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Research Article

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A family of singular functions and its relation to harmonic fractal analysis and fuzzy logic

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Abstract: We study a parameterized family of singular functions which appears in a paper by H. Okamoto and M. Wunsch (2007). Various properties are revisited from the viewpoint of fractal geometry and probabilistic techniques. Hausdorff dimensions are calculated for several sets related to these functions, and new properties close to fractal analysis and strong negations are explored.

Keywords: Singular function, Hausdorff dimension, Katok foliation, Strong negation

MSC: 26A30, 26A24, 26A48, 11A67

1 Introduction and preliminaries

Examples of singular functions, that is, monotone increasing continuous functions whose derivatives vanish almost everywhere, have been known since the end of the 19th century (see [1]). Since then, these functions have been studied from a wide variety of fields very distant from one another. Some classes of these functions have been considered in Probability Theory (see [2–4]) as well as in Number Theory, where what is known as Minkowski's question mark function is specially relevant (see [5–8]). It relates to the alternate dyadic and continuous fraction systems of representation. Another example of singular function that relates to representation number systems can be found in [9]. Possibly the best known and most widely studied singular function is Cantor's (see [10] and the references therein) which can be studied with the aid of the 2- or 3-base representation systems, although it is often geometrically built as the limit of a sequence of functions with polygonal graphs. This is also the case of the functions firstly studied by Cahen [11], which Salem [7] introduced using geometric ideas similar to those of Cantor. Other references related to these functions can be found in [7, 12–16].

In recent times, a parameterized family of continuous functions has been considered by Okamoto in [17], and revisited in [18] to see if they are also singular. They contain Bourbaki and Perkins' nowhere differentiable functions as well as Cantor's singular function. In this paper, we study a wide family of two-parametric singular functions $f_{a,b}$ and explore new properties, several of them closely related to fractal analysis and strong negations. We borrow a few ideas from [18, Sec.4], starting with an exact definition of what we mean by $f_{a,b}$.

Definition 1.1. *Let* $a, b \in\,]0,1[$ *and* $f_{a,b}$ *be defined on the unit interval* $[0,1]$ *by iterations of piecewise affine functions* $\{f_n\}_{n=0}^{\infty}$, *as follows:*

Let $f_0(x) = x$, *and suppose that* f_n *has been properly defined on the whole unit interval. Then, for* f_{n+1} *we define*

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$$\begin{cases} f_{n+1}\left(\frac{k}{3^n}\right) := f_n\left(\frac{k}{3^n}\right), & k = 0, \ldots, 3^n, \\ f_{n+1}\left(\frac{3k+1}{3^{n+1}}\right) := (1-a)\, f_n\left(\frac{k}{3^n}\right) + a f_n\left(\frac{k+1}{3^n}\right), & k = 0, \ldots, 3^n - 1, \\ f_{n+1}\left(\frac{3k+2}{3^{n+1}}\right) := (1-b)\, f_n\left(\frac{k}{3^n}\right) + b f_n\left(\frac{k+1}{3^n}\right), & k = 0, \ldots, 3^n - 1, \end{cases}$$

and complete its definition for each $x \in \left[\frac{k}{3^n}, \frac{k+1}{3^n}\right]$ *as the segment that joints the points* $\left(\frac{k}{3^n}, f\left(\frac{k}{3^n}\right)\right)$ *and* $\left(\frac{k+1}{3^n}, f\left(\frac{k+1}{3^n}\right)\right)$ *in its graph.*

Now, let $f_{a,b}(x) = \lim_{n\to\infty} f_n(x)$ *for all* $x \in [0, 1]$.

Let us mention that $f_{\frac{1}{2},\frac{1}{2}}$ is the Cantor function and $f_{\frac{2}{3},\frac{1}{3}}$ is the function defined by Bourbaki (see [17, 19]). We will consider $0 < a < b < 1$ such that $(a, b) \neq (\frac{1}{3}, \frac{2}{3})$.

Theorem 1.2 ([18, Th.5]). *$f_{a,b}$ is a continuous, strictly monotone, and singular function.*

This paper is structured in the following way: In Section 2 we show new properties of $f_{a,b}$, from the viewpoint of fractal geometry theory. We prove that the graph of $f_{a,b}$ is a compact set (an attractor) that appears as the fixed point of a suitable contraction mapping, which directly implies its monotony and continuity. Moreover, we characterize $f_{a,b}$ as the unique bounded function satisfying a given system of functional equations. We will provide further proof of the singularity of $f_{a,b}$ with the sole aid of probabilistic techniques, using the result that the sequence of Fourier coefficients of its associated measure $\mathrm{d}f_{a,b}$ does not converge to zero (for a given monotone function S, $\mathrm{d}S$ denotes its Stieltjes measure). In the same section, we also establish the Hausdorff dimension of sets related to $f_{a,b}$, one of them is a measure zero set whose image by $f_{a,b}$ has measure one, and its dimension is obtained as an application of the Besicovitch-Eggleston theorem. To this end, we previously introduced a representation system called a, b-system. In Section 3 we generalize all the results to a 4-parametric family of functions. Once more, we calculate Hausdorff dimensions for the sets associated with these functions again. The last section is devoted to applications of the family of singular functions studied. With these and the representation system referred to above, we find a bi-parametric family of Katok foliations. In addition, we establish its relation to harmonic analysis on fractals and to strong negations in fuzzy logic.

2 Properties of $f_{a,b}$

2.1 Analytic properties

First, as we pointed out above, we examine new properties of $f_{a,b}$ from a geometric point of view. But the tool we shall use is the Hausdorff-Pompeiu metric (see [20, 21]). A straightforward way to introduce this metric is as follows:

Let $\mathcal{K}\left([0,1]^2\right)$ denote the space of compact subsets in $[0, 1]^2$. For each A and B in $\mathcal{K}\left([0,1]^2\right)$ we put

$$\rho(A, B) := \sup\{\mathrm{dist}(b, A) : b \in B\},$$

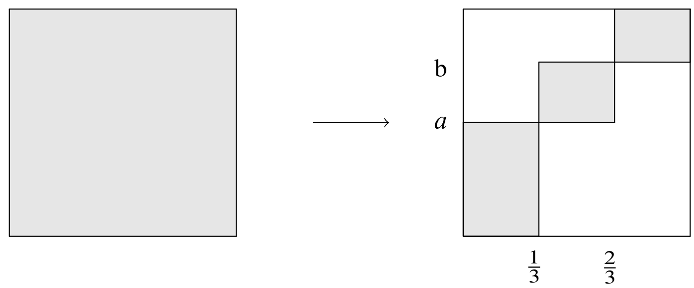
and then $\mathcal{K}\left([0,1]^2\right)$ is complete metric space endowed with the metric

$$D(A, B) := \max\{\rho(A, B), \rho(B, A)\}.$$

Thus, the graph of $f_{a,b}$ is a self-affine subset of the unit square $[0, 1]^2$, obtained as the fixed point of a suitable contraction C in the space $\mathcal{K}\left([0,1]^2\right)$, via the Contaction Mapping Theorem.

The underlying idea is graphically expressed in Fig. 1.

Fig. 1. The transform of the unit square by C



The graph of the polygonal f_n is defined by iteration as the image, by a contraction C that is defined below, of the graph of f_{n-1} with the start $f_0 =$ identity.

For convenience, let us put $a_1 = a;\ a_2 = b - a;\ a_3 = 1 - b$.

Proposition 2.1. *The mapping*

$$C : \mathcal{K}\left([0,1]^2\right) \longrightarrow \mathcal{K}\left([0,1]^2\right)$$
$$C(T) := C_1 t\,(T) \cup C_2\,(T) \cup C_3\,(T)\,,$$

where C_1, C_2 and C_3 are contractions given by

$$\begin{array}{ll} C_1 : [0,1]^2 \longrightarrow [0,1]^2\,, & C_1\,(x,y) = \left(\frac{x}{3}, ay\right) \\ C_2 : [0,1]^2 \longrightarrow [0,1]^2\,, & C_2\,(x,y) = \left(\frac{1+x}{3}, a + a_2 y\right) \\ C_3 : [0,1]^2 \longrightarrow [0,1]^2\,, & C_3\,(x,y) = \left(\frac{2+x}{3}, b + a_3 y\right), \end{array} \tag{1}$$

$(a, b, a_2, a_3 \in\,]0,1[)$ is a contraction for the Hausdorff metric in $\mathcal{K}\left([0,1]^2\right)$.

The graph of $f_{a,b}$ is the unique invariant set for the iterated function system defined by (1), that is the fixed point of C. The next result follows as a consequence of the definition of $f_{a,b}$ and its graph is a compact set.

Corollary 2.2. *The function $f_{a,b}$ is monotonic and continuous.*

Now, since each affine contraction represents a functional equation, according to the Banach fixed point theorem, we have $f_{a,b}$ determined by the following functional equations.

Theorem 2.3. *The function $f_{a,b}$ is the unique bounded solution of the system of functional equations*

$$\begin{cases} h\left(\frac{x}{3}\right) = ah(x) \\ h\left(\frac{1+x}{3}\right) = a + a_2 h(x) \\ h\left(\frac{2+x}{3}\right) = b + a_3 h(x) \end{cases} \tag{2}$$

Corollary 2.4. *The area under the graph of $f_{a,b}$ is $\int_0^1 f_{a,b}(x)\mathrm{d}x = \frac{2a_1+a_2}{2}$.*

In [22], for Okamoto's one-parameterized function it is proved that $f_a = f_{a,1-a}$ is non-differentiable almost everywhere at the critical parameter value. In [19], the performance of f_a' is explored almost everywhere for different values of a. For $f_{a,b}$, we have the following general result.

Theorem 2.5. *The derivative $f'_{a,b}(x)$, when it exists, can only vanish.*

Proof. If $f'_{a,b}(0^+)$ exists, then it must vanish. To prove it, we take into account that $f_{a,b}\left(\frac{1}{3^n}\right) = a^n$ and $f_{a,b}\left(\frac{2}{3^n}\right) = ba^{n-1}$. This implies that the slope of the segments joining the point $(0,0)$ with $\left(\frac{1}{3^n}, a^n\right)$, $\left(\frac{2}{3^n}, ba^{n-1}\right)$

and $\left(\frac{1}{3^{n-1}}, a^{n-1}\right)$ are, repectively, $(3a)^n$, $\frac{3b(3a)^{n-1}}{2}$ and $(3a)^{n-1}$. If $f'_{a,b}(0^+)$ exists and does not vanish, then the limits of the quotients $\frac{(3a)^n}{(3a)^{n-1}} = 3a$ and $\frac{3b(3a)^{n-1}}{2(3a)^{n-1}} = \frac{3b}{2}$ exist and are equal to 1. That is, $(a,b) = \left(\frac{1}{3}, \frac{2}{3}\right)$ which is a contradiction.

A similar argument allows us to obtain $f'_{a,b}(1^-) = 0$, if it exists.

By repeting this reasoning or using the system of functional equations (2), it follows that if exist, the numbers $f'_{a,b}(\frac{m}{3^k}^-)$ and $f'_{a,b}(\frac{m}{3^k}^+)$ are zero.

Let us suppose that x cannot be expanded in the form $\frac{m}{3^k}$. If $f'_{a,b}(x)$ exists and does not vanish,

$$\lim_{k\to\infty} \frac{f_{a,b}(y_{k+1}) - f_{a,b}(x_{k+1})}{y_{k+1} - x_{k+1}} : \frac{f_{a,b}(y_k) - f_{a,b}(x_k)}{y_k - x_k} = 1$$

for sequences (x_k) and (y_k) that converge to x such that $x_k \le x < y_k$.

Let us take

$$x_k := \max\left\{\frac{j}{3^k} : j = 0, \ldots, 3^k \text{ and } \frac{j}{3^k} \le x\right\},$$

and

$$y_k := \min\left\{\frac{j}{3^k} : j = 0, \ldots, 3^k \text{ and } x < \frac{j}{3^k}\right\},$$

then we have

$$\frac{f_{a,b}(y_{k+1}) - f_{a,b}(x_{k+1})}{y_{k+1} - x_{k+1}} : \frac{f_{a,b}(y_k) - f_{a,b}(x_k)}{y_k - x_k} \in \{3a_i : i = 1, 2, 3\}.$$

Thus, if $3a_i \ne 1$ for the three values, this is a contradiction. If some of them equal 1, then the other two differ from 1, because we have avoided the case $a = 1/3$ and $b = 2/3$. Therefore, if we obtain $3a_i = 1$, we shall change that quotient by

$$\frac{f_{a,b}\left(y_k + \frac{1}{3^{k+1}}\right) - f_{a,b}(x_{k+1})}{y_k + \frac{1}{3^{k+1}} - x_{k+1}} : \frac{f_{a,b}(y_k) - f_{a,b}(x_k)}{y_k - x_k},$$

or by

$$\frac{f_{a,b}(y_k) - f_{a,b}\left(x_k - \frac{1}{3^{k+1}}\right)}{y_k - x_k - \frac{1}{3^{k+1}}} : \frac{f_{a,b}(y_{k+1}) - f_{a,b}(x_k)}{y_{k+1} - x_k}.$$

This quotient is $\frac{3(a_i + a_j)}{2}$, with $i \ne j$, and differs from 1.

For each k we have described the possibility of taking the quotient on a finite set (where 1 is not included). Consequently, the limit $f'_{a,b}(x)$ must be zero. □

Note that the theorem above provides a new proof for the singularity of $f_{a,b}$.

2.2 Singular functions as the convolution of distribution functions

We shall prove that $f_{a,b}$ can be obtained as an infinite convolution S of atomic probabilities, and that the sequence of Fourier coefficients of $\mathrm{d}S$ does not converge to zero. A number of definitions and results to be used later are recorded bellow.

If F_1 and F_2 are distribution functions, then the function $F(x) = \int_{\mathbb{R}} F_1(y-x)\mathrm{d}F_2(y)$ is called the convolution of the distribution functions F_1 and F_2. This is a new distribution function denoted as $F = F_1 * F_2$. The convolution $F_1 * F_2$ provides the distribution function of the sum of two independent random variables with distribution functions F_1 and F_2.

Let F be a distribution function, then its characteristic function $\widetilde{F}$ is the expected value of e^{ixt}, that is,

$$\widetilde{F}(t) = E\left[e^{ixt}\right] = \int_{-\infty}^{+\infty} e^{ixt}\mathrm{d}F(x),$$

the nth moment M_n of F is defined as $\int_{-\infty}^{+\infty} x^n \mathrm{d}F(x)$ y $\sigma^2 = M_2 - M_1^2$.

By imposing conditions on the distribution functions, Jessen and Wintner in their paper [3], obtain convergence criteria for the convolution of distribution functions. To be more specific, in [3] they obtain the following results.

Lemma 2.6 ([3, Th. 4]). *If $M_2(F_n)$ is finite for every n, then the convergence of two series $\sum_{n=1}^{\infty} E(F_n)$ and $\sum_{n=1}^{\infty} \sigma^2(F_n)$ implies the weak convergence of $H_n(x) = F_1 * F_2 * \cdots * F_n(x)$ (i.e. $H_n(x) \longrightarrow H(x)$ for each x at which H is continuous).*

Theorem 2.7 ([3, Th. 35]). *If $F = F_1 * F_2 * \cdots$ is a convergent infinite convolution of distribution functions F_n each of which is purely discontinuous, then F is either purely discontinuous, or singular, or absolutely continuous.*

The next result can be found in [23, pg. 46].

Theorem 2.8. *Let $\gamma_n = \max_z\{F_n(z+) - F_n(z-)\}$, then the infinite convolution $F(x) = (F_1 * F_2 * \cdots)(x)$ is continuous if and only if the series $\sum_{n\geq 1}(1-\gamma_n)$ diverges.*

Finally, in [26] the following useful result is proven.

Lemma 2.9. *If F_n and F have characteristic functions $\widetilde{F}_n$ and $\widetilde{F}$, respectively, and (F_n) is weakly convergent to F, then $\widetilde{F}_n(x) \to \widetilde{F}(x)$ for each x.*

Definition 2.10. *Let be the infinite convolution function $S := F_1 * F_2 * F_3 * \cdots$, where $F_1, F_2, F_3, \ldots$ are distribution functions given, for each positive integeer n, by*

$$F_n(x) = \begin{cases} 0, & x < 0 \\ a, & 0 \leq x < \frac{1}{3^n} \\ b, & \frac{1}{3^n} \leq x < \frac{2}{3^n} \\ 1, & x \geq \frac{2}{3^n} \end{cases}$$

This function S is well defined. In fact, let us note that, for every n:

$$E(F_n) = \frac{a_2 + 2a_3}{3^n},\ M_2(F_n) = \frac{a_2 + 4a_3}{3^{2n}},\ \sigma^2(F_n) = \frac{a_2 + 4a_3}{3^{2n}} - \frac{(a_2 + 2a_3)^2}{3^{2n}}.$$

The convergence of the corresponding series ensures, by Lemma 2.6, the weak convergence of the convolution, that is, $S(x)$ exists for every x.

Theorem 2.11. *With the notation as in the above definition, we have:*

a) *The characteristic function of S is $\widetilde{S}(t) = \prod_{n=1}^{\infty}\left(a_1 + a_2 e^{i\frac{t}{3^n}} + a_3 e^{i\frac{2t}{3^n}}\right)$.*
b) *The sequence of Fourier coefficients of $\widetilde{S}$, i.e. $S_p = \int_0^1 e^{2\pi p i x}\mathrm{d}S(x)$, does not converge to zero.*
c) *S is a singular function.*

Proof. a) The characteristic function of F_n is $a_1 + a_2 e^{i\frac{t}{3^n}} + a_3 e^{i\frac{2t}{3^n}}$. If $H = F * G$, then their respective characteristic functions satisfy the relation $\widetilde{H} = \widetilde{F}\widetilde{G}$, and as a consequence, by Lemma 2.6, we obtain that the characteristic of S is $\widetilde{S}(t) = \prod_{n=1}^{\infty}\left(a_1 + a_2 e^{i\frac{t}{3^n}} + a_3 e^{i\frac{2t}{3^n}}\right)$.

b) Let us denote by $S_p := \widetilde{S}(2\pi p)$ the Fourier coefficients of $\mathrm{d}S$. The expression of $\widetilde{S}$ allows us to obtain that $S_{3^q p} = S_p$ for all $q \geq 0$. Since the measure is not equal to the Lebesgue measure, then there exists $n > 0$ such that $S_p \neq 0$, and, as a consequence, the Fourier coefficients do not converge to zero.

c) The functions in the convolution are purely discontinuous, thus S is also pure. But it is neither discontinuous, by Theorem 2.8, nor absolutely continuous, because by b) it does not satisfy the Riemann-Lebesgue theorem. Therefore, it must be singular. □

For a nonnegative integer $m < 3^n$, let us consider its expansion in the base 3 $m = r_0 3^0 + r_1 3^1 + r_2 3^2 + \cdots + r_{n-1} 3^{n-1}$, with $r_i \in \{0, 1, 2\}$. Set $c(m)$ (resp., $u(m)$ and $d(m)$) the number of times that the digit 0 appears (resp., 1s and 2s) in the above expansion of m.

Lemma 2.12. *The distribution function $F_1 * F_2 * \cdots * F_n$ has the following associated probability*

$$P_n\left(\frac{m}{3^n}\right) = a_1^{c(m)} a_2^{u(m)} a_3^{d(m)}.$$

Proof. The proof is reached by induction. The statement is true for the first values. Let us suppose that it is true for $n-1$, and we will prove it for n. (Let us denote by $\overline{P_n}$ the associated probability with F_n.)

If m is congruent with zero module 3, i.e. $m = 3k$, then

$$k = r_1 3^0 + r_2 3^1 + r_3 3^2 + \cdots\cdots + r_{n-1} 3^{n-2},$$
$$m = 0 + r_1 3^1 + r_2 3^2 + r_3 3^3 + \cdots\cdots + r_{n-1} 3^{n-1},$$

and

$$P_n\left(\frac{m}{3^n}\right) = P_n\left(\frac{3k}{3^n}\right) = P_{n-1}\left(\frac{k}{3^{n-1}}\right) \cdot \overline{P_n}(0) = a_1 P_{n-1}\left(\frac{k}{3^{n-1}}\right).$$

Taking into account that the expansion of m has one less 0 than the expansion of k, and the same number of 1s and 2s, by applying the induction hypothesis we obtain the desired result.

For the cases $m \equiv 1 \pmod 3$ and $m \equiv 2 \pmod 3$ we proceed in a similar way. □

Applying the above lemma and making an inductive hypothesis we obtain the following result.

Lemma 2.13.

a) $F_1 * F_2 * \cdots * F_n\left(\frac{3^{n-1}-1}{3^n}\right) = a.$

b) $F_1 * F_2 * \cdots * F_n\left(\frac{2\cdot 3^{n-1}-1}{3^n}\right) = b.$

Now, we show the relationship between $f_{a,b}$ and S.

Theorem 2.14. *$f_{a,b}$ is the infinite convolution function S.*

Proof. We have to prove that the graph of $G_n = F_1 * F_2 * \cdots * F_n$ and $C^n(d)$ are the same set, where d denotes the segment $\overline{(0,1)\,(1,1)}$ and C is given by (2.1), but for a finite set of points. Namely, the intersection of $C^n(d)$ and the line $x = \frac{k}{3^n}$ for $1 \le k \le 3^n - 1$ has two points: $\left(\frac{k}{3^n}, G_n\left(\frac{k}{3^n}\right)\right)$ and $\left(\frac{k}{3^n}, G_n\left(\frac{k}{3^n}^-\right)\right)$, where the minus sign means left-side limit. For the other values of x, the graph of G_n and the set $C^n(d)$ coincide.

Set $C^n(x) := \max\{y : (x, y) \in C^n(d)\}$. It is immediate that $G_1(\frac{1}{3}) = C^1(\frac{1}{3})$, $G_1(\frac{2}{3}) = C^1(\frac{2}{3})$. Thus we will show that for n, it follows that $C^n(\frac{k}{3^n}) = G_n(\frac{k}{3^n})$ for $0 \le k < 3^n$. Let us suppose that it is also true for $n-1$.

For $0 \le m < 3^{n-1}$, we consider these possibilities:

a) $G_n\left(\frac{m}{3^n}\right) = \sum_{k=0}^{m} P_n\left(\frac{k}{3^n}\right) = \sum_{k=0}^{m} a_1 P_{n-1}\left(\frac{k}{3^{n-1}}\right) = a_1 G_{n-1}\left(\frac{m}{3^n}\right) = a_1 C_{n-1}\left(\frac{m}{3^n}\right) = C_n\left(\frac{m}{3^n}\right)$, where the second equality is true because when we write k with one more digit, this must be zero.

b) If $G_n\left(\frac{m+3^{n-1}}{3^n}\right) = a_1 + a_2 G_{n-1}\left(\frac{m}{3^{n-1}}\right) = a_1 + a_2 C_{n-1}\left(\frac{m}{3^{n-1}}\right) = C_n\left(\frac{m}{3^{n-1}}\right)$, the proof is similar to the above.

c) If $G_n\left(\frac{m+2.3^{n-1}}{3^n}\right) = a_1 + a_2 + a_3 G_{n-1}\left(\frac{m}{3^{n-1}}\right) = a_1 + a_2 + a_3 C_{n-1}\left(\frac{m}{3^{n-1}}\right) = C_n\left(\frac{m}{3^{n-1}}\right)$, the proof also follows similarly.

This ensures that the statement is also true for n.

If G_n are step functions that coincide with $f_{a,b}$ at points in the form $\frac{m}{3^n}$ with $0 \le m < 3^n$, then $\lim_{n\to\infty} G_n(x) = f_{a,b}(x)$ for all $x \in [0, 1]$. □

Remark 2.15. *Theorems 2.11 and 2.14 give further proof of Okamoto and Wunsch's result [18, Th. 5] using exclusively probability theory methods. In fact, $f_{a,b}$ is a singular function and additionally we know its characteristic function, as well as the fact that the sequence of its Fourier coefficients does not converge to zero.*

3 Fractal sets associated with $f_{a,b}$

This section is devoted to describing several sets related to the function $f_{a,b}$ and to computing their Hausdorff dimensions. We need some additional results.

We are going to introduce a representation system of real numbers in $[0,1]$.The method employed is based on the imitation of the action of $f_{a,b}$ on the Y axis. With notations already used above, let us divide the unit interval into subintervals $[0,a]$, $[a,b]$ and $[b,1]$. Thus, if $y \in [0,1]$, then y can be written as one of the following expressions:

$$y = \begin{cases} b_1 + a_1 z \\ b_2 + a_2 z \\ b_3 + a_3 z \end{cases},$$

with $b_1 = 0,\ b_2 = a,\ b_3 = b$, and $z \in [0,1]$. We can now apply the above algorithm to $z \in [0,1]$, and by iteration we have the formal relation

$$y = d_1 + s_1 d_2 + s_1 s_2 d_3 + s_1 s_2 s_3 d_4 + \cdots$$

with $d_i = b_j$, $j = 1,2,3$, and $s_i = a_k$ for a suitable j and k, such that if $d_i = b_j$, then $s_i = a_j$.

Observe that if $y := b_2 + a_2 z$, with $z \in \{0,1\}$, then y has two formal equalities. But this fact is of no relevance to our task because it is true on a denumerable set, hence on a set of measure zero. The nature of the construction also ensures that for different points their corresponding formal equalities are different, as well.

Proposition 3.1. *The series $d_1 + s_1 d_2 + s_1 s_2 d_3 + s_1 s_2 s_3 d_4 + \cdots$ converges to y, that is, the formal equality above is in fact an equality.*

Proof. Note that $s_1 \ldots s_n d_{n+1} \le \max\{a_1, a_2, a_3, b\}^{n+1}$. Thus, by construction

$$\begin{aligned} 0 &\le y - (d_1 + s_1 d_2 + s_1 s_2 d_3 + s_1 s_2 s_3 d_4 + \cdots + s_1 s_2 \ldots s_n d_{n+1}) \\ &= O\left(\max\{a_1, a_2, a_3, b\}^n\right), \end{aligned}$$

which implies that the series converges to y. □

Next, the family of parameterized singular functions is described with the help of the a,b-representation system.

Definition 3.2. *The series in the proposition above is named as the a,b-representation of $y \in [0,1]$.*

Theorem 3.3. *The random variables $d_n(x)$ are independent and equally distributed with probability function*

$$P(d_n = 0) = a_1,\ P(d_n = a) = a_2,\ P(d_n = b) = a_3.$$

Applying the Law of Large Numbers we obtain the following result.

Corollary 3.4. *With the above notation the set of points whose proportion of times that $d_n = 0$ is a_1, for $d_n = a$ is a_2 times or $d_n = b$ is a_3, has measure 1.*

We are now going to exhibit a set of measure zero whose image under $f_{a,b}$ is of measure one and vice versa.

Theorem 3.5. *There exists a set of measure zero and Hausdorff dimension*

$$-\frac{a_1 \ln a_1 + a_2 \ln a_2 + a_3 \ln a_3}{\ln 3}$$

that is mapped on a set of measure one by $f_{a,b}$.

Proof. The nature of the construction of the representation for a point $x = \sum_{n=1}^{\infty} \frac{c_n}{3^n}$ in $[0,1]$ and its image $f_{a,b}(x)$ entails that if $c_n = 0,1,2$, then $d_n = 0, a, b$, respectively. Therefore, the set of points whose proportions of 0s, 1s and 2s are a_1, a_2 and a_3, respectively, has measure zero and is mapped on a set of measure 1.

The Hausdorff dimension is calculated in [20, Chp. 10]. □

In addition, we shall give another analogue property using the measure of the normal numbers in $[0, 1]$.

We recall that a number is normal in the 3-base if the proportions of 0s, 1s and 2s in this base is 1/3 for each case, and that the set of normal numbers in the 3-base is of measure 1 (see, for instance, [24] or [25]).

Theorem 3.6. *There exists a set of measure one that is mapped by $f_{a,b}$ on a set of measure zero and Hausdorff dimension*

$$\frac{\ln 27}{-\ln\left(a_1 a_2 a_3\right)}.$$

Proof. A normal number in the base 3 has the same proportion for each of the three digits in its expansion. Their images will have the same proportion in the random variables d_n. Corollary 3.4 shows that the proportions must be a_1, a_2 and a_3, on a set of measure 1. Thus, the image of normal numbers is a measure zero set.

To calculate the Hausdorff dimension for this zero-measure set, we use that the measure $\mathrm{d} f_{a,b}$ concentrates its mass on it. Thus, applying [20, Lemma 4.9], by substitution of balls by neigbourhoods in the following way: If

$$x = d_1 + s_1 d_2 + s_1 s_2 d_3 + s_1 s_2 s_3 d_4 + \cdots,$$

then we use the representation method and, in the n-th iteration, we do $z = 0$ and $z = 1$, obtaining the extremes of an interval containing x, namely:

$$\left[d_1 + \cdots + s_1 s_2 \ldots s_{n-1} d_n, d_1 + \cdots + s_1 s_2 \ldots s_{n-1} d_n + s_1 s_2 \ldots s_n\right].$$

Clearly, its Lebesgue measure is $s_1 s_2 \ldots s_n$, and, by construction, its $\mathrm{d} f_{ba}^{-1}$-measure is $\frac{1}{3^n}$.

Therefore, the Hausdorff dimension is given by the number

$$\sup\left\{\beta > 0 : \lim_{n\to\infty} \frac{1/3^n}{\left(s_1 s_2 \cdots s_n\right)^\beta} < +\infty\right\}.$$

Taking logs, because it must be finite,

$$\lim_{n\to+\infty} n \ln\left(3\left(a_1 a_2 a_3\right)^{\frac{\beta}{3}+o(1)}\right) < +\infty,$$

and the supremum β of these values is $\frac{\ln 27}{-\ln(a_1 a_2 a_3)}$. □

4 Generalization

Although their description is complex, a geometric generalization of the preceding functions can be easily carried out. We directly proceed in the following way, avoiding proofs in the results that follow. They can be derived from the techniques and ideas already used in the proofs of the preceding results.

Definition 4.1. *Let $a, b, a', b' \in \,]0, 1[$ such that $a < b$ and $a' < b'$. We set the function $f_{aba'b'} = f_{a',b'} \circ f_{a,b}^{-1}$.*

Theorem 4.2 (Functional Equations)**.** *$f_{aba'b'}$ is the only bounded function defined in $[0, 1]$ satisfying*

$$\begin{cases} h\left(ax\right) = a' h(x), \\ h\left(a + a_2 x\right) = a' + a_2' h(x), \\ h\left(b + a_3 x\right) = b' + a_3' h(x). \end{cases} \tag{3}$$

Theorem 4.3. *The function $f_{aba'b'}$ is continuous.*

Corollary 4.4.

$$\int_0^1 f_{aba'b'}(x)\mathrm{d}x = \frac{a_1 a_1' + a_2 a_2' + a_2 a_1'}{1 - \left(a_1 a_1' + a_2 a_2' + a_3 a_3'\right)}.$$

Theorem 4.5. *$f_{aba'b'}$ is an increasing and singular function whose derivative $f'_{aba'b'}(x)$, when it exists, can only vanish.*

Theorem 4.6.

i) *There exists a set of measure zero and Hausdorff dimension $\frac{a'_1 \ln a'_1 + a'_2 \ln a'_2 + a'_3 \ln a'_3}{a'_1 \ln a_1 + a'_2 \ln a_2 + a'_3 \ln a_3}$ that is mapped on a set of measure one by $f_{aba'b'}$.*

ii) *A set of measure one exists that is mapped by $f_{aba'b'}$ on a set of measure zero and Hausdorff dimension $\frac{a_1 \ln a_1 + a_2 \ln a_2 + a_3 \ln a_3}{a_1 \ln a'_1 + a_2 \ln a'_2 + a_3 \ln a'_3}$.*

5 Applications

In this section we apply the main results obtained in this paper to provide nontrivial examples in Measure Theory and Fractal Analysis. On the one hand, we will construct an example of Katok foliation, and on the other hand, we study singular functions and their interaction with harmonic functions defined on a self-similar set.

5.1 Katok Foliation

The first example of a pathological foliation was constructed by Katok. A different version of this construction on the square appeared in Milnor's work [27], which showed examples of foliations of the unit square such that a full measure set intersects each leaf of the foliation at exactly one point.

Here we introduce all the necessary notions required for the precise formulation of our results.

Definition 5.1. *A pair (E, f_α) is a Katok foliation if:*

a. *$E \subset [0,1]^2$ is a set of measure 1;*
b. *f_α is a family of analytic functions from $]0,1[$ to $[0,1]$;*
c. *the graphs of the functions f_α fill the interior of $[0,1]^2$;*
d. *the graphs of the functions f_α are pairwise disjoint;*
e. *the graph of each function f_α intersects with E at one point at most.*

Lemma 5.2. *For each $x \in]0,1[$, the function $g_x(t) := f_{t^2,t}(x)$ is analytic at $t \in]0,1[$ and $g_x(0) = 0, g_x(1) = 1$.*

Proof. If $x = d_1 + s_1 d_2 + s_1 s_2 d_3 + s_1 s_2 s_3 d_4 + \cdots$, then $g_x(t)$ is obtained by the substitution of d_i, s_i with one of the values $0,\ t,\ t^2,\ 1-t,\ t(1-t)$, depending on d_i. For each substitution, we obtain an expansion series in powers of t. It is clear that $g_x(0) = 0$ and $g_x(1) = 1$: the series expansion begins with a term in the t^n power, thus $g_x(0) = 0$. For the latter, we can write in the form $t^n +$ a series of positive terms, with $0 \leq g_x(t) \leq 1$. Hence, by continuity, the equality follows. □

Lemma 5.3. *For $x, x' \in]0,1[$, $x \neq x'$, the graphs of g_x and $g_{x'}$ in $]0,1[$ are disjoint.*

Proof. If $(t, g_x(t))$ and $(t, g_{x'}(t))$ are equal, the series expansion in the t^2, t-representation system of x and x' is the same. But this would imply $x = x'$, a contradiction. □

Proposition 5.4. *The functions $f_{t^2,t}$ are singular, and their graphs fill the interior of the unit square $[0,1]^2$.*

Proof. Singularity follows from the above results. Let $x \in]0,1[$. The function g_x is continuous with $g_x(0) = 0$ and $g_x(1) = 1$, and as a consequence, it takes all the values $0 < y < 1$. □

Theorem 5.5. *The pair* (E, g_x) *, where*

$$E := \left\{ \begin{array}{l} (t, y) : \text{the proportion of variables } d_i \text{ in} \\ \qquad\quad \text{the } t^2, t\text{-representation of } y \text{ is } t^2, t(1-t), 1-t \end{array} \right\}$$

and g_x *defined as above, is a Katok foliation.*

Proof. E is a set of measure 1 by Fubini's theorem. The analyticity was already obtained above. To deduce that the graph of each function in the foliation intersects at most at one point is a consequence of the fact that the variables d_i (that only depend on x) appear in the same proportion in the image of g_x. Because this proportion is a limit, it is possible that it does not exist. If it exists, then we find a value t with the desired proportions. □

Remark 5.6. *In exchange of* t^2, t *for* t^u, t^v *with* $u > v$*, one can obtain a bi-parametric family of foliations.*

5.2 Fractal Analysis

Applications of fractal sets in physics phenomena have brought about an important development of fractal techniques in the last twenty years, specifically in harmonic functions in fractals. A good introduction to this topic can be found in [28]**.** A study of the harmonic functions on the Sierpinski triangle can be found in [29]**.** In this subsection we study the harmonic functions on these sets and their relationship with singular functions which are the object of our research in this paper.

We focus our study on a self-similar subset in the unit square $[0,1]^2$, which is determined as the fixed point of a suitable contraction in the metric space $\mathcal{K}\left([0,1]^2\right)$, endowed with the Hausdorff metric.

Definition 5.7. *For* r *such that* $0 < r < 1/2$*, let us set functions:*

$$\begin{cases} G_1 : [0,1]^2 \longrightarrow [0,1]^2 \,,\ G_1(x,y) = (rx, ry)\,, \\ G_2 : [0,1]^2 \longrightarrow [0,1]^2 \,,\ G_2(x,y) = (1-r+rx, ry)\,, \\ G_3 : [0,1]^2 \longrightarrow [0,1]^2 \,,\ G_3(x,y) = (r+(1-2r)\,x, r+(1-2r)\,y)\,, \\ G_4 : [0,1]^2 \longrightarrow [0,1]^2 \,,\ G_4(x,y) = (rx, 1-r+ry)\,, \\ G_5 : [0,1]^2 \longrightarrow [0,1]^2 \,,\ G_5(x,y) = (1-r+rx, 1-r+ry)\,, \end{cases}$$

and let

$$G : \mathcal{K}\left([0,1]^2\right) \longrightarrow \mathcal{K}\left([0,1]^2\right), \qquad G(T) = \bigcup_{i=1}^{5} G_i(T).$$

The only fixed point of G *will be denoted by* N_r*.*

N_r is a fractal set, and we will study harmonic functions on it. This type of sets has already been used to show the first example of copulas with fractal support (see for instance [30] or [31]). Let us recall that the fixed point N_r in the Banach contraction mapping is obtained as the limit by iterations of G starting at any choosen point. Taking as the starting point the compact unit square, for $r = 1/3$, the first two are as Figure 2 shows.
We name the vertices of $[0,1]^2$: $p_1 = (0,1)$, $p_2 = (1,1)$, $p_3 = (0,0)$ and $p_4 = (1,0)$, and for the vertices in the first iteration:

$$\begin{array}{llll} q_1 = (r,1) & q_2 = (1-r,1) & q_3 = (0,1-r) & q_4 = (r,1-r) \\ q_5 = (1-r,1-r) & q_6 = (1,1-r) & q_7 = (0,r) & q_8 = (r,r) \\ q_9 = (1-r,r) & q_{10} = (1,r) & q_{11} = (r,0) & q_{12} = (1-r,0) \end{array}$$

(see Figure 3).

Fig. 2. The two first iterations of the unit square in the case $r = 1/3$

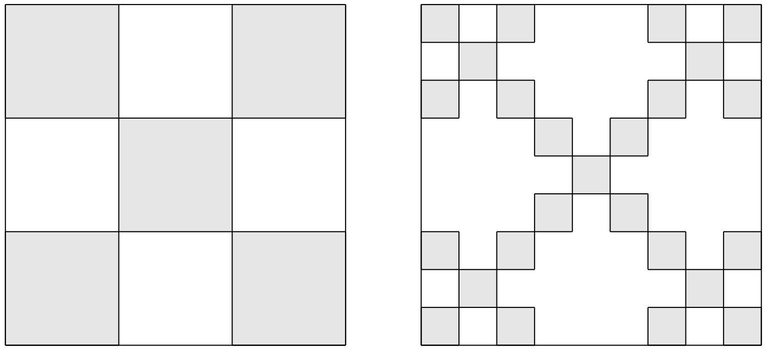
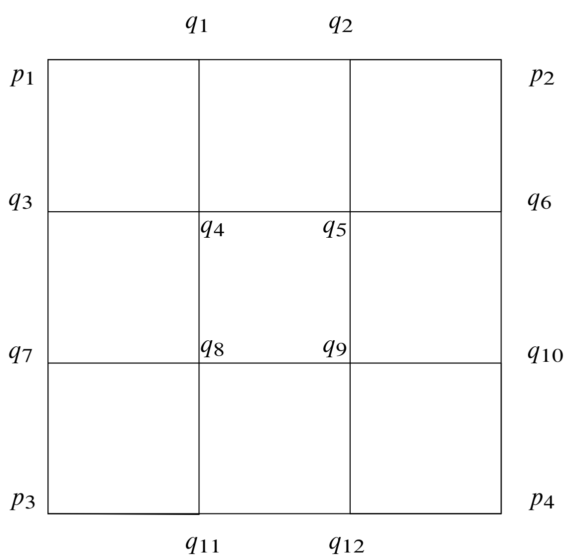


Fig. 3. Vertices considered in the unit square



For the sake of our paper, we state that a function h defined in C is harmonic if

$$h(p_1) = s_1,\ h(p_2) = s_2,\ h(p_3) = s_3,\ h(p_4) = s_4$$

implies that

$$\begin{cases} h(q_1) = \frac{22s_1+3s_2+3s_3+2s_4}{30},\ h(q_2) = \frac{3s_1+22s_2+2s_3+3s_4}{30}, \\ h(q_3) = \frac{22s_1+3s_2+3s_3+2s_4}{30},\ h(q_4) = \frac{14s_1+6s_2+6s_3+4s_4}{30}, \\ h(q_5) = \frac{6s_1+14s_2+4s_3+6s_4}{30},\ h(q_6) = \frac{3s_1+22s_2+2s_3+3s_4}{30}, \\ h(q_7) = \frac{3s_1+2s_2+22s_3+3s_4}{30},\ h(q_8) = \frac{6s_1+4s_2+14s_3+6s_4}{30}, \\ h(q_9) = \frac{4s_1+6s_2+6s_3+14s_4}{30},\ h(q_{10}) = \frac{2s_1+3s_2+3s_3+22s_4}{30}, \\ h(q_{11}) = \frac{3s_1+2s_2+22s_3+3s_4}{30},\ h(q_{12}) = \frac{2s_1+3s_2+3s_3+22s_4}{30}. \end{cases}$$

And the same occurs on each of the squares that appear at each stage. G can be extended to C by continuity, because the set of vertices of the squares form a dense set. These functions will be denoted by $h_{s_1 s_2 s_3 s_4}$. These functions and the rest that follows in this section, depend on the parameter r, but we omit it for simplicity. We will study their performance on the diagonal of the set, and we will do it for the particular case of $h_{\alpha 10 \alpha}$, which we denote by h_α for short. Actually, the only condition we impose is that $s_1 = s_4 = \alpha$, because it is necessary to take it into account:

$$h_{s_1 s_2 s_3 s_4}((x, y)) = s_3 + (s_2 - s_3)\, h_{\left(\frac{s_1 - s_3}{s_2 - s_3}\right) 10 \left(\frac{s_4 - s_3}{s_2 - s_3}\right)}((x, y)).$$

When we are restricted to the diagonal, it is necessary to study the subsets $G_1([0,1]^2)$, $G_3([0,1]^2)$ and $G_5([0,1]^2)$: they are the new squares having a non-empty intersection with it. The self-similarity of the set and the way the function takes the values at the vertices of the new squares, allow us to write the following equations:

$$\begin{cases} h_\alpha\left((rx, rx)\right) = h_{\frac{2+6\alpha}{30}\,\frac{4+12\alpha}{30}\,0\,\frac{2+6\alpha}{30}}\left((x,x)\right), \\ h_\alpha\left((r+(1-2r)\,x, r+(1-2r)\,x)\right) = h_{\frac{18\alpha+6}{30}\,\frac{14+12\alpha}{30}\,\frac{4+12\alpha}{30}\,\frac{18\alpha+6}{30}}\left((x,x)\right), \\ h_\alpha\left((1-r+rx, r+1-r+rx)\right) = h_{\frac{22+6\alpha}{30}\,1\,\frac{14+12\alpha}{30}\,\frac{22+6\alpha}{30}}\left((x,x)\right), \end{cases}$$

that can be rewritten as:

$$\begin{cases} h_\alpha\left((rx, rx)\right) = \frac{4+12\alpha}{30} h_{\frac{1}{2}}\left((x,x)\right), \\ h_\alpha\left((r+(1-2r)\,x, r+(1-2r)\,x)\right) = \frac{4+12\alpha}{30} + \frac{1}{3} h_{\frac{6\alpha+2}{10}}\left((x,x)\right), \\ h_\alpha\left((1-r+rx, r+1-r+rx)\right) = \frac{14+12\alpha}{30} + \frac{16-12\alpha}{30} h_{\frac{1}{2}}\left((x,x)\right). \end{cases}$$

Definition 5.8. *Set* $g_\alpha : [0,1] \longrightarrow [0,1]$, $g_\alpha(x) := h_\alpha\left((x,x)\right)$ *for all* $x \in [0,1]$.

The above equalities for these functions give:

$$\begin{cases} g_\alpha\left(rx\right) = \frac{4+12\alpha}{30} g_{\frac{1}{2}}\left(x\right), \\ g_\alpha\left(r+(1-2r)\,x\right) = \frac{4+12\alpha}{30} + \frac{1}{3} g_{\frac{6\alpha+2}{10}}\left(x\right), \\ g_\alpha\left(1-r+rx\right) = \frac{14+12\alpha}{30} + \frac{16-12\alpha}{30} g_{\frac{1}{2}}\left(x\right), \end{cases}$$

and if $\alpha = 1/2$, then they become the functional equations for $g_{\frac{1}{2}}$:

$$\begin{cases} g_{\frac{1}{2}}\left(rx\right) = \frac{1}{3} g_{\frac{1}{2}}\left(x\right), \\ g_{\frac{1}{2}}\left(r+(1-2r)\,x\right) = \frac{1}{3} + \frac{1}{3} g_{\frac{1}{2}}\left(x\right), \\ g_{\frac{1}{2}}\left(1-r+rx\right) = \frac{2}{3} + \frac{1}{3} g_{\frac{1}{2}}\left(x\right). \end{cases}$$

Now attending to the functional equations of $f_{r,1-r}$ (with $0 < r < 1/2$) we have the following result.

Proposition 5.9. $f^{-1}_{r,1-r}$ *and* $g_{\frac{1}{2}}$ *coincide on the unit interval.*

Corollary 5.10. g_α *is a singular function.*

Proof. The equalities already established for the family of functions g_α ensure the way to divide the unit interval $[0,1]$ into subintervals where the corresponding function is the scale copy of $g_{\frac{1}{2}}$. Thus, we complete the proof, because the inverse of a singular function is also singular. □

Corollary 5.11. *There exists a set of measure zero and Hausdorff dimension* $\frac{\ln 27}{-\ln r^2(1-2r)}$ *that is mapped on a set of measure one by* g_α.

Corollary 5.12. *There exists a set of measure one that is mapped on a set of measure zero by* $g_{\alpha,r}$ *and Hausdorff dimension*

$$-\frac{2r\ln r + (1-2r)\ln(1-2r)}{\ln 3}.$$

Remark 5.13. *This study can be generalized to the set* C' *determined as the fixed point of the mapping generated by the following functions:*

$$\begin{cases} G_1 : [0,1]^2 \longrightarrow [0,1]^2,\ G_1(x,y) = (rx, ry), \\ G_2 : [0,1]^2 \longrightarrow [0,1]^2,\ G_2(x,y) = (r' + (1-r')x, ry), \\ G_3 : [0,1]^2 \longrightarrow [0,1]^2,\ G_3(x,y) = (r + (r'-r)\,x, r + (r'-r)\,y), \\ G_4 : [0,1]^2 \longrightarrow [0,1]^2,\ G_4(x,y) = (rx, r' + (1-r')\,y), \\ G_5 : [0,1]^2 \longrightarrow [0,1]^2,\ G_5(x,y) = (r' + (1-r')\,x, r' + (1-r')\,y), \end{cases}$$

with $0 < r < r' < 1$.

The results are analogous, but the functions we now obtain are the inverses of $f_{r,r'}$, *and the fractal dimensions of the related fractal sets are* $\frac{\ln 27}{-\ln rr'(1-r'-r)}$ *and* $-\frac{r\ln r + r'\ln r' + (1-r-r')\ln(1-r-r')}{\ln 3}$, *respectively.*

5.3 Strong Negations

In the framework of Fuzzy Logic, a strictly decreasing mapping $n : [0,1] \longrightarrow [0,1]$ is a *strong negation* if $n^2(x) := n \circ n\,(x) = x$ for all x in $[0,1]$ (see [32]). Next, we are going to introduce a family of strong negations related to the functions we have studied.

Theorem 5.14. *For* $0 < a < b < 1$, *the function* $n_{a,b} : [0,1] \longrightarrow [0,1]$ *given by*

$$n_{a,b}(x) := f_{ab(1-b)(1-a)}(1-x)$$

for all x *in* $[0,1]$ *is a strong negation.*

Proof. Continuity and monotony are evident. Through functional equations (3), we deduce that $n_{a,b}$ is the only function defined in $[0,1]$ that is bounded and satisfies the relations:

$$\begin{cases} h\left((1-b)\,x\right) = 1-a+ah(x) \\ h\left(1-b+(b-a)\,x\right) = 1-b+(b-a)h(x) \\ h\left(1-a+ax\right) = (1-b)h(x). \end{cases} \tag{4}$$

By the definition of $n_{a,b}$ and properties of $f_{ab(1-b)(1-a)}$:

$$\begin{aligned} n_{a,b}\left((1-b)\,x\right) &= f_{ab(1-b)(1-a)}(b+(1-b)(1-x)) \\ &= 1-a+af_{ab(1-b)(1-a)}(1-x) \\ &= 1-a+an_{a,b}(x). \end{aligned}$$

The others follow is an analogous way. Finally, its uniqueness follows from that of $f_{ab(1-b)(1-a)}$ in (3) or from the fixed point Banach theorem.

To show that $n^2_{a,b}(x) = x$ it is sufficient to use equations (4). From them, we deduce that $n^2_{a,b}$ satisfies the functional equations

$$\begin{cases} h\left((1-b)\,x\right) = (1-b)\,h(x), \\ h\left(1-b+(b-a)\,x\right) = 1-b+(b-a)h(x), \\ h\left(1-a+ax\right) = 1-a+ah(x). \end{cases} \tag{5}$$

However, the solution of this system is unique in the unit interval, and this is the identity function. Thus, $n^2_{a,b}$ equals identity in $[0,1]$. □

6 Conclusions

We have studied a wide family of two-parametric singular functions. To this end, we introduce a biparametric representation system that allows us to give an explicit expression of these functions and to calculate the Hausdorff dimension of several sets with distinguished properties for $f_{a,b}$.

In addition, we have examined some applications of the study of these families of singular functions obtaining Katok foliations, results concerning its relations with fractal harmonic analysis and with strong negations.

For further investigations we are interested in the study of these functions in the context of homeomorphisms between the supports of copulas with fractal support [31], as they appear in the disintegration of the measure associated to some self-similar copulas, and in the study of functions generated as $f_{a,b}$, but with random a and b.

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