Physics Letters B 750 (2015) 331–337

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Heavy baryons in the large N_c limit

C. Albertus [a], E. Ruiz Arriola [a], I.P. Fernando [b,c], J.L. Goity [b,c,*]

[a] Departamento de Física Atómica, Molecular y Nuclear and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, E-18071 Granada, Spain
[b] Department of Physics, Hampton University, Hampton, VA 23668, USA
[c] Thomas Jefferson National Accelerator Facility, Newport News, VA 23606, USA

ARTICLE INFO

Article history:
Received 20 July 2015
Received in revised form 14 September 2015
Accepted 14 September 2015
Available online 16 September 2015
Editor: W. Haxton

ABSTRACT

It is shown that in the large N_c limit heavy baryon masses can be estimated quantitatively in a $1/N_c$ expansion using the Hartree approximation. The results are compared with available lattice calculations for different values of the ratio between the square root of the string tension and the heavy quark mass $\sqrt{\sigma}/m_Q$. These estimates implement important $1/N_c$ corrections and assume a string tension independent of N_c. Using a potential adjusted to agree with the one obtained in lattice QCD, a variational analysis of the ground state spin averaged baryon mass is performed using Gaussian Hartree wave functions. Relativistic corrections through the quark kinetic energy are included. The results provide good estimates for the first sub-leading in $1/N_c$ corrections.

1. Introduction

QCD in the large N_c limit becomes a non-trivial theory in terms of an arbitrary and fixed 't Hooft coupling $\lambda = \alpha_s N_c$ [1]. In that limit, baryons [2], unlike mesons, remain as complicated structures (for a recent review see e.g. [3,4] and references therein). This is the result of the strong coupling of mesons to baryons $\mathcal{O}(\sqrt{N_c})$, giving baryons a light meson cloud which contributes to its mass at leading order in N_c. In the world of QCD with only heavy quarks, the meson cloud becomes suppressed in Λ_{QCD}/m_Q, m_Q being the heavy quark mass, and baryonic states become amenable to a treatment based on non-relativistic QCD. Thus, heavy baryons are a good laboratory to study the $1/N_c$ expansion. This simpler setting of QCD permits a straightforward application of the mean field approach, which will be used in the present work and which should provide a good description of baryons in the large N_c and large quark mass limits.

The quantitative understanding of the $1/N_c$ expansion has become possible in the light meson sector [5], where meson masses have been determined in lattice QCD (LQCD) calculations at different values of N_c and in the quenched approximation, where the leading $\mathcal{O}(1/N_c)$ corrections are absent, and moderate N_c values allow for a safe extrapolation to the large N_c limit. In addition, estimates based on short distance constraints provide an analytical understanding of those results [6]. More recently, LQCD calculations of low lying baryon masses for $N_c = 3$, 5 and 7 [7,8] have opened the door for a quantitative test of the $1/N_c$ expansion in baryons as well. Those pioneering calculations, which are in the quenched approximation, have quark masses in the light to moderately heavy range. The present work is largely motivated by the possibility that such LQCD calculations could be extended to heavier quark masses, where the framework presented here would become realistically applicable.

In his seminal paper, Witten [2] discussed specifically heavy baryons in the large N_c limit and invoked the mean field Hartree approximation. For heavy quarks, it is built from the simple two-body Hamiltonian, where the interaction is the OGE (one gluon exchange) (see [9] for details) for the short range part of the interaction. In addition, there are the long range confining forces, whose effects become suppressed as m_Q grows, and also short distance radiative corrections must be taken into account (running of α_s) (see [10]). Furthermore, the effects of three-body interactions are of potential interest; for a recent discussion in the quark model see Ref. [11]. They will be discussed briefly in this work.

At leading order in the $1/N_c$ expansion, the ground state of the heavy baryon will be described by a wave function which is the direct product of single-quark wave functions. Since the hyperfine interactions have spin-flavor non-singlet effects which are $\mathcal{O}(1/N_c)$, it is clear that at leading order the ground state baryon is in the totally symmetric spin-flavor state, and the baryon has a spin-flavor contracted symmetry [12,13], which holds in the limit $N_c \to \infty$ at fixed quark mass. The effect of removing the center of mass (CM) motion is sub-leading in $1/N_c$, and can be implemented

* Corresponding author.
E-mail addresses: albertus@ugr.es (C. Albertus), earriola@ugr.es (E. Ruiz Arriola), ishara@jlab.org (I.P. Fernando), goity@jlab.org (J.L. Goity).

http://dx.doi.org/10.1016/j.physletb.2015.09.030

using standard techniques such as the Peierls–Yoccoz projection (for a review see e.g. [14,15] and references therein).

The mean field for heavy quarks at large N_c has been studied in Refs. [16,17] along with possible implications for baryonic matter. This work builds on that one and compares to recent lattice calculations for $N_c = 3,\ 5,\ 7$ [7,8] after including some important $1/N_c$ effects such as the CM correction. Brief discussions of the role of hyperfine splittings as well as the expected corrections of many-body forces are also given. A previous large N_c analysis has been conducted in Ref. [18].

Note that in order to have low lying baryons with different spins it is necessary to have more than one flavor of heavy quark. The mass of the baryon will then have an $\mathcal{O}(1/N_c)$ hyperfine contribution (dependent on the spin S of the baryon). The masses of ground state baryons take the form of a rotational band,

$$M_B(S) = N_c m_0 + \frac{C_{HF}}{N_c}(S(S+1) - \frac{3}{4}N_c) + \mathcal{O}\left(1/N_c^2\right), \tag{1}$$

where m_0 and C_{HF} are $\mathcal{O}\left(N_c^0\right)$ and have an expansion in $1/N_c$, and depend on the quark mass m_Q. The hyperfine independent component of the baryon mass given by m_0 is obtained by the following combination of baryon masses:

$$m_0 = \frac{2}{N_c^2(N_c+1)(N_c+3)^2} \times \sum_{S=\frac{1}{2}}^{\frac{N_c}{2}} (3 + N_c(3N_c+2) - 8(N_c-3)S)\, M_B(S)\,. \tag{2}$$

The baryon masses studied here will be the ones with the hyperfine effects removed, i.e., $\mathring{M}_B \equiv N_c m_0$. These will be later compared with the available LQCD results of Refs. [7,8,18].

Of course, for any different value of N_c one has a different theory. Thus, in order to relate them one must assume that some observables are N_c independent. Actually, on general grounds one has that:

$$\frac{m_0}{\sqrt{\sigma}} = F(N_c, \frac{m_Q}{\sqrt{\sigma}}), \tag{3}$$

where σ sets the scale of QCD and can be identified for instance with the string tension, and m_Q is the heavy quark mass. F is a universal function $\mathcal{O}\left(N_c^0\right)$ which admits an expansion in $1/N_c$, and which for large m_Q can be more conveniently expressed as $F(N_c, \frac{m_Q}{\sqrt{\sigma}}) = \frac{m_Q}{\sqrt{\sigma}} f(N_c, \frac{m_Q}{\sqrt{\sigma}})$.

The present work goes beyond Refs. [16,17] by analyzing the main $1/N_c$ contributions such as the CM effect, and relativistic corrections, and actually compares to available LQCD results. For $N_c = 3$, triply heavy baryons have been studied on the lattice as a Ω_{bbb} state [19], and also re-addressed in quark models within several schemes [10,11,20] which, however, have not addressed larger N_c values.

One important goal on the lattice has been to make the quarks as light as possible. Actually, quarkonium studies based on LQCD proceed always through the determination of the $\bar{Q}Q$ potential, and a subsequent solution of the non-relativistic Schrödinger equation (see e.g. [21]). The present work takes a similar point of view as a N_c-body problem. It should be emphasized that studying heavy baryons at varying values of N_c will help with the understanding of the $1/N_c$ expansion in a setting where an analytic approach with small model dependencies can be applied.

2. Color singlet states

The starting point is the Hamiltonian for heavy quarks. Using non-relativistic heavy quark field operators $Q(x)$, the Hamiltonian is given by:

$$H = \int d^3x \left[-\frac{1}{2m_Q} Q^\dagger(x)\, \nabla^2 Q(x) + m_Q\, Q^\dagger(x) Q(x) \right] + \frac{1}{2}\int d^3x d^3x'\, Q^\dagger(x)\frac{\lambda_a}{2} Q(x)\, Q^\dagger(x')\frac{\lambda^a}{2} Q(x') V(x-x'), \tag{4}$$

where λ^a are the $SU(N_c)$ generators in the fundamental representation, and in perturbation theory $V(r) = \alpha_s/r$ is the OGE interaction. Here, only two-body interactions are included. The role of many body interactions is commented below. An equivalent representation for the case of a heavy baryon is the Hamiltonian

$$H = \sum_i \left[m_Q + \frac{p_i^2}{2m_Q} \right] + \frac{1}{4}\sum_{i<j}^{N_c} \lambda_a(i) \otimes \lambda^a(j) V(x_i - x_j) \tag{5}$$

The $\lambda \otimes \lambda$ interaction implies exact Casimir scaling of the potential energy. Casimir scaling for the $Q\bar{Q}$ potential holds perturbatively up to two loops (there are three-loop violations) [22] and numerically on the lattice [23].

For a color singlet state the wave function is completely symmetric in the orbital and spin-flavor quantum numbers, and the baryon behaves effectively as a bosonic system. In particular, for ground state baryons the wave function is the product of a symmetric spacial wave function and a symmetric spin-flavor wave function and reads as follows:

$$\Psi(x_1, \ldots, x_N) = \psi(x_1, \ldots, x_N)\chi_{SF}, \tag{6}$$

where χ_{SF} is the spin-flavor wave function. For excited baryon states, spin-flavor and spatial mixed symmetry states also occur. The color matrix elements for arbitrary N_c in the ground state can be computed as follows. Starting with the quadratic Casimir operator for the fundamental representation given by ($F^a = \lambda^a/2$)

$$\vec{F}_q \cdot \vec{F}_q = \vec{F}_{\bar{q}} \cdot \vec{F}_{\bar{q}} = \frac{N_c^2 - 1}{2N_c}, \tag{7}$$

for a baryon (color singlet) state one obtains:

$$\begin{aligned} 0 &= \langle B|(\sum_{i=1}^{N_c} \vec{F}_i)^2|B\rangle \\ &= \langle B|\sum_{i=1}^{N_c}(\vec{F}_i)^2|B\rangle + 2\sum_{i<j}\langle B|\vec{F}_i \cdot \vec{F}_j|B\rangle \\ &= N_c\langle B|(\vec{F}_q)^2|B\rangle + N_c(N_c-1)\langle B|\vec{F}_q \cdot \vec{F}_{q'}|B\rangle, \end{aligned} \tag{8}$$

and likewise for a meson state one obtains:

$$\begin{aligned} 0 &= \langle M|(\vec{F}_q + \vec{F}_{\bar{q}})^2|M\rangle \\ &= 2\langle M|(\vec{F}_q)^2|M\rangle + 2\langle M|\vec{F}_q \cdot \vec{F}_{\bar{q}}|M\rangle \end{aligned} \tag{9}$$

These equations lead to

$$\langle B|\vec{F}_q \cdot \vec{F}_{q'}|B\rangle = -\frac{1}{2}\left(1 + \frac{1}{N_c}\right) \tag{10}$$

$$\langle M|\vec{F}_q \cdot \vec{F}_{\bar{q}}|M\rangle = -\frac{N_c^2 - 1}{2N_c} \tag{11}$$

At very short distances the potential between a heavy quark and antiquark should be described with perturbative QCD, and approximately given by an N_c-independent expression at leading order (LO) in terms of the running strong coupling $\alpha_s^{N_c}(r)$,

$$V_{Q\bar{Q}}^{N_c,\mathrm{LO}}(r) = -\frac{N_c^2-1}{2N_c}\frac{\alpha_s^{N_c}(r)}{r} = \frac{1}{r}\frac{6}{11\log(r\Lambda_{\overline{\mathrm{MS}}})}. \tag{12}$$

At long distances it is of linear confining form and the corresponding string tension σ is determined in LQCD. For $N_c = 3$ the $\bar{Q}Q$ potential has been computed in LQCD in the quenched approximation [24], and for $N_c > 3$ also [7,8]. For $N_c = 3$, it is well described by the bosonic string model [25], namely:

$$V_{Q\bar{Q}}^{N_c=3}(r) = -\frac{\pi}{12r} + \sigma r. \tag{13}$$

The Coulomb term on the RHS is what results from the fluctuations of the string. It is remarkable that it provides the bulk of the Coulomb interaction down to the lattice spacings used in present day calculations. Using $\Lambda_{\overline{\mathrm{MS}}}/\sqrt{\sigma} = 0.503(2)(40) + 0.33(3)(3)/N_c^2 + \mathcal{O}(N_c^{-4})$ obtained in [26] one gets that at $r\sqrt{\sigma} \sim 0.2$ the $1/r$ term in Eqs. (12) and (13) coincide. For the heavy quark mass corresponding to Compton wave lengths much smaller than present lattice spacings, where the long distance potential plays a minor role, the Coulomb interaction will increasingly become the one predicted by perturbative QCD, Eq. (12).

At arbitrary N_c, $V_{Q\bar{Q}}^{N_c}$ will only receive corrections $\mathcal{O}\left(1/N_c^2\right)$, as required by the $1/N_c$ expansion in pure gluodynamics. Assuming the leading scaling in N_c for α_s and σ, and Eq. (11), the potential becomes:

$$\begin{aligned} V_{Q\bar{Q}}^{N_c}(r) &= \frac{9}{8}\frac{N_c^2-1}{N_c^2} V_{Q\bar{Q}}^{N_c=3}(r) \\ &= (1+\mathcal{O}\left(1/N_c^2\right)) V_{Q\bar{Q}}^{N_c=3}(r). \end{aligned} \tag{14}$$

This N_c dependence will be loosely named "Casimir scaling". This is verified by the 't Hooft coupling $\lambda = 4\pi N_c \alpha_s$ used in Refs. [7,8]. Clearly this follows only if the above assumption is made, and with the present calculation at $N_c > 3$ it can be verified, as discussed below.

As mentioned earlier, the $1/N_c$ expansion requires definition because it compares different theories. The most obvious way to proceed is to require that certain quantities are independent of N_c, e.g., the string tension and quark masses at a given scale. Since the LQCD results of Refs. [7,8] have the property that the string tension is approximately independent of N_c, i.e., $\sigma = \frac{9}{8}\frac{N_c^2-1}{N_c^2}\sigma(3) \sim \mathrm{const}$, this condition is adopted in what follows. The result from Fig. 1 vividly shows the N_c independence of the $Q\bar{Q}$ potential within the current lattice uncertainties and the astonishing agreement with the bosonic string model [25]. Thus, generalizing the N_c lattice findings [24] the potential for all N_c will be taken to be:

$$V_{Q\bar{Q}}^{N_c}(r) = V_{Q\bar{Q}}^{N_c=3}(r) = -\frac{\pi}{12r} + \sigma r \tag{15}$$

From Eqs. (10)–(15) the two-body interaction potential in the baryon becomes:

$$V_{QQ}^{N_c}(r) = \frac{V_{Q\bar{Q}}^{N_c}(r)}{N_c-1} = \frac{1}{N_c-1}\left(-\frac{\pi}{12r} + \sigma r\right) \tag{16}$$

3. Mean field approximation and beyond

3.1. Mean field approximation

The calculation for different values of $N_c = 3,\ 5,\ 7, \dots$ of the baryon mass with the Hamiltonian Eq. (5) requires solving separate

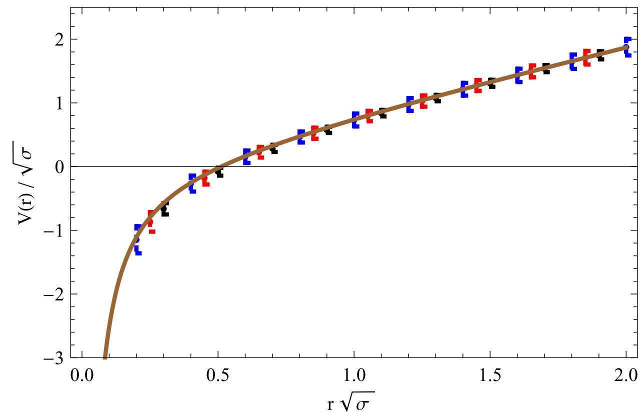

Fig. 1. Quark–antiquark potential on the lattice in units of the string tension for $N_c = 3, 5, 7$ compared with the bosonic string model [25] (full line). The values for different N_c: 3 (blue), 5 (red) and 7 (black), have been transported to avoid cluttering of points. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

few body problems with their inherent technical complications. In the large N_c limit, however, an important simplification arises as a mean field approach becomes valid. Due to its color antisymmetry, the N_c-quark wave function in the baryon must be totally symmetric under simultaneous permutations of position and spin-flavor indices. In the ground state of the baryon, the N_c-quark spatial wave function is given in the Hartree approximation by [2]:

$$\psi(x_1, \dots, x_N) = \prod_{i=1}^{N} \phi(x_i). \tag{17}$$

For a single baryon, the baryon mass $\mathring{M}_B = \langle\psi|H|\psi\rangle \equiv \langle H\rangle_\psi$ is given by:

$$\begin{aligned} \mathring{M}_B &= N_c m_Q + N_c \int d^3x \frac{1}{2m_Q}|\nabla\phi(x)|^2 \\ &+ \frac{N_c(N_c-1)}{2}\int d^3x d^3x' |\phi(x')|^2|\phi(x)|^2 V_{QQ}(x-x'). \end{aligned} \tag{18}$$

The large N_c scaling becomes obvious after the relation, Eq. (14) is used. It is useful to define the effective mean field potential $\bar{V}(x)$ generated by $N_c - 1$ quarks

$$\begin{aligned} \bar{V}(x) &= (N_c-1)\int d^3x' V_{QQ}(x-x')|\phi(x')|^2 \\ &= \int d^3x' V_{Q\bar{Q}}(x-x')|\phi(x')|^2, \end{aligned} \tag{19}$$

where the Casimir scaling assumption provided by Eq. (16) has been used. The mean field potential is the self-energy of a quark within the hadron which sees the remaining $N_c - 1$ quarks (which are coupled into the anti-fundamental representation $\bar{F}$).

The mean field equations are then obtained by minimizing with respect to a normalized $\phi(x)$ leading to the eigenvalue problem:

$$-\frac{1}{2m_Q}\nabla^2\phi(x) + \bar{V}(x)\phi(x) = \epsilon\phi(x). \tag{20}$$

3.2. Numerical and variational solution

The mean field equations Eqs. (20) and (19) can be solved by iterations until self-consistency solution is obtained. Actually, for the case $\sigma = 0$ the system can be written as a coupled Schrödinger–Newton equation, which was already solved in Ref. [27]. A Gaussian ansatz of the form

$$\phi(r) = \left(\frac{2}{\pi b^2}\right)^{\frac{3}{4}} e^{-r^2/b^2} \tag{21}$$

yields a good approximation to this solution and allows for a simple analytical discussion.[1]

3.3. CM corrections and mass formula

One standard and well documented problem of the mean field approximation in nuclear physics is the violation of Galilean invariance [15,14] which is a symmetry of the starting Hamiltonian, Eq. (5), namely the invariance under the boost operation with velocity v, $\Psi(x_1,\ldots,x_N) \to e^{im_Q \mathrm{v}\cdot\sum_i x_i}\Psi(x_1,\ldots,x_N)$, which implies the energy of the moving system to be given by $E(P) = M + P^2/2m_Q N_c$ where the rest mass differs from the inertial mass $M \neq N_c m_Q$.

Since the interest here is to include $1/N_c$ corrections in the calculation, it is important to build a wave function that is an eigenfunction of the momentum. This is achieved by implementing, e.g., the Peierls–Yoccoz projection method [15,14].[2] However, for the simple Gaussian single particle wave function, Eq. (21), this corresponds just to replace $N_c \to N_c - 1$ in the kinetic energy contribution. Thus, the projection becomes trivial to deal with, and one obtains for a moving baryon of momentum P:

$$\mathring{M}_B = N_c m_Q + \frac{P^2}{2m_Q N_c} + \frac{3(N_c-1)}{2b^2 m_Q} + \frac{N_c}{b\sqrt{\pi}}\left(-\lambda^2 + b^2\sigma\right), \tag{22}$$

where $\lambda^2 = \pi/12$. Minimizing with respect to b (b_0) yields the baryon mass at rest. At large m_Q, b_0 and the baryon mass become:

$$\begin{aligned} b_0 &= \frac{3\sqrt{\pi}}{\lambda^2 m_Q}\frac{N_c-1}{N_c}\left(1 - 9\pi\left(\frac{N_c-1}{N_c}\right)^2 \frac{\sigma}{\lambda^6 m_Q^2}\right) \\ &\quad + \mathcal{O}\left(1/m_Q^5\right) \\ \mathring{M}_B &= N_c m_Q + \frac{P^2}{2m_Q N_c} \\ &\quad - (N_c-1)\left(\left(\frac{N_c}{N_c-1}\right)^2 \frac{\lambda^4 m_Q}{6\pi} - \frac{3\sigma}{\lambda^2 m_Q}\right) \\ &\quad + \mathcal{O}\left(1/m_Q^2\right), \end{aligned} \tag{23}$$

which shows a delayed onset of the heavy quark regime due to large numerical factors, namely in b_0 the factor $9\pi(\frac{N_c-1}{N_c})^2\frac{1}{\lambda^6}\frac{\sigma}{m_Q^2} = 1575(\frac{N_c-1}{N_c})^2\frac{\sigma}{m_Q^2}$ and in $\mathring{M}_B$ the factor $\frac{3\sigma}{\lambda^2 m_Q} = 11.46\frac{\sigma}{m_Q}$. Thus, one should expect relativity to play a role even for moderately heavy quarks.

3.4. Relativistic corrections

Of course, a full relativistic treatment implies particle creation as implied by locality, and Poincaré invariant Hamiltonian methods with a fixed number of particles exhibit well known features (see e.g. Ref. [28] and included references). While this can be improved, here only an estimate of the relativistic corrections is considered by the standard replacement at the single particle level, $m_Q + p_i^2/2m_Q \to \sqrt{p_i^2 + m_Q^2}$, which leads remarkably to an analytical expression for the zero momentum projected variational energy

$$\begin{aligned} \mathring{M}_B^{\mathrm{rel}} &= \frac{1}{\sqrt{2\pi}\, b}\Bigg(\frac{b^2 m_Q^2\, {N_c}^{3/2} e^{\frac{b^2 m_Q^2 N_c}{4(N_c-1)}} K_1\left(\frac{b^2 m_Q^2 N_c}{4(N_c-1)}\right)}{\sqrt{N_c-1}} \\ &\quad + \sqrt{2}\, N_c\left(-\lambda^2 + b^2\sigma\right)\Bigg), \end{aligned} \tag{24}$$

which reproduces from the simple non-relativistic CM rule $N_c \to N_c - 1$ in the kinetic energy in the heavy quark limit.[3] The scheme as in the mean field case of minimizing with respect to the oscillator parameter b yields the final baryon mass at any N_c.[4] This case will be used in order to compare with the LQCD results in Refs. [7, 8], where the largest quark masses used are still not in the heavy regime.

3.5. Ground state correlations

As expected Eqs. (20) and (19) are N_c independent and correspond to the leading order approximation. These equations have corrections corresponding to different physical effects. Within the Gaussian ansatz for the single particle states Eq. (21) a Harmonic oscillator shell model interpretation applies since the baryon is in a $(1s)^{N_c}$ state. In this picture, ground state correlations correspond to virtual excitations to higher shell states $(n_1 l_1)\ldots(n_{N_c} l_{N_c})$.

In order to quantify the accuracy of the Hartree approximation within the large N_c framework, one evaluates the variance of the Hamiltonian defined by $\Delta H_\psi^2 = \langle H^2\rangle - \langle H\rangle^2$ where $\langle O\rangle \equiv \langle\psi|O|\psi\rangle$. When solving the equation approximately, as it is done here using a variational wave function, it turns out that $\Delta H_{\mathrm{var}}/\langle H\rangle = \mathcal{O}\left(1/\sqrt{N_c}\right)$ typical of statistical fluctuations. Straightforward calculation, explicitly using the mean field equation Eq. (20), shows that[5]

$$\Delta H_\psi^2 = \frac{N_c(N_c-1)}{2}\left[\langle V_{QQ'}\rangle^2 + \langle V_{QQ'}^2\rangle - 2\langle V_{QQ'}V_{Q'Q''}\rangle\right]. \tag{25}$$

Only when the self-consistent Hartree mean field equation is exactly satisfied and due to the Casimir scaling assumption, Eq. (16), one has $\Delta H_{\mathrm{Hartree}} = \mathcal{O}\left(N_c^0\right)$, which means $\Delta H_\psi/\langle H\rangle = \mathcal{O}(1/N_c)$ for the correction relative to the baryon mass.

3.6. Multiquark interactions

In general, there are multiquark interactions which contribute to the baryon mass at the nominal leading $\mathcal{O}(N_c)$. For heavy

[1] In the $\sigma = 0$ case one has $\mathring{M}_B - 3m_Q = -0.00034\alpha_s^2 m_Q$ [27] vs $\mathring{M}_B - 3m_Q = -0.00031\alpha_s^2 m_Q$ from Eq. (21). For the case $\sigma \neq 0$ more sophisticated ansätze were tried embodying better short and long distance behaviors, but improvement is at the per cent level since the quarks are located in the mid-range region. Discussion of several possibilities will be given elsewhere.

[2] Semiclassical collective quantization methods provide an alternative after due attention to zero modes is paid [15,14].

[3] Note that here one projects and does not boost the mean field solution. In the relativistic case the rest and inertial masses ought to coincide due to Poincaré invariance. The necessary identity between boosting and projecting onto linear momentum only holds for exact solutions [29]. At the mean field level the identity is guaranteed at the mean field solution [30].

[4] Note that the direct extrapolation of Eq. (24) to light quarks $m_q \to 0$ leads to the rest mass $\mathring{M}_B/(N_c\sqrt{\sigma}) = 1.81 - 0.50/N_c - 0.19/N_c^2 + \ldots$, which is the crude estimate for the multiplet center in the quenched approximation.

[5] Here the notation corresponds to

$$\langle V_{QQ'}\rangle \equiv \int d^3x\, d^3y\, V_{QQ}(x-y)|\phi(x)|^2|\phi(y)|^2$$

$$\langle V_{QQ'}V_{Q'Q''}\rangle \equiv \int d^3x\, d^3y\, d^3z\, V_{QQ}(x-y)V_{QQ}(y-z)|\phi(x)|^2|\phi(y)|^2|\phi(z)|^2$$

quarks one expects that in the baryon only n-body interactions with $n \leq N_c$ are of any significance. For $N_c = 3$ there is a long history of studying the 3-quark interactions, where there are two competing alternatives to confining forces of quarks in baryons, the Δ (pairwise triangle shape) and the Y (junction shape) inspired by string models [31].

Three body interactions have been addressed perturbatively [32] for arbitrary N_c. In the present case, the non-perturbative effect of 3-body interactions can be visualized with one example. Consider a 3-body potential of the form:

$$V_3(x_1, x_2, x_3) = \sum_{i=1}^{3} \mathrm{v}_3(x_i - X)\, d_{abc}\, \lambda^a \otimes \lambda^b \otimes \lambda^c, \tag{26}$$

where X is the CM position of the three quarks. The expectation value of V_3 in the baryon ground state at rest can be evaluated explicitly choosing $\mathrm{v}_3(r) = \frac{1}{N_c^2}\left(-\frac{\lambda_3^2}{r} + \sigma_3 r\right)$ where λ_3 and σ_3 are $\mathcal{O}\left(N_c^0\right)$, one obtains for the Gaussian wave function:

$$\langle V_3 \rangle = 2\sqrt{\frac{3}{\pi}}\left(N_c - \frac{5}{N_c} + \frac{4}{N_c^3}\right)\left(-\frac{\lambda_3^2}{b} + \sigma_3 b\right), \tag{27}$$

where the color matrix element for the baryon was used,

$$\langle d_{abc}\, \lambda^a \otimes \lambda^b \otimes \lambda^c \rangle = 4\frac{(N_c - 3)!}{N_c!}(N_c^3 - 5N_c + \frac{4}{N_c}). \tag{28}$$

Note that the expectation value of the 2-body interaction Eq. (22) and the one of the 3-body interaction studied here have the same form except that their N_c scalings differ by terms which are of relative order $1/N_c^2$. Therefore, the 3-body forces cannot be distinguished from the 2-body ones unless those higher order terms in the expansion are taken into account. This is in a sense direct consequence of the mean field approximation, which naturally "hides" the n-body nature of the interactions. Other n-body forces are in principle possible for a large N_c baryon, whose color structure is given by $1/N_c^{n-1}\, d_{a_1 \cdots a_n} \lambda^{a_1} \otimes \cdots \otimes \lambda^{a_n}$, where $d_{a_1 \cdots a_n}$ is the rank n invariant symmetric tensor of $SU(N_c)$. A simple calculation shows that they contribute to the baryon mass with an overall factor $N_c/n!$, which implies that even for very large N_c, n-body forces with $n > 5$ become very suppressed.

3.7. Hyperfine effects

The simple OGE potential contains hyperfine components $\mathcal{O}(m_Q^{-2})$, which have implications on meson spectra (see e.g. Ref. [33]), as they contribute at $\mathcal{O}\left(N_c^0\right)$ in mesons, but contribute to hyperfine splitting in baryons only at $\mathcal{O}(1/N_c)$. They can be easily evaluated as perturbations using the wave function obtained here. A quick calculation generalizing the $N_c = 3$ result [34] to arbitrary N_c gives for the hyperfine mass shifts:

$$\delta M_B^{HF}(S) = \frac{8}{3\sqrt{\pi}}\frac{\alpha_s^{N_c}(m_Q)}{m_Q^2 b^3}(S(S+1) - \frac{3}{4}N_c). \tag{29}$$

They play no role for the spin-weighted average baryon mass Eq. (2).

4. Towards relating to LQCD results

Following the motivation of this work, the aim here is to compare the mean field description including relativistic and CM corrections with results from LQCD. At present, the only available LQCD results for ground state baryon masses at several N_c values are those of Refs. [7,8] (slightly updated in Ref. [18]), where

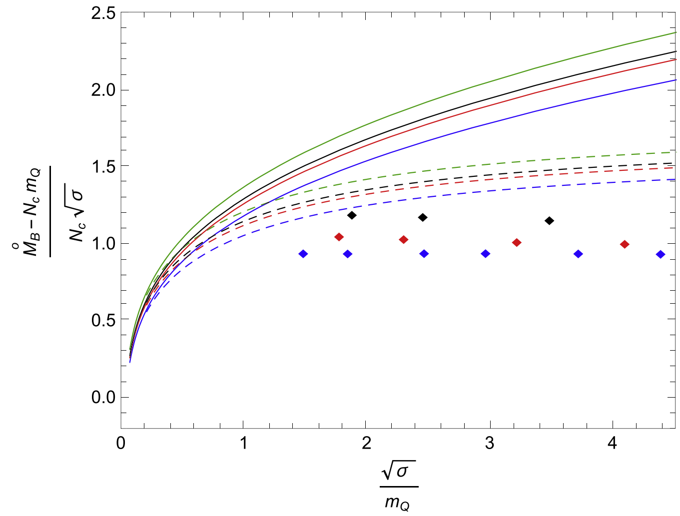

Fig. 2. Baryon mass as a function of the string tension for Gaussian wave function. Depicted are the results for non-relativistic (full curves) and relativistic (dashed) calculations, and the lattice QCD results for m_0 defined by Eq. (2) (diamonds) [18]. The color coding is that of Fig. 1, and in green the limit $N_c \to \infty$. The string tension corresponding to the lattice QCD results was obtained as explained in the text. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

quenched calculations have been undertaken at several values of the quark mass and at $N_c = 3,\ 5$, and 7. While the purpose there was to pursue the light quark limit, here the opposite situation is emphasized where simplifications are expected and the quenched approximation is better fulfilled.

As discussed earlier, the explicit N_c dependence is inferred from taking σ to be N_c independent. The lattice results displayed in Refs. [7,8,18] are given in lattice units, with a the lattice spacing. Using the form of the quark–quark potential the Sommer parameter r_1 is determined by the standard definition

$$-r_1^2 V'_{Q\bar{Q}}(r_1) = -1\,, \tag{30}$$

yielding in the present case

$$r_1^2 \sigma = 1 - \frac{\pi}{12}\,. \tag{31}$$

This value, namely $r_1\sqrt{\sigma} = 0.859$, is roughly valid for the LQCD calculations with $N_c = 3,\ 5$ and 7, where the respective results from Table I of Ref. [7] are 0.856(5), 0.850(4) and 0.845(2). Using the values of r_1/a in the same Table one obtains respectively $\sqrt{\sigma}\, a = 0.219(2)$, 0.225(2), and 0.216(1). For the level of precision of the present comparison it is therefore sufficient to take $\sqrt{\sigma}\, a = 0.22$ for all N_c. While the main goal of [7,8] was to pursue the lowest quark mass limit, some moderately high quark masses were included. These are now used to compare with the results of this work.

The numerical results are presented in Fig. 2. As expected, the relativistic effects start becoming significant for $2\sqrt{\sigma} \sim m_Q$. The lattice data of Refs. [7,8] stop at twice larger values, so it would be highly interesting to extend the lattice calculations to the non-relativistic regime, where the comparison of the approach used and LQCD becomes more realistic. Qualitatively, it seems that there is a trend of the model and the LQCD results towards some agreement. The LQCD qualitative feature that $\mathring{M}_B(N_c)/N_c$ increases with N_c is also shown by the model, although it is not in good quantitative agreement.

At this point it is important to mention the issues involved in comparing the model with LQCD results. The main obstacle

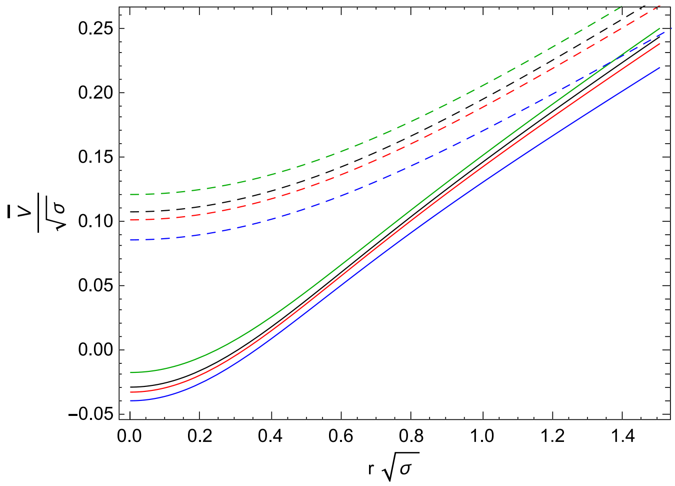

Fig. 3. Effective potential for a heavy quark in a heavy baryon with $m_Q = 2\sqrt{\sigma}$ (dashed) and $10\sqrt{\sigma}$ (full). The same color coding as in Fig. 2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

pertains to the quark mass m_Q. While in the model it is a parameter (identifiable with a constituent quark mass), in LQCD it is a genuine QCD parameter which depends on the renormalization scheme [35,36]. Thus, the comparison made in Fig. 2 is only qualitative as it assumes that the masses are identical. For a more rigorous comparison one could proceed in different ways. Perhaps the best one would be to consider the spectrum of baryons for each N_c, adjusting the relation between the model and LQCD quark mass to best fit the LQCD results. While this would give a very realistic comparison, it seems unlikely that the spectrum of excited baryons at $N_c > 3$ could be calculated in LQCD in the foreseeable future. An intermediate approach is to compare the differences $(\mathring{M}_B(N_c)/N_c - \mathring{M}_B(N_c')/N_c')$, also adjusting the masses to a best fit (as mentioned earlier, it can be done with the present available LQCD results, but due to the rather small m_Q values of those results it is still unrealistic to compare). A rigorous approach along the lines discussed here would entail the use of an effective non-relativistic QCD theory for the heavy baryon [37], similar to the one for heavy quarkonium [38], where in principle it is possible to relate within QCD the quark mass of the effective theory to that of LQCD. Finally, an approach with immediate physical meaning would be to write the heavy baryon masses as a function of the corresponding quarkonium masses avoiding in this way having to relate quark masses and the use of the string tension as a fundamental parameter. Unfortunately, there are no direct LQCD evaluations of quarkonium masses; their masses are calculated via the use of the LQCD determined potential, similarly to what has been done in the present work for baryons.

The mean field approximation is visualized through the mean field potential $\bar{V}(r)$ created by the $N_c - 1$ quarks, see Eq. (19). In the present case, for zero momentum states and the Gaussian profile, Eq. (21) one obtains:

$$\bar{V}(r) = \frac{b_0\,\sigma}{\sqrt{2\pi}} e^{-\frac{2r^2}{b_0{}^2}} - \left(\lambda^2 - \sigma\left(\frac{b_0^2}{4} + r^2\right)\right)\frac{1}{r}\,\mathrm{erf}\left(\frac{\sqrt{2}r}{b_0}\right), \tag{32}$$

which is shown for illustration, in Fig. 3 for different values of N_c and m_Q. Improvements to this behavior correct for long distance behavior and will be discussed in a forthcoming publication.

5. Conclusions

In the present work, a scheme is put forward where the large N_c expansion of baryon masses in the lattice can be described in terms of the mean field approximation as originally advocated by Witten and $1/N_c$ corrections thereof. The quark–quark potential is assumed to follow Casimir scaling at arbitrary N_c and hence proportional to the quark–antiquark potential, which to good accuracy as per current LQCD calculations is N_c-independent. This provides a universal N_c independent scheme where the ratio of the baryon mass to $N_c\sqrt{\sigma}$ can be numerically evaluated.

It was shown that the corrections to the mean field energy are generically $\mathcal{O}(\sqrt{N_c})$, but become $\mathcal{O}(N_c^0)$, when the mean field energy takes its minimum value. This accuracy is the result of the density of quarks in the baryon growing as proportional with N_c. Among the estimated corrections are the leading in N_c relativistic $\mathcal{O}(m_Q^{-3})$ and subleading $\mathcal{O}(N_c^0)$ CM corrections. Hyperfine splittings are removed by suitably averaging over spin states. When compared with available LQCD calculations, the present results account within 20% for the dimensionless ratio $(\mathring{M}_B - N_c m_Q)/(N_c\sqrt{\sigma})$ which is of natural size. This is encouraging, as it suggests to push the LQCD calculations to heavier quark masses and also to refine the calculations in the present work. The comparison undertaken here is so far just qualitative, as discussed in Section 4, due to the ambiguities in matching the model calculations to LQCD. More progress in this regard is needed in order to draw more rigorous comparisons.

One of the obvious benefits of the present investigation is the possibility of going beyond the ground state and extend these ideas to the excited baryon spectrum, where lattice calculations are admittedly more involved and less accurate. LQCD calculations of excited baryons for $N_c > 3$ may still be an unreachable goal. However, it is likely that this will be achieved first with heavy quarks, and in that case the approach followed here can be easily used to make predictions of excited states. Finally, other heavy baryon properties, such as form factors, are easily derived with the wave functions obtained here.

Acknowledgements

Useful discussions and correspondence with Thomas DeGrand, Marco Panero and Joan Soto are greatly appreciated. This work was supported in part by DOE Contract No. DE-AC05-06OR23177 under which JSA operates the Thomas Jefferson National Accelerator Facility (J.L.G.), by the National Science Foundation through grant PHY-1307413 (I.P.F. and J.L.G.) and the Spanish Mineco (grant FIS2014-59386-P) and Junta de Andalucía (grant FQM225) (C.A.T. and E.R.A.). C.A.T. acknowledges a contract from the CPAN.

References

[1] G. 't Hooft, A planar diagram theory for strong interactions, Nucl. Phys. B 72 (1974) 461, http://dx.doi.org/10.1016/0550-3213(74)90154-0.
[2] E. Witten, Baryons in the 1/n expansion, Nucl. Phys. B 160 (1979) 57, http://dx.doi.org/10.1016/0550-3213(79)90232-3.
[3] B. Lucini, M. Panero, SU(N) gauge theories at large N, Phys. Rep. 526 (2013) 93–163, http://dx.doi.org/10.1016/j.physrep.2013.01.001, arXiv:1210.4997.
[4] B. Lucini, M. Panero, Introductory lectures to large-N QCD phenomenology and lattice results, Prog. Part. Nucl. Phys. 75 (2014) 1–40, http://dx.doi.org/10.1016/j.ppnp.2014.01.001, arXiv:1309.3638.
[5] G.S. Bali, F. Bursa, L. Castagnini, S. Collins, L. Del Debbio, et al., Mesons in large-N QCD, J. High Energy Phys. 1306 (2013) 071, http://dx.doi.org/10.1007/JHEP06(2013)071, arXiv:1304.4437.
[6] T. Ledwig, J. Nieves, A. Pich, E. Ruiz Arriola, J. Ruiz de Elvira, Large-N_c naturalness in coupled-channel meson–meson scattering, Phys. Rev. D 90 (11) (2014) 114020, http://dx.doi.org/10.1103/PhysRevD.90.114020, arXiv:1407.3750.
[7] T. DeGrand, Lattice baryons in the 1/N expansion, Phys. Rev. D 86 (2012) 034508, http://dx.doi.org/10.1103/PhysRevD.86.034508, arXiv:1205.0235.

[8] T. DeGrand, Lattice calculations of the spectroscopy of baryons with broken flavor SU(3) symmetry and 3, 5, or 7 colors, Phys. Rev. D 89 (1) (2014) 014506, http://dx.doi.org/10.1103/PhysRevD.89.014506, arXiv:1308.4114.

[9] J.L. Goity, $1/N_c$ countings in baryons, Phys. At. Nucl. 68 (2005) 624–633, http://dx.doi.org/10.1134/1.1903092, arXiv:hep-ph/0405304.

[10] F.J. Llanes-Estrada, O.I. Pavlova, R. Williams, A first estimate of triply heavy baryon masses from the pNRQCD perturbative static potential, Eur. Phys. J. C 72 (2012) 2019, http://dx.doi.org/10.1140/epjc/s10052-012-2019-9, arXiv:1111.7087.

[11] J. Vijande, A. Valcarce, H. Garcilazo, Heavy-baryon quark model picture from lattice QCD, Phys. Rev. D 90 (9) (2014) 094004, http://dx.doi.org/10.1103/PhysRevD.90.094004.

[12] R.F. Dashen, A.V. Manohar, Baryon-pion couplings from large N(c) QCD, Phys. Lett. B 315 (1993) 425–430, http://dx.doi.org/10.1016/0370-2693(93)91635-Z, arXiv:hep-ph/9307241.

[13] R.F. Dashen, A.V. Manohar, 1/N(c) corrections to the baryon axial currents in QCD, Phys. Lett. B 315 (1993) 438–440, http://dx.doi.org/10.1016/0370-2693(93)91637-3, arXiv:hep-ph/9307242.

[14] P. Ring, P. Schuck, The Nuclear Many-Body Problem, Springer, 2004.

[15] J.-P. Blaizot, G. Ripka, Quantum Theory of Finite Systems, vol. 3, MIT Press, Cambridge, 1986.

[16] T.D. Cohen, N. Kumar, K.K. Ndousse, Baryons and baryonic matter in the large Nc and heavy quark limits, Phys. Rev. C 84 (2011) 015204, http://dx.doi.org/10.1103/PhysRevC.84.015204, arXiv:1102.2197.

[17] P. Adhikari, T.D. Cohen, Mean-field theory of baryonic matter for QCD in the large N_c and heavy quark mass limits, Phys. Rev. C 88 (5) (2013) 055202, http://dx.doi.org/10.1103/PhysRevC.88.055202, arXiv:1307.7725.

[18] A.C. Cordón, T. DeGrand, J. Goity, N_c dependencies of baryon masses: analysis with lattice QCD and effective theory, Phys. Rev. D 90 (1) (2014) 014505, http://dx.doi.org/10.1103/PhysRevD.90.014505, arXiv:1404.2301.

[19] S. Meinel, Prediction of the Ω_{bbb} mass from lattice QCD, Phys. Rev. D 82 (2010) 114514, http://dx.doi.org/10.1103/PhysRevD.82.114514, arXiv:1008.3154.

[20] J. Flynn, E. Hernandez, J. Nieves, Triply heavy baryons and heavy quark spin symmetry, Phys. Rev. D 85 (2012) 014012, http://dx.doi.org/10.1103/PhysRevD.85.014012, arXiv:1110.2962.

[21] A. Laschka, N. Kaiser, W. Weise, Charmonium potentials: matching perturbative and lattice QCD, Phys. Lett. B 715 (2012) 190–193, http://dx.doi.org/10.1016/j.physletb.2012.07.049, arXiv:1205.3390.

[22] C. Anzai, Y. Kiyo, Y. Sumino, Violation of Casimir scaling for static QCD potential at three-loop order, Nucl. Phys. B 838 (2010) 28–46, http://dx.doi.org/10.1016/j.nuclphysb.2010.05.012, arXiv:1004.1562.

[23] G.S. Bali, Casimir scaling of SU(3) static potentials, Phys. Rev. D 62 (2000) 114503, http://dx.doi.org/10.1103/PhysRevD.62.114503, arXiv:hep-lat/0006022.

[24] S. Necco, R. Sommer, The $N_f = 0$ heavy quark potential from short to intermediate distances, Nucl. Phys. B 622 (2002) 328–346, http://dx.doi.org/10.1016/S0550-3213(01)00582-X, arXiv:hep-lat/0108008.

[25] M. Lüscher, P. Weisz, Quark confinement and the bosonic string, J. High Energy Phys. 0207 (2002) 049, http://dx.doi.org/10.1088/1126-6708/2002/07/049, arXiv:hep-lat/0207003.

[26] C. Allton, M. Teper, A. Trivini, On the running of the bare coupling in SU(N) lattice gauge theories, J. High Energy Phys. 0807 (2008) 021, http://dx.doi.org/10.1088/1126-6708/2008/07/021, arXiv:0803.1092.

[27] E. Ruiz Arriola, J. Soler, Asymptotic behaviour for the 3-d Schrödinger–Poisson system in the attractive case with positive energy, Appl. Math. Lett. 12 (8) (1999) 1–6.

[28] B. Keister, W. Polyzou, Relativistic Hamiltonian dynamics in nuclear and particle physics, Adv. Nucl. Phys. 20 (1991) 225–479.

[29] M. Betz, R. Goldflam, Boosting the bag, Phys. Rev. D 28 (1983) 2848, http://dx.doi.org/10.1103/PhysRevD.28.2848.

[30] P. Pobylitsa, E. Ruiz Arriola, T. Meissner, F. Grummer, K. Goeke, W. Broniowski, Pushing the Nambu–Jona–Lasinio soliton and the zero point energy, J. Phys. G 18 (1992) 1455–1466, http://dx.doi.org/10.1088/0954-3899/18/9/008.

[31] G.S. Bali, QCD forces and heavy quark bound states, Phys. Rep. 343 (2001) 1–136, http://dx.doi.org/10.1016/S0370-1573(00)00079-X, arXiv:hep-ph/0001312.

[32] N. Brambilla, J. Ghiglieri, A. Vairo, The Three-quark static potential in perturbation theory, Phys. Rev. D 81 (2010) 054031, http://dx.doi.org/10.1103/PhysRevD.81.054031, arXiv:0911.3541.

[33] J. Segovia, D. Entem, F. Fernandez, E. Ruiz Arriola, Renormalization approach to constituent quark models of quarkonium, Phys. Rev. D 85 (2012) 074001, http://dx.doi.org/10.1103/PhysRevD.85.074001, arXiv:1108.0208.

[34] N. Isgur, G. Karl, Ground state baryons in a quark model with hyperfine interactions, Phys. Rev. D 20 (1979) 1191–1194, http://dx.doi.org/10.1103/PhysRevD.20.1191.

[35] A. Pineda, The static potential: lattice versus perturbation theory in a renormalon based approach, J. Phys. G 29 (2003) 371–385, http://dx.doi.org/10.1088/0954-3899/29/2/313, arXiv:hep-ph/0208031.

[36] G.S. Bali, C. Bauer, A. Pineda, The static quark self-energy at $O(\alpha^2 0)$ in perturbation theory, PoS LATTICE2013 (2014) 371, arXiv:1311.0114.

[37] N. Brambilla, A. Vairo, T. Rosch, Effective field theory Lagrangians for baryons with two and three heavy quarks, Phys. Rev. D 72 (2005) 034021, http://dx.doi.org/10.1103/PhysRevD.72.034021, arXiv:hep-ph/0506065.

[38] N. Brambilla, A. Pineda, J. Soto, A. Vairo, Effective field theories for heavy quarkonium, Rev. Mod. Phys. 77 (2005) 1423, http://dx.doi.org/10.1103/RevModPhys.77.1423, arXiv:hep-ph/0410047.