,, & t > u_1(x), \\ u_1^q(x)\,, & t \le u_1(x), \end{cases}$$

and by $0 < u_1$ we denote the minimal solution for a fixed $\lambda_1 \in (0,\lambda)$ which is obtained in Theorem 10.1.1. We point out that, $\widehat{u}_\lambda$, critical point of $\widehat{G}_\lambda$ corresponds to a supersolution of problem (10.10) with $f = \lambda_1 u_1^q$. Hence, by Proposition 10.3.2, it follows that $\widehat{u}_\lambda \ge u_1$. Moreover, if $\lambda > \lambda_1$ we obtain $\widehat{u}_\lambda > u_1$ and then it is also a critical point of G_λ.

In order to use the Mountain Pass theorem, as usual, a preliminary step is to show the existence of a bounded Palais-Smale sequence at the mountain pass level and then prove that it posses a convergent subsequence. We recall that a Palais-Smale sequence for the functional $\widehat{G}_\lambda$ at level $c(\lambda) \in \mathbb{R}$ is a sequence $\{u_n\} \subset \mathcal{W}(\Omega)$ verifying $\lim_n \widehat{G}_\lambda(u_n) = c(\lambda)$ and $\lim_n \widehat{G}'_\lambda(u_n) = 0$ in $\mathcal{W}(\Omega)'$. We start by showing that bounded Palais-Smale sequences have a subsequence converging strongly in $\mathcal{W}(\Omega)$. Note that we have to assume that the sequence is bounded, since it is not clear how to obtain boundedness in $\mathcal{W}(\Omega)$ using only that $\lim_n \widehat{G}_\lambda(u_n) = c(\lambda)$ and $\lim_n \widehat{G}'_\lambda(u_n) = 0$. This difficulty (showing that Palais-Smale sequences are bounded) forces us to use Jeanjean's ideas (Jeanjean (1999)) and hence obtain existence of a second solution for almost every $\lambda \in (0,\lambda^*)$.

Lemma 10.4.7 *Let $\{u_n\} \subset \mathcal{W}(\Omega)$ be a sequence satisfying*

(i) $\{u_n\}$ bounded in $\mathcal{W}(\Omega)$,

(ii) $\widehat{G}_\lambda(u_n)$ bounded,

(iii) $\widehat{G}'_\lambda(u_n) \to 0$ in $\mathcal{W}'(\Omega)$.

Then, $\{u_n\}$ has a convergent subsequence in $\mathcal{W}(\Omega)$.

Proof: From *(i)* there exists a subsequence $\{u_{n_k}\}$ and $u \in \mathcal{W}(\Omega)$, such that $u_{n_k} \rightharpoonup u$ in $\mathcal{W}(\Omega)$ and, by the embedding $\mathcal{W}(\Omega) \subset W_0^{1,2}(\Omega) \subset L^r(\Omega)$, $\forall r \in [1, 2^*)$, it holds $u_{n_k} \to u$ strongly in $L^r(\Omega)$.

Let now $\varepsilon_{n_k} = \|\widehat{G}'_\lambda(u_{n_k})\|_{\mathcal{W}'(\Omega)}$. By *(iii)* it holds $\varepsilon_{n_k} \to 0$. Furthermore

$$\left|\widehat{G}'_\lambda(u_{n_k})(v)\right| \le \varepsilon_{n_k}[v]_{\mathcal{W}(\Omega)}, \qquad \forall v \in \mathcal{W}(\Omega),\ k \in \mathbb{N}. \tag{10.34}$$

Choosing $v = u_{n_k} - u$ in (10.34) and taking into account that

$$\int_\Omega \widehat{H}(x, u_{n_k}(x))(u_{n_k} - u)(x) \to 0$$

(because $u_{n_k} \to u$ strongly in $L^{q+1}(\Omega)$, since $q+1 < 2^*$), we have from (10.34) the following inequality

$$\int_{D_1} \nabla u_{n_k} \nabla(u_{n_k} - u) + \int_{D_2} |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \nabla(u_{n_k} - u) \le \varepsilon_{n_k}[u_{n_k} - u]_{\mathcal{W}(\Omega)}.$$

And, since $\{u_n\}$ is bounded in norm $[\,\cdot\,]_{\mathcal{W}(\Omega)}$, it follows that

$$\int_{D_1} \nabla u_{n_k} \nabla(u_{n_k} - u) + \int_{D_2} |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \nabla(u_{n_k} - u) \to 0, \quad k \to \infty. \tag{10.35}$$

Let's show that (10.35) implies the existence of a subsequence of $\{u_{n_k}\}$ which converges strongly in $\mathcal{W}(\Omega)$.

We set the operator $S : \mathcal{W}(\Omega) \to [0,\infty)$ as

$$S(v) = \frac{1}{2}\|\nabla v\|^2_{L^2(D_1)} + \frac{1}{p}\|\nabla v\|^p_{L^p(D_2)},$$

namely,

$$S(v) = \widehat{G}_\lambda(v) + \lambda \int_\Omega \widehat{H}(x,u).$$

It is easy to check that S is convex and weakly lower semicontinuous. First, we claim that

$$\lim_{k\to\infty} S(u_{n_k}) = S(u). \tag{10.36}$$

Indeed, by *(ii)* and by the strong convergence of $\{u_{n_k}\}$ in $L^{q+1}(\Omega)$, we get that the sequence $\{S(u_{n_k})\}$ is bounded. Thus, up to a subsequence, $S(u_{n_k}) \to a \in \mathbb{R}$. Moreover, since S is weakly lower semicontinuous, we obtain

$$a = \lim_{k\to\infty} \inf S(u_{n_k}) \geq S(u).$$

By the other hand, due to convexity of S, i.e.

$$S(u) \geq S(u_{n_k}) + S'(u_{n_k})(u - u_{u_k})$$

and keeping in mind, by (10.35), that $S'(u_{n_k})(u - u_{u_k}) \to 0$, we obtain (taking limits)

$$S(u) \geq a$$

and the claim (10.36) is proved.

Then, to show that there exists a subsequence of $\{u_{n_k}\}$ which converges strongly to u in $\mathcal{W}(\Omega)$, we argue by contradiction. We consider a subsequence $\{u_{n_{k_l}}\}$ and $\delta > 0$ such that $[u_{n_{k_l}} - u]_{\mathcal{W}(\Omega)} \geq \delta$. In particular, there is a $\tilde{\delta} > 0$ such that $S(u_{n_{k_l}} - u) \geq \tilde{\delta}$.

We have

$$\frac{u_{n_{k_l}} + u}{2} \rightharpoonup u$$

and, by using again that S is weakly lower semicontinuous, it holds

$$S(u) \leq \liminf S\left(\frac{u_{n_{k_l}} + u}{2}\right). \tag{10.37}$$

On the other hand, due to Clarkson's inequality:

$$\left|\frac{z+w}{2}\right|^r + \left|\frac{z-w}{2}\right|^r \leq \frac{1}{2}|z|^r + \frac{1}{2}|w|^r, \quad z, w \in \mathbb{R},\, 2 \leq r < \infty.$$

it is easy to check that

$$\begin{aligned} S\left(\frac{u_{n_{k_l}} + u}{2}\right) &\leq \frac{1}{2}S(u_{n_{k_l}}) + \frac{1}{2}S(u) - S\left(\frac{u_{n_{k_l}} - u}{2}\right) \\ &\leq \frac{1}{2}S(u_{n_{k_l}}) + \frac{1}{2}S(u) - \frac{\tilde{\delta}}{2^p}. \end{aligned}$$

Finally, taking superior limits and taking into account (10.36), we have

$$\limsup S\left(\frac{u_{n_{k_l}} + u}{2}\right) \leq S(u) - \frac{\tilde{\delta}}{2^p}$$

which, together with (10.37), leads to the following contradiction

$$S(u) \leq \liminf S\left(\frac{u_{n_{k_l}} + u}{2}\right) \leq \limsup S\left(\frac{u_{n_{k_l}} + u}{2}\right) \leq S(u) - \frac{\tilde{\delta}}{2^p}.$$

□

Now we are ready to find a second solution.

Proof:[Proof of Theorem 10.1.2] For every fixed $\lambda \in (0, \lambda^*)$, we consider

$$\Gamma(\lambda) := \{\gamma \in \mathcal{C}([0,1], \mathcal{W}(\Omega)) \,:\, \gamma(0) = \tilde{u}_\lambda,\, \gamma(1) = Tw\}.$$

Here $\tilde{u}_\lambda$ is the local minimum of the functional G_λ obtained in Theorem 10.4.6. In addition, by construction, $\tilde{u}_\lambda$ is greater that u_1, the minimal positive solution for $0 < \lambda_1 < \lambda$ obtained in Theorem 10.1.1. Therefore, $\tilde{u}_\lambda$ is also a local minimum from $\widehat{G}_\lambda$. On the other hand, $0 < w \in \mathcal{C}_c^\infty(D_1)$ and $T = T(\lambda) > 0$ big enough to ensure that $Tw > u_1$ in D_1 and $\widehat{G}_\lambda(\tilde{u}_\lambda) > \widehat{G}_\lambda(Tw)$.

Let's also consider

$$c(\lambda) := \inf_{\gamma\in\Gamma(\lambda)} \max_{t\in[0,1]} \widehat{G}_\lambda(\gamma(t)).$$

Obviously, $c(\lambda) \geq \max\{\widehat{G}_\lambda(\tilde{u}_\lambda), \widehat{G}_\lambda(Tw)\} = \widehat{G}_\lambda(\tilde{u}_\lambda) = G_\lambda(\tilde{u}_\lambda)$. Where in the last equality we have used the fact that $u_1 < \tilde{u}_\lambda$.

We distinguish between two possible cases:

If $c(\lambda) = \widehat{G}_\lambda(\tilde{u}_\lambda)$. In this case, since $\tilde{u}_\lambda$ is a local minimizer of $\widehat{G}_\lambda$, there is $\delta > 0$ such that $\widehat{G}_\lambda(\tilde{u}_\lambda) \leq \widehat{G}_\lambda(v)$ for all v belongs in the ball $B_\delta(\tilde{u}_\lambda) = \{v \in \mathcal{W}(\Omega) : [v - \tilde{u}_\lambda]_{\mathcal{W}(\Omega)} < \delta\}$. In the case that there is a $v_0 \in B_\delta(\tilde{u}_\lambda) \setminus \{\tilde{u}_\lambda\}$ with $\widehat{G}_\lambda(\tilde{u}_\lambda) = \widehat{G}_\lambda(v_0)$, then v_0 will be another minimum (in fact, there will be infinity many minimums) and the proof is finished. Therefore, we can suppose

$$\widehat{G}_\lambda(\tilde{u}_\lambda) < \widehat{G}_\lambda(v), \qquad \forall v \in B_\delta(\tilde{u}_\lambda) \setminus \{\tilde{u}_\lambda\}.$$

In particular, for all $r \in (0, \delta)$, it holds

$$c(\lambda) = \widehat{G}_\lambda(\tilde{u}_\lambda) < \widehat{G}_\lambda(v), \qquad \text{if } [\tilde{u}_\lambda - v]_{\mathcal{W}(\Omega)} = r.$$

Then, applying the refinement of the Mountain Pass Theorem dues to Ghoussoub-Preiss (Ghoussoub and Preiss, 1989, Theorem 1) with the closed subset

$$F_r = \{v \in \mathcal{W}(\Omega) : [v - \tilde{u}_\lambda]_{\mathcal{W}(\Omega)} = r\} \subset \mathcal{W}(\Omega)$$

we obtain the existence of a sequence $\{u_n\} \subset \mathcal{W}(\Omega)$ verifying:

$$\lim_n \operatorname{dist}(u_n, F_r) = 0, \qquad \lim_n \widehat{G}_\lambda(u_n) = c(\lambda) \qquad \text{and} \qquad \lim_n \|\widehat{G}'_\lambda(u_n)\|_{\mathcal{W}(\Omega)'} = 0.$$

Then, $\{u_n\}$ is bounded (because F_r is bounded and the distance of u_n to F_r goes to zero) and by Lemma 10.4.7 our functional satisfies the Palais-Smale condition for bounded sequences. Consequently, there exists a critical point of $\widehat{G}_\lambda$ on F_r with critical value $c(\lambda)$ (see (Ghoussoub and Preiss, 1989, Theorem 1. bis)). Then, this critical point is a nontrivial weak solution to our problem (10.2) (that is in fact strictly greater than u_1). Note that we can apply this reasoning for every closed subset F_r with $r \in (0, \delta)$, and to conclude the existence of infinite critical points of G_λ in $B_\delta(\tilde{u}_\lambda)$.

If $c(\lambda) > \widehat{G}_\lambda(\tilde{u}_\lambda)$, for some $\lambda = \hat{\lambda} \in (0, \lambda^*)$. Let $\lambda_1 < \hat{\lambda}$ and u_1 the minimal solution in the construction of $\widehat{G}_{\hat{\lambda}}$ in (10.33). In this way, we consider the interval $[\hat{\lambda} - \varepsilon_0, \hat{\lambda}]$, with $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < \min\left\{\frac{(q+1)\varepsilon_1}{\|\tilde{u}_{\hat{\lambda}}\|^{q+1}_{L^{q+1}(\Omega}}, \hat{\lambda} - \lambda_1\right\},$$

where $\varepsilon_1 = c(\hat{\lambda}) - \widehat{G}_{\hat{\lambda}}(\tilde{u}_{\hat{\lambda}}) > 0$. Obviously, $[\hat{\lambda} - \varepsilon_0, \hat{\lambda}] \subset (0, \lambda^*)$ since $\varepsilon_0 < \hat{\lambda}$. Then, for this (u_1, λ_1) fixed, we define $\widehat{G}_\lambda$ for $\lambda \in [\hat{\lambda} - \varepsilon_0, \hat{\lambda}]$. Of course, $\widehat{G}_\lambda$ is non-increasing with respect to λ. Furthermore, we get for every $\lambda \in [\hat{\lambda} - \varepsilon_0, \hat{\lambda}]$:

$$\begin{aligned} c(\lambda) \geq c(\hat{\lambda}) &= \widehat{G}_{\hat{\lambda}}(\tilde{u}_{\hat{\lambda}}) + \varepsilon_1 \\ &= \widehat{G}_{\hat{\lambda}-\varepsilon_0}(\tilde{u}_{\hat{\lambda}}) + \varepsilon_1 - \frac{\varepsilon_0}{q+1}\int_\Omega \tilde{u}_{\hat{\lambda}}^{q+1} \\ &> \widehat{G}_{\hat{\lambda}-\varepsilon_0}(\tilde{u}_{\hat{\lambda}}) \\ &\geq \widehat{G}_\lambda(\tilde{u}_{\hat{\lambda}}), \end{aligned}$$

where we have used the fact that $\widehat{G}_\lambda(\tilde{u}_{\hat{\lambda}}) = G_\lambda(\tilde{u}_{\hat{\lambda}})$ for $\lambda \in [\hat{\lambda} - \varepsilon_0, \hat{\lambda}]$.

Summarizing, we have

$$c(\lambda) > \max\{\widehat{G}_\lambda(\tilde{u}_{\hat{\lambda}}), \widehat{G}_\lambda(Tw)\}, \qquad \text{for all } \lambda \in [\hat{\lambda} - \varepsilon_0, \hat{\lambda}].$$

Finally, applying Jeanjean's result (Jeanjean, 1999, Theorem 1.1), there exists a bounded Palais-Smale sequence at the level $c(\lambda)$ for almost every $\lambda \in [\hat{\lambda} - \varepsilon_0, \hat{\lambda}]$. This Palais-Smale sequence, due Lemma 10.4.7, has a subsequence that converges strongly. In this setting, by the Mountain Pass theorem due to Ambrosetti and Rabinowitz (Ambrosetti and Rabinowitz (1973)) there exists a critical point of $\widehat{G}_\lambda$ at level $c(\lambda)$ (hence different from the minimum $\tilde{u}_\lambda$) for almost every $\lambda \in [\hat{\lambda} - \varepsilon_0, \hat{\lambda}]$. Arguing as in the previous case, we obtain a positive critical point of G_λ.

Then, we conclude that there exists a second positive solution of problem (10.2) for almost every $\lambda \in (0, \lambda^*)$. □

10.5 Appendix

We include here a proof of the fact that Palais-Smale sequences are bounded when we assume an Ambrosetti-Rabinowitz type condition with $\kappa > p$. We remark again that this condition does not hold here, but we include this simple computation for the sake of completeness.

Lemma 10.5.1 *Consider the functional $F : \mathcal{W}(\Omega) \to \mathbb{R}$ defined as follows:*

$$F(u) = \int_{D_1} \frac{|\nabla u|^2}{2}\, dx + \int_{D_2} \frac{|\nabla u|^p}{p}\, dx - \lambda \int_\Omega H(x, u(x))\, dx,$$

with H such that there exists $\kappa > p$ satisfying

$$0 \le \kappa H(x, s) \le s h(x, s), \qquad s \ge 0,\, x \in \Omega, \tag{10.38}$$

where $H(x, s) = \int_0^s h(x, t) dt$.

Then, Palais-Smale sequences for F are bounded.

Proof: Let $\{u_n\} \subset \mathcal{W}(\Omega)$ be a Palais-Smale sequence. That is, $|F(u_n)| \le C$ and $F'(u_n) \to 0$ in $\mathcal{W}(\Omega)'$. Then

$$\begin{aligned}
C &\ge \int_{D_1} \frac{|\nabla u_n|^2}{2} + \int_{D_2} \frac{|\nabla u_n|^p}{p} - \lambda \int_\Omega H(x, u_n)\, dx, \\
&\ge \int_{D_1} \frac{|\nabla u_n|^2}{2} + \int_{D_2} \frac{|\nabla u_n|^p}{p} - \frac{\lambda}{\kappa} \int_\Omega u_n h(x, u_n) dx \\
&= \left(\frac{1}{2} - \frac{1}{\kappa}\right) \int_{D_1} |\nabla u_n|^2 + \left(\frac{1}{p} - \frac{1}{\kappa}\right) \int_{D_2} |\nabla u_n|^p + \frac{1}{\kappa} F'(u_n)(u_n) \\
&\ge \left(\frac{1}{p} - \frac{1}{\kappa}\right) \left(\int_{D_1} |\nabla u_n|^2 + \int_{D_2} |\nabla u_n|^p\right) - \frac{\varepsilon_n}{\kappa} [u_n]_{\mathcal{W}(\Omega)},
\end{aligned}$$

where $\varepsilon_n \to 0$. This leads to the boundedness of $\{u_n\}$ in $\mathcal{W}(\Omega)$. □

We remark that the condition (10.38) can be relaxed imposing the inequality for $|s| \ge R > 0$.

Resumen

En esta memoria de tesis doctoral tratamos varias cuestiones relevantes en la teoría de las ecuaciones diferenciales tanto locales como no locales. Los resultados presentados en este manuscrito se concentran en tres partes. Cada parte está dividida en capítulos. Cada capítulo corresponde a un trabajo aceptado o *sometido a publicación* tal como se detalla a continuación:

Parte (I): **Problemas de difusión no local**;

- J.A. Cañizo y A. Molino. Improved Energy Methods for Nonlocal Diffusion Problems, *Discrete and Continuous Dynamical System. Serie A*, **18** no. 3, Art. 17 (2018).
- A. Molino y J.D. Rossi. Nonlocal diffusion problems that approximate a parabolic equation with spatial dependence, *Z. Angew. Math. Phys*, **67** no. 3, Art. 41, 14 pp. (2016).
- A. Molino y J.D. Rossi. Nonlocal approximations to Fokker-Planck equations, aparecerá en *Funkcialaj Ekvacioj*, (2017).
- T. Leonori, A. Molino y S. Segura de León. Parabolic equations with natural growth approximated by nonlocal equations, *sometido a publicación* (2017).

Part (II): **Ecuaciones elípticas con singularidad en el término del gradiente al cuadrado y problemas tipo Gelfand**;

- J. Carmona, A. Molino y L. Moreno-Mérida. Existence of a continuum of solutions for a quasilinear elliptic singular problem, *J. Math. Anal. Appl.*, **436** no. 2, 1048-1062, (2016).
- J. Carmona, A. Molino y J.D. Rossi. The Gelfand problem for the $1-$homogeneous p–Laplacian, aparecerá en *Adv. Nonlinear Anal.* (2017).
- A. Molino. Gelfand type problem for singular quadratic quasilinear equations, *NoDEA. Nonlinear Differential Equations and Applications*, **23** no. 5, Art. 56, 20, (2016).

Parte (III): **Algunos resultados en ecuaciones elípticas modeladas por el operador $p-$laplaciano**;

- A. Molino y S. Segura de León. Elliptic equations involving the 1–Laplacian and a subcritical source term, *sometido a publicación* (2017).
- D. Arcoya, A. Molino y L. Moreno-Mérida. Existence and regularizing effect of degenerate lower order terms in elliptic equations beyond the Hardy constant, *sometido a publicación* (2017).
- A. Molino y J.D. Rossi. A concave-convex problem with a variable operator, *sometido a publicación* (2017).

Así pues la memoria está dividida en diez capítulos, cada uno de los cuales contiene los resultados que se han obtenido. Los capítulos son autocontenidos y se pueden leer de forma independiente, exceptuando la incorporación de una bibliografía completa al final del manuscrito. Aunque cada capítulo contiene su propia introducción concerniente al problema, se ha considerado conveniente presentar en el siguiente resumen todos los resultados obtenidos en esta memoria. Por último, se hace notar que la metodología, objetivos y conclusiones de esta tesis se encuentran comprendidos en cada capítulo.

PARTE I: Problemas de difusión no local

Empezamos este resumen considerando la siguiente ecuación diferencial de difusión no local:

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} \left(K(x,y)u(y,t) - K(y,x)u(x,t)\right) \mathrm{d}y, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{10.39}$$

Donde u_0 es el dato en tiempo inicial y $K : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ es el llamado *núcleo de difusión* satisfaciendo la siguiente propiedad:

$$\text{existen } R, r > 0, \text{ tales que } K(x,y) \geq r \text{ cuando } |x-y| \leq R. \tag{10.40}$$

Hipótesis adicionales de u_0 y K se irán incorporando a lo largo de esta sección.

La ecuación (10.39) es formalmente la ecuación de Kolmogorov para procesos de Markov con probabilidad de salto K y u la densidad del proceso (Ethier and Kurtz (1986, Chapter 4.2)). Una interpretación física de la ecuación (10.39) sería la siguiente: si entendemos $u(x,t)$ como la densidad de población de individuos de una cierta especie en el lugar x y tiempo t (cuya densidad inicial es $u_0(x)$) y $K(x,y)$ como la función de distribución de probabilidad que un individuo salte desde el lugar y hasta el lugar x, entonces la tasa de individuos que llegan al lugar x desde otras localizaciones es $\int_{\mathbb{R}^N} K(x,y)u(y,t)\,\mathrm{d}y$. Por otro lado, la tasa de individuos que abandonan el lugar x para desplazarse a otros lugares es $-\int_{\mathbb{R}^N} K(y,x)u(x,t)\,\mathrm{d}y$. De esta manera, en ausencia de causas internas o externas, la densidad $u(x,t)$ satisface la ecuación (10.39). Obsérvese que la hipótesis (10.40) implica $K(x,x) > 0$ en un entorno de x. Es decir, desde el enfoque de la dinámica de poblaciones, significa que los individuos tienen probabilidad positiva de saltar cerca de donde se encuentran. Por lo tanto, este tipo de ecuaciones de difusión no local es relevante en el estudio de dispersión biológica de especies. Así como también en el procesamiento de imágenes, modelos de elasticidad y coagulación, sistemas de partículas, etc. Véanse por ejemplo los trabajos de Bobaru et al. (2009); Bodnar and Velazquez (2006); Carrillo and Fife (2005); Fife (2003); Fournier and Laurençot (2006) y Hutson et al. (2003).

En esta línea, cabría destacar también el siguiente modelo unidimensional propuesto por Cortázar et al. (2007)

$$u_t(x,t) = \int_{\mathbb{R}} \tilde{J}\left(\frac{x-y}{g(y)}\right) \frac{u(y,t)}{g(y)}\,\mathrm{d}y - u(x,t), \quad x \in \mathbb{R}, t > 0, \tag{10.41}$$

con dato inical $u_0(x)$. Donde $\tilde{J}$ es una función par, no negativa y suave con integral igual a 1 y con soporte en el intervalo $[-1,1]$. En cuanto a g es una función positiva y continua la cual afecta a la distancia de dispersión ya que depende del punto de partida. Así pues, g modela la heterogeneidad del entorno afectando la distribución de las especies (véanse también Cortázar et al. (2011); Cortázar et al. (2015) y Cortázar et al. (2016)). Obsérvese que si tomamos

$$K(x,y) = \tilde{J}\left(\frac{x-y}{g(y)}\right)\frac{1}{g(y)},$$

se tiene que $\int_{\mathbb{R}} K(y,x)\,\mathrm{d}y = 1$ y por tanto la ecuación (10.41) entra en el marco del modelo (10.39). Un ejemplo mucho más general sería el caso

$$K(x,y) = J\big(\mathcal{M}(y)(x-y)\big)\det\mathcal{M}(y),$$

donde $\mathcal{M}(y)$ es una matriz real $N \times N$.

Por otro lado, en el caso de la ecuación (10.39) con núcleo simétrico, $K(x,y) = K(y,x)$, es decir, los individuos tienen la misma probabilidad de saltar desde x hacia y, que a la inversa, se obtiene el siguiente problema de difusión no local:

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} K(x,y)(u(y,t) - u(x,t))\,\mathrm{d}y, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{10.42}$$

Este tipo de ecuaciones se han tratado en los últimos años, así como también su versión en dominios acotados con condiciones de Dirichlet en la frontera, esto es, problemas del tipo

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} K(x,y)(u(y,t)-u(x,t))dy, & x \in \Omega,\ t>0, \\ u(x,t) = g(x,t), & x \notin \Omega,\ t>0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{10.43}$$

donde g y u_0 son datos dados. Para un amplio estudio de este tipo de problemas consúltese el libro Andreu-Vaillo et al. (2010). Nótese que en el caso particular de

$$K(x,y) = \frac{1}{|x-y|^{N+2s}}, \tag{10.44}$$

las ecuaciones (10.42) y (10.43) se corresponden con los problemas del tipo laplaciano fraccionario

$$u_t(x,t) = -(-\Delta)^s u(x,t),$$

donde la integral del núcleo singular (10.44) coresponde al valor principal de la integral. Véase Valdinoci (2009) para una completa descripción de ecuaciones que involucran este tipo de operadores.

Comportamiento asintótico

El Capítulo 1 de esta tesis está dedicado al comportamiento asintótico en norma de las soluciones del problema (10.39) mediante la técnica de *métodos de energía*. Esta estrategia se basa en conseguir, para una cierta función $\mathcal{F}$, una desigualdad funcional del tipo

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|_p \le \mathcal{F}\left(\|u(\cdot,t)\|_p\right),$$

donde $\|u(\cdot,t)\|_p$ es la norma en $L^p(\mathbb{R}^N)$. La resolución posterior de dicha inecuación diferencial ordinaria permitirá deducir el comportamiento asintótico en norma de las soluciones. Este tipo de estrategia es muy parecida a la muy exitosa técnica de *métodos de entropía*, la cual se basa en comparar la derivada en tiempo de un funcional de Lyapunov con el mismo funcional de Lyapunov y así obtener una cierta tasa de decaimiento de las soluciones (véase el reciente libro de Jüngel (2016) y los trabajos de Arnold et al. (2004); Bakry and Émery (1985); Carrillo et al. (2001); Gross (1975); Otto and Villani (2000); Villani (2002) y Desvillettes and Villani (2004)). Varias son las ventajas del uso de los *métodos de energía*. Una de ellas es que siguen siendo válidos si perturbamos ligeramente el operador, es decir, son bastante robustos bajo ciertas modificaciones en el operador. Otra de las ventajas es que se pueden aplicar a ecuaciones que no son explícitamente resolubles mediante transformadas de Fourier. Un ejemplo modelo en el cual la ecuación (10.39) sí es resoluble por transformada de Fourier es cuando $K(x,y) = J(x-y)$ donde $J \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ es no negativa, radial con integral igual a 1 y tal que $J(0) > 0$. En este caso la ecuación que quedaría es

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y,t)\,\mathrm{d}y - u(x,t), & x \in \mathbb{R}^N, t>0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{10.45}$$

Estas ecuaciones de difusión no locales son en forma de convolución, ya que puede reescribirse como $u_t(x,t) = J * u - u(x,t)$. Así pues, aplicando la transformada de Fourier se deduce que $\hat{u}_t(\xi,t) = \hat{u}(\xi,t)(\hat{J}(\xi)-1)$. Por tanto $\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\hat{u}_0(\xi)$, pudiéndose obtener, a partir de esto, su comportamiento asintótico (ver (Andreu-Vaillo et al., 2010, Chapter 1)).

El uso de *métodos de energía* para la ecuación (10.39) no es nuevo. En el caso de núcleos simétricos Ignat and Rossi (2008) obtuvieron, entre otros, el siguiente resultado:

Theorem 1 *Sea $N \geq 3$, $K : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ simétrico satisfaciendo (10.40) y $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Entonces, toda solución del problema (10.42) satisface*

$$\|u(\cdot,t)\|_p \leq C\, t^{-\frac{N(p-1)}{2p}},$$

para todo $p \in [1, \infty)$ y t suficientemente grande, siendo C una costante positiva que depende de $R, r, N, p, \|u_0\|_1$ y $\|u_0\|_\infty$.

Así pues, uno de los objetivos del Capítulo 1 es completar el resultado anterior en dos direcciones: que no haya ninguna restricción sobre la dimensión del espacio N, así como que sea aplicable para núcleos no simétricos (ecuación (10.39)). Concretamente, se obtiene el siguiente teorema:

Theorem 2 *Sea $N \geq 1$. Supongamos que existe $u_\infty : \mathbb{R}^N \to (0, \infty)$, solución de equilibrio de (10.39), tal que $1/m \leq u_\infty \leq m$ para algún $m > 0$ y sea u, solución de (10.39) con dato inicial $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, $1 < p < \infty$. Entonces existe una constante positiva $C = C(r, R, N, m, p, \|u_0\|_1, \|u_0\|_p)$ tal que*

$$\|u(\cdot,t)\|_p \leq C(1+t)^{-\frac{N(p-1)}{2p}},$$

para todo $t \geq 0$.

Recuérdese que una solución de equilibrio es una solución que no depende de la variable temporal t. De esta manera, en el caso de K simétrico cualquier constante positiva es solución de equilibrio. El Teorema 2 amplía y mejora los resultados obtenidos por Ignat and Rossi (2008) para núcleos simétricos, obteniendo la misma tasa de decaimiento pero en este caso para toda dimensión N.

Una aplicación directa del Teorema 2 es el comportamiento asintótico de las soluciones del problema (10.41). En efecto, en Cortázar et al. (2007) los autores prueban la existencia de una solución de equilibrio u_∞ positiva y acotada tanto superior como inferiormente. Por tanto, bajo las hipótesis del Teorema 2 podemos afirmar que

$$\|u(\cdot,t)\|_p \leq C(1+t)^{-\frac{p-1}{2p}}, \qquad \textit{para todo } t \geq 0.$$

Otro de los resultados del primer capítulo corresponde al caso $K(x,y) = J(x-y)$ siendo $J : \mathbb{R}^N \to [0, \infty)$ radialmente simétrica e integrable. Por consiguiente, el problema (10.39) se reduce a

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))\, \mathrm{d}y, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{10.46}$$

Nótese que la ecuación (10.45) es un caso particular. Obsérvese que la hipótesis (10.40) equivale a imponer que $J(z) \geq r$ siempre y cuando $|z| < R$, que obviamente se cumple si J es continua en el cero y $J(0) > 0$. Para este tipo de núcleos, se obtiene un decaimiento en norma de las soluciones mucho más preciso,

Theorem 3 *Considérese u solución de la ecuación (10.46) con dato inicial $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. Entonces, existe una constante $C = C(N,p)$ tal que*

$$\|u(\cdot,t)\|_p^p \leq \begin{cases} \|u_0\|_p^p, & 0 \leq t \leq t_0, \\ \left(\|u_0\|_p^{-p\gamma} + C\gamma r R^{N+2} \|u_0\|_1^{-p\gamma}(t-t_0)\right)^{-\frac{1}{\gamma}}, & t \geq t_0, \end{cases}$$

donde $\gamma := \frac{2}{N(p-1)}$ y

$$t_0 = \max\left\{0, \ \frac{1}{CrR^N} \log\left(R^{\frac{2}{\gamma}} \|u_0\|_1^{-p} \|u_0\|_p^p)\right)\right\}.$$

Véase también el Teorema 1.1.4 para el decaimiento de las derivadas de las soluciones. En este Teorema 3 se puede apreciar que a partir de un tiempo $t \geq t_0$, el decaimiento en norma de las soluciones de (10.46) es el mismo que el de la ecuación del calor (Giga et al. (2010))

$$\|u\|_p^p \leq \left(\|u_0\|_p^{-p\gamma} + C\|u_0\|_1^{-p\gamma} t\right)^{-\frac{1}{\gamma}}, \qquad \text{para todo } t \geq 0. \tag{10.47}$$

Esto nos hace pensar la analogía que tiene la ecuación (10.46) con la ecuación del calor $u_t = \Delta u$. Este parecido se hace más visible cuando se reescala el núcleo J. Siendo más precisos, es bien conocido que si tomamos el siguiente reescalamiento J_ε como

$$J_\varepsilon(z) := \frac{C(J)}{\varepsilon^{2+N}} J\left(\frac{z}{\varepsilon}\right), \qquad \text{con } C(J)^{-1} = \frac{1}{2}\int_{\mathbb{R}^N} J(z) z_N^2 \, \mathrm{d}z, \tag{10.48}$$

donde hay que imponer que J tiene momento de segundo orden finito para que $C(J)$ no sea trivial, entonces u^ε solución de

$$\partial_t u^\varepsilon(x,t) = \int_{\mathbb{R}^N} J_\varepsilon(x-y)((u^\varepsilon(y,t) - u^\varepsilon(x,t))\, \mathrm{d}y, \qquad x \in \mathbb{R}^N,\ t > 0, \tag{10.49}$$

con dato inicial $u_0 \in \mathcal{C}(\mathbb{R}^N)$, al tender $\varepsilon \to 0$ cumple $u^\varepsilon \to v$ uniformemente sobre compactos de $\mathbb{R}^N \times [0,\infty)$, siendo v la solución de la ecuación del calor $v_t = \Delta v$ con el mismo dato inicial $v(x,0) = u_0(x)$ (véase por ejemplo Andreu-Vaillo et al. (2010) y Rey and Toscani (2013)). Por tanto, si las soluciones u^ε tienden a la solución del calor, cabe preguntarse si existirá algún ε_0 tal que el comportamiento asintótico de u^ε sea exactamente la expresión (10.47) para todo $\varepsilon < \varepsilon_0$ y $t \geq 0$. A continuación damos una respuesta afirmativa poniendo fin al resumen del primer capítulo:

Theorem 4 *Sea u^ε solución de (10.49) con dato inicial $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ y $p \in [2,\infty)$. Entonces se tiene,*

$$\|u^\varepsilon(t,\cdot)\|_p^p \leq \left(\|u_0\|_p^{-p\gamma} + C_1\|u_0\|_1^{-p\gamma}(t-t_0)\right)^{-\frac{1}{\gamma}} \qquad \textit{para } t \geq t_0,$$

donde $C_1 = C(N,p)\gamma r R^{N+2} C(J)$ no depende de ε y

$$t_0 = \max\left\{0,\ \frac{\varepsilon^2}{CrR^N C(J)} \log\left(\varepsilon^{\frac{2}{\gamma}} R^{\frac{2}{\gamma}} \|u_0\|_1^{-p} \|u_0\|_p^p)\right)\right\}.$$

En particular, $t_0 = 0$ para todo $\varepsilon < \varepsilon_0 = \|u_0\|_1^{\frac{\gamma p}{2}} / \left(R\|u_0\|_p^{\frac{\gamma p}{2}}\right)$.

Reescalamiento de núcleos

Como se ha mencionado anteriormente, con el reescalamiento (10.48), las soluciones u^ε de (10.49) convergen uniformemente a la solución de la ecuación del calor cuando $\varepsilon \to 0$. Una pregunta natural es si existen soluciones de otros tipos de reescalamiento de manera que converjan a soluciones de ecuaciones parabólicas locales más generales que la del calor. La respuesta a esta pregunta la tenemos en los Capítulos 2 y 3. Más aún, en una primera parte del segundo capítulo veremos que si$\Omega \subset \mathbb{R}^N$ es un dominio acotado, $A(x) = (a_{ij}(x))$ es una matriz con coeficientes diferenciables en $\overline{\Omega}$, simétrica y definida positiva, $g \in L^1\left(\left(\mathbb{R}^N \setminus \Omega\right) \times (0,\infty)\right)$ y el dato inicial $u_0 \in L^1(\Omega)$, las soluciones *suaves* del problema de Dirichlet de ecuaciones parabólicas locales en forma de divergencia:

$$\begin{cases} v_t(x,t) = \operatorname{div}\left(A(x)\nabla v(x,t)\right), & x \in \Omega, t > 0, \\ v(x,t) = g(x,t), & x \in \partial\Omega, t > 0, \\ v(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{10.50}$$

pueden ser aproximadas uniformemente por soluciones del problema no local (10.43), bajo un reescalamiento adecuado del núcleo. Es importante hacer notar que hacemos un uso *a priori* de la existencia de soluciones *suaves* de (10.50). De hecho podemos asumir que, bajo suposiciones de regularidad tanto del dato en la frontera g, como de la frontera del dominio Ω y de la condición inicial u_0, se tiene que las soluciones de (10.50) están en $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\Omega}\times[0,T]\right)$ (véase por ejemplo Lieberman (1996)). Así pues, el resultado en cuestión es el siguiente:

Theorem 5 *Sea* $v \in \mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\Omega}\times[0,T]\right)$ *con* $0<\alpha<1$*, solución de* (10.50)*. Consideremos, para cada* $\varepsilon>0$*,* u^{ε} *solución de*

$$\begin{cases} u_t^{\varepsilon}(x,t) = \displaystyle\int_{\mathbb{R}^N} K_{\varepsilon}(x,y)(u^{\varepsilon}(y,t)-u^{\varepsilon}(x,t))dy, & x\in\Omega,\ t>0,\\ u^{\varepsilon}(x,t) = g(x,t), & x\notin\Omega,\ t>0,\\ u^{\varepsilon}(x,0)=u_0(x), & x\in\Omega, \end{cases}$$

donde

$$K_{\varepsilon}(x,y)=\frac{C(J)}{\varepsilon^{N+2}}G\left(B^{-1}(x)\frac{x-y}{\varepsilon}\right)G\left(B^{-1}(y)\frac{x-y}{\varepsilon}\right), \qquad (10.51)$$

siendo $G^2(s)=J(s)$ *(con* J *función suave no negativa, radialmente simétrica y con soporte compacto), y* $B(x)=(b_{ij}(x))$ *es una matriz* $N\times N$ *tal que*

$$\det(B(x))B(x)B^t(x)=A(x). \qquad (10.52)$$

Entonces, se tiene

$$\|v-u^{\varepsilon}\|_{L^{\infty}(\Omega\times[0,T])}\to 0, \qquad \textit{cuando } \varepsilon\to 0.$$

Este teorema viene a decirnos que toda solución $v\in\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\overline{\Omega}\times[0,T]\right)$ de (10.50), siempre se puede aproximar por una familia de soluciones de problemas no locales con el reescalamiento (10.51). Obsérvese que tanto la descomposición matricial (10.52) como la existencia de la matriz $B^{-1}(x)$ es siempre viable al ser $A(x)$ una matriz simétrica definida positiva.

Se puede apreciar también que $K_{\varepsilon}(x,y)$ es simétrico. Esta propiedad era deseable ya que, cuando se tienen núcleos simétricos, en general uno obtiene la siguiente *fórmula de integración por partes*

$$\begin{aligned}&\iint K(x,y)(u(y)-u(x))\varphi(x)dydx\\ &\qquad = \frac{-1}{2}\iint K(x,y)(u(y)-u(x))(\varphi(y)-\varphi(x))dydx.\end{aligned}$$

Que es el análogo a la usual fórmula de integración por partes que se obtiene con los operadores en forma de divergencia,

$$\int \operatorname{div}(A(x)\nabla v(x))\varphi(x)dx = -\int A(x)\nabla v(x)\nabla\varphi(x)dx.$$

Otra consecuencia del teorema es que si consideramos el problema de Dirichlet para la ecuación del calor, es decir, el problema (10.50) con $A(x)$ la matriz identidad, entonces obtenemos que el reescalamiento adecuado es

$$K_{\varepsilon}(x,y)=\frac{C(J)}{\varepsilon^{2+N}}J\left(\frac{x-y}{\varepsilon}\right),$$

resultado que ya fue obtenido por Cortázar et al. (2009).

Por otro lado, en el Theorem 2.1.1 del Capítulo 2 se obtiene también un resultado análogo al Teorema 5 pero para ecuaciones parabólicas en general, es decir, que no están en forma de divergencia.

$$\begin{cases} v_t(x,t) = \displaystyle\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 v(x,t)}{\partial x_i \partial x_j} + \sum_{i}^{N} b_i(x)\frac{\partial v(x,t)}{\partial x_i}, & x \in \Omega, t > 0, \\ v(x,t) = g(x,t), & x \in \partial\Omega, t > 0, \\ v(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Donde en este caso el reescalamiento adecuado es

$$K_\varepsilon(x,y) = \frac{C(x)}{\varepsilon^{N+2}}\, a\Big(x - E(x)(x-y)\Big)\, J\left(L^{-1}(x)\frac{x-y}{\varepsilon}\right), \tag{10.53}$$

siendo a una función dada por $a(s) = \sum_i (s_i + M)$, con M constante positiva suficientemente grande para garantizar que $a(x) \geq \beta > 0$ para un cierto β. La matriz $L(x)$ es el conocido factor de Cholesky de la matriz $A(x)$, esto es, satisface la igualdad $A(x) = L(x)L^t(x)$. En cuanto a la matriz $E(x)$ involucra los coeficientes $(a_{ij}(x))$ y $b_i(x)$ y por último $C(x)$ es una cierta función normalizadora (véase el apartado 2.3 para una precisa definición de estos términos). Como era de esperar, el núcleo (10.53) no es simétrico al tratarse de operadores que no están en forma de divergencia.

En cuanto al Capítulo 3, se consideran núcleos de la forma

$$K(x,y) = J\big(\mathcal{M}(y)(x-y)\big)\det \mathcal{M}(y).$$

Donde $J : \mathbb{R}^N \to \mathbb{R}$ es una función radial no negativa tal que

$$J \in \mathcal{C}_c(\mathbb{R}^N) \quad \text{y} \quad \int_{\mathbb{R}^N} J(z)dz = 1\,, \tag{10.54}$$

y siendo $\mathcal{M}(y)$ una matriz real $N \times N$ con coeficientes diferenciables y acotados tal que $\det \mathcal{M}(y) \geq \gamma > 0$. Nótese que este tipo de núcleos preservan la masa, esto es,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\big(\mathcal{M}(y)(x-y)\big)\det \mathcal{M}(y)\, u(y)dydx = \int_{\mathbb{R}^N} u(x)dx, \qquad \forall u \in \mathcal{C}(\mathbb{R}^N).$$

Por lo que problemas del tipo

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J\big(\mathcal{M}(y)(x-y)\big)\det \mathcal{M}(y) u(y,t)dy - u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

se encuadran de nuevo en el modelo (10.39). En el caso particular de un múltiplo de la matriz identidad, es decir, $\mathcal{M}(y) = g(y)^{-1}\,\mathrm{Id}$, siendo g una función escalar positiva, la ecuación anterior toma la forma de

$$u_t(x,t) = \int_{\mathbb{R}^N} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g^N(y)}dy - u(x,t).$$

Como ya se comentó anteriormente, este tipo de núcleos de difusión fueron introducidos por Cortázar et al. (2007) a la hora de modelar procesos de dispersión no homogéneos, véase también Coville (2010) y Cortázar et al. (2015).

En el Capítulo 3 veremos que, con un adecuado reescalamiento de este tipo de núcleos, sus soluciones convergen a la solución clásica local de la ecuación de Fokker-Planck

$$\begin{cases} v_t(x,t) = \displaystyle\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ij}(x)v(x,t)\right), & x \in \mathbb{R}^N, t \in [0,T], \\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{10.55}$$

siendo $A(x) = (a_{ij}(x))$ una matriz $N \times N$ real y definida positiva. Siendo más precisos, dado el siguiente reescalamiento

$$K_\varepsilon(x,y) = \frac{1}{\varepsilon^N} J\Big(B^{-1}(y)\frac{(x-y)}{\varepsilon}\Big) \det B^{-1}(y),$$

donde la matriz B es tal que $BB^t = A$ y J satisface (10.54), obtenemos el siguiente resultado principal del capítulo

Theorem 6 *Sea $v \in \mathcal{C}^{2+\alpha,1+\alpha/2}\big(\mathbb{R}^N,[0,T]\big)$ la solución clásica de la ecuación de Fokker-Planck (10.55) con dato inicial $v_0 \in \mathcal{C}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Para todo $\varepsilon > 0$, consideramos u^ε la solución de la ecuación no local*

$$\begin{cases} u_t^\varepsilon = \dfrac{C}{\varepsilon^2}\left\{\displaystyle\int_{\mathbb{R}^N} K_\varepsilon(x,y)u(y,t)dy - u(x,t)\right\}, & x \in \mathbb{R}^N, t \in [0,T], \\ u^\varepsilon(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{10.56}$$

donde $C^{-1} = \frac{1}{2}\int J(z)z_N^2 dz$. Entonces,

$$\sup_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - v(\cdot,t)\|_{L^\infty} \to 0,$$

cuando $\varepsilon \to 0$.

Obsérvese, que el caso particular $B(y) = g(y)\,\mathrm{Id}$, la ecuación (10.56) toma la forma

$$u_t^\varepsilon(x,t) = \frac{C}{\varepsilon^2}\left\{\int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} J\Big(\frac{x-y}{\varepsilon g(y)}\Big)\frac{u(y,t)}{g^N(y)}dy - u(x,t)\right\}.$$

Y por tanto, sus soluciones convergen a la ecuación local

$$v_t(x,t) = \sum_i (g^2(x)v(x,t))_{x_i x_i}.$$

Este resultado también fue obtenido por Sun et al. (2011).

Aproximación de la ecuación Kardar-Parisi-Zhang por ecuaciones no locales

En el Capítulo 4 consideramos problemas no locales del tipo:

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)\big(u(y,t)-u(x,t)\big)\,\mathcal{G}\big(x,u(y,t)-u(x,t)\big)\,dy, \tag{10.57}$$

donde J cumple (10.54) y $\mathcal{G} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ es una función auxiliar no negativa de Carathéodory (es decir, $\mathcal{G}(\cdot,s)$ es medible para cada $s \in \mathbb{R}$ y $\mathcal{G}(x,\cdot)$ es continua para casi todo $x \in \mathbb{R}^N$) y que además satisface la siguiente condición:

$$\exists\ \alpha_2 \geq \alpha_1 > 0 : \qquad \alpha_1 \leq \frac{\mathcal{G}(x,s)s - \mathcal{G}(x,\sigma)\sigma}{s-\sigma} \leq \alpha_2, \tag{10.58}$$

para todo s,$\sigma \in \mathbb{R}$ con $s \neq \sigma$ y para casi todo $x \in \mathbb{R}^N$. En el caso particular $G(x,s) \equiv 1$, recobramos nuevamente las ecuaciones de difusión no locales en forma de convolución (10.45).

Obsérvese que esta condición implica que $\mathcal{G}$ es una función positiva y acotada, ya que si tomamos $\sigma = 0$ se tiene

$$0 < \alpha_1 \leq \mathcal{G}(x,s) \leq \alpha_2, \qquad \text{para todo } s \in \mathbb{R} \text{ y para casi todo } x \in \mathbb{R}^N.$$

En el capítulo se aborda tanto el problema de Cauchy de la ecuación (10.57)

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y;x,t)\ \mathcal{G}(x,u(y;x,t))\,dy & \text{en } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{en } \mathbb{R}^N, \end{cases} \tag{10.59}$$

donde se ha denotado $u(y;x,t) := u(y,t) - u(x,t)$ y $u_0 \in \mathcal{C}(\mathbb{R}^N)$. Así como el problema de Dirichlet de la ecuación (10.57)

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y;x,t)\,\mathcal{G}(x,u(y;x,t))\,dy, & \text{en}\Omega \times (0,T), \\ u(x,t) = h(x,t), & \text{en } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{en}\Omega, \end{cases} \tag{10.60}$$

para $T \in \mathbb{R}^+ \cup \{\infty\}$, $h \in L^1\left((\mathbb{R}^N \setminus \Omega) \times (0,T)\right)$ y $u_0 \in L^1(\Omega)$. Siendo Ω un dominio acotado de $\mathbb{R}^N$ con $N \geq 1$.

En concreto, para el problema de Cauchy (10.59) con u_0 acotada se obtiene unicidad y existencia de solución en $\mathcal{C}\left([0,T);\mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\right)$ (Theorem 4.2.12) además de un principio de comparación (Theorem 4.2.14). De la misma manera, para el problema de Dirichlet (10.60) con $u_0 \in \mathcal{C}(\overline{\Omega})$ y $h \in \mathcal{C}((\mathbb{R}^N \setminus \overline{\Omega}) \times [0,T))$, se demuestra la existencia y unicidad de solución en $\mathcal{C}(\overline{\Omega} \times (0,T))$ (Theorem 4.2.3) así como también un principio de comparación (Theorem 4.2.5).

Otro de los aspectos que abordamos en el capítulo es el vínculo entre la ecuación no local (10.59) y la ecuación local determista de Kardar-Parisi-Zhang (ecuación KPZ a partir de ahora)

$$\begin{cases} u_t - \Delta u = \mu(x)|\nabla u|^2 & \text{en } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{en } \mathbb{R}^N. \end{cases} \tag{10.61}$$

Esta ecuación, al menos para $\mu(x) = \mu > 0$, fue propuesta por Kardar et al. (1986) en el estudio de la teoría física del crecimiento y la rugosidad de las superficies. Véase también Barabási and Stanley (1995) para otras aplicaciones en la física y el reciente estudio completo de dicha ecuación en Wio et al. (2011). Desde un punto de vista de las Ecuaciones en Derivadas Parciales, la ecuación de KPZ posee un término de gradiente al cuadrado, también llamado como *crecimiento natural en el gradiente*, que ha sido extensamente estudiado en las últimas décadas partiendo de los trabajos pioneros de Ladyzenskaja et al. (1968) y Aronson and Serrin (1967) como también de los resultados de Boccardo, Murat y Puel en Boccardo et al. (1989). Véase también la Parte II de este resumen.

Concretamente, en el Theorem 4.2.15 se establece que el problema de Cauchy (10.59) con dato inicial $u_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ y bajo el rescalamiento usual del núcleo J

$$J_\varepsilon(z) = \frac{C(J)}{\varepsilon^{2+N}} J\left(\frac{z}{\varepsilon}\right), \tag{10.62}$$

tiene una única solución u^ε (para cada $\varepsilon > 0$) que además converge uniformemente cuando $\varepsilon \to 0$ a una solución clásica de la ecuación KPZ (10.61) con

$$\mu(x) = \frac{2\mathcal{G}_s'(x,0)}{\mathcal{G}(x,0)}. \tag{10.63}$$

Más aún, se prueba que toda solución clásica de la ecuación KPZ (10.61) con dato inicial $u_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ se puede aproximar uniformemente por soluciones de la ecuación no local (10.59) con el reescalamiento (10.62) y con el mismo dato inicial u_0, tomando como función auxiliar

$$\mathcal{G} \equiv \mathcal{G}_\mu(x,s) = 1 + \frac{\mu(x)\,s}{2(1+\mu(x)^2 s^2)}.$$

Obsérvese que $\mathcal{G}_\mu$ cumple la hipótesis (10.58) y además satisface (10.63). Resultados análogos se obtienen en el Theorem 4.2.8 para el caso del problema de Dirichlet (10.60).

Para finalizar, cabe mencionar que sobre el comportamiento asintótico de las soluciones se obtienen dos tipos de resultados: en el caso del problema de Dirichlet se prueba que las soluciones convergen uniformemente a la solución estacionaria. Por otra parte, en el problema de Cauchy, se prueba que la norma L^2 de las soluciones tiene un decaimento en tiempo que depende de la naturaleza (absorción o reacción) de $\mathcal{G}$ (Theorem 4.2.16 y Theorem 4.2.17).

PARTE II: Ecuaciones elípticas con singularidad en el término del gradiente al cuadrado y problemas tipo Gelfand

En la segunda parte de la memoria se estudian diferentes problemas de valores iniciales asociados a una ecuación casilineal elíptica de segundo orden que involucra o bien una nolinealidad tipo Gelfand (λe^u) o bien un operador diferencial con términos de orden inferior que presentan un crecimiento cuadrático en el gradiente y singularidad en la incógnita ($|\nabla u|^2/u^\gamma$).

Los problemas tipo Gelfand han sido extensivamente estudiados en el campo de las ecuaciones elípticas semilineales. Podemos citar, en el estudio de problemas de auto ignición térmica de una mezcla de gases químicamente activa, los siguientes trabajos clásicos: Chandrasekhar (1957); Gel'fand (1963); Joseph and Sparrow (1970); Keller and Cohen (1967) así como las referencias en ellos contenidas.

Recordemos el problema de Gelfand clásico:

$$\begin{cases} -\Delta u = \lambda\, e^u\,, & \text{en}\Omega, \\ u \geq 0\,, & \text{en}\Omega, \\ u = 0\,, & \text{en } \partial\Omega, \end{cases}$$

donde$\Omega \subset \mathbb{R}^N$ es un abierto acotado con frontera suave, $N \geq 1$ y $\lambda \geq 0$. Aunque esencialmente, el término no lineal e^u se puede sustituir por una función regular $f(u)$, positiva, creciente y convexa con $f(0) > 0$.

Un cambio de variable formal del tipo $u = \ln(1+v)$ transforma el problema anterior en

$$\begin{cases} -\Delta v + \dfrac{|\nabla v|^2}{1+v} = \lambda\,(1+v)^2\,, & \text{en } \Omega, \\ v \geq 0\,, & \text{en } \Omega, \\ v = 0\,, & \text{en } \partial\Omega. \end{cases}$$

Este ejemplo pone de manifiesto la relación directa entre problemas semilineales y problemas casilineales con términos de orden inferior que tienen dependencia cuadrática en el gradiente. Este tipo de problemas, al constituir una clase cerrada para cambios de variable, se suelen denominar problemas con crecimiento natural en el gradiente. Además, suelen aparecer de manera natural en el Cálculo de Variaciones. De hecho, la ecuación de Euler-Lagrange asociada al funcional

$$I(u) = \frac{1}{2}\int_\Omega a(x,u)|\nabla u|^2 - \int_\Omega f_0(x)u,$$

viene dada formalmente por

$$-\text{div}\,(a(x,u)\nabla u) + \frac{1}{2}a'_u(x,u)|\nabla u|^2 = f_0(x).$$

Obsérvese además que para $a(x,u) = 1 + |u|^\delta$ con $\delta \in (0,1)$ la ecuación de Euler-Lagrange contiene un término singular además de la dependencia cuadrática en el gradiente. Algunas aplicaciones donde aparecen este tipo de ecuaciones singulares se describen en Barenblatt et al. (2000); Berestycki et al. (2001) y Kardar et al. (1986).

El estudio general de operadores diferenciales con crecimiento natural comenzó a desarrollarse en Aronson and Serrin (1967); Ladyzenskaja et al. (1968) y posteriormente en Boccardo et al. (1982, 1983) y desde entonces se han estudiado multitud de problemas de contorno asociados. Por ejemplo, el problema de la existencia de solución para

$$\begin{cases} -\Delta u + \mu(x)g(u)\,|\nabla u|^2 = f_0(x) & \text{en } \ ,\Omega \\ u = 0 & \text{en } \partial\Omega, \end{cases}$$

es considerado en Bensoussan et al. (1988), Boccardo and Gallouët (1992) y Boccardo et al. (1982), siendo $\mu \in L^\infty(\Omega)$ y g una función continua en $\mathbb{R}$.

Singularidad en el término del gradiente

En el Capítulo 5 se considera un problema con crecimiento cuadrático en el gradiente, que además presenta singularidad en la variable dependiente, confrontado con un término no lineal de tipo potencia. Un problema modelo es

$$\begin{cases} -\Delta u + \mu(x)\,\dfrac{|\nabla u|^2}{u^\gamma} = \lambda u^p + f_0(x) & \text{en} \quad, \Omega \\ u = 0 & \text{en } \partial\Omega, \end{cases}$$

siendo $\gamma, p \geq 0$, $\mu \in L^\infty(\Omega)$ no negativa y $0 \lneqq f_0 \in L^q(\Omega)$ para algún $q > N/2$.

El estudio de este tipo de problemas con términos de orden inferior que presentan crecimiento cuadrático y singularidad fueron introducidos en Arcoya and Martínez-Aparicio (2008) donde abordan el caso $\lambda = 0$, $\gamma = 1$. Posteriormente el caso $\lambda = 0$, $\gamma \leq 1$ ha sido ampliamente estudiado en Arcoya and Segura de León (2010); Boccardo (2008); Martínez-Aparicio (2009) y Giachetti and Murat (2009). Se deduce en dichos trabajos, la existencia de solución para $\gamma \leq 1$ y unicidad si $\gamma < 1$ (véase además Arcoya et al. (2017)).

El caso $\gamma > 1$ requiere una restricción sobre la nolinealidad f_0. Concretamente en Arcoya et al. (2009b) se prueba existencia de solución si y solo si $\gamma < 2$ imponiendo que

$$\operatorname{ess\,inf}\{f_0(x) : x \in \omega\} > 0\,, \ \forall \omega \subset\subset \Omega\,.$$

Además, como se observa en Zhou et al. (2012), para $\gamma \geq 2$ se tiene que $\frac{|\nabla u|^2}{u^\gamma} \notin L^1(\Omega)$ para toda $u \in W_0^{1,2}(\Omega)$ y por tanto no puede existir solución.

Las dificultades del caso $\gamma > 1$ quedan patentes cuando se observa que la unicidad de solución sigue siendo un problema abierto en la actualidad. Dicha unicidad tampoco es un problema completamente resuelto para $\gamma = 1$ aunque algunos avances se han conseguido en Carmona and Leonori (2017).

El problema fue estudiado para $\lambda \neq 0$ simultáneamente en Arcoya et al. (2011); Boccardo et al. (2011) cuando $\mu(x)$ es una función constante, $\gamma < 1$ y además los exponentes verifican la restricción $\gamma + p < 2$. La diferencia entre ambos trabajos radica en las técnicas empleadas, mientras que en Arcoya et al. (2011) se emplean métodos de grado topológico, en Boccardo et al. (2011) se utilizan convenientes esquemas iterativos. En ambos casos dichas técnicas no se pueden aplicar directamente para estudiar el caso $\mu(x)$ no constante o bien $p < 1 \leq \gamma < 2$. Los resultados del capítulo 5 vienen a completar a los anteriores precisamente en estos casos. Además, se consideran términos de orden inferior generales del tipo $\mu(x)g(u)|\nabla u|^2$, con g singular en cero, y se muestra que el conjunto de valores del parámetro λ para los que existe solución, no solo se ve afectado por la singularidad en cero de g, sino también por su comportamiento en infinito. Para diferenciar dicho comportamiento en cero y en infinito, tomamos $\gamma \leq \beta$ y consideramos el problema modelo

$$\begin{cases} -\Delta u + \mu(x)\dfrac{|\nabla u|^2}{u^\gamma + u^\beta} = \lambda u^p + f_0(x) & \text{en} \quad, \Omega \\ u = 0 & \text{en } \partial\Omega, \end{cases} \tag{Q_λ}$$

El resultado principal en el caso $\mu(x)$ constante es el siguiente.

Theorem 7 *Sea $\mu(x) = \mu$ una función constante y sea $f_0 \in L^q(\Omega)$ con $q > \frac{N}{2}$ y verificando que* $\operatorname{ess\,inf}\{f_0(x) : x \in \omega\} > 0\,, \ \forall \omega \subset\subset \Omega\,.$ *Entonces*

i) Si $1 \leq \gamma < 2$ y $0 < p < 1$ el problema (Q_λ) admite al menos una solución para cada $\lambda \geq 0$.

ii) Si $\gamma < 1 < \beta$ y $1 \leq p$, entonces existen valores del parámetro $\lambda_*, \lambda^* > 0$ tales que (Q_λ) no admite solución para $\lambda > \lambda^*$ y admite al menos una solución cuando $0 \leq \lambda < \lambda_*$.

Además, existe un continuo (cerrado y conexo) no acotado Σ en

$$\{(\lambda, u) \in [0, +\infty) \times C(\overline{\Omega}) : u \text{ solución de } (Q_\lambda)\}\,,$$

cuya proyección al eje de λ corresponde con el intervalo de valores del parámetro para los que, en los items i) y ii), se obtiene la existencia de solución.

Para el caso en que $\mu(x)$ no sea necesariamente constante, pero se encuentre entre dos constantes positivas, podemos abordar el problema con las mismas técnicas si además $\beta \leq 1$. Concretamente se prueba el siguiente teorema.

Theorem 8 *Supongamos que* $0 < \gamma \leq \beta \leq 1$, $0 < p < 2 - \beta$, $f_0 \in L^q(\Omega)$ *con* $q > \frac{N}{2}$ *y* $m \leq \mu(x) \leq M$, *a.e.* $x \in \Omega$ *(con* $M < 2$ *si* $\alpha = \beta = 1$*). Existe un continuo no acotado* Σ *de soluciones de* (Q_λ) *cuya proyección al eje de* λ *es el intervalo* $[0, +\infty)$.

Este resultado no solo extiende a funciones $\mu(x)$ no constantes, los resultados previos de Arcoya et al. (2011) y Boccardo et al. (2011) sino que, al trabajar con una función $g(s) = 1/(s^\gamma + s^\beta)$ con diferente comportamiento en cero y en infinito, se constata que la hipótesis $p < 2-\beta$ es una restricción sobre el comportamiento de g en infinito más que sobre la singularidad en el origen.

Por otra parte nuestros argumentos combinan la aproximación, como en Boccardo et al. (2011), por problemas donde es posible usar técnicas de grado topológico de Leray Schauder y el teorema de continuación de Rabinowitz, como en Arcoya et al. (2011). Así obtenemos la existencia de continuos Σ_n. Posteriormente, un conveniente lema topológico nos permite pasar al límite enΣ_n y obtener nuestro continuo de soluciones.

Problema de Gelfand

En los Capítulos 6 y 7 consideramos problemas tipo Gelfand asociados a diferentes operadores diferenciales (p-Laplaciano 1-homogéneo y operadores con términos de orden inferior con crecimiento cuadrático y singularidad) para los que, al menos para nuestro conocimiento, no habían sido tratados en la literatura hasta ahora.

Recordemos que el problema

$$\begin{cases} -\Delta u = \lambda f(u)\,, & \text{en}\Omega, \\ u \geq 0\,, & \text{en}\Omega, \\ u = 0\,, & \text{en } \partial\Omega, \end{cases} \tag{G_λ}$$

para una función regular $f(u)$, positiva, creciente y convexa con $f(0) > 0$ fue estudiado en Crandall and Rabinowitz (1975) (véase también Mignot and Puel (1980) y las citas que contienen). En este trabajo, para una función f que sea superlineal en infinito, es decir, $\lim_{s\to+\infty} \frac{f(s)}{s} = \infty$, prueban el siguiente resultado.

Proposición 10.5.2 *Crandall and Rabinowitz (1975) Existe un valor del parámetro* $\lambda^* > 0$*, denominado parámetro extremal de manera que*

- *Si* $\lambda < \lambda^*$ *el problema* (G_λ) *admite una solución minimal acotada* w_λ.
- *Si* $\lambda > \lambda^*$ *el problema* (G_λ) *no admite solución.*

Parece razonable pensar que la existencia de solución minimal se puede extender para problemas tipo Gelfand asociados a operadores diferenciales para los que se verifique un conveniente principio de comparación. Así, en el Capítulo 6, probamos un principio de comparación para el p-laplaciano 1-homogéneo que generaliza los conocidos en Barles and Busca (2001); Martínez-Aparicio et al. (2014a). Por otra parte, en Capítulo 7, usamos el principio de comparación contenido en Arcoya and Segura de León (2010) (véase también Arcoya et al. (2014, 2017)).

Por otra parte, en Crandall and Rabinowitz (1975) se prueba además que el conjunto de soluciones minimales $\{w_\lambda\}$ de (G_λ) es no decreciente en λ. Más aún, dichas soluciones minimales son estables en el sentido de satisfacer la siguiente condición

$$\int_\Omega \left(|\nabla\xi|^2 - \lambda f'(w_\lambda)\xi^2\right) \geq 0, \qquad \forall \xi \in \mathcal{C}_c^\infty(\Omega).$$

Dicha condición de estabilidad juega un papel fundamental en el estudio de la existencia y regularidad de lo que se conoce habitualmente como solución extremal, $u^* := \lim_{\lambda\to\lambda^*} w_\lambda$. En concreto, se ha usado para determinar resultados óptimos de regularidad de la solución extremal en términos de la dimensión N. Así, en el caso $f(s) = e^s$ se obtiene regularidad de la solución extremal para $N < 10$ y si $f(s) = (1+s)^p$ para $N < 4 + 2(1-1/p) + 4\sqrt{1-1/p}$ (véase Crandall and Rabinowitz (1975)).

Dicha condición de estabilidad parece que se podría extender a operadores con estructura variacional. Este no es el caso de los considerados en los Capítulos 6 y 7 donde el operador no tiene estructura variacional. Sin embargo en el Capítulo 7 si que será posible extender la noción de solución estable permitiendo así la obtención de resultados relativos a la solución extremal.

Antes de pasar a describir los resultados de ambos capítulos debemos recordar nuevamente que los problemas tipo Gelfand son un problema clásico de la literatura y por tanto las referencias sobre el tema son numerosas, citaremos entre otros los trabajos de Arcoya et al. (2014); Cabré and Capella (2006); Cabré and Sanchón (2013); Gel'fand (1963) y las referencias que contienen.

Concretamente en el Capítulo 6 consideramos el problema

$$\begin{cases} -\Delta_p^N u = \lambda\, e^u\,, & \text{en}\Omega, \\ u = 0\,, & \text{en } \partial\Omega, \end{cases} \tag{$P_{\lambda,p}$}$$

en un dominio acotado regular$\Omega \subset \mathbb{R}^N$ y para $p \in [2,\infty]$ notamos por Δ_p^N al operador p-laplaciano 1-homogéneo. Este viene definido, para $p < \infty$ como

$$\Delta_p^N u := \frac{1}{p-1}|\nabla u|^{2-p}\,\mathrm{d}iv\left(|\nabla u|^{p-2}\nabla u\right) = \frac{1}{p-1}\Delta u + \frac{p-2}{p-1}\Delta_\infty u,$$

mientras que para $p = \infty$ viene dado por

$$\Delta_\infty u \equiv \Delta_\infty^N u = \frac{\nabla u}{|\nabla u|}\cdot\left(D^2 u\frac{\nabla u}{|\nabla u|}\right),$$

conocido como infinito laplaciano 1-homogéneo.

Este operador aparece en el estudio de juegos Tug-of-War con ruido en Manfredi et al. (2012); Peres and Sheffield (2008); Peres et al. (2009), donde se analiza el problema de Poisson asociado. Además, en Martínez-Aparicio et al. (2014a) y Martínez-Aparicio et al. (2014b) ha sido confrontado con un término no lineal sublineal del tipo λu^q con $0 < q \le 1$.

Nuestro primer resultado es el siguiente:

Theorem 9 *Para cada $p \in [2,+\infty]$ existe un valor extremal del parámetro $\lambda^* = \lambda^*(\Omega, N, p)$ de manera que:*

- *Si $\lambda < \lambda^*$ el problema ($P_{\lambda,p}$) admite una solución minimal positiva w_λ.*
- *Si $\lambda > \lambda^*$ el problema ($P_{\lambda,p}$) no admite solución.*

Además, el conjunto de soluciones minimales $\{w_\lambda\}$ es no decreciente en λ.

Otra de las novedades que aporta el Capítulo 6 es la utilización de técnicas de teoría de grado para el estudio de problemas asociados al p-laplaciano 1-homogéneo. Esto presenta dificultades no triviales debido a la falta de regularidad. No obstante, es posible usar algunos argumentos de Charro et al. (2013) para obtener la compacidad necesaria para usar dichas técnicas. Así, se obtiene la existencia de continuos de soluciones tanto cuando el parámetro es λ como p. Para ello notaremos, para cada p fijado,

$$\mathscr{S}_p = \{(\lambda, u) \in [0, \lambda^*(\Omega, N, p)] \times \mathcal{C}(\overline{\Omega}) : u \text{ solución de } (P_{\lambda,p})\},$$

Analogamente, para cada λ fijado, notamos

$$\mathcal{S}_\lambda = \{(p, u) \in [2,\infty] \times \mathcal{C}(\overline{\Omega}) : u \text{ solución de } (P_{\lambda,p})\}.$$

Theorem 10 *Para cada $p \in [2, \infty]$ existe un continuo de soluciones no acotado $\mathcal{C} \subset \mathscr{S}_p$ que emana de la solución trivial para $\lambda = 0$, es decir $(0,0) \in \mathcal{C}$. Además, existe $\lambda_0 \in (0, \lambda^*)$ de manera que, para cada $\lambda < \lambda_0$ existe un continuo de soluciones $\mathcal{D} \subset \mathcal{S}_\lambda$, cuya proyección sobre el eje p es $[2, +\infty]$.*

En el Capítulo 7 abordamos el estudio de problemas tipo Gelfand asociados en este caso a problemas con singularidades en el término del gradiente cuadrado. Concretamente consideramos el problema

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = \lambda\, f(u)\,, & \text{en}\Omega, \\ u > 0\,, & \text{en}\Omega, \\ u = 0\,, & \text{en } \partial\Omega, \end{cases}$$

en un abierto y acotado$\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ con frontera suave, $\lambda > 0$, f estrictamente creciente y derivable en $[0, \infty)$, $f(0) > 0$ y g una función no negativa y no trivial que o bien es continua en $[0, \infty)$ o bien lo es $(0, \infty)$ y presenta una singularidad integrable en cero. Los casos modelo son $g(s) = \frac{1}{s^\gamma}$ con $\gamma \in (0,1)$ y $f(s) = e^s$.

Recientemente, el caso g continua en $[0, +\infty)$ ha sido estudiado en Arcoya et al. (2014). Además de la existencia de soluciones minimales en un intervalo acotado maximal $(0, \lambda^*)$, los autores analizan la existencia y regularidad de solución extremal caracterizando las soluciones minimales como aquellas que satisfacen una determinada condición de estabilidad (véase también Brézis and Vázquez (1997) para el caso semilineal). Concretamente, en Arcoya et al. (2014) se define que una solución al problema es estable si

$$\int_\Omega |\nabla \phi|^2 \geq \lambda \int_\Omega (f'(u) - g(u) f(u))\phi^2$$

para cada $\phi \in W_0^{1,2}(\Omega)$. Esta es la noción de estabilidad que se adopta en el Capítulo 7 en el caso g singular en cero y no solo se extienden los resultados de Arcoya et al. (2014) al caso singular, sino que además se mejoran las condiciones impuestas en el caso continuo. Por ejemplo, se puntualiza que la condición $f'(s) - g(s)f(s)$ estrictamente creciente, impuesta en Arcoya et al. (2014), solamente se necesita para probar que las soluciones estables son minimales.

Así, los resultados obtenidos en dicho capítulo particularizados al caso $g(s) = \frac{c}{s^\gamma}$ con $0 < \gamma < 1$ permiten considerar funciones $f(s)$ no convexas. De hecho, tomando $f(s) = e^{\frac{s^{1-\gamma}}{1-\gamma} + (s+\delta)^{1-\gamma}}$ con δ suficientemente pequeño, entonces $f'(s) - g(s)f(s)$ es decreciente. Sin embargo, existe $\lambda^* > 0$ de manera que el problema admite solución minimal acotada w_λ para cada $\lambda < \lambda^*$ y no existe solución para $\lambda > \lambda^*$. Incluso se puede probar la existencia de solución extremal $u^* = \lim\limits_{\lambda \to \lambda^*} w_\lambda$ que además es solución estable, en el sentido anterior, para $\lambda = \lambda^*$ (dicha solución extremal no es necesariamente minimal). Más aún, para dimensiones

$$N < \frac{6(1-\gamma) + 2c + 4\sqrt{(c+1-\gamma)(1-\gamma)}}{c+1-\gamma},$$

se obtiene que la solución extremal es acotada.

PARTE III: Algunos resultados en ecuaciones elípticas modeladas por el $p-$laplaciano

Se considera la siguiente familia de ecuaciones diferenciales elípticas que involucran el operador $p-$laplaciano y con condiciones de Dirichlet en la frontera de un dominio acotado$\Omega \subset \mathbb{R}^N$,

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x,u), & \text{en}\Omega\ , \\ u = 0, & \text{en } \partial\Omega, \end{cases} \tag{10.64}$$

para $p > 1$ y siendo el dato fuente $f(x,s)$, con $(x,s) \in \Omega \times \mathbb{R}$, una cierta función que a continuación detallaremos.

Seguidamente presentamos tres resultados clásicos relacionados con la ecuación anterior:

- El problema con dato subcrítico: $f(x,s) \le C_1|s|^q + C_2$ con exponente $0 < q < p^* - 1$. Existen al menos dos soluciones no triviales $v \le 0 \le w$ para todo $p > 1$ (véase por ejemplo Dinca et al. (1995)).
- El problema con el potencial de Hardy: $f(x,s) = \dfrac{\lambda|s|^{p-2}s}{|x|^p}$ y $0 \in \Omega$. No existe solución para $\lambda > ((N-p)/p)^p$, $1 < p < N$ (véase García Azorero and Peral Alonso (1998)).
- El problema cóncavo-convexo: $f(x,s) = |s|^{r-1}s + \lambda|s|^{q-1}s$ con $0 < q < p-1 < r < p^* - 1$ y $\lambda > 0$. Existe $\lambda^* > 0$ tal que hay al menos dos soluciones positivas para $\lambda < \lambda^*$ y no hay solución positiva para $\lambda > \lambda^*$ (véase García Azorero et al. (2000)).

En esta tercera parte de la memoria se pretende ampliar el estudio de estos problemas bien extendiendo el operador o bien extendiendo el dato, siempre sin perder la naturaleza del problema clásico. En concreto, en el Capítulo 8 se estudia el problema subcrítico para el $1-$Laplaciano ($p = 1$) en el cual demostramos la existencia de 2 soluciones no triviales para $0 < q < 1^* = 1/(N-1)$ y que además están acotadas. Otro de los resultados notables del capítulo es la existencia de una identidad tipo Pohožaev para este tipo de operadores. El operador $1-$Laplaciano fue originalmente tratado en Kawohl (1991, 1990), Demengel (1999) y Andreu et al. (2001) dando lugar a una gran literatura desde entonces. Una de sus aplicaciones más interesantes es el empleo de modelos variacionales en la restauración de imágenes, véase Andreu-Vaillo et al. (2004) y recientemente Martín et al. (2017).

En el Capítulo 9 se considera un problema con un potencial de Hardy para el operador laplaciano ($p = 2$). Probamos que la presencia en la ecuación de términos de orden inferior $h(x)u(x)^\gamma$ ($h \in L^1_{loc}(\Omega)$, $\gamma > 1$) produce un efecto regularizante al obtener solución para valores de λ mayores que el crítico $\frac{(N-2)^2}{4}$, inclusive si h se anula en subconjuntos de Ω. Además, dicho término provoca que las soluciones sean más regulares. El problema de Hardy para $p = 2$ fue estudiado primeramente por Baras and Goldstein (1984). Los autores observaron que como $\frac{\lambda}{|x|^2} \in L^r_{loc}(\Omega)$ sí y solo sí $1 \le r \le N/2$, las teorías clásicas de unicidad y regularidad no se podían aplicar. Prueban que el comportamiento asintótico de las soluciones depende de los valores de λ, determinando un valor crítico $\mathcal{H} = (N-2)^2/4$ también llamado constante de Hardy. Más tarde, en García Azorero and Peral Alonso (1998) los autores realizan un estudio más exhaustivo de la ecuación para todo valor de $1 < p < N$ donde efectivamente revelan que el comportamiento de las soluciones depende del valor crítico $\lambda^* = ((N-p)/p)^p$, obteniendo soluciones para $\lambda < \lambda^*$. A partir de entonces ha surgido un gran número de trabajos relacionados.

En el Capítulo 10 se estudia el problema cóncavo-convexo pero, en lugar de realizar un efecto cóncavo-convexo a la nolinealidad $f(x,s)$, se provocará tal efecto al operador. Es decir, el operador en consideración es $-\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ siendo $p(x)$ la función constante 2 en una región del dominio $D_1 \subset \Omega$, y la función constante $p > 2$ en la región restante del dominio $D_2 = \Omega \setminus D_1$. En cuanto a la nolinealidad tomamos $f(x,s) = \lambda|s|^q$ con $1 < q < p-1$. Nótese que estos valores de q inducen en la ecuación un efecto convexo en la región D_1 y un efecto cóncavo en la región D_2. Los problemas cóncavo-convexo han recibido un gran interés en la literatura de las ecuaciones diferenciales desde los trabajos pioneros de Lions (1982); García Azorero and Peral Alonso (1991); Ambrosetti et al. (1994) y

Boccardo et al. (1995). Por otro lado, el estudio de operadores $p(x)-$laplaciano con $p(x)$ una función discontinua ha cobrado una gran atención en los últimos años al modelar el flujo de corriente en los diodos orgánicos que emiten luz (OLED) usados en las pantallas de dispositivos pórtatiles, véanse los trabajos de Bulíček et al. (2016); Fischer et al. (2014) y Bulíček et al. (2017). En este capítulo demostramos la existencia de un valor crítico λ^* tal que para $\lambda > \lambda^*$ no existe solución positiva, y para $\lambda < \lambda^*$ hay una solución positiva y minimal. Además, si $p < 2N/(N-2)$ entonces existe una segunda solución para casi todo $\lambda < \lambda^*$.

La técnica para afrontar los problemas de esta tercera parte de la memoria es principalmente el cálculo de variaciones. Obsérvese que el problema (10.64) tiene asociado el siguiente funcional de energía $\mathcal{I} : W_0^{1,p}(\Omega) \to \mathbb{R}$ definido como

$$\mathcal{I}(u) = \int_\Omega \frac{|\nabla u|^p}{p} - \int_\Omega F(x,u)\,,$$

siendo $F(x,s) = \int_0^s f(x,t)dt$, en el sentido que los puntos críticos de $\mathcal{I}$ son las soluciones del problema (10.64). A la hora de afrontar los problemas citados anteriormente un paso importante es sustituir el espacio $W_0^{1,p}(\Omega)$ por otro más conveniente, cambiando así la geometría del funcional $\mathcal{I}$ y siendo más accesible encontrar sus puntos críticos. Esto tendrá sus ventajas como se verá en los Capítulos 9 y 10 convirtiendo el funcional en coercivo, y sus desventajas cuando el espacio elegido no es reflexivo como es el caso del Capítulo 8, no pudiendo así aplicar resultados tan conocidos como la compacidad en las sucesiones de Palais-Smale.

Problema subcrítico para el $1-$Laplaciano

En el Capítulo 8 tratamos sobre la existencia y regularidad de soluciones del problema de Dirichlet para una ecuación diferencial elíptica con operador el $1-$Laplaciano y dato subcrítico, cuyo modelo es

$$\begin{cases} -\text{div}\left(\dfrac{Du}{|Du|}\right) = |u|^{q-1}u, & \text{en}\Omega\ , \\ \\ u = 0 & \text{en } \partial\Omega, \end{cases} \tag{10.65}$$

siendo$\Omega\ \subset \mathbb{R}^N$ $(N \geq 2)$ un dominio acotado con frontera Lipschitz y donde $0 < q < \frac{1}{N-1}$.

El espacio natural para tratar este tipo de operadores es el espacio de funciones de variación acotada $BV(\Omega)$, esto es, funciones que pertenecen al espacio $L^1(\Omega)$ tales que su gradiente distribucional es una medida de Radon finita. Además, $BV(\Omega)$ es un espacio de Banach con la norma

$$\|u\|_{BV(\Omega)} = \int_\Omega |Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1},$$

donde $\mathcal{H}^{N-1}$ es la medida $(N-1)$–dimensional de Hausdorff (véase por ejemplo Ambrosio et al. (2000)).

Sin embargo el espacio $BV(\Omega)$, no es reflexivo ni separable. Así pues el hecho de tratar con el operador $1-$Laplaciano conlleva una dificultad extra. Otra de las dificultades es dotar de sentido al cociente $\frac{Du}{|Du|}$, ya que tanto Du como $|Du|$ son medidas de Radon finitas. Para salvar este obstáculo se usa la Teoría de Anzellotti (véase Anzellotti (1983)) en la cual se considera un campo vectorial $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ tal que $\|\mathbf{z}\|_\infty \leq 1$ y $(\mathbf{z}, Du) = |Du|$, de esta manera $\mathbf{z}$ juega el papel del cociente $\frac{Du}{|Du|}$. Por otro lado, el significado de la condición impuesta en la frontera debe ser precisado. Para ello se define la traza débil en $\partial\Omega$ de la componente normal de $\mathbf{z}$ como la aplicación $[\mathbf{z}, \nu] : \partial\Omega \to \mathbb{R}$, siendo ν el vector normal exterior unitario de $\partial\Omega$, tal que $[\mathbf{z}, \nu] \in L^\infty(\partial\Omega)$ y $\|\,[\mathbf{z}, \nu]\,\|_{L^\infty(\partial\Omega)} \leq \|\mathbf{z}\|_{L^\infty(\Omega;\mathbb{R}^N)}$. Observemos que para $\mathbf{z} \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^N)$, esta definición coincide con la clasica $[\mathbf{z}, \nu] = \mathbf{z} \cdot \nu$.

Así pues, diremos que $u \in BV(\Omega)$ es solución del problema (10.65) si existe un campo vectorial $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ con $\|\mathbf{z}\|_\infty \leq 1$ y tal que

(1) $-\operatorname{div}\mathbf{z} = |u|^{q-1}u$, en el sentido de las distribuciones $\mathcal{D}'(\Omega)$,

(2) $(\mathbf{z}, Du) = |Du|$, como medidas en Ω ,

(3) $[\mathbf{z}, \nu] \in \operatorname{sign}(-u)$, en $\partial\Omega$.

A la hora de considerar la ecuación (10.65) en un marco variacional, en el Lemma 8.2.6 establecemos que soluciones del problema coinciden con puntos críticos del funcional $\mathcal{I} : BV(\Omega) \to \mathbb{R}$ definido como

$$\mathcal{I}(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} - \frac{1}{q+1}\int_\Omega |u|^{q+1}.$$

Se recuerda que el método variacional para encontrar soluciones no triviales en el problema subcrítico para el $p-$laplaciano ($p > 1$), radica en el uso del Teorema de Paso de Montaña (véase Dinca et al. (1995)). Concretamente, primero se prueba que la solución trivial es un mínimo local de su correspondiente funcional de energía. Después, debido a que el funcional tiene una geometría de paso de montaña y satisface la condición de compacidad de Palais-Smale, se encuentran dos puntos críticos (uno positivo y otro negativo). Hay que resaltar que la condición de Palais-Smale se basa en la reflexividad del espacio $W_0^{1,p}(\Omega)$. Como se ha comentado anteriormente el espacio de funciones de variación acotada no es reflexivo. La estrategia consiste en tomar w_p solución no trivial y positiva (de igual forma se realiza para la negativa) obtenida por el paso de montaña del problema subcrítico para el $p-$laplaciano y en cierto sentido hacer tender $p \to 1^+$. Un paso importante y delicado es probar que dicho límite no es la solución trivial. De esta manera, demostramos en el Theorem 8.1.1 la existencia de al menos dos soluciones no triviales $v \le 0 \le w$ del problema (10.65). Además, se demuestra que dichas soluciones están acotadas, para tal fin resulta crucial la existencia de una constante positiva C, independiente de p tal que la solución w_p verifica

$$\|w_p\|_{W_0^{1,1}(\Omega)} \le C\,,$$

para todo $p > 1$.

En la última parte del capítulo presentamos una desigualdad tipo Pohožaev para soluciones que están en $W^{1,1}(\Omega)$ (Proposition 8.4.1). Además, damos ejemplos explícitos donde se hace constar la existencia de soluciones sin restricción del exponente q.

Efecto regularizante de términos de orden inferior en problemas elípticos que involucran un potencial de Hardy

En el Capítulo 9 tratamos sobre el efecto regularizante que proporciona la inclusión de términos de orden inferior en ecuaciones del tipo (10.64) que involucran un potencial de Hardy. Siendo más específicos, consideramos ecuaciones del tipo

$$\begin{cases} -\Delta u + h(x)|u|^{p-1}u = \lambda \dfrac{u}{|x|^2} + f(x) & \text{en} \quad , \ \Omega \\ u = 0 & \text{en } \partial\Omega, \end{cases} \tag{10.66}$$

donde $p > 1$, $\lambda \in \mathbb{R}$, $0 \le h \in L^1(\Omega)$ y $f \in L^{\frac{p+1}{p}}(\Omega; hdx)$, es decir, $|f|^{\frac{p+1}{p}} h \in L^1(\Omega)$. Obsérvese que al ser h integrable, se tiene

$$L^m(\Omega; hdx) \subset L^{\frac{p+1}{p}}(\Omega; hdx), \qquad \text{para todo } m \ge \frac{p+1}{p}.$$

En el caso de no incluir el término regularizante, es decir $h \equiv 0$, es conocido que existe solución para toda $f \in W^{-1,2}(\Omega)$ siempre que

$$\lambda < \mathcal{H} = \frac{(N-2)^2}{4}\,, \tag{10.67}$$

véase García Azorero and Peral Alonso (1998). Visto desde una perspectiva variacional la condición (10.67) implica que gracias a la desigualdad de Hardy

$$\int_\Omega \frac{|u|^2}{|x|^2}dx \le \mathcal{H}^{-1}\int_\Omega |\nabla u|^2\,,$$

el funcional de energía asociado es coercivo en $W_0^{1,2}(\Omega)$.

El hecho de incluir el término $h(x)|u|^{p-1}u$ al problema con un potencial de Hardy no es nuevo, véase Adimurthi et al. (2017); Porzio (2007); Wei and Du (2017) y Wei and Feng (2015). En estos dos últimos trabajos los autores estudian el comportamiento asintótico de la solución en 0 para el caso $f \equiv 0$ y $h(x) = |x|^\sigma$ con $\sigma > -2$. Por otro lado, Porzio (2007) y recientemente Adimurthi et al. (2017) tratan el caso $h(x) \equiv h_0 > 0$ obteniendo el siguiente resultado

Theorem 11 *Sea $p > 2^* - 1$, $h(x) \equiv h_0 > 0$ y $f \in L^m(\Omega)$ con $\frac{p+1}{p} \leq m < \frac{N}{2}\frac{p-1}{p}$. Entonces, existe solución del problema (10.66) para todo $\lambda \geq 0$. Además, la solución pertenece al espacio $W_0^{1,2}(\Omega) \cap L^{pm}(\Omega)$.*

Destacamos que la solución proporcionada por este teorema se obtiene como límite de soluciones de una sucesión de problemas aproximantes y además que la regularidad en $L^{pm}(\Omega)$ se prueba únicamente para esa específica solución.

En este capítulo se mejora el Teorema 11 en dos sentidos. Primero, probamos que la solución puede obtenerse como un mínimo del funcional asociado y además obtenemos regularidad para cualquier solución. Como segunda mejora, señalamos que podemos considerar el caso $h \in L^1(\Omega)$ no necesariamente constante y que puede anularse en subconjuntos de Ω. Así por ejemplo, probamos la existencia y regularidad de solución cuando h se anula en$\Omega_\delta = \{x \in \Omega : \text{ dist } (x,\partial\Omega) < \delta\}$ para valores de δ suficientemente pequeños. Más aún, en cuanto a la existencia de soluciones es suficiente que $h \in L^1_{loc}(\Omega)$.

Como se comentó al principio de la introducción, ya que el problema (10.66) posee una caracterización variacional, la elección de un espacio de funciones adecuado será ventajoso a la hora de encontrar sus puntos críticos. En efecto, consideremos el espacio $E = W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega; hdx)$ y el funcional de energía $\mathcal{I}_\lambda : E \to \mathbb{R}$ definido por

$$\mathcal{I}_\lambda(u) = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{p+1}\int_\Omega |u|^{p+1}\, h - \frac{\lambda}{2}\int_\Omega \frac{u^2}{|x|^2} - \int_\Omega f\, u\, h, \ \forall u \in E\,.$$

Obsérvese que para cada función $f \in L^{\frac{p+1}{p}}(\Omega; h(x)dx)$ se tiene un funcional asociado φ_f en el espacio dual E^* dado por

$$\langle \varphi_f, g\rangle = \int_\Omega fgh, \ \forall\, g \in L^{p+1}(\Omega; h(x)dx).$$

Probamos que $\mathcal{I}_\lambda$ es coercivo y acotado inferiormente. Posteriormente, usando el Principio Variacional de Ekeland una sucesión minimizante es débilmente convergente hacia un punto crítico en E. De esta manera, en el Theorem 9.2.1, establecemos la existencia de soluciones, bajo la condición de integrabilidad

$$\int_{\Omega\setminus\Omega_\delta} |x|^{\frac{2(p+1)}{1-p}}\, h(x)^{\frac{2}{1-p}} < \infty\,. \tag{10.68}$$

Nótese que la condición (10.68) para $h(x) \equiv h_0 > 0$ equivale a $p > 2^* - 1$, impuesta en el Teorema 11. Además, bajo una condición algo más fuerte que (10.68): existe $\bar{s} \in (2, p+1)$ tal que

$$\int_{\Omega\setminus\Omega_\delta} |x|^{\frac{2\bar{s}}{2-\bar{s}}}\, h(x)^{\frac{2\bar{s}}{(p+1)(2-\bar{s})}} < \infty,$$

el funcional $\mathcal{I}_\lambda$ es débilmente inferiomente semicontinuo, por tanto, la solución es mínimo del funcional.

En cuanto a la regularidad de las soluciones en el Theorem 9.3.1 establecemos que toda solución del problema (10.66) pertenece a $W_0^{1,2}(\Omega) \cap L^{pm}(\Omega; h(x)dx)$ supuesto que se cumplen las hipótesis:

i) $h \in L^1(\Omega)$ y $h(x) > 0$ para casi todo $x \in \Omega$,

ii) $|x|^{\frac{2pm}{1-p}} h^{1-\frac{pm}{p-1}} \in L^1(\Omega)$,

iii) $f \in L^m(\Omega; h(x)dx)$ con $m \geq \frac{p+1}{p}$.

Una vez más, queda patente el efecto regularizante del término $h(x)|u|^{p-1}u$, ya que en un principio las soluciones se encuentran en $W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega; h(x)dx)$.

Por último, un caso interesante donde se aplica el resultado anterior es $h(x) \geq \dfrac{\mu}{|x|^\beta}$ con $\mu > 0$ y $0 \leq \beta < N$. Donde obtenemos que la solución se encuentra en $W_0^{1,2}(\Omega) \cap L^{pm}(\Omega; h(x)dx)$ con

- $m \in \left[\frac{p+1}{p}, \frac{(N-\beta)(p-1)}{(2-\beta)p}\right)$, si $0 \leq \beta < 2$,
- $m \in \left[\frac{p+1}{p}, \infty\right)$, si $2 \leq \beta < N$.

Así pues, en el caso $\beta = 0$ (que corresponde a h constante) obtenemos el resultado de reguaridad del Teorema 11 pero en esta ocasión para toda solución, en lugar de para una solución obtenida como límite de soluciones de problemas aproximados.

Problema Cóncavo-Convexo con un operador discontinuo

Como se explicó anteriormente, en el Capítulo 10 estudiamos la existencia de soluciones positivas del siguiente problema

$$\begin{cases} -\Delta_{p(x)} u = \lambda u^q, & \text{en}\Omega\ , \\ u = 0, & \text{en } \partial\Omega, \end{cases} \tag{10.69}$$

donde $\lambda > 0$, $1 < q < p-1$,$\Delta_{\ p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ y $p(x)$ es una función discontinua dada por

$$p(x) = \begin{cases} 2 & \text{si } x \in D_1, \\ p & \text{si } x \in D_2, \end{cases}$$

siendo $p > 2$ y D_1 y D_2 dominios con frontera suave y tales que

$$\overline{\Omega} = \overline{D_1 \cup D_2}, \qquad D_1 \cap D_2 = \emptyset\,.$$

Llamaremos Γ a la interfaz o superficie que delimita las dos regiones, $\Gamma = \ \partial D_1 \cap \Omega = \partial D_2 \cap \Omega$, la cual supondremos suave con medida de Hausdorff($N-1$)-dimensional finita.

Para plantear de forma variacional el problema (10.69), descompondremos la ecuación diferencial en dos ecuaciones diferenciales, una en cada subdominio D_i $(i = 1, 2)$. Con tal fin, debemos proporcionar una "continuidad" de la solución cuando atraviesa de una región a otra. Es decir, que la traza de u en Γ coincide tanto "si viene" de D_1 como de D_2, y también respecto al flujo al cruzar Γ. Así pues, consideramos soluciones del problema (10.69) como soluciones débiles del siguiente problema:

$$\begin{cases} -\Delta u = \lambda u^q, & & \text{en } D_1, \\ -\Delta_p u = \lambda u^q, & & \text{en } D_2, \\ \dfrac{\partial u}{\partial \eta} = |\nabla u|^{p-2}\dfrac{\partial u}{\partial \eta}, & u|_{D_1} = u|_{D_2}, & \text{en}\Gamma\ , \\ u = 0, & & \text{en } \partial\Omega, \end{cases} \tag{10.70}$$

siendo η el vector normal unitario normal de Γ que apunta hacia afuera de D_1. El espacio adecuado para encontrar soluciones débiles es

$$\mathcal{W}(\Omega) = \left\{ v \in W_0^{1,2}(\Omega) : \int_{D_2} |\nabla v|^p < \infty \right\},$$

dotado con la norma

$$[v]_{\mathcal{W}(\Omega)} := \|\nabla v\|_{L^2(D_1)} + \|\nabla v\|_{L^p(D_2)}\,,$$

es un espacio de Banach reflexivo y separable (Lemma 10.2.1). De esta manera, demostramos que las soluciones de (10.70) corresponden a puntos críticos del funcional de energía $\mathcal{F}_\lambda : \mathcal{W}(\Omega) \to \mathbb{R}$ definido como

$$\mathcal{F}_\lambda(u) = \int_{D_1} \frac{|\nabla u|^2}{2}\,dx + \int_{D_2} \frac{|\nabla u|^p}{p}\,dx - \lambda \int_\Omega \frac{|u|^{q+1}}{q+1}\,dx.$$

A continuación, usando el método de sub y super-solución, para el cual es necesario un principio de comparación (véase Proposition 10.3.2), demostramos la existencia de $\lambda^* > 0$ tal que para $0 < \lambda < \lambda^*$ existe w_λ solución minimal y positiva. Además, w_λ es única y creciente respecto a λ. Por otro lado, si $\lambda > \lambda^*$ entonces no existe solución positiva. Para este último resultado resulta esencial que el problema parabólico $u_t = \Delta u + \lambda u^q$ en la región D_1, con un dato inicial $u(x,0) = u_0(x)$ suficientemente grande, explota en tiempo finito (Theorem 10.1.1).

En el Theorem 10.1.2 establecemos, bajo las hipótesis adicionales $p < 2^*$ y $D_2 \subset\subset \Omega$, la existencia de una segunda solución para casi todo $0 < \lambda < \lambda^*$. La demostración del teorema se divide en dos partes: primero, usando métodos variacionales y las ideas de Ambrosetti et al. (1994); Brézis and Nirenberg (1993); García Azorero et al. (2000) probamos que el funcional de energía $\mathcal{F}_\lambda$ tiene un mínimo local (véase Theorem 10.4.6). Para este resultado, como el operador $p(x)-$laplaciano, con $p(x)$ discontinuo, actúa de manera diferente en D_1 y en D_2, lo máximo que podemos aspirar es que las soluciones sean localmente Hölder (véase Acerbi and Fusco (1994)). Así pues, para demostrar que hay un mínimo local en $\mathcal{W}(\Omega)$ imponemos que $D_2 \subset\subset \Omega$ para así obtener regularidad $\mathcal{C}^1$ de la solución cerca de $\partial\Omega$ (donde actúa el operador laplaciano). Como consecuencia deducimos que existe un mínimo en la topología $\mathcal{C}^1(F_\delta) \cap \mathcal{C}(\overline{\Omega})$, donde F_δ es una pequeña banda alrededor de la frontera

$$F_\delta = \{x \in \Omega \,:\, \text{dist}(x,\partial\,\Omega) < \delta\}.$$

Seguidamente, usando un delicado argumento de regularidad tipo Stampacchia, relajamos el mínimo a la topología de $\mathcal{W}(\Omega)$. Es en ese último paso cuando debemos exigir la condición $p < 2^*$ (Proposition 10.4.5). En cuanto a la segunda parte de la demostración, para probar la existencia de una segunda solución, nótese que el funcional no tiene un mínimo global. En efecto, si tomamos $v \in \mathcal{W}(\Omega)$ con soporte compacto en D_1, como $q > 1$, se obtiene que

$$\mathcal{F}_\lambda(tv) = t^2 \int_{D_1} \frac{|\nabla v|^2}{2}\,dx - t^{q+1}\lambda \int_{D_1} \frac{|v|^{q+1}}{q+1}\,dx \to -\infty$$

cuando $t \to \infty$. Por tanto, $\mathcal{F}_\lambda$ tiene una geometría de paso de montaña. Si embargo, para aplicar el Teorema de Paso de Montaña, la gran dificultad radica en la compacidad del funcional o más concretamente en probar que las sucesiones de Palais-Smale están acotadas en $\mathcal{W}(\Omega)$. Hasta ahora, éste sigue siendo un problema abierto. Recordemos que, para encontrar puntos críticos de paso de montaña para problemas semilineales del tipo

$$\begin{cases} -\Delta u = f(x,u), & \text{en}\Omega\ , \\ u = 0, & \text{en } \partial\Omega, \end{cases}$$

usualmente se asume que se satisfaga la condición de Ambrosetti-Rabinowitz

$$\exists\, \kappa > 2 \ \text{ tal que } \ 0 \le \kappa F(x,s) \le s f(x,s), \ \ \forall\, s \ge 0 \ \text{ y a.e. } \ x \in \Omega\,. \tag{A-R}$$

Esta condición implica que todas las sucesiones de Palais-Smale al nivel del paso de montaña están acotadas. De forma análoga para nuestro operador variable$\Delta_{\ p(x)}$ se puede comprobar que si $f(x,s)$ satisface la propiedad (A-R) para $\kappa > p$, entonces se tiene que las sucesiones de Palais-Smale están acotadas (véase Apéndice 10.5). Sin embargo, en nuestro marco concreto $f(x,s) = \lambda s^q$ no cumple la condición (A-R) para $\kappa > p$, ya que $q + 1 < p$.

Para superar esta dificultad de la compacidad del funcional, combinamos los resultados clásicos de Ambrosetti and Rabinowitz (1973); De Figueiredo (1989) con una técnica de Jeanjean (1999) que prueba la existencia de una sucesión de Palais-Smale acotada a nivel del paso de montaña para casi todo $0 < \lambda < \lambda^*$.

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