Translating solitons of the mean curvature flow

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Contents

Abstract and conclusions

The evolution of geometric objects with respect to a time variable (t) is a field of intense study in the framework of Riemannian Geometry. A good example is the work of Perelman which led to the proof of the *geometrization conjecture of Thurston* and, consequently, to the Poincaré conjecture. In general, two types of geometric objects are distinguished: the intrinsic ones, such as the metric of a manifold, and the extrinsic ones, such as an embedding (or immersion) of a manifold into another one. These two types of objects give rise to two types of flows: the intrinsic ones, such as the Ricci flow, and the extrinsic ones, such as the mean curvature flow. In these particular flows the object that evolves is precisely the metric and the immersion respectively.

The mean curvature flow is maybe the most important geometric evolution equation of submanifolds in Riemannian manifolds. Intuitively, a family of smooth submanifolds evolves under mean curvature flow if the velocity at each point of the submanifold is given by the mean curvature vector at this point, that is, the submanifolds move at each point in the direction of the corresponding normal unit vector and with speed equals to its scalar mean curvature. For example, a sphere in the Euclidean space evolves under mean curvature flow by shrinking inward until it collapses in a finite time to a point.

In the literature there are different approaches to the mean curvature flow: from the geometric measure theory, from level set flows and from partial differential equations. We will follow the latter approach, which began with the pioneer work of Huisken [Hui84].

The evolution equation can develop *singularities*, that is, solutions may become non-smooth in finite time. For example, it is known (see for instance

[Smo12]) that closed submanifolds (i.e., compact submanifolds without boundary) of the Euclidean space remain smooth for a finite time during their evolution under mean curvature flow and then they develop a singularity. Thus, a particularly interesting topic in this theory is to study the behavior of singularities. In order to do that, following Huisken's work [Hui90], two types of singularities are often distinguished: type I singularities, which are those cases with a better control of the growth of the norm of the second fundamental form —in a sense which is made precise in [Hui90]—, and type II singularities, which are all the other singularities. As the definition suggests, type II singularities are more difficult to treat than type I singularities.

In the case of hypersurfaces, Huisken and Sinestrari [HS99] proved that if the initial surface is mean convex (that is, $H \geq 0$ in the whole surface, where H is the mean curvature) and it develops a type II singularity, then the *limit surface* obtained with a certain technique of *rescaling* of the flow is convex and satisfies the equation $\mathbf{H} = \mathrm{v}^{\perp}$, where $\mathbf{H}$ is the mean curvature vector, v is a constant vector and the superscript $\perp$ denotes projection onto the normal bundle. A surface whose mean curvature vector satisfies the above equation is called translating soliton of the mean curvature flow. Geometrically it evolves moving in the direction of v with speed $|\mathrm{v}|$, that is, with fixed direction and constant speed given by the vector v. Thus, it does not change its shape during the evolution, it simply translates. Therefore, translating solitons are eternal solutions of the flow, that is, their evolution exists for all times $-\infty < t < +\infty$.

Translating solitons arise not only in the study of singularities, but also in the general investigation of the mean curvature flow. For example, again in the case of hypersurfaces in the Euclidean space, Hamilton [Ham95] proved that any strictly convex eternal solution to the mean curvature flow where the mean curvature assumes its maximum value at a point in space-time must be a translation soliton.

Moreover, translating solitons are interesting examples of the mean curvature flow because they are precise solutions in the sense that their evolution is known, which is very hard to determine in general. In the first chapter of this thesis, after a brief introduction to the mean curvature flow and translating solitons, we present the classic examples of the latter ones (see section 1.4).

It is well known that translating solitons are related to minimal surfaces [Ilm94]. Obviously, this relationship is important because it allows to use classical results of the theory of minimal surfaces to study translating solitons. In this spirit, the maximum principle, stated as its geometric counterpart, the tangency principle, is the main tool of the second chapter of the thesis, which begins with the proof of the results of non-existence of translating solitons. We prove that there are no non-compact translating solitons contained in a solid cylinder (Theorem 2.1.2). We also rule out the existence of certain compact embedded translating solitons with two boundary components (Theorem 2.1.5). Then, by comparison with a tilted grim reaper cylinder, we obtain an estimate of the maximum height that a compact translating soliton embededd in $\mathbb{R}^3$ can achieve; this estimate is in terms of the diameter of the boundary curved of the translator (Theorem 2.2.1). Another application to the tangency principle is to study *graphical perturbations* of translating solitons, which allows us to easily prove the characterization of the translating paraboloid given in [MSHS15, Theorem A]. On the other hand, we use the method of moving planes to show that a compact embedded translating soliton contained in a slab and with boundary components given by two convex curves in the parallel planes determining the slab inherits all the symmetries of its boundary (Theorem 2.4.1).

The main result of the thesis is presented in the third and last chapter and it is a characterization of grim reaper cylinders as properly embedded translators with uniformly bounded genus and asymptotic to two half-planes whose boundaries are contained in the boundary of a solid cylinder with axis perpendicular to the direction of translation (Theorem 3.0.2). The proof is quite elaborated and heavily uses analytic tools developed by Brian White: a compactness theorem for sequences of minimal surfaces properly embedded into three-dimensional manifolds with locally uniformly bounded area and genus, as well as a barrier principle. As mentioned above, the key ingredient to use these results of White is to consider translating solitons as minimal surfaces in the so-called Ilmanen's metric and to establish the good relation between these surfaces in both (usual Euclidean and Ilmanen) metrics, in particular with respect to their asymptotic behavior.

Resumen y conclusiones

La evolución de objetos geométricos respecto a una variable temporal (t) es un área de estudio intenso en el marco de la Geometría Riemanniana. Un buen ejemplo lo constituye el trabajo de Perelman que ha permitido probar la *conjetura de geometrización de Thurston* y, en consecuencia, la conjetura de Poincaré. En general, se distinguen dos tipos de objetos geométricos: los intrínsecos, como la métrica de una variedad, y los extrínsecos, como el embebimiento (o inmersión) de una variedad en otra. Estos dos tipos de objetos dan lugar a dos tipos de flujos: intrínsecos, como por ejemplo el llamado flujo de Ricci, y extrínsecos, como por ejemplo es el caso del flujo de la curvatura media. En estos flujos concretos el objeto que evoluciona es precisamente la métrica y la inmersión respectivamente.

El flujo de la curvatura media es quizás la ecuación de evolución geométrica más importante de subvariedades en variedades de Riemann. Intuitivamente, una familia de subvariedades diferenciables evoluciona según el flujo de la curvatura media si la velocidad en cada punto de la subvariedad viene dada por el vector curvatura media en ese punto, es decir, las subvariedades se mueven en cada punto siguiendo la dirección del correspondiente vector normal unitario y con rapidez igual a su curvatura media. Un ejemplo habitual es el de las esferas en el espacio euclídeo, las cuales evolucionan según el flujo de la curvatura media contrayéndose concéntricamente hasta que colapsan en un tiempo finito en un punto, el centro común de las esferas.

De las diferentes aproximaciones al flujo de la curvatura media que hay en la literatura, a saber, desde la teoría geométrica de la medida, desde los flujos de conjuntos de nivel y desde las ecuaciones en derivadas parciales, seguiremos este último enfoque, que se inició con el trabajo pionero de Huisken de 1984 [Hui84].

La ecuación de evolución puede desarrollar *singularidades*, esto es, la solución puede dejar de ser diferenciable, en tiempo finito. Por ejemplo, se sabe que las subvariedades cerradas (es decir, compactas y sin borde) del espacio euclídeo evolucionan siguiendo el flujo de la curvatura media como subvariedades diferenciables durante un tiempo finito tras el cual se desarrollan una singularidad [Smo12]. Así, un tema particularmente interesante en esta teoría es estudiar el comportamiento de las singularidades. Para ello a menudo se distinguen dos tipos de singularidades, distinción introducida originalmente por Huisken [Hui90]: singularidades de tipo I, que son aquellas en las que se tiene —en cierto sentido que se hace preciso por ejemplo en [Hui90]— el mejor control del crecimiento de la norma de la segunda forma fundamental, y singularidades de tipo II, que son el resto. Como cabe esperar de la propia definición, las singularidades de tipo II son mucho más difíciles de tratar que las de tipo I.

En el caso de hipersuperficies, Huisken y Sinestrari [HS99] probaron que si la superficie inicial es convexa en media (esto es, $H \geq 0$ en toda la hipersuperficie, siendo H la curvatura media) y tiene una singularidad de tipo II, entonces toda *superficie límite* (obtenida mediante cierta técnica de *reescalamiento* del flujo) es convexa y satisface la ecuación $\mathbf{H} = \mathrm{v}^{\perp}$, siendo $\mathbf{H}$ el vector curvatura media, v un vector constante y el superíndice $\perp$ indica proyección sobre el fibrado normal. Las superficies que verifican la ecuación anterior ($\mathbf{H} = \mathrm{v}^{\perp}$) reciben el nombre de solitones de traslación del flujo de la curvatura media. Son soluciones eternas del flujo de la curvatura media, esto es, su evolución existe para todo tiempo $-\infty < t < +\infty$. Geométricamente, un solitón de traslación se mueve siguiendo el flujo de la curvatura media trasladándose en la dirección de v con velocidad $|\mathrm{v}|$, es decir, con dirección y velocidad constantes dadas por el vector v. No cambia su forma, solo se traslada (con velocidad y dirección fijas) en el espacio euclídeo.

Los solitones de traslación no solo aparecen en el estudio de las singularidades, sino en general en el estudio del flujo de la curvatura media. Por ejemplo, de nuevo en el caso de hipersuperficies, Hamilton [Ham95] probó que una solución del flujo de la curvatura media que sea eterna, estrictamente convexa, y para la que la curvatura media alcance su valor máximo en un punto del espacio-tiempo, es un solitón de traslación.

Además, los solitones de traslación forman ejemplos interesantes del flujo de la curvatura media ya que constituyen soluciones precisas del mismo, las cuales en general son muy difíciles de obtener. En el primer capítulo de esta tesis, tras una breve introducción al flujo de la curvatura media y a los solitones de traslación, presentamos los ejemplos clásicos de los mismos (véase la sección 1.4).

Es conocido que los solitones de traslación del espacio euclídeo también están relacionados con las superficies mínimas [Ilm94]. Obviamente, esta relación es importante ya que permite usar resultados de la teoría de superficies mínimas para estudiar los solitones de traslación. En esta línea, el principio del máximo, enunciado en su versión geométrica, el principio de tangencia, es la herramienta principal del segundo capítulo de la tesis, el cual comienza con la demostración de resultados de no existencia de solitones de traslación. Por ejemplo, probamos que no existen solitones de traslación no compactos contenidos en un cilindro sólido (Teorema 2.1.2). También descartamos la existencia de ciertos solitones de traslación compactos y embebidos con frontera compuesta por dos componentes compactas (Teorema 2.1.5). A continuación, comparando con un cilindro *grim reaper* inclinado obtenemos una estimación de la altura de un solitón de traslación compacto en términos del diámetro de su curva frontera (Teorema 2.2.1). Una última aplicación directa que realizamos del principio de tangencia consiste en estudiar *perturbaciones gráficas* de solitones de traslación, lo que nos permite demostrar fácilmente la caracterización del paraboloide de traslación dada en [MSHS15, Teorema A]. Por otro lado, utilizamos la técnica de los planos en movimiento (*moving planes*) de Alexandrov para probar que un solitón de traslación compacto, embebido y contenido en una banda que tenga por componentes frontera dos curvas convexas situadas en los planos paralelos de la banda hereda todas las simetrías de la frontera (Teorema 2.4.1).

El resultado principal de esta tesis se presenta en el tercer y último capítulo y es una caracterización del cilindro *grim reaper* a partir de su comportamiento asintótico (Teorema 3.0.2). La demostración es bastante elaborada y utiliza fuertemente herramientas analíticas desarrolladas por Brian White: un teorema de compacidad para las sucesiones de superficies mínimas propiamente embebidas en variedades tridimensionales con área y género ambos localmente uniformemente acotados, así como un principio de barrera. El ingrediente clave para poder aplicar estos resultados de White es considerar los solitones de traslación como superficies mínimas con la métrica de Ilmanen y establecer la buena relación que hay entre estas superficies en ambas métricas (la euclídea usual y la de Ilmanen), en particular en lo que respecta a su comportamiento asintótico.

Chapter 1

Introduction

1.1 Mean curvature flow

A one-parameter family of smooth hypersurfaces evolves (or moves) by mean curvature flowif the velocity vector coincides with the mean curvature vector at each point.

Definition 1.1.1. *Let $F_0 : M \to \mathbb{R}^{m+1}$ be an orientable hypersurface. Let $F_t(\cdot) := F(\cdot, \cdot) : M \times [0, T) \to \mathbb{R}^{m+1}$ be a smooth family of immersions of M in $\mathbb{R}^{m+1}$, where $T \in (0, +\infty]$. We say that F_t is a mean curvature flow (MCF for short) or, equivalently, that M evolves under the mean curvature flow, if*

$$\left(\frac{\partial}{\partial t} F_t\right)^{\perp} = \mathbf{H}, \tag{1.1.1}$$

where

- $\left(\frac{\partial}{\partial t} F_t\right)^{\perp}$ *is the normal component of* $\frac{\partial}{\partial t} F_t$,
- $\mathbf{H}$ *is the mean curvature vector.*

In terms of scalar quantities,

$$H = -\left\langle \frac{\partial}{\partial t} F_t, \nu \right\rangle, \tag{1.1.2}$$

- H is the scalar mean curvature of F,
- ν is a unit normal vector of F,

where we follow the usual convention in this field:

$$\mathbf{H} = -H\,\nu, \tag{1.1.3}$$

that is, the scalar mean curvature is computed with respect to $-\nu$. Note that the unit normal vector is defined up to a sign, but the mean curvature vector $\mathbf{H}$ is independent of such choice. Therefore, with this convention we choose a normal vector, the opposite vector to the one given by ν.

In order to abbreviate notation:

- We will often specify the t-argument as a subscript, as in $F(x,t) = F_t(x)$. And we will proceed this way with all the objects derived from the family $\{f_t\}_{t\in[0,T)}$. For instance,

  $$M_t \coloneqq f(M,t), \quad \mathbf{H}_t(p) \coloneqq \mathbf{H}(p,t), \quad \text{etc.}$$

  Furthermore, we will suppress the subscript t if the context does not cause confusion.

- *Einstein summation convention*: when an index variable appears twice (and only twice) in a term of an expression, it implies summation of that term over all the values of the index.

1.2 Graphical solutions

Mean curvature flow has been widely studied in certain families such as graphical hypersurfaces or lagrangian submanifolds. For our purposes it will be very useful to study graphical solutions of mean curvature flow.

Let $u\colon \mathbb{R}^m \to \mathbb{R}$ be a smooth function. It is very well known that then the graph of u, $\operatorname{Graph}(u) = \{(x,u(x))\colon x \in \mathbb{R}^m\}$, is a smooth hypersurface of

$\mathbb{R}^{m+1}$. Our goal is to study a solution to the mean curvature flow where the evolving hypersurfaces are all graphical, that is, we look for a smooth map

$$F : \mathbb{R}^m \times [0,T) \to \mathbb{R}^{m+1}, \; F(x,t) := (x(t), u(x(t),t)) .$$

such that

$$\begin{cases} \dfrac{\partial F}{\partial t} = H\,\nu, \\ F(\cdot,0) = \mathrm{Graph}(u). \end{cases}$$

Note that, in general, our setting to study mean curvature flow is a family of immersions $F : M \times [0,T) \to \mathbb{R}^{m+1}$ that satisfies the mean curvature equation (1.1.1). However, since here $M = \mathrm{Graph}(u)$, we can simply consider the map F defined on $\mathbb{R}^m \times [0,T)$.

Thus, we will need to know the geometry of the graphical hypersurfaces, which is condensed in the following lemma.

Lemma 1.2.1 (**Graphical hypersurfaces**)**.** *Let $u : \mathbb{R}^m \to \mathbb{R}$ be a smooth map. Then the (induced) metric, the downwards unit normal vector, the scalar second fundamental form and the mean curvature of* $\mathrm{Graph}(u) \subset \mathbb{R}^{m+1}$ *are given by:*

$$\begin{aligned} g_{ij} &= \delta_{ij} + u_i u_j, \\ g^{ij} &= \delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}, \\ \nu &= \frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}}, \\ A_{ij} &= \frac{u_{ij}}{\sqrt{1+|\nabla u|^2}}, \\ H &= \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right). \end{aligned}$$

Notation 1.2.2. As usual, the subscripts of the first and second fundamental forms denote evaluation of these forms at the corresponding vector fields of a local basis $\left\{ \dfrac{\partial}{\partial x_i} \right\}_{i=1}^m$ of the tangent bundle, i.e., the components of these tensors in a local basis:

$$g_{ij} := g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right), \qquad A_{ij} := A\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right).$$

In the case of the smooth function u, the subscripts denote first or second partial derivative:

$$u_i := u_{x_i} = \partial_i u = \frac{\partial u}{\partial x_i}, \qquad u_{ij} := u_{x_i x_j} = \partial_{ij} u = \frac{\partial^2 u}{\partial x_i x_j}.$$

Proof.

1. Considering the parametrization

$$F : \mathbb{R}^m \to \text{Graph}(u) \subset \mathbb{R}^{m+1}, F(x) := (x, u(x)),$$

the induced metric of Graph(u) is

$$g_{ij} = \langle F_i, F_j \rangle = \langle (e_i, u_i), (e_j, u_j) \rangle = \delta_{ij} + u_i u_j,$$

where e_i are the vectors of the canonical basis of $\mathbb{R}^{m+1}$.

2. It holds that

$$\begin{aligned} g_{ik} g^{kj} &= (\delta_{ik} + u_i u_k) \left(\delta_{kj} - \frac{u_k u_j}{1 + |\nabla u|^2} \right) = \\ &= \delta_{ik}\delta_{kj} - \delta_{ik} \frac{u_k u_j}{1 + |\nabla u|^2} + u_i u_k \delta_{kj} - u_i u_k \frac{u_k u_j}{1 + |\nabla u|^2} = \\ &= \delta_{ij} - \frac{u_i u_j}{1 + |\nabla u|^2} + u_i u_j - |\nabla u|^2 \frac{u_i u_j}{1 + |\nabla u|^2} = \\ &= \delta_{ij} - (1 + |\nabla u|^2) \frac{u_i u_j}{1 + |\nabla u|^2} + u_i u_j = \\ &= \delta_{ij} - u_i u_j + u_i u_j = \delta_{ij}, \end{aligned}$$

which means that g^{ij} is indeed the inverse matrix of g_{ij}.

3. The vectors $F_i = (e_i, u_i)$, $i = 1, \dots, m$, form a basis of the tangent space to Graph(u). On the other hand, the vector $(\nabla u, -1)$ is orthogonal to all these vectors:

$$\langle F_i, (\nabla u, -1) \rangle = \langle (e_i, u_i), (\nabla u, -1) \rangle = u_i - u_i = 0.$$

Therefore, normalizing it, we obtain the downwards unit normal vector

$$\nu = \frac{(\nabla u, -1)}{\sqrt{1 + |\nabla u|^2}}.$$

4. It is a straightforward computation from the definition of the scalar second fundamental form:

$$A_{ij} = \langle F_{ij}, -\nu\rangle = \left\langle (0, u_{ij}), \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}} \right\rangle = \frac{u_{ij}}{\sqrt{1+|\nabla u|^2}}.$$

   Note that the scalar second fundamental form is considered with respect to $-\nu$, according to our convention (1.1.3).

5. To establish the formula $H = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$, we will compute both sides of the equality independently.

   On the one side,

$$\begin{aligned}
\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) &= \frac{\partial}{\partial x_i}\left(\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right) \\
&= \frac{\sum_{i=1}^n u_{ii}}{\sqrt{1+|\nabla u|^2}} - \frac{1}{2}u_i \frac{1}{(1+|\nabla u|^2)^{3/2}} \partial_i \left(|\nabla u|^2\right) \\
&= \frac{\Delta u}{\sqrt{1+|\nabla u|^2}} - \frac{1}{2}u_i \frac{1}{(1+|\nabla u|^2)^{3/2}} \partial_i \left(\sum_j (u_j)^2\right) \\
&= \frac{\Delta u}{\sqrt{1+|\nabla u|^2}} - \frac{u_i u_j u_{ij}}{(1+|\nabla u|^2)^{3/2}}.
\end{aligned}$$

   On the other hand,

$$\begin{aligned}
H = g^{ij}A_{ij} &= \left(\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}\right)\frac{u_{ij}}{\sqrt{1+|\nabla u|^2}} \\
&= \delta_{ij}\frac{u_{ij}}{\sqrt{1+|\nabla u|^2}} - \frac{u_i u_j u_{ij}}{(1+|\nabla u|^2)^{3/2}} \\
&= \frac{\Delta u}{\sqrt{1+|\nabla u|^2}} - \frac{u_i u_j u_{ij}}{(1+|\nabla u|^2)^{3/2}}.
\end{aligned}$$

   Hence,

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

□

Remark 1.2.3. In the last item of the previous lemma we have seen that

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{1}{\sqrt{1+|\nabla u|^2}}\left(\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}\right) u_{ij}.$$

This expression will be helpful below.

Now we can use Lemma 1.2.1 to deduce a partial differential equation for u (instead of F) for the evolution of graphical hypersurfaces.

Recall that our starting point is the family of immersions

$$F : \mathbb{R}^m \times [0,T) \to \mathbb{R}^{m+1},\ F(x,t) := (x(t), u(x(t),t)),$$

satisfying the scalar mean curvature equation (1.1.2):

$$\left\langle \frac{\partial F}{\partial t}, -\nu \right\rangle = H.$$

We start the computation by differentiating F with respect to the t-argument and applying the chain rule, we have that

$$\frac{\partial F}{\partial t} = \left(\frac{\partial x}{\partial t}, \frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial t} + \frac{\partial u}{\partial t}\right) = \left(\frac{\partial x}{\partial t}, \left\langle \nabla u, \frac{\partial x}{\partial t}\right\rangle + \frac{\partial u}{\partial t}\right),$$

so the scalar mean curvature equation becomes

$$\left\langle \left(\frac{\partial x}{\partial t}, \left\langle \nabla u, \frac{\partial x}{\partial t}\right\rangle + \frac{\partial u}{\partial t}\right), -\nu \right\rangle = H. \tag{1.2.1}$$

Using Lemma 1.2.1 to express ν, the left-hand side of this equation is:

$$\begin{aligned}
&\left\langle \left(\frac{\partial x}{\partial t}, \left\langle \nabla u, \frac{\partial x}{\partial t}\right\rangle + \frac{\partial u}{\partial t}\right), -\nu \right\rangle \\
&= \left\langle \left(\frac{\partial x}{\partial t}, \left\langle \nabla u, \frac{\partial x}{\partial t}\right\rangle\right) + \left(0, \frac{\partial u}{\partial t}\right),\ \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}} \right\rangle \\
&= \frac{1}{\sqrt{1+|\nabla u|^2}}\left[\left\langle \left(\frac{\partial x}{\partial t}, \left\langle \nabla u, \frac{\partial x}{\partial t}\right\rangle\right),\ (-\nabla u, 1)\right\rangle + \left\langle \left(0, \frac{\partial u}{\partial t}\right),\ (-\nabla u, 1)\right\rangle\right] \\
&= \frac{1}{\sqrt{1+|\nabla u|^2}}\frac{\partial u}{\partial t},
\end{aligned}$$

And by Remark 1.2.3, the left-hand side of (1.2.1) is:

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{1}{\sqrt{1+|\nabla u|^2}}\left(\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}\right)u_{ij}.$$

Therefore, equation (1.2.1) reads

$$\frac{\partial u}{\partial t} = \sqrt{1+|\nabla u|^2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \tag{1.2.2}$$

or, in expanded form,

$$\frac{\partial u}{\partial t} = \left(\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}\right)u_{ij}, \tag{1.2.3}$$

where, following Einstein notation, recall that we are summing over repeated indices.

1.3 Special solutions: translators

In general, evolution equations often have special solutions, called solitons, that evolve over time by a conformal transformation of the ambient space. Consequently, these solutions keep their shape during their evolution. Two very important classes of solitons in mean curvature flow are self-shrinkers and translating solutions, which evolve by an homothety or a translation respectively.

This thesis is about translating solutions of the mean curvature flow. Therefore, we introduce them in more detail.

Translators. Let $F_0 : M \to \mathbb{R}^{m+1}$ be an orientable hypersurface. For $T > 0$, consider a family of immersions $\{F_t\}_{t\in[0,T)}$, where $F_t := M \to \mathbb{R}^{m+1}$. Assume that it represents a translating soliton of the mean curvature flow, i.e., that the family of hypersurfaces moves by translation. Then there must exist a vector depending only on time, $\mathrm{v}(t) \in \mathbb{R}^{m+1}$, such that

$$F_t(p) = F_0(p) + \mathrm{v}(t),$$

Observe also that, under these hypothesis, the solution exists for all times, that is, $T = +\infty$, in which case it is said that the solution is eternal.
On the other hand, as a solution of the mean curvature flow, it holds

$$\left(\frac{\partial}{\partial t}F_t\right)^{\perp} = \mathbf{H} \Leftrightarrow (\mathrm{v}'(t))^{\perp} = \mathbf{H},$$

Since the family of immersions is a family of translations of M, the mean curvature vector $\mathbf{H} = \mathbf{H}(p,t)$ is independent of time, which means that $(\mathrm{v}'(t))^{\perp}$ is a constant vector in $\mathbb{R}^{m+1}$.
In summary, for a translating soliton of the mean curvature flow there exists a constant vector in the ambient space such that the mean curvature vector coincides with its normal component at each point, and vice versa.

The previous discussion motivates the following definition.

Definition 1.3.1. *An oriented smooth hypersurface $F : M \to \mathbb{R}^{m+1}$ is called* translating soliton *of the mean curvature flow (*translator *for short) if its mean curvature vector field* $\mathbf{H}$ *satisfies the differential equation*

$$\mathbf{H} = \mathrm{v}^{\perp}, \tag{1.3.1}$$

where $\mathrm{v} \in \mathbb{R}^{m+1}$ *is a fixed vector and* $\mathrm{v}^{\perp}$ *stands for the orthogonal projection of* v *to the normal bundle of the immersion* F.

For simplicity we will assume that all translators to be considered in this thesis are translating in the direction of $\mathrm{v} = e_{m+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{m+1}$, unless, unless otherwise stated.

Graphical translators. By Graphical translators we mean translating solitons of the mean curvature flow that are graphs over the $x_1 \ldots x_m$-hyperplane, i.e., usual graphs, and that are translating in the direction of $e_{m+1} \in \mathbb{R}^{m+1}$.
Therefore, the family of immersions associated to a graph translator is

$$F_t(x) = F_0(x) + te_{m+1} = (x, u(x,0)) + t(0, \ldots, 0, 1) = (x, u(x,0) + t)\,.$$

Remark 1.3.2. When we work with graphical translators, the t-argument can be suppressed and reintroduced at conveniency simply taking into account that

$$u(x,t) = u(x,0) + t.$$

Observe also that, since

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial\left(u(x,0)+t\right)}{\partial t} = 1,$$

the mean curvature equation for graphical translators, (1.2.3), simplifies to

$$1 = \left(\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}\right)u_{ij}. \tag{1.3.2}$$

Translators and minimal hypersurfaces. Due to a result of Ilmanen [Ilm94], there is a duality between translators in the euclidean space $\mathbb{R}^{m+1}$ and minimal surfaces in $(\mathbb{R}^{m+1}, \mathrm{g})$, where g is the conformally flat Riemannian metric

$$\mathrm{g}(\cdot\,,\cdot) := e^{\frac{2}{m}\langle p,\mathrm{v}\rangle}\langle\cdot\,,\cdot\rangle,$$

and $\langle\cdot\,,\cdot\rangle$ stands for the euclidean inner product of $\mathbb{R}^{m+1}$. The metric g will be called Ilmanen's metric. Specifically, every translator in the euclidean space $\mathbb{R}^{m+1}$ is a minimal surface in $(\mathbb{R}^{m+1}, \mathrm{g})$ and vice versa.

Once this duality is known, it is not difficult to prove it:

Lemma 1.3.3. *Let $f : M^m \to \mathbb{R}^{m+1}$ be an immersed oriented smooth hypersurface. Then $M \equiv f(M^m)$ is a translator in $(\mathbb{R}^{m+1}, \langle\cdot,\cdot\rangle)$ with translating velocity* v *if and only if M is a* g*-minimal hypersurface in $(\mathbb{R}^{m+1}, \mathrm{g})$.*

Proof. We provide a direct proof computing the relation between the mean curvatures of the hypersurface with respect to each of the two metrics.

First, we derive the relation between the mean curvatures in the general case, that is, for an arbitrary riemannian metric g in M and for an arbitrary conformal change of metric $\tilde{g} := e^{2f}g$, where f is a smooth function in M. Our goal is to compute the corresponding mean curvatures H and $\tilde{H}$. To this end, we use the expression of the mean curvature as the trace (with respect to the metric) of the scalar second fundamental form:

$$H = \sum_{i,j=1}^{m} g^{ij}A_{ij},$$

$$\tilde{H} = \sum_{i,j=1}^{m} \tilde{g}^{ij}\tilde{A}_{ij}. \tag{1.3.3}$$

Obviously, $\tilde{g}^{ij} = e^{-2f} g^{ij}$. On the other hand,

$$\tilde{A}_{ij} = \tilde{g}\big(\tilde{\mathbf{A}}(\partial_i, \partial_j), -\tilde{\nu}\big) = e^{2f} g\big(\tilde{\nabla}_{\partial_i}\partial_j, -\tilde{\nu}\big) = e^{f} g\big(\tilde{\nabla}_{\partial_i}\partial_j, -\nu\,\big), \tag{1.3.4}$$

where $\tilde{\mathbf{A}}$ is the second fundamental form, $-\tilde{\nu}$ is the Gauß map with respect to the second fundamental form is considered (recall the convention (1.1.3)), and $\tilde{\nabla}$ is the Levi-Civita connection of $(M, \tilde{g})$. We denote the coordinate vector fields by $\partial_i \equiv \frac{\partial}{\partial x_i}$. In the last equality we used that $\tilde{\nu} = e^{-f}\,\nu$.

Claim 1. *The Levi-Civita connection ∇ of (M, g) and the Levi-Civita connection $\tilde{\nabla}$ of $(M, \tilde{g})$ are related by the following identity*

$$\tilde{\nabla}_X Y = \nabla_X Y + (Xf)Y + (Yf)X - g(X, Y)\,\mathrm{grad}\, f,$$

for any X, Y smooth vector fields in M.

The proof of Claim 1 is immediate from the Koszul formula. Now, using the relation given by Claim 1, (1.3.4) reads

$$\tilde{A}_{ij} = e^{f}\big(A_{ij} - g_{ij}\langle \mathrm{grad}\, f, -\,\nu\rangle\big) \tag{1.3.5}$$

And using this expression in (1.3.3), together with $\tilde{g}^{ij} = e^{-2f} g^{ij}$, we get

$$\tilde{H} = \sum_{i,j=1}^{m} e^{-f} g^{ij}\big(h_{ij} - g_{ij}\langle \mathrm{grad}\, f, -\,\nu\rangle\big) = e^{-f}\big(H + m\langle \mathrm{grad}\, f, \nu\rangle\big), \tag{1.3.6}$$

which is the relation between the mean curvatures.

Finally, note that in our specific case (Ilmanen's metric) the smooth function $f : M \to \mathbb{R}$ is

$$f(p) = \frac{1}{m}\langle p, \mathrm{v}\rangle \Rightarrow \mathrm{grad}\, f(p) = \frac{1}{m}\,\mathrm{v}\,.$$

Thus

$$\tilde{H} = e^{\frac{1}{m}\langle p, \mathrm{v}\rangle}\big(H + \langle \mathrm{v}, \nu\rangle\big).$$

Therefore,

$$\tilde{H} = 0 \Leftrightarrow H = -\langle \mathrm{v}, \nu\rangle,$$

as claimed. $\square$

1.4 Classical examples of translators

The simplest examples of solutions to the mean curvature flow are minimal hypersurfaces, that is, solutions for which H vanishes identically. Dynamically, these are solutions that do not move at all. If, moreover, we look for the translating ones, the translating equation must be satisfied:

$$\mathbf{0} = \mathbf{H} = \mathrm{v}^{\perp},$$

that is, v must be tangential to the translator. Therefore, the simplest examples of translators are hyperplanes containing the direction of translation v.

According to their simplicity, the next examples are the grim reaper cylinder, which is explicity known, and the rotationally symmetric translators, which are solutions to an ODE. Because of their importance in our work, we introduce them in detail in the following subsections.

1.4.1 The grim reaper cylinder

In order to introduce the grim reaper cylinder, first it is convenient to mention the *curve shortening flow*, the analogy of the mean curvature flow in ambient dimension 2, considering the curvature of the curve instead of the scalar mean curvature of the hypersurface. It is very well known that the curve given by

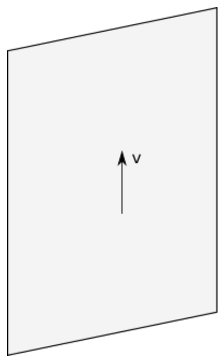
$$\gamma : (-\pi/2, \pi/2) \to \mathbb{R}^2, \quad \gamma(t) = (t, -\log\cos t),$$

is a translating solution of the curve shortening flow. It is called the grim reaper curve, and it is the unique translating curve up to homotheties and rigid motions.

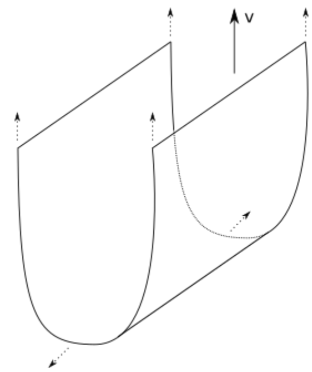
The *grim reaper cylinder* $\mathcal{G}$ is the product of the grim reaper curve and $\mathbb{R}^{m-1}$. Then, it can be parametrized by

$$F : (-\pi/2, \pi/2)\times\mathbb{R}^m \to \mathbb{R}^{m+1},\ F(x_1, \dots, x_m) = (x_1, x_2, \dots, x_m, -\log\cos x_1).$$

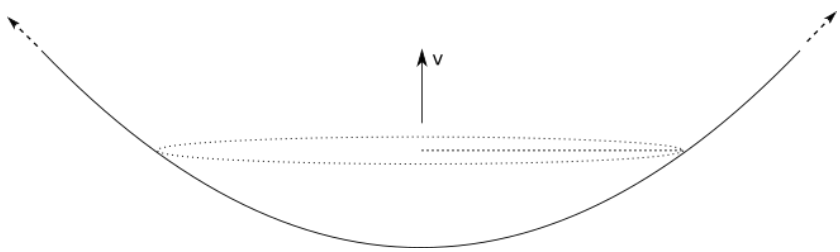
Consequently, it is a graphical translator which is invariant under translations in the direction of e_i, $i \in \{2, 3, \dots, m\}$. Note also that this parametrization is an embedding, so we will usually identify the grim reaper cylinder $\mathcal{G}$ with its image by this parametrization.



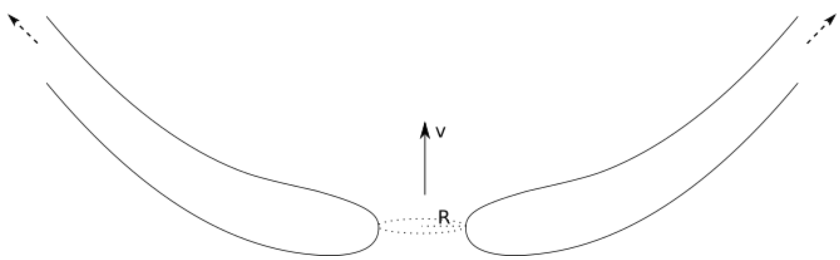
(a) A plane tangential to v



(b) A grim reaper cylinder



(c) A translating paraboloid or bowl soliton



(d) A translating catenoid or winglike translator

Figure 1.1: Classic examples of translators

In the following lemma we collect a few computations about the grim reaper cylinder that will be useful later.

Lemma 1.4.1. *Using the above parametrization of the grim reaper cylinder* $\mathcal{G}$*, we have that, at any point* $(x_1, x_2, \ldots, x_m, -\log\cos x_1) \in \mathcal{G}$*, an orthonormal frame* $\{E_i\}_{i=1}^m$ *of the tangent space, the downwards unit normal vector* ν

and the scalar mean curvature H are given respectively by:

$$E_1 = \cos x_1 \cdot e_1 + \sin x_1 \cdot e_{m+1}, \quad E_i = e_i \;\; \textit{for any } i = 2, \ldots, m,$$

$$\nu = \sin x_1 \cdot e_1 - \cos x_1 \cdot e_{m+1},$$
$$H = \cos x_1,$$

where the e_i, $i \in \{1, \ldots, m+1\}$, are the vectors of the canonical basis of $\mathbb{R}^{m+1}$.

Proof. The grim reaper cylinder is a graphical hypersurface, so we can apply Lemma 1.2.1. We will use the same notation as in Lemma 1.2.1, except that here we will denote by e_i, $i = 1, \ldots, m+1$, the vectors of the canonical basis of $\mathbb{R}^{m+1}$ instead of $\mathbb{R}^m$. In particular, we will denote the above parametrization by

$$F(x_1, \ldots, x_m) = (x_1, x_2, \ldots, x_m, u(x_1, \ldots, x_m))$$

where $u(x_1, \ldots, x_m) := -\log \cos x_1$. Then,

$$u_1 = \tan x_1, \quad u_i = 0 \qquad \text{for any } i = 2, \ldots, m,$$

$$u_{11} = 1 + \tan^2 x_1 = \frac{1}{\cos^2 x_1}, \quad u_{ij} = 0 \quad \text{for any other } i, j.$$

By Lemma 1.2.1, we know that the vectors $\{F_{x_i}\}_{i=1}^m$ form a basis of the tangent space:

$$\begin{aligned} F_{x_1} &= (1, 0, \ldots, 0, u_1) = e_1 + \tan x_1 \cdot e_{m+1} \\ F_{x_i} &= (1, 0, \ldots, 0, u_i) = e_i \quad \text{for any } i = 2, \ldots, m, \end{aligned}$$

and the induced metric g of $\mathcal{G}$ is

$$g_{ij} = \delta_{ij} + u_i u_j = \delta_{ij} + \delta_{1i} \tan^2 x_1 \quad \text{for any } i, j \in \{1, 2, \ldots, m\}.$$

Then an orthornomal frame of the tangent space is

$$\begin{aligned} E_1 &= \frac{F_{x_1}}{\sqrt{\langle F_{x_1}, F_{x_1} \rangle}} = \cos x_1 \cdot F_{x_1} = \cos x_1 \cdot (e_1 + \tan x_1 \cdot e_{m+1}) \\ &= \cos x_1 \cdot e_1 + \sin x_1 \cdot e_{m+1}, \\ E_i &= F_{x_i} = e_i \quad \text{for any } i = 2, \ldots, m. \end{aligned}$$

The downwards unit normal ν is

$$\begin{aligned}\nu &= \frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}} = \frac{(-\tan x_1, 0, \dots, 0, -1)}{\sqrt{1+\tan^2(x_1)}} \\ &= \cos x_1 \cdot (\tan x_1, 0, \dots, 0, -1) = (\sin x_1, 0, \dots, 0, -\cos x_1) \\ &= \sin x_1 \cdot e_1 - \cos x_1 \cdot e_{m+1}.\end{aligned}$$

Finally, the mean curvature is given by

$$\begin{aligned}H &= \delta_{ij}\frac{u_{ij}}{\sqrt{1+|\nabla u|^2}} - \frac{u_i u_j u_{ij}}{(1+|\nabla u|^2)^{3/2}} = -\frac{u_{11}}{\sqrt{1+u_1^2}} - \frac{u_1^2 u_{11}}{(1+u_1^2)^{3/2}} \\ &= \sqrt{u_{11}} - \frac{u_1^2}{\sqrt{u_{11}}} = \frac{u_{11}-u_1^2}{\sqrt{u_{11}}} = \frac{1}{\sqrt{u_{11}}} = \cos x_1,\end{aligned}$$

where we used the derivatives of u computed above and the fact that, in this particular case, $1+u_1^2 = u_{11} \Leftrightarrow u_{11} - u_1^2 = 1$.

Nevertheless, we can obtain further information if we compute the principal curvatures. For instance, this can be done easily through the shape operator.
The partial derivatives of ν are:

$$\begin{aligned}\nu_1 &= (-\cos x_1, 0, \dots, 0, -\sin x_1) = -\cos x_1 \cdot e_1 - \sin x_1 \cdot e_{m+1} = -E_1 \\ \nu_i &= (0, \dots, 0).\end{aligned}$$

Thus, in the above basis $\{F_{x_i}\}_{i=1}^m$ of the tangent space, the derivative of the Gauß map, $d\,\nu$, is

$$\begin{aligned}d\,\nu(F_{x_1}) &= \nu_1 = -E_1 = -\cos x_1 \cdot F_{x_1}, \\ d\,\nu(F_{x_i}) &= \nu_i = (0, \dots, 0) = 0 \cdot F_{x_i} \quad \text{for any } i = 2, \dots, m,\end{aligned}$$

so the shape operator $S := -d\,\nu$ (this definition preserves our convention (1.1.3) on the sign of H with respect to $-\nu$ in mean curvature flow) is

$$\begin{aligned}S(F_{x_1}) &= -d\,\nu(F_{x_1}) = \cos x_1 \cdot F_{x_1}, \\ S(F_{x_i}) &= -d\,\nu(F_{x_i}) = 0 \cdot F_{x_i} \quad \text{for any } i = 2, \dots, m.\end{aligned}$$

Therefore, the principal curvatures are $\lambda_1 = \cos x_1$ and $\lambda_i = 0$ for any $i = 2, \dots, m$, and the scalar mean curvature is

$$H = \sum_{i=1}^m \lambda_i = \cos x_1.$$

□

Remark 1.4.2. The grim reaper cylinder $\mathcal{G}$ is strictly mean convex, that is,

$$H = \cos x_1 > 0 \quad \text{for all points in } \mathcal{G},$$

since $x_1 \in (-\pi/2, \pi/2)$.

Tilted grim reaper cylinders. Applying a suitable dilation and rotation to a grim reaper cylinder, new examples of graphical translators with the same translating velocity e_{m+1} can be constructed. These examples are known as tilted grim reaper cylinders and they were introduced in [Lee12] by Lee using his correspondence from null curves in $\mathbb{C}^3$ to translators in $\mathbb{R}^3$ [Lee12, Theorem 4]. We will follow a different approach based on the invariance of the grim reaper cylinder under translations in the direction of e_i, $i \in \{2, 3, \ldots, m\}$.

Consider the grim reaper cylinder $\mathcal{G}$ and transform it as follows:

1. Dilation.
   Apply to $\mathcal{G}$ a dilation of factor $\lambda > 1$.
   Note that with this dilation the translating velocity changes from e_{m+1} to $(1/\lambda)e_{m+1}$; this follows, for instance, from the scalar translating equation: initially the equation is

   $$H = -\langle e_{m+1}, \nu \rangle,$$

   and after the dilation of factor λ the unit normal vector $\hat{\nu}$ of $\lambda\mathcal{G}$ is the same one but the mean curvature $\hat{H}$ is scaled by $1/\lambda$, so denoting by w to the translating velocity of $\lambda\mathcal{G}$ we have that:

   $$\hat{H} = -\langle w, \hat{\nu} \rangle \Leftrightarrow \frac{1}{\lambda} H = -\langle w, \nu \rangle \Leftrightarrow H = -\langle \lambda w, \nu \rangle,$$

   thus

   $$-\langle e_{m+1}, \nu \rangle = -\langle \lambda w, \nu \rangle \Leftrightarrow w = \frac{1}{\lambda} e_{m+1}.$$

2. Translation in order to obtain unitary translating velocity.
   Since $\lambda\mathcal{G}$ is invariant under translations in the direction of e_i, $i \in \{2, 3, \ldots, m\}$, applying a translation of vector ae_2, where $a \in \mathbb{R}$, $\lambda\mathcal{G}$ can be considered as a translator in the direction of

   $$(1/\lambda)\,\mathrm{v} + ae_2 = (0, a, 0, \ldots, 0, 1/\lambda) \in \mathbb{R}^{m+1}, \text{ for any } a \in \mathbb{R}.$$

In particular, for $a_0 = \sqrt{1-(1/\lambda)^2}$ we have that

$$\tilde{\mathrm{v}} = (0, \sqrt{1-(1/\lambda)^2}, 0, \dots, 0, 1/\lambda) \in \mathbb{R}^{m+1}$$

is a unit vector. Observe that $\lambda > 1$ is necessary in this step.

3. Rotation.
Finally, a rotation around the x_1-axis is performanced in order to transform $\tilde{\mathrm{v}}$ into e_{m+1}. Specifically, the angle α of rotation must be the one between the vectors $\tilde{v}$ and e_{m+1}, that is,

$$\frac{1}{\lambda} = \langle \tilde{\mathrm{v}}, e_{m+1} \rangle = |\tilde{\mathrm{v}}||e_{m+1}| \cos\alpha = \cos\alpha \Rightarrow \alpha = \arccos\left(\frac{1}{\lambda}\right).$$

In this way we obtain a dilated, slanted grim reaper cylinder translating with velocity e_{m+1} and defined over a strip of width $\lambda\pi > \pi$, called tilted grim reaper cylinder.

1.4.2 Rotationally symmetric translators

The aim of this section is to introduce the rotationally symmetric translators described in [CSS07]. We will follow the notation in this reference, except for the dimension, which we denote by m instead of n. In this way we can easily keep track of the results exposed there and develop their arguments.

First, assume that M is a rotationally symmetric graphical e_{m+1}-translator given by $u : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}, (x,t) \mapsto u(x,t)$. In fact, the rotationally symmetric property implies that the translator is, in a certain way, one-dimensional: it can be described more easily via the map $V : (0, +\infty) \times \mathbb{R} \to \mathbb{R}, (r,t) \mapsto V(r,t)$, where $r = |x| = \sqrt{\sum_{i=1}^m x_i^2}$. In this case the translating equation reads

$$1 = \frac{V''}{1+(V')^2} + (m-1)\frac{V'}{r}, \tag{1.4.1}$$

where r' denotes derivatives with respect to r.

Proof. Recall the equation (1.3.2) for a graphical translator:

$$1 = \left(\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}\right) u_{ij}.$$

By the rotational symmetry, there exists $V : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ defined by

$$u(x,t) = V(|x|, t) = V\left(\left(\sum_{i=1}^{m} x_i^2 \right)^{1/2}, t \right) = V(r,t).$$

To make the computation using this extra hypothesis, let us introduce the function

$$\tilde{u} : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R},\ (x,t) \mapsto \tilde{u}(x,t) := V\left(\left(\sum_{i=1}^{m} x_i^2 \right)^{1/2}, t \right).$$

Hence,

$$\tilde{u}_i = \frac{\partial V}{\partial r} \frac{1}{2r} 2x_i = \frac{1}{r} x_i \frac{\partial V}{\partial r},$$

$$|\nabla \tilde{u}|^2 = \sum_{i=1}^{m} \tilde{u}_i^2 = \frac{1}{r^2} r^2 \left(\frac{\partial V}{\partial r} \right)^2 = \left(\frac{\partial V}{\partial r} \right)^2,$$

$$\begin{aligned} \tilde{u}_{ij} &= \frac{\partial}{\partial x_j} \left(\frac{x_i}{r} \frac{\partial V}{\partial r} \right) = \frac{\delta_{ij} r - \frac{x_i x_j}{r}}{r^2} \frac{\partial V}{\partial r} + \frac{x_i}{r} \frac{\partial^2 V}{\partial r^2} \frac{1}{r} x_j \\ &= \frac{1}{r^2} \left(\left(\delta_{ij} r - \frac{x_i x_j}{r} \right) \frac{\partial V}{\partial r} + x_i x_j \frac{\partial^2 V}{\partial r^2} \right). \end{aligned}$$

Then,

$$
\begin{aligned}
\tilde{u}_i\tilde{u}_j\tilde{u}_{ij} &= \frac{1}{r^4}\left(\left(\delta_{ij}x_ix_jr - \frac{(x_ix_j)^2}{r}\right)\left(\frac{\partial V}{\partial r}\right)^3 + (x_ix_j)^2\left(\frac{\partial V}{\partial r}\right)^2\frac{\partial^2 V}{\partial r^2}\right)\\
&= \frac{1}{r^4}\left(\left(r^2r - \frac{r^2r^2}{r}\right)\left(\frac{\partial V}{\partial r}\right)^3 + r^2r^2\left(\frac{\partial V}{\partial r}\right)^2\frac{\partial^2 V}{\partial r^2}\right)\\
&= \left(\frac{\partial V}{\partial r}\right)^2\frac{\partial^2 V}{\partial r^2}, \qquad (1.4.2)
\end{aligned}
$$

$$
\begin{aligned}
\delta_{ij}\tilde{u}_{ij} &= \frac{1}{r^2}\left(\left(\delta_{ij}r - \frac{\delta_{ij}x_ix_j}{r}\right)\frac{\partial V}{\partial r} + \delta_{ij}x_ix_j\frac{\partial^2 V}{\partial r^2}\right)\\
&= \frac{1}{r^2}\left(\left(mr - \frac{r^2}{r}\right)\frac{\partial V}{\partial r} + r^2\frac{\partial^2 V}{\partial r^2}\right)\\
&= \frac{1}{r^2}\left((m-1)\,r\frac{\partial V}{\partial r} + r^2\frac{\partial^2 V}{\partial r^2}\right)\\
&= \frac{1}{r}(m-1)\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2},
\end{aligned}
$$

$$
\begin{aligned}
\left(1+|\nabla\tilde{u}|^2\right)\delta_{ij}\tilde{u}_{ij} &= \left(1+\left(\frac{\partial V}{\partial r}\right)^2\right)\left(\frac{1}{r}(m-1)\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2}\right)\\
&= \frac{1}{r}(m-1)\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}(m-1)\left(\frac{\partial V}{\partial r}\right)^3\\
&\quad + \left(\frac{\partial V}{\partial r}\right)^2\frac{\partial^2 V}{\partial r^2}. \qquad (1.4.3)
\end{aligned}
$$

Thus, subtracting (1.4.3) and (1.4.2), one gets

$$
\begin{aligned}
&\frac{1}{r}(m-1)\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}(m-1)\left(\frac{\partial V}{\partial r}\right)^3\\
&= \frac{1}{r}(m-1)\frac{\partial V}{\partial r}\left(1+\left(\frac{\partial V}{\partial r}\right)^2\right) + \frac{\partial^2 V}{\partial r^2},
\end{aligned}
$$

and finally dividing by $1+|\nabla\tilde{u}|^2 = 1+\left(\frac{\partial V}{\partial r}\right)^2$, the graphical translating equation (1.3.2) reads

$$1 = \frac{1}{r}(m-1)\frac{\partial V}{\partial r} + \frac{\frac{\partial^2 V}{\partial r^2}}{1+\left(\frac{\partial V}{\partial r}\right)^2},$$

as claimed. □

Our goal now is to study the existence of solutions to equation 1.4.1. Note that this is a second order ODE. Considering $\varphi := V'$, it becomes a first order ODE:

$$1 = \frac{\varphi'}{1+\varphi^2} + (m-1)\frac{\varphi}{r} \Leftrightarrow \varphi' = (1+\varphi^2)\left(1-(m-1)\frac{\varphi}{r}\right). \quad (1.4.4)$$

Lemma 1.4.3 (Lemma 2.1 in [CSS07])**.** *For any $R > 0$ and $\varphi_0 \in \mathbb{R}$, the boundary value problem*

$$\begin{cases} \varphi'(r) = (1+\varphi^2)\left(1-(m-1)\frac{\varphi}{r}\right), \\ \varphi(R) = \varphi_0, \end{cases}$$

has a unique C^∞-solution φ on $[R, +\infty)$. Moreover, as $r \to +\infty$, we have the asymptotic expansion

$$\varphi(r) = \frac{r}{m-1} - \frac{1}{r} + O(r^{-2}). \quad (1.4.5)$$

Existence easily follows from a discussion about the sign of the derivative of φ, using equation (1.4.4).

The asymptotic behaviour is more tricky to establish. The basic idea is to use a computer to conjecture the form of the solution, and then proving the asymptotic behaviour suggested by this solution using this information (the coefficients on it) and equation (1.4.4).

Existence of the translating paraboloid.

Lemma 1.4.4 (Lemma 2.2 in [CSS07])**.** *There exists an entire rotationally symmetric, strictly convex graphical e_{m+1}-translator, $U : \mathbb{R}^m \times [0, +\infty) \to \mathbb{R}, m \geq 2$. We have the following asymptotic expansion as r approaches infinity:*

$$U(r,t) = t + \frac{r^2}{2(m-1)} - \log r + O(r^{-1}).$$

The existence was shown by Altschuler and Wu in [AW94], as well as the fact that these translators are asymptotic to the paraboloid $\frac{r^2}{2(m-1)}$, which justify the name *translating paraboloid*. The finer asymptotic behaviour at infinity comes from Lemma 1.4.3.

These translators, which are unique up to a rigid motion, are also known as the *bowl soliton* since their shape reminds of the one of these objects. See Figure 1.1c.

Existence of the translating catenoid.

Lemma 1.4.5 (Lemma 2.3 in [CSS07])**.** *For every $R > 0$, there exist rotationally symmetric graphical e_{m+1}-translators, $W_R^+, W_R^- : (\mathbb{R}^m \setminus B_R) \times [0, +\infty) \to \mathbb{R}, m \geq 2$. We have the following asymptotic expansion as r approaches infinity:*

$$U(r,t) = t + \frac{r^2}{2(m-1)} - \log r + O(r^{-1}) + C^{\pm}.$$

Proof. We split the construction of this translator into two steps.

Step 1: Construction of a small piece of the translator considering it as a graph over the e_{m+1}-axis.

At points where the tangent space is not horizontal (i.e., it is not orthogonal to e_{m+1}), the translator can be represented locally as a graph over the e_{m+1}-axis:

$$\bigcup_{x_{m+1}} h(x_{m+1}, t) \cdot \mathbb{S}^{m-1} \times \{x_{m+1}\},$$

where $h : (a, b) \times (0, +\infty) \to \mathbb{R}$ is a function defined over a certain interval (a, b) of the e_{m+1}-axis, which represents the radius of the surface given its last component, i.e, the "height".

Claim 2. *The function h satisfies the ODE*

$$h'' = \left(\frac{m-1}{h} - h'\right)(1 + h'^2),$$

where $'$ denotes differentiation with respect to x_{m+1}.

Proof of Claim 1. We distinguish two cases:

Case A. At points where the tangent space is not vertical (i.e., it is not parallel to e_{m+1}), the translator can be represented locally as a graph V over the $e_1 \dots e_m$-hyperplane. Observe that the functions h and V are inverse (the first one represents the radius of the translator given its height and the other one is precisely the reverse): $V \circ h = \mathrm{id}$. Applying the chain rule to this identity, we have that $V'(h)h' = 1$, that is,

$$V'(h) = \frac{1}{h'}. \tag{1.4.6}$$

Hence,

$$V''(h)h' = [V'(h)]' = \left[(h')^{-1}\right]' = -\frac{1}{(h')^2}h'',$$

where the first equailty follows from the chain rule and the second one from (1.4.6). We conclude that

$$V''(h) = -\frac{h''}{(h')^3}. \tag{1.4.7}$$

On the other hand, the hypothesis imply that V satisfies the translator equation for rotationally symmetric graphical translators (1.4.1). Evaulating this equation at points $h = h(z)$, we have that

$$1 = \frac{V''(h)}{1 + (V'(h))^2} + (m-1)\frac{V'(h)}{h},$$

Using the previous computations,

$$\frac{V''(h)}{1 + (V'(h))^2} = -\frac{h''}{(h')^3}\frac{1}{1 + \left(\frac{1}{h'}\right)^2} = -\frac{h''}{(h')^3}\frac{(h')^2}{1 + (h')^2} = -\frac{h''}{h'}\frac{1}{1 + (h')^2},$$

$$(m-1)\frac{V'(h)}{h} = (m-1)\frac{1}{hh'}.$$

Hence,

$$1 = \frac{V''(h)}{1 + (V'(h))^2} + (m-1)\frac{V'(h)}{h} \Leftrightarrow 1 = -\frac{h''}{h'}\frac{1}{1 + (h')^2} + (m-1)\frac{1}{hh'},$$

or equivalently

$$h'' = \left(\frac{m-1}{h} - h'\right)(1 + (h')^2),$$

as claimed.

Case B. At points where the tangent space is vertical, we can consider the two branches of the translator and argue separately with each of them as in the previous Case A. □

Now fix $z_0 \in \mathbb{R}$; a different choice of z_0 corresponds to translating the hypersurface in the e_{m+1}-direction.

Claim 3. *Starting with $h'(z_0) = 0, h(z_0) = R > 0$, we obtain a strictly convex solution h in a small interval around z_0.*

Proof of Claim 2. It is straightforward. By Claim 2 and the initial conditions at the point z_0, we have that

$$\begin{aligned} &h'(z_0) = 0, \\ &h''(z_0) = \left(\frac{m-1}{h(z_0)} - h'(z_0)\right)(1 + h'^2(z_0)) = \frac{m-1}{h(z_0)} = \frac{m-1}{R} > 0, \end{aligned}$$

then z_0 is a local minimum of h, which in the coordinate system considered means precisely that h is strictly convex in a small interval around z_0. □

Step 2: Construction of the rest of the translator considering it as a graph over the $e_1 \dots e_m$-hyperplane.

Therefore, we return now to our original coordinate system. From this point of view, the translator constructed so far has two branches. Lemma 1.4.3 allows us to extend both branches all the way to infinity. Indeed, to this end, consider two points in the interior of the constructed translator and apply Lemma 1.4.3 with the corresponding initial data (i.e., the corresponding slope in the points considered. It follows that both branches can be extended uniquely to the infinity and, moreover, both branches have the same asymptotic behaviour at infinity, the one given in Lemma 1.4.3, as claimed.

□

Remark 1.4.6. The extension of the translator constructed in Lemma 1.4.5 is carried out through a *superposition procedure*, not through a *glue argument*.

These translators, which are unique up to a rigid motion, are also known as the *winglike* translator because of their wing-shape. See Figure 1.1d.

Chapter 2

Translators and the tangency principle

The aim of this chapter is to use the classic examples of translators described in 1.4 and the tangency principle (see section 2.1) to deduce interesting consequences on translating solitons of the mean curvature flow.

In section 2.1, we use the tangency principle to derive two non-existence results for translators. In section 2.2 we provide a height estimate for compact translators. In section 2.3 it is shown that a graphical perturbation of a graph translator of revolution M which is asymptotic to M, remains a hypersurface of revolution. As an immediate consequence, we give an alternative proof of the uniqueness theorem for complete embedded translating solitons with a single end that are asymptotic to a translating paraboloid [MSHS15, Theorem A]. Finally, in section 2.4, using the Alexandrov's reflection principle we prove that if a compact translator lies between two parallel planes P_1 and P_2 which are orthogonal to v, and its boundary consists of two strictly convex curves contained respectively in P_1 and P_2, then the translator inherits the symmetries of its boundary.

Notation. We will often discuss the hypothesis in our results using pieces of these examples, in which case the following notation will be very useful: for any $a \in \mathbb{R}$, we denote the corresponding closed upper and lower half-space in $\mathbb{R}^{m+1}$, respectively, by

$$Z_a^+ = \{(x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : x_{m+1} \geq a\},$$
$$Z_a^- = \{(x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : x_{m+1} \leq a\}.$$

Remark 2.0.1. After I submitted the preprint [PG16], whose content led to this chapter, I learned that Pyo also studied compact translating solitons with non-empty planar boundary in the interesting paper [Pyo16], using similar techniques, that is, the tangency principle. In particular, he also used the Alexandrov's reflection principle in order to prove that compact translators spanning two horizontal planar Jordan curves inherits the symmetries of its boundary. The only difference is that our proof follows the approach in [MSHS15, Section 3] and Pyo's proof follows the one in [Lóp]; that is why we assume that the boundary curves are strictly convex. Moreover, Pyo notes in his paper the interesting fact that there exist plenty of compact translators with plane boundary, as a consequence of a result of Serrin [Ser69] which says that the necessary and sufficient condition for the existence of a solution of the graphical translating equation (1.3.2) in a domain D in a horizontal plane is that D is mean convex.

2.1 Non-existence of translators

We begin with the statement of our main tool throughout this paper, the tangency principle.

Theorem 2.1.1 (Tangency principle). *Let Σ_1 and Σ_2 be embedded connected translators in $\mathbb{R}^{m+1}$ with boundaries $\partial\Sigma_1$ and $\partial\Sigma_2$.*

(a) (**Interior principle**) *Suppose that there exists a common point x in the interior of Σ_1 and Σ_2 where the corresponding tangent spaces coincide and such that Σ_1 lies at one side of Σ_2. Then Σ_1 coincides with Σ_2.*

(b) (**Boundary principle**) *Suppose that the boundaries $\partial\Sigma_1$ and $\partial\Sigma_2$ lie in the same hyperplane Π and that the intersection of Σ_1, Σ_2 with Π is transversal. Assume that Σ_1 lies at one side of Σ_2 and that there exists a common point of $\partial\Sigma_1$ and $\partial\Sigma_2$ where the surfaces Σ_1 and Σ_2 have the same tangent space. Then Σ_1 coincides with Σ_2.*

Roughly speaking, this maximum principle says that two different translators cannot "touch" each other at one interior or boundary point. Thanks to the fact that translating solitons are minimal hypersurfaces in a conformally changed Riemannian metric [Ilm94], the proof is based on the well-known tangency principle for minimal hypersurfaces. For more details, please see [MSHS15, Theorem 2.1].

In [MSHS15, Remark 3.1 (c)], the authors pointed out that the tangency principle implies that there are no complete and embedded translators contained in a solid half-cylinder. In [Møl14], Møller combined knowledge of explicit examples of translators with a maximum principle and he proved a geometric obstruction (the so-called *funnel condition*) that generalized previously known non-existence conditions such as the above-mentioned cylindrical boundedness.

Following these ideas, we provide an easy proof for the non-existence of translators inside a cylinder:

Theorem 2.1.2. *Let $f : M^m \to \mathbb{R}^{m+1}$ be a non-compact embedded connected translator with compact boundary (possiby empty). Then M cannot be contained in any cylinder.*

Remark 2.1.3. By compact boundary we mean that the boundary consists of finite connected components, each of them compact.

Proof. We argue by contradiction. Suppose that $M \equiv f(M^m)$ is contained in a cylinder $\mathcal{C}_{r_0}$. We distinguish two cases:

Case 1: The axis of $\mathcal{C}_{r_0}$ is parallel to the direction of translation v.

Consider first a winglike translator $\mathcal{W}_{R_0}$ with center in the axis of $\mathcal{C}_{r_0}$ and with radius $R_0 > r_0$, so that, in particular, $\mathcal{W}_{R_0} \cap M = \emptyset$. Consider next the family of winglike translators $\{\mathcal{W}_R\}_{0<R\leq R_0}$. Since $\mathcal{W}_{R_0} \cap M = \emptyset$, there must be a $R_1 \in (0, R_0]$ such that $\mathcal{W}_{R_1}$ intersects M for the first time. Without loss of generality, we can assume that this first point of contact is an interior point of both surfaces, otherwise it is at the boundary of M, in which case it is sufficient to consider the initial winglike translator $\mathcal{W}_{R_0}$ located at a higher height (recall that the boundary of M is compact by hypothesis). Therefore, by the interior tangency principle, $M \subset \mathcal{W}_{R_1}$, which contradicts that M is a non-compact surface contained in $\mathcal{C}_{r_0}$.

Case 2: The axis of $\mathcal{C}_{r_0}$ is not parallel to v.

In this case the argument is similar but comparing with a grim reaper cylinder. Let us see it in detail. Due to the compactness of the boundary and the non-compactness of the translator, there exists a real number a such that $\overline{S} \cap \partial M = \emptyset$, where $S := (-\pi + a, \pi + a) \times \mathbb{R}^m$. Let $\hat{\mathcal{G}}$ be the grim reaper cylinder located in this slab S at a large height so that it does not intersect M. Then translate it down until it "touches" M for the first time. Observe that this procedure is feasible because $\overline{S} \cap M$ is compact, since by hypothesis the cylinder is tilted. Moreover, as $\overline{S} \cap \partial M = \emptyset$, this point of contact must be in the interior of M. Hence, by the interior tangency principle, $M \subset \hat{\mathcal{G}}$, a contradiction. $\square$

Remark 2.1.4. Let us make here some remarks concerning the previous Theorem 2.1.2.

a) The result is not true if the translator (with boundary) is compact. A counterexample is the piece of translating paraboloid $\mathcal{P}$ obtained by cutting this surface with a horizontal plane at any arbitrary but fixed height $a > 0$ and considering the lower part, that is, $\mathcal{P} \cap Z_a^-$.

b) The compactness of the boundary is also necessary. A counterexample is the intersection of the grim reaper cylinder $\mathcal{G}$ with a cylinder of arbitrary but fixed radius $R > 0$ and axis the x_2-axis; this surface is contained, for instance, in the cylinder of radius $2R$ and axis the x_2-axis.

In the following result we prove that there are no translators *that resemble a handle* (see figure 2.1). More precisely,

Theorem 2.1.5. *There do not exist a connected compact embedded translator in $\mathbb{R}^{m+1}$ whose boundary is contained in a hyperplane orthogonal to the direction of translation* v *and consists of two strictly convex Jordan curves located at distance greater or equal than π and such that one of them is not contained in the region enclosed by the other one.*

Proof. We will denote by $f : M \to \mathbb{R}^{m+1}$ to an embedding of M, and by P to the hyperplane that contained the boundary of $M \equiv f(M)$.

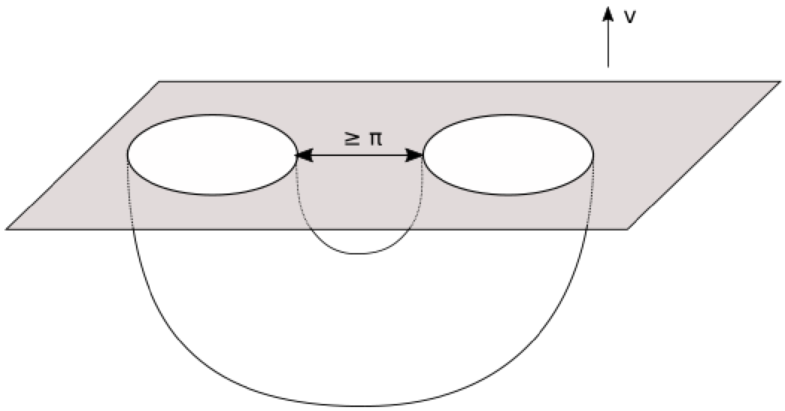


Figure 2.1: A surface under the conditions of Theorem 2.1.5

First, note that M must be below the plane P. Otherwise, by compactness of M, the height function of M, $u := \langle f, \mathrm{v} \rangle$, would attain a global maximum. But recall [MSHS15, Lemma 2.1 (d)] that this height function satisfies the equation $\Delta u + |\nabla u|^2 = 1$, so u does not admit any local maxima in the interior, a contradiction.

Now consider the segment s realizing the distance between the two boundary curves of M. The length of this segment is greater or equal than π by hypothesis. Let l be the straight line in the direction of v passing through the middle point of the segment s. Place a grim reaper cylinder $\hat{\mathcal{G}}$ in such a way that its lower generatix coincides with l. Observe that $\hat{\mathcal{G}}$ is strictly contained in a slab S defined as the cartesian product of the segment s times the line l. Initially $\hat{\mathcal{G}}$ does not intersect M because M is below the hyperplane P. Then translate $\hat{\mathcal{G}}$ down following the direction of translation v until it intersects M for the first time, which necessarily occurs in an interior point of M because any of these translations of $\hat{\mathcal{G}}$ is strictly contained in the slab S. Then, by the interior tangency principle, $M \subset \hat{\mathcal{G}}$, which is absurd. □

Remark 2.1.6.

a) In Theorem 2.1.5, it is necessary that the boundary curves lie in the same hyperplane. Otherwise the result is not true, as the following example shows: the piece of the translating paraboloid which is between two horizontal hyperplanes: $Z_a^+ \cap \mathcal{P} \cap Z_b^-$, where $0 < a < b$.

b) Moreover, if it is allowed that one of the boundary curves is in the region enclosed by the other one, then there exist translators under the rest of the hypothesis of the theorem 2.1.5. For instance, the intersection of a winglike translator with a lower half-space, $\mathcal{W}_{\mathcal{R}} \cap Z_a^-$, where, obviously, a is large enough so that this intersection is non-empty.

2.2 A height estimate

Our aim in this section is to develop a geometric argument for obtaining an upper bound to the maximum height that a compact embedded translator in $\mathbb{R}^3$ can achieve.

Theorem 2.2.1. *Let $M \subset \mathbb{R}^3$ be a connected compact embedded translator whose boundary is a connected curve Γ contained in a plane P orthogonal to* v. *Assume that the diameter of Γ is $d > 0$. Then, for all $p \in M$, the distance in $\mathbb{R}^3$ from p to P is less or equal than*

$$\begin{cases} -\log\cos\left(\frac{d}{2}\right) & 0 < d < \pi \\ \min\limits_{1 < s \leq s_0} C(s) & d \geq \pi \end{cases}$$

where $C : (1, +\infty) \to (0, +\infty)$ is the function given by

$$C(s) := -\left(\frac{d}{\pi}s\right)^2 \log\cos\left(\frac{\pi/2}{s}\right) + \frac{d}{2}\sqrt{\left(\frac{d}{\pi}s\right)^2 - 1},$$

and

$$s_0 := \frac{\pi}{2}\frac{1}{\arctan\left(\frac{4-\sqrt{2}}{2}\right)} \approx 1.722.$$

Proof. The idea is to compare M with an appropiate grim reaper cylinder.

First suppose that $0 < d < \pi$. Without loss of generality, assume that the diameter of lenght d coincides with

$$\{(x, y, z) \in \mathbb{R}^3 : -d/2 \leq x \leq d/2, y = 0, z = z_0\},$$

for an arbitrary but fixed $z_0 \in \mathbb{R}$. Consider a grim reaper cylinder $\mathcal{G}$ and observe that, since $d < \pi$, the region between the two parallel planes asymptotic to $\mathcal{G}$ contains Γ. Hence, this grim reaper cylinder can be translated down until it does not intersect M. Now traslate it up until their first point of contact occurs. By the tangency principle, this must happen at a boundary point of M. Observe also that for any point $(x_0, y_0, -\log\cos x_0)$ of the grim reaper cylinder, the width between its two "wings" is precisely $2x_0$, so the width is d when the height is $-\log\cos\left(\frac{d}{2}\right)$. In conclusion, this argument shows that M must be contained in the compact region enclosed by the intersection of the horizontal plane P and a grim reaper cylinder whose lowest point is at distance $-\log\cos\left(\frac{d}{2}\right)$ from P, which proves the boundedness if $0 < d < \pi$.

Second, suppose that $d \geq \pi$. In this case a grim reaper cylinder $\mathcal{G}$ cannot contained Γ. It is necessary a dilation of factor $\lambda > \frac{d}{\pi} > 1$. But then the velocity changes, which does not allow us to use the tangency principle anymore. To overcome this problem, we use tilted grim reaper cylinders. Following the description in 1.4.1, consider the grim reaper cylinder

$$\mathcal{G} = \{(x, y, -\log\cos x) : (x, y) \in (-\pi/2, \pi/2) \times \mathbb{R}\},$$

and apply to $\mathcal{G}$ a dilation of factor $\lambda > 1$ such that $\lambda\pi > d$,

$$\lambda\mathcal{G} = \{\lambda(x, y, -\log\cos x) : (x, y) \in (-\pi/2, \pi/2) \times \mathbb{R}\},$$

so that Γ fits in the slab determined by the dilated grim reaper cylinder $\lambda\mathcal{G}$. Observe that there are infinite factors of dilation with this property. A way to parametrize them is to consider $\lambda(s) := \frac{d}{\pi}s$, where $s > 1$. For brevity, we will usually omit the parameter s.
Then, following the procedure and notation in 1.4.1, the suitable rotation is given by the matrix

$$R_x(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\lambda & -\sqrt{1-(1/\lambda)^2} \\ 0 & \sqrt{1-(1/\lambda)^2} & 1/\lambda \end{pmatrix}$$

Therefore, a parametrization of the tilted grim reaper cylinder is

$$R_x(\alpha)(\lambda\mathcal{G}) = \Big\{ \Big(\lambda x, y + \sqrt{\lambda^2 - 1}\log\cos x, \sqrt{\lambda^2-1}y - \log\cos x\Big) : \\ (x, y) \in (-\pi/2, \pi/2) \times \mathbb{R}\Big\}. \tag{2.2.1}$$

For brevity, we will denote $R_x(\alpha)(\lambda\mathcal{G})$ by $\mathcal{G}_{\lambda,\alpha}$.
Now the idea is to translate $\mathcal{G}_{\lambda,\alpha}$ until it does not intersect M and translate it back until they intersect each other for the first time. By the tangency principle, the first point of contact must be at the boundary of M. To make the computations it is convenient to consider the following static situation, which is equivalent: to determine the intersection of $\mathcal{G}_{\lambda,\alpha}$ with the cylinder $\mathcal{C}$ of diameter d,

$$\mathcal{C} = \mathcal{C}_{d/2} := \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = \left(\frac{d}{2}\right)^2 \right\}, \tag{2.2.2}$$

and compute the global minimum and maximun of the third coordinate function of the parametrization of this intersection.
Combining (2.2.1) and (2.2.2), we obtain that a parametrization of the intersection of $\mathcal{G}_{\lambda,\alpha}$ and $\mathcal{C}$ is $\gamma_\pm : \left[-\frac{d/2}{\lambda}, \frac{d/2}{\lambda}\right] \to \mathbb{R}^3$ given by

$$\begin{aligned}\gamma_\pm(x) := \Bigg(&\lambda x, \pm\sqrt{(d/2)^2 - (\lambda x)^2},\\ &- \lambda^2 \log\cos x \pm \sqrt{\lambda^2 - 1}\sqrt{(d/2)^2 - (\lambda x)^2}\Bigg).\end{aligned}$$

The critical points of $\gamma_\pm$ correspond to $x = 0$:

$$\left(0, -\frac{d}{2}, -\frac{d}{2}\sqrt{\lambda^2 - 1}\right), \quad \left(0, \frac{d}{2}, \frac{d}{2}\sqrt{\lambda^2 - 1}\right).$$

On the other hand, the points on the boundary of $\mathcal{G}_{\lambda,\alpha} \cap \mathcal{C}$ are

$$\left(-\frac{d}{2}, 0, -\lambda^2 \log\cos\frac{d/2}{\lambda}\right), \quad \left(\frac{d}{2}, 0, -\lambda^2 \log\cos\frac{d/2}{\lambda}\right).$$

Therefore, evaluating at all these points we conclude that the global maximum and minimum of the third coordinate function of $\gamma_\pm$ are $-\lambda^2 \log\cos\frac{d/2}{\lambda}$ and $-\frac{d}{2}\sqrt{\lambda^2 - 1}$, respectively. Hence, the boundedness is given in this case by their difference, which is precisely $C(s)$, as claimed.

Now observe that the function $C(s)$ is positive and

$$\lim_{s \to 1^+} C(s) = \lim_{s \to +\infty} C(s) = +\infty,$$

hence $C(s)$ has a global minimum. The problem is that it cannot be computed analytically. Indeed, the critical points of $C(s)$ are the zeros of

$$C'(s) = -2\frac{d^2}{\pi^2} s \log\cos\left(\frac{\pi/2}{s}\right) - \frac{d^2}{2\pi}\tan\left(\frac{\pi/2}{s}\right) + \frac{d^3}{2\pi^2}\frac{s}{\sqrt{\left(\frac{d}{\pi}s\right)^2 - 1}}.$$

Nevertheless, we can determine an $s_0 > 1$ such that $C(s)$ is increasing in $(s_0, +\infty)$. Specifically, for all $s > 1$,

$$\begin{aligned}
C'(s) &= -2\frac{d^2}{\pi^2} s \log\cos\left(\frac{\pi/2}{s}\right) - \frac{d^2}{2\pi}\tan\left(\frac{\pi/2}{s}\right) + \frac{d^3}{2\pi^2}\frac{s}{\sqrt{\left(\frac{d}{\pi}s\right)^2 - 1}} \\
&> -\frac{d^2}{2\pi}\tan\left(\frac{\pi/2}{s}\right) + \frac{d^3}{2\pi^2}\frac{s}{\sqrt{\left(\frac{d}{\pi}s\right)^2 - 1}} \\
&= \frac{d^2}{2\pi}\left(-\tan\left(\frac{\pi/2}{s}\right) + \frac{d}{\pi}\frac{s}{\sqrt{\left(\frac{d}{\pi}s\right)^2 - 1}}\right) \\
&\geq \frac{d^2}{2\pi}\left(-\tan\left(\frac{\pi/2}{s}\right) + \frac{d}{\pi}\frac{s}{\sqrt{\left(\frac{d}{\pi}s\right)^2}}\right) \\
&= \frac{d^2}{2\pi}\left(-\tan\left(\frac{\pi/2}{s}\right) + 1\right).
\end{aligned}$$

Since

$$-\tan\left(\frac{\pi/2}{s}\right) + 1 \geq 0 \Leftrightarrow s \geq 2,$$

then

$$\min_{s\in(1,+\infty)} C(s) = \min_{1<s\leq 2} C(s).$$

Once we know that C is increasing for $s > 2$, we can easily improve the above lower bound of C':

$$\begin{aligned} C'(s) &= \frac{d^2}{2\pi}\left(\frac{4}{\pi}s\left(-\log\cos\left(\frac{\pi/2}{s}\right)\right) - \tan\left(\frac{\pi/2}{s}\right) + \frac{d}{\pi}\frac{s}{\sqrt{\left(\frac{d}{\pi}s\right)^2 - 1}}\right) \\ &> \frac{d^2}{2\pi}\left(1\left(1 - \cos\left(\frac{\pi/2}{s}\right)\right) - \tan\left(\frac{\pi/2}{s}\right) + 1\right) \\ &= \frac{d^2}{2\pi}\left(2 - \cos\left(\frac{\pi/2}{s}\right) - \tan\left(\frac{\pi/2}{s}\right)\right). \end{aligned}$$

Now, observe that

$$2 - \cos\left(\frac{\pi/2}{s}\right) - \tan\left(\frac{\pi/2}{s}\right) \geq 0 \Leftrightarrow \cos\left(\frac{\pi/2}{s}\right) + \tan\left(\frac{\pi/2}{s}\right) \leq 2,$$

and taking into account that, due to our previous computations, we can restrict our estimation of C' to the interval $(1, 2]$, we have that it is sufficient to find an s such that

$$\cos\left(\frac{\pi/2}{s}\right) + \tan\left(\frac{\pi/2}{s}\right) \leq \cos\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi/2}{s}\right) \leq 2,$$

that is,

$$\frac{\sqrt{2}}{2} + \tan\left(\frac{\pi/2}{s}\right) \leq 2 \Leftrightarrow s \geq \frac{\pi}{2}\frac{1}{\arctan\left(\frac{4-\sqrt{2}}{2}\right)},$$

and the proof is complete.

□

Remark 2.2.2. The height estimate is valid in a more general setting: in the statement of the Theorem 2.2.1, instead of considering the diameter d of Γ, we can assume that the curve Γ is strictly contained in a slab of width $d > 0$, and the proof remains exactly the same.

2.3 Graphical perturbations of translators

Definition 2.3.1 (Graphical perturbation)**.** *Let N be a connected graph hypersurface given by $u : U \subset \mathbb{R}^m \to \mathbb{R}^{m+1}$. Let M be a hypersurface in $\mathbb{R}^{m+1}$. We say that M is a graphical perturbation of N if there exists a function $\varphi : U \to \mathbb{R}$ such that M can be represented as the graph of $u + \varphi$, that is,*

$$M = \text{Graph}\,(u + \varphi).$$

We will say that M is an asymptotic graphical perturbation of N if M is a graphical perturbation of N and, moreover, for every sequence $\{x_i\}_{i=1}^{+\infty}$ in U such that $\lim_{i\to+\infty} u(x_i) = \infty$ it holds that $\lim_{i\to+\infty} \varphi(x_i) = 0$.

Finally, we will say that M is a (an asymptotic) graphical perturbation of N outside a compact set $\mathcal{K} \subset \mathbb{R}^{m+1}$ if $M - \mathcal{K}$ is a (an asymptotic) graphical perturbation of $N - \mathcal{K}$.

Remark 2.3.2. Note that if there exists a function φ as in the previous definition, then it is smooth because it is the difference of two graph hypersurfaces, which are always assumed to be smooth in this paper.

Roughly speaking, the asymptotic behaviour here means that, outside a bounded region in M, M is arbitrarily close to N. An example is shown in figure 2.2.

The goal of this section is to show that if we have two hypersurfaces that are graphically asymptotic outside a compact set, then there are some interesting common properties between them, such as being a graph hypersurface or a hypersurface of revolution. This is the content of our next theorem. Before stating it precisely, let us present the idea of the proof. Basically it is another consequence of the tangency principle, comparing the translator *with a transformation of itself* according to the following scheme:

1) Consider $\hat{M}$ a "copy" of M;

2) Translate $\hat{M}$ up, $\hat{M} \mapsto \hat{M} + t_0 v$ for some $t_0 > 0$, until it does not intersect M;

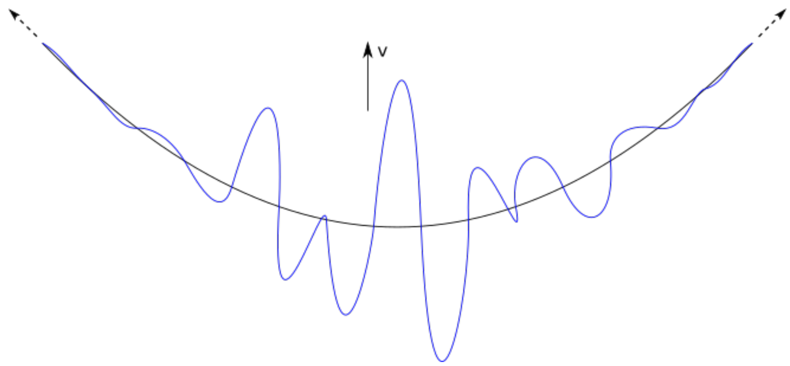


Figure 2.2: A profile view of an example of an asymptotic graphical perturbation of a translating paraboloid

3) Apply an isometry $i : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ of the ambient space to $\hat{M} + t_0 v$ so that $i(\hat{M} + t_0 v) \cap M = \emptyset$;

4) Move $i(\hat{M} + t_0 v)$ down until it "touches" M for the first time.

Then, by the interior tangency principle, $i(\hat{M}) = M$. Hence, M is invariant under the isometry i.

We will follow this scheme in the next results.

For example,

- To show that M is a graph hypersurface, take $i =$ identity;
- To show that M is a hypersurface of revolution, consider as i any arbitrary but fixed rotation around the axis of symmetry.

Here by a hypersurface (or set, in general) of revolution (or, equivalently, "rotationally symmetric hypersurface") in Euclidean space $\mathbb{R}^{m+1}$ we mean a hypersurface (set) of $\mathbb{R}^{m+1}$ which is invariant by the action of $\mathrm{SO}_l(m + 1)$, the subgroup of the special orthogonal group $\mathrm{SO}(m + 1)$ that fixes a given straight line l. We will assume that all the sets of revolution that appear together are sets of revolution with respect to the same axis unless otherwise stated.

Lemma 2.3.3. *Let N be a connected graph translator in $\mathbb{R}^{m+1}$. Assume that $M \subset \mathbb{R}^{m+1}$ is, outside a compact set $\mathcal{K} \subset \mathbb{R}^{m+1}$, a translator which is a graphical perturbation of N. Suppose that the boundary of M is graphical (possibly empty). Then M is graphical.*

Proof. Obviously, by definition of graphical perturbation, $M - \mathcal{K}$ is graphical. We have to prove that $M \cap \mathcal{K}$ also is. To this end, just apply the scheme presented above, which clearly works because we deal with a compact region. To avoid contact at the boundary of $M \cap \mathcal{K}$ during step 4, we work from the very beginning with a bigger compact set $\overline{B_r(0)} \supset \mathcal{K}$ with $r > 0$ sufficiently large so that the boundary created intersecting M with $\overline{B_r(0)}$ is graphical. Therefore, the contact at the boundary can occur only when $i(\hat{M})$ comes back to its original position, in which case M is graphical, as claimed. □

Corollary 2.3.4. *If the definitions given in 2.3.1 hold outside a compact set, they hold everywhere.*

Proof. The case of *graphical perturbation* is precisely the content of the previous lemma 2.3.3.

With respect to *asymptotic graphical perturbation*, observe simply that this definition is indepedent of what happens in compact regions since it deals with the behaviour of the surfaces at infinity. □

Theorem 2.3.5. *Let N be a connected graph translator of revolution in $\mathbb{R}^{m+1}$. Suppose that M is, outside a compact set $\mathcal{K} \subset \mathbb{R}^{m+1}$, a translator of $\mathbb{R}^{m+1}$ which is an asymptotic graphical perturbation of N. Assume that the boundary of M is graphical (possibly empty) and a set of revolution. Then M is a hypersurface of revolution.*

Remark 2.3.6. Under the hypothesis of Theorem 2.3.5, if there exists the boundary of M, then it is not necessarily connected. For instance, the intersection of a winglike translator with two different and parallel horizontal planes. But, in general, due to the rotational symmetry hypothesis on the boundary of M, each connected component of the boundary of M must be contained in a horizontal hyperplane, and indeed it must be a circumference.

Proof. We will show that the previous scheme works for any arbitrary but fix element i of $\mathrm{SO}_l(m+1)$, where l is the axis of symmetry. Indeed, l must be parallel to the direction of translation $\mathrm{v} = (0,\ldots,0,1)$, otherwise M would not be graphical. Let us consider such an isometry i.

First, by corollary 2.3.4, M is an asymptotic graphical perturbation of N everywhere. Then, there exists $d < \infty$ (for instance, $d := \max_U |\varphi|$) such that

$$|\varphi(x)| < d \text{ for all } x \in U.$$

Geometrically, this means that M is contained in a slab S of diameter d centered at N:

$$S := \{(s_1,\ldots,s_m,s_{m+1}) \in U \times \mathbb{R} : |s_{m+1} - u(s_1,\ldots,s_m)| \leq d, x \in N\},$$

and $M \subset S$.
Since N is a hypersurface of revolution by hypothesis, then S is a set of revolution.

We can easily argue now that our scheme works:

- Step 2 is trivially possible to do because M is graphical;
- Step 3 is achievable because

$$\begin{aligned} &M \subset S \text{ (by construction of } S), \ (\hat{M} + t_0 v) \cap S = \emptyset \text{ (by step 2)} \\ &\Rightarrow i(\hat{M} + t_0 v) \cap M \subset i(\hat{M} + t_0 v) \cap S = i\big((\hat{M} + t_0 v) \cap S\big) = \emptyset; \end{aligned}$$

- Step 4:
  The first point of contact cannot be at infinity because φ tends to zero at infinity.
  Since the boundary is a graphical set of revolution, the first point of contact cannot be at the boundary unless the hypersurface returns to its original position, in which case $i(\hat{M}) = M$, as claimed.

□

Corollary 2.3.7. [MSHS15, Theorem A] *Let $f : M^m \to \mathbb{R}^{m+1}$ be a complete embedded translating soliton of the mean curvature flow with a single end that is smoothly asymptotic to a translating paraboloid. Then, $M = f(M^m)$ is a translating paraboloid.*

Proof. It is a consequence of our previous theorem 2.3.5, taking N as the translating paraboloid.
Observe that the meaning of *smoothly asymptotic* in [MSHS15, Theorem A] is that there exists a sufficiently large $r > 0$ such that $M - B_r(0)$ can be written as the graph of a function g such that

$$g(x) = \frac{1}{2}||x|| - \frac{1}{2}\log\big(||x||\big) + O\bigg(\frac{1}{||x||}\bigg), \tag{2.3.1}$$

where $||\cdot||$ denotes the usual euclidean norm in $\mathbb{R}^m$.
Now, taking into account that the translating paraboloid is a graph hypersurface for a function $f \in C^\infty(\mathbb{R}^m)$ satisfying the same asymptotic behaviour

$$f(x) = \frac{1}{2}||x|| - \frac{1}{2}\log\big(||x||\big) + O\bigg(\frac{1}{||x||}\bigg), \tag{2.3.2}$$

then being *smoothly asymptotic* clearly implies being *an asymptotic graphical perturbation*. Indeed, from the relation $g = f + \varphi$ and from (2.3.1) and (2.3.2), we deduce that

$$\varphi = g - f = O\bigg(\frac{1}{||x||}\bigg).$$

That is, it is sufficient to consider as φ any smooth function such that $\varphi = O\Big(\frac{1}{||x||}\Big)$ (as $||x|| \to \infty$), i.e., $\varphi(x) \leq \frac{C}{||x||}$ for all $||x|| > r$ and for some constant $C \in \mathbb{R}$. □

2.4 Compact translators with symmetric boundary

In this section we apply the method of moving planes [Ale56, Sch84, Lóp] to study compact translators with symmetric boundary.

Theorem 2.4.1. *Let M be a connected compact embedded translator in $\mathbb{R}^{m+1}$ whose boundary consists of two strictly convex curves Γ_1 and Γ_2 contained, respectively, in two parallel planes P_1 and P_2 which are orthogonal to* v. *Assume that M lies between the two planes P_1 and P_2, and suppose that the curves Γ_1 and Γ_2 are symmetric with respect to a plane Π containing the direction of translation* v. *Then M is symmetric with respect to the plane Π.*

Proof. Without loss of generality, up to a rigid motion, we can assume that

$$P_1 = \{(x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : x_{m+1} = 0\}$$

and

$$\Pi = \{(x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : x_1 = 0\}.$$

We will apply the Alexandrov's method of moving planes (see [Ale56, Sch84]). We will follow the application of this method in [MSHS15, Section 3], including the notation, which we recall briefly:

The family of planes $\{\Pi(t)\}_{t\in\mathbb{R}}$ is given by

$$\Pi(t) := \big\{(x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : x_1 = t\big\},$$

and given a subset A of $\mathbb{R}^{m+1}$, for any $t \in \mathbb{R}$ we define the sets

$$\begin{aligned}
\delta_t(A) &:= \big\{(x_1, \dots, x_{m+1}) \in A : x_1 = t\big\} = A \cap \Pi(t),\\
A_+(t) &:= \big\{(x_1, \dots, x_{m+1}) \in A : x_1 \geq t\big\},\\
A_-(t) &:= \big\{(x_1, \dots, x_{m+1}) \in A : x_1 \leq t\big\},\\
A_+^*(t) &:= \big\{(2t - x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : (x_1, \dots, x_{m+1}) \in A_+(t)\big\},\\
A_-^*(t) &:= \big\{(2t - x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : (x_1, \dots, x_{m+1}) \in A_-(t)\big\}.
\end{aligned}$$

Note that $A_+^*(t)$ and $A_-^*(t)$ are the image of $A_+(t)$ and $A_-(t)$ by the reflection respect to the plane $\Pi(t)$.

Consider the set

$$\mathcal{A} := \{t \in [0, t_0] : M_+(t) \text{ is a graph over } \Pi \text{ and } M_+^*(t) \geq M_-(t)\},$$

where $t_0 := \max\{t > 0 : M \cap \Pi(t) \neq \emptyset\}$ is a positive real number that exists because of the compactness of M. Indeed, $q_0 := M \cap \Pi(t_0)$, the first point of contact between M and a vertical plane coming from $+\infty$, must be a boundary point of M, otherwise M would coincide with $\Pi(t_0)$ by the interior tangency principle, which is absurd.

Our goal is to prove that $0 \in \mathcal{A}$. The proof of this fact will be divided into 3 claims.

Claim 1. The set $\mathcal{A} - \{t_0\}$ is not empty. Moreover, if $s \in \mathcal{A}$, then $[s, t_0] \in A$.

To show that $\mathcal{A} - \{t_0\}$, we prove that there exists an $\varepsilon > 0$ such that $(t_0 - \varepsilon, t_0] \subset \mathcal{A}$.
First note that $\Gamma_1 \cup \Gamma_2$ is a bi-graph over its plane of symmetry Π because, by hypothesis, both boundary curves are strictly convex plane curves. Then, in a neighborhood around $q_0 \in \Gamma_1 \cup \Gamma_2$, M is a graph over Π. Otherwise, as M lies between the planes P_1 and P_2, a neighborhood of M around q_0 would be contain in the plane P_i, for some $i \in \{1, 2\}$, that is, M would not be locally around q_0 a translator in the direction of v, which is absurd. In other words, since q_0 is in $\Gamma_1 \cup \Gamma_2$ and it is the first point of contact between M and $\Pi(t_0)$, by continuity this implies that there exists a sufficiently small $\varepsilon > 0$ such that $M_+(t)$ is a graph over $\Pi(t)$ for every $t \in (t_0 - \varepsilon, t_0]$. Moreover, as M is embedded, considering $\varepsilon > 0$ smaller if necessary, it holds that $M_+^*(t) \geq M_-(t)$ for every $t \in (t_0 - \varepsilon, t_0]$.
For the second part of the claim, let $\tilde{t}$ be an arbitrary but fixed number in the interval (s, t_0). Our goal is to prove that $\tilde{t} \in \mathcal{A}$. According to the definition of the set $\mathcal{A}$, there are two conditions to be checked, so the proof falls naturally into two parts or steps.

Step 1: $M_+(\tilde{t})$ is a graph over Π.
As $s \in \mathcal{A}$, then $M_+(s)$ is a graph over Π. Therefore, $M_+(t)$ is a graph over Π for every $t \in [s, t_0]$. In particular, $M_+(\tilde{t})$ is a graph over Π.

Step 2: $M_+^*(\tilde{t}) \geq M_-(\tilde{t})$.
On the contrary, if $M_+^*(\tilde{t}) \ngeq M_-(\tilde{t})$, then, by compactness of M, there exists a number $t_1 \in [\tilde{t}, t_0 - \varepsilon)$ such that $M_+^*(t_1) - \delta_{t_1}(M)$ and $M_-(t_1) - \delta_{t_1}(M)$ intersect for the first time. Furthermore, this first point of contact is an interior point of $M_+^*(t_1)$ and $M_-(t_1)$ because the boundary of M consists of two strictly convex plane curves symmetric with respect to Π. Then $M_+^*(t_1) = M_-(t_1)$ by the interior tangency principle. Thus, $\Pi(t_1) \neq \Pi$ would be a plane of symmetry of M, hence, in particular, it would be a plane of symmetry of the curves Γ_1 and Γ_2, a contradiction.

Claim 2. $\mathcal{A}$ is a closed set of the interval $[0, t_0]$.
The argument is identical to the one in [MSHS15, Theorem A]: it is proved by contradiction, using the sequential characterization of closed sets; first it is assumed that the graphical condition in $\mathcal{A}$ is not satisfied, which contradicts Claim 1; then the graphical condition and the continuity gives the reflection condition in $\mathcal{A}$.

Claim 3. The minimum of the set $\mathcal{A}$ is 0.
We argue by contradiction. Suppose $s_0 := \min \mathcal{A} > 0$. We will show that then there exists $\varepsilon_0 > 0$ such that $s_0 - \varepsilon_0 \in \mathcal{A}$, contradicting that s_0 is the minimum of $\mathcal{A}$.
Again, we divide the proof into two steps.

Step 1: There exists $\varepsilon_1 \in (0, s_0)$ such that $M_+^*(s_0 - \varepsilon_1)$ is a graph over Π. Since $s_0 \in \mathcal{A}$, $M_+(s_0)$ is a graph over Π. Moreover, there is no point in $M_+(s_0)$ with normal vector included in Π. If there were such a point, let us say that its first coordinate is $\tilde{t} \in [s_0, t_0)$, then by the tangency principle at the boundary, $M_+^*(\tilde{t}) = M_-(\tilde{t})$, that is, $\Pi(\tilde{t})$ would be a plane of symmetry of M. In particular, $\Pi(\tilde{t})$ would be a plane of symmetry of the curves Γ_1 and Γ_2, which contradicts that $\Pi \neq \Pi(\tilde{t})$ also is. Thus,

$$\xi\{M_+(s_0)\} \cap \Pi = \emptyset.$$

As M is compact, we have that there exists $\varepsilon_1 \in (0, s_0)$ such that

$$\xi\{M_+(s_0)\} \cap \Pi = \emptyset \quad \text{for all } t \in [s_0 - \varepsilon_1, s_0].$$

From this fact, together with the compactness of M, it follows that $M_+(t)$ can be represented as a graph over the plane Π for every $t \in [s_0 - \varepsilon_1, s_0]$. In particular, $M_+^*(s_0 - \varepsilon_1)$ is a graph over Π and the proof of this step is complete.

Step 2: There exists $\varepsilon_0 \in (0, \varepsilon_1)$ such that $M_+^*(s_0 - \varepsilon_0) \geq M_-(s_0 - \varepsilon_0)$. We are going to show that there exists $\varepsilon_0 \in (0, \varepsilon_1]$ such that

$$M_+^*(t) \cap M_-(t) = \delta_t(M) \quad \text{for all } t \geq s_0 - \varepsilon_0,$$

which in particular implies that $M_+^*(s_0 - \varepsilon_0) \geq M_-(s_0 - \varepsilon_0)$, and this step will be proved.
We argue by contradiction. If it were not true, then there would exist an increasing sequence $\{t_n\}_{n \in \mathbb{N}}$ converging to s_0 such that

$$\big(M_+^*(t_n) \cap M_-(t_n)\big) - \delta_{t_n}(M) \neq \emptyset.$$

For each natural n, denote by $P_n = (p_1^n, p_2^n, p_3^n)$ a point in the above set. At this point, we make two key observations:

$$\big(M_+^*(t) \cap M_-(t)\big) - \delta_t(M) \subset M_-(s_0 - \varepsilon_1) \quad \text{for all } t \in [s_0 - \varepsilon_1, t_0], \quad (2.4.1)$$

$$M_+^*(s_0) \cap M_-(s_0) = \delta_{s_0}(M). \tag{2.4.2}$$

(2.4.1) follows from Step 1, that is, from the fact that $M_+^*(s_0-\varepsilon_1)$ is a graph over Π for every $t \in [s_0-\varepsilon_1, t_0]$. Therefore,

$$\Big(\big(M_+^*(t) \cap M_-(t)\big) - \delta_t(M)\Big) \cap S = \emptyset,$$

where $S := \{(x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} : s_0 - \varepsilon_1 \leq x_1 \leq t_0\}$, simply because, in plain language,

"*the reflection of a graph over a plane* Π *with respect to this plane* Π *is always on the right hand side of the left part of the graph*",

where orientation (right and left) is considered with respect to the plane Π. This is a direct consequence of the definitions, in particular from the meaning of being a graph over a plane.
On the other hand, (2.4.2) follows from the fact that $s_0 \in \mathcal{A}$. Indeed, if $M_+^*(s_0) \cap M_-(s_0)$ were a set bigger than $\delta_{s_0}(M)$, then, as $M_+^*(s_0) \geq M_-(s_0)$, there would be a first point of contact between $M_+^*(s_0)$ and $M_-(s_0)$, which would be in the interior because the boundary of M consists of two strictly convex plane curves symmetric with respect to Π. Then by the interior tangency principle both surfaces would coincide, hence $\Pi(s_0)$ would be a symmetric plane of M, a contradiction.
Let us come back to the sequence $\{P_n\}_{n\in\mathbb{N}}$. By the compactness of M, we can assume without loss of generality that this sequence converges to a point $P_\infty = (p_1^\infty, p_2^\infty, p_3^\infty) \in M$. Indeed, since $t_n \nearrow s_0$, $P_\infty \in M_+^*(s_0) \cap M_-(s_0) = \delta_{s_0}(M)$, where the last equality is by (2.4.2). On the other hand, from (2.4.1) we see that $p_1^n \leq s_0 - \varepsilon_1$ for each n. Thus, $p_1^\infty \leq s_0 - \varepsilon_1 < s_0$, which contradicts that $P_\infty \in \delta_{s_0}(M)$. $\square$

Remark 2.4.2. The assumption that M must be between the two parallel planes P_1 and P_2 cannot be dropped, as the following counterexample shows. Consider the intersection of a winglike solution $\mathcal{W}$ with two horizontal parallel planes P_1 and P_2, so that the lower one, P_1, contains the radius of $\mathcal{W}$. Observe that the intersection of each of these planes with $\mathcal{W}$ consists of two concentric circles. The counterexample is the piece of $\mathcal{W}$ between these two planes and whose boundary is the inner circle in P_1 and the outer circle in P_2.

Corollary 2.4.3. *In the setting of the previous theorem 2.4.1, if* Γ_1 *and* Γ_2 *are concentric circles, then* M *is a hypersurface of revolution.*

Proof. Simply note that, by theorem 2.4.1, M is symmetric with respect to any plane Π containing the direction of translation v. Therefore, M is a hypersurface of revolution around v. □

Corollary 2.4.4. *Theorem 2.4.1 remains true if the boundary of the translator* M *is assumed to be only one strictly convex plane curve* Γ*.*

Proof. Observe that in this case the translator lies below the plane P that contains the curve Γ because M is compact and, as a translator, its height function cannot attain a local maximum [MSHS15, Lemma 2.1 (d)]. Hence, the same argument using the Alexandrov's method proves the corollary. □

Corollary 2.4.5. *Let* M *be a connected compact embedded translator in* $\mathbb{R}^{m+1}$ *whose boundary* Γ *is a* $(m-1)$*-sphere contained in a hyperplane* P *orthogonal to* v*. Then* M *is the compact piece of the translating paraboloid whose boundary coincides with* Γ*.*

Proof. Let $\mathcal{P}$ be the compact piece of the translating paraboloid whose boundary coincides with Γ. Place $\mathcal{P}$ above the plane P so that its vertex lies on the same line as the center of Γ. Then translate it down until they "touch" for the first time. There are two possibilities: either they intersect for the first time in an interior point or they do it in a boundary point. In any case, the interior or boundary tangency principle tells us they coincide. □

Chapter 3

A characterization of the grim reaper cylinder

The content of this chapter is published in [MPGSHS16]. We prove that a connected and properly embedded translating soliton in $\mathbb{R}^3$ with uniformly bounded genus on compact sets which is C^1-asymptotic to two planes outside a cylinder, either is flat or coincides with the grim reaper cylinder. Before stating this result rigorously, let us set up the notation and provide some definitions.

Definition 3.0.1. *Let $\mathcal{H}$ be an open half-plane in $\mathbb{R}^3$ and w the unit inward pointing normal of $\partial\mathcal{H}$. For a fixed positive number δ, denote by $\mathcal{H}_\delta$ the set given by*

$$\mathcal{H}_\delta := \big\{p + t\,\mathrm{w} : p \in \partial\mathcal{H} \quad \textit{and} \quad t > \delta\big\}.$$

(a) *We say that a smooth surface M is C^k-asymptotic to the open half-plane $\mathcal{H}$ if M can be represented as the graph of a C^k-function $\varphi : \mathcal{H} \to \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ so that for any $j \in \{1, 2, \dots, k\}$ it holds*

$$\sup_{\mathcal{H}_\delta}|\varphi| < \varepsilon \quad \textit{and} \quad \sup_{\mathcal{H}_\delta}|D^j\varphi| < \varepsilon.$$

(b) *A smooth surface M is called C^k-asymptotic outside a cylinder to two half-planes $\mathcal{H}_1$ and $\mathcal{H}_2$ if there exists a solid cylinder $\mathcal{C}$ such that:*

(b_1) *the solid cylinder* $\mathcal{C}$ *contains the boundaries of the half-planes* $\mathcal{H}_1$ *and* $\mathcal{H}_2$,

(b_2) *the set* $M - \mathcal{C}$ *consists of two connected components* M_1 *and* M_2 *that are* C^1*-asymptotic to* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, respectively.*

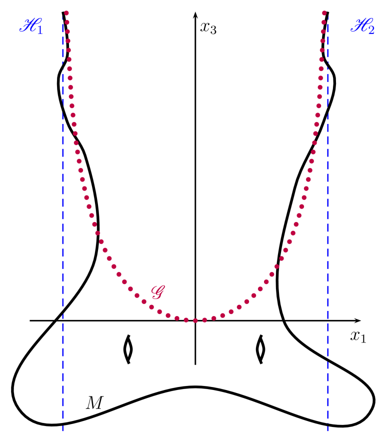


Figure 3.1: Asymptotic behavior

For example the grim reaper cylinder $\mathcal{G}$ is asymptotic to the parallel half-planes

$$\mathcal{H}_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > r_0 > 0,\ x_1 = -\pi/2\}$$

and

$$\mathcal{H}_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > r_0 > 0,\ x_1 = +\pi/2\}$$

outside the solid cylinder

$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_3^2 \leq r_0^2 + \pi^2/4\},$$

where here r_0 is a positive real constant.

Let us now state the main result of this chapter.

Theorem 3.0.2. *Let* $f : M^2 \to \mathbb{R}^3$ *be a connected, properly embedded*[1] *translating soliton with uniformly bounded genus on compact sets of* $\mathbb{R}^3$ *and* $\mathcal{C}$ *be a solid cylinder whose axis is perpendicular to the direction of translation of* $M := f(M^2)$*. Assume that* M *is* C^1*-asymptotic outside the cylinder* $\mathcal{C}$ *to two half-planes whose boundaries belongs on* $\partial\mathcal{C}$*. Then either*

[1]Here by embedded we only mean that M has no self-intersections.

(a) *both half-planes are contained in the same vertical plane* Π *and* $M = \Pi$, *or*

(b) *the half-planes are included in different parallel planes and* M *coincides with a grim reaper cylinder.*

Remark 3.0.3.

(a) Notice that in the above theorem infinite genus a priori could be possible. The assumption that M has uniformly bounded genus on compact sets of $\mathbb{R}^3$ means that for any positive r there exists $m(r)$ such that for any $p \in M$ it holds

$$\text{genus}\left\{M \cap \mathbb{B}_r(p)\right\} \leq m(r),$$

where $\mathbb{B}_r(p)$ is the ball of radius r in $\mathbb{R}^3$ centered at the point p. Roughly speaking, the above condition says that as we approach infinity the "size of the holes'' of M is not becoming arbitrary small and furthermore they are not getting arbitrary close to each other.

(b) We would like to mention here that Nguyen [Ngu15, Ngu13, Ngu09] constructed examples of complete embedded translating solitons in the euclidean space $\mathbb{R}^3$ with infinite genus. Outside a cylinder, these examples look like a family of parallel half-planes. This means that the hypothesis about the number of half-planes is sharp. Very recently, Dávila, Del Pino & Nguyen [DdPN15] and, independently, Smith [Smi15] constructed examples of complete embedded translators with finite non-trivial topology. For an exposition of examples of translators see also [MSHS15, Subsection 2.2].

(c) Ilmanen constructed a one-parameter family of complete convex translators, defined on strips, connecting the grim reaper cylinder with the bowl soliton [Whi02]. Note that the level sets of these translators are closed curves. This means that our hypothesis of being asymptotic to two planes outside a cylinder is natural and cannot be removed.

Let us describe now the general idea and the steps of the proof. Without loss of generality we will assume that the translating velocity of M is $\mathrm{v} = (0, 0, 1)$; then, again without loss of generality, we can choose the x_2-axis as the axis of rotation of $\mathcal{C}$. First we show that the half-planes must be parallel to each other, that they should be also parallel to the translating direction and that both wings of M outside the cylinder must point in the direction of v. Then, after a translation in the direction of the x_1-axis, if necessary, we prove that the asymptotic half-planes $\mathcal{H}_1$ and $\mathcal{H}_2$ are subsets of the parallel planes

$$\Pi(-\pi/2) = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = -\pi/2\big\}$$

and

$$\Pi(+\pi/2) = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = +\pi/2\big\},$$

respectively, and that M is contained in the slab between the planes $\Pi(-\pi/2)$ and $\Pi(+\pi/2)$. To prove this claim we study the x_1-coordinate function of M in order to control its range. By the strong maximum principle we conclude that the x_1-coordinate function cannot attain local maxima or minima. To prove that $\sup_M x_1 = \pi/2 = -\inf_M x_1$ we perform a "blow-down" argument based on a compactness theorem of White [Whi15b] for sequences of properly embedded minimal surfaces in Riemannian 3-manifolds. The next step is to show that M is a bi-graph over $\Pi(+\pi/2)$ and that the plane

$$\Pi(0) = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0\big\}$$

is a plane of symmetry for M. This is proven using Alexandrov's method of moving planes. In the sequel we show that M must be a graph over a slab of the x_1x_2-plane. Thus, M must have zero genus and it must be strictly mean convex. To achieve this goal we carefully investigate the set of the local maxima and minima of the profile curve

$$\Gamma = M \cap \Pi(0) \subset \mathcal{C}.$$

Performing again a "blow-down" argument along the ends of the curve Γ we deduce that M looks like a grim reaper cylinder at infinity. To finish the proof, we consider the function ξ_2 which measures the x_2-coordinate of the Gauß map ξ of M. Then, by applying the strong maximum principle to $\xi_2 H^{-1}$, we deduce that ξ_2 is identically zero. This implies that the Gauß curvature of M is zero and then M must coincide with a grim reaper cylinder (see [MSHS15, Theorem B]).

The structure of the chapter is as follows. In Section 3.1 we introduce the compactness and the strong barrier principle of White [Whi15a, Whi15b]. In

Section 3.2 we show that the smooth asymptotic behaviour with respect to the so-called Ilmanen's metric implies the smooth asymptotic behaviour with respect to the euclidean metric, an important fact that we need at the end of our proof. In Section 3.3 we present a lemma that will play a crucial role in the proof of our theorem. This lemma (Lemma 3.3.1) asserts that every complete, properly embedded translating soliton in $\mathbb{R}^3$ with the asymptotic behavior of two half-planes has a surprising amount of internal dynamical periodicity. The main theorem is proved in Section 3.4.

3.1 A compactness theorem and a strong barrier principle of Brian White

We will introduce here the main tools that we will use in the proofs. We will consider sequences of translators and we will take limits (up to a subsequence) thanks to a compactness theorem and a strong barrier principle, both results of Brian White [Whi15a, Whi15b]. Another crucial tool will be the tangency principle, already presented in section 2.1.

Let Σ be a surface in a 3-manifold (Ω, g). Given $p \in \Sigma$ and $r > 0$ we denote by

$$D_r(p) := \{w \in T_p\Sigma : |w| < r\}$$

the tangent disc of radius r. Consider now $T_p\Sigma$ as a vector subspace of $T_p\Omega$ and let ν be the unit normal vector of $T_p\Sigma$ in $T_p\Omega$. Fix a sufficiently small $\varepsilon > 0$ and denote by $W_{r,\varepsilon}(p)$ the solid cylinder around p, that is

$$W_{r,\varepsilon}(p) := \{ \exp_p(q + t\nu_q) : q \in D_r(p) \text{ and } |t| \leq \varepsilon\},$$

where exp stands for the exponential map of the ambient Riemannian 3-manifold (Ω, g). Given a function $u : D_r(p) \to \mathbb{R}$, the set

$$\text{Graph}(u) := \{ \exp_p(q + u(q)\nu_q) : q \in D_r(p)\}$$

is called the graph of u over $D_r(p)$.

Definition 3.1.1 (Convergence in the C^∞-topology). *Let (Ω, g) be a Riemannian 3-manifold and $\{M_i\}_{i\in\mathbb{N}}$ a sequence of connected embedded surfaces. The sequence $\{M_i\}_{i\in\mathbb{N}}$ converges in the C^∞-topology with finite multiplicity to a smooth embedded surface M_∞ if:*

(a) *M_∞ consists of accumulation points of $\{M_i\}_{i\in\mathbb{N}}$, that is for each $p \in M_\infty$ there exists a sequence of points $\{p_i\}_{i\in\mathbb{N}}$ such that $p_i \in M_i$, for each $i \in \mathbb{N}$, and $p = \lim_{i\to\infty} p_i$.*

(b) *For all $p \in M_\infty$ there exist $r, \varepsilon > 0$ such that $M_\infty \cap W_{r,\varepsilon}(p)$ can be represented as the graph of a function u over $D_r(p)$.*

(c) *For all large $i \in \mathbb{N}$, the set $M_i \cap W_{r,\varepsilon}(p)$ consists of a finite number k, independent of i, of graphs of functions $u_i^1, \ldots, u_i^k$ over $D_r(p)$ which converge smoothly to u.*

The multiplicity of a given point $p \in M_\infty$ is defined to be the number of graphs in $M_i \cap W_{r,\varepsilon}(p)$, for i large enough.

Remark 3.1.2. Note that although each surface of the sequence $\{M_i\}_{i\in\mathbb{N}}$ is connected, the limiting surface M_∞ is not necessarily connected. However, the multiplicity remains constant on each connected component Σ of M_∞. For more details we refer to [PR02, CS85].

Definition 3.1.3. *Let $\{M_i\}_{i\in\mathbb{N}}$ be a sequence of embedded surfaces in a Riemannian 3-manifold (Ω, g).*

(a) *We say that $\{M_i\}_{i\in\mathbb{N}}$ has uniformly bounded area on compact subsets of Ω if*

$$\limsup_{i\to\infty} \operatorname{area}\{M_i \cap K\} < \infty,$$

for any compact subset K of Ω.

(b) *We say that $\{M_i\}_{i\in\mathbb{N}}$ has uniformly bounded genus on compact subsets of Ω if*

$$\limsup_{i\to\infty} \operatorname{genus}\left\{M_i \cap K\right\} < \infty,$$

for any compact subset K of Ω.

Theorem 3.1.4 (White's compactness theorem). *Let (Ω, g) be an arbitrary Riemannian 3-manifold. Suppose that $\{M_i\}_{i\in\mathbb{N}}$ is a sequence of connected properly embedded minimal surfaces. Assume that the area and the genus of $\{M_i\}_{i\in\mathbb{N}}$ are uniformly bounded on compact subsets of Ω. Then, after passing to a subsequence, $\{M_i\}_{i\in\mathbb{N}}$ converges to a smooth properly embedded minimal surface $M_\infty \subset \Omega$. The convergence is smooth away from a discrete set denoted by* Sing. *Moreover, for each connected component Σ of M_∞, either*

(a) *the convergence to* Σ *is smooth everywhere with multiplicity* 1*, or*

(b) *the convergence is smooth, with some multiplicity greater than one, away from* $\Sigma \cap \mathrm{Sing}$.

Now suppose that Ω *is an open subset of* $\mathbb{R}^3$ *while the metric* g *is not necessarily flat. If* $p_i = (p_{1i}, p_{2i}, p_{3i}) \in M_i$, $i \in \mathbb{N}$, *converges to* $p \in M_\infty$ *then, after passing to a further subsequence, either* $T_{p_i}M_i \to T_pM$ *or there exists a sequence of real number* $\{\lambda_i\}_{i\in\mathbb{N}}$ *tending to* ∞ *such that the sequence of surfaces* $\{\lambda_i(M_i - p_i)\}_{i\in\mathbb{N}}$*, where*

$$\lambda_i(M_i - p_i) = \big\{\lambda_i(x_1 - p_{1i}, x_2 - p_{2i}, x_3 - p_{3i}) \in \mathbb{R}^3 : (x_1, x_2, x_3) \in M\big\},$$

converge smoothly and with multiplicity 1 *to a non-flat, complete and properly embedded minimal surface* M^*_∞ *of finite total curvature and with ends parallel to* T_pM_∞.

A crucial assumption in the compactness theorem of White is that the sequence has uniformly bounded area on compact subsets of Ω. Let us denote by

$$\mathscr{X} := \big\{p \in \Omega : \limsup_{i\to\infty}\mathrm{area}\{M_i \cap \mathbb{B}_r(p)\} = \infty \text{ for every } r > 0\big\},$$

the set where the area blows up. Clearly $\mathscr{X}$ is a closed set. It will be useful to have conditions that will imply that the set $\mathscr{X}$ is empty. In this direction, White [Whi15a, Theorem 2.6 and Theorem 7.4] shows that under some natural conditions the set $\mathscr{X}$ satisfies the same maximum principle as properly embedded minimal surfaces without boundary.

Theorem 3.1.5 (White's strong barrier principle). *Let* (Ω, g) *be a Riemannian 3-manifold and* $\{M_i\}_{i\in\mathbb{N}}$ *a sequence of properly embedded minimal surfaces, with boundaries* $\{\partial M_i\}_{i\in\mathbb{N}}$ *in* (Ω, g)*. Suppose that:*

(a) *The lengths of* $\{\partial M_i\}_{i\in\mathbb{N}}$ *are uniformly bounded on compact subsets of* Ω*, that is*

$$\limsup_{i\to\infty}\mathrm{length}\{\partial M_i \cap K\} < \infty,$$

for any relatively compact subset K *of* Ω.

(b) *The set* $\mathscr{L}$ *of* $\{M_i\}_{i\in\mathbb{N}}$ *is contained in a closed region* N *of* Ω *with smooth, connected boundary* ∂N *such that* $g\big(H_{\partial N}, \xi\big) \geq 0$*, at every point of* ∂N*, where* $H_{\partial N}(p)$ *is the mean curvature vector of* ∂N *at* p *and* $\xi(p)$ *is the unit normal at* p *to the surface* ∂N *that points into* N*.*

If the set $\mathscr{L}$ *contains any point of* ∂N*, then it contains all of* ∂N*.*

Remark 3.1.6. The above theorem is a sub-case of a more general result of White. In fact the strong barrier principle of White holds for sequences of embedded hypersurfaces of n-dimensional Riemannian manifolds which are not necessarily minimal but they have uniformly bounded mean curvatures. For more details we refer to [Whi15a].

3.2 Distance in Ilmanen's metric

Our aim in this section is to show that the smooth asymptotic behaviour of Definition 3.0.1 when the metric involved is the so-called Ilmanen's metric g implies the smooth asymptotic behaviour with respect to the euclidean metric $(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$, an important fact that we need at the end of our proof. Recall that in this chapter the ambient space is $\mathbb{R}^3$ and the translating velocity of the translator is $\mathrm{v} = e_3 = (0,0,1)$, so the specific Ilmanen's metric is

$$\mathrm{g} = e^{x_3}\langle\cdot\,,\cdot\rangle.$$

The strategy is to find a suitable relation between the distance with respect to the Euclidean metric and the distance with respect to the Ilmanen's metric. For this purpose, we begin by computing the geodesics of $(\mathbb{R}^3, \mathrm{g})$.

Proposition 3.2.1. *Vertical straight lines and "grim-reaper-type" curves, that is, images of smooth curves* $\gamma : (-\pi,\pi) \to (\mathbb{R}^3, \mathrm{g})$ *of the form* $\gamma(t) = \big(t, 0, -2\log\cos\frac{t}{2}\big)$*, are geodesics with respect to the Ilmanen's metric.*

Proof. Let the parametrized curve $\gamma : I \to \mathbb{R}^3$ be a geodesic. The geodesic equation of γ in a system of coordiantes where the curve can be written as $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ is

$$\frac{d^2 x_k}{dt} + \tilde{\Gamma}^k_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt} = 0, \quad k \in \{1,2,3\}, \tag{3.2.1}$$

where $\tilde{\Gamma}_{ij}^k$ are the Christoffel symbols in $(\mathbb{R}^3, \mathrm{g})$. Thus

$$\tilde{\Gamma}_{ij}^k = \frac{1}{2}\left(\frac{\partial}{\partial x_i}\,\mathrm{g}_{jm} + \frac{\partial}{\partial x_j}\,\mathrm{g}_{im} - \frac{\partial}{\partial x_m}\,\mathrm{g}_{ij}\right)\mathrm{g}^{mk}.$$

Since the Ilmanen's metric is $\mathrm{g}_{ij} = e^{x_3}\delta_{ij}$, we have that

$$\mathrm{g}^{ij} = e^{-x_3}\delta_{ij} \quad \text{and} \quad \frac{\partial}{\partial x_m}\,\mathrm{g}_{ij} = \frac{\partial}{\partial x_m}(e^{x_3}\delta_{ij}) = e^{x_3}\delta_{m3}\delta_{ij}.$$

Then

$$\begin{aligned}
\tilde{\Gamma}_{ij}^k &= \frac{1}{2}\left(\frac{\partial}{\partial x_i}\,\mathrm{g}_{jm} + \frac{\partial}{\partial x_j}\,\mathrm{g}_{im} - \frac{\partial}{\partial x_m}\,\mathrm{g}_{ij}\right)\mathrm{g}^{mk} \\
&= \frac{1}{2}\left(e^{x_3}\delta_{i3}\delta_{jm} + e^{x_3}\delta_{j3}\delta_{im} - e^{x_3}\delta_{m3}\delta_{ij}\right)e^{-x_3}\delta_{mk} \\
&= \frac{1}{2}\left(\delta_{i3}\delta_{jm}\delta_{mk} + \delta_{j3}\delta_{im}\delta_{mk} - \delta_{3m}\delta_{mk}\delta_{ij}\right) \\
&= \frac{1}{2}\left(\delta_{i3}\delta_{jk} + \delta_{j3}\delta_{ik} - \delta_{3k}\delta_{ij}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde{\Gamma}_{ij}^1 &= \frac{1}{2}\left(\delta_{i3}\delta_{j1} + \delta_{j3}\delta_{i1} - \delta_{31}\delta_{ij}\right) = \frac{1}{2}\left(\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1}\right), \\
\tilde{\Gamma}_{ij}^2 &= \frac{1}{2}\left(\delta_{i3}\delta_{j2} + \delta_{j3}\delta_{i2} - \delta_{32}\delta_{ij}\right) = \frac{1}{2}\left(\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2}\right), \\
\tilde{\Gamma}_{ij}^3 &= \frac{1}{2}\left(\delta_{i3}\delta_{j3} + \delta_{j3}\delta_{i3} - \delta_{33}\delta_{ij}\right) = -\frac{1}{2}\left(\delta_{ij} - 2\delta_{i3}\delta_{j3}\right).
\end{aligned}$$

Therefore, the second order system of EDOs (3.2.1) reads

$$\left.\begin{aligned}
\frac{d^2x_1}{dt} + \frac{dx_1}{dt}\frac{dx_3}{dt} &= 0 \\
\frac{d^2x_2}{dt} + \frac{dx_2}{dt}\frac{dx_3}{dt} &= 0 \\
\frac{d^2x_3}{dt} - \frac{1}{2}\left(\frac{dx_1}{dt}\frac{dx_1}{dt} + \frac{dx_2}{dt}\frac{dx_2}{dt} - \frac{dx_3}{dt}\frac{dx_3}{dt}\right) &= 0
\end{aligned}\right\} \tag{3.2.2}$$

Let us prove that the curves mentioned are solutions of this system, arguing how they may be found.

First of all, by looking at the two first equations we conjecture that $\gamma = (A, B, x_3)$, where A and B are arbitrary constants in $\mathbb{R}$, is a solution. Trivially it satisfies the two first equations. With respect to the third one, it becomes

$$\frac{d^2x_3}{dt} + \frac{1}{2}\frac{dx_3}{dt}\frac{dx_3}{dt} = 0$$

or, equivalently, writing $f(t) := \frac{dx_3}{dt}$ and denoting by f' the derivative of f with respect to t, it reads

$$f' + \frac{1}{2}f^2 = 0 \Rightarrow -\frac{f'}{f^2} = \frac{1}{2} \Rightarrow \frac{1}{f} = \frac{1}{2}t + C \Rightarrow \frac{dx_3}{dt} = f = \frac{2}{t+2C}$$
$$\Rightarrow x_3 = 2\log|t+2C| + D,$$

for arbitrary constans $C, D \in \mathbb{R}$. Since C is an arbitrary real constant, we can substitute $2C$ by C without loss of generality. Thus, a solution of the system (3.2.2) is the curve $\gamma : I \to \mathbb{R}^3$, defined —assuming that $C \geq 0$— in an open interval $I \subseteq (-C, +\infty)$ or $I \subseteq (-\infty, -C)$, given by

$$\gamma(t) = (A, B, 2\log|t+C| + D), \quad \text{for constants } A, B, C, D \in \mathbb{R},$$

Furthermore, we impose that γ is arc-lenght parametrized:

$$\begin{aligned} 1 &= \left|\frac{d\gamma}{dt}\right|_{\mathrm{g}_{\gamma(t)}} = \left|\left(0,0,2|t+C|^{-1}\right)\right|_{\mathrm{g}_{\gamma(t)}} \\ &= \left[\mathrm{g}_{\gamma(t)}\left((0,0,2|t+C|^{-1}),(0,0,2|t+C|^{-1})\right)\right]^{1/2} \\ &= e^{(2\log|t+C|+D)/2}\frac{2}{|t+C|} = e^{D/2}|t+C|\frac{2}{|t+C|} = 2e^{D/2} \\ &\Rightarrow D = 2\log\left(\frac{1}{2}\right). \end{aligned}$$

Then, a solution of (3.2.2), parametrized by the arc-length, is

$$\gamma(t) = \left(A, B, \log\frac{(t+C)^2}{4}\right), \quad \text{for arbitrary constants } A, B, C \in \mathbb{R}.$$

Second, we conjecture that another particular solution γ of the system is a grim reaper curve, that is,

$$\gamma : (-\pi/2, \pi/2) \to \mathbb{R}^3, \ \gamma(t) = (t, 0, -\log\cos(t)).$$

Then, we check immediately that the second equation of the system is trivially satisfied but the first one does not hold. So this is not a solution. This is not surprising if we recall that a necessary condition for a curve to be a geodesic is that the length of its tangent vector $\frac{d\gamma}{dt}$ must be constant. Hence, our next step is to consider the arc-length parameter $s = s(t)$ of the curve γ, or rather, the inverse of this function $t(s)$, in order to consider the arc-length parametrization of γ:

$$\gamma(s) = (t(s), 0, \log\cos(t(s)))\,,$$

where $t(s)$ is a smooth function such that:

$$\left|\frac{d\gamma}{ds}\right|_{\mathrm{g}_{\gamma(s)}} = 1.$$

If we now come back to the first equation of the system, it turns out that this $\gamma(s)$ does not satisfy it. But there exists the possibility to include another degree of freedom considering a new real constant K, to be determined later, while $\gamma(s)$ keeps the form of a grim reaper curve. Let us consider simply

$$\gamma(s) = \left(t(s), 0, -\frac{1}{K}\log\cos(Kt(s))\right).$$

where $t(s)$ is a smooth function such that:

$$\begin{aligned}
1 = \left|\frac{d\gamma}{ds}\right|^2_{\mathrm{g}_{\gamma(s)}} &= \left|\big(t'(s), 0, -t'(s)\tan(Kt(s))\big)\right|^2_{\mathrm{g}_{\gamma(s)}} \\
&= \mathrm{g}_{\gamma(s)}\Big(\big(t'(s), 0, -t'(s)\tan(Kt(s))\big), \big(t'(s), 0, -t'(s)\tan(Kt(s))\big)\Big) \\
&= e^{-\frac{1}{K}\log\cos(Kt(s))}t'(s)\left(1+\tan^2(Kt(s))\right) \\
&= t'(s)\cos^{-1/K}(Kt(s))\frac{1}{\cos^2(Kt(s))} = t'(s)\frac{1}{\cos^{2+\frac{1}{K}}(Kt(s))},
\end{aligned}$$

that is, such that:

$$t'(s) = \cos^{1+\frac{1}{2K}}(Kt(s)). \tag{3.2.3}$$

We will also need $t''(s)$:

$$\begin{aligned}
t''(s) &= -\left(1+\frac{1}{2k}\right)kt'(s)\cos^{\frac{1}{2K}}(Kt(s))\sin(Kt(s)) \\
&= -\left(K+\frac{1}{2}\right)\sin(Kt(s))\cos^{1+\frac{1}{K}}(Kt(s)).
\end{aligned} \tag{3.2.4}$$

Let us try this curve $\gamma(s)$ as a solution of the system (3.2.2). We have that:

$$\begin{aligned}\gamma(s) &= \left(t(s), 0, -\frac{1}{K}\log\cos(Kt(s))\right)\\ \Rightarrow \gamma'(s) &= (t'(s), 0, t'(s)\tan(Kt(s)))\\ \Rightarrow \gamma''(s) &= \left(t''(s), 0, t''(s)\tan(Kt(s)) + \frac{K(t'(s))^2}{\cos^2(Kt(s))}\right).\end{aligned}$$

Then, the first equation of the system is

$$\begin{aligned}0 &= \frac{d^2x_1}{ds} + \frac{dx_1}{ds}\frac{dx_3}{ds} = t''(s) + [t'(s)]^2\tan(Kt(s))\\ &= -\left(K + \frac{1}{2}\right)\sin(Kt(s))\cos^{1+\frac{1}{K}}(Kt(s)) + \cos^{2+\frac{1}{K}}(Kt(s))\frac{\sin(Kt(s))}{\cos(Kt(s))}\\ &= \left(\frac{1}{2} - K\right)\sin(Kt(s))\cos^{1+\frac{1}{K}}(Kt(s)).\end{aligned}$$

Consequently, this equation holds if and only if $K = 1/2$. Then

$$\gamma(s) = \left(t(s), 0, -2\log\cos\left(\frac{t(s)}{2}\right)\right).$$

On the other hand, the second equation of the system trivially holds because the second component of the curve $\gamma(s)$ vanishes identically. Finally, the third equation is:

$$\begin{aligned}&\frac{d^2x_3}{ds} - \frac{1}{2}\left(\frac{dx_1}{ds}\frac{dx_1}{ds} + \frac{dx_2}{ds}\frac{dx_2}{ds} - \frac{dx_3}{ds}\frac{dx_3}{ds}\right)\\ &= t''(s)\tan\left(\frac{t(s)}{2}\right) + \frac{(t'(s))^2}{2\cos^2\left(\frac{t(s)}{2}\right)} - \frac{1}{2}\left([t'(s)]^2 - [t'(s)\tan\left(\frac{t(s)}{2}\right)]^2\right)\\ &= [t'(s)]^2\left(\frac{t''(s)}{[t'(s)]^2}\tan\left(\frac{t(s)}{2}\right) + \frac{1}{2\cos^2\left(\frac{t(s)}{2}\right)} - \frac{1}{2}\left(1 - \tan\left(\frac{t(s)}{2}\right)^2\right)\right)\\ &= [t'(s)]^2\left(-\tan^2\left(\frac{t(s)}{2}\right) + \frac{1}{2}\left(1 + \tan^2\left(\frac{t(s)}{2}\right)\right) - \frac{1}{2}\left(1 - \tan\left(\frac{t(s)}{2}\right)^2\right)\right)\\ &= [t'(s)]^2 0 = 0.\end{aligned}$$

Therefore, the curve $\gamma(s) = \left(t(s), 0, -2\log\cos\left(\frac{t(s)}{2}\right)\right)$ is another solution of the system (3.2.2), as claimed. $\square$

Remark 3.2.2. In the above proof, once we know that $K = 1/2$, we can compute explicitly $t(s)$, the inverse of the arc-length parameter of our second geodesic: it is the solution of $t'(s) = \cos^2(Kt(s))$, i.e., $t(s) = 2\arctan\left(\frac{s+A}{2}\right)$, where A is any real number. However, we will not need it.

The following proposition allows us to construct all the geodesics of $(\mathbb{R}^3, \mathrm{g})$ from the previous ones. Although we assume throughout the whole chapter that $\mathrm{v} = e_3$, in the next result we work with an arbitrary translating velocity v because this way the proof is even easier to understand and its geometric sense is fully preserved.

Proposition 3.2.3. *Rotations and parallel transports in* $(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ *that preserve* v*, also preserve the geodesics of* $(\mathbb{R}^3, \mathrm{g})$*.*

Proof. Taking into account that rotations and parallel transports in $(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ are isometries of $(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$, we are going to prove a more general result: any isometry Φ of the Euclidean space $(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ that preserves v (i.e., $\Phi(\mathrm{v}) = \mathrm{v}$), also preserve the geodesics of $(\mathbb{R}^3, \mathrm{g})$.

It is very well known that isometries preserve geodesics. Then it is sufficient to show that Φ is an isometry of $(\mathbb{R}^3, \mathrm{g}) = (\mathbb{R}^3, e^{\langle p,\mathrm{v}\rangle}\langle\cdot\,,\cdot\rangle)$, that is, for all $p \in \mathbb{R}^3$ and for all $w_1, w_2 \in T_p\mathbb{R}^3$ we have

$$\mathrm{g}_p(w_1, w_2) = \mathrm{g}_{\Phi(p)}(d\Phi_p(w_1), d\Phi_p(w_2)).$$

Using the hypothesis, this is a straightforward computation:

$$\begin{aligned} \mathrm{g}_{\Phi(p)}(d\Phi_p(w_1), d\Phi_p(w_2)) &= e^{\langle\Phi(p),\mathrm{v}\rangle}\langle d\Phi_p(w_1), d\Phi_p(w_2)\rangle = e^{\langle\Phi(p),\Phi(\mathrm{v})\rangle}\langle w_1, w_2\rangle \\ &= e^{\langle p,\mathrm{v}\rangle}\langle w_1, w_2\rangle = \mathrm{g}_p(w_1, w_2). \end{aligned}$$

□

Now we can argue geometrically to deduce all the geodesics of $(\mathbb{R}^3, \mathrm{g})$. The basic geodesic curves that we already computed are:

$$\gamma(t) = \left(0, 0, \log\left(\frac{t^2}{4}\right)\right),$$
$$\gamma(s) = \left(t(s), 0, -2\log\cos\left(\frac{t(s)}{2}\right)\right),$$

where recall that $t(s)$ is the inverse of the arc-length parameter function $s(t)$. Their derivatives are, respectively:

$$\gamma'(t) = \left(0, 0, \frac{2}{t}\right),$$
$$\gamma'(s) = \left(t'(s), 0, t'(s)\tan\left(\frac{t(s)}{2}\right)\right),$$

Consequently, vertical straight lines as above can be used to obtain geodesics with any initial vertical velocity (i.e., velocities proportional to e_3) and "grim-reaper-type" curves can be used to obtain geodesics with any non-vertical velocity in the plane $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = 0\}$. On the other hand, rotations around the e_3-axis allow us to obtain geodesics with any initial velocity in general. And the parallel transport serves to establish any point p as the the initial one of the geodesic. In summary, we proved the following result:

Corollary 3.2.4. *All the geodesics of* $(\mathbb{R}^3, \mathrm{g})$ *can be constructed using the above geodesics (Proposition 3.2.1) and transformations (Proposition 3.2.3).*

Finally, we can establish now a relation between the distance of a point from a vertical plane in $(\mathbb{R}^3, \mathrm{g})$ and in $(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$, which allows us to obtain a relation between the smooth asymptotic behaviour described in Definition 3.0.1 in these two ambient spaces.

Let δ be a sufficiently small positive number and $p = (p_1, p_2, p_3)$ a point in $\mathbb{R}^3$ such that $p_1 \in (-\delta, 0)$ and $p_3 > 0$. Let us denote by $\mathrm{dist}_{\mathrm{g}}(p, \Pi(0))$ the distance of p from the plane

$$\Pi(0) = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0\right\}.$$

with respect to the Ilmanen's metric and by $\mathrm{dist}(p, \Pi(0)) = -p_1$ the euclidean distance of the point p from the plane $\Pi(0)$. According to our previous discussion about the geodesics in $(\mathbb{R}^3, \mathrm{g})$, and since geodesics locally minimize the arc length, the distance $\mathrm{dist}_{\mathrm{g}}(p, \Pi(0))$ is given as the length with respect to the Ilmanen's metric of the smooth curve $l : (p_1, 0) \to (\mathbb{R}^3, \mathrm{g})$ given by

$$l(t) = \left(t, p_2, -2\log\cos\tfrac{t}{2} + 2\log\cos\tfrac{p_1}{2} + p_3\right).$$

This length is:

$$\begin{aligned}
\mathrm{dist}_{\mathrm{g}}(p,\Pi(0)) &= \int_{p_1}^{0}\left|\frac{dl}{dt}\right|_{\mathrm{g}_{\gamma(t)}} dt = \int_{p_1}^{0} e^{\frac{p_3}{2}}\cdot\frac{\cos\frac{p_1}{2}}{\cos\frac{t}{2}}\cdot\sqrt{1+\left(\tan\frac{t}{2}\right)^2}dt \\
&= e^{\frac{p_3}{2}}\cdot\cos\left(\frac{p_1}{2}\right)\int_{p_1}^{0}\frac{1}{\cos^2\left(\frac{t}{2}\right)}dt = e^{\frac{p_3}{2}}\cdot\cos\left(\frac{p_1}{2}\right)\cdot\left[2\tan\left(\tfrac{t}{2}\right)\right]_{p_1}^{0} \\
&= 2e^{\frac{p_3}{2}}\cdot\left(-\sin\tfrac{p_1}{2}\right) = 2e^{\frac{p_3}{2}}\cdot\sin\tfrac{\mathrm{dist}(p,\Pi(0))}{2}. \qquad (3.2.5)
\end{aligned}$$

Remark 3.2.5. As we saw in the proof of Proposition 3.2.1, the curve l is not a geodesic because its tangent vector does not have constant length. However, we proved there that its image coincide with the image of a geodesic. Therefore, it can be used to compute the distance between sufficiently close points since the length of a curve is invariant under reparametrization.

The above formula gives us a smooth bijection between $\mathrm{dist}_{\mathrm{g}}(p,\Pi(0))$ and $\mathrm{dist}(p,\Pi(0))$ for points $p=(p_1,p_2,p_3)\in\mathbb{R}^3$ such that $p_1\in(-\delta,0)$ and $p_3>0$, where δ is a sufficiently small positive number:

$$\mathrm{dist}_{\mathrm{g}}(p,\Pi(0)) = 2e^{p_3/2}\cdot\sin\tfrac{\mathrm{dist}(p,\Pi(0))}{2} \qquad (3.2.6)$$

$$\Leftrightarrow \mathrm{dist}(p,\Pi(0)) = 2\arcsin\left(\frac{e^{-p_3/2}}{2}\cdot\mathrm{dist}_{\mathrm{g}}(p,\Pi(0))\right). \qquad (3.2.7)$$

(3.2.7) implies that, as $p_1\to 0$ and $p_3\to+\infty$, if $\mathrm{dist}_{\mathrm{g}}(p,\Pi(0))\to 0$, then $\mathrm{dist}(p,\Pi(0))\to 0$.

Remark 3.2.6. Observe that (3.2.6) does not imply the reciprocal since an indeterminate form is obtained when the limit is evaluated as $p_1\to 0$ and $p_3\to+\infty$. Indeed, this indeterminate form can be evaluated using the boundedness $x-\frac{x^3}{3!}<\sin x<x$ for $x>0$ and the fact that the exponential grows faster than any polynomial. Hence, the limit is $+\infty$. Therefore, it is not true that as $p_1\to 0$ and $p_3\to+\infty$, if $\mathrm{dist}(p,\Pi(0))\to 0$, then $\mathrm{dist}_{\mathrm{g}}(p,\Pi(0))\to 0$.

Bearing in mind Definition 3.0.1 (asymptotic behaviour) and the definition of the graph of a function over an open disc of the tangent plane of a surface given at the beginning of section 3.1 (in this definition, the role of the tangent plane is played here by $\Pi(0)$), using the same notation we have that the function φ at the point $l(0) \in \Pi(0)$ coincides with the distance $\mathrm{dist}(l(p_1), \Pi(0)) = \mathrm{dist}(p, \Pi(0))$:

$$\varphi(l(0)) = \mathrm{dist}(l(p_1), \Pi(0)) = \mathrm{dist}(p, \Pi(0)).$$

Analogously with φ_{g} and $\mathrm{dist}_{\mathrm{g}}(p, \Pi(0))$, where we indicate the dependence of φ on the ambient space by writing the metric as a subscript. See figure 3.2, where $\gamma(q, \nu_q)$ denotes the corresponding geodesic passing through the point $q \in \Pi(0)$ with initial velocity ν_q, the unit normal vector of $\Pi(0)$.

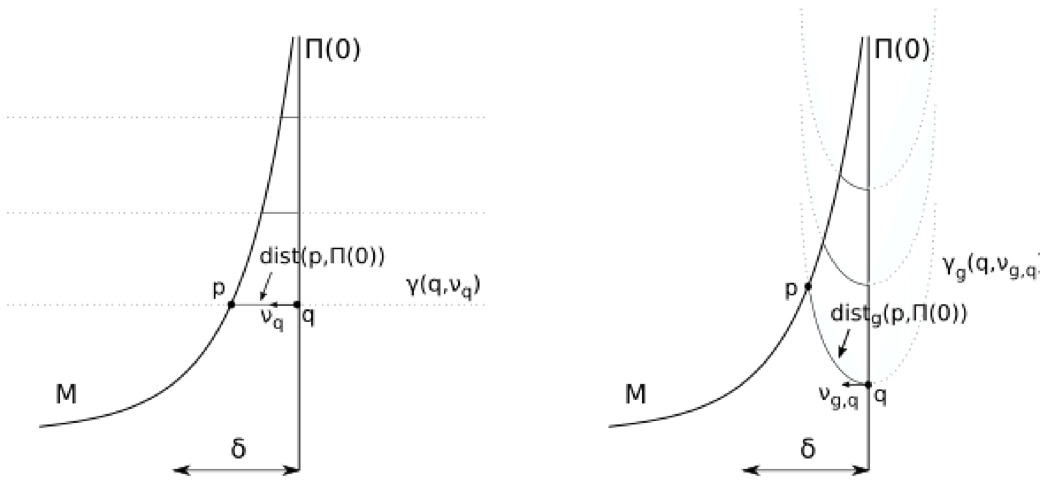


Figure 3.2: Euclidean and Ilmanen distances of a point p in the surface M from the vertical point $\Pi(0)$, respectively. Profile view in the x_1x_3-plane

In order to show the smoothness of the asymptotic behaviour in $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$, assuming that it is smooth in $(\mathbb{R}^3, \mathrm{g})$, first of all observe that p_2 and p_3 can be considered as coordinates in the plane $T_p\Pi(0) = \Pi(0)$ via

$$l(0) : (p_2, p_3) \mapsto \left(0, p_2, 2 \log \cos \tfrac{p_1}{2} + p_3\right).$$

We have that:

$$\begin{aligned}
\varphi(l(0)) &= \varphi\left(0, p_2, 2 \log \cos \tfrac{p_1}{2} + p_3\right) = \mathrm{dist}(p, \Pi(0)) \\
&= 2 \arcsin\left(\frac{e^{-p_3/2}}{2} \cdot \mathrm{dist}_{\mathrm{g}}(p, \Pi(0))\right) \\
&= 2 \arcsin\left(\frac{e^{-p_3/2}}{2} \cdot \varphi_{\mathrm{g}}(l(0))\right).
\end{aligned}$$

Then

$$\frac{\partial\varphi(l(0))}{\partial p_2} = \left(1 - \frac{e^{-p_3}}{4}\cdot\varphi_{\mathrm{g}}^2(l(0))\right)^{-1/2}\cdot\frac{e^{-p_3/2}}{2}\cdot\frac{\partial\varphi_{\mathrm{g}}(l(0))}{\partial p_2},$$

$$\frac{\partial\varphi(l(0))}{\partial p_3} = \left(1 - \frac{e^{-p_3}}{4}\cdot\varphi_{\mathrm{g}}^2(l(0))\right)^{-1/2}\left(\frac{e^{-p_3/2}}{2}\cdot\frac{\partial\varphi_{\mathrm{g}}(l(0))}{\partial p_3} - \frac{e^{-p_3/2}}{4}\cdot\varphi_{\mathrm{g}}(l(0))\right)$$

By hypothesis $\varphi_{\mathrm{g}}(l(0))$ and all its partial derivatives tend to 0 as $p_1 \to 0$ and $p_3 \to +\infty$. So the first partial derivatives of $\varphi(l(0))$ also do. Moreover, the higher-order partial derivatives also do because by the derivative rules (product and potential rules) and the functions involved ($\varphi_{\mathrm{g}}(l(0))$ and exponential function with negative exponent), they keep the form of the previous derivatives. Consequently, $|D^j\varphi|$ also does for every j.

These considerations proves the following result, which will be very useful in the last step of the proof of our main theorem of this chapter.

Lemma 3.2.7. *Suppose that M, regarded as a minimal surface in $(\mathbb{R}^3\,\mathrm{g})$, is C^∞-asymptotic to two parallel vertical half-planes $\mathcal{H}_1$ and $\mathcal{H}_2$ outside the cylinder $\mathcal{C}$. Then the translator M is also smoothly asymptotic to the above mentioned half-planes outside $\mathcal{C}$ with respect to the Euclidean metric.*

3.3 A compactness result and its first consequences

The translating property is preserved if we act on M via isometries of $\mathbb{R}^3$ which preserves the translating direction. Therefore, if (a, b, c) is a vector of $\mathbb{R}^3$ then the surface

$$M + (a, b, c) = \left\{(x_1 + a, x_2 + b, x_3 + c) \in \mathbb{R}^3 : (x_1, x_2, x_3) \in M\right\}$$

is again a translator. Based on White's compactness theorem, we can prove a convergence result for some special sequences of translating solitons. More precisely, we show the following:

Lemma 3.3.1. *Let M be a surface as in our theorem. Suppose that $\{b_i\}_{i\in\mathbb{N}}$ is a sequence of real numbers and let $\{M_i\}_{i\in\mathbb{N}}$ be the sequence of surfaces given by $\big\{M_i := M + (0, b_i, 0)\big\}_{i\in\mathbb{N}}$. Then, after passing to a subsequence, $\{M_i\}_{i\in\mathbb{N}}$ converges smoothly with multiplicity one to a properly embedded connected translating soliton M_∞ which has the same asymptotic behavior as M.*

Proof. Recall that any translator $M \subset \mathbb{R}^3$ can be regarded as a minimal surface of $(\Omega = \mathbb{R}^3, \mathrm{g})$ where g is the Ilmanen's metric. Notice that each element of the sequence $\{M_i\}_{i\in\mathbb{N}}$ has the same asymptotic behavior as M. Without loss of generality, we can arrange the coordinate system such that

$$\mathcal{C} = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_3^2 \le r_0^2\big\}.$$

By assumption our surface M is C^1-asymptotic outside $\mathcal{C}$ to two half-planes $\mathcal{H}_1$, $\mathcal{H}_2$ (see Fig. 3.3). Let now w_1, w_2 be the unit inward pointing vectors

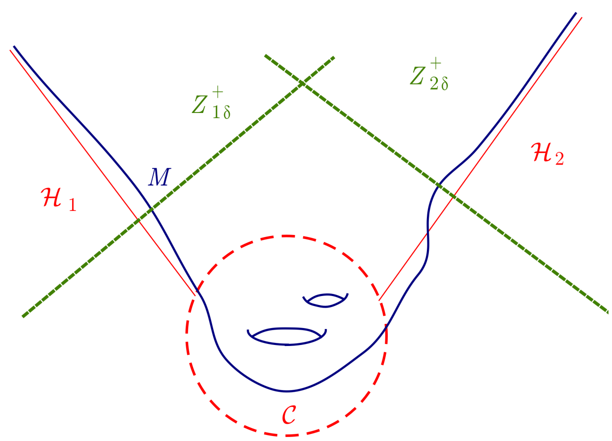


Figure 3.3: Asymptotic behaviour with tilted half-planes

of $\partial\mathcal{H}_1$, $\partial\mathcal{H}_2$, respectively. For any $\delta > 0$ consider the closed half-planes

$$\mathcal{H}_k(\delta) = \{p + t\mathrm{w}_k : p \in \partial\mathcal{H}_k \text{ and } t \ge \delta\},$$

for $k \in \{1, 2\}$ and denote by $Z^+_{k\delta}$, $k \in \{1, 2\}$, the closed half-space of $\mathbb{R}^3$ containing $\mathcal{H}_k(\delta)$ and with boundary containing $\partial\mathcal{H}_k(\delta)$ and being perpendicular to w_k. Moreover, consider the closed half-spaces

$$Z^-_{k\delta} = \big(\mathbb{R}^3 - Z^+_{k\delta}\big) \cup \partial Z^+_{k\delta},$$

for any $k \in \{1,2\}$.

In the case where the sequence $\{b_i\}_{i\in\mathbb{N}}$ is bounded, we can consider a subsequence such that $\lim b_i = b_\infty \in \mathbb{R}$. Then obviously $\{M_i\}_{i\in\mathbb{N}}$ converges smoothly with multiplicity one to the properly embedded translating soliton

$$M_\infty = M + (0, b_\infty, 0).$$

Clearly M_∞ has the same asymptotic behavior with M.

Let us examine now the case where the sequence $\{b_i\}_{i\in\mathbb{N}}$ is not bounded. Split each surface M_i of the surface into the parts

$$M^+_{1i}(\delta) := M_i \cap Z^+_{1\delta},\ M^+_{2i}(\delta) := M_i \cap Z^+_{2\delta} \text{ and } M^-_i(\delta) := M_i \cap Z^-_{1\delta} \cap Z^-_{2\delta}.$$

Claim 1. *The sequences $\{M^+_{1i}(\delta)\}_{i\in\mathbb{N}}$ and $\{M^+_{2i}(\delta)\}_{i\in\mathbb{N}}$ have uniformly bounded area on compact sets.*

Proof of the claim. Let K be a compact subset of Ω and $\mathbb{B}_r(0)$ a ball of radius r centered at the origin of $\mathbb{R}^3$ containing K. Denote by V_i the projection of the surface $M^+_{1i}(\delta) \cap K$ to the closed half-plane $\mathcal{H}_1(\delta)$. Hence we can parametrize $M^+_{1i}(\delta)$ by a map $\Phi_i : V_i \to \mathbb{R}^3$ of the form

$$\begin{aligned}
\Phi_i(s,t) &= (c_1,c_2,c_3) + s\mathrm{e}_2 + t\mathrm{w}_1 + \varphi(s-b_i,t)\mathrm{e}_2 \wedge \mathrm{w}_1 \\
&= \big\{c_1 + (\cos\alpha)t + (\sin\alpha)\varphi(s-b_i,t)\big\}\mathrm{e}_1 + \big\{c_2 + s\big\}\mathrm{e}_2 \\
&\quad + \big\{c_3 + (\sin\alpha)t - (\cos\alpha)\varphi(s-b_i,t)\big\}\mathrm{e}_3,
\end{aligned}$$

where $i \in \mathbb{N}$, $\big\{\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3\big\}$ is the standard basis of $\mathbb{R}^3$, α is the angle between the vectors e_1 and w_1 and (c_1,c_2,c_3) is a fixed point on $\partial\mathcal{H}_1(\delta)$. By taking δ very large we can make sure that $|\varphi|$ and $|D\varphi|$ are bounded by a universal constant ε. Hence, for any index $i \in \mathbb{N}$ we have that

$$\begin{aligned}
\mathrm{area_g}\big\{M^+_{1i}(\delta) \cap K\big\} &= \int_{V_i} e^{c_3 + (\sin\alpha)t - (\cos\alpha)\varphi(s-b_i,t)}\sqrt{1+|D\varphi|^2}\, dsdt \\
&\le \int_{V_i} e^{c_3 + c(r) + \varepsilon}\sqrt{1+\varepsilon^2}\, dsdt \\
&= e^{c_3 + c(r) + \varepsilon}\sqrt{1+\varepsilon^2}\ \mathrm{area_{euc}}(V_i),
\end{aligned}$$

where $c(r)$ is a constant depending on r and $\mathrm{area_{euc}}(V_i)$ is the euclidean area of V_i. Note that $\mathrm{area_{euc}}(V_i)$ is less or equal than the euclidean area of the projection of K to the plane containing $\mathcal{H}_1(\delta)$. Thus there exists a number $m(K)$ depending only on K such that

$$\text{area}_{\text{g}}\big\{M_{1i}^{+}(\delta)\cap K\big\} \leq m(K).$$

Consequently, $\big\{M_{1i}^{+}(\delta)\big\}_{i\in\mathbb{N}}$ has uniformly bounded area. Similarly, we show that $\big\{M_{2i}^{+}(\delta)\big\}_{i\in\mathbb{N}}$ has uniformly bounded area and this concludes the proof of the claim.

Claim 2. *The sequence of surfaces* $\big\{M_i^{-}(\delta)\big\}_{i\in\mathbb{N}}$ *has uniformly bounded area on compact sets.*

Proof of the claim. Let us show a first that the sequence $\big\{\partial M_i^{-}(\delta)\big\}_{i\in\mathbb{N}}$ has uniformly bounded length on compact sets. Following the notation introduced in the above claim, each connected component of $\partial M_i^{-}(\delta)$ can be represented as the image of the curve $\gamma_i:\mathbb{R}\to\mathbb{R}^3$ given by

$$\begin{aligned}\gamma_i(s) &= \big\{c_1+(\cos\alpha)\delta+(\sin\alpha)\varphi(s-b_i,\delta)\big\}\mathrm{e}_1\\ &\quad+\big\{c_2+s\big\}\mathrm{e}_2+\big\{c_3+(\sin\alpha)\delta-(\cos\alpha)\varphi(s-b_i,\delta)\big\}\mathrm{e}_3,\end{aligned}$$

for any index $i\in\mathbb{N}$. Let K be a compact set of Ω, $\mathbb{B}_r(0)$ a ball of radius r centered at the origin and containing K. Denote by I_i the projection of $\partial M_i^{-}(\delta)\cap K$ to $\partial\mathcal{H}_1(\delta)$. Estimating as in Claim 1, we get that

$$\text{length}_{\text{g}}\big\{\partial M_i^{-}(\delta)\cap K\big\} \leq \int_{I_i} e^{\frac{c_3+c(r)+\varepsilon}{2}}\sqrt{1+\varepsilon^2}\,ds,$$

where $c(r)$ is a constant depending on r. Thus, there exists a constant $n(K)$ depending only on the compact set K such that

$$\text{length}_{\text{g}}\big\{\partial M_i^{-}(\delta)\cap K\big\} \leq n(K).$$

Hence, the sequence $\big\{\partial M_i^{-}(\delta)\big\}_{i\in\mathbb{N}}$ has uniformly bounded length on compact sets.

Recall now that the set $\mathscr{Z}$ is closed. From Claim 1 it follows that $\mathscr{Z}$ is contained inside a cylinder. Consider now a translating paraboloid and translate it in the direction of the x_3-axis until it has no common point with $\mathscr{Z}$. Then move back the translating paraboloid until it intersects for the first time the set $\mathscr{Z}$ (see Fig. 3.4).

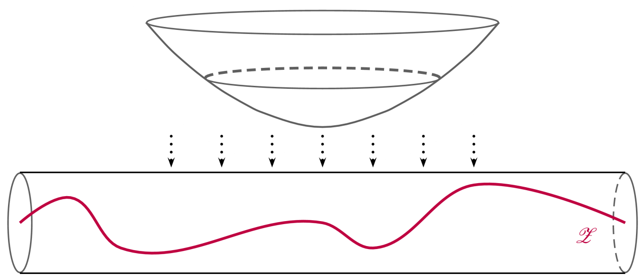


Figure 3.4: The area blow-up set $\mathscr{Z}$

From the strong barrier principle of White (Theorem 3.1.5), the translating paraboloid is contained in $\mathscr{Z}$. But this leads to a contradiction, because now the area blow-up set $\mathscr{Z}$ is not contained inside a cylinder. Thus, $\mathscr{Z}$ must be empty and consequently $\{M_i^-(\delta)\}_{i\in\mathbb{N}}$ has uniformly bounded area.

Since the parts $\{M_{1i}^+(\delta)\}_{i\in\mathbb{N}}$, $\{M_{2i}^+(\delta)\}_{i\in\mathbb{N}}$, $\{M_i^-(\delta)\}_{i\in\mathbb{N}}$ have uniformly bounded area, we see that the whole sequence $\{M_i\}_{i\in\mathbb{N}}$ has uniformly bounded area. From our assumptions, also the genus of the sequence is uniformly bounded. The convergence to a smooth properly embedded translator M_∞ follows from Theorem 3.1.4 of White. Since each $M_{ki}^+(\delta)$, $k \in \{1,2\}$, is a graph and each M_i is connected, we deduce that the multiplicity is one everywhere. Thus, the convergence is smooth. Moreover, observe that each component of $M_\infty \cap Z_{k\delta}^+$, $k \in \{1,2\}$, can be represented as the graph of a smooth function φ_∞ which is the limit of the sequence of graphs generated by the smooth functions

$$\varphi_i(s,t) = \varphi(s - b_i, t)$$

for any $i \in \mathbb{N}$. Thus, the limiting surface M_∞ has the same asymptotic behavior as M. The limiting surface M_∞ must be connected since otherwise there should exist a properly embedded connected component Σ of M lying inside $\mathcal{C}$. But then, the x_3-coordinate function of Σ must be bounded from above, which is absurd. This concludes the proof. $\square$

As a first application of the above compactness result we show that the half-planes $\mathcal{H}_1$ and $\mathcal{H}_2$ must be parallel to each other.

Lemma 3.3.2. *Let M be a translating soliton as in our theorem. Then, the half-planes $\mathcal{H}_1$ and $\mathcal{H}_2$ must be parallel to the translating direction. Moreover, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are parts of the same plane Π, then M should coincide with Π.*

Proof. We follow the notation introduced in the last lemma. Assume to the contrary that the half-plane

$$\mathcal{H}_1 = \big\{p + t\,\mathrm{w}_1 : p \in \partial\mathcal{H}_1 \text{ and } t > 0\big\}$$

is not parallel to the translating direction v. Let us suppose at first that the cosine of angle between the unit inward pointing normal w_1 of $\partial\mathcal{H}_1$ and e_1 is positive. Consider the strip S_{t_0} given by

$$S_{t_0} := (t_0 - \pi/2, t_0 + \pi/2) \times \mathbb{R} \times \mathbb{R}.$$

For sufficiently large t_0 this slab does not intersects the cylinder $\mathcal{C}$. For fixed real numbers t, l let $\mathcal{G}^{t,l}$ be the grim reaper cylinder

$$\mathcal{G}^{t,l} := \big\{(x_1, x_2, l + \log\cos(x_1 - t)) \in \mathbb{R}^3 : |x_1 - t| < \pi/2, x_2 \in \mathbb{R}\big\}.$$

By our assumptions, as δ becomes larger the wing $M_\delta := M \cap Z^+_{1\delta}$ of M is getting closer to $\mathcal{H}_1$. By the asymptotic behavior of M to two half-planes, there exists $t_0, l_0 \in \mathbb{R}$ large enough such that $\mathcal{G}^{t_0,l_0}$ does not intersect M_δ. Then translate this grim reaper cylinder in the direction of $-$ v. Since $\mathcal{H}_1$ is not parallel to v, after some finite time l_1 either there will be a first interior point of contact between the surface M_δ and $\mathcal{G}^{t_0,l_0-l_1}$ or there will exist a sequence of points $\{p_i = (p_{1i}, p_{2i}, p_{3i})\}_{i\in\mathbb{N}}$ in the interior of M_δ, with $\{p_{3i}\}_{i\in\mathbb{N}}$ bounded and $\{p_{2i}\}_{i\in\mathbb{N}}$ unbounded, such that

$$\lim_{i\to\infty} \operatorname{dist}(p_i, \mathcal{G}^{t_0,l_0-l_1}) = 0.$$

The first possibility contradicts the asymptotic behavior of M. So let us examine the second possibility. Consider the sequence of surfaces $\{M_i\}_{i\in\mathbb{N}}$ given by $M_i = M + (0, -p_{2i}, 0)$, for any $i \in \mathbb{N}$. By Lemma 3.3.1, after passing to a subsequence, $\{M_i\}_{i\in\mathbb{N}}$ converges smoothly to a connected and properly embedded translator M_∞ which has the same asymptotic behavior as M. But now there exists an interior point of contact between M_∞ and $\mathcal{G}^{t_0,l_0-l_1}$, which is absurd. Similarly we treat the case where the cosine of the angle between w_1 and e_1 is negative. Hence both half-planes must be parallel to the translating direction v.

Suppose now that the half-planes $\mathcal{H}_1$ and $\mathcal{H}_2$ are contained in the same vertical plane Π. Without loss of generality we may assume that $\Pi = \Pi(0)$. Suppose to the contrary that the translator M does not coincide with Π. Observe that in this case the x_1-coordinate function attains a non-zero supremum or a non-zero infimum along a sequence $\{p_i = (p_{1i}, p_{2i}, p_{3i})\}_{i\in\mathbb{N}}$ in the interior of M, with $\{p_{3i}\}_{i\in\mathbb{N}}$ bounded and $\{p_{2i}\}_{i\in\mathbb{N}}$ unbounded. Performing a limiting process as in the previous case we arrive to a contradiction. Therefore, the x_1-coordinate function must be zero constant and thus M must be planar. □

Another application of the above compactness result is the following strong maximum principle.

Lemma 3.3.3. *Let M be a translating soliton as in our theorem and assume that the half-planes $\mathcal{H}_1$ and $\mathcal{H}_2$ are distinct. Consider a portion Σ of M (not necessarily compact) with non-empty boundary $\partial\Sigma$ such that the x_3-coordinate function of Σ is bounded. Then the supremum and the infimum of the x_1-coordinate function of Σ are reached along the boundary of Σ i.e., there exists no sequence $\{p_i\}_{i\in\mathbb{N}}$ in the interior of Σ such that $\lim_{i\to\infty} \operatorname{dist}(p_i, \partial\Sigma) > 0$ and $\lim_{i\to\infty} x_1(p_i) = \sup_\Sigma x_1$ or $\lim_{i\to\infty} x_1(p_i) = \inf_\Sigma x_1$.*

Proof. Recall that from the above lemma the half-planes $\mathcal{H}_1$ and $\mathcal{H}_2$ must be parallel to each other and to the direction v of translation. From our assumptions the x_1-coordinate function of the surface M is bounded. Moreover, the extrema of x_1 cannot be attained at an interior point of Σ, since otherwise from the tangency principle Σ should be a plane. This would imply that M is a plane, something that contradicts the asymptotic assumptions. So, let us suppose that there exists a sequence of points $\{p_i = (p_{1i}, p_{2i}, p_{3i})\}_{i\in\mathbb{N}}$ in the interior of Σ such that $\lim_{i\to\infty} \operatorname{dist}(p_i, \partial\Sigma) > 0$ and $x_1(p_i)$ is tending to its supremum or infimum. Then, consider the sequence of surfaces $\{M_i\}_{i\in\mathbb{N}}$ given by $M_i = M + (0, -p_{2i}, 0)$, for any $i \in \mathbb{N}$. By Lemma 3.3.1, after passing to a subsequence, $\{M_i\}_{i\in\mathbb{N}}$ converges smoothly to a connected and properly embedded translator M_∞ which has the same asymptotic behavior as M. But now there exists a point in M_∞ where its x_1-coordinate function reaches its local extremum, which is absurd. □

Remark 3.3.4. The x_1-coordinate function of M satisfies the partial differential equation $\Delta x_1 + \langle \nabla x_1, \nabla x_3 \rangle = 0$. However, Lemma 3.3.3 is not a direct

consequence of the strong maximum principle for elliptic PDE's because in general Σ is not bounded.
Let us see that x_1 satisfies that PDE. Let $\{E_1, E_2\}$ be an orthonormal frame defined on an open neighborhood of M. We have that

$$\Delta\,\mathrm{x}_1 = \sum_i \{E_i\left(E_i(\mathrm{x}_1)\right) - \left(\nabla_{E_i} E_i\right)(\mathrm{x}_1)\}.$$

We begin by differentiating $\mathrm{x}_1 = \langle f, e_1\rangle$ with respect to E_i:

$$d\,\mathrm{x}_1(E_i) = E_i(\mathrm{x}_1) = E_i(\langle f, e_1\rangle) = \langle D_{E_i} f, e_1\rangle = \langle df(E_i), e_1\rangle. \tag{3.3.1}$$

Then

$$\begin{aligned} E_i\left(E_i(\mathrm{x}_1)\right) &\overset{*}{=} E_i\left(\langle df(E_i), e_1\rangle\right) = \langle D_{E_i} df(E_i), e_1\rangle = \langle D_{df(E_i)} df(E_i), e_1\rangle \\ &= \langle \mathbf{A}(E_i, E_i) + df\left(\nabla_{E_i} E_i\right), e_1\rangle, \end{aligned}$$

$$\left(\nabla_{E_i} E_i\right)(\mathrm{x}_1) = d\,\mathrm{x}_1\left(\nabla_{E_i} E_i\right) \overset{*}{=} \langle df\left(\nabla_{E_i} E_i\right), e_1\rangle,$$

where we used (3.3.1) in the equalities marked with $*$, and that $\mathbf{A}(X, Y) = D_{df(X)} df(Y) - df\left(\nabla_X Y\right)$ for any smooth vector fields X, Y of M by definition of the second fundamental form $\mathbf{A}$ of f.
Therefore,

$$\begin{aligned} \Delta\,\mathrm{x}_1 &= \sum_i \{E_i\left(E_i(\mathrm{x}_1)\right) - \left(\nabla_{E_i} E_i\right)(\mathrm{x}_1)\} \\ &= \sum_i \{\langle \mathbf{A}(E_i, E_i) + df\left(\nabla_{E_i} E_i\right), e_1\rangle - \langle df\left(\nabla_{E_i} E_i\right), e_1\rangle\} \\ &= \sum_i \langle A(E_i, E_i), e_1\rangle = \langle \textstyle\sum_i \mathbf{A}(E_i, E_i), e_1\rangle = \langle \mathbf{H}, e_1\rangle \\ &= \langle e_3^{\perp}, e_1\rangle = \langle e_3 - e_3^{\top}, e_1\rangle = -\langle e_3^{\top}, e_1\rangle = -\langle \nabla\,\mathrm{x}_3, e_1\rangle, \end{aligned}$$

where in the last equality we used [MSHS15, Lemma 2.1 (a)]; note that the notation in this reference for the third coordinate function of M is u instead of x_3. We denote by $e_3^{\perp}$ the orthogonal projection of e_3 onto the normal bundle of f and by $e_3^{\top}$ the orthogonal projection of e_3 onto the tangent bundle of f. This establishes the PDE.

3.4 Proof of the theorem

We have to deal only with the case where $\mathcal{H}_1$ and $\mathcal{H}_2$ are distinct and parallel to v. We can arrange the coordinates such that $\mathrm{v} = (0, 0, 1)$ and such that the x_2-axis is the axis of rotation of our cylinder

$$\mathcal{C} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_3^2 \leq r^2\}.$$

Following the setting in [MSHS15], let us define the family of planes $\{\Pi(t)\}_{t \in \mathbb{R}}$, given by

$$\Pi(t) := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = t\}.$$

Moreover, given a subset A of $\mathbb{R}^3$, for any $t \in \mathbb{R}$ we define the sets

$$\begin{aligned}
A_+(t) &:= \{(x_1, x_2, x_3) \in A : x_1 \geq t\}, \\
A_-(t) &:= \{(x_1, x_2, x_3) \in A : x_1 \leq t\}, \\
A^+(t) &:= \{(x_1, x_2, x_3) \in A : x_3 \geq t\}, \\
A^-(t) &:= \{(x_1, x_2, x_3) \in A : x_3 \leq t\}, \\
A_+^*(t) &:= \{(2t - x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2, x_3) \in A_+(t)\}, \\
A_-^*(t) &:= \{(2t - x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2, x_3) \in A_-(t)\}.
\end{aligned}$$

Note that $A_+^*(t)$ and $A_-^*(t)$ are the image of $A_+(t)$ and $A_-(t)$ by the reflection respect to the plane $\Pi(t)$.

STEP 1: We claim that both parts of M outside the cylinder point in the direction of v. We argue indirectly. Let us suppose that one part of $M - \mathcal{C}$ is asymptotic to

$$\mathcal{H}_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > r_1 > 0,\ x_1 = -\delta\}$$

and the other part is asymptotic to

$$\mathcal{H}_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < r_2 < 0,\ x_1 = +\delta\},$$

for some $\delta > 0$ (see Fig. 3.5). Fix real numbers t, l and let $\mathcal{G}^{t,l}$ be the grim reaper cylinder

$$\mathcal{G}^{t,l} := \{(x_1, x_2, l + \log\cos(x_1 - t)) \in \mathbb{R}^3 : |x_1 - t| < \pi/2, x_2 \in \mathbb{R}\}.$$

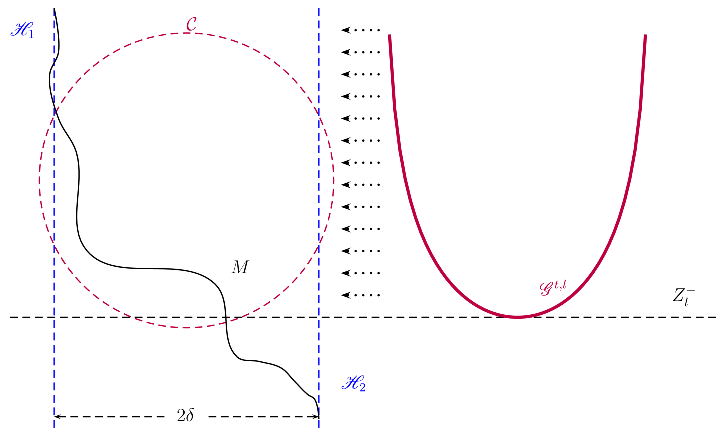


Figure 3.5: Comparison with a grim reaper cylinder

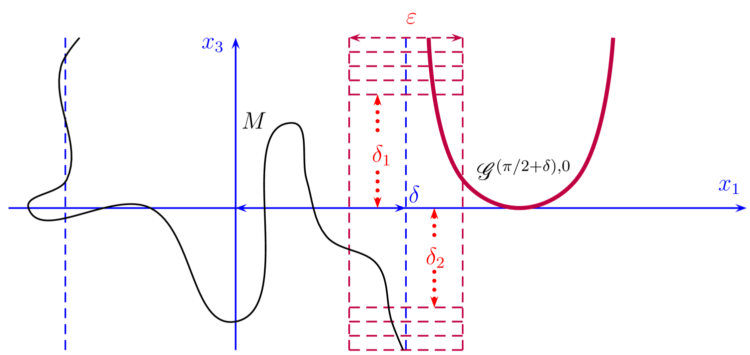


Figure 3.6: Comparison with a grim reaper cylinder

The idea is to obtain a contradiction by comparing the surface M with an appropriate grim reaper cylinder $\mathcal{G}^{t,l}$. Let us start with the grim reaper cylinder $\mathcal{G}^{\pi/2+\delta,0}$. Note that $\mathcal{G}^{\pi/2+\delta,0}$ lies outside the strip $(-\delta,\delta)\times\mathbb{R}^2$ and it is asymptotic to two half-planes contained in $\Pi(\delta)$ and $\Pi(\delta+\pi)$.

Fix $\varepsilon \in (0, 2\delta)$. Because outside a cylinder the grim reaper cylinder $\mathcal{G}^{\pi/2+\delta,0}$ is asymptotic to two half-planes, there exists $\delta_1 > 0$, depending on ε, such that $\mathcal{G}^{\pi/2+\delta,0} \cap Z^+_{\delta_1}$ is inside the region

$$(\delta, \delta+\varepsilon/2)\times\mathbb{R}\times(\delta_1, +\infty).$$

Moreover, there exists $\delta_2 > 0$, depending on ε, such that $M \cap Z^-_{-\delta_2}$ is inside the region

$$(\delta - \varepsilon/2, \delta + \varepsilon/2) \times \mathbb{R} \times (-\infty, -\delta_2).$$

Consider now the grim reaper cylinder $\mathcal{G}^{\pi/2+\delta+t,-\delta_1-\delta_2-1}$ and choose t large enough so that

$$\mathcal{G}^{\pi/2+\delta+t,-\delta_1-\delta_2-1} \cap M = \emptyset.$$

Translate the above grim reaper cylinder in the direction of $(-1, 0, 0)$. Since $\varepsilon \in (0, 2\delta)$, we see that after some finite time t_0 either there will be a first interior point of contact between M and $\mathcal{G}^{\pi/2+\delta+t_0,-\delta_1-\delta_2-1}$ or there will exist a sequence $\{p_i = (p_{1i}, p_{2i}, p_{3i})\}_{i\in\mathbb{N}}$ of points in M, with $\{p_{3i}\}_{i\in\mathbb{N}}$ bounded and $\{p_{2i}\}_{i\in\mathbb{N}}$ unbounded, such that

$$\lim_{i\to\infty} \operatorname{dist}(p_i, \mathcal{G}^{\pi/2+\delta+t_0,-\delta_1-\delta_2-1}) = 0.$$

As in Lemma 3.3.3, we deduce that both cases contradict the asymptotic behavior of M. Therefore, both parts of $M - \mathcal{C}$ must point in the direction of v.

STEP 2: We claim now that M lies in the slab $S := \big(-\delta, +\delta\big) \times \mathbb{R}^2$. Assume at first that $\lambda := \sup_M x_1 > \delta$. Consider now the surface (see Fig. 3.7)

$$\Sigma := \{(x_1, x_2, x_3) \in M : x_1 \geq \delta/2 + \lambda/2\}.$$

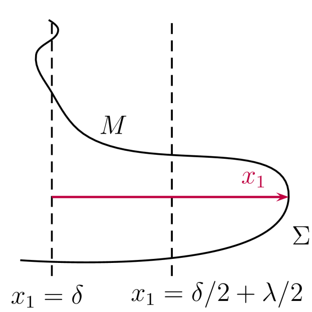


Figure 3.7: A slice of Σ

The asymptotic assumptions on M imply that the x_3-coordinate of Σ is bounded. Therefore, due to Lemma 3.3.3,

$$\sup_\Sigma x_1 = \sup_{\partial\Sigma} x_1.$$

But since

$$\partial\Sigma \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = \delta/2 + \lambda/2\},$$

we have that

$$x_1(p) = \delta/2 + \lambda/2 < \lambda = \sup_\Sigma x_1,$$

for any $p \in \partial\Sigma$, which is absurd. Thus $\sup_M x_1 \leq \delta$. Observe that if equality holds, then a contradiction is reached comparing M and the plane $\Pi(\delta)$ using the tangency principle. Hence $\sup_M x_1 < \delta$. Similarly, we can prove that $\inf_M x_1 > -\delta$. Consequently, M should lie inside the slab S.

STEP 3: Using the same arguments we will prove now that $2\delta = \pi$. Indeed, suppose at first that $2\delta > \pi$. We can then place a grim reaper cylinder $\mathcal{G}^{0,l}$ inside the slab S, by taking l sufficiently large, so that $\mathcal{G}^{0,l} \cap M = \emptyset$ (see Fig. 3.8).

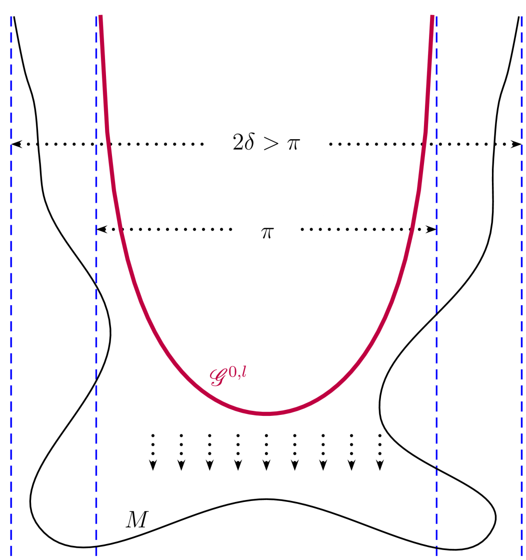


Figure 3.8: Comparison with a grim reaper cylinder from inside

Consider now the set

$$\mathscr{A} := \{l > 0 : M \cap \mathcal{G}^{0,l} = \emptyset\}.$$

Let $l_0 := \inf \mathscr{A}$. Assume at first that $l_0 \notin \mathscr{A}$. Because $M \cap \mathcal{G}^{0,l_0} \neq \emptyset$, it follows that there is an interior point of contact between M and $\mathcal{G}^{0,l_0}$. But then

$M \equiv \mathcal{G}^{0,l_0}$ which leads to a contradiction with the asymptotic assumptions on M. Let us treat now the case where $l_0 \in \mathscr{A}$. In this case dist $\{M, \mathcal{G}^{0,l_0}\} = 0$. Therefore, there exists a sequence of points $\{p_i = (p_{1i}, p_{2i}, p_{3i})\}_{i\in\mathbb{N}}$ in M such that

$$\lim_{i\to\infty} p_{1i} = p_{1\infty} \in \mathbb{R},\ \lim_{i\to\infty} p_{2i} = \infty,\ \lim_{i\to\infty} p_{3i} = p_{3\infty} \in \mathbb{R}$$

and

$$\lim_{i\to\infty} \text{dist}\left(p_i, \mathcal{G}^{0,l_0}\right) = 0.$$

Consider the sequence

$$\left\{M_i = M + (0, -p_{2i}, 0)\right\}_{i\in\mathbb{N}}.$$

By Lemma 3.3.1 we know that after passing to a subsequence, $\{M_i\}_{i\in\mathbb{N}}$ converges to a connected properly embedded translator M_∞ which has the same asymptotic behavior as M. On the other hand M_∞ has an interior point of contact with $\mathcal{G}^{0,l_0}$ and thus they must coincide. But this contradicts again the assumption on the asymptotic behavior of M. Thus 2δ must be less or equal than π. We exclude also the case where $2\delta < \pi$ by comparing M with a grim reaper cylinder from outside (see Fig. 3.9).

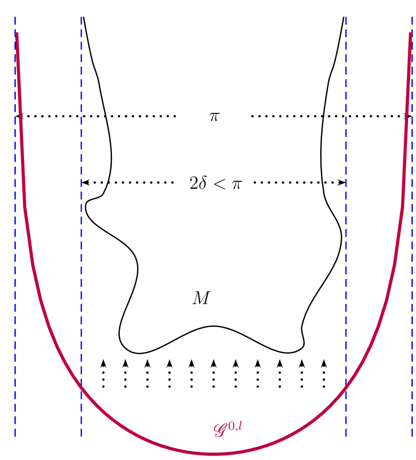


Figure 3.9: Comparison with a grim reaper cylinder from outside

Consequently, $2\delta = \pi$.

STEP 4: We will prove here two auxiliary results that will be very useful in the rest of the proof.

Claim 4. *The inequality*

$$-\pi/2 < \inf_{\partial M^-(t)} x_1 \leq \inf_{M^-(t)} x_1 \leq \sup_{M^-(t)} x_1 \leq \sup_{\partial M^-(t)} x_1 < \pi/2,$$

holds for any any real number t such that $M^-(t) \neq \emptyset$.

Proof of the claim. Recall that

$$M^-(t) = \{(x_1, x_2, x_3) \in M : x_3 \leq t\}.$$

Hence, from Lemma 3.2, we have that

$$\operatorname{dist}\big(M^-(t), \Pi(\pi/2)\big) = \operatorname{dist}\big(\partial M^-(t), \Pi(\pi/2)\big).$$

Suppose now to the contrary that

$$\operatorname{dist}\big(\partial M^-(t), \Pi(\pi/2)\big) = 0.$$

Then, there exists a sequence $\{p_i = (p_{1i}, p_{2i}, t)\}_{i\in\mathbb{N}}$ of points of $\partial M^-(t)$ such that

$$\lim_{i\to\infty} p_{1i} = \pi/2 \quad \text{and} \quad \lim_{i\to\infty} p_{2i} = \infty.$$

Consider the sequence of surfaces $\{M_i := M + (0, -p_{2i}, 0)\}_{i\in\mathbb{N}}$. From Lemma 3.1 we know that $\{M_i\}_{i\in\mathbb{N}}$ converges to a connected properly embedded translator M_∞ which has the same asymptotic behavior as M. On the other hand, there is an interior point of contact between M_∞ and $\Pi(\pi/2)$, which is a contradiction. Thus,

$$\operatorname{dist}\big(\partial M^-(t), \Pi(\pi/2)\big) > 0.$$

which implies that $\sup_{M^-(t)} x_1 < \pi/2$. In the same way, we can prove that $\inf_{M^-(t)} x_1 > -\pi/2$. This completes the proof of the claim.

Claim 5. *There exists a sufficiently large number t such that the parts of $M^+(t)$ are graphs over the x_1x_2-plane, and there exists a sufficiently small $\delta > 0$ such that $M_+(\pi/2 - \delta)$ is a graph over the x_1x_2-plane.*

Proof of the claim. First note that the proof is complete if we see the second part of the claim, i.e., that there exists a sufficiently small $\delta > 0$ such that $M_+(\pi/2-\delta)$ is a graph over the x_1x_2-plane. Assuming this, the hypothesis on the asymptotic behavior of M implies that there exists a sufficiently large number t such that $M^+(t) \subset M_-(-\pi/2+\delta) \cup M_+(\pi/2-\delta)$, so both connected components of $M^+(t)$ are graphs over the x_1x_2-plane, as claimed.

In order to prove the existence of $\delta > 0$ such that $M_+(\pi/2-\delta)$ is a graph over the x_1x_2-plane, recall that from STEP 3 we know that M lies inside the slab

$$S = (-\pi/2, \pi/2) \times \mathbb{R}^2.$$

Since $\mathcal{G}$ and $M - \mathcal{C}$ are C^1-asymptotic to $\Pi\left(\frac{\pi}{2}\right)$, we can represent each wing of $M - \mathcal{C}$ as a graph over $\mathcal{G}$. Fix a sufficiently small positive number ε. Then, there exists $\delta > 0$ such that the interior of the right wing $M_+(\pi/2-\delta)$ of $M - \mathcal{C}$ (and analogously with the left wing $M_-(-\pi/2+\delta)$; indeed, the whole argument below is completely analogous for the left wing) can be parametrized by a smooth map

$$\tilde{F} : T_\delta := (\pi/2-\delta, \pi/2) \times \mathbb{R} \to \mathbb{R}^3, \quad \tilde{F} = F + \varphi\, \nu_F, \tag{3.4.1}$$

where the map $F(x_1, x_2) = (x_1, x_2, -\log\cos x_1)$ describes the position vector of $\mathcal{G}$, $\nu_F(x_1, x_2) = (\sin x_1, 0, -\cos x_1)$ is the downwards unit normal of F (see Lemma 1.4.1) and $\varphi : (\pi/2-\delta, \pi/2) \times \mathbb{R} \to \mathbb{R}$ is a smooth function such that

$$\sup_{T_\delta}|\varphi| < \varepsilon \quad \text{and} \quad \sup_{T_\delta}|D\varphi| < \varepsilon. \tag{3.4.2}$$

We are going to see that this $\delta > 0$ works.

Denote by $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the usual projection map defined by $\pi(x_1, x_2, x_3) = (x_1, x_2)$. Consider the following domain and range restriction of π:

$$\tilde{\pi} : \operatorname{int}(M_+(\pi/2-\delta)) \subset M \subset \mathbb{R}^3 \to T_\delta = (\pi/2-\delta, \pi/2) \times \mathbb{R} \subset \mathbb{R}^2,$$

where int denotes the interior of $M_+(\pi/2-\delta)$. Note that it is well-defined because, by definition of $M_+(\pi/2-\delta)$, the first coordinate function of $M_+(\pi/2-\delta)$ is greater or equal than $\pi/2-\delta$, and by STEP 3 the first coordinate function of $M \supset M_+(\pi/2-\delta)$ is less than $\pi/2$.
We finish if we prove that $\tilde{\pi}$ is a global homeomorphism. The scheme of the proof is:

1. $\tilde{\pi}$ is a covering map.
   This follow from [Lee03, Proposition 2.19], whose hypothesis are:

   (a) $\tilde{\pi}$ is a proper local diffeomorphism.

   (b) $\mathrm{int}\,(M_+(\pi/2-\delta))$ and T_δ are connected smooth manifolds.

2. $\tilde{\pi}$ is a global homeomorphism.
   This follow from [dC76, Corollary of Proposition 5, section 5-6 A], whose hypothesis, besides $\tilde{\pi}$ being a covering map, are:

   (a) $\mathrm{int}\,(M_+(\pi/2-\delta))$ is arcwise connected;

   (b) T_δ is simply connected.

Let us prove all these statements.

First, let us see that $\tilde{\pi}$ is a proper map, that is, for every compact $K \subset T_\delta$ the inverse image $\tilde{\pi}^{-1}(K)$ is compact. This follows from the asymptotic behaviour of M. Indeed, by definition of the projection map, $\tilde{\pi}^{-1}(K) = K \times A$, where A is a subset of $\mathbb{R}$. Observe that A is closed because π is continuous and K is closed, and A must be bounded since otherwise M could not be asymptotic to $\Pi(\pi/2)$ in the sense of definition 3.0.1.

Furthermore, $\tilde{\pi}$ is a local diffeomorphism. Indeed, it is sufficient to show that $\tilde{\pi}$ is a local bijection; then, being a local diffeomorphism comes from the fact that $\tilde{\pi}$ is a chart of the smooth surface $\mathrm{int}\,(M_+(\pi/2-\delta))$. The strategy to prove that $\tilde{\pi}$ is a local bijection is to use the parametrization $\tilde{F}$ introduced in (3.4.1) in order to show that $\mathrm{int}\,(M_+(\pi/2-\delta))$ is strictly mean convex. Since it is a translator, $0 < H = -\langle \nu_{\tilde{F}}, \mathrm{v}\rangle$. Hence $\langle \nu_{\tilde{F}}, \mathrm{v}\rangle < 0$. Thus, each point of $\mathrm{int}\,(M_+(\pi/2-\delta))$ has an open neighborhood that can be represented as a graph over the x_1x_2-plane, that is, $\tilde{\pi}$ is a local bijection, as claimed.
We begin by computing the downwards unit normal $\nu_{\tilde{F}}$ of $\tilde{F}$:

$$\nu_{\tilde{F}} = \frac{-\varphi_{x_1}\cos^2 x_1 F_{x_1} - (1+\varphi\cos x_1)\varphi_{x_2}F_{x_2} + (1+\varphi\cos x_1)\,\nu_F}{\sqrt{\varphi_{x_1}^2\cos^2 x_1 + (1+\varphi\cos x_1)^2(1+\varphi_{x_2}^2)}}. \quad (3.4.3)$$

Proof. First, let us express the tangent vectors of $\tilde{F}$ in the orthonormal basis of $\mathbb{R}^3$ (see Lemma 1.4.1)

$$\{E_1 = (\cos x_1, 0, \sin x_1),\ E_2 = (0,1,0),\ \nu_F = (\sin x_1, 0, -\cos x_1)\}.$$

In that Lemma 1.4.1 we also computed the tangent vectors F_{x_1} and F_{x_2}:

$$F_{x_1} = (1, 0 \tan x_1),\ F_{x_2} = (0, 1, 0),$$

and the partial derivatives of ν_F:

$$\begin{aligned}(\nu_F)_{x_1} &= (\cos x_1, 0, \sin x_1) = E_1,\\ (\nu_F)_{x_2} &= (0, 0, 0).\end{aligned}$$

Observe that $E_1 = \cos x_1 F_{x_1},\ E_2 = F_{x_2}$.
Then

$$\begin{aligned}\tilde{F}_{x_1} &= F_{x_1} + \varphi_{x_1}\, \nu_F + \varphi(\nu_F)_{x_1} = (1 + \varphi \cos x_1) F_{x_1} + \varphi_{x_1}\, \nu_F\\ &= \left(\tfrac{1}{\cos x_1} + \varphi\right) E_1 + \varphi_{x_1}\, \nu_F = \left(\tfrac{1}{\cos x_1} + \varphi, 0, \varphi_1\right),\\ \tilde{F}_{x_2} &= F_{x_2} + \varphi_{x_2}\, \nu_F + \varphi(\nu_F)_{x_2} = E_2 + \varphi_{x_2}\, \nu_F = (0, 1, \varphi_2),\end{aligned}$$

and the unit downwards normal vector $\nu_{\tilde{F}}$ of $\tilde{F}$ can be computed using the cross product:

$$\begin{aligned}\tilde{F}_{x_1} \times \tilde{F}_{x_2} &= \begin{vmatrix} E_1 & E_2 & \nu_F \\ \frac{1}{\cos x_1} + \varphi & 0 & \varphi_{x_1} \\ 0 & 1 & \varphi_{x_2} \end{vmatrix}\\ &= -\varphi_{x_1} E_1 - \left(\tfrac{1}{\cos x_1} + \varphi\right) \varphi_{x_2} E_2 + \left(\tfrac{1}{\cos x_1} + \varphi\right) \nu_F\end{aligned}$$

Then

$$|\tilde{F}_{x_1} \times \tilde{F}_{x_2}| = \sqrt{\varphi_{x_1}^2 + \left(\tfrac{1}{\cos x_1} + \varphi\right)^2 (1 + \varphi_{x_2}^2)},$$

hence

$$\begin{aligned}\nu_{\tilde{F}} &= \frac{\tilde{F}_{x_1} \times \tilde{F}_{x_2}}{|\tilde{F}_{x_1} \times \tilde{F}_{x_2}|} = \frac{-\varphi_{x_1} E_1 - \left(\frac{1}{\cos x_1} + \varphi\right) \varphi_{x_2} E_2 + \left(\frac{1}{\cos x_1} + \varphi\right) \nu_F}{\sqrt{\varphi_{x_1}^2 + \left(\frac{1}{\cos x_1} + \varphi\right)^2 (1 + \varphi_{x_2}^2)}}\\ &= \frac{-\varphi_{x_1} \cos x_1 F_{x_1} - \left(\frac{1}{\cos x_1} + \varphi\right) \varphi_{x_2} F_{x_2} + \left(\frac{1}{\cos x_1} + \varphi\right) \nu_F}{\sqrt{\varphi_{x_1}^2 + \left(\frac{1}{\cos x_1} + \varphi\right)^2 (1 + \varphi_{x_2}^2)}}\\ &= \frac{-\varphi_{x_1} \cos^2 x_1 F_{x_1} - (1 + \varphi \cos x_1) \varphi_{x_2} F_{x_2} + (1 + \varphi \cos x_1)\, \nu_F}{\sqrt{\varphi_{x_1}^2 \cos^2 x_1 + (1 + \varphi \cos x_1)^2 (1 + \varphi_{x_2}^2)}}. \qquad (3.4.4)\end{aligned}$$

□

Now, since $\tilde{F}$ is a translator, we have that its mean curvature is

$$H = -\langle \nu_{\tilde{F}}, \mathrm{v} \rangle = \frac{\cos x_1 (1 + \varphi \cos x_1 + \varphi_{x_1} \sin x_1)}{\sqrt{\varphi_{x_1}^2 \cos^2 x_1 + (1 + \varphi \cos x_1)^2 (1 + \varphi_{x_2}^2)}}. \qquad (3.4.5)$$

Proof. We have that

$$\begin{aligned}
\langle F_{x_1}, \mathrm{v} \rangle &= \langle (1, 0, \tan x_1), (0, 0, 1) \rangle = \tan x_1, \\
\langle F_{x_2}, \mathrm{v} \rangle &= \langle (0, 1, 0), (0, 0, 1) \rangle = 0, \\
\langle \nu_F, \mathrm{v} \rangle &= \langle (\sin x_1, 0, -\cos x_1), (0, 0, 1) \rangle = -\cos x_1.
\end{aligned}$$

By linearity of the metric and using expression (3.4.4), we obtain the desired formula (3.4.5). □

Then, taking into account that $x_1 \in (\pi/2 - \delta, \pi/2)$, thus $\cos x_1 > 0$, and (3.4.2), which means that φ and its first partial derivatives are small, we have that $H > 0$, as claimed.

With respect to the connectedness of $\operatorname{int}(M_+(\pi/2 - \delta))$ and T_δ, this is trivially true in the case of $T_\delta = (\pi/2 - \delta, \pi/2) \times \mathbb{R}$. And, by Lemma 3.3.3, $M_+(\pi/2 - \delta)$ must be connected. Indeed, assume to the contrary that $M_+(\pi/2 - \delta)$ has more than one connected component. Let Σ be a connected component different from the one whose x_3-coordinate function is not bounded (there is at least one by assumption). Then due to Lemma 3.3.3 the infimum and the supremum of the x_1-coordinate function of Σ are reached along the boundary, that is, Σ is contained in the plane $\Pi(\pi/2 - \delta)$, so the whole surface M must coincide with this plane by the interior tangency principle, which is a contradiction. Once we know that $M_+(\pi/2 - \delta)$ is connected, then its interior $\operatorname{int}(M_+(\pi/2 - \delta))$ is also connected; this is not true in general for any arbitrary set, but observe that in this case the boundary of $M_+(\pi/2 - \delta)$ is a connected infinite non-self-intersecting curve. If this boundary were finite, then the asymptotic behaviour of M would not be true since it implies, according to its definition and the hypothesis on Theorem 3.0.2, that for sufficiently small $\delta > 0$, as it is the case, $M_+(\pi/2 - \delta)$ can be represented as a graph over a whole half-plane of $\Pi(\pi/2)$ with boundary parallel to the x_2-axis. This shows also that it is non-self-intersecting, which also follows from the hypothesis that M is embedded. Moreover, the boundary of $M_+(\pi/2 - \delta)$ is connected, otherwise, since by hypothesis M is connected and, bearing in mind its asymptotic behaviour, then necessarily the number of times that M intersects the plane $\Pi(\pi/2-\delta)$ is finite and odd. Then, if there were three or more intersections, this would mean that $M_+(\pi/2-\delta)$ is not connected, a contradiction. Therefore, the boundary of $M_+(\pi/2-\delta)$ is a single infinite non-self-intersecting curve, so when it is removed, the resulting set, $\operatorname{int}(M_+(\pi/2-\delta))$, remains connected, as claimed.

So far, we have established that $\tilde{\pi}$ is a covering map.

On the other hand, let us prove now that $\operatorname{int}(M_+(\pi/2-\delta))$ is arcwise connected. This is terminology of [dC76], where it is also proved that, in the case of (regular) surfaces, connectedness and arcwise connectedness are equivalent properties, although this is not true in general (see [dC76, Appendix, part B, Proposition 10]). Therefore, since we already proved $\operatorname{int}(M_+(\pi/2-\delta))$ is connected, then in fact it is arcwise connected.

Finally, it remains to prove that $T_\delta=(\pi/2-\delta,\pi/2)\times\mathbb{R}$ is simply connected, but this is trivially true.

STEP 5: We shall prove now that M is symmetric with respect to

$$\Pi(0)=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1=0\right\}$$

and that M is a bi-graph over this plane. The main tool used in the proof is the method of moving planes of Alexandrov (see [Ale56, Sch84]). Let us define

$$\mathcal{A}:=\{t\in[0,\pi/2) : M_+(t) \text{ is a graph over } \Pi(0) \text{ and } M_+^*(t)\geq M_-(t)\}.$$

Recall from [MSHS15, Definition 3.1] that the relation $M_+^*(t)\geq M_-(t)$ means that $M_+^*(t)$ is on the right hand side of $M_-(t)$. We will prove that $0\in\mathcal{A}$. In this case we have that $M_+^*(0)\geq M_-(0)$. By a symmetric argument we can show that $M_+(0)\geq M_-^*(0)$. Thus $M_+^*(0)\equiv M_-(0)$ and the proof of this step will be completed. The steps of the proof are the same as in [MSHS15, Proof of Theorem A] with the difference that here we have to control the behavior of the Gauß map in the direction of the x_2-axis.

Claim 6. *The minimum of the set $\mathcal{A}$ is 0. In particular, $\mathcal{A}=[0,\pi/2)$.*

Proof of the claim. Due to Claim 5 it follows that given a sufficiently small number ε, there exists a positive number t such that the surface $M_+(t)$

can be represented as a graph over $\Pi(0)$ as well as a graph over the x_1x_2-plane. Hence one can easily show that $\mathcal{A}$ is a non-empty set. Following the same arguments as in [MSHS15, Section 3, Proof of Theorem A], we can show that $\mathcal{A}$ is a closed subset of $[0, \pi/2)$. Moreover if $s \in \mathcal{A}$, then $[s, \pi/2) \subset \mathcal{A}$. Suppose now that $s_0 := \min \mathcal{A} > 0$. Then we will get at a contradiction, i.e., we will show that there exists a positive number ε such that $s_0 - \varepsilon \in \mathcal{A}$.

Condition 1: We will show at first that there exists a positive constant $\varepsilon_1 < s_0$ such that $M_+(s_0 - \varepsilon_1)$ is a graph over the plane $\Pi(0)$. Take a positive number α and consider the sets

$$M_+^+(s) := \{(x_1, x_2, x_3) \in M_+(s) : x_3 > \alpha\},$$

$$M_-^+(s) := \{(x_1, x_2, x_3) \in M_-(s) : x_3 > \alpha\},$$

and

$$M_+^-(s) := \{(x_1, x_2, x_3) \in M_+(s) : x_3 \leq \alpha\},$$

$$M_-^-(s) := \{(x_1, x_2, x_3) \in M_-(s) : x_3 \leq \alpha\}.$$

Since $M_+(s_0)$ is a graph over $\Pi(0)$, there exists α large enough such that

$$\operatorname{dist}\left[\xi\left(M_+^+(s_0)\right), \Pi(0)\right] > 0. \tag{3.4.6}$$

We fix such an α. From (3.4.6) it follows that there exists $\varepsilon_0 > 0$ such that $M_+^+(s_0 - \varepsilon_0)$ can be represented as a graph over the plane $\Pi(0)$ and furthermore

$$M_+^{+*}(s_0 - \varepsilon_0) \geq M_-^+(s_0 - \varepsilon_0). \tag{3.4.7}$$

Let us now investigate the lower part of our surface $M_+^-(s_0)$. Because $s_0 \in \mathcal{A}$, we can represent $M_+^-(s_0)$ as a graph over the plane $\Pi(0)$. Note that there is no point in $M_+^-(s_0)$ with normal vector included in the plane $\Pi(0)$ since otherwise $M_+^-(s_0)$ and its reflection with respect to $\Pi(s_0)$ would have the same tangent plane at that point so by the tangency principle at the boundary M would have been symmetric to a plane parallel to $\Pi(0)$. But this contradicts the asymptotic behavior of M. Consequently,

$$\xi\left(M_+^-(s_0)\right) \cap \Pi(0) = \emptyset. \tag{3.4.8}$$

Assertion. *There exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that, for all $t \in [s_0 - \varepsilon_1, s_0]$,*

$$\xi\left(M_+^-(t)\right) \cap \Pi(0) = \emptyset. \tag{3.4.9}$$

Proof of the assertion. Suppose to the contrary that such ε_1 does not exist. This implies that for all $i \in \mathbb{N}$ there exists $t_i \in [s_0 - 1/i, s_0]$ such that

$$\xi\left(M_+^-(t_i)\right) \cap \Pi(0) \neq \emptyset.$$

Then there exists a sequence $\{q_i\}_{i\in\mathbb{N}} \subset M_+^-(t_i)$ such that $\xi(q_i) \in \Pi(0)$. Only two situations can occur, namely either the sequence $\{q_i\}_{i\in\mathbb{N}}$ is bounded or it is unbounded. We will show that both cases lead to a contradiction.

If $\{q_i\}_{i\in\mathbb{N}}$ is bounded, then it should have a convergent subsequence that we do not relabel for simplicity. Denote its limit by q_∞. Note that q_∞ belongs to the closure of $M_+^-(s_0)$. Hence, by the continuity of the Gauß map

$$\Pi(0) \supset \mathbb{S}^1 \ni \xi(q_i) \to \xi(q_\infty) \in \mathbb{S}^1 \subset \Pi(0).$$

Then

$$\xi\left(M_+^-(s_0)\right) \cap \Pi(0) \neq \emptyset,$$

which contradicts the relation (3.4.8).

Let us now examine the case where the sequence $\{q_i = (q_{1i}, q_{2i}, q_{3i})\}_{i\in\mathbb{N}}$ is not bounded. The first coordinate $\{q_{1i}\}_{i\in\mathbb{N}}$ of $\{q_n\}_{n\in\mathbb{N}}$ is bounded. The last coordinate $\{q_{3i}\}_{i\in\mathbb{N}}$ of $\{q_i\}_{i\in\mathbb{N}}$ is also bounded. Therefore, the second coordinate $\{q_{2i}\}_{i\in\mathbb{N}}$ of the sequence must be unbounded. Consider now the sequence $\{M_i = M + (0, -q_{2i}, 0)\}_{i\in\mathbb{N}}$. Due to Lemma 3.3.1, we have that after passing to a subsequence, $\{M_i\}_{i\in\mathbb{N}}$ converges smoothly to a properly embedded connected translator M_∞ which has the same asymptotic behavior as M. Furthermore, the limiting surface M_∞ has the following additional properties:

(a) The surface $(M_\infty)_+(s_0)$ can be represented as a graph over the plane $\Pi(0)$.

(b) The inequality $(M_\infty)_+^*(s_0) \geq (M_\infty)_-(s_0)$ holds true.

(c) There exists a point in M_∞ in which the Gauß map belongs to the plane $\Pi(0)$.

Applying the tangency principle at the boundary of $(M_\infty)_+^*(s_0)$ and $(M_\infty)_-(s_0)$ we deduce that $\Pi(s_0)$ is a plane of symmetry for M_∞, something that contradicts the asymptotic behavior of M_∞. This completes the proof of our assertion.

The relation (3.4.9) implies that, for every $t \in [s_0 - \varepsilon_1, s_0]$, the surface $M_+^-(t)$ can be represented as a graph over $\Pi(0)$. Consequently, $M_+(t)$ is a graph over $\Pi(0)$ for all $t \geq s_0 - \varepsilon_1$. Hence the first condition in the definition of the set $\mathcal{A}$ is verified.

Condition 2: Reasoning again as in [MSHS15, Proof of Theorem A] and with the help of Lemma 3.1 we can prove the inequality $M_+^*(s_0 - \varepsilon_1) \geq M_-(s_0 - \varepsilon_1)$.

Therefore, by Conditions 1 and 2, we have that $s_0 - \varepsilon \in \mathcal{A}$. This contradicts the fact that s_0 is the infimum of $\mathcal{A}$. So, $s_0 = 0$ and this concludes the proof of STEP 5.

STEP 6: Let us explore the asymptotic behavior of our translating soliton M as its x_2-coordinate function tends to infinity.

Claim 7. *Consider the profile curve $\Gamma = M \cap \Pi(0)$. If the coordinate function $x_3|_\Gamma$ attains its global extremum on Γ (maximum or minimum), then M is a grim reaper cylinder.*

Proof of the claim. We will distinguish two cases. The idea is to compare M with a "half-grim reaper cylinder" at the level where x_3 attains its extremum.

Case A: Suppose at first that there exists a point $p \in \Gamma$ (see Fig. 3.10) such that

$$l := x_3(p) = \max_\Gamma x_3.$$

Observe that

$$\partial M_+(0) \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \leq l\}.$$

For a fixed real number t consider the "half-grim reaper cylinder" (see Fig. 3.11) given by

$$\mathcal{G}_+^{t,l} = \big\{\big(x_1, x_2, l + \log\cos(x_1 - t)\big) \in \mathbb{R}^3 : x_1 \in [t, \pi/2 + t), x_2 \in \mathbb{R}\big\}.$$

Define now the set

$$\mathcal{Q} := \big\{t \in (-\infty, 0) : \mathcal{G}_+^{t,l} \cap M_+(0) = \emptyset\big\}$$

Obviously, $\mathcal{Q}$ is a non-empty set. Moreover, if $t \in \mathcal{Q}$ then $(-\infty, t) \subset \mathcal{Q}$. Let $t_0 := \sup \mathcal{Q}$.

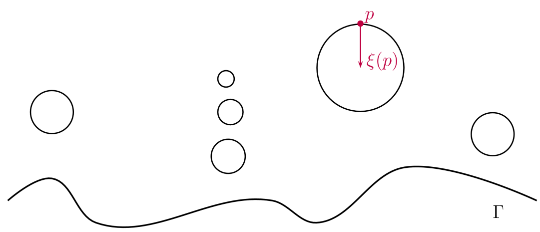


Figure 3.10: The profile curve Γ

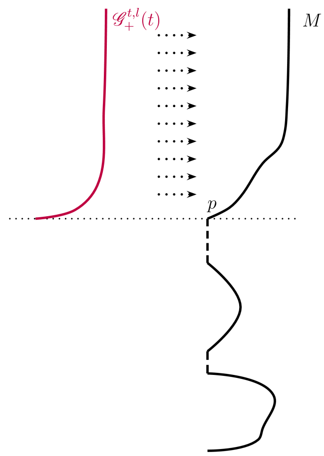


Figure 3.11: Comparing with a plane

We claim that $t_0 = 0$. Suppose this is not true. If $t_0 \notin \mathcal{Q}$, then there would be an interior point of contact (notice that the boundaries of both surfaces do not touch when $t < 0$). This implies that $M = \mathcal{G}^{t_0,l}$, which contradicts the assumption on the asymptotic behavior of M. Let us consider now the case where $t_0 \in \mathcal{Q}$. In this case there exists a divergent sequence $\{p_i = (p_{1i}, p_{2i}, p_{3i})\}_{i\in\mathbb{N}} \subset M_+(0)$ such that

$$\lim_{i\to\infty} \operatorname{dist}\big(p_i, \mathcal{G}_+^{t_0,l}\big) = 0.$$

Because the asymptotic behavior of $\mathcal{G}_+^{t_0,l}$ and $M_+(0)$ is different and the distance between their boundaries is positive, then one can find constants a_0 and a_1 such that $a_0 < x_3(p_i) < a_1$, for all $i \in \mathbb{N}$. So, $\{p_{2i}\}_{i\in\mathbb{N}}$ tends to infinity. Now we can apply Lemma 3.3.1 in order to deduce that the limit of the sequence $\{M_i\}_{i\in\mathbb{N}}$, given by

$$M_i := M - (0, p_{2i}, 0),$$

exists and has the same asymptotic behavior as M. Let us call this limit M_∞. But now M_∞ and $\mathcal{G}_+^{t_0,l}$ have an interior point of contact and thus they must coincide. This leads again to a contradiction because M_∞ and $\mathcal{G}_+^{t_0,l}$ do not have the same asymptotic behavior. Hence, $t_0 = 0$. Consequently, $\mathcal{G}_+^{0,l}$ and $M_+(0)$ have a boundary contact at p. Observe that the tangent plane at p of both surfaces is horizontal by STEP 5, and therefore by the boundary tangency principle they must coincide.

Case B: Suppose now that there exists $q \in \Gamma$ such that

$$\mu = x_3(q) = \min_\Gamma x_3.$$

In this case, we compare $M_+(0)$ with the family of "half-grim reaper cylinders" $\left\{\mathcal{G}_+^{t,\mu}\right\}_{t\geq 0}$ and we proceed exactly as in the proof of Case A.

Claim 8. *The surface M is a graph over the x_1x_2-plane.*

Proof of the claim: Recall that the profile curve $\Gamma = \Pi(0) \cap M$ lies inside the cylinder $\mathcal{C}$. Let

$$\alpha := \limsup_{x_2\to+\infty} (x_3|_\Gamma) \quad \text{and} \quad \beta := \liminf_{x_2\to-\infty} (x_3|_\Gamma).$$

Take sequences $\{p_i = (0, p_{2i}, p_{3i})\}_{i\in\mathbb{N}}$ and $\{q_i = (0, q_{2i}, q_{3i})\}_{i\in\mathbb{N}}$ along the curve Γ such that

$$\lim_{i\to\infty} p_{2i} = +\infty, \ \lim_{i\to\infty} q_{2i} = -\infty, \ \lim_{i\to\infty} p_{3i} = \alpha \text{ and } \lim_{i\to\infty} q_{3i} = \beta.$$

and define the sequences of translators $\{M_i^\alpha\}_{i\in\mathbb{N}}$, $\{M_i^\beta\}_{i\in\mathbb{N}}$ given by

$$M_i^\alpha := M - (0, p_{2i}, 0) \quad \text{and} \quad M_j^\beta := M - (0, q_{2j}, 0).$$

From Lemma 3.3.1 we deduce that

$$M_i^\alpha \to M_\infty^\alpha \quad \text{and} \quad M_i^\beta \to M_\infty^\beta,$$

where M_∞^α and M_∞^β are connected properly embedded translators with the same asymptotic behavior as our surface M.

Consider the points $(0,0,\alpha) \in M_\infty^\alpha$ and $(0,0,\beta) \in M_\infty^\beta$. Taking into account the way in which we have constructed our limits, we have that

$$\alpha = \max_{M_\infty^\alpha \cap \Pi(0)} x_3 \quad \text{and} \quad \beta = \min_{M_\infty^\beta \cap \Pi(0)} x_3.$$

At this point, we can use Claim 7 to conclude that the limits M_∞^α and M_∞^β are grim reaper cylinders, possibly displayed at different heights. From the definition of the limit and the second part of Theorem 3.1.4, it follows that for large enough values $i \geq i_0$ there exist:

(a) strictly increasing sequences of positive numbers $\{m_{1i}\}_{i\in\mathbb{N}}$, $\{m_{2i}\}_{i\in\mathbb{N}}$, $\{n_{1i}\}_{i\in\mathbb{N}}$ and $\{n_{2i}\}_{i\in\mathbb{N}}$ satisfying

$$m_{1i} < m_{2i} \quad \text{and} \quad -n_{1i} < -n_{2i},$$

for every $i \geq i_0$,

(b) real smooth functions $\varphi_i : (-\pi/2, \pi/2) \times (m_{1i}, m_{2i}) \to \mathbb{R}$ and $\vartheta_i : (-\pi/2, \pi/2) \times (-n_{1i}, -n_{2i}) \to \mathbb{R}$ satisfying the conditions

$$|\varphi_i| < 1/i,\ |\vartheta_i| < 1/i,\ |D\varphi_i| < 1/i \text{ and } |D\vartheta_i| < 1/i,$$

for any $i \geq i_0$,

such that the surfaces

$$R_i := \big\{(x_1, x_2, x_3) \in M : m_{1i} < x_2 < m_{2i}\big\}$$

and

$$L_i := \big\{(x_1, x_2, x_3) \in M : -n_{1i} < x_2 < -n_{2i}\big\}$$

can be represented as graphs over grim reaper cylinders that are generated by the functions φ_i and ϑ_i, respectively.

Now we prove that R_i and L_i can be represented globally as graphs over rectangles of the x_1x_2-plane. Formally, consider an arbitrary but fixed $i \geq i_0$, and let

$$\tilde{\pi} : R_i \subset M \subset \mathbb{R}^3 \to T_i := (-\pi/2, \pi/2) \times (m_{1i}, m_{2i}) \subset \mathbb{R}^2,$$

be the domain and range restriction of the usual projection map $\pi : \mathbb{R}^3 \to \mathbb{R}^2$. In order to prove that R_i (the argument is analogous with L_i) is a graph over the corresponding rectangle $(-\pi/2, \pi/2) \times (m_{1i}, m_{2i})$ of the x_1x_2-plane, we can follow the strategy of Claim 5. Indeed, we can make the same argument, taking larger i_0 if necessary; the only difference here is the proof of the connectedness of R_i. In order to prove it, consider a sufficiently large $t > 0$ such that the two wings of $R_i^+(t)$ are graphs over the x_1x_2-plane. By Claim 5, there exists such a t and, moreover, the two wings of $R_i^+(t)$ are connected. Thus, we finish if we prove that $R_i^-(t)$ is connected. But thanks to the compactness of $R_i^-(t)$, what we can easily prove is that $R_i^-(t)$ is contained in a connected set. This is a consequence of the following proposition of Topology: if X_α is connected for all α and $\cap_\alpha X_\alpha \neq \emptyset$, then $\cup_\alpha X_\alpha$ is connected. Indeed, since we already know that the limit M_∞^α of the above sequence $\{M_i^\alpha\}_{i\in\mathbb{N}}$ is a grim reaper cylinder, from the definition of the limit of a surface, keeping the same notation (see Definition 3.1.1), we have that:

$$R_i = \bigcup_{p\in M_\infty^\alpha} R_i \cap W_{r,\varepsilon}(p) = \bigcup_{p\in M_\infty^\alpha} \text{Graph}\left(\left(u_i^1\right)_{|D_r(p)}\right)$$

where each $u_i^1(D_r(p))$ is connected because u_i^1 is continuous and the disk $D_r(p)$ is connected for all p. Then, by the compactness of $R_i^-(t)$, we can express $R_i^-(t)$ as a finite union of connected sets and, furthermore, we can argue with the above property as follows:

- (Step 1) Consider initially just two non-disjoint connected sets between the ones in the above union that cover $R_i^-(t)$; then their union, denoted by U_1, is connected by the above property.

- (Step 2) Take now another set non-disjoint with U_1; again, their union, U_2, is connected by the above property.

- (Step i) And so on: in the i-th step just two sets are considered, the union of the two previous ones, U_{i-1}, and another set non-disjoint with it. Their union, U_i, is connected by the above property.

By the compactness of $R_i^-(t)$, this process ends after a finite number of steps, let us say s steps. Therefore, $U_s \supseteq R_i^-(t)$ is a connected set.
Finally,

$$R_i = R_i^+(t) \cup U_s,$$

where $R_i^+(t)$ and U_s are non-disjoint connected sets. Then, again by the above property, their union, R_i, is connected, which is the desired conclusion.

Consider now the compact exhaustion $\{\Lambda_i\}_{i\geq i_0}$ (see Fig. 3.12) of the surface M given by

$$\Lambda_i := \big\{(x_1, x_2, x_3) \in M : -a_i \leq x_2 \leq b_i,\ x_3 \leq i\big\}$$

where

$$a_i = (n_{1i} + n_{2i})/2 \quad \text{and} \quad b_i = (m_{1i} + m_{2i})/2.$$

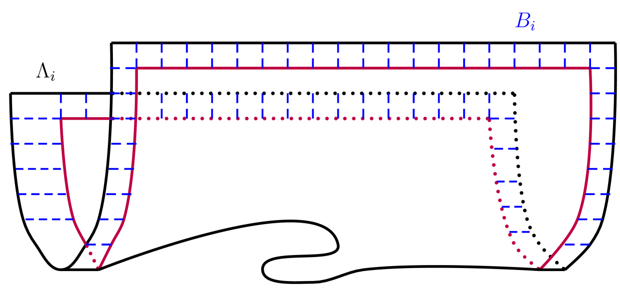


Figure 3.12: The exhaustion set Λ_i

The boundary of each Λ_i is piecewise smooth and consists of two lateral curves that converge to grim reapers and two top curves that converge to two parallel horizontal lines. Observe that we just proved that in a strip B_i around $\partial\Lambda_i$ (see again Fig. 3.12) the surface Λ_i is a graph over the x_1x_2-plane. The proof will be concluded if we prove that there exists $i_1 \in \mathbb{N}$ such that each Λ_i is a graph over the x_1x_2-plane, for any $i \geq i_1$.
We will argue by contradiction. The idea is to find a suitable $i_1 \in \mathbb{N}$ and assume that there exists $i \geq i_1$ such that Λ_i is not a graph over the x_1x_2-plane. Then we compare Λ_i with a vertical translation of itself, and translate it back to reach a contradiction using the interior tangency principle, as we did in Section 2.3.
In order to formalize the above argument, first note that by claim 5 we can fix a large height t_0 such that $M^+(t_0)$ is a graph over the x_1x_2-plane. From Claim 4 we know that

$$\text{dist}\,\big(M^-(t_0), \Pi(\pi/2)\big) = \text{dist}\,\big(\partial M^-(t_0), \Pi(\pi/2)\big) =: \delta.$$

From the asymptotic behavior of M we know that there exists a number $t_1 > t_0$ such that

$$\operatorname{dist}\left(M^-(t_1), \Pi(\pi/2)\right) = \operatorname{dist}\left(\partial M^-(t_1), \Pi(\pi/2)\right) = \delta/2.$$

Now fix an integer $i_1 > \max\{i_0, t_1\}$. Let us see that this integer works. Suppose to the contrary that there is $i \geq i_1$ such that Λ_i is not a graph over the x_1x_2-plane. We will derive a contradiction. Let

$$\Lambda_i(s) := \Lambda_i + (0, 0, s)$$

be the translation of Λ_i in direction of v. Take a number s_0 such that

$$\Lambda_i(s_0) \cap \Lambda_i = \emptyset.$$

Start to move back $\Lambda_i(s_0)$ in the direction of $-$v. Then there exists $s_1 > 0$ where $\Lambda_i(s_1)$ intersects Λ_i. From the choice of i_1 we see that the intersection points must be interior points of contact. But then, from the tangency principle, it follows that $\Lambda_i(s_1) = \Lambda_i$, which is a contradiction. Therefore, for each $i > i_1$ the surface Λ_i must be a graph over the x_1x_2-plane. Because $\{\Lambda_i\}_{i\in\mathbb{N}}$ is a compact exhaustion of M we deduce that M itself must be a graph over the x_1x_2-plane. In particular, genus$(M) = 0$.

STEP 7: From Claim 8 we see that our surface M must be strictly mean convex. Consider now the x_2-coordinate of the Gauß map, i.e., the smooth function $\nu_2 : M \to \mathbb{R}$ given by $\nu_2 = \langle \nu, e_2 \rangle$, where $e_2 = (0, 1, 0)$. Let us see that ν_2 and H satisfy the following partial differential equations

$$\Delta \nu_2 + \langle \nabla \nu_2, \nabla x_3 \rangle + |A|^2 \nu_2 = 0 \tag{3.4.10}$$

and

$$\Delta H + \langle \nabla H, \nabla x_3 \rangle + |A|^2 H = 0, \tag{3.4.11}$$

where $|A|^2$ stands for the squared norm of the second fundamental form of M.

Proof. Equation (3.4.11) is proved in [MSHS15, Lemma 2.1 (f)]. We will follow a similar approach to establish equation (3.4.10).

To simplify our computations, we introduce normal coordinates: for an arbitrary but fixed point $p \in M$, there exists a smooth frame field $\{E_1, E_2\}$

such that $g_{ij}(p) := g(E_i, E_j)(p) = \delta_{ij}$ for all $1 \leq i, j \leq 2$ and $\nabla_{E_i} E_j = 0$ for all $1 \leq i, j \leq 2$. We do all the calculations at an arbitrary but fixed point $p \in M$ using normal coordinates at p.

Let us compute $\Delta \nu_2 = \sum_i E_i \left(E_i \left(\nu_2 \right) \right)$. First,

$$\begin{aligned} E_i \left(\nu_2 \right) &= E_i \left(\langle \nu, e_2 \rangle \right) = \langle D_{E_i} \nu, e_2 \rangle = \langle d\nu(E_i), e_2 \rangle = \langle d\nu(E_i), e_2^\top \rangle \\ &= A(E_i, e_2^\top). \end{aligned} \tag{3.4.12}$$

Recall that, in general, for any vector $v \in \mathbb{R}^3$ we denote by $v^\perp$ the orthogonal projection of v onto the normal bundle of f and by $v^\top$ the orthogonal projection of v onto the tangent bundle of f.
Thus,

$$\Delta \nu_2 = \sum_i E_i \left(E_i \left(\nu_2 \right) \right) = \sum_i E_i \left(A(E_i, e_2^\top) \right).$$

Our next step is to *interchange* the derivative E_i and $e_2^\top$, that is, to apply the Codazzi equation to these two indices: $\left(\nabla_{E_i} A \right) (E_i, e_2^\top) = \left(\nabla_{e_2^\top} A \right) (E_i, E_i)$. Since in normal coordinates around p it holds that $\left(\nabla_{E_i} E_i \right) (p) = 0$,

$$\begin{aligned} & E_i \left(A(E_i, e_2^\top) \right) \\ &= \left(E_i \left(A(E_i, e_2^\top) \right) - A \left(\nabla_{E_i} E_i, e_2^\top \right) - A \left(E_i, \nabla_{E_i} e_2^\top \right) \right) + A \left(E_i, \nabla_{E_i} e_2^\top \right) \\ &= \left(\nabla_{E_i} A \right) (E_i, e_2^\top) + A \left(E_i, \nabla_{E_i} e_2^\top \right) = \left(\nabla_{e_2^\top} A \right) (E_i, E_i) + A \left(E_i, \nabla_{E_i} e_2^\top \right) \\ &= e_2^\top \left(A(E_i, E_i) \right) + A \left(E_i, \nabla_{E_i} e_2^\top \right). \end{aligned}$$

Then

$$\Delta \nu_2 = \sum_i e_2^\top \left(A(E_i, E_i) \right) + \sum_i A \left(E_i, \nabla_{E_i} e_2^\top \right).$$

On the one hand,

$$\sum_i e_2^\top \left(A(E_i, E_i) \right) = e_2^\top \left(\sum_i A(E_i, E_i) \right) = e_2^\top (H) = \left\langle \nabla H, e_2^\top \right\rangle,$$

and, by [MSHS15, Lemma 2.1 (e)], $\left\langle \nabla H, e_2^\top \right\rangle = -A(\nabla \mathrm{x}_3, e_2^\top)$, so

$$\begin{aligned}\sum_i e_2^\top \left(A(E_i, E_i)\right) &= -A(\nabla \mathrm{x}_3, e_2^\top) = -A\left(\sum_j E_j(\mathrm{x}_3)E_j, e_2^\top\right)\\ &= -\sum_j E_j(\mathrm{x}_3)A(E_j, e_2^\top) = -\sum_j E_j(\mathrm{x}_3)E_j(\nu_2)\\ &= -\left\langle \nabla \nu_2, \nabla \mathrm{x}_3 \right\rangle,\end{aligned}$$

where we used (3.4.12) in the penultimate equality.

On the other hand, in order to compute $\sum_i A\left(E_i, \nabla_{E_i} e_2^\top\right)$, we are going to follow a similar approach to the one in [MSHS15, Lemma 2.1] in order to deduce item (f). To this end, first note that $e_2^\top = \nabla \mathrm{x}_2$:

$$\langle E_i, \nabla \mathrm{x}_2\rangle = E_i(\mathrm{x}_2) = E_i(\langle f, e_2\rangle) = \langle D_{E_i} f, e_2\rangle = \langle df(E_i), e_2\rangle = \langle df(E_i), e_2^\top\rangle,$$

where D denotes the Levi-Civita connection in $\mathbb{R}^{m+1}$.
Now recall that $\left(\nabla_{E_i} E_i\right)(p) = 0$ because we are using normal coordinates around p, so:

$$\begin{aligned}\nabla_{E_i} e_2^\top &= \nabla_{E_i}\left(\sum_k E_k(\mathrm{x}_2)E_k\right) = \sum_k \left(E_i(E_k(\mathrm{x}_2))E_k + E_k(\mathrm{x}_2)\nabla_{E_i} E_k\right)\\ &= \sum_k E_i(E_k(\mathrm{x}_2))E_k = \sum_k \nabla^2 \mathrm{x}_2(E_i, E_k)E_k.\end{aligned}$$

And the Hessian $\nabla^2 \mathrm{x}_2$ fulfills a similar equation to the one that satisfies $\nabla^2 \mathrm{x}_3$ in [MSHS15, Lemma 2.1 (c)]:

$$\begin{aligned}\nabla^2 \mathrm{x}_2(E_i, E_j) &= E_i\left(E_j(\mathrm{x}_2)\right) = E_i\left(E_j(\langle f, e_2\rangle)\right) = E_i\left(\langle df(E_j), e_2\rangle\right)\\ &= \langle D_{df(E_i)} df(E_j), e_2\rangle = \langle \mathbf{A}(E_i, E_j), e_2\rangle\\ &= A(E_i, E_j)\langle -\nu, e_2\rangle = -A(E_i, E_j)\,\nu_2,\end{aligned}$$

where we used that $\mathbf{A}(E_i, E_j) = D_{df(E_i)} df(E_j) - df(\nabla_{E_i} E_j) = D_{df(E_i)} df(E_j)$, being D the Levi-Civita connection in $\mathbb{R}^{m+1}$.
Thus

$$\nabla_{E_i} e_2^\top = \sum_k \nabla^2 \mathrm{x}_2(E_i, E_k)E_k = -\nu_2 \sum_k A(E_i, E_k)E_k.$$

Then

$$\sum_i A\left(E_i, \nabla_{E_i} e_2^\top\right) = \sum_i A\left(E_i, -\nu_2 \sum_k A(E_i, E_k)E_k\right)$$
$$= -\nu_2 \sum_{i,k} A(E_i, E_k)A(E_i, E_k) = -\nu_2\,|A|^2.$$

Therefore,

$$\Delta\,\nu_2 = \sum_i e_2^\top\left(A(E_i, E_i)\right) + \sum_i A\left(E_i, \nabla_{E_i} e_2^\top\right) = -\left\langle \nabla\,\nu_2, \nabla\,\mathrm{x}_3\right\rangle - \nu_2\,|A|^2,$$

as claimed. □

Define now the function $h := \nu_2\,H^{-1}$. Combining the equations (3.4.10) and (3.4.11) we deduce that h satisfies the following differential equation

$$\Delta h + \langle \nabla h, \nabla(x_3 + 2\log H)\rangle = 0. \tag{3.4.13}$$

Proof. In order to prove it, we simply compute Δh and $\langle \nabla h, \nabla(x_3 + 2\log H)\rangle$ and check that they are opposite each other.

$$\begin{aligned}
\Delta h &= \Delta\left(\nu_2 \cdot \frac{1}{H}\right) = \frac{1}{H}\Delta\,\nu_2 + \nu_2\,\Delta\left(\frac{1}{H}\right) + 2\left\langle \nabla\,\nu_2, \nabla\left(\frac{1}{H}\right)\right\rangle \\
&= \frac{1}{H}\Delta\,\nu_2 + \frac{2\,\nu_2}{H^3}\langle \nabla H, \nabla H\rangle - \frac{\nu_2}{H^2}\Delta H - \frac{2}{H^2}\left\langle \nabla\,\nu_2, \nabla H\right\rangle \\
&= -\frac{1}{H}\langle \nabla\,\nu_2, \nabla\,\mathrm{x}_3\rangle - |A|^2\frac{\nu_2}{H} + \frac{2\,\nu_2}{H^3}\langle \nabla H, \nabla H\rangle + \frac{\nu_2}{H^2}\langle \nabla H, \nabla\,\mathrm{x}_3\rangle \\
&\quad + |A|^2\frac{\nu_2}{H} - \frac{2}{H^2}\left\langle \nabla\,\nu_2, \nabla H\right\rangle \\
&= -\frac{1}{H}\langle \nabla\,\nu_2, \nabla\,\mathrm{x}_3\rangle + \frac{2\,\nu_2}{H^3}\langle \nabla H, \nabla H\rangle + \frac{\nu_2}{H^2}\langle \nabla H, \nabla\,\mathrm{x}_3\rangle - \frac{2}{H^2}\left\langle \nabla\,\nu_2, \nabla H\right\rangle
\end{aligned}$$

where we used that for any smooth functions u and v in M the Laplace-Beltrami operator satisfies $\Delta(uv) = v\Delta u + u\Delta v + 2\langle \nabla u, \nabla v\rangle$ and, from here, taking $u = \tilde{u}/\tilde{v}$ and $v = \tilde{v}$, for any smooth functions $\tilde{u}$ and $\tilde{v} \neq 0$ in M, it holds that

$$\Delta\left(\frac{\tilde{u}}{\tilde{v}}\right) = \frac{1}{\tilde{v}}\Delta\tilde{u} - \frac{2}{\tilde{v}}\left\langle \nabla\left(\frac{\tilde{u}}{\tilde{v}}\right), \nabla\tilde{u}\right\rangle - \frac{\tilde{u}}{\tilde{v}^2}\Delta\tilde{v}.$$

We also used that $\nabla(1/H) = -1/H^2 \nabla H$ and equations (3.4.10) and (3.4.11).

Now, in order to compute $\langle \nabla h, \nabla(\mathrm{x}_3 + 2\log H)\rangle$, let us calculate first ∇h and $\nabla \log H$:

$$\nabla h = \nabla\left(\nu_2 \cdot \frac{1}{H}\right) = \nu_2 \nabla\left(\frac{1}{H}\right) + \frac{1}{H}\nabla \nu_2 = -\frac{\nu_2}{H^2}\nabla H + \frac{1}{H}\nabla \nu_2,$$
$$\nabla \log H = \frac{1}{H}\nabla H,$$

where in the last equality we used the chain rule; observe that $\log H$ is well defined because $H > 0$. Then

$$\begin{aligned}
&\langle \nabla h, \nabla(\mathrm{x}_3 + 2\log H)\rangle \\
&\quad = \langle \nabla h, \nabla \mathrm{x}_3\rangle + 2\langle \nabla h, \nabla \log H\rangle \\
&\quad = -\frac{\nu_2}{H^2}\langle \nabla H, \nabla \mathrm{x}_3\rangle + \frac{1}{H}\langle \nabla \nu_2, \nabla \mathrm{x}_3\rangle - \frac{2\,\nu_2}{H^3}\langle \nabla H, \nabla H\rangle + \frac{2}{H^2}\left\langle \nabla \nu_2, \nabla H\right\rangle.
\end{aligned}$$

This is the desired conclusion. □

Claim 9. *The surface M is smoothly asymptotic outside a cylinder to the grim reaper cylinder.*

Proof of the claim. Consider the sequence of surfaces $\{M_i\}_{i\in\mathbb{N}}$ given by $M_i := M + (0, 0, -i)$, for any $i \in \mathbb{N}$. One can readily see that for any compact set K of $\mathbb{R}^3$, it holds

$$\limsup_{i\to\infty} \text{area}\big\{M_i \cap K\big\} < \infty \quad \text{and} \quad \limsup_{i\to\infty} \text{genus}\big\{M_i \cap K\big\} < \infty.$$

From the compactness theorem of White, the sequence of surfaces $\{M_i\}_{i\in\mathbb{N}}$ converges smoothly (with respect to the Ilmanen's metric) to the union $\Pi(-\pi/2) \cup \Pi(\pi/2)$. Hence, due to Lemma 3.2.7, the wings of the translator M outside the cylinder must be smoothly asymptotic to the corresponding wings of the grim reaper cylinder. This completes the proof of the claim.

Claim 10. *The function h tends to zero as we approach infinity of our surface M.*

Proof of the claim. Consider the compact exhaustion $\{\Lambda_i\}_{i>i_1}$ defined in the STEP 6. The boundary of each Λ_i consists of four parts, namely:

$$\begin{aligned}
\Lambda_{1i} : &= \big\{(x_1,x_2,x_3) \in M : x_1 > 0,\ -a_i \le x_2 \le b_i,\ x_3 = i\big\},\\
\Lambda_{2i} : &= \big\{(x_1,x_2,x_3) \in M : x_1 < 0,\ -a_i \le x_2 \le b_i,\ x_3 = i\big\},\\
\Lambda_{3i} : &= \big\{(x_1,x_2,x_3) \in M : x_2 = -a_i,\ x_3 \le i\big\},\\
\Lambda_{4i} : &= \big\{(x_1,x_2,x_3) \in M : x_2 = b_i,\ x_3 \le i\big\}.
\end{aligned}$$

Bearing in mind the asymptotic behavior of M, we deduce that around each boundary curve line there exists a tubular neighborhood that can be represented as the graph of a smooth function over a slab of the grim reaper cylinder. If φ is such a function then, from the equations (3.4.3) and (3.4.5), we can represent h in the form

$$h = -\frac{\varphi_{x_2}}{\cos x_1} \cdot \frac{1 + \varphi \cos x_1}{1 + \varphi \cos x_1 + \varphi_{x_1} \sin x_1}. \tag{3.4.14}$$

Let us examine the behavior of h along Λ_{1i}. Note that these curves belong to the wings of M outside the cylinder. Fix a sufficiently small $\varepsilon > 0$. Then, there exists $\delta_2 > 0$ and large enough index i_2 such that

$$M \cap \big\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 \ge i_2\big\}$$

can be written as the graph over the grim reaper cylinder of a smooth function φ defined in the domain $T_{\delta_2} := \big(\pi/2 - \delta_2, \pi/2\big) \times \mathbb{R}$ satisfying

$$\sup_{T_{\delta_2}}|\varphi| < \varepsilon, \quad \sup_{T_{\delta_2}}|D\varphi| < \varepsilon \quad \text{and} \quad \sup_{T_{\delta_2}}|D^2\varphi| < \varepsilon.$$

Because for any fixed x_2 we have

$$\lim_{x_1 \to \pi/2^-} \varphi = \lim_{x_1 \to \pi/2^-} |D\varphi| = 0,$$

we get

$$\begin{aligned}
|\varphi_{x_2}(x_1,x_2)| &= \Big| - \int_{x_1}^{\frac{\pi}{2}} \varphi_{x_2 x_1}(x_1,x_2)dx_1 \Big| \le \big(\pi/2 - x_1\big) \Big|\sup_{T_{\delta_2}} \varphi_{x_1 x_2}\Big|\\
&\le \big(\pi/2 - x_1\big)\varepsilon.
\end{aligned}$$

Hence, for any $i \ge i_2$, from equation (3.4.14) we see that $\sup_{\Lambda_{1i}}|h| < \varepsilon$. Because of the symmetry we immediately get that $\sup_{\Lambda_{2i}}|h| < \varepsilon$. On the other hand, recall that the strips R_i and L_i are getting C^1-close to the corresponding grim reaper cylinders. Hence, there exists an index $i_3 \geq i_2$ such that for $i \geq i_3$ we can represent

$$R_i \cap \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \leq i_3\right\}$$

as the graph over a grim reaper cylinder of a smooth function φ_i defined in a slab of the form $G_{\delta_3 i} := (-\pi/2 + \delta_3, \pi/2 - \delta_3) \times (m_{1i}, m_{2i})$, where here δ_3 depends only on i_3, satisfying the properties

$$\sup_{G_{\delta_3 i}} |\varphi_i| < \varepsilon \quad \text{and} \quad \sup_{G_{\delta_3 i}} |D\varphi_i| < \varepsilon.$$

Exactly the same estimate can be obtained along the strips L_i. Note that in this case the x_1-coordinate is not tending to $\pm\pi/2$ and so $\cos x_1$ is bounded from below by a positive number. Going now back to equation (3.4.14) we obtain that for $i \geq i_3$ we have

$$\sup_{\Lambda_{4i}} |h| < \varepsilon \quad \text{and} \quad \sup_{\Lambda_{3i}} |h| < \varepsilon.$$

Therefore $h|_{\partial\Lambda_i}$ becomes arbitrary small as i tends to infinity. This completes the proof of the claim.

Finally observe that, from Claim 10, there exists an interior point where h attains a local maximum or a local minimum. From the strong maximum principle of Hopf we deduce that h must be identically zero. Consequently, $\nu_2 = 0$ and thus $e_2 = (0, 1, 0)$ is a tangent vector of M. Now differentiating the equation $\nu_2 = 0$ with respect to the tangent vector e_2, we deduce that $A(e_2, e_2) = 0$:

$$0 = e_2(0) = e_2(\nu_2) = e_2(\langle \nu, e_2 \rangle) = \langle D_{e_2} \nu, e_2 \rangle = \langle d\nu(e_2), e_2 \rangle = A(e_2, e_2).$$

Thus, $\det A = 0$, hence $K = 0$ and so $|A|^2 = |A|^2 + 2K = H^2$. Since $|A|^2 = H^2$ we deduce from [MSHS15, Theorem B] that M must be a grim reaper cylinder and the proof is complete.

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