Higher categorical structures in Algebraic Topology: Classifying spaces and homotopy coherence

TESIS DOCTORAL

Por

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Programa de doctorado en Física y Matemáticas (FisyMat)

Universidad de Granada

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26 de febrero de 2015

Editorial: Universidad de Granada. Tesis Doctorales
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ISBN: 978-84-9125-141-5
URI: http://hdl.handle.net/10481/40300

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Prólogo

Esta memoria de tesis doctoral es presentada por D. Benjamín Alarcón Heredia para optar al título de Doctor en Matemáticas por la Universidad de Granada, dentro del programa oficial de Doctorado en Física y Matemáticas (FisyMat). Se realiza por tanto de acuerdo con las normas que regulan las enseñanzas oficiales de Doctorado y del Título de Doctor en la Universidad de Granada, aprobadas por Consejo de Gobierno de la Universidad en su sesión de 2 de Mayo de 2012, donde se especifica que *"la tesis doctoral consistirá en un trabajo original de investigación elaborado por el candidato en cualquier campo del conocimiento que se enmarcará en alguna de las líneas del programa de doctorado en el que está matriculado. Para garantizar, con anterioridad a su presentación formal, la calidad del trabajo desarrollado se aportará, al menos, una publicación aceptada o publicada en un medio de impacto en el ámbito de conocimiento de la tesis doctoral firmada por el doctorando, que incluya parte de los resultados de la tesis. La tesis podrá ser desarrollada y, en su caso, defendida, en los idiomas habituales para la comunicación científica en su campo de conocimiento. Si la redacción de la tesis se realiza en otro idioma, deberá incluir un resumen en español."*.

La presente memoria ha sido redactada en base a cuatro artículos de investigación, todos ellos publicados entre los años 2012-2015 [38, 45–47], que se han seleccionado tenido en cuenta sobre todo su coherencia temática, pero también su extensión en orden a que la tesis tenga un tamaño razonable. Todas estas publicaciones han aparecido en revistas de relevancia internacional en los ámbitos de la Teoría de Categorías, la Topología Algebraica y la Teoría de Homotopía y estructuras algebraicas asociadas, referenciadas todas ellas en el Journal of Citations Reports e incluidas en las bases de datos MathSciNet (American Mathematical Society) y Zentralblatt für Mathematik (European Mathematical Society).

Para optar a la mención internacional en el título de doctor, la mayor parte de la memoria está escrita en inglés, idioma que actualmente es de mayoritario uso en la comunicación científica en el ámbito de las matemáticas, respetando así el idioma en que los artículos de investigación recopilados han sido publicados. Al redactarse en una lengua no oficial, sin embargo, incluimos un resumen también en español.

Los resultados novedosos presentados en la memoria han sido obtenidos a lo largo de los últimos años bajo la supervisión del Dr. Antonio Martínez Cegarra en el Departamento de Álgebra de la Universidad de Granada. En este tiempo, el doctorando ha sido alumno del Programa Oficial de Doctorado en Física y Matemáticas (FisyMat); desde Septiembre de 2011 ha disfrutado de una Beca de Formación de Profesorado Universitario (FPU: AP2010-3521), financiada por el Ministerio de Educación, Cultura y Deportes español, y ha realizado sus investigaciones en el marco del Grupo de Investigación FQM-1668, financiado por la Junta de Andalucía, y del Proyecto de Investigación MTM2011-22554, financiado por la Dirección General de Investigación del gobierno de España. Durante los meses de Julio, Agosto y Septiembre de 2013, el doctorando realizó una estancia de investigación en el Centre of Australian Category Theory, en la Macquarie University (Sídney, Australia), y durante los meses de Septiembre, Octubre y Noviembre de 2014, realizó una otra estancia en la School of Mathematics and Statistics, University of Sheffield (Reino Unido).

Declaración del doctorando

Benjamín Alarcón Heredia,

CERTIFICA:

Que la tesis titulada *Higher categorical structures in Algebraic Topology: Classifying spaces and homotopy coherence*, presentada para optar al Grado de Doctor en Matemáticas, ha sido realizada por él mismo, bajo la supervisión del Dr. Antonio Martínez Cegarra, en el Departamento de Álgebra de la Universidad de Granada.

Granada, 26 de febrero de 2015

Benjamín Alarcón Heredia

Declaración del director

Antonio Martínez Cegarra, doctor en Matemáticas y catedrático de Álgebra de la Universidad de Granada

CERTIFICA:

Que la tesis titulada *Higher categorical structures in Algebraic Topology: Classifying spaces and homotopy coherence*, presentada por Benjamín Alarcón Heredia para optar al Grado de Doctor en Matemáticas, ha sido realizada bajo su supervisión, en el Departamento de Álgebra de la Universidad de Granada.

Granada, 26 de febrero de 2015

Antonio Martínez Cegarra

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El doctorando D. Benjamín Alarcón Heredia y el director de la tesis D. Antonio Martínez Cegarra, garantizan que, hasta donde su conocimiento alcanza, en la realización de la presente tesis doctoral se han respetado los derechos de otros autores a ser citados, cuando se han utilizado sus resultados o publicaciones.

Granada, 26 de febrero de 2015

Doctorando	Director de la tesis
Benjamín Alarcón Heredia	Antonio Martínez Cegarra

Agradecimientos

En primer lugar, me gustaría agradecer a mi director Antonio Martínez Cegarra, por ser un gran profesor, por introducirme en el tema en el que he estado trabajando todos estos años y por su apoyo constante en cada momento del camino.

A mi compañera, colaboradora y amiga María Calvo Cervera, por haber compartido conmigo dudas, alegrías, demostraciones y problemas aún pendientes. Por ayudarme a resolver cuestiones atascadas, por plantearme problemas interesantes y por aguantarme cuando hablaba sin fin sobre mi tesis.

A Esperanza López Centella, por estar de becaria conmigo desde el principio, compartiendo despacho, burocracia y todas esas partes de una tesis que son más duras en solitario.

A Josué Remedios, con el que he trabajado en dos artículos de los cuatro que componen la tesis.

Thanks to Nick Gurski, for receiving me and tutoring me during my visit to Sheffield, and for all the interesting conversations we had there. Thanks also to Ross Street and Richard Garner for helping me during my visit to Sydney. Special thanks also to George Maltsiniotis, for inviting me to give a talk in Paris, and giving me good advice every time I have seen him.

During both visits, and in all the category theory conferences I have attended, I have met a lot of fascinating people and had many awesome conversations about math and more. I wish to thank all of them but specially to Lukas Buhné, Mitchell Buckley, Jonathan Chiche, John Bourke, Eduardo Pareja Tobes, Alex Corner, Rémy Tuyéras, Jonathan Elliott, Matthew Burke, Daniel Schäppi, Isar Goyvaerts, Guilherme Frederico Lima and Christina Vasilakopoulou.

Thanks to all the anonymous referees that improved through their comments the papers that are part of this thesis.

Al resto de estudiantes de doctorado de la Universidad de Granada, por los buenos momentos durante el café: Mario, Stefano, Lourdes y Rafa. También a los miembros del Departamento de Álgebra, por hacer que los becarios nos sintamos acogidos y a gusto.

Fuera del mundo de las matemáticas tengo que agradecer a mis padres Juan y Josefina, por trasmitirme su pasión por aprender. Y sobre todo por regalarme tantísimos libros sobre matemáticas que me engancharon y ya no me dejaron escapar.

A mis ‘padres granadinos’ Adela y Rodolfo, por cuidarme y acogerme siempre que lo he necesitado, y también cuando no lo he necesitado. Por hacer de vuestra casa un sitio de reunión para todos al que siempre estamos deseando de volver.

A mi hermana Aixa, por ir siempre delante mía, abriendo el camino, enseñándome hasta dónde podía llegar.

A mis amigos, que me han soportado durante tantos años, Laura, Paco, Patricio, Guillermo, David, Magda, Vicky, Josillo, Carlos, Omar, Rocío, Dani, Jesús, Joe, Chris y todos los que me dejo. Muy especialmente tengo que agradecer a Óscar, por compartir conmigo toda una vida juntos, y a Inma, porque no se puede tener una amiga mejor.

A Simon, por haber mantenido nuestra amistad a pesar de la distancia, y por todas las partidas de dominó y go que hemos tenido.

Finalmente, a Mari por acompañarme durante estos últimos años. Por su cariño y ternura, y por todos los buenos momentos que hemos pasado juntos y que nos quedan por pasar.

A todos, muchas gracias.

Benjamín

Abstract

After the seminal paper by Quillen [109] in 1973, the homotopy theory of categorical structures has become a relevant part of the machinery of algebraic topology and algebraic K-theory, and this work contributes to clarifying several relationships between certain higher categories, the homotopy types of their classifying spaces (or geometric realizations) and some classical homotopical constructions applied to these homotopy types.

Higher categorical structures provide a powerful tool for the study of several areas of mathematics. See the recent book *Toward Higher Categories* [8], which provides a useful background for this subject. They also find applications in areas such as theoretical physics and computer science, since they appear in the study of topological quantum field theories (see for example [6]), or more recently, they serve as a starting point for the Univalent Foundations program in the study of homotopy type theory [122].

This thesis consists of four main chapters presenting the results obtained, and a conclusion chapter in Spanish. All chapters can be read independently, although most of the terminology and some technical arguments are shared between them. Apart from a few minor notational changes that have been made to unify our presentation, and that the full bibliography has been collected at the end of the thesis, Chapter 1 has appeared as [46] in the journal *Applied Categorical Structures* (2012), Chapter 2 as [45] in *Algebraic and Geometric Topology* (2014), Chapter 3 as [38] in *Journal of Homotopy and Related Structures* (2014), and Chapter 4 as [47] in *Theory and Applications of Categories* (2015).

In Chapter 1, we deal with certain double categories which model homotopy 2-types. A small *double category* (defined by Ehresmann around 1963 [62, 63]) can be interpreted as a set of 'squares', whose vertices are objects and whose edges are two different kinds of morphisms –vertical and horizontal–, with two category compositions –the vertical and horizontal ones–, together with compatible category compositions of the morphisms, obeying several conditions. Any (small) double category admits a double nerve construction, which in a very standard way can be made into a simplicial set, and therefore into a topological space, its *geometric realization* or *classifying space.* The simplicial set obtained this way is not a Kan complex, and to some extent it is hard to work with it. Nevertheless, a necessary and sufficient condition on a double category to obtain a Kan complex through its double nerve is actually very simple to formulate: it must be a *double groupoid* satisfying the *filling condition*. This condition says that any pair of arrows, one vertical and the other horizontal, with a common vertex occurs in the boundary of a square of the double groupoid. This fact can be seen as a higher version of the well-known fact that the nerve of a category is a Kan complex if and only if the category is a groupoid.

Any such double groupoid with filling condition characteristically has associated to it *homotopy groups*, which are defined using only its algebraic structure. Thus arises the notion of *weak equivalence* between such double groupoids, and a corresponding *homotopy category* is defined. A main result here states that the geometric realization functor induces an equivalence between the homotopy category of double groupoids with filling condition and the category of homotopy 2-types (that is, the homotopy category of all topological spaces with the property that the n^{th} homotopy group at any base point vanishes for $n \geq 3$). A quasi-inverse functor is explicitly given by means of a new *homotopy double groupoid* construction for topological spaces.

Chapter 2 deals with small *tricategories*, as introduced by Gordon, Power, and Street in their 1995 AMS Memoir paper [69]. They were aware that strict 3-groupoids do not model homotopy 3-types, and thus the aim of their work was to create an explicit definition of a weak 3-category, which would not be equivalent (in the appropriate three-dimensional sense) to that of a strict 3-category. Our results here contribute to the study of classifying spaces for (small) tricategories, with applications to the homotopy theory of monoidal categories, bicategories, braided monoidal categories, and monoidal bicategories. Any tricategory has associated various simplicial or pseudo-simplicial objects, and we explore the relationship between three of them: the pseudo-simplicial bicategory so-called *Grothendieck nerve* of the tricategory, the simplicial bicategory termed its *Segal nerve*, and the simplicial set called its *Street geometric nerve*. We prove the fact that the geometric realizations of all of these 'nerves of the tricategory' are homotopy equivalent. Any one of these realizations could therefore be taken as the classifying space of the tricategory. Nevertheless, each of them clarifies different aspects of the theory. These nerves have been used recently by Buckley, Garner, Lack and Street in their work on skew-monoidal categories [34].

The Grothendieck nerve construction serves as a generalization for the triple nerve of a strict 3-category. The Segal nerve allows us to prove that, under natural requirements, the classifying space of a monoidal bicategory is, in a precise way, a loop space. With the use of Street geometric nerves, we obtain simplicial sets whose simplices have a pleasing geometrical description in terms of the cells of the tricategory, and we can make precise the form in which the classifying space construction transports tricategorical coherence to homotopy coherence. We also prove that, via this construction, bicategorical groups are a convenient algebraic model for path-connected homotopy 3-types, that is, spaces whose n^{th} homotopy groups vanish for $n \geq 4$.

In the next Chapters 3 and 4, we change from modeling homotopy types to use our algebraic models in some homotopy constructions usually applied to them. More precisely, we are interested in homotopy pullbacks. General limits and colimits associated with a diagram, which include the notion of pullback, are a powerful tool in category

theory, with uncountable applications. Sadly, they behave poorly in homotopy theory. That is, if we replace our original diagram by a weak homotopy equivalent diagram, the corresponding limits, or colimits, are not necessarily weak homotopy equivalent. That is why homotopy limits and colimits are studied.

Theorems A and B (due to Quillen [109]) are the starting point for Quillen's homotopy-theoretic description of higher algebraic K-theory, and they are now two of the most important theorems in the foundations of homotopy theory. Chapter 3 of the thesis focuses on generalizations of these theorems to lax functors between bicategories (defined by Bénabou around 1967 [15]), which cover both (lax) monoidal functors between monoidal categories and (lax) functors between 2-categories. In our theorems, we use a construction of *homotopy-fiber bicategories* of a lax functor between bicategories, a naive bicategorical emulation of the topological construction of homotopy fibers of continuous maps, which is, however, subtle. In fact, we prove that, under natural necessary conditions, the classifying space of each homotopy-fiber bicategory is actually homotopy equivalent to the homotopy-fiber of the induced continuous map on classifying spaces by the lax functor (Theorem B). In particular, when all homotopy-fiber bicategories have contractible classifying spaces, the continuous map induced by the lax functor is a homotopy equivalence (Theorem A).

We should stress that the process of taking homotopy-fiber bicategories of lax functors is more complicated than for categories and ordinary functors, since we are forced to deal with *lax bidiagrams of bicategories* with the shape of a given bicategory, which are a type of trihomomorphism from the shape bicategory into the tricategory of bicategories. A higher *Grothendieck construction* on such bidiagrams leads us to state and prove a higher version of Quillen's Homotopy Lemma [109] that, as it happens for the ordinary case of functors between categories, is a key result here.

Going further, in Chapter 4 we deal with general homotopy pullbacks. For any cospan of bicategories in which one leg is a lax functor, and the other one an oplax functor, we construct a *homotopy-fiber product bicategory* and prove that, under reasonable necessary conditions, the classifying space of this homotopy-fiber product bicategory is naturally homotopy equivalent to the ordinary homotopy-fiber product space of the induced cospan of spaces obtained by taking classifying spaces. Our main theorem here generalizes the bicategorical Quillen's Theorem B of Chapter 3, and extends recent similar results for spans of categories by Cisinski (2006) [55] and Barwick and Kan (2011) [13]. We should mention that the category of (strict) 2-categories and 2-functors has a Thomason model structure, as was first announced by Worytkiewicz, Hess, Parent and Tonks (2007) [125] and fully proved by Ara and Maltsiniotis (2014) [2], such that the classifying space functor induces an equivalence of homotopy theories between 2-categories and topological spaces. Hence, our results when restricted to 2-categories find a natural interpretation in terms of homotopy pullbacks relative to its Thomason model structuture. Similarly, thanks to the equivalence between the category of crossed modules (over groupoids) and the category of 2-groupoids, they find an application in this setting in terms of the model structure of crossed complexes by Brown and Golasinski (1989) [26]. Also, since any monoidal category can be

regarded as a bicategory with only one 0-cell, our results are applicable to monoidal categories and we dedicate a part of the chapter to do this.

Chapter 4 also includes some new results concerning classifying spaces of bicategories, which are needed here to obtain the aforementioned results about homotopy-fiber products. The development of the work above is a great example of how useful it is to establish the relationship between the different nerves of bicategories, in order to be able to work both with lax and oplax functors at the same time. The Street geometric nerve of a bicategory is usually the simplest one to work with, but it is only functorial with respect to lax functors. There is however an op-geometric nerve, which is functorial with respect to oplax functors. Then, in the Appendix of Chapter 4, we complete the work by Carrasco, Cegarra and Garzón [41] by proving new naturality results for the comparisons of nerves of bicategories. The results here are obtained following ideas used in Chapter 2 for the study of nerves of tricategories.

Chapter 1

Double groupoids and homotopy 2-types

1.1 Introduction and summary.

Higher-dimensional categories provide a suitable setting for the treatment of an extensive list of subjects of recognized mathematical interest. The construction of nerves and classifying spaces of higher categorical structures discovers ways to transport categorical coherence to homotopic coherence, and it has shown its relevance as a tool in algebraic topology, algebraic geometry, algebraic K-theory, string field theory, conformal field theory, and in the study of geometric structures on low-dimensional manifolds.

Double groupoids, that is, groupoid objects in the category of groupoids, were introduced by Ehresmann [62, 63]) in the late fifties and later studied by several people because of their connection with several areas of mathematics. Roughly, a double groupoid consists of *objects*, *horizontal* and *vertical morphisms*, and *squares.* Each square, say α, has objects as vertices and morphisms as edges, as in

$$\begin{tikzcd} \cdot & \cdot \arrow[l] \\ \cdot \arrow[u] \arrow[ur, phantom, "\alpha"] & \cdot , \arrow[l] \arrow[u] \end{tikzcd}$$

together with two groupoid compositions -the *vertical* and *horizontal compositions*- of squares, and compatible groupoid compositions of the edges, obeying several conditions (see Section 1.3 for details). Any double groupoid $\mathcal{G}$ has a *geometric realization* (or *classifying space*) $\mathrm{B}\mathcal{G}$, which is the topological space defined by first taking the double nerve $\mathrm{N}^{(2}\mathcal{G}$, which is a bisimplicial set, and then realizing the diagonal to obtain a space: $\mathrm{B}\mathcal{G} = |\mathrm{diag}\mathrm{N}^{(2}\mathcal{G}|$. In this chapter, we address the homotopy types obtained in this way from double groupoids satisfying a natural *filling condition*: Any filling problem

$$\begin{tikzcd}
\cdot & \cdot \arrow[l] \\
{} \arrow[u, dotted] \arrow[ur, phantom, "\exists ?"] & \cdot \arrow[u] \arrow[l, dotted]
\end{tikzcd}$$

finds a solution in the double groupoid. This filling condition on double groupoids is often assumed in the case of double groupoids arising in different areas of mathematics, such as in differential geometry or in weak Hopf algebra theory (see the papers by Mackenzie [102] and Andruskiewitsch and Natale [1], for example), and it is satisfied for those double groupoids that have emerged with an interest in algebraic topology, mainly thanks to the work of Brown, Higgins, Spencer, *et al.*, where the connection of double groupoids with crossed modules and a higher Seifert-van Kampen Theory has been established (see, for instance, the survey paper [24] and references therein). Thus, the filling condition is easily proven for edge symmetric double groupoids (also called special double groupoids) with connections (see, for example [27, 31] or [25, 30, 32], for more recent instances), for double groupoid objects in the category of groups (also termed cat^2-groups, [37, 98, 107]), or, for example, for 2-groupoids (regarded as double groupoids where one of the side groupoids of morphisms is discrete [106],[79]).

When a double groupoid $\mathcal{G}$ satisfies the filling condition, then there are characteristically associated to it 'homotopy groups', $\pi_i(\mathcal{G}, a)$, which we define using only the algebraic structure of $\mathcal{G}$, and which are trivial for integers $i \geq 3$. A first major result states that:

> *If $\mathcal{G}$ is a double groupoid with filling condition, then, for each object a, there are natural isomorphisms $\pi_i(\mathcal{G}, a) \cong \pi_i(\mathrm{B}\mathcal{G}, \mathrm{B}a)$, $i \geq 0$.*

The proof of this result requires a prior recognition of the significance of the filling condition on double groupoids in the homotopy theory of simplicial sets; namely, we prove that

> *A double category $\mathcal{C}$ is a double groupoid with filling condition if and only if the simplicial set diagonal of its double nerve,* $\mathrm{diag}\mathrm{N}^{(2}\mathcal{C}$*, is a Kan complex.*

This fact can be seen as a higher version of the well-known fact that the nerve of a category is a Kan complex if and only if the category is a groupoid (see [85], for example).

Once we have defined the homotopy category of double groupoids satisfying the filling condition $\mathrm{Ho}(\mathbf{DG}_{\mathrm{fc}})$, to be the localization of the category of these double groupoids, with respect to the class of *weak equivalences* or double functors $F : \mathcal{G} \to \mathcal{G}'$ inducing isomorphisms $\pi_i F : \pi_i(\mathcal{G}, a) \cong \pi_i(\mathcal{G}', Fa)$ on the homotopy groups, we then obtain an induced functor

$$\mathrm{B} : \mathrm{Ho}(\mathbf{DG}_{\mathrm{fc}}) \to \mathrm{Ho}(\mathbf{Top}), \quad \mathcal{G} \mapsto \mathrm{B}\mathcal{G}\,,$$

where Ho(**Top**) is the localization of the category of topological spaces with respect to the class of weak equivalences. Furthermore, we show a new functorial construction of a *homotopy double groupoid* $\Pi^{(2}X$, for any topological space X, that induces a functor

$$\mathrm{Ho}(\mathbf{Top}) \to \mathrm{Ho}(\mathbf{DG}_{\mathrm{fc}}), \quad X \mapsto \Pi^{(2}X.$$

A main goal in this chapter is to prove the following result, whose proof is somewhat indirect since it is given through an explicit description of a left adjoint functor, $\mathrm{P}^{(2} \dashv \mathrm{N}^{(2}$, to the double nerve functor $\mathcal{G} \mapsto \mathrm{N}^{(2}\mathcal{G}$:

> *The functors $\mathcal{G} \mapsto \mathrm{B}\mathcal{G}$ and $X \mapsto \Pi^{(2}X$ induce mutually quasi-inverse equivalences*
>
> $$\mathrm{Ho}(\mathbf{DG}_{\mathrm{fc}}) \simeq \mathrm{Ho}(\mathbf{2\text{-}types}),$$

where Ho(**2-types**) is the full subcategory of the homotopy category of topological spaces given by those spaces X with $\pi_i(X, a) = 0$ for any integer $i > 2$ and any base point a. From the point of view of this fact, the use of double groupoids and their classifying spaces in homotopy theory goes back to Whitehead [124] and Mac Lane-Whitehead [101] since double groupoids where one of the side groupoids of morphisms is discrete with only one object (= strict 2-groups, in the terminology of Baez [7]) are the same as crossed modules (this observation is attributed to Verdier in [31]). In this context, we should mention the work by Brown-Higgins [27] and Moerdijk-Svensson [106] since crossed modules over groupoids are essentially the same thing as 2-groupoids and double groupoids where one of the side groupoids of morphisms is discrete. Along the same line, our result is also a natural 2-dimensional version of the well-known equivalence between the homotopy category of groupoids and the homotopy category of 1-types (for a useful survey of groupoids in topology, see [23]).

The plan of this chapter is, briefly, as follows. After this introductory Section 1.1, the chapter is organized in six sections. Section 1.2 aims to make this chapter as self-contained as possible; hence, at the same time as fixing notations and terminology, we also review necessary aspects and results from the background of (bi)simplicial sets and their geometric realizations that will be used throughout the thesis. However, the material in Section 1.2 is quite standard, so the expert reader may skip most of it. The most original part is in Subsection 1.2.2, related to the extension condition on bisimplicial sets. In Section 1.3, after recalling the notion of a double groupoid and fixing notations, we mainly introduce the homotopy groups $\pi_i(\mathcal{G}, a)$, at any object a of a double groupoid with filling condition $\mathcal{G}$. Section 1.4 is dedicated to showing in detail the construction of the homotopy double groupoid $\Pi^{(2}X$, characteristically associated to any topological space X. Here, we prove that a continuous map $X \to Y$ is a weak homotopy 2-equivalence (i.e., it induces bijections on the homotopy groups π_i for $i \leq 2$) if and only if the induced double functor $\Pi^{(2}X \to \Pi^{(2}Y$ is a weak equivalence. Next, in Section 1.5, we give a manageable description for the bisimplices in $\mathrm{N}^{(2}\mathcal{G}$, the double nerve of a double groupoid, and then we determine the homotopy type of the geometric realization $\mathrm{B}\mathcal{G}$ of a double groupoid with filling condition. Specifically, we prove that the homotopy groups of $\mathrm{B}\mathcal{G}$ are the same as those of $\mathcal{G}$. Our goal in Section 1.6 is to prove that the double nerve functor, $\mathcal{G} \mapsto \mathrm{N}^{(2}\mathcal{G}$, embeds, as a reflexive subcategory, the category of double groupoids satisfying the filling condition into a certain category of bisimplicial sets. The reflector functor $K \mapsto \mathrm{P}^{(2}K$ works as a bisimplicial version of Brown's construction in [25, Theorem 2.1]. Furthermore, as we will prove, the resulting double groupoid $\mathrm{P}^{(2}K$ always represents the homotopy 2-type of the input bisimplicial set K, in the sense that there is a natural weak 2-equivalence $|K| \to \mathrm{BP}^{(2}K$. This result becomes crucial in the final Section 1.7 where, bringing into play all the previous work, the equivalence of categories $\mathrm{Ho}(\mathbf{DG}_{\mathrm{fc}}) \simeq \mathrm{Ho}(\mathbf{2\text{-}types})$ is achieved.

1.2 Some preliminaries on bisimplicial sets.

This section aims to make this chapter as self-contained as possible; therefore, while fixing notations and terminology, we also review necessary aspects and results from the background of (bi)simplicial sets and their geometric realizations used throughout the thesis. However, the material in this section is quite standard and, in general, we employ the standard symbolism and nomenclature to be found in texts on simplicial homotopy theory, mainly in [68] and [104], so the expert reader may skip most of it. The most original part is in Subsection 1.2.2, related to the extension condition and the bihomotopy relation on bisimplicial sets.

1.2.1 Kan complexes: Fundamental groupoids and homotopy groups.

We start by fixing some notations. In the simplicial category[1] Δ, the generating coface and codegeneracy maps are denoted by $d^i\colon [n-1] \to [n]$ and $s^i\colon [n+1] \to [n]$ respectively. However, for $L : \Delta^{\mathrm{op}} \to \mathbf{Set}$ any simplicial set, we write $d_i = L(d^i) : L_n \to L_{n-1}$ and $s_i = L(s^i) : L_n \to L_{n+1}$ for its corresponding face and degeneracy maps.

The *standard n-simplex* is $\Delta[n] = \Delta(-, [n])$ and, as is usual, we identify any simplicial map $x : \Delta[n] \to L$ with the simplex $x(1_{[n]}) \in L_n$, the image by x of the basic simplex $1_{[n]} = id : [n] \to [n]$ of $\Delta[n]$. Thus, for example, the i^{th}-*face* of $\Delta[n]$ is $d^i = \Delta(-, d^i) : \Delta[n-1] \to \Delta[n]$, the simplicial map with $d^i(1_{[n-1]}) = d_i(1_{[n]})$. Similarly, $s^i = \Delta(-, s^i) : \Delta[n+1] \to \Delta[n]$ is the simplicial map that we identify with the degenerated simplex $s_i(1_{[n]})$ of $\Delta[n]$.

The *boundary* $\partial\Delta[n] \subset \Delta[n]$ is the smallest simplicial subset containing all the faces $d^i : \Delta[n-1] \to \Delta[n]$, $0 \leq i \leq n$, of $\Delta[n]$. Similarly, for any given k with

---

[1] Throughout this chapter we use the convention that there is a morphism $j \to i$ for every $i \leq j$ in $[n]$. This convention is different in the others chapter of the thesis and it only amounts to changing certain subindices. We decided to maintain it as in the original published paper to avoid introducing potential errors in the formulas.

$0 \leq k \leq n$, the k^{th}-*horn*, $\Lambda^k[n] \subset \Delta[n]$, is the smallest simplicial subset containing all the faces $d^i : \Delta[n-1] \to \Delta[n]$ for $0 \leq i \leq n$ and $i \neq k$. For a more geometric (and useful) description of these simplicial sets, recall that there are coequalizers

$$\coprod_{0 \leq i < j \leq n} \Delta[n-2] \rightrightarrows \coprod_{0 \leq i \leq n} \Delta[n-1] \to \partial\Delta[n], \tag{1.1}$$

and

$$\coprod_{\substack{0 \leq i < j \leq n \\ i \neq k \neq j}} \Delta[n-2] \rightrightarrows \coprod_{\substack{0 \leq i \leq n \\ i \neq k}} \Delta[n-1] \to \Lambda^k[n], \tag{1.2}$$

given by the relations $d^j d^i = d^i d^{j-1}$ if $i < j$.

A simplicial set L is a *Kan complex* if it satisfies the so-called *extension condition.* Namely, for any simplicial diagram

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & L \\
\Delta[n] \arrow[ur, dotted] &
\end{tikzcd}$$

there is a map $\Delta[n] \to L$ (the dotted arrow) making the diagram commute.

In a Kan complex L, two simplices $x, x' : \Delta[n] \to L$ are said to be *homotopic* whenever they have the same faces and there is a *homotopy* from x to x', that is, a simplex $y \colon \Delta[n+1] \to L$ making this diagram commutative

$$\begin{tikzcd}
\partial\Delta[n+1] \arrow[rr, "{(xd^0 s^{n-1}, \cdots, xd^{n-1} s^{n-1}, x, x')}"] \arrow[d, hook] & & L. \\
\Delta[n+1] \arrow[urr, "y"'] & &
\end{tikzcd}$$

Being homotopic establishes an equivalence relation on the simplices of L, and we write

$$[x] \tag{1.3}$$

for the homotopy class of a simplex x. A useful result is the following:

Fact 1.1 *Let* $y, y' \colon \Delta[n+1] \to L$ *be two simplices such that* $[yd^i] = [y'd^i]$ *for all* $i \neq k$*; then* $[yd^k] = [y'd^k]$.

The *fundamental groupoid* of L, denoted by

$$\mathrm{P}L, \tag{1.4}$$

also called its Poincaré groupoid, has objects the vertices $a \colon \Delta[0] \to L$, and a morphism $[x] \colon a \to b$ is the homotopy class of a simplex $x \colon \Delta[1] \to L$ with $xd^0 = a$ and $xd^1 = b$. The composition in $\mathrm{P}L$ is defined by

$$[x] \circ [x'] = [yd^1],$$

where $y : \Delta[2] \to L$ is any simplex with $yd^2 = x$ and $yd^0 = x'$, and the identities are $1_a = [as^0]$.

The set of *path components* of L, denoted by $\pi_0 L$, is the set of connected components of $\mathrm{P}L$, so it consists of all homotopy classes of the 0-simplices of L. For any given vertex of the Kan complex $a\colon\Delta[0] \to L$, $\pi_0(L, a)$ is the set $\pi_0 L$, pointed by $[a]$, the component of a. The group of automorphisms of a in the fundamental groupoid of L is $\pi_1(L, a)$, the *fundamental group* of L at a. Furthermore, denoting every composite map $\Delta[m] \to \Delta[0] \overset{a}{\to} L$ by a as well, the n^{th} *homotopy group* $\pi_n(L, a)$ of L at a consists of homotopy classes of simplices $x\colon\Delta[n] \to L$ for all simplices x with faces $xd^i = a$, for $0 \leq i \leq n$. The multiplication in the (abelian, for $n \geq 2$) group $\pi_n(L, a)$ is given by

$$[x] \circ [x'] = [yd^n],$$

where $y\colon\Delta[n+1] \to L$ is a (any) solution to the extension problem

$$\begin{tikzcd}
\Lambda^n[n+1] \arrow[r, "{(a,\cdots,a,x',-,x)}"] \arrow[d, hook] & L. \\
\Delta[n+1] \arrow[ur, dotted, "y"'] &
\end{tikzcd}$$

The following fact is used several times throughout the chapter:

Fact 1.2 *Let L be a Kan complex and n an integer such that the homotopy groups $\pi_n(L, a)$ vanish for all base vertices a. Then, every extension problem*

$$\begin{tikzcd}
\partial\Delta[n+1] \arrow[r] \arrow[d, hook] & L \\
\Delta[n+1] \arrow[ur, dotted, "\exists ?"'] &
\end{tikzcd}$$

has a solution. In particular, any two n-simplices with the same faces are homotopic.

We shall end this preliminary subsection by recalling that two simplicial maps $f, g\colon L \to L'$ are *homotopic* whenever there is a map $L \times \Delta[1] \to L'$ which is f on $L \times 0$ and g on $L \times 1$. The resulting homotopy relation becomes a congruence on the category **KC** of Kan complexes, and the corresponding quotient category is the *homotopy category of Kan complexes*, Ho(**KC**). A map between Kan complexes is a *homotopy equivalence* if it induces an isomorphism in the homotopy category. The following result is known as Whitehead's theorem for Kan complexes:

Fact 1.3 *A simplicial map between Kan complexes, $L \to L'$, is a homotopy equivalence if and only if it induces an isomorphism $\pi_i(L, a) \cong \pi_i(L', fa)$ for all base vertex a of L and any integer $i \geq 0$.*

1.2.2 Bisimplicial sets: The extension condition and the bihomotopy relation.

It is often convenient to view a bisimplicial set $K\colon \Delta^{\rm op}\times\Delta^{\rm op}\to \mathbf{Set}$ as a (horizontal) simplicial object in the category of (vertical) simplicial sets. For this case, we write $d_i^{\rm h}=K(d^i,1)\colon K_{p,q}\to K_{p-1,q}$ and $s_i^{\rm h}=K(s^i,1) : K_{p,q}\to K_{p+1,q}$ for the horizontal face and degeneracy maps, and, similarly $d_j^{\rm v}=K(1,d^j)$ and $s_j^{\rm v}=K(1,s^j)$ for the vertical ones.

For simplicial sets X and Y, let $X\tilde{\times}Y$ be the bisimplicial set with $(X\tilde{\times}Y)_{p,q}=X_p\times Y_q$. The *standard* (p,q)*-bisimplex* is

$$\Delta[p,q] := \Delta\times\Delta(-,([p],[q])) = \Delta[p]\tilde{\times}\Delta[q],$$

the bisimplicial set represented by the object $([p],[q])$, and usually we identify any bisimplicial map $x:\Delta[p,q]\to K$ with the (p,q)-bisimplex $x(1_{[p]},1_{[q]})\in K$. The functor $([p],[q])\mapsto\Delta[p,q]$ is then a co-bisimplicial bisimplicial set, whose cofaces and codegeneracy operators are denoted by $d_{\rm h}^i$, $d_{\rm v}^j$, and so on, as in the diagram

$$\begin{tikzcd}
\Delta[p-1,q] \arrow[r, shift left, "d^i_{\rm h}=d^i\tilde{\times}1"] & \Delta[p,q] \arrow[l, shift left, "s^i_{\rm h}=s^i\tilde{\times}1"] \arrow[r, shift right, swap, "s^j_{\rm v}=1\tilde{\times}s^j"] & \Delta[p,q-1] \arrow[l, shift right, swap, "d^j_{\rm v}=1\tilde{\times}d^j"]
\end{tikzcd} .$$

The $(k,l)^{\rm th}$-*horn* $\Lambda^{k,l}[p,q]$, for any integers $0\leq k\leq p$ and $0\leq l\leq q$, is the bisimplicial subset of $\Delta[p,q]$ generated by the horizontal and vertical faces $\Delta[p-1,q]\overset{d^i_{\rm h}}{\hookrightarrow}\Delta[p,q]$ and $\Delta[p,q-1]\overset{d^j_{\rm v}}{\hookrightarrow}\Delta[p,q]$ for all $i\neq k$ and $j\neq l$. There is a natural pushout diagram

$$\begin{tikzcd}
\Lambda^k[p]\tilde{\times}\Lambda^l[q] \arrow[r, hook] \arrow[d, hook] & \Delta[p]\tilde{\times}\Lambda^l[q] \arrow[d, hook] \\
\Lambda^k[p]\tilde{\times}\Delta[q] \arrow[r, hook] & \Lambda^{k,l}[p,q],
\end{tikzcd}$$

which, taking into account the coequalizers (1.2), states that the system of data to define a bisimplicial map $x:\Lambda^{k,l}[p,q]\to K$ consists of a list of bisimplices

$$x=(x_0,\dots,x_{k-1},-,x_{k+1},\dots,x_p;x'_0,\dots,x'_{l-1},-,x'_{l+1},\dots,x'_q),$$

where $x_i:\Delta[p-1,q]\to K$ and $x'_j:\Delta[p,q-1]\to K$, such that the following compatibility conditions hold:

- $x_jd^i_{\rm h}=x_id^{j-1}_{\rm h}$, for all $0\leq i<j\leq p$ with $i\neq k\neq j$,
- $x'_jd^i_{\rm v}=x'_id^{j-1}_{\rm v}$, for all $0\leq i<j\leq q$ with $i\neq l\neq j$,
- $x'_jd^i_{\rm h}=x_id^j_{\rm v}$, for all $0\leq i\leq p$, $0\leq j\leq q$ with $i\neq k$, $j\neq l$.

Definition 1.1 *A bisimplicial set K satisfies the* extension condition *if all simplicial sets $K_{p,*}$ and $K_{*,q}$ are Kan complexes, that is, if any of the extension problems*

$$\begin{tikzcd}
\Delta[p]\widetilde{\times}\Lambda^l[q] \arrow[r] \arrow[d, hook] & K & \Lambda^k[p]\widetilde{\times}\Delta[q] \arrow[r] \arrow[d, hook] & K \\
\Delta[p,q] \arrow[ur, dotted, "\exists ?"'] & & \Delta[p,q] \arrow[ur, dotted, "\exists ?"'] &
\end{tikzcd}$$

has a solution, and, moreover, if there is also a solution for any extension problem of the form

$$\begin{tikzcd}
\Lambda^{k,l}[p,q] \arrow[r] \arrow[d, hook] & K \\
\Delta[p,q]\,. \arrow[ur, dotted, "\exists ?"'] &
\end{tikzcd}$$

When a bisimplicial set K satisfies the extension condition, then every bisimplex $x : \Delta[p,q] \to K$, which can be regarded both as a simplex of the vertical Kan complex $K_{p,*}$ and as a simplex of the horizontal Kan complex $K_{*,q}$, defines both a *vertical homotopy class* and a *horizontal homotopy class*, denoted respectively by

$$[x]_{\mathrm{v}}, \quad [x]_{\mathrm{h}}. \tag{1.5}$$

The following lemma is needed.

Lemma 1.1 *Let $x, x' : \Delta[p,q] \to K$ be bisimplices of bisimplicial set K, which satisfies the extension condition. The following conditions are equivalent:*

i) There exists $y : \Delta[p,q] \to K$ such that $[x]_{\mathrm{h}} = [y]_{\mathrm{h}}$ and $[y]_{\mathrm{v}} = [x']_{\mathrm{v}}$,

ii) There exists $z : \Delta[p,q] \to K$ such that $[x]_{\mathrm{v}} = [z]_{\mathrm{v}}$ and $[z]_{\mathrm{h}} = [x']_{\mathrm{h}}$.

Proof: We only prove that *i*) implies *ii*) since the proof for the other implication is similar. Let $\alpha : \Delta[p{+}1, q] \to K$ be a horizontal homotopy (i.e., a homotopy in the Kan complex $K_{*,q}$) from x to y, and let $\beta : \Delta[p, q{+}1] \to K$ be a vertical homotopy from y to x'. Since K satisfies the extension condition, a bisimplicial map $\Gamma : \Delta[p{+}1, q{+}1] \to K$ can be found such that the diagram below commutes.

$$\begin{tikzcd}
\Lambda^{p,q+1}[p+1,q+1] \arrow[r, "{(\beta d_{\mathrm{h}}^0 s_{\mathrm{h}}^{p-1},\dots,\beta d_{\mathrm{h}}^{p-1} s_{\mathrm{h}}^{p-1},-,\beta;\alpha d_{\mathrm{v}}^0 s_{\mathrm{v}}^{q-1},\dots,\alpha d_{\mathrm{v}}^{q-1} s_{\mathrm{v}}^{q-1},\alpha,-)}"] \arrow[d, hook] & K \\
\Delta[p+1,q+1] \arrow[ur, "\Gamma"'] &
\end{tikzcd}$$

Then, by taking $\alpha' = \Gamma d_{\mathrm{v}}^{q+1} : \Delta[p+1,q] \to K$, $\beta' = \Gamma d_{\mathrm{h}}^{p} : \Delta[p,q+1] \to K$, and $z = \alpha' d_{\mathrm{h}}^{p} = \beta' d_{\mathrm{v}}^{q+1} : \Delta[p,q] \to K$, one sees that α' becomes a horizontal homotopy (i.e., a homotopy in $K_{*,q}$) from z to x' and β' becomes a vertical homotopy from x to z. Therefore, $[x]_{\mathrm{v}} = [z]_{\mathrm{v}}$ and $[z]_{\mathrm{h}} = [x']_{\mathrm{h}}$, as required. □

The two simplices $x, x' : \Delta[p,q] \to K$ in the above Lemma 1.1 are said to be *bihomotopic* if the equivalent conditions i) and ii) hold.

Lemma 1.2 *If K is a bisimplicial set satisfying the extension condition, then 'to be bihomotopic' is an equivalence relation on the bisimplices of bidegree (p,q) of K, for any $p,q \geq 0$.*

Proof: The relation is obviously reflexive, and it is symmetric thanks to Lemma 1.1. For transitivity, suppose $x, x', x'' : \Delta[p,q] \to K$ such that x and x' are bihomotopic as well as x' and x'' are. Then, for some $y, y' : \Delta[p,q] \to K$, we have $[x]_{\mathrm{h}} = [y]_{\mathrm{h}}$, $[y]_{\mathrm{v}} = [x']_{\mathrm{v}}$, $[x']_{\mathrm{h}} = [y']_{\mathrm{h}}$, and $[y']_{\mathrm{v}} = [x'']_{\mathrm{v}}$. Also, again by Lemma 1.1, there is $z : \Delta[p,q] \to K$ such that $[y]_{\mathrm{h}} = [z]_{\mathrm{h}}$ and $[z]_{\mathrm{v}} = [y']_{\mathrm{v}}$. It follows that $[x]_{\mathrm{h}} = [z]_{\mathrm{h}}$ and $[z]_{\mathrm{v}} = [x'']_{\mathrm{v}}$, whence x and x'' are bihomotopic. □

We will write

$$[[x]] \tag{1.6}$$

for the bihomotopic class of a bisimplex $x : \Delta[p,q] \to K$.

Lemma 1.3 *Let K be any bisimplicial set satisfying the extension condition. There are four well-defined mappings such that $[[x]] \mapsto [xd^i_{\mathrm{h}}]_{\mathrm{v}}$, $[[x]] \mapsto [xd^j_{\mathrm{v}}]_{\mathrm{h}}$, $[x]_{\mathrm{h}} \mapsto [[xs^j_{\mathrm{v}}]]$, and $[x]_{\mathrm{v}} \mapsto [[xs^i_{\mathrm{h}}]]$ respectively, for any $x : \Delta[p,q] \to K$, $0 \leq i \leq p$ and $0 \leq j \leq q$.*

Proof: Suppose that $[[x]] = [[x']]$. Then, $[x]_{\mathrm{h}} = [y]_{\mathrm{h}}$ and $[y]_{\mathrm{v}} = [x']_{\mathrm{v}}$, for some $y : \Delta[p,q] \to K$. It follows that $xd^i_{\mathrm{h}} = yd^i_{\mathrm{h}}$ and there is a vertical homotopy, say $z : \Delta[p,q+1] \to K$, from y to x'. As $zd^i_{\mathrm{h}} : \Delta[p-1,q+1] \to K$ is then a vertical homotopy from yd^i_{h} to $x'd^i_{\mathrm{h}}$, we conclude that $[xd^i_{\mathrm{h}}]_{\mathrm{v}} = [x'd^i_{\mathrm{h}}]_{\mathrm{v}}$. The proof that $[xd^i_{\mathrm{v}}]_{\mathrm{h}} = [x'd^i_{\mathrm{v}}]_{\mathrm{h}}$ is similar. For the third mapping, note that any horizontal homotopy $y : \Delta[p+1,q] \to K$ from x to x' yields the horizontal homotopy $ys^j_{\mathrm{v}} : \Delta[p+1,q+1] \to K$ from xs^j_{v} to $x's^j_{\mathrm{v}}$. Therefore, $[xs^j_{\mathrm{v}}]_{\mathrm{h}} = [x's^j_{\mathrm{v}}]_{\mathrm{h}}$, whence $[[xs^j_{\mathrm{v}}]] = [[x's^j_{\mathrm{v}}]]$, as required. Similarly, we see that $[x]_{\mathrm{v}} = [x']_{\mathrm{v}}$ implies $[[xs^i_{\mathrm{h}}]] = [[x's^i_{\mathrm{h}}]]$. □

We shall end this subsection by remarking that any bisimplicial set K, satisfying the extension condition, has associated *horizontal fundamental groupoids* $\mathrm{P}K_{*,q}$, one for each integer $q \geq 0$, whose objects are the bisimplices $x : \Delta[0,q] \to K$ and morphisms $[y]_{\mathrm{h}} : x' \to x$ horizontal homotopy classes of bisimplices $y : \Delta[1,q] \to K$ with $yd^0_{\mathrm{h}} = x'$ and $yd^1_{\mathrm{h}} = x$. The composition in these groupoids $\mathrm{P}K_{*,q}$ is written using the symbol $\circ_{\mathrm{h}}$, so the composite of $[y]_{\mathrm{h}}$ with $[y']_{\mathrm{h}} : x'' \to x'$ is

$$[y]_{\mathrm{h}} \circ_{\mathrm{h}} [y']_{\mathrm{h}} = [\gamma d^1_{\mathrm{h}}]_{\mathrm{h}},$$

where $\gamma : \Delta[2,q] \to K$ is a (any) bisimplex with $\gamma d^2_{\mathrm{h}} = y$ and $\gamma d^0_{\mathrm{h}} = y'$. The identities are denoted by 1^{h}_x, that is, $1^{\mathrm{h}}_x = [xs^0_{\mathrm{h}}]_{\mathrm{h}}$. Similarly, K also has associated *vertical fundamental groupoids* $\mathrm{P}K_{p,*}$, whose morphisms $[z]_{\mathrm{v}} : zd^0_{\mathrm{v}} \to zd^1_{\mathrm{v}}$ are vertical homotopy classes of bisimplices $z : \Delta[p,1] \to K$. For these, we use the symbol $\circ_{\mathrm{v}}$ for denoting the composition and 1^{v} for identities.

1.2.3 Weak homotopy types: Some related constructions.

Let **Top** denote the category of spaces and continuous maps. A map $X \to X'$ in **Top** is a *weak equivalence* if it induces an isomorphism $\pi_i(X,a) \cong \pi_i(X',fa)$ for all base points a of X and $i \geq 0$. The *category of weak homotopy types* is defined as the localization of the category of spaces with respect to the class of weak equivalences [82, 108] and, for any given integer n, the *category of homotopy n-types* is its full subcategory given by those spaces X with $\pi_i(X,a) = 0$ for any integer $i > n$ and any base point a.

There are various constructions on (bi)simplicial sets that traditionally aid in the algebraic study of homotopy n-types. Below is a brief review of the constructions used in this work.

Segal's *geometric realization* functor [111], for simplicial spaces $K\colon \Delta^{\mathrm{op}} \to \mathbf{Top}$, is denoted by $K \mapsto |K|$. Recall that it is defined as the left adjoint to the functor that associates to a space X the simplicial space $[n] \mapsto X^{\Delta_n}$, where

$$\Delta_n = \{(t_0,\dots,t_n)\in\mathbb{R}^{n+1} \,|\, \Sigma\, t_i{=}1,\ 0{\leq}t_i{\leq}1\}$$

denotes the affine simplex having $[n]$ as its set of vertices and X^{Δ_n} is the the function space of continuous maps from Δ_n to X, given the compact-open topology. The underlying simplicial set is the *singular complex* of X, denoted by

$$\mathrm{S}X. \tag{1.7}$$

For instance, by regarding a set as a discrete space, the (Milnor's) geometric realization of a simplicial set $L : \Delta^{\mathrm{op}} \to \mathbf{Set}$ is

$$|L|, \tag{1.8}$$

which is a CW-complex whose n-cells are in one-to-one correspondence with the n-simplices of L which are nondegenerate. The following six facts are well-known:

Facts 1.4 *1. For any space X, $\mathrm{S}X$ is a Kan complex.*

2. For any Kan complex L, there are natural isomorphisms $\pi_i(L,a) \cong \pi_i(|L|,|a|)$, for all base vertices $a : \Delta[0] \to L$ and $n \geq 0$.

3. A simplicial map between Kan complexes $L \to L'$ is a homotopy equivalence if and only if the induced map on realizations $|L| \to |L'|$ is a homotopy equivalence.

4. For any Kan complex L, the unit of the adjunction $L \to S|L|$ is a homotopy equivalence.

5. A continuous map $X \to Y$ is a weak homotopy equivalence if and only if the induced $\mathrm{S}X \to \mathrm{S}Y$ is a homotopy equivalence.

6. For any space X, the counit $|\mathrm{S}X| \to X$ is a weak homotopy equivalence.

When a bisimplicial set $K:\Delta^{\rm op}\times\Delta^{\rm op}\to \mathbf{Set}$ is regarded as a simplicial object in the simplicial set category and one takes geometric realizations, then one obtains a simplicial space $\Delta^{\rm op}\to\mathbf{Top}$, $[p]\mapsto|K_{p,*}|$, whose Segal realization is taken to be $|K|$, the geometric realization of K. As there are natural homeomorphisms [109, Lemma on page 86]

$$|[p]\mapsto|K_{p,*}||\cong|\mathrm{diag}K|\cong|[q]\mapsto|K_{*,q}||,$$

where $\mathrm{diag}K$ is the simplicial set obtained by composing K with the diagonal functor $\Delta\to\Delta\times\Delta$, $[n]\mapsto([n],[n])$, one usually takes

$$|K|=|\mathrm{diag}K|. \tag{1.9}$$

Composing with the ordinal sum functor or : $\Delta\times\Delta\to\Delta$, $([p],[q])\mapsto[p+1+q]$, gives Illusie's *total* Dec functor, $L\mapsto \mathrm{Dec}L$, from simplicial to bisimplicial sets [85, VI, 1.5]. More specifically, for any simplicial set L, $\mathrm{Dec}L$ is the bisimplicial set whose bisimplices of bidegree (p,q) are the $(p+1+q)$-simplices of L, $x:\Delta[p+1+q]\to L$, and whose simplicial operators are given by $xd^i_{\rm h}=xd^i$, $xs^i_{\rm h}=xs^i$, for $0\leq i\leq p$, and $xd^j_{\rm v}=d^{p+1+j}$, $xs^j_{\rm v}=xs^{p+1+j}$, for $0\leq j\leq q$. The functor Dec has a right adjoint [59]

$$\mathrm{Dec}\dashv\overline{W}, \tag{1.10}$$

often called the *codiagonal* functor, whose description is as follows [3, III]: for any bisimplicial set K, an n-simplex of $\overline{W}K$ is a bisimplicial map

$$\bigsqcup_{p=0}^{n}\Delta[p,n-p]\xrightarrow{(x_0,\dots,x_n)}K$$

such that $x_pd^0_{\rm v}=x_{p+1}d^{p+1}_{\rm h}$, for $0\leq p<n$, whose faces and degeneracies are given by

$$\begin{aligned}(x_0,\dots,x_n)d^i&=(x_0d^i_{\rm v},\dots,x_{i-1}d^1_{\rm v},x_{i+1}d^i_{\rm h},\dots,x_nd^i_{\rm h}),\\(x_0,\dots,x_n)s^i&=(x_0s^i_{\rm v},\dots,x_is^0_{\rm v},x_is^i_{\rm h},\dots,x_ns^i_{\rm h})\,.\end{aligned}$$

The unit and the counit of the adjunction, $\mathrm{u}:L\to\overline{W}\,\mathrm{Dec}L$ and $\mathrm{v}:\mathrm{Dec}\overline{W}K\to K$, are respectively defined by

$$\begin{array}{ll}\mathrm{u}(y)=(ys^0,\dots,ys^n) & (y:\Delta[n]\to L)\\ \mathrm{v}(x_0,\dots,x_{p+1+q})=x_{p+1}d^0_{\rm h} & ((x_0,\dots,x_{p+1+q})\!:\!\Delta[p,q]\to\mathrm{Dec}\overline{W}X)\,.\end{array}$$

The following facts are used in our development below:

Facts 1.5 *1. For each $n\geq0$, there is a natural Alexander-Whitney type diagonal approximation*

$$\begin{aligned}&\phi:\mathrm{Dec}\Delta[n]\to\Delta[n,n],\\&(\Delta[p+1+q]\overset{x}{\to}\Delta[n])\ \mapsto\ (\Delta[p]\overset{x(d^{p+1})^q}{\longrightarrow}\Delta[n],\Delta[q]\overset{x(d^0)^{p+1}}{\longrightarrow}\Delta[n])\end{aligned}$$

such that, for any bisimplicial set K, the induced simplicial map $\phi^*: \mathrm{diag}K \to \overline{W}K$ determines a homotopy equivalence

$$|\mathrm{diag}K| \simeq |\overline{W}K|$$

on the corresponding geometric realizations [49, Theorem 1.1].

2. *For any simplicial map $f: L \to L'$, the induced $|f|: |L| \to |L'|$ is a homotopy equivalence if and only if the induced $|\mathrm{Dec}f|: |\mathrm{Dec}L| \to |\mathrm{Dec}L'|$ is a homotopy equivalence* [49, Corollary 7.2].

3. *For any simplicial set L and any bisimplicial set K, both induced maps $|\mathrm{u}|: |L| \to |\overline{W}\mathrm{Dec}L|$ and $|\mathrm{v}|: |\mathrm{Dec}\overline{W}K| \to |K|$ are homotopy equivalences* [49, Proposition 7.1 and discussion below].

4. *If K is any bisimplicial set satisfying the extension condition, then $\overline{W}K$ is a Kan complex* [51, Proposition 2].

5. *If L is a Kan complex, then* Dec$L$ *satisfies the extension condition* (the proof is a straightforward application of [104, Lemma 7.4]) or [51, Lemma 1]) .

1.3 Double groupoids satisfying the filling condition: Homotopy groups.

A (small) double groupoid [31, 62, 63, 92] is a groupoid object in the category of small groupoids. In general, we employ the standard nomenclature concerning double categories but, for the sake of clarity, we shall fix some terminology and notations below.

A (small) category can be described as a system $(M, O, \mathrm{s}, \mathrm{t}, 1, \circ)$, where M is the set of morphisms, O is the set of objects, $\mathrm{s}, \mathrm{t}: M \to O$ are the source and target maps, respectively, $1: O \to M$ is the identities map, and $\circ: M_{\mathrm{s}}\times_{\mathrm{t}} M \to M$ is the composition map, subject to the usual associativity and identity axioms. Therefore, a *double category* provides us with the following data: a set O of *objects*, a set H of *horizontal morphisms*, a set V of *vertical morphisms*, and a set C of *squares*, together with four category structures, namely, the *category of horizontal morphisms* $(H, O, \mathrm{s}^{\mathrm{h}}, \mathrm{t}^{\mathrm{h}}, 1^{\mathrm{h}}, \circ_{\mathrm{h}})$, the *category of vertical morphisms* $(V, O, \mathrm{s}^{\mathrm{v}}, \mathrm{t}^{\mathrm{v}}, 1^{\mathrm{v}}, \circ_{\mathrm{v}})$, the *horizontal category of squares* $(C, V, \mathrm{s}^{\mathrm{h}}, \mathrm{t}^{\mathrm{h}}, 1^{\mathrm{h}}, \circ_{\mathrm{h}})$, and the *vertical category of squares* $(C, H, \mathrm{s}^{\mathrm{v}}, \mathrm{t}^{\mathrm{v}}, 1^{\mathrm{v}}, \circ_{\mathrm{v}})$. These are subject to the following three axioms:

$$\textbf{Axiom 1}\ \begin{cases} \text{(i)} & \mathrm{s}^{\mathrm{h}}\mathrm{s}^{\mathrm{v}} = \mathrm{s}^{\mathrm{v}}\mathrm{s}^{\mathrm{h}},\ \mathrm{t}^{\mathrm{h}}\mathrm{t}^{\mathrm{v}} = \mathrm{t}^{\mathrm{v}}\mathrm{t}^{\mathrm{h}},\ \mathrm{s}^{\mathrm{h}}\mathrm{t}^{\mathrm{v}} = \mathrm{t}^{\mathrm{v}}\mathrm{s}^{\mathrm{h}},\ \mathrm{s}^{\mathrm{v}}\mathrm{t}^{\mathrm{h}} = \mathrm{t}^{\mathrm{h}}\mathrm{s}^{\mathrm{v}}, \\ \text{(ii)} & \mathrm{s}^{\mathrm{h}}1^{\mathrm{v}} = 1^{\mathrm{v}}\mathrm{s}^{\mathrm{h}},\ \mathrm{t}^{\mathrm{h}}1^{\mathrm{v}} = 1^{\mathrm{v}}\mathrm{t}^{\mathrm{h}},\ \mathrm{s}^{\mathrm{v}}1^{\mathrm{h}} = 1^{\mathrm{h}}\mathrm{s}^{\mathrm{v}},\ \mathrm{t}^{\mathrm{v}}1^{\mathrm{h}} = 1^{\mathrm{h}}\mathrm{t}^{\mathrm{v}}, \\ \text{(iii)} & 1^{\mathrm{h}}1^{\mathrm{v}} = 1^{\mathrm{v}}1^{\mathrm{h}}. \end{cases}$$

Equalities in **Axiom 1** allow a square $\alpha \in C$ to be depicted in the form

$$\begin{tikzcd}
d & b \arrow[l, "g"'] \\
c \arrow[u, "w"] \arrow[u, phantom, shift left=4, "\alpha"] & a \arrow[l, "f"] \arrow[u, "u"']
\end{tikzcd} \tag{1.11}$$

where $\mathrm{s}^{\mathrm{h}}\alpha = u$, $\mathrm{t}^{\mathrm{h}}\alpha = w$, $\mathrm{s}^{\mathrm{v}}\alpha = f$ and $\mathrm{t}^{\mathrm{v}}\alpha = g$, and the four vertices of the square representing α are $\mathrm{s}^{\mathrm{h}}\mathrm{s}^{\mathrm{v}}\alpha = a$, $\mathrm{t}^{\mathrm{h}}\mathrm{t}^{\mathrm{v}}\alpha = d$, $\mathrm{s}^{\mathrm{h}}\mathrm{t}^{\mathrm{v}}\alpha = b$ and $\mathrm{s}^{\mathrm{v}}\mathrm{t}^{\mathrm{h}}\alpha = c$. Moreover, if we represent identity morphisms by the symbol $=\!=$, then, for any horizontal morphism f, any vertical morphism u, and any object a, the associated identity squares 1^{v}_f, 1^{h}_u and $1_a := 1^{\mathrm{h}}_{1^{\mathrm{v}}_a} = 1^{\mathrm{v}}_{1^{\mathrm{h}}_a}$ are respectively given in the form

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "f"'] \\
\cdot \arrow[u, equal] \arrow[u, phantom, shift left=4, "1^{\mathrm{v}}_f"] & \cdot \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot & \cdot \arrow[l, equal] \\
\cdot \arrow[u, "u"] \arrow[u, phantom, shift left=4, "1^{\mathrm{h}}_u"] & \cdot \arrow[l, equal] \arrow[u, "u"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot & \cdot \arrow[l, equal] \\
\cdot \arrow[u, equal] \arrow[u, phantom, shift left=4, "1_a"] & \cdot \arrow[l, equal] \arrow[u, equal]
\end{tikzcd}$$

The equalities in **Axiom 2** below show the squares are compatible with the boundaries, whereas **Axiom 3** establishes the necessary coherence between the two vertical and horizontal compositions of squares.

$$\textbf{Axiom 2} \left\{ \begin{array}{ll}
\text{(i)} & \mathrm{s}^{\mathrm{v}}(\alpha \circ_{\mathrm{h}} \beta) = \mathrm{s}^{\mathrm{v}}\alpha \circ_{\mathrm{h}} \mathrm{s}^{\mathrm{v}}\beta, \ \ \mathrm{t}^{\mathrm{v}}(\alpha \circ_{\mathrm{h}} \beta) = \mathrm{t}^{\mathrm{v}}\alpha \circ_{\mathrm{h}} \mathrm{t}^{\mathrm{v}}\beta, \\
\text{(ii)} & \mathrm{s}^{\mathrm{h}}(\alpha \circ_{\mathrm{v}} \beta) = \mathrm{s}^{\mathrm{h}}\alpha \circ_{\mathrm{v}} \mathrm{s}^{\mathrm{h}}\beta, \ \ \mathrm{t}^{\mathrm{h}}(\alpha \circ_{\mathrm{v}} \beta) = \mathrm{t}^{\mathrm{h}}\alpha \circ_{\mathrm{v}} \mathrm{t}^{\mathrm{h}}\beta, \\
\text{(iii)} & 1^{\mathrm{v}}_{f \circ_{\mathrm{h}} f'} = 1^{\mathrm{v}}_f \circ_{\mathrm{h}} 1^{\mathrm{v}}_{f'}, \ \ 1^{\mathrm{h}}_{u \circ_{\mathrm{v}} u'} = 1^{\mathrm{h}}_u \circ_{\mathrm{v}} 1^{\mathrm{h}}_{u'}.
\end{array} \right.$$

Axiom 3 *In the situation*

$$\begin{tikzcd}
\cdot & \cdot \arrow[l] & \cdot \arrow[l] \\
\cdot \arrow[u] \arrow[u, phantom, shift left=4, "\alpha"] & \cdot \arrow[l] \arrow[u] \arrow[u, phantom, shift left=4, "\beta"] & \cdot \arrow[l] \arrow[u] \\
\cdot \arrow[u] \arrow[u, phantom, shift left=4, "\gamma"] & \cdot \arrow[l] \arrow[u] \arrow[u, phantom, shift left=4, "\delta"] & \cdot \arrow[l] \arrow[u]
\end{tikzcd}$$

the interchange law holds, that is, $(\alpha \circ_{\mathrm{h}} \beta) \circ_{\mathrm{v}} (\gamma \circ_{\mathrm{h}} \delta) = (\alpha \circ_{\mathrm{v}} \gamma) \circ_{\mathrm{h}} (\beta \circ_{\mathrm{v}} \delta)$.

A *double groupoid* is a double category such that all the four component categories are groupoids. We shall use the following notation for inverses in a double groupoid: $f^{-1_{\mathrm{h}}}$ denotes the inverse of a horizontal morphism f, and $u^{-1_{\mathrm{v}}}$ denotes the inverse of a vertical morphism u. For any square α as in (1.11), the first one of

$$\begin{tikzcd}
b & d \arrow[l, "g^{-1_{\mathrm{h}}}"'] \\
a \arrow[u, "u"] \arrow[u, phantom, shift left=5, "\alpha^{-1_{\mathrm{h}}}"] & c, \arrow[l, "f^{-1_{\mathrm{h}}}"] \arrow[u, "w"']
\end{tikzcd}
\qquad
\begin{tikzcd}
c & a \arrow[l, "f"'] \\
d \arrow[u, "w^{-1_{\mathrm{v}}}"] \arrow[u, phantom, shift left=5, "\alpha^{-1_{\mathrm{v}}}"] & b, \arrow[l, "g"] \arrow[u, "u^{-1_{\mathrm{v}}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a & c \arrow[l, "f^{-1_{\mathrm{h}}}"'] \\
b \arrow[u, "u^{-1_{\mathrm{v}}}"] \arrow[u, phantom, shift left=5, "\alpha^{-1}"] & d, \arrow[l, "g^{-1_{\mathrm{h}}}"] \arrow[u, "w^{-1_{\mathrm{v}}}"']
\end{tikzcd} \tag{1.12}$$

is the inverse of α in the horizontal groupoid of squares, the second one denotes the inverse of α in the vertical groupoid of squares, and the third one is the square $(\alpha^{-1_{\mathrm{h}}})^{-1_{\mathrm{v}}} = (\alpha^{-1_{\mathrm{v}}})^{-1_{\mathrm{h}}}$, which is denoted simply by α^{-1}.

The double groupoids we are interested in satisfy the condition below.

Filling condition : *Any filling problem*

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "g"'] \\
\cdot \arrow[u, dotted] \arrow[ur, phantom, "\exists?"] & \cdot \arrow[u, "u"'] \arrow[l, dotted]
\end{tikzcd}$$

> *has a solution; that is, for any horizontal morphism g and any vertical morphism u such that $\mathrm{s}^{\mathrm{h}}g = \mathrm{t}^{\mathrm{v}}u$, there is a square α with $\mathrm{s}^{\mathrm{h}}\alpha = u$ and $\mathrm{t}^{\mathrm{v}}\alpha = g$.*

As we recalled in the introduction, this filling condition on double groupoids is often satisfied for those double groupoids arising in algebraic topology. However, we should stress the existence of double groupoids which do not satisfy the filling condition (see [50, Example] for instances). Later in Sections 1.4 and 1.6, we show two new homotopical double groupoid constructions: one, $\Pi^{(2}X$, for topological spaces X, and the other, $\mathrm{P}^{(2}K$, for bisimplicial sets K, both yielding double groupoids satisfying the filling condition.

The remainder of this section is devoted to defining *homotopy groups*, $\pi_i(\mathcal{G}, a)$, for double groupoids $\mathcal{G}$ satisfying the filling condition. The useful observation below is a direct consequence of [1, Lemma 1.12].

Lemma 1.4 *A double groupoid $\mathcal{G}$ satisfies the filling condition if and only if any filling problem such as the one below has a solution.*

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, dotted] \\
\cdot \arrow[u, "w"] \arrow[ur, phantom, "\exists?"] & \cdot \arrow[u, dotted] \arrow[l, "f"]
\end{tikzcd}, \quad
\begin{tikzcd}
\cdot & \cdot \arrow[l, dotted] \\
\cdot \arrow[u, dotted] \arrow[ur, phantom, "\exists?"] & \cdot \arrow[u, "u"'] \arrow[l, "f"]
\end{tikzcd}, \quad
\begin{tikzcd}
\cdot & \cdot \arrow[l, "g"'] \\
\cdot \arrow[u, "w"] \arrow[ur, phantom, "\exists?"] & \cdot \arrow[u, dotted] \arrow[l, dotted]
\end{tikzcd},$$

Hereafter, we assume $\mathcal{G}$ is a double groupoid satisfying the filling condition.

1.3.1 The pointed sets $\pi_0(\mathcal{G}, a)$.

We state that two objects a, b of $\mathcal{G}$ are *connected* whenever there is a pair of morphisms (g, u) in $\mathcal{G}$ of the form

$$\begin{tikzcd}
b & \cdot \arrow[l, "g"'] \\
 & a \arrow[u, "u"']
\end{tikzcd},$$

that is, where g is a horizontal morphism and u a vertical morphism such that $\mathrm{s}^{\mathrm{h}}g = \mathrm{t}^{\mathrm{v}}u$, $\mathrm{t}^{\mathrm{h}}g = b$, and $\mathrm{s}^{\mathrm{v}}u = a$. Because of the filling condition, this is equivalent to saying that there is a square in $\mathcal{G}$ of the form

$$\begin{tikzcd}
b & \cdot \arrow[l, "g"'] \\
\cdot \arrow[u, "w"] \arrow[ur, phantom, "\alpha"] & a \arrow[u, "u"'] \arrow[l, "f"]
\end{tikzcd},$$

and it is also equivalent to saying that there is a pair of matching morphisms (w, f) as

$$\begin{tikzcd} b & \\ \cdot \arrow[u, "w"] & a. \arrow[l, "f"'] \end{tikzcd}$$

If a and b are recognized as being connected by means of the pair of morphisms (g, u) as above, then the pair $(u^{\text{-}1_\mathrm{v}}, g^{\text{-}1_\mathrm{h}})$ shows that b is connected to a. Hence, being connected is a symmetric relation on the set of objects of $\mathcal{G}$. This relation is clearly reflexive thanks to the identity morphisms $({1^\mathrm{h}}_a, {1^\mathrm{v}}_a)$, and it is also transitive. Suppose a is connected with b, which itself is connected with another object c. Then, we have morphisms u, f, v, g as in the diagram

$$\begin{tikzcd} c & \cdot \arrow[l, "g"'] & \cdot \arrow[l, dotted, "g'"'] \\ & b \arrow[u, "v"] \arrow[ur, phantom, "\beta"] & \cdot \arrow[l, "f"] \arrow[u, dotted, "u'"'] \\ & & a \arrow[u, "u"'] \end{tikzcd}$$

where β is any square with $\mathrm{t}^\mathrm{h}\beta = v$ and $\mathrm{s}^\mathrm{v}\beta = f$, and the dotted g' and u' are the other sides of β. Consequently, on considering the pair of composites $(g \circ_\mathrm{h} g', u' \circ_\mathrm{v} u)$, we see that a and c are connected.

Therefore, being connected establishes an equivalence relation on the objects of the double groupoid and, associated to $\mathcal{G}$, we take

$$\pi_0\mathcal{G} = \textit{the set of connected classes of objects of } \mathcal{G},$$

and we write $\pi_0(\mathcal{G}, a)$ for the set $\pi_0\mathcal{G}$ pointed with the class $[a]$ of an object a of $\mathcal{G}$.

1.3.2 The groups $\pi_1(\mathcal{G}, a)$

Let a be any given object of $\mathcal{G}$, and let

$$\mathcal{G}(a) = \left\{ \begin{tikzcd} a & \cdot \arrow[l, "g"'] \\ & a \arrow[u, "u"'] \end{tikzcd} \right\}$$

be the set of all pairs of morphisms (g, u), where g is a horizontal morphism and u a vertical morphism in $\mathcal{G}$ such that $\mathrm{t}^\mathrm{h}g = a = \mathrm{s}^\mathrm{v}u$ and $\mathrm{s}^\mathrm{h}g = \mathrm{t}^\mathrm{v}u$.

Define a relation $\sim$ on $\mathcal{G}(a)$ by the rule $(g, u) \sim (g', u')$ if and only if there are two squares α and α' in $\mathcal{G}$ of the form

$$\begin{tikzcd} a & \cdot \arrow[l, "g"'] \\ \cdot \arrow[u, "w"] \arrow[ur, phantom, "\alpha"] & a \arrow[l, "f"] \arrow[u, "u"'] \end{tikzcd} \qquad \begin{tikzcd} a & \cdot \arrow[l, "g'"'] \\ \cdot \arrow[u, "w"] \arrow[ur, phantom, "\alpha'"] & a \arrow[l, "f"] \arrow[u, "u'"'] \end{tikzcd}$$

that is, such that $\mathrm{t}^{\mathrm{h}}\alpha = \mathrm{t}^{\mathrm{h}}\alpha'$, $\mathrm{s}^{\mathrm{v}}\alpha = \mathrm{s}^{\mathrm{v}}\alpha'$, $\mathrm{s}^{\mathrm{h}}\alpha = u$, $\mathrm{s}^{\mathrm{h}}\alpha' = u'$, $\mathrm{t}^{\mathrm{v}}\alpha = g$, and $\mathrm{t}^{\mathrm{v}}\alpha' = g'$.

Lemma 1.5 *The relation $\sim$ is an equivalence.*

Proof: Since $\mathcal{G}$ satisfies the filling condition, the relation is clearly reflexive, and it is obviously symmetric. To prove transitivity, suppose $(g, u) \sim (g', u') \sim (g'', u'')$, so that there are squares α, α', β and β' as below.

$$\begin{tikzcd}
a & \cdot \arrow[l, "g"'] & a & \cdot \arrow[l, "g'"'] & a & \cdot \arrow[l, "g'"'] & a & \cdot \arrow[l, "g''"'] \\
\cdot \arrow[u, "w"] \arrow[ur, phantom, "\alpha"] & a \arrow[l, "f"] \arrow[u, "u"'] & \cdot \arrow[u, "w"] \arrow[ur, phantom, "\alpha'"] & a \arrow[l, "f"] \arrow[u, "u'"'] & \cdot \arrow[u, "w'"] \arrow[ur, phantom, "\beta"] & a \arrow[l, "f'"] \arrow[u, "u'"'] & \cdot \arrow[u, "w'"] \arrow[ur, phantom, "\beta'"] & a \arrow[l, "f'"] \arrow[u, "u''"']
\end{tikzcd}$$

Then, we have the horizontally composable squares

$$\begin{tikzcd}
a & \cdot \arrow[l, "g'"'] & a \arrow[l, "g'^{\text{-}1_{\mathrm{h}}}"'] & \cdot \arrow[l, "g"'] \\
\cdot \arrow[u, "w'"] \arrow[ur, phantom, "\beta"] & a \arrow[l, "f'"] \arrow[u] \arrow[ur, phantom, "\alpha'^{\text{-}1_{\mathrm{h}}}"] & \cdot \arrow[l, "f^{\text{-}1_{\mathrm{h}}}"] \arrow[u] \arrow[ur, phantom, "\alpha"] & a \arrow[l, "f"] \arrow[u, "u"']
\end{tikzcd}$$

whose composition $\beta \circ_{\mathrm{h}} \alpha'^{\text{-}1_{\mathrm{h}}} \circ_{\mathrm{h}} \alpha$ and β' show that $(g, u) \sim (g'', u'')$. $\square$

We write $[g, u]$ for the $\sim$-equivalence class of $(g, u) \in \mathcal{G}(a)$. Now we define a product on

$$\pi_1(\mathcal{G}, a) := \mathcal{G}(a) / \sim$$

as follows: given $[g_1, u_1]$, $[g_2, u_2] \in \pi_1(\mathcal{G}, a)$, by the filling condition on $\mathcal{G}$, we can choose a square γ with $\mathrm{s}^{\mathrm{v}}\gamma = g_2$ and $\mathrm{t}^{\mathrm{h}}\gamma = u_1$ so that we have a configuration in $\mathcal{G}$ of the form

$$\begin{tikzcd}
a & \cdot \arrow[l, "g_1"'] & \cdot \arrow[l, "g"'] \\
 & a \arrow[u, "u_1"] \arrow[ur, phantom, "\gamma"] & \cdot \arrow[l, "g_2"] \arrow[u, "u"'] \\
 & & a \arrow[u, "u_2"']
\end{tikzcd}$$

where $g = \mathrm{t}^{\mathrm{v}}\gamma$ and $u = \mathrm{s}^{\mathrm{h}}\gamma$. Then we define

$$[g_1, u_1] \circ [g_2, u_2] = [g_1 \circ_{\mathrm{h}} g, u \circ_{\mathrm{v}} u_2]$$

Lemma 1.6 *The product is well defined.*

Proof: Let $[g_1, u_1] = [g_1', u_1']$, $[g_2, u_2] = [g_2', u_2']$ be elements of $\pi_1(\mathcal{G}, a)$. Then, there are squares

$$\begin{tikzcd}
a & \cdot \arrow[l, "g_1"'] & a & \cdot \arrow[l, "g_1'"'] & a & \cdot \arrow[l, "g_2"'] & a & \cdot \arrow[l, "g_2'"'] \\
\cdot \arrow[u, "w_1"] \arrow[ur, phantom, "\alpha"] & a \arrow[l, "f_1"] \arrow[u, "u_1"'] & \cdot \arrow[u, "w_1"] \arrow[ur, phantom, "\alpha'"] & a \arrow[l, "f_1"] \arrow[u, "u_1'"'] & \cdot \arrow[u, "w_2"] \arrow[ur, phantom, "\beta"] & a \arrow[l, "f_2"] \arrow[u, "u_2"'] & \cdot \arrow[u, "w_2"] \arrow[ur, phantom, "\beta'"] & a \arrow[l, "f_2"] \arrow[u, "u_2'"']
\end{tikzcd}$$

and choosing squares γ and γ' as in

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "g"'] \\
a \arrow[u, "u_1"] \arrow[ru, phantom, "\gamma"] & \cdot \arrow[l, "g_2"] \arrow[u, "u"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot & \cdot \arrow[l, "g'"'] \\
a \arrow[u, "u'_1"] \arrow[ru, phantom, "\gamma'"] & \cdot \arrow[l, "g'_2"] \arrow[u, "u'"']
\end{tikzcd}$$

we have $[g_1, u_1] \circ [g_2, u_2] = [g_1 \circ_{\mathrm{h}} g, u \circ_{\mathrm{v}} u_2]$ and $[g'_1, u'_1] \circ [g'_2, u'_2] = [g'_1 \circ_{\mathrm{h}} g', u' \circ_{\mathrm{v}} u'_2]$. Now, letting θ be any square with $\mathrm{t}^{\mathrm{v}}\theta = f_1$ and $\mathrm{s}^{\mathrm{h}}\theta = w_2$, we have squares as in

$$\begin{tikzcd}
a & \cdot \arrow[l, "g_1"'] & \cdot \arrow[l, "g"'] \\
\cdot \arrow[u, "w_1"] \arrow[ru, phantom, "\alpha"] & a \arrow[l] \arrow[u] \arrow[ru, phantom, "\gamma"] & \cdot \arrow[l] \arrow[u, "u"'] \\
\cdot \arrow[u] \arrow[ru, phantom, "\theta"] & \cdot \arrow[l] \arrow[u] \arrow[ru, phantom, "\beta"] & a \arrow[l, "f_2"] \arrow[u, "u_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a & \cdot \arrow[l, "g'_1"'] & \cdot \arrow[l, "g'"'] \\
\cdot \arrow[u, "w_1"] \arrow[ru, phantom, "\alpha'"] & a \arrow[l] \arrow[u] \arrow[ru, phantom, "\gamma'"] & \cdot \arrow[l] \arrow[u, "u'"'] \\
\cdot \arrow[u] \arrow[ru, phantom, "\theta"] & \cdot \arrow[l] \arrow[u] \arrow[ru, phantom, "\beta'"] & a \arrow[l, "f_2"] \arrow[u, "u'_2"']
\end{tikzcd}$$

whose corresponding composites $(\alpha \circ_{\mathrm{h}} \gamma) \circ_{\mathrm{v}} (\theta \circ_{\mathrm{h}} \beta)$ and $(\alpha' \circ_{\mathrm{h}} \gamma') \circ_{\mathrm{v}} (\theta \circ_{\mathrm{h}} \beta')$ show that $[g_1 \circ_{\mathrm{h}} g, u \circ_{\mathrm{v}} u_2] = [g'_1 \circ_{\mathrm{h}} g', u' \circ_{\mathrm{v}} u'_2]$, as required. ☐

Lemma 1.7 *The given multiplication turns $\pi_1(\mathcal{G}, a)$ into a group.*

Proof: To see the associativity, let $[g_1, u_1]$, $[g_2, u_2]$, $[g_3, u_3] \in \pi_1(\mathcal{G}, a)$, and choose γ, γ' and γ'' any three squares as in the diagram (1.13) below. Then we have

$$([g_1, u_1] \circ [g_2, u_2]) \circ [g_3, u_3] = [g_1 \circ_{\mathrm{h}} g \circ_{\mathrm{h}} g', u \circ_{\mathrm{v}} u' \circ_{\mathrm{v}} u_3] = [g_1, u_1] \circ ([g_2, u_2] \circ [g_3, u_3]).$$

$$\begin{tikzcd}
a & \cdot \arrow[l, "g_1"'] & \cdot \arrow[l, "g"'] & \cdot \arrow[l, "g'"'] \\
 & a \arrow[u, "u_1"] \arrow[ru, phantom, "\gamma"] & \cdot \arrow[l, "g_2"] \arrow[u] \arrow[ru, phantom, "\gamma'"] & \cdot \arrow[l] \arrow[u, "u"'] \\
 & & a \arrow[u, "u_2"] \arrow[ru, phantom, "\gamma''"] & \cdot \arrow[l, "g_3"] \arrow[u, "u'"'] \\
 & & & a \arrow[u, "u_3"']
\end{tikzcd} \tag{1.13}$$

The identity of $\pi_1(\mathcal{G}, a)$ is $[1^{\mathrm{h}}_a, 1^{\mathrm{v}}_a]$. In effect, if $[g, u] \in \pi_1(\mathcal{G}, a)$, then the diagrams

$$\begin{tikzcd}
a & x \arrow[l, "g"'] \arrow[r, equal] & x \\
 & a \arrow[u, "u"] \arrow[r, equal] \arrow[ru, phantom, "1^{\mathrm{h}}_u"] & a \arrow[u, "u"'] \arrow[d, equal] \\
 & & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, equal] & a \arrow[d, equal] & x \arrow[l, "g"'] \arrow[d, equal] \\
 & a \arrow[ru, phantom, "1^{\mathrm{v}}_g"] & x \arrow[l, "g"] \\
 & & a \arrow[u, "u"']
\end{tikzcd}$$

show that

$$[g, u] \circ [1^{\mathrm{h}}_a, 1^{\mathrm{v}}_a] = [g \circ_{\mathrm{h}} 1^{\mathrm{h}}_x, u \circ_{\mathrm{v}} 1^{\mathrm{v}}{}_a] = [g, u] = [1^{\mathrm{h}}_a \circ_{\mathrm{h}} g, 1^{\mathrm{v}}{}_x \circ_{\mathrm{v}} u] = [1^{\mathrm{h}}_a, 1^{\mathrm{v}}_a] \circ [g, u].$$

Finally, to see the existence of inverses, let $[g,u] \in \pi_1(\mathcal{G},a)$. By choosing any square α with $\mathrm{t}^{\mathrm{h}}\alpha = u^{-1_{\mathrm{v}}}$ and $\mathrm{s}^{\mathrm{v}}\alpha = g^{-1_{\mathrm{h}}}$, that is, of the form

$$\begin{tikzcd} a & \cdot \arrow[l, "f"'] \\ \cdot \arrow[u, "u^{-1_{\mathrm{v}}}"] & a \arrow[l, "g^{-1_{\mathrm{h}}}"] \arrow[u, "v"'] \arrow[ul, phantom, "\alpha"] \end{tikzcd}$$

we find $[f,v] := [\mathrm{t}^{\mathrm{v}}\alpha, \mathrm{s}^{\mathrm{h}}\alpha] \in \pi_1(\mathcal{G},a)$. Since the diagrams

$$\begin{tikzcd} a & \cdot \arrow[l, "g"'] & a \arrow[l, "g^{-1_{\mathrm{h}}}"'] \\ & a \arrow[u, "u"] & \cdot \arrow[l, "f"] \arrow[u, "v^{-1_{\mathrm{v}}}"'] \arrow[ul, phantom, "\alpha^{-1_{\mathrm{v}}}"] \\ & & a \arrow[u, "v"'] \end{tikzcd} \qquad \begin{tikzcd} a & \cdot \arrow[l, "f"'] & a \arrow[l, "f^{-1_{\mathrm{h}}}"'] \\ & a \arrow[u, "v"] & \cdot \arrow[l, "g"] \arrow[u, "u^{-1_{\mathrm{v}}}"'] \arrow[ul, phantom, "\alpha^{-1_{\mathrm{h}}}"] \\ & & a \arrow[u, "u"'] \end{tikzcd}$$

show that $[g,u] \circ [f,v] = [1^{\mathrm{h}}_a, 1^{\mathrm{v}}_a] = [f,v] \circ [g,u]$, we have $[g,u]^{-1} = [f,v]$. ☐

1.3.3 The abelian groups $\pi_i(\mathcal{G},a)$, $i \geq 2$.

These are easier to define than the previous ones. For $i = 2$, as in [31, Section 2], we take

$$\pi_2(\mathcal{G},a) = \left\{ \begin{tikzcd} a \arrow[r, equal] \arrow[d, equal] & a \arrow[d, equal] \\ a \arrow[r, equal] \arrow[ur, phantom, "\alpha"] & a \end{tikzcd} \right\}$$

the set of all squares $\alpha \in \mathcal{G}$ whose boundary edges are $\mathrm{s}^{\mathrm{h}}\alpha = \mathrm{t}^{\mathrm{h}}\alpha = 1^{\mathrm{v}}_a$ and $\mathrm{s}^{\mathrm{v}}\alpha = \mathrm{t}^{\mathrm{v}}\alpha = 1^{\mathrm{h}}_a$.

By the general Eckman-Hilton argument, it is a consequence of the interchange law that, on $\pi_2(\mathcal{G},a)$, operations $\circ_{\mathrm{h}}$ and $\circ_{\mathrm{v}}$ coincide and are commutative. In effect, for $\alpha, \beta \in \pi_2(\mathcal{G},a)$,

$$\begin{aligned} \alpha \circ_{\mathrm{h}} \beta &= (\alpha \circ_{\mathrm{v}} 1_a) \circ_{\mathrm{h}} (1_a \circ_{\mathrm{v}} \beta) = (\alpha \circ_{\mathrm{h}} 1_a) \circ_{\mathrm{v}} (1_a \circ_{\mathrm{h}} \beta) \\ &= \alpha \circ_{\mathrm{v}} \beta \\ &= (1_a \circ_{\mathrm{h}} \alpha) \circ_{\mathrm{v}} (\beta \circ_{\mathrm{h}} 1_a) = (1_a \circ_{\mathrm{v}} \beta) \circ_{\mathrm{h}} (\alpha \circ_{\mathrm{v}} 1_a) \\ &= \beta \circ_{\mathrm{h}} \alpha. \end{aligned}$$

Therefore, $\pi_2(\mathcal{G},a)$ is an abelian group with product

$$\alpha \circ_{\mathrm{h}} \beta = \alpha \circ_{\mathrm{v}} \beta \ ,$$

identity $1_a = 1^{\mathrm{v}}_{1^{\mathrm{h}}_a}$, and inverses $\alpha^{-1_{\mathrm{h}}} = \alpha^{-1_{\mathrm{v}}}$.

The higher homotopy groups of the double groupoid are defined to be trivial, that is,

$$\pi_i(\mathcal{G},a) = 0 \quad \text{if} \quad i \geq 3.$$

1.3.4 Weak equivalences.

A *double functor* $F : \mathcal{G} \to \mathcal{G}'$ between double categories takes objects, horizontal and vertical morphisms, and squares in $\mathcal{G}$ to objects, horizontal and vertical morphisms, and squares in $\mathcal{G}'$, respectively, in such a way that all the structure categories are preserved.

Clearly, each double functor $F : \mathcal{G} \to \mathcal{G}'$, between double groupoids satisfying the filling condition, induces maps (group homomorphisms if $i > 0$)

$$\pi_i F : \pi_i(\mathcal{G}, a) \to \pi_i(\mathcal{G}', Fa)$$

for $i \geq 0$ and a any object of $\mathcal{G}$. Call such a double functor a *weak equivalence* if it induces isomorphisms $\pi_i F$ for all integers $i \geq 0$.

1.4 A homotopy double groupoid for topological spaces.

Our aim here is to provide a new construction of a double groupoid for a topological space that, as we will see later, captures the homotopy 2-type of the space. For any given space X, the construction of this *homotopy double groupoid*, denoted by $\Pi^{(2}X$, is as follows:

The objects in $\Pi^{(2}X$ are the paths in X, that is, the continuous maps $u : I = [0,1] \to X$.

The groupoid of horizontal morphisms in $\Pi^{(2}X$ is the category with a unique morphism between each pair (u', u) of paths in X such that $u'(1) = u(1)$, and, similarly, the groupoid of vertical morphisms in $\Pi^{(2}X$ is the category having a unique morphism between each pair (v, u) of paths in X such that $v(0) = u(0)$.

A square in $\Pi^{(2}X$, $[\alpha]$, with a boundary as in

$$\begin{tikzcd} v' & v \arrow[l] \\ u' \arrow[u] \arrow[r, phantom, "{[\alpha]}"] & u \arrow[l] \arrow[u] \end{tikzcd} \tag{1.14}$$

is the equivalence class, $[\alpha]$, of a map $\alpha : I^2 \to X$ whose effect on the boundary $\partial(I^2)$ is such that $\alpha(x, 0) = u(x)$, $\alpha(0, y) = v(y)$, $\alpha(1, 1-y) = u'(y)$, and $\alpha(1-x, 1) = v'(x)$, for $x, y \in I$. We call such an application a "square in X" and draw it as

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "v'"'] \arrow[d, "u'"] \\ \cdot \arrow[u, "v"] \arrow[r, "u"'] \arrow[r, phantom, shift left=4, "\alpha"] & \cdot \end{tikzcd}$$

Observe that the arrows representing the paths u' and v' above are reversed from those that might be expected, namely, their corresponding inverses $y \mapsto \alpha(1, y) = u'(1-y) = u'^{-1}(y)$ and $x \mapsto \alpha(x, 1) = v'(1-x) = v'^{-1}(x)$, respectively. The reason for that is precisely to avoid the use of inverse paths, which could be confusing in our context (where a path is a vertex).

Two such mappings α, α' are equivalent, and then represent the same square in $\Pi^{(2}X$, whenever they are related by a homotopy relative to the sides of the square, that is, if there exists a continuous map $H : I^2 \times I \to X$ such that $H(x, y, 0) = \alpha(x, y)$, $H(x, y, 1) = \alpha'(x, y)$, $H(x, 0, t) = u(x)$, $H(0, y, t) = v(y)$, $H(x, 1, t) = v'(1 - x)$ and $H(1, y, t) = u'(1 - y)$, for $x, y, t \in I$.

Given the squares in $\Pi^{(2}X$

$$\begin{tikzcd}
 & w' & w \arrow[l] \\
v'' & v' \arrow[u] \arrow[ur, phantom, "{[\beta]}"] & v \arrow[l] \arrow[u] \\
u'' \arrow[u] \arrow[ur, phantom, "{[\alpha']}"] & u' \arrow[l] \arrow[u] \arrow[ur, phantom, "{[\alpha]}"] & u \arrow[l] \arrow[u]
\end{tikzcd}$$

the corresponding composite squares

$$\begin{tikzcd}
v'' & & v \arrow[ll] & w' & & w \arrow[ll] \\
u'' \arrow[u] \arrow[urr, phantom, "{[\alpha']\circ_{\mathrm{h}}[\alpha]}"] & & u \arrow[ll] \arrow[u] & u' \arrow[u] \arrow[urr, phantom, "{[\beta]\circ_{\mathrm{v}}[\alpha]}"] & & u \arrow[ll] \arrow[u]
\end{tikzcd}$$

are defined to be those represented by the squares in X

$$\begin{tikzcd}
\cdot & & \cdot \arrow[ll, "v''"'] \arrow[dd, "u''"] & & \cdot & & \cdot \arrow[ll, "w'"'] \arrow[dl, "v'"'] \arrow[dd, "u'"] \\
 & \cdot \arrow[ul, "v'"'] \arrow[dr, "u'"] \arrow[ur, phantom, "\alpha'"] \arrow[dl, phantom, "\alpha"] & & & & \cdot \arrow[ul, phantom, "\beta"] \arrow[dr, phantom, "\alpha"] & \\
\cdot \arrow[uu, "v"] \arrow[rr, "u"'] & & \cdot & & \cdot \arrow[uu, "w"] \arrow[rr, "u"'] \arrow[ur, "v"] & & \cdot
\end{tikzcd}$$

obtained, respectively, by pasting α' with α, and β with α, along their common pair of sides. That is,

$$[\alpha'] \circ_{\mathrm{h}} [\alpha] = [\alpha' \circ_{\mathrm{h}} \alpha], \qquad [\beta] \circ_{\mathrm{v}} [\alpha] = [\beta \circ_{\mathrm{v}} \alpha],$$

where

$$(\alpha' \circ_{\mathrm{h}} \alpha)(x, y) = \begin{cases} \alpha(2x, x + y) & \text{if } \ x \leq y, \ x + y \leq 1, \\ \alpha(x + y, 2y) & \text{if } \ x \geq y, \ x + y \leq 1, \\ \alpha'(x + y - 1, 2y - 1) & \text{if } \ x \leq y, \ x + y \geq 1, \\ \alpha'(2x - 1, x + y - 1) & \text{if } \ x \geq y, \ x + y \geq 1, \end{cases} \tag{1.15}$$

$$(\beta \circ_{\mathrm{v}} \alpha)(x, y) = \begin{cases} \alpha(2x - 1, 1 - x + y) & \text{if } \ x \geq y, \ x + y \geq 1, \\ \alpha(x - y, 2y) & \text{if } \ x \geq y, \ x + y \leq 1, \\ \beta(1 + x - y, 2y - 1) & \text{if } \ x \leq y, \ x + y \geq 1, \\ \beta(2x, y - x) & \text{if } \ x \leq y, \ x + y \leq 1. \end{cases}$$

It is not hard to see that both the horizontal and vertical compositions of squares in $\Pi^{(2}X$ are well defined. For example, to prove that $[\alpha] = [\alpha_1]$ and $[\alpha'] = [\alpha'_1]$ imply

$[\alpha' \circ_{\mathrm{h}} \alpha] = [\alpha'_1 \circ_{\mathrm{h}} \alpha_1]$, let $H, H' : I^2 \times I \to X$ be homotopies ($rel\ \partial(I^2)$) from α to α_1 and from α' to α'_1 respectively. Then, a homotopy $F : I^2 \times I \to X$ is defined by

$$F(x,y,t) = \begin{cases} H(2x, x+y, t) & \text{if } \ x \leq y, \ x+y \leq 1, \\ H(x+y, 2y, t) & \text{if } \ x \geq y, \ x+y \leq 1, \\ H'(x+y-1, 2y-1, t) & \text{if } \ x \leq y, \ x+y \geq 1, \\ H'(2x-1, x+y-1, t) & \text{if } \ x \geq y, \ x+y \geq 1, \end{cases}$$

showing that $\alpha' \circ_{\mathrm{h}} \alpha$ and $\alpha'_1 \circ_{\mathrm{h}} \alpha_1$ represent the same square in $\Pi^{(2}X$.

The horizontal identity square on a vertical morphism (v, u) is

$$1^{\mathrm{h}}_{(v,u)} = \begin{tikzcd} v \arrow[r, equal] & v \\ u \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "{[e^{\mathrm{h}}]}"] & u \arrow[u] \end{tikzcd}$$

where

$$e^{\mathrm{h}} = e^{\mathrm{h}}(v,u) = \begin{tikzcd} \cdot & \cdot \arrow[l, "v"'] \arrow[d, "u"] \\ \cdot \arrow[u, "v"] \arrow[r, "u"'] \arrow[ur, equal] & \cdot \end{tikzcd}$$

is defined by

$$e^{\mathrm{h}}(x,y) = \begin{cases} v(y-x) & \text{if } \ x \leq y, \\ u(x-y) & \text{if } \ x \geq y, \end{cases}$$

whereas, for any horizontal morphism (u', u), its corresponding vertical identity square is

$$1^{\mathrm{v}}_{(u',u)} = \begin{tikzcd} u' & u \arrow[l] \arrow[d, equal] \\ u' \arrow[u, equal] \arrow[ur, phantom, "{[e^{\mathrm{v}}]}"] & u \arrow[l] \end{tikzcd}$$

where

$$e^{\mathrm{v}} = e^{\mathrm{v}}(u',u) = \begin{tikzcd} \cdot \arrow[dr, equal] & \cdot \arrow[l, "u'"'] \arrow[d, "u'"] \\ \cdot \arrow[u, "u"] \arrow[r, "u"'] & \cdot \end{tikzcd}$$

is defined by

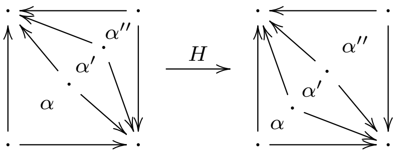
$$e^{\mathrm{v}}(x,y) = \begin{cases} u(x+y) & \text{if } \ x+y \leq 1, \\ u'(2-x-y) & \text{if } \ x+y \geq 1. \end{cases} \tag{1.16}$$

Theorem 1.1 $\Pi^{(2}X$ *is a double groupoid satisfying the filling condition.*

Proof: The horizontal composition of squares in $\Pi^{(2}X$ is associative since, for any three composable squares, say

$$\begin{tikzcd}
\cdot & \cdot \arrow[l] & \cdot \arrow[l] & \cdot \arrow[l] \\
\cdot \arrow[u] \arrow[ur, phantom, "{[\alpha'']}"] & \cdot \arrow[l] \arrow[u] \arrow[ur, phantom, "{[\alpha']}"] & \cdot \arrow[l] \arrow[u] \arrow[ur, phantom, "{[\alpha]}"] & \cdot \arrow[l] \arrow[u]
\end{tikzcd},$$

a relative homotopy

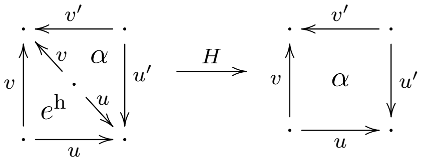


$(\alpha''\circ_{\mathrm{h}}\alpha')\circ_{\mathrm{h}}\alpha$ $\alpha''\circ_{\mathrm{h}}(\alpha'\circ_{\mathrm{h}}\alpha)$

is given by the formula
$H(x,y,t) =$

$$\begin{cases}
\alpha\big(\frac{4x}{2-t}, \frac{(2+t)x+(2-t)y}{2-t}\big) & \text{if} \quad x\leq y,\ (2-t)(1-y)\geq(2+t)x, \\
\alpha\big(\frac{(2-t)x+(2+t)y}{2-t}, \frac{4y}{2-t}\big) & \text{if} \quad x\geq y,\ (2-t)(1-x)\geq(2+t)y, \\
\alpha'\big(t(1+x-y)+2(x+y-1), x+3y-2+t(1+x-y)\big) & \text{if} \quad x\leq y, (2-t)(1-y)\leq(2+t)x, (1+t)x\leq(3-t)(1-y), \\
\alpha'\big(3x+y-2+t(1-x+y), t(1-x+y)+2(x+y-1)\big) & \text{if} \quad x\geq y, (2-t)(1-x)\leq(2+t)y, (1+t)y\leq(3-t)(1-x), \\
\alpha''\big(\frac{x+3y-3+t(1+x-y)}{1+t}, \frac{t-3+4y}{1+t}\big) & \text{if} \quad x\leq y,\ (1+t)x\geq(1-y)(3-t), \\
\alpha''\big(\frac{t-3+4x}{1+t}, \frac{3x+y-3+t(1-x+y)}{1+t}\big) & \text{if} \quad x\geq y,\ (1+t)y\geq(3-t)(1-x).
\end{cases}$$

And, similarly, we prove the associativity for the vertical composition of squares in $\Pi^{(2}X$. For identities, let $[\alpha]$ be any square in $\Pi^{(2}X$ as in (1.14). Then, a relative homotopy



between $\alpha\circ_{\mathrm{h}}e^{\mathrm{h}}$ and α is given by the formula

$$H(x,y,t) = \begin{cases}
v\big(y-x\big) & \text{if} \quad x\leq y,\ x\leq\frac{1}{2}(1-t)(1+x-y), \\
u\big(x-y\big) & \text{if} \quad x\geq y,\ x\leq\frac{1}{2}(1-t)(1+x-y), \\
\alpha\big(\frac{x+y-1+t(1+x-y)}{1+t}, \frac{2y+t-1}{1+t}\big) & \text{if} \quad \frac{1}{2}(1-t)(1+x-y)\leq x\leq y, \\
\alpha\big(\frac{2x+t-1}{1+t}, \frac{x+y-1+t(1-x+y)}{1+t}\big) & \text{if} \quad \frac{1}{2}(1-t)(1-x-y)\leq y\leq x.
\end{cases}$$

Therefore, $[\alpha] \circ_{\mathrm{h}} 1^{\mathrm{h}}_{(v,u)} = [\alpha]$; and similarly we prove the remaining needed equalities:

$$[\alpha] = 1^{\mathrm{h}} \circ_{\mathrm{h}} [\alpha] = [\alpha] \circ_{\mathrm{v}} 1^{\mathrm{v}} = 1^{\mathrm{v}} \circ_{\mathrm{v}} [\alpha].$$

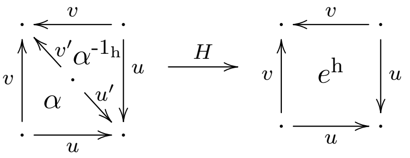
Let us now describe inverse squares in $\Pi^{(2}X$. For any given square $[\alpha]$ as in (1.14), its respective horizontal and vertical inverses

$$\begin{tikzcd} v & v' \arrow[l] & u' & u \arrow[l] \\ u \arrow[u] \arrow[ur, phantom, "{[\alpha]^{\text{-}1_{\mathrm{h}}}}"] & u' \arrow[l] \arrow[u], & v' \arrow[u] \arrow[ur, phantom, "{[\alpha]^{\text{-}1_{\mathrm{v}}}}"] & v \arrow[l] \arrow[u], \end{tikzcd}$$

are represented by the squares in X, $\alpha^{\text{-}1_{\mathrm{h}}}, \alpha^{\text{-}1_{v}} : I^2 \to X$, defined respectively by the formulas

$$\alpha^{\text{-}1_{\mathrm{h}}}(x,y) = \alpha(1-y, 1-x), \qquad \alpha^{\text{-}1_{\mathrm{v}}}(x,y) = \alpha(y,x).$$

The equality $[\alpha^{\text{-}1_{\mathrm{h}}}] \circ_{\mathrm{h}} [\alpha] = 1^{\mathrm{h}}_{(v,u)}$ holds, thanks to the homotopy



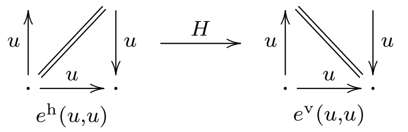
defined by

$$H(x,y,t) = \begin{cases} \alpha\big(2x(1-t),(1-2t)x+y\big) & \text{if} \quad x\le y,\ x+y\le 1, \\ \alpha\big(x+(1-2t)y,2y(1-t)\big) & \text{if} \quad x\ge y,\ x+y\le 1, \\ \alpha\big(2(ty-t-y+1),2(ty-t+1)-x-y\big) & \text{if} \quad x\le y,\ x+y\ge 1, \\ \alpha\big((2x-2)t+2-x-y,(2x-2)t+2-2x\big) & \text{if} \quad x\ge y,\ x+y\ge 1. \end{cases}$$

And, similarly, one sees the remaining equalities:

$$[\alpha] \circ_{\mathrm{h}} [\alpha]^{\text{-}1_{\mathrm{h}}} = 1^{\mathrm{h}},\ [\alpha] \circ_{\mathrm{v}} [\alpha]^{\text{-}1_{\mathrm{v}}} = 1^{\mathrm{v}},\ [\alpha]^{\text{-}1_{\mathrm{v}}} \circ_{\mathrm{v}} [\alpha] = 1^{\mathrm{v}}.$$

By construction of $\Pi^{(2}X$, conditions (i) and (ii) in **Axiom 1** are clearly satisfied. For (iii) in **Axiom 1**, we need to prove that, for any path $u : I \to X$, the equality $1^{\mathrm{h}}_{(u,u)} = 1^{\mathrm{v}}_{(u,u)}$ holds. This follows from the relative homotopy



defined by

$$H(x,y,t)=\begin{cases} u\big(y{-}x\big) & \text{if} \quad x{\leq}y,\ (1{-}t)(1{-}y){\geq}(1{+}t)x, \\ u\big(2y{-}1{+}t(1{+}x{-}y)\big) & \text{if} \quad x{\leq}y,\ x{+}y{\leq}1,\ (1{-}t)(1{-}y){\leq}(1{+}t)x, \\ u\big(x{-}y\big) & \text{if} \quad x{\geq}y,\ (1{-}t)(1{-}x){\geq}(1{+}t)y, \\ u\big(2x{-}1{+}t(1{-}x{+}y)\big) & \text{if} \quad x{\geq}y,\ x{+}y{\leq}1,\ (1{-}t)(1{-}x){\leq}(1{+}t)y, \\ u\big(1{-}2x{+}t(1{+}x{-}y)\big) & \text{if} \quad x{\leq}y,\ x{+}y{\geq}1,\ (1{+}t)(1{-}y){\geq}(1{-}t)x, \\ u\big(y{-}x\big) & \text{if} \quad x{\leq}y,\ x{+}y{\geq}1,\ (1{+}t)(1{-}y){\leq}(1{-}t)x, \\ u\big(1{-}2y{+}t(1{-}x{+}y)\big) & \text{if} \quad x{\geq}y,\ x{+}y{\geq}1,\ (1{+}t)(1{-}x){\geq}(1{-}t)y, \\ u\big(x{-}y\big) & \text{if} \quad x{\geq}y,\ (1{+}t)(1{-}x){\leq}(1{-}t)y. \end{cases}$$

The given definition of how squares in $\Pi^{(2}X$ compose makes the conditions (i) and (ii) in **Axiom 2** clear, and the remaining condition (iii) holds since, for any three paths $u,u',u'':I\to X$ with $u(1)=u'(1)=u''(1)$, there is a relative homotopy

$$\begin{tikzcd}
\cdot & & \cdot \arrow[ll, "u''"'] \arrow[dd, "u''"] & & & \cdot \arrow[ddrr, equal] & & \cdot \arrow[ll, "u''"'] \arrow[dd, "u''"] \\
& \cdot \arrow[ul, "u'"'] \arrow[dr, "u'"] & & {} \arrow[r, "H"] & {} & & & \\
\cdot \arrow[uu, "u"] \arrow[rr, "u"] & & \cdot & & & \cdot \arrow[uu, "u"] \arrow[rr, "u"] & & \cdot, \\
& e^{\mathrm{v}}(u'',u')\circ_{\mathrm{h}}e^{\mathrm{v}}(u',u) & & & & & e^{\mathrm{v}}(u'',u) &
\end{tikzcd}$$

defined by

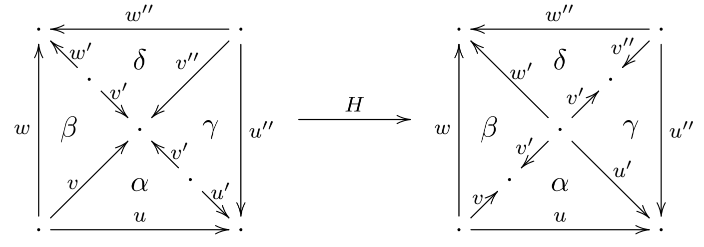
$$H(x,y,t)=\begin{cases} u\big(y{-}x{+}\frac{4x}{1+t}\big) & \text{if} \quad x{\leq}y,\ (1{+}t)(1{-}y){\geq}(3{-}t)x, \\ u'\big(2{-}3x{-}y{+}t(1{+}x{-}y)\big) & \text{if} \quad x{\leq}y,\ x{+}y{\leq}1,\ (1{+}t)(1{-}y){\leq}(3{-}t)x, \\ u\big(x{-}y{+}\frac{4y}{1+t}\big) & \text{if} \quad x{\geq}y,\ (1{+}t)(1{-}x){\geq}(3{-}t)y, \\ u'\big(2{-}x{-}3y{+}t(1{-}x{+}y)\big) & \text{if} \quad x{\geq}y,\ x{+}y{\leq}1,\ (1{+}t)(1{-}x){\leq}(3{-}t)y, \\ u'\big(x{+}3y{-}2{+}t(1{+}x{-}y)\big) & \text{if} \quad x{\leq}y,\ x{+}y{\geq}1,\ (3{-}t)(1{-}y){\geq}(1{+}t)x, \\ u''\big(y{-}x{+}\frac{4(y-1)}{1+t}\big) & \text{if} \quad x{\leq}y,\ (3{-}t)(1{-}y){\leq}(1{+}t)x, \\ u'\big(3x{+}y{-}2{+}t(1{-}x{+}y)\big) & \text{if} \quad x{\geq}y,\ x{+}y{\geq}1,\ (3{-}t)(1{-}x){\geq}(1{+}t)y, \\ u''\big(x{-}y{+}\frac{4(1-x)}{1+t}\big) & \text{if} \quad x{\geq}y,\ (3{-}t)(1{-}x){\leq}(1{+}t)y. \end{cases}$$

Whence $1^{\mathrm{v}}_{(u'',u')}\circ_{\mathrm{h}}1^{\mathrm{v}}_{(u',u)}=1^{\mathrm{v}}_{(u'',u)}$. Similarly, for any three paths in X, $u,v,w:I\to X$ with $u(0)=v(0)=w(0)$, one proves the equality $1^{\mathrm{h}}_{(w,v)}\circ_{\mathrm{v}}1^{\mathrm{h}}_{(v,u)}=1^{\mathrm{h}}_{(w,u)}$.

Then, it only remains to prove the interchange law in **Axiom 3**. To do so, let

$$\begin{tikzcd}
w'' & w' \arrow[l] & w \arrow[l] \\
v'' \arrow[u] \arrow[ur, phantom, "{[\delta]}"] & v' \arrow[l] \arrow[u] \arrow[ur, phantom, "{[\beta]}"] & v \arrow[l] \arrow[u] \\
u'' \arrow[u] \arrow[ur, phantom, "{[\gamma]}"] & u' \arrow[l] \arrow[u] \arrow[ur, phantom, "{[\alpha]}"] & u \arrow[l] \arrow[u]
\end{tikzcd}$$

be squares in $\Pi^{(2}X$. Then, the required equality follows from the existence of the relative homotopy



$(\delta\circ_{\mathrm{h}}\beta)\circ_{\mathrm{v}}(\gamma\circ_{\mathrm{h}}\alpha)$ $(\delta\circ_{\mathrm{v}}\gamma)\circ_{\mathrm{h}}(\beta\circ_{\mathrm{v}}\alpha)$

defined by the map $H : I^2 \times I \to X$ such that

$$H(x,y,t) =$$

- $\alpha\big(x+y-2ty,4y\big)$ if $1-x+2ty\geq 5y,\ x-3y\geq 2ty,$
- $\alpha\big(\frac{2(x-y)}{1+t}, \frac{2(x-tx+y+3ty)}{2+t-t^2}\big)$ if $2+t-t^2-6x+4tx+2y\geq 8ty,\ (3+2t)y\geq x\geq y,$
- $\alpha\big(\frac{t^2-2t+2x-2y+4ty}{1-2t+2t^2}, \frac{3t^2-t(1+4x)+2(x+y)}{2-4t+4t^2}\big)$ if $\begin{array}{l} t^2+t(4x-3)\geq 2(x+y-1),\\ t^2-2x+6y\geq t(4x+8y-3),\\ t^2+6x+t(8y-4x-1)\geq 2(1+y),\end{array}$
- $\alpha\big(\frac{t-2(x+y)}{t-2}, \frac{2t(x+3y-1)-8y}{t^2-t-2}\big)$ if $\begin{array}{l} 1\geq x+y,\ x-1\geq(2t-5)y,\\ 2x-t^2-6y\geq t(3-4x-8y),\end{array}$
- $v'\big(3-t-4x\big)$ if $x\geq y,\ 1\geq x+y,\ 2(x+y-1)\geq t^2+t(4x-3),$
- $\gamma\big(4x-3, x+y-1-2t(x-1)\big)$ if $5x+y-5\geq 2t(x-1), 2t(x-1)+3x\geq y+2,$
- $\gamma\big(\frac{6+t^2-8x+t(6x+2y-7)}{t^2-t-2}, \frac{2(x+y-1)}{2-t}\big)$ if $\begin{array}{l} x+y\geq 1, 5+2t(x-1)\geq 5x+y,\\ 9t+6x\geq 4+t^2+8tx+2y+4ty,\end{array}$
- $\gamma\big(\frac{t+t^2-4ty+2(x+y-1)}{2-4t+4t^2}, \frac{1+t^2+4t(x-1)-2x+2y}{1-2t+2t^2}\big)$ if $\begin{array}{l} t+t^2+2(x+y-1)\geq 4ty,\\ t^2+2(1+x-3y)\geq t(8x-y-3),\\ 4+t^2-6x+2y\geq t(9-8x-4y),\end{array}$
- $\gamma\big(\frac{t^2-2(x+y-1)+t(3-6x+2y)}{t^2-t-2}, \frac{1+t-2x+2y}{1+t}\big)$ if $\begin{array}{l} 8tx+6y-4ty\geq 2+3t+t^2+2x,\\ x\geq y,\ 2+y\geq 2t(x-1)+3x,\end{array}$
- $v'\big(4y-1-t\big)$ if $x\geq y,\ x+y\geq 1, 2+4ty\geq t+t^2+2x+2y,$
- $\beta\big(4x,x+y-2tx\big)$ if $1+2tx\geq y+5x,\ y\geq 3x+2tx,$
- $\beta\big(\frac{2t(y+3x-1)-8x}{t^2-t-2}, \frac{t-2(x+y)}{t-2}\big)$ if $\begin{array}{l} 1\geq x+y,\ y+(5-2t)x\geq 1,\\ 2y+t(3-4y-8x)-6x\geq t^2,\end{array}$
- $\beta\big(\frac{3t^2-t(1+4y)+2(x+y)}{2-4t+4t^2}, \frac{t^2-2t+2y-2x+4tx}{1-2t+2t^2}\big)$ if $\begin{array}{l} t^2+t(4y-3)\geq 2(x+y-1),\\ t^2+t(3-4y-8x)+6x\geq 2y,\\ t^2+6y-2(1+x)\geq t(1+4y-8x),\end{array}$
- $\beta\big(\frac{2(y-ty+x+3tx)}{2+t-t^2}, \frac{2(y-x)}{1+t}\big)$ if $2+t+4ty+2x\geq t^2+6y+8tx, (3+2t)x\geq y\geq x,$
- $v'\big(3-t-4y\big)$ if $y\geq x,\ 1\geq x+y,\ 2(x+y-1)\geq t^2+t(3-4y),$
- $\delta\big(2t(1-y)+x+y-1,4y-3\big)$ if $5y+x\geq 5+2t(y-1),\ 2t(y-1)+3y\geq x+2,$
- $\delta\big(\frac{2(x+y-1)}{2-t}, \frac{6+t^2-8y+t(6y+2x-7)}{t^2-t-2}\big)$ if $\begin{array}{l} x+y\geq 1, 9t+6y-8ty-2x-4tx\geq 4+t^2,\\ 5+2t(y-1)\geq 5y+x,\end{array}$

- $\delta(\frac{1+t^2+4t(y-1)-2y+2x}{1-2t+2t^2}, \frac{t+t^2-4tx+2(x+y-1)}{2-4t+4t^2})$ if $t+t^2-4tx\geq 2(1-x-y)$, $t^2+2(1+y-3x)\geq t(8y-4x-3)$, $4+t^2-6y+2x\geq t(9-8y-4x)$,
- $\delta(\frac{1+t-2y+2x}{1+t}, \frac{t^2-2(x+y-1)+t(3-6y+2x)}{t^2-2-t})$ if $8ty+6x-4tx\geq 2+3t+t^2+2y$, $y\geq x, 2-3y+x\geq 2t(y-1)$,
- $v'(4y-1-t)$ if $y\geq x,\ x+y\geq 1, 2+4tx\geq t+t^2+2y+2x$.

Finally, we observe that $\Pi^{(2}X$ satisfies the filling condition. Suppose a configuration of morphisms in $\Pi^{(2}X$

$$\begin{tikzcd} v' & v \arrow[l] \\ & u \arrow[u] \end{tikzcd}$$

is given. This means we have paths $u, v, v' : I \to X$ with $u(0) = v(0)$ and $v(1) = v'(1)$. Since the inclusion $\partial I \hookrightarrow I$ is a cofibration, the map $f : (\{0\}\times I)\cup(I\times\partial I) \to X$ with $f(0,t) = v(t)$, $f(t,0) = u(t)$ and $f(t,1) = v'(1-t)$ for $0 \leq t \leq 1$, has an extension to a map $\alpha : I \times I \to X$, which precisely represents a square in $\Pi^{(2}X$ of the form

$$\begin{tikzcd} v' & v \arrow[l] \\ u' \arrow[u] & u \arrow[l] \arrow[u] \arrow[ul, phantom, "{[\alpha]}"] \end{tikzcd}$$

where $u' : I \to X$ is the path $u'(t) = \alpha(1, 1-t)$. Hence, $\Pi^{(2}X$ verifies the filling condition. $\square$

In the previous Section 1.3 we introduced homotopy groups for double groupoids satisfying the filling condition. The next proposition provides greater specifics on the relationship between the homotopy groups of the associated homotopy double groupoid $\Pi^{(2}X$ to a topological space X and the corresponding for X.

Theorem 1.2 *For any space X, any path $u : I \to X$, and $0 \leq i \leq 2$, there is an isomorphism*

$$\pi_i(\Pi^{(2}X, u) \cong \pi_i(X, u(0)).$$

Proof: For any two points $x, y \in X$, the constant paths c_x and c_y are in the same connected component of $\Pi^{(2}X$ if and only if there is a pair of morphisms in $\Pi^{(2}X$ of the form

$$\begin{tikzcd} c_y & u \arrow[l] \\ & c_x \arrow[u] \end{tikzcd}$$

or, equivalently, if and only if there is a path $u : I \to X$ in X such that $u(1) = y$ and $u(0) = x$. Then, we have an injective map

$$\pi_0 X \to \pi_0\Pi^{(2}X, \qquad [x] \mapsto [c_x],$$

which is also surjective since, for any path u in X, we have a vertical morphism $u \leftarrow c_{u(0)}$ in $\Pi^{(2}X$; whence the announced bijection $\pi_0 X \cong \pi_0\Pi^{(2}X$.

Next, we prove that there is an isomorphism $\pi_1(\Pi^{(2}X, u) \cong \pi_1(X, u(0))$ for any given path $u : I \to X$. To do so, we shall use the fundamental groupoid ΠX of the space X; that is, the groupoid whose objects are the points of X and whose morphisms are the (relative to ∂I) homotopy classes $[v]$ of paths $v : I \to X$. Simply by checking the construction, we see that an element $[(u, v), (v, u)] \in \pi_1(\Pi^{(2}X, u)$ is determined by a path $v : I \to X$, with $v(0) = u(0)$ and $v(1) = u(1)$. Moreover, for any other such $v' : I \to X$, it holds that $[(u, v), (v, u)] = [(u, v'), (v', u)]$ in $\pi_1(\Pi^{(2}X, u)$ if and only if there are squares in $\Pi^{(2}X$ of the form

$$\begin{tikzcd} u & v \arrow[l] \\ w \arrow[u] & u \arrow[l] \arrow[u] \arrow[ul, phantom, "{[\alpha]}"] \end{tikzcd} \qquad \begin{tikzcd} u & v' \arrow[l] \\ w \arrow[u] & u \arrow[l] \arrow[u] \arrow[ul, phantom, "{[\alpha']}"] \end{tikzcd}$$

or, equivalently, if and only if there are squares in X, $\alpha, \alpha' : I^2 \to X$ with boundaries as in

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "u"'] \arrow[d, "w"] \\ \cdot \arrow[u, "v"] \arrow[r, "u"'] \arrow[ur, phantom, "\alpha"] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot & \cdot \arrow[l, "u"'] \arrow[d, "w"] \\ \cdot \arrow[u, "v'"] \arrow[r, "u"'] \arrow[ur, phantom, "\alpha'"] & \cdot \end{tikzcd}$$

Since this last condition simply means that, in the fundamental groupoid ΠX, the equality $[v] = [v']$ holds, we conclude with bijections

$$\begin{tikzcd} \pi_1(\Pi^{(2}X, u) \arrow[r, phantom, "\cong"] & \mathrm{Hom}_{\Pi X}(u(0), v(1)) \arrow[r, phantom, "\cong"] & \pi_1(X, u(0)) \\ {[(u,v),(v,u)]} \arrow[r, mapsto] & {[v]} \arrow[r, mapsto] & {[u]^{-1} \circ [v]} \end{tikzcd}$$

To see that the composite bijection $\phi : [(u, v), (v, u)] \mapsto [u]^{-1} \circ [v]$ is actually an isomorphism, let $v_1, v_2 : I \to X$ be paths in X, both from $u(0)$ to $u(1)$. Then, $[(u, v_1), (v_1, u)] \circ [(u, v_2), (v_2, u)] = [(u, v), (v, u)]$, where v occurs in a configuration such as

$$\begin{tikzcd} u & v_1 \arrow[l] & v \arrow[l] \\ & u \arrow[u] & v_2 \arrow[l] \arrow[u] \arrow[ul, phantom, "{[\gamma]}"] \\ & & u \arrow[u] \end{tikzcd}$$

for some (any) square $\gamma : I^2 \to X$ in X with boundary as below.

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "v_1"'] \arrow[d, "u"] \\ \cdot \arrow[u, "v"] \arrow[r, "v_2"'] \arrow[ur, phantom, "\gamma"] & \cdot \end{tikzcd}$$

It follows that, in ΠX, $[v] = [v_1] \circ [u]^{-1} \circ [v_2]$ and therefore

$$\begin{aligned} \phi[(u, v_1), (v_1, u)] \circ \phi[(u, v_2), (v_2, u)] &= [u]^{-1} \circ [v_1] \circ [u]^{-1} \circ [v_2] = [u]^{-1} \circ [v] \\ &= \phi([(u, v_1), (v_1, u)] \circ [(u, v_2), (v_2, u)]). \end{aligned}$$

Finally, we consider the case $i = 2$. Let $u : I \to X$ be any path with $u(0) = x$. Then, the mapping $[\alpha] \mapsto 1^{\mathrm{h}}_{(c_x,u)} \circ_{\mathrm{v}} [\alpha] \circ_{\mathrm{v}} 1^{\mathrm{h}}_{(u,c_x)}$, which carries a square $[\alpha] \in \pi_2(\Pi^{(2}X, u)$, to the composite of

$$\begin{tikzcd}
c_x \arrow[r, equal] & c_x \\
u \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "{[e^{\mathrm{h}}(c_x,u)]}"] & u \arrow[u] \\
u \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "{[\alpha]}"] & u \arrow[u, equal] \\
c_x \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "{[e^{\mathrm{h}}(u,c_x)]}"] & c_x \arrow[u]
\end{tikzcd}$$

establishes an isomorphism $\pi_2(\Pi^{(2}X, u) \cong \pi_2(\Pi^{(2}X, c_x)$. Now, it is clear that both $\pi_2(\Pi^{(2}X, c_x)$ and $\pi_2(X, x)$ are the same abelian group of relative to ∂I^2 homotopy classes of maps $I^2 \to X$ which are constant x along the four sides of the square. $\square$

The construction of the double groupoid $\Pi^{(2}X$ from a space X is easily seen to be functorial and, moreover, the isomorphisms in Theorem 1.2 above become natural. Then, we have the next corollary.

Corollary 1.1 *A continuous map $f : X \to Y$ is a weak homotopy 2-equivalence if and only if the induced double functor $\Pi^{(2}f : \Pi^{(2}X \to \Pi^{(2}Y$ is a weak equivalence.*

1.5 The geometric realization of a double groupoid.

If $\mathcal{A}$ and $\mathcal{B}$ are categories, then let $\mathcal{A}\tilde{\times}\mathcal{B}$ denote the double category whose objects are pairs (a, b), where a is an object of $\mathcal{A}$ and b is an object of $\mathcal{B}$; horizontal morphisms are pairs $(f, b) : (a, b) \to (c, b)$, with $f : a \to c$ a morphism in $\mathcal{A}$; vertical morphisms are pairs $(a, u) : (a, b) \to (a, d)$ with $u : b \to d$ in $\mathcal{B}$; and a square in $\mathcal{A}\tilde{\times}\mathcal{B}$ is given by each morphism $(f, u) : (a, b) \to (c, d)$ in the product category $\mathcal{A} \times \mathcal{B}$, by stating its boundary as in

$$\begin{tikzcd}
(c,d) & (a,d) \arrow[l, "{(f,d)}"'] \\
(c,b) \arrow[u, "{(c,u)}"] \arrow[ur, phantom, "{(f,u)}"] & (a,b) \arrow[l, "{(f,b)}"] \arrow[u, "{(a,u)}"']
\end{tikzcd}$$

Compositions in $\mathcal{A}\tilde{\times}\mathcal{B}$ are defined in the evident way.

Hereafter, we shall regard each ordered set $[n]$ as the category with exactly one arrow $j \to i$ when $0 \le i \le j \le n$. Then, a non-decreasing map $[n] \to [m]$ is the same as a functor.

The geometric realization, or classifying space, of a category $\mathcal{C}$, [109], is $\mathrm{B}\mathcal{C} := |\mathrm{N}\mathcal{C}|$, the geometric realization of its *nerve* [73]

$$\mathrm{N}\mathcal{C} : \Delta^{\mathrm{op}} \to \mathbf{Set}, \qquad [n] \mapsto \mathrm{Func}([n], \mathcal{C}),$$

that is, the simplicial set whose n-simplices are the functors $F : [n] \to \mathcal{C}$, or tuples of arrows in $\mathcal{C}$

$$F = \big(F_i \xleftarrow{F_{i,j}} F_j\big)_{0\leq i\leq j\leq n}$$

such that $F_{i,j} \circ F_{j,k} = F_{i,k}$ and $F_{i,i} = 1_{F_i}$. If $\mathcal{G}$ is a double category, then its geometric realization, $\mathrm{B}\mathcal{G}$, is

$$\mathrm{B}\mathcal{G} := |\mathrm{N}^{(2}\mathcal{G}|,$$

the geometric realization of its *double nerve*

$$\mathrm{N}^{(2}\mathcal{G} : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}, \quad ([p],[q]) \mapsto \mathrm{DFunc}([p]\widetilde{\times}[q], \mathcal{G}),$$

that is, the bisimplicial set whose (p,q)-bisimplices are the double functors $F : [p]\widetilde{\times}[q] \to \mathcal{G}$ or configurations of squares in $\mathcal{G}$ of the form

$$\left(\begin{tikzcd} F^r_i & F^r_j \arrow[l, "F^r_{i,j}"'] \\ F^s_i \arrow[u, "F^{r,s}_i"] \arrow[ur, phantom, "F^{r,s}_{i,j}"] & F^s_j \arrow[l, "F^s_{i,j}"] \arrow[u, "F^{r,s}_j"'] \end{tikzcd}\right)_{\substack{0\leq i\leq j\leq p\\ 0\leq r\leq s\leq q}},$$

such that $F^{r,s}_{i,j} \circ_{\mathrm{h}} F^{r,s}_{j,k} = F^{r,s}_{i,k}$, $F^{r,s}_{i,j} \circ_{\mathrm{v}} F^{s,t}_{i,j} = F^{r,t}_{i,j}$, $F^{r,s}_{i,i} = 1^{\mathrm{h}}_{F^{r,s}_i}$, and $F^{r,r}_{i,j} = 1^{\mathrm{v}}_{F^r_{i,j}}$.

Note that the double category $[p]\widetilde{\times}[q]$ is free on the bigraph

$$\left(\begin{tikzcd} (j-1,r-1) & (j,r-1) \arrow[l] \\ (j-1,r) \arrow[u] & (j,r) \arrow[l] \arrow[u] \end{tikzcd}\right)_{\substack{0\leq i\leq j\leq p\\ 0\leq r\leq s\leq q}},$$

and therefore, giving a double functor $F : [p]\widetilde{\times}[q] \to \mathcal{G}$ as above is equivalent to specifying the $p \times q$ configuration of squares in $\mathcal{G}$

$$\left(\begin{tikzcd} F^{r-1}_{j-1} & F^{r-1}_j \arrow[l, "F^{r-1}_{j-1,j}"'] \\ F^r_{j-1} \arrow[u, "F^{r-1,r}_{j-1}"] \arrow[ur, phantom, "F^{r-1,r}_{j-1,j}"] & F^r_j \arrow[l, "F^r_{j-1,j}"] \arrow[u, "F^{r-1,r}_j"'] \end{tikzcd}\right)_{\substack{1\leq j\leq p\\ 1\leq r\leq q}}.$$

Thus, each vertical simplicial set $\mathrm{N}^{(2}\mathcal{G}_{p,*}$ is the nerve of the "vertical" category having as objects strings of p-composable horizontal morphisms $a_0 \leftarrow a_1 \leftarrow \cdots \leftarrow a_p$, whose arrows consist of p horizontally composable squares as in

$$\begin{tikzcd} b_0 & b_1 \arrow[l] & \cdot \arrow[l] \arrow[r, phantom, "\cdots"] & \cdot & b_p \arrow[l] \\ a_0 \arrow[u] & a_1 \arrow[l] \arrow[u] & \cdot \arrow[l] \arrow[u] \arrow[r, phantom, "\cdots"] & \cdot \arrow[u] & a_p \arrow[l] \arrow[u] \end{tikzcd}$$

And, similarly, each horizontal simplicial set $\mathrm{N}^{(2}\mathcal{G}_{*,q}$ is the nerve of the "horizontal" category whose objects are the length q sequences of composable vertical morphisms of $\mathcal{G}$, with length q sequences of vertically composable squares as morphisms between them.

For instance, if $\mathcal{A}$ and $\mathcal{B}$ are categories, then $\mathrm{N}^{(2}(\mathcal{A}\tilde{\times}\mathcal{B}) = \mathrm{N}\mathcal{A}\tilde{\times}\mathrm{N}\mathcal{B}$. In particular,

$$\mathrm{N}^{(2}([p]\tilde{\times}[q]) = \Delta[p]\tilde{\times}\Delta[q] = \Delta[p,q],$$

is the standard (p,q)-bisimplex.

It is a well-known fact that the nerve $\mathrm{N}\mathcal{C}$ of a category $\mathcal{C}$ satisfies the Kan extension condition if and only if $\mathcal{C}$ is a groupoid, and, in such a case, every (k,n)-horn $\Lambda^k[n] \to \mathrm{N}\mathcal{C}$, for $n \geq 2$, has a unique extension to an n-simplex of $\mathrm{N}\mathcal{C}$

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & \mathrm{N}\mathcal{C} \\
\Delta[n] \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$

(see [85, Propositions 2.6.1], for example). For double categories $\mathcal{G}$, we have the following:

Theorem 1.3 *Let $\mathcal{G}$ be a double category. The following statements are equivalent:*

(i) $\mathcal{G}$ is a double groupoid satisfying the filling condition.

(ii) The bisimplicial set $\mathrm{N}^{(2}\mathcal{G}$ satisfies the extension condition.

(iii) The simplicial set $\mathrm{diag}\mathrm{N}^{(2}\mathcal{G}$ is a Kan complex.

Proof: (i)⇒ (ii) Since $\mathcal{G}$ is a double groupoid, all simplicial sets $\mathrm{N}^{(2}\mathcal{G}_{p,*}$ and $\mathrm{N}^{(2}\mathcal{G}_{*,q}$ are nerves of groupoids. Therefore, every extension problem of the form

$$\begin{tikzcd}
\Delta[p]\tilde{\times}\Lambda^l[q] \arrow[r] \arrow[d, hook] & \mathrm{N}^{(2}\mathcal{G} \\
\Delta[p,q] \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
\Lambda^k[p]\tilde{\times}\Delta[q] \arrow[r] \arrow[d, hook] & \mathrm{N}^{(2}\mathcal{G} \\
\Delta[p,q] \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$

has a solution and it is unique. Suppose then an extension problem of the form

$$\begin{tikzcd}
\Lambda^{k,l}[p,q] \arrow[r] \arrow[d, hook] & \mathrm{N}^{(2}\mathcal{G} \\
\Delta[p,q] \arrow[ur, dotted] &
\end{tikzcd} \tag{1.17}$$

If $p \geq 2$, then the restricted map $\Lambda^k[p]\tilde{\times}\Delta[q] \hookrightarrow \Lambda^{k,l}[p,q] \to \mathrm{N}^{(2}\mathcal{G}$ has a unique extension to a bisimplex $\Delta[p,q] \to \mathrm{N}^{(2}\mathcal{G}$, which is a solution to (1.17) (which in fact has a unique solution if $p \geq 2$ or $q \geq 2$). Hence, we reduce the proof to the case in which $p = 1 = q$, with the four possibilities $k = 0, 1$ and $l = 0, 1$. But any such extension problem has a solution thanks to Lemma 1.4. For example, let us discuss the case $k = 0 = l$: A bisimplicial map $\Lambda^{0,0}[1,1] \xrightarrow{(-,w;-,g)} \mathrm{N}^{(2}\mathcal{G}$ consists of two bisimplicial maps $w : \Delta[0,1] \to \mathrm{N}^{(2}\mathcal{G}$ and $g : \Delta[1,0] \to \mathrm{N}^{(2}\mathcal{G}$, such that $wd_v^1 = gd_h^1$. That is, a vertical morphism w of $\mathcal{G}$ and a horizontal morphism g of $\mathcal{G}$, such that both have the same target. By Lemma 1.4, there is a square α in $\mathcal{G}$ of the form

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "g"'] \\
\cdot \arrow[u, "w"] \arrow[ur, phantom, "\alpha"] & \cdot \arrow[l, dotted] \arrow[u, dotted]
\end{tikzcd}$$

which defines a bisimplicial map $F : \Delta[1,1] \to \mathrm{N}^{(2}\mathcal{G}$ such that $F_{0,1}^{0,1} = \alpha$. Then $Fd_h^1 = w$, $Fd_v^1 = g$, and the diagram below commutes, as required.

$$\begin{tikzcd}
\Lambda^{0,0}[1,1] \arrow[r, "{(-,w;-,g)}"] \arrow[d, hook] & \mathrm{N}^{(2}\mathcal{G} \\
\Delta[1,1] \arrow[ur, "F"'] &
\end{tikzcd}$$

(ii) ⇒ (i) The simplicial sets $\mathrm{N}^{(2}\mathcal{G}_{0,*}$, $\mathrm{N}^{(2}\mathcal{G}_{*,0}$, $\mathrm{N}^{(2}\mathcal{G}_{1,*}$, and $\mathrm{N}^{(2}\mathcal{G}_{*,1}$ are respectively the nerves of the four component categories of the double category $\mathcal{G}$. Since all these simplicial sets satisfy the Kan extension condition, it follows that the four category structures involved are groupoids; that is, $\mathcal{G}$ is a double groupoid. Furthermore, for any given filling problem in $\mathcal{G}$,

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "g"'] \\
\cdot \arrow[u, dotted] \arrow[ur, phantom, "\exists?"] & \cdot \arrow[l, dotted] \arrow[u, "u"']
\end{tikzcd}$$

we can solve the extension problem

$$\begin{tikzcd}
\Lambda^{1,0}[1,1] \arrow[r, "{(u,-;-,g)}"] \arrow[d, hook] & \mathrm{N}^{(2}\mathcal{G} \\
\Delta[1,1] \arrow[ur, dotted, "F"'] &
\end{tikzcd}$$

and the square $F_{0,1}^{0,1}$ has u as horizontal source and g as vertical target. Thus $\mathcal{G}$ satisfies the filling condition.

(i) ⇒ (iii) The higher dimensional part of the proof is in the following lemma, that we establish for future reference.

Lemma 1.8 *If $\mathcal{G}$ is any double groupoid and n is any integer such that $n \geq 3$, then every extension problem*

$$\begin{tikzcd}
\Lambda^{k}[n] \arrow[r] \arrow[d, hook] & \mathrm{diag}\mathrm{N}^{(2}\mathcal{G} \\
\Delta[n] \arrow[ur, dotted, "\exists?"'] &
\end{tikzcd}$$

has a solution and it is unique.

Proof: Let $F = (F^{r,s}_{i,j}) : [n]\widetilde{\times}[n] \to \mathcal{G}$ denote the double functor we are looking for solving the given extension problem. Recall that to give such an F is equivalent to specifying the $n \times n$ configuration of squares

$$\left(\begin{tikzcd} F^{r-1}_{j-1} \arrow[d, phantom] & F^{r-1}_{j} \arrow[l, "F^{r-1}_{j-1,j}"'] \\ F^{r}_{j-1} \arrow[u, "F^{r-1,r}_{j-1}"] & F^{r}_{j} \arrow[l, "F^{r}_{j-1,j}"] \arrow[u, "F^{r-1,r}_{j}"'] \arrow[ul, phantom, "F^{r-1,r}_{j-1,j}"] \end{tikzcd}\right)_{\substack{1 \le j \le n \\ 1 \le r \le n}}.$$

We claim that F exists and, moreover, that it is completely determined by any three of its (known) faces $[n-1]\widetilde{\times}[n-1] \xrightarrow{d^m\widetilde{\times}d^m} [n]\widetilde{\times}[n] \xrightarrow{F} \mathcal{G}$, $m \neq k$; therefore, by the input data $\Lambda^k[n] \to \mathrm{diagN}^{(2}\mathcal{G}$. In effect, since each m^{th}-face consists of all squares $F^{r,s}_{i,j}$ such that $m \notin \{i, j, r, s\}$, once we have selected any three integers m, p, q with $m < p < q$ and $k \notin \{m, p, q\}$, we know explicitly all squares $F^{r,s}_{i,j}$ except those in which m, p and q appear in the labels, that is: $F^{m,p}_{q,j}$, $F^{m,q}_{p,j}$, $F^{j,p}_{m,q}$, and so on. In the case where $k \geq 3$, if we take $\{m, p, q\} = \{0, 1, 2\}$ then we have given all squares $F^{r,s}_{i,j}$, except those with $\{0, 1, 2\} \subseteq \{r, s, i, j\}$. In particular, we have all $F^{r,r+1}_{i,i+1}$, except four of them, namely, $F^{0,1}_{2,3}$, $F^{0,1}_{1,2}$, $F^{1,2}_{0,1}$, and $F^{2,3}_{0,1}$, which, however, are uniquely determined by the equations

$$F^{0,1}_{2,3} \circ_{\mathrm{v}} F^{1,2}_{2,3} = F^{0,2}_{2,3},\ F^{2,3}_{0,1} \circ_{\mathrm{h}} F^{2,3}_{1,2} = F^{2,3}_{0,2},\ F^{0,1}_{1,2} \circ_{\mathrm{h}} F^{0,1}_{2,3} = F^{0,1}_{1,3},\ F^{1,2}_{0,1} \circ_{\mathrm{v}} F^{2,3}_{0,1} = F^{1,3}_{0,1},$$

that is, $F^{0,1}_{2,3} = F^{0,2}_{2,3} \circ_{\mathrm{v}} (F^{1,2}_{2,3})^{-1_{\mathrm{v}}}$, and so on. The other possibilities for k are discussed in a similar way: If $k = 2$, then we select $\{m, p, q\} = \{0, 1, n\}$ and determine F completely by taking into account the two equations $F^{0,1}_{n-1,n} \circ_{\mathrm{v}} F^{1,2}_{n-1,n} = F^{0,2}_{n-1,n}$, $F^{n-1,n}_{0,1} \circ_{\mathrm{h}} F^{n-1,n}_{1,2} = F^{n-1,n}_{0,2}$.

If $k = 1$, then we take $\{m, p, q\} = \{0, 2, 3\}$ and find the unknown squares $F^{2,3}_{0,1}$ and $F^{0,1}_{2,3}$ by the equations $F^{1,2}_{0,1} \circ_{\mathrm{v}} F^{2,3}_{0,1} = F^{1,3}_{0,1}$ and $F^{0,1}_{1,2} \circ_{\mathrm{h}} F^{0,1}_{2,3} = F^{0,1}_{1,3}$, respectively.

Finally, in the case where $k = 0$, we take $\{m, p, q\} = \{n-2, n-1, n\}$ and we find the non-given four squares of the family $(F^{r,r+1}_{i,i+1})$, that is, $F^{n-2,n-1}_{n-1,n}$, $F^{n-1,n}_{n-2,n-1}$, $F^{n-3,n-2}_{n-1,n}$, and $F^{n-1,n}_{n-3,n-2}$ by means of the four equations $F^{n-3,n-2}_{n-3,n-1} \circ_{\mathrm{h}} F^{n-3,n-2}_{n-1,n} = F^{n-3,n-2}_{n-3,n}$, $F^{n-3,n-1}_{n-3,n-2} \circ_{\mathrm{v}} F^{n-1,n}_{n-3,n-2} = F^{n-3,n}_{n-3,n-2}$, $F^{n-3,n-2}_{n-1,n} \circ_{\mathrm{v}} F^{n-2,n-1}_{n-1,n} = F^{n-3,n-1}_{n-1,n}$, and $F^{n-1,n}_{n-3,n-2} \circ_{\mathrm{h}} F^{n-1,n}_{n-2,n-1} = F^{n-1,n}_{n-3,n-1}$. This completes the proof of the lemma. □

We now return to the proof of (i) ⇒ (iii) in Theorem 1.3. Following on from Lemma 1.8 above, it remains to prove that every extension problem

$$\begin{tikzcd} \Lambda^k[2] \arrow[r] \arrow[d, hook] & \mathrm{diagN}^{(2}\mathcal{C} \\ \Delta[2] \arrow[ur, dotted, "\exists ?"'] & \end{tikzcd}$$

for $k = 0, 1, 2$, has a solution. In the case where $k = 0$, the data for a simplicial map $(-, \tau, \sigma) : \Lambda^0[2] \to \mathrm{diagN}^{(2}\mathcal{G}$ consists of a couple of squares in $\mathcal{G}$ of the form

$$\begin{tikzcd}
a & \cdot \arrow[l] & & a & \cdot \arrow[l] \\
\cdot \arrow[u] & \cdot \arrow[l] \arrow[u] \arrow[ul, phantom, "\sigma"] & & \cdot \arrow[u] & \cdot \arrow[l] \arrow[u] \arrow[ul, phantom, "\tau"]
\end{tikzcd}$$

and an extension solution $\Delta[2] \dashrightarrow \mathrm{diagN}^{(2}\mathcal{G}$ amounts to a diagram of squares as in

$$\begin{tikzcd}
a & \cdot \arrow[l] & \cdot \arrow[l, dotted] \\
\cdot \arrow[u] & \cdot \arrow[l] \arrow[u] \arrow[ul, phantom, "\sigma"] & \cdot \arrow[l, dotted] \arrow[u, dotted] \arrow[ul, phantom, "x"] \\
\cdot \arrow[u, dotted] & \cdot \arrow[l, dotted] \arrow[u, dotted] \arrow[ul, phantom, "y"] & \cdot \arrow[l, dotted] \arrow[u, dotted] \arrow[ul, phantom, "z"]
\end{tikzcd}$$

such that $(\sigma \circ_{\mathrm{h}} x) \circ_{\mathrm{v}} (y \circ_{\mathrm{h}} z) = \tau$. To see that such squares x, y, and z exist, we first select squares α and β such that $\mathrm{t}^{\mathrm{h}}\alpha = \mathrm{s}^{\mathrm{h}}\sigma^{-1}$, $\mathrm{s}^{\mathrm{v}}\alpha = \mathrm{t}^{\mathrm{v}}\tau$, $\mathrm{s}^{\mathrm{h}}\beta = \mathrm{t}^{\mathrm{h}}\tau$, and $\mathrm{t}^{\mathrm{v}}\beta = \mathrm{s}^{\mathrm{v}}\sigma^{-1}$. The filling condition in $\mathcal{G}$ assures that these α and β can be found. The squares σ, $\sigma^{-1_{\mathrm{h}}}$,$\sigma^{-1_{\mathrm{v}}}$, σ^{-1}, α, β, $\alpha^{-1_{\mathrm{v}}}$, $\beta^{-1_{\mathrm{h}}}$, and τ fit together in the configuration (actually, a 3-simplex of $\mathrm{diagN}^{(2}\mathcal{G}$)

$$\begin{tikzcd}
a & \cdot \arrow[l] & \cdot \arrow[l] & \cdot \arrow[l] \\
\cdot \arrow[u] & \cdot \arrow[l] \arrow[u] \arrow[ul, phantom, "\sigma"] & \cdot \arrow[l] \arrow[u] \arrow[ul, phantom, "\sigma^{-1_{\mathrm{h}}}"] & \cdot \arrow[l, dotted] \arrow[u, dotted] \arrow[ul, phantom, "\alpha^{-1_{\mathrm{v}}}"] \\
\cdot \arrow[u] & \cdot \arrow[l] \arrow[u] \arrow[ul, phantom, "\sigma^{-1_{\mathrm{v}}}"] & a \arrow[l] \arrow[u] \arrow[ul, phantom, "\sigma^{-1}"] & \cdot \arrow[l] \arrow[u, dotted] \arrow[ul, phantom, "\alpha"] \\
\cdot \arrow[u, dotted] & \cdot \arrow[l, dotted] \arrow[u, dotted] \arrow[ul, phantom, "\beta^{-1_{\mathrm{h}}}"] & \cdot \arrow[l, dotted] \arrow[u] \arrow[ul, phantom, "\beta"] & \cdot \arrow[l] \arrow[u] \arrow[ul, phantom, "\tau"]
\end{tikzcd}$$

and then, we take $x = \sigma^{-1_{\mathrm{h}}} \circ_{\mathrm{h}} \alpha^{-1_{\mathrm{v}}}$, $y = \sigma^{-1_{\mathrm{v}}} \circ_{\mathrm{v}} \beta^{-1_{\mathrm{h}}}$, and $z = (\sigma^{-1} \circ_{\mathrm{h}} \alpha) \circ_{\mathrm{v}} (\beta \circ_{\mathrm{h}} \tau)$.

The case in which $k = 2$ is dual of the case $k = 0$ above, and the case when $k = 1$ is easier: A simplicial map $(\sigma, -, \tau) : \Lambda^1[2] \to \mathrm{diagN}^{(2}\mathcal{G}$ amounts to a couple of squares in $\mathcal{G}$ of the form

$$\begin{tikzcd}
a & \cdot \arrow[l] & & \cdot & \cdot \arrow[l] \\
\cdot \arrow[u] & \cdot \arrow[l] \arrow[u] \arrow[ul, phantom, "\sigma"] & & \cdot \arrow[u] & a \arrow[l] \arrow[u] \arrow[ul, phantom, "\tau"]
\end{tikzcd}$$

and an extension solution $\Delta[2] \dashrightarrow \mathrm{diagN}^{(2}\mathcal{G}$ is given by any configuration of squares in $\mathcal{G}$ of the form

$$\begin{tikzcd}
\cdot & \cdot \arrow[l] & \cdot \arrow[l, dotted] \\
\cdot \arrow[u] & a \arrow[l] \arrow[u] \arrow[ul, phantom, "\tau"] & \cdot \arrow[l] \arrow[u, dotted] \arrow[ul, phantom, "x"] \\
\cdot \arrow[u, dotted] & \cdot \arrow[l, dotted] \arrow[u] \arrow[ul, phantom, "y"] & \cdot \arrow[l] \arrow[u] \arrow[ul, phantom, "\sigma"]
\end{tikzcd}$$

Since $\mathcal{G}$ satisfies the filling condition (recall Lemma 1.4), it is clear that filling squares x and y as above exist, and therefore the required extension map exists.

(iii) ⇒ (i) By [51, Theorem 8], all simplicial sets $\mathrm{N}^{(2}\mathcal{G}_{p,*}$ and $\mathrm{N}^{(2}\mathcal{G}_{*,q}$ satisfy the Kan extension condition. In particular, the nerves of the four component categories of the double category $\mathcal{G}$, that is, the simplicial sets $\mathrm{N}^{(2}\mathcal{G}_{0,*}$, $\mathrm{N}^{(2}\mathcal{G}_{*,0}$,$\mathrm{N}^{(2}\mathcal{G}_{1,*}$, and $\mathrm{N}^{(2}\mathcal{G}_{*,1}$ are all Kan complexes. By [85, Propositions 2.6.1], it follows that the four category structures involved are groupoids, and so $\mathcal{G}$ is a double groupoid.

To see that $\mathcal{G}$ satisfies the filling condition, suppose that a filling problem

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "g"'] \\
\cdot \arrow[u, dotted] \arrow[ur, phantom, "\exists?"] & \cdot \arrow[l, dotted] \arrow[u, "u"']
\end{tikzcd}$$

is given. Since the simplicial map $\Lambda^1[2] \xrightarrow{({1^{\mathrm{h}}}_u, -, {1^{\mathrm{v}}}_g)} \mathrm{diagN}^{(2}\mathcal{C}$ has an extension to a 2-simplex $\Delta[2] \dashrightarrow \mathrm{diagN}^{(2}\mathcal{G}$, we conclude the existence of a diagram of squares in $\mathcal{G}$ of the form

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "g"'] & \cdot \arrow[l, dotted] \\
\cdot \arrow[u, equal] \arrow[ur, phantom, "1^{\mathrm{v}}_g"] & \cdot \arrow[l, "g"] \arrow[u, equal] & \cdot \arrow[l, equal] \arrow[u, dotted] \\
\cdot \arrow[u, dotted] \arrow[ur, phantom, "\alpha"] & \cdot \arrow[l, dotted] \arrow[u, "u"] \arrow[ur, phantom, "1^{\mathrm{h}}_u"] & \cdot \arrow[l, equal] \arrow[u, "u"']
\end{tikzcd}$$

and then, particularly, the existence of a square α as is required. □

We now state our main result in this section.

Theorem 1.4 *Let $\mathcal{G}$ be a double groupoid satisfying the filling condition. Then, for each object a of $\mathcal{G}$, there are natural isomorphisms*

$$\pi_i(\mathcal{G}, a) \cong \pi_i(\mathrm{B}\mathcal{G}, \mathrm{B}a), \ i \geq 0. \tag{1.18}$$

Proof: By taking into account Fact 1.4 (2), we shall identify the homotopy groups of $\mathrm{B}\mathcal{G}$ with those of the Kan complex (by Theorem 1.3) $\mathrm{diagN}^{(2}\mathcal{G}$, which are defined, as we noted in the preliminary Section 1.2, using only its simplicial structure.

To compare the π_0 sets, observe that the 0-simplices $a \in \mathrm{diagN}^{(2}\mathcal{G}_0 = \mathrm{N}^{(2}\mathcal{G}_{0,0}$ are precisely the objects of $\mathcal{G}$. Furthermore, two 0-simplices a, b are in the same connected component of $\mathrm{diagN}^{(2}\mathcal{G}$ if and only if there is a square (i.e., a 1-simplex) of the form

$$\begin{tikzcd}
b & \cdot \arrow[l, dotted] \\
\cdot \arrow[u, dotted] \arrow[ur, phantom, "\exists?"] & a, \arrow[l, dotted] \arrow[u, dotted]
\end{tikzcd}$$

that is, since $\mathcal{G}$ satisfies the filling condition, if and only if a and b are connected in $\mathcal{G}$ (see Subsection 1.3.1). Thus, $\pi_0\mathrm{B}\mathcal{G} = \pi_0\mathcal{G}$.

We now compare the π_1 groups. An element $[\alpha] \in \pi_1(\mathrm{B}\mathcal{G}, \mathrm{B}a)$ is the equivalence class of a square α in $\mathcal{G}$ of the form

$$\begin{tikzcd}
a & \cdot \arrow[l, "g"'] \\
\cdot \arrow[u] \arrow[ur, phantom, "\alpha"] & a \arrow[l] \arrow[u, "u"']
\end{tikzcd}$$

and $[\alpha]=[\alpha']$ if and only if there is a configuration of squares in $\mathcal{G}$ of the form

$$\begin{tikzcd}
a & \cdot \arrow[l, "g"'] & \cdot \arrow[l, "g'"'] \\
\cdot \arrow[u] \arrow[ur, phantom, "\alpha"] & a \arrow[l] \arrow[u, "u"'] \arrow[r, equal] \arrow[ur, phantom, "x"] & a \arrow[u, "u'"'] \\
\cdot \arrow[u] \arrow[ur, phantom, "y"] & a \arrow[l] \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "1_a"] & a \arrow[u, equal]
\end{tikzcd}$$

such that $(\alpha \circ_{\mathrm{h}} x) \circ_{\mathrm{v}} y = \alpha'$. By recalling now the definition of the homotopy group $\pi_1(\mathcal{G},a)$, we observe that, if $[\alpha]=[\alpha']$ in $\pi_1(\mathrm{B}\mathcal{G},\mathrm{B}a)$, then, by the existence of the squares α and $\alpha \circ_{\mathrm{h}} x$, we have $[g,u]=[g\circ_{\mathrm{h}} g',u']$ in $\pi_1(\mathcal{G},a)$; that is,$[\mathrm{t}^{\mathrm{v}}\alpha,\mathrm{s}^{\mathrm{h}}\alpha]=[\mathrm{t}^{\mathrm{v}}\alpha',\mathrm{s}^{\mathrm{h}}\alpha']$. It follows that there is a well-defined map

$$\begin{array}{rcl}\Phi:\pi_1(\mathrm{B}\mathcal{G},\mathrm{B}a) & \longrightarrow & \pi_1(\mathcal{G},a), \\ {[\alpha]} & \longmapsto & [g,u]=[\mathrm{t}^{\mathrm{v}}\alpha,\mathrm{s}^{\mathrm{h}}\alpha]\end{array}$$

which is actually a group homomorphism. To see that, let

$$\begin{tikzcd}
a & \cdot \arrow[l, "g_1"'] & a & \cdot \arrow[l, "g_2"'] \\
\cdot \arrow[u] \arrow[ur, phantom, "\alpha_1"] & a \arrow[l] \arrow[u, "u_1"'] & \cdot \arrow[u] \arrow[ur, phantom, "\alpha_2"] & a \arrow[l] \arrow[u, "u_2"']
\end{tikzcd}$$

be squares representing elements $[\alpha_1]$, $[\alpha_2]\in\pi_1(\mathrm{B}\mathcal{G},\mathrm{B}a)$. Then, its product in the homotopy group $\pi_1(\mathrm{B}\mathcal{G},\mathrm{B}a)$ is $[\alpha_1]\circ[\alpha_2]=[(\alpha_1\circ_{\mathrm{h}}\beta)\circ_{\mathrm{v}}(\gamma\circ_{\mathrm{h}}\alpha_2)]$, where β and γ are any squares in $\mathcal{G}$ defining a configuration of the form (i.e., a 2-simplex of $\mathrm{diagN}^{(2}\mathcal{G}$)

$$\begin{tikzcd}
a & \cdot \arrow[l, "g_1"'] & \cdot \arrow[l, "g"'] \\
\cdot \arrow[u] \arrow[ur, phantom, "\alpha_1"] & a \arrow[l] \arrow[u] \arrow[ur, phantom, "\beta"] & \cdot \arrow[l] \arrow[u, "u"'] \\
\cdot \arrow[u] \arrow[ur, phantom, "\gamma"] & \cdot \arrow[l] \arrow[u] \arrow[ur, phantom, "\alpha_2"] & a \arrow[l] \arrow[u, "u_2"']
\end{tikzcd}$$

Hence,

$$\Phi([\alpha_1]\circ[\alpha_2])=[g_1\circ_h g,u\circ_v u_2]=[g_1,u_1]\circ[g_2,u_2]=\Phi([\alpha_1])\circ\Phi([\alpha_2]),$$

and therefore Φ is a homomorphism.

From the filling condition on $\mathcal{G}$, it follows that Φ is a surjective map. To prove that it is also injective, suppose $\Phi[\alpha_1]=\Phi[\alpha_2]$, where $[\alpha_1]$, $[\alpha_2]\in\pi_1(\mathrm{B}\mathcal{G},\mathrm{B}a)$ are as above. This means that there are squares in $\mathcal{G}$, say x_1 and x_2, of the form

$$\begin{tikzcd}
a & \cdot \arrow[l, "g_1"'] & a & \cdot \arrow[l, "g_2"'] \\
\cdot \arrow[u, "w"] \arrow[ur, phantom, "x_1"] & a \arrow[l, "f"] \arrow[u, "u_1"'] & \cdot \arrow[u, "w"] \arrow[ur, phantom, "x_2"] & a \arrow[l, "f"] \arrow[u, "u_2"']
\end{tikzcd}$$

with which we can form the following three 2-simplices of $\mathrm{diagN}^{(2}\mathcal{G}$

$$\begin{tikzcd}
\cdot & \cdot \arrow[l] & \cdot \arrow[l, equal] \\
\cdot \arrow[u] \arrow[ur, phantom, "x_1"] & \cdot \arrow[l] \arrow[u] \arrow[ur, phantom, "1^{\mathrm{h}}_{u_1}"] & \cdot \arrow[l, equal] \arrow[u] \\
\cdot \arrow[u] \arrow[ur, phantom, "x_1^{-1^{\mathrm{v}}}\circ_{\mathrm{v}}\alpha_1"] & \cdot \arrow[l] \arrow[u, equal] \arrow[ur, phantom, "1_a"] & \cdot \arrow[l, equal] \arrow[u, equal]
\end{tikzcd},
\quad
\begin{tikzcd}
\cdot & \cdot \arrow[l] & \cdot \arrow[l, equal] \\
\cdot \arrow[u] \arrow[ur, phantom, "x_2"] & \cdot \arrow[l] \arrow[u] \arrow[ur, phantom, "1^{\mathrm{h}}_{u_2}"] & \cdot \arrow[l, equal] \arrow[u] \\
\cdot \arrow[u] \arrow[ur, phantom, "x_2^{-1^{\mathrm{v}}}\circ_{\mathrm{v}}\alpha_2"] & \cdot \arrow[l] \arrow[u, equal] \arrow[ur, phantom, "1_a"] & \cdot \arrow[l, equal] \arrow[u, equal]
\end{tikzcd},
\quad
\begin{tikzcd}
\cdot & \cdot \arrow[l] & \cdot \arrow[l] \\
\cdot \arrow[u] \arrow[ur, phantom, "x_1"] & \cdot \arrow[l] \arrow[u] \arrow[ur, phantom, "x_1^{-1^{\mathrm{h}}}\circ_{\mathrm{h}}x_2"] & \cdot \arrow[l, equal] \arrow[u] \\
\cdot \arrow[u, equal] \arrow[ur, phantom, "1^{\mathrm{v}}_f"] & \cdot \arrow[l] \arrow[u, equal] \arrow[ur, phantom, "1_a"] & \cdot \arrow[l, equal] \arrow[u, equal]
\end{tikzcd}.$$

The first one shows that $[x_1] = [\alpha_1]$ in the group $\pi_1(\mathrm{B}\mathcal{G}, \mathrm{B}a)$, the second that $[x_2] = [\alpha_2]$, and the third that $[x_1] = [x_2]$. Whence $[\alpha_1] = [\alpha_2]$, as required.

Finally, we show the isomorphisms $\pi_i(\mathrm{B}\mathcal{G}, \mathrm{B}a) \cong \pi_i(\mathcal{G}, a)$, for $i \geq 2$. For $i \geq 3$, it follows from Lemma 1.8 that $\pi_i(\mathrm{B}\mathcal{G}, a) = 0$, and the result becomes obvious. For the case $i = 2$, it is also a consequence of the afore-mentioned Lemma 1.8 that the homotopy relation between 2-simplices in $\mathrm{diagN}^{(2}\mathcal{G}$ is trivial. Then, the group $\pi_2(\mathrm{B}\mathcal{G}, \mathrm{B}a)$ consists of all 2-simplices in $\mathrm{diagN}^{(2}\mathcal{G}$ of the form

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, equal] & \cdot \arrow[l, equal] \\
\cdot \arrow[u, equal] \arrow[ur, phantom, "1_a"] & \cdot \arrow[l, equal] \arrow[u, equal] \arrow[ur, phantom, "\sigma"] & \cdot \arrow[l, equal] \arrow[u, equal] \\
\cdot \arrow[u, equal] \arrow[ur, phantom, "\sigma^{-1}"] & \cdot \arrow[l, equal] \arrow[u, equal] \arrow[ur, phantom, "1_a"] & \cdot \arrow[l, equal] \arrow[u, equal]
\end{tikzcd}$$

for $\sigma \in \pi_2(\mathcal{G}, a)$, whence the isomorphism becomes clear. ☐

Corollary 1.2 *A double functor $F\colon\mathcal{G} \to \mathcal{G}'$ is a weak equivalence if and only if the induced cellular map on realizations $\mathrm{B}F\colon\mathrm{B}\mathcal{G} \to \mathrm{B}\mathcal{G}'$ is a homotopy equivalence.*

1.6 A left adjoint to the double nerve functor.

Recall from Theorem 1.3 (ii) that the double nerve $\mathrm{N}^{(2}\mathcal{G}$, of any double groupoid satisfying the filling condition, is a bisimplicial set which satisfies the extension condition. Moreover, since both simplicial sets $\mathrm{N}^{(2}\mathcal{G}_{*,0}$ and $\mathrm{N}^{(2}\mathcal{G}_{0,*}$ are nerves of groupoids, all homotopy groups $\pi_2\mathrm{N}^{(2}\mathcal{G}_{*,0}$ and $\pi_2\mathrm{N}^{(2}\mathcal{G}_{0,*}$ vanish. Our main goal in this section is to prove that any bisimplicial set K that satisfies the extension condition and such that the homotopy groups $\pi_2K_{*,0}$ and $\pi_2K_{0,*}$ vanish, has functorially associated a double groupoid with filling condition, denoted by $\mathrm{P}^{(2}K$ and called its *homotopy double groupoid*, such that: (i) $\mathrm{P}^{(2}K$ always represents the same homotopy 2-type as K does, (ii) If $K = \mathrm{N}^{(2}\mathcal{G}$, for a double groupoid $\mathcal{G}$, then $\mathrm{P}^{(2}K = \mathcal{G}$. (see Theorems 1.6 and 1.7, for details).

Let K be any given bisimplicial set K, under the assumption that it satisfies the extension condition of Definition 1.1 and both the Kan complexes $K_{*,0}$ and $K_{0,*}$ have trivial groups π_2. The construction of its homotopy double groupoid

$$\mathrm{P}^{(2}K,$$

which works as a bisimplicial version of Brown's construction in [25, Theorem 2.1], is as follows:

The objects of $\mathrm{P}^{(2}K$ are the vertices $a : \Delta[0,0] \to K$ of K.

The groupoid of horizontal morphisms is the horizontal fundamental groupoid $\mathrm{P}K_{*,0}$, and the groupoid of vertical morphisms is the vertical fundamental groupoid $\mathrm{P}K_{0,*}$ (see the last part of Subsection 1.2.2). Thus, a horizontal morphism $[f]_{\mathrm{h}} : a \to b$ is the horizontal homotopy class of a bisimplex $f : \Delta[1,0] \to K$ with $fd^0_{\mathrm{h}} = a$ and $fd^1_{\mathrm{h}} = b$, whereas a vertical morphism in $\mathrm{P}^{(2}K$, $[u]_{\mathrm{v}} : a \to b$, is the vertical homotopy class of a bisimplex $u : \Delta[0,1] \to K$ with $ud^0_{\mathrm{v}} = a$ and $ud^1_{\mathrm{v}} = b$.

A square of $\mathrm{P}^{(2}K$ is the bihomotopy class $[[x]]$ of a bisimplex $x : \Delta[1,1] \to K$, with boundary

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "{[xd^1_{\mathrm{v}}]_{\mathrm{h}}}"'] \\
\cdot \arrow[u, "{[xd^1_{\mathrm{h}}]_{\mathrm{v}}}"] & \cdot \arrow[l, "{[xd^0_{\mathrm{v}}]_{\mathrm{h}}}"] \arrow[u, "{[xd^0_{\mathrm{h}}]_{\mathrm{v}}}"']
\end{tikzcd}$$

which is well defined thanks to Lemma 1.3.

The horizontal composition of squares in $\mathrm{P}^{(2}K$ is the only one making the correspondence

$$\Big(xd^0_{\mathrm{h}} \overset{[x]_{\mathrm{h}}}{\to} xd^1_{\mathrm{h}}\Big) \overset{[\ ]}{\longmapsto} \Big([xd^0_{\mathrm{h}}]_{\mathrm{v}} \overset{[[x]]}{\to} [xd^1_{\mathrm{h}}]_{\mathrm{v}}\Big)$$

a surjective fibration of groupoids from the horizontal fundamental groupoid $\mathrm{P}K_{*,1}$ to the horizontal groupoid of squares in $\mathrm{P}^{(2}K$. To define this composition, we shall need the following:

Lemma 1.9 *Let $x, y : \Delta[1,1] \to K$ be bisimplices such that $[xd^0_{\mathrm{h}}]_{\mathrm{v}} = [yd^1_{\mathrm{h}}]_{\mathrm{v}}$. Then, there is a bisimplex $x' : \Delta[1,1] \to K$ such that:*

(a) $[x']_{\mathrm{v}} = [x]_{\mathrm{v}}$,

(b) $x'd^0_{\mathrm{h}} = yd^1_{\mathrm{h}}$.

For any such bisimplex x' satisfying (a) and (b), the following equalities hold:

(d) $[[x']] = [[x]]$,

(e) $x'd^0_{\mathrm{v}} = xd^0_{\mathrm{v}}, \quad x'd^1_{\mathrm{v}} = xd^1_{\mathrm{v}}$.

Proof: Once any vertical homotopy from xd^0_{h} to yd^1_{h} is selected, say $\alpha : \Delta[0,2] \to K$, let $\beta : \Delta[1,2] \to K$ be any bisimplex solving the extension problem

$$\begin{tikzcd}
\Lambda^{1,2}[1,2] \arrow[r, "{(\alpha,-;xd^0_{\mathrm{v}}s^0_{\mathrm{v}},x,-)}"] \arrow[d, hook] & K. \\
\Delta[1,2] \arrow[ur, dotted, "\beta"']
\end{tikzcd}$$

Note that such a simplex β exists since it is assumed that the bisimplicial set K satisfies the extension condition in Definition 1.1. Then, we take $x' = \beta d^2_{\mathrm{v}} : \Delta[1,1] \to K$. Since β becomes a vertical homotopy from x to x', we have $[x]_{\mathrm{v}} = [x']_{\mathrm{v}}$, and then the equalities $[[x']] = [[x]]$, $x'd^0_{\mathrm{v}} = xd^0_{\mathrm{v}}$, and $x'd^1_{\mathrm{v}} = xd^1_{\mathrm{v}}$, follow. Moreover, $x'd^0_{\mathrm{h}} = \beta d^2_{\mathrm{v}} d^0_{\mathrm{h}} = \beta d^0_{\mathrm{h}} d^2_{\mathrm{v}} = \alpha d^2_{\mathrm{v}} = y d^1_{\mathrm{h}}$, as required. ☐

Now define the horizontal composition of squares in $\mathrm{P}^{(2}K$ by

$$[[x]] \circ_{\mathrm{h}} [[y]] = [[x']_{\mathrm{h}} \circ_{\mathrm{h}} [y]_{\mathrm{h}}] \quad \text{if} \quad [x]_{\mathrm{v}} = [x']_{\mathrm{v}} \text{ and } x'd^0_{\mathrm{h}} = yd^1_{\mathrm{h}}, \tag{1.19}$$

where $[x']_{\mathrm{h}} \circ_{\mathrm{h}} [y]_{\mathrm{h}}$ is the composite in the fundamental groupoid $\mathrm{P}K_{*,1}$, that is,

$$[[x]] \circ_{\mathrm{h}} [[y]] = [[\gamma d^1_{\mathrm{h}}]]$$

for $\gamma : \Delta[2,1] \to K$ any bisimplex with $\gamma d^2_{\mathrm{h}} = x'$ and $\gamma d^0_{\mathrm{h}} = y$.

In view of Lemma 1.9, our product is given for all squares $[[x]]$ and $[[y]]$ with $\mathrm{s}^{\mathrm{h}}[[x]] = \mathrm{t}^{\mathrm{h}}[[y]]$. We also have the lemma below, where it is crucial in our argument the hypothesis of $\pi_2(K_{0,*}, a)$ being trivial.

Lemma 1.10 *The horizontal composition of squares in* $\mathrm{P}^{(2}K$ *is well defined.*

Proof: We first prove that the square in (1.19) does not depend on the choice of x'. To do so, suppose $x'' : \Delta[1,1] \to K$ is another bisimplex such that $[x]_{\mathrm{v}} = [x'']_{\mathrm{v}}$ and $x''d^0_{\mathrm{h}} = yd^1_{\mathrm{h}}$, and let $\beta, \beta' : \Delta[1,2] \to K$ be vertical homotopies from x to x' and from x to x'' respectively. Then, both bisimplices $\beta d^0_{\mathrm{h}} : \Delta[0,2] \to K$ and $\beta' d^0_{\mathrm{h}} : \Delta[0,2] \to K$ have the same vertical faces. Since the 2nd homotopy groups of the Kan complex $K_{0,*}$ vanish, it follows that βd^0_{h} and $\beta' d^0_{\mathrm{h}}$ are vertically homotopic (Fact 1.2). Choose $\omega : \Delta[0,3] \to K$ any vertical homotopy from βd^0_{h} to $\beta' d^0_{\mathrm{h}}$, and then let $\Gamma : \Delta[1,3] \to K$ be a solution to the extension problem

$$\begin{tikzcd}
\Lambda^{1,3}[1,3] \arrow[r, "{(\omega,-;xd^0_{\mathrm{v}}s^0_{\mathrm{v}}s^1_{\mathrm{v}},xs^1_{\mathrm{v}},\beta,-)}"] \arrow[d, hook] & K\,. \\
\Delta[1,3] \arrow[ur, dotted, "\Gamma"'] &
\end{tikzcd}$$

Then, the bisimplex $\widetilde{\beta} = \Gamma d^3_{\mathrm{v}} : \Delta[1,2] \to K$ has vertical faces

$$\begin{aligned}
\widetilde{\beta} d^0_{\mathrm{v}} &= \Gamma d^3_{\mathrm{v}} d^0_{\mathrm{v}} = \Gamma d^0_{\mathrm{v}} d^2_{\mathrm{v}} = x d^0_{\mathrm{v}} s^0_{\mathrm{v}} s^1_{\mathrm{v}} d^2_{\mathrm{v}} = x d^0_{\mathrm{v}} s^0_{\mathrm{v}}, \\
\widetilde{\beta} d^1_{\mathrm{v}} &= \Gamma d^3_{\mathrm{v}} d^1_{\mathrm{v}} = \Gamma d^1_{\mathrm{v}} d^2_{\mathrm{v}} = x s^1_{\mathrm{v}} d^2_{\mathrm{v}} = x, \\
\widetilde{\beta} d^2_{\mathrm{v}} &= \Gamma d^3_{\mathrm{v}} d^2_{\mathrm{v}} = \Gamma d^2_{\mathrm{v}} d^2_{\mathrm{v}} = \beta d^2_{\mathrm{v}} = x',
\end{aligned}$$

so that $\widetilde{\beta}$ is another vertical homotopy from x to x', and moreover

$$\widetilde{\beta} d^0_{\mathrm{h}} = \Gamma d^3_{\mathrm{v}} d^0_{\mathrm{h}} = \Gamma d^0_{\mathrm{h}} d^3_{\mathrm{v}} = \omega d^3_{\mathrm{v}} = \beta' d^0_{\mathrm{h}},$$

that is, $\widetilde{\beta}$ and β' have both the same horizontal 0-face, say α. Now let $\Phi : \Delta[1,3] \to K$ and $\theta : \Delta[2,2] \to K$ be solutions to the following extension problems

$$\begin{tikzcd}
\Lambda^{1,3}[1,3] \arrow[r, two heads, "(\alpha s^1_{\mathrm{v}},-;x d^0_{\mathrm{v}} s^0_{\mathrm{v}} s^1_{\mathrm{v}},\widetilde{\beta},\beta',-)"] \arrow[d, hook] & K & \Lambda^{1,2}[2,2] \arrow[r, two heads, "(y s^1_{\mathrm{v}},-,\Phi d^3_{\mathrm{v}};\gamma d^0_{\mathrm{v}} s^0_{\mathrm{v}},\gamma,-)"] \arrow[d, hook] & K \\
\Delta[1,3] \arrow[ur, dotted, two heads, "\Phi"'] & & \Delta[2,2] \arrow[ur, dotted, two heads, "\theta"'] &
\end{tikzcd}$$

where $\gamma : \Delta[2,1] \to K$ is any bisimplex such that $\gamma d^2_{\mathrm{h}} = x'$ and $\gamma d^0_{\mathrm{h}} = y$. Then, θ is actually a vertical homotopy from γ to $\gamma' = \theta d^2_{\mathrm{v}}$, and this bisimplex γ' satisfies that

$$\begin{aligned}
\gamma' d^2_{\mathrm{h}} &= \theta d^2_{\mathrm{v}} d^2_{\mathrm{h}} = \theta d^2_{\mathrm{h}} d^2_{\mathrm{v}} = \Phi d^3_{\mathrm{v}} d^2_{\mathrm{v}} = \Phi d^2_{\mathrm{v}} d^2_{\mathrm{v}} = \beta' d^2_{\mathrm{v}} = x'', \\
\gamma' d^0_{\mathrm{h}} &= \theta d^2_{\mathrm{v}} d^0_{\mathrm{h}} = \theta d^0_{\mathrm{h}} d^2_{\mathrm{v}} = y s^1_{\mathrm{v}} d^2_{\mathrm{v}} = y.
\end{aligned}$$

Hence, $[x']_{\mathrm{h}} \circ_{\mathrm{h}} [y]_{\mathrm{h}} = [\gamma d^1_{\mathrm{h}}]_{\mathrm{h}}$ whereas $[x'']_{\mathrm{h}} \circ_{\mathrm{h}} [y]_{\mathrm{h}} = [\gamma' d^1_{\mathrm{h}}]_{\mathrm{h}}$. Since the bisimplex $\theta d^1_{\mathrm{h}} : \Delta[1,2] \to K$ is a vertical homotopy from γd^1_{h} to $\gamma' d^1_{\mathrm{h}}$, we conclude that $[[\gamma d^1_{\mathrm{h}}]] = [[\gamma' d^1_{\mathrm{h}}]]$, that is, $[[x']_{\mathrm{h}} \circ_{\mathrm{h}} [y]_{\mathrm{h}}] = [[x'']_{\mathrm{h}} \circ_{\mathrm{h}} [y]_{\mathrm{h}}]$, as required.

Suppose now $x_0, x_1, y : \Delta[1,1] \to K$ bisimplices with $[[x_0]] = [[x_1]]$ and $[x_0 d^0_{\mathrm{h}}]_{\mathrm{v}} = [y d^1_{\mathrm{h}}]_{\mathrm{v}}$. Then, for some $x : \Delta[1,1] \to K$, we have $[x_0]_{\mathrm{v}} = [x]_{\mathrm{v}}$ and $[x]_{\mathrm{h}} = [x_1]_{\mathrm{h}}$. Let $x'_0 : \Delta[1,1] \to K$ be any bisimplex with $[x'_0]_{\mathrm{v}} = [x]_{\mathrm{v}}$ and $x'_0 d^0_{\mathrm{h}} = y d^1_{\mathrm{h}}$. Since $[x'_0]_{\mathrm{v}} = [x_0]_{\mathrm{v}}$, we have

$$[[x_0]] \circ_{\mathrm{h}} [[y]] = [[x'_0]_{\mathrm{h}} \circ [y]_{\mathrm{h}}]. \tag{1.20}$$

Letting $\beta : \Delta[1,2] \to K$ be any vertical homotopy from x to x'_0 and $\delta : \Delta[2,1] \to K$ be any horizontal homotopy from x_1 to x, we can choose $\theta : \Delta[2,2] \to K$, a bisimplex making commutative the diagram

$$\begin{tikzcd}
\Lambda^{1,2}[2,2] \arrow[r, two heads, "(\beta d^0_{\mathrm{h}} s^0_{\mathrm{h}},-,\beta;\delta d^0_{\mathrm{v}} s^0_{\mathrm{v}},\delta,-)"] \arrow[d, hook] & K \\
\Delta[2,2] \arrow[ur, two heads, "\theta"'] &
\end{tikzcd}$$

Then, $\beta_1 = \theta d^1_{\mathrm{h}} : \Delta[1,2] \to K$ is a vertical homotopy from x_1 to $x'_1 := \beta_1 d^2_{\mathrm{v}}$, and since

$$x'_1 d^0_{\mathrm{h}} = \beta_1 d^2_{\mathrm{v}} d^0_{\mathrm{h}} = \beta_1 d^0_{\mathrm{h}} d^2_{\mathrm{v}} = \theta d^1_{\mathrm{h}} d^0_{\mathrm{h}} d^2_{\mathrm{v}} = \theta d^0_{\mathrm{h}} d^0_{\mathrm{h}} d^2_{\mathrm{v}} = \beta d^0_{\mathrm{h}} d^2_{\mathrm{v}} = \beta d^2_{\mathrm{v}} d^0_{\mathrm{h}} = x'_0 d^0_{\mathrm{h}} = y d^1_{\mathrm{h}},$$

we have

$$[[x_1]] \circ_{\mathrm{h}} [[y]] = [[x'_1]_{\mathrm{h}} \circ_{\mathrm{h}} [y]_{\mathrm{h}}]. \tag{1.21}$$

As $\theta d^2_{\mathrm{v}} : \Delta[2,1] \to K$ is a horizontal homotopy from x'_1 to x'_0, we have $[x'_0]_{\mathrm{h}} = [x'_1]_{\mathrm{h}}$. Therefore, comparing (1.20) with (1.21), we obtain the desired conclusion, that is,

$$[[x_0]] \circ_{\mathrm{h}} [[y]] = [[x_1]] \circ_{\mathrm{h}} [[y]].$$

Finally, suppose x, y_0, $y_1 : \Delta[1,1] \to K$ with $[[y_0]] = [[y_1]]$ and $[xd^0_{\mathrm{h}}]_{\mathrm{v}} = [y_0 d^1_{\mathrm{h}}]_{\mathrm{v}}$. Then, $[y_0]_{\mathrm{v}} = [y]_{\mathrm{v}}$, $[y]_{\mathrm{h}} = [y_1]_{\mathrm{h}}$, for some $y : \Delta[1,1] \to K$. Let $x' : \Delta[1,1] \to K$ be such that $[x]_{\mathrm{v}} = [x']_{\mathrm{v}}$ and $x'd^0_{\mathrm{h}} = yd^1_{\mathrm{h}}$. Since $x'd^0_{\mathrm{h}} = y_1 d^1_{\mathrm{h}}$, we have

$$[[x]] \circ_{\mathrm{h}} [[y_1]] = [[x']_{\mathrm{h}} \circ_{\mathrm{h}} [y_1]_{\mathrm{h}}] = [[x']_{\mathrm{h}} \circ_{\mathrm{h}} [y]_{\mathrm{h}}] = [[\gamma d^1_{\mathrm{h}}]], \tag{1.22}$$

for $\gamma : \Delta[2,1] \to K$ any bisimplex with $\gamma d^2_{\mathrm{h}} = x'$ and $\gamma d^0_{\mathrm{h}} = y$. Now, as $[y_0]_{\mathrm{v}} = [y]_{\mathrm{v}}$, we can select a vertical homotopy $\delta : \Delta[1,2] \to K$ from y to y_0, and then a bisimplex $\beta_0 : \Delta[1,2] \to K$ making commutative the diagram

$$\begin{tikzcd}
\Lambda^{1,2}[1,2] \arrow[r, "(\delta d^1_{\mathrm{h}}, -; x'd^0_{\mathrm{v}} s^0_{\mathrm{v}}, x', -)"] \arrow[d, hook] & K. \\
\Delta[1,2] \arrow[ur, "\beta_0"'] &
\end{tikzcd}$$

This bisimplex β_0 becomes a vertical homotopy from x' to $x'_0 := \beta_0 d^2_{\mathrm{v}}$, and this x'_0 verifies that $x'_0 d^0_{\mathrm{h}} = y_0 d^1_{\mathrm{h}}$. Hence,

$$[[x]] \circ_{\mathrm{h}} [[y_0]] = [[x'_0]_{\mathrm{h}} \circ_{\mathrm{h}} [y_0]_{\mathrm{h}}].$$

But, by taking $\theta : \Delta[2,2] \to K$ any bisimplex solving the extension problem

$$\begin{tikzcd}
\Lambda^{1,2}[2,2] \arrow[r, "(\delta, -, \beta_0; \gamma d^0_{\mathrm{v}} s^0_{\mathrm{v}}, \gamma, -)"] \arrow[d, hook] & K, \\
\Delta[2,2] \arrow[ur, "\theta"'] &
\end{tikzcd}$$

we obtain a bisimplex $\gamma_0 := \theta d^2_{\mathrm{v}} : \Delta[2,1] \to K$ satisfying that $\gamma_0 d^0_{\mathrm{h}} = y_0$ and $\gamma_0 d^2_{\mathrm{h}} = x'_0$, whence

$$[[x]] \circ_{\mathrm{h}} [[y_0]] = [[\gamma_0 d^1_{\mathrm{h}}]]. \tag{1.23}$$

As the bisimplex $\theta d^1_{\mathrm{h}} : \Delta[1,2] \to K$ is easily recognized to be a vertical homotopy from γd^1_{h} to $\gamma_0 d^1_{\mathrm{h}}$, we conclude $[[\gamma d^1_{\mathrm{h}}]] = [[\gamma_0 d^1_{\mathrm{h}}]]$. Consequently, the required equality

$$[[x]] \circ_{\mathrm{h}} [[y_0]] = [[x]] \circ_{\mathrm{h}} [[y_1]]$$

follows by comparing (1.22) with (1.23). □

Simply by exchanging the horizontal and vertical directions in the foregoing discussion, we also have a well-defined vertical composition of squares $[[x]]$ and $[[y]]$ in $\mathrm{P}^{(2}K$, whenever $[xd^0_{\mathrm{v}}]_{\mathrm{h}} = [yd^1_{\mathrm{v}}]_{\mathrm{h}}$, which is given by

$$[[x]] \circ_{\mathrm{v}} [[y]] = [[x']_{\mathrm{v}} \circ_{\mathrm{v}} [y]_{\mathrm{v}}] \quad \text{if} \quad [x]_{\mathrm{h}} = [x']_{\mathrm{h}} \text{ and } x'd^0_{\mathrm{v}} = yd^1_{\mathrm{v}},$$

where $[x']_{\mathrm{v}} \circ_{\mathrm{v}} [y]_{\mathrm{v}}$ is the composite in the fundamental groupoid $\mathrm{P}K_{1,*}$, that is,

$$[[x]] \circ_{\mathrm{v}} [[y]] = [[\gamma d^1_{\mathrm{v}}]]$$

for $\gamma : \Delta[1,2] \to K$ any bisimplex with $\gamma d^2_{\mathrm{v}} = x'$ and $\gamma d^0_{\mathrm{v}} = y$.

Theorem 1.5 $\mathrm{P}^{(2}K$ *is a double groupoid satisfying the filling condition.*

Proof: We first observe that, with both defined horizontal and vertical compositions, the squares in $\mathrm{P}^{(2}K$ form groupoids. The associativity for the horizontal composition of squares in $\mathrm{P}^{(2}K$ follows from the associativity of the composition of morphisms in the fundamental groupoid $\mathrm{P}K_{*,1}$. In effect, let $[[x]]$, $[[y]]$ and $[[z]]$ be three horizontally composable squares in $\mathrm{P}^{(2}K$. By changing representatives if necessary, we can assume that $xd_{\mathrm{h}}^0 = yd_{\mathrm{h}}^1$ and $yd_{\mathrm{h}}^0 = zd_{\mathrm{h}}^1$. Then,

$$\begin{array}{rclcl} [[x]] \circ_{\mathrm{h}} ([[y]] \circ_{\mathrm{h}} [[z]]) & = & [[x]] \circ_{\mathrm{h}} [[y]_{\mathrm{h}} \circ_{\mathrm{h}} [z]_{\mathrm{h}}] & = & [[x]_{\mathrm{h}} \circ_{\mathrm{h}} ([y]_{\mathrm{h}} \circ_{\mathrm{h}} [z]_{\mathrm{h}})] \\ & = & [([x]_{\mathrm{h}} \circ_{\mathrm{h}} [y]_{\mathrm{h}}) \circ_{\mathrm{h}} [z]_{\mathrm{h}}] & = & [[x]_{\mathrm{h}} \circ_{\mathrm{h}} [y]_{\mathrm{h}}] \circ_{\mathrm{h}} [[z]] \\ & = & ([[x]] \circ_{\mathrm{h}} [[y]]) \circ_{\mathrm{h}} [[z]]. & & \end{array}$$

The horizontal identity square on the vertical morphism represented by a bisimplex $u\colon\Delta[0,1] \to K$ is

$$1^{\mathrm{h}}_{[u]_{\mathrm{v}}} = [[us_{\mathrm{h}}^0]]$$

(recall Lemma 1.3), as can be easily deduced from the fact that $[us_{\mathrm{h}}^0]_{\mathrm{h}}$ is the identity morphism on u in the groupoid $\mathrm{P}K_{*,1}$. Thus, for example, for any $x : \Delta[1,1] \to K$,

$$[[x]] \circ_{\mathrm{h}} 1^{\mathrm{h}}_{[xd_{\mathrm{h}}^0]_{\mathrm{v}}} = [[x]_{\mathrm{h}} \circ_{\mathrm{h}} [xd_{\mathrm{h}}^0 s_{\mathrm{h}}^0]_{\mathrm{h}}] = [[x]_{\mathrm{h}}] = [[x]].$$

The horizontal inverse in $\mathrm{P}^{(2}K$ of a square $[[x]]$ is $[[x]]^{\text{-}1_{\mathrm{h}}} = [[x]_{\mathrm{h}}^{\text{-}1}]$, where $[x]_{\mathrm{h}}^{\text{-}1}$ is the inverse of $[x]_{\mathrm{h}}$ in $\mathrm{P}K_{*,1}$, as is easy to verify:

$$[[x]] \circ_{\mathrm{h}} [[x]_{\mathrm{h}}^{\text{-}1}] = [[x]_{\mathrm{h}} \circ_{\mathrm{h}} [x]_{\mathrm{h}}^{\text{-}1}] = [[xd_{\mathrm{h}}^1 s_{\mathrm{h}}^0]] = 1^{\mathrm{h}}_{[xd_{\mathrm{h}}^1]_{\mathrm{v}}}.$$

Similarly, we see that the associativity for the vertical composition of squares in $\mathrm{P}^{(2}K$ follows from the associativity of the composition in the fundamental groupoid $\mathrm{P}K_{1,*}$, that the vertical identity square on the horizontal morphism represented by a bisimplex $f : \Delta[1,0] \to K$ is $1^{v}_{[f]_{\mathrm{h}}} = [[fs_{\mathrm{v}}^0]]$, and that the vertical inverse in $\mathrm{P}^{(2}K$ of a square $[[x]]$ is $[[x]_{\mathrm{v}}^{-1}]$, where $[x]_{\mathrm{v}}^{\text{-}1}$ denotes the inverse of $[x]_{\mathrm{v}}$ in $\mathrm{P}K_{1,*}$.

We are now ready to prove that $\mathrm{P}^{(2}K$ is actually a double groupoid. **Axiom 1** is easily verified. Thus, for example, given any $x : \Delta[1,1] \to K$,

$$\mathrm{s}^{\mathrm{h}}\mathrm{s}^{\mathrm{v}}[[x]] = \mathrm{s}^{\mathrm{h}}[xd_{\mathrm{v}}^0]_{\mathrm{h}} = xd_{\mathrm{v}}^0 d_{\mathrm{h}}^0 = xd_{\mathrm{h}}^0 d_{\mathrm{v}}^0 = \mathrm{s}^{\mathrm{v}}[xd_{\mathrm{h}}^0]_{\mathrm{v}} = \mathrm{s}^{\mathrm{v}}\mathrm{s}^{\mathrm{h}}[[x]],$$

or, given any $f : \Delta[1,0] \to K$,

$$\mathrm{s}^{\mathrm{h}}1^{\mathrm{v}}_{[f]_{\mathrm{h}}} = \mathrm{s}^{\mathrm{h}}[[fs_{\mathrm{v}}^0]] = [fs_{\mathrm{v}}^0 d_{\mathrm{h}}^0]_{\mathrm{v}} = [fd_{\mathrm{h}}^0 s_{\mathrm{v}}^0]_{\mathrm{v}} = 1^{\mathrm{v}}_{fd_{\mathrm{h}}^0} = 1^{\mathrm{v}}_{\mathrm{s}^{\mathrm{h}}[f]_{\mathrm{h}}},$$

and so on. Also, for any $a : \Delta[0,0] \to K$,

$$1^{\mathrm{h}}_{1^{\mathrm{v}}_a} = 1^{\mathrm{h}}_{[as_{\mathrm{v}}^0]_{\mathrm{v}}} = [[as_{\mathrm{v}}^0 s_{\mathrm{h}}^0]] = [[as_{\mathrm{h}}^0 s_{\mathrm{v}}^0]] = 1^{\mathrm{v}}_{[as_{\mathrm{h}}^0]_{\mathrm{h}}} = 1^{\mathrm{v}}_{1^{\mathrm{h}}_a}.$$

For **Axiom 2** (i), let $[[x]]$ and $[[y]]$ be two horizontally composable squares in $\mathrm{P}^{(2}K$. We can assume that $xd_{\mathrm{h}}^0 = yd_{\mathrm{h}}^1$, and then $[[x]] \circ_{\mathrm{h}} [[y]] = [[\gamma d_{\mathrm{h}}^1]]$, for any $\gamma : \Delta[2,1] \to K$ with $\gamma d_{\mathrm{h}}^2 = x$ and $\gamma d_{\mathrm{h}}^0 = y$. Hence,

$$\begin{aligned}
\mathrm{s}^{\mathrm{v}}([[x]] \circ_{\mathrm{h}} [[y]]) &= [\gamma d_{\mathrm{h}}^1 d_{\mathrm{v}}^0]_{\mathrm{h}} = [\gamma d_{\mathrm{v}}^0 d_{\mathrm{h}}^1] = [\gamma d_{\mathrm{v}}^0 d_{\mathrm{h}}^2]_{\mathrm{h}} \circ_{\mathrm{h}} [\gamma d_{\mathrm{v}}^0 d_{\mathrm{h}}^0]_{\mathrm{h}} \\
&= [\gamma d_{\mathrm{h}}^2 d_{\mathrm{v}}^0]_{\mathrm{h}} \circ_{\mathrm{h}} [\gamma d_{\mathrm{h}}^0 d_{\mathrm{v}}^0] = [xd_{\mathrm{v}}^0]_{\mathrm{h}} \circ_{\mathrm{h}} [yd_{\mathrm{v}}^0]_{\mathrm{h}} = \mathrm{s}^{\mathrm{v}}[[x]] \circ_{\mathrm{h}} \mathrm{s}^{\mathrm{v}}[[y]], \\
\mathrm{t}^{\mathrm{v}}([[x]] \circ_{\mathrm{h}} [[y]]) &= [\gamma d_{\mathrm{h}}^1 d_{\mathrm{v}}^1]_{\mathrm{h}} = [\gamma d_{\mathrm{v}}^1 d_{\mathrm{h}}^1]_{\mathrm{h}} = [\gamma d_{\mathrm{v}}^1 d_{\mathrm{h}}^2]_{\mathrm{h}} \circ_{\mathrm{h}} [\gamma d_{\mathrm{v}}^1 d_{\mathrm{h}}^0]_{\mathrm{h}} \\
&= [\gamma d_{\mathrm{h}}^2 d_{\mathrm{v}}^1]_{\mathrm{h}} \circ_{\mathrm{h}} [\gamma d_{\mathrm{h}}^0 d_{\mathrm{v}}^1]_{\mathrm{h}} = [xd_{\mathrm{v}}^1]_{\mathrm{h}} \circ_{\mathrm{h}} [yd_{\mathrm{v}}^1]_{\mathrm{h}} = \mathrm{t}^{\mathrm{v}}[[x]] \circ_{\mathrm{h}} \mathrm{t}^{\mathrm{v}}[[y]].
\end{aligned}$$

Axiom 2 (ii) is proved analogously, and for (iii), let f, $f' : \Delta[1,0] \to K$ be maps with $fd_{\mathrm{h}}^0 = f'd_{\mathrm{h}}^1$. Then, $[f]_{\mathrm{h}} \circ_{\mathrm{h}} [f']_{\mathrm{h}} = [\gamma d_{\mathrm{h}}^1]_{\mathrm{h}}$, for $\gamma : \Delta[2,0] \to K$ any bisimplex with $\gamma d_{\mathrm{h}}^2 = f$ and $\gamma d_{\mathrm{h}}^0 = f'$, and we have the equalities:

$$1^{\mathrm{v}}_{[f]_{\mathrm{h}} \circ_{\mathrm{h}} [f']_{\mathrm{h}}} = 1^{\mathrm{v}}_{[\gamma d_{\mathrm{h}}^1]_{\mathrm{h}}} = [[\gamma d_{\mathrm{h}}^1 s_{\mathrm{v}}^0]] = [[\gamma s_{\mathrm{v}}^0 d_{\mathrm{h}}^1]] = [[\gamma s_{\mathrm{v}}^0 d_{\mathrm{h}}^2]] \circ_{\mathrm{h}} [[\gamma s_{\mathrm{v}}^0 d_{\mathrm{h}}^0]] = 1^{\mathrm{v}}_{[f]_{\mathrm{h}}} \circ_{\mathrm{h}} 1^{\mathrm{v}}_{[f']_{\mathrm{h}}}.$$

And similarly one sees that $1^{\mathrm{h}}_{[u]_{\mathrm{v}} \circ_{\mathrm{v}} [u']_{\mathrm{v}}} = 1^{\mathrm{h}}_{[u]_{\mathrm{v}}} \circ_{\mathrm{v}} 1^{\mathrm{h}}_{[u']_{\mathrm{v}}}$ for any u, $u' : \Delta[0,1] \to K$ with $ud_{\mathrm{v}}^0 = u'd_{\mathrm{v}}^1$.

To verify **Axiom 3**, that is, to prove that the interchange law holds in $\mathrm{P}^{(2}K$, let

$$\begin{tikzcd}
\cdot & \cdot \arrow[l] & \cdot \arrow[l] \\
\cdot \arrow[u] \arrow[ur, phantom, "{[[x]]}"] & \cdot \arrow[l] \arrow[u] \arrow[ur, phantom, "{[[x']]}"] & \cdot \arrow[l] \arrow[u] \\
\cdot \arrow[u] \arrow[ur, phantom, "{[[y]]}"] & \cdot \arrow[l] \arrow[u] \arrow[ur, phantom, "{[[y']]}"] & \cdot \arrow[l] \arrow[u]
\end{tikzcd}$$

be squares in $\mathrm{P}^{(2}K$. By an iterated use of Lemmas 1.9 and 1.10 (and their corresponding versions for vertical direction), we can assume that $xd_{\mathrm{h}}^0 = x'd_{\mathrm{h}}^1$, $xd_{\mathrm{v}}^0 = yd_{\mathrm{v}}^1$, $x'd_{\mathrm{v}}^0 = y'd_{\mathrm{v}}^1$ and $yd_{\mathrm{h}}^0 = y'd_{\mathrm{h}}^1$. Let $\alpha : \Delta[2,1] \to K$ and $\beta : \Delta[1,2] \to K$ be bisimplicial maps such that $\alpha d_{\mathrm{h}}^2 = y$, $\alpha d_{\mathrm{h}}^0 = y'$, $\beta d_{\mathrm{v}}^2 = x'$ and $\beta d_{\mathrm{v}}^0 = y'$; therefore, $[[y]] \circ_{\mathrm{h}} [[y']] = [[\alpha d_{\mathrm{h}}^1]]$ and $[[x']] \circ_{\mathrm{v}} [[y']] = [[\beta d_{\mathrm{v}}^1]]$. Now we select bisimplices $\gamma : \Delta[1,2] \to K$ and $\delta : \Delta[2,1] \to K$ as respective solutions to the following extension problems:

$$\begin{tikzcd}
\Lambda^{1,1}[1,2] \arrow[r, "{(\beta d_{\mathrm{h}}^1,-;y,-,x)}"] \arrow[d, hook] & K \\
\Delta[1,2] \arrow[ur, dotted, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^{1,1}[2,1] \arrow[r, "{(x',-,x;\alpha d_{\mathrm{v}}^1,-)}"] \arrow[d, hook] & K \\
\Delta[2,1] \arrow[ur, dotted, "\delta"'] &
\end{tikzcd}$$

Then $[[x]] \circ_{\mathrm{v}} [[y]] = [[\gamma d_{\mathrm{v}}^1]]$, $[[x]] \circ_{\mathrm{h}} [[x']] = [[\delta d_{\mathrm{h}}^1]]$ and, moreover, we can find a bisimplex $\theta : \Delta[2,2] \to K$ making the triangle below commutative.

$$\begin{tikzcd}
\Lambda^{1,1}[2,2] \arrow[r, "{(\beta,-,\gamma;\alpha,-,\delta)}"] \arrow[d, hook] & K \\
\Delta[2,2] \arrow[ur, dotted, "\theta"'] &
\end{tikzcd}$$

Letting $\phi = \theta d^1_{\mathrm{h}} : \Delta[1,2] \to K$ and $\psi = \theta d^1_{\mathrm{v}} : \Delta[2,1] \to K$, we have the equalities:

$$\begin{aligned} \phi d^2_{\mathrm{v}} &= \theta d^2_{\mathrm{v}} d^1_{\mathrm{h}} = \delta d^1_{\mathrm{h}}, \quad \phi d^0_{\mathrm{v}} = \theta d^0_{\mathrm{v}} d^1_{\mathrm{h}} = \alpha d^1_{\mathrm{h}}, \\ \psi d^2_{\mathrm{h}} &= \theta d^2_{\mathrm{h}} d^1_{\mathrm{v}} = \gamma d^1_{\mathrm{v}}, \quad \psi d^0_{\mathrm{h}} = \theta d^0_{\mathrm{h}} d^1_{\mathrm{v}} = \beta d^1_{\mathrm{v}}, \end{aligned}$$

whence,

$$\begin{aligned} ([[x]] \circ_{\mathrm{h}} [[x']]) \circ_{\mathrm{v}} ([[y]] \circ_{\mathrm{h}} [[y']]) &= [[\delta d^1_{\mathrm{h}}]] \circ_{\mathrm{v}} [[\alpha d^1_{\mathrm{h}}]] = [[\phi d^1_{\mathrm{v}}]], \\ ([[x]] \circ_{\mathrm{v}} [[y]]) \circ_{\mathrm{h}} ([[x']] \circ_{\mathrm{h}} [[y']]) &= [[\gamma d^1_{\mathrm{v}}]] \circ_{\mathrm{h}} [[\beta d^1_{\mathrm{v}}]] = [[\psi d^1_{\mathrm{h}}]]. \end{aligned}$$

Since $\phi d^1_{\mathrm{v}} = \theta d^1_{\mathrm{h}} d^1_{\mathrm{v}} = \theta d^1_{\mathrm{v}} d^1_{\mathrm{h}} = \psi d^1_{\mathrm{h}}$, the interchange law follows.

Thus, $\mathrm{P}^{(2}K$ is a double groupoid and, moreover, it satisfies the filling condition: given morphisms

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "{[g]_{\mathrm{h}}}"'] \\ & \cdot \arrow[u, "{[u]_{\mathrm{v}}}"'] \end{tikzcd}$$

represented by bisimplices $u : \Delta[0,1] \to K$ and $g : \Delta[1,0] \to K$ with $gd^0_{\mathrm{h}} = ud^1_{\mathrm{v}}$, if $x : \Delta[1,1] \to K$ is any solution to the extension problem

$$\begin{tikzcd} \Lambda^{0,1}[1,1] \arrow[r, "{(-,g;u,-)}"] \arrow[d, hook] & K \\ \Delta[1,1] \arrow[ur, dotted, "x"'] & \end{tikzcd}$$

then the bihomotopy class of x is a square in $\mathrm{P}^{(2}K$

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "{[g]_{\mathrm{h}}}"'] \\ \cdot \arrow[u] \arrow[ur, phantom, "{[[x]]}"] & \cdot \arrow[u, "{[u]_{\mathrm{v}}}"'] \arrow[l] \end{tikzcd}$$

as required. □

The construction of the double groupoid $\mathrm{P}^{(2}K$ is clearly functorial on K, and we have the following:

Theorem 1.6 *The double nerve construction, $\mathcal{G} \mapsto \mathrm{N}^{(2}\mathcal{G}$, embeds, as a reflexive subcategory, the category of double groupoids satisfying the filling condition into the category of those bisimplicial sets K that satisfy the extension condition and such that $\pi_2(K_{*,0}, a) = 0 = \pi_2(K_{0,*}, a)$ for all vertices $a \in K_{0,0}$. The reflector functor for such bisimplicial sets is given by the above described homotopy double groupoid construction*

$$K \mapsto \mathrm{P}^{(2}K.$$

Thus, $\mathrm{P}^{(2}\mathrm{N}^{(2}\mathcal{G} = \mathcal{G}$, *and there are natural bisimplicial maps*

$$\epsilon(K) : K \to \mathrm{N}^{(2}\mathrm{P}^{(2}K, \tag{1.24}$$

such that $\mathrm{P}^{(2}\epsilon = \mathrm{id}$ *and* $\epsilon\mathrm{N}^{(2} = \mathrm{id}$.

Proof: From Theorem 1.3(ii), if $\mathcal{G}$ is any double groupoid satisfying the filling condition, then its double nerve $\mathrm{N}^{(2}\mathcal{G}$ satisfies the extension condition and, since both simplicial sets $\mathrm{N}^{(2}\mathcal{G}_{*,0}$ and $\mathrm{N}^{(2}\mathcal{G}_{0,*}$ are nerves of groupoids, all homotopy groups $\pi_2(\mathrm{N}^{(2}\mathcal{G}_{*,0}, a)$ and $\pi_2(\mathrm{N}^{(2}\mathcal{G}_{0,*}, a)$ vanish. Moreover, since the bihomotopy relation is trivial on the bisimplices $\Delta[p, q] \to \mathrm{N}^{(2}\mathcal{G}$, for $p \geq 1$ or $q \geq 1$, it is easy to see that $\mathrm{P}^{(2}\mathrm{N}^{(2}\mathcal{G} = \mathcal{G}$.

For any bisimplicial set K in the hypothesis of the theorem, there is a natural bisimplicial map

$$\epsilon = \epsilon(K) : K \to \mathrm{N}^{(2}\mathrm{P}^{(2}K,$$

that takes a bisimplex $x : \Delta[p, q] \to K$, of K, to the bisimplex $\epsilon x : [p]\widetilde{\times}[q] \to \mathrm{P}^{(2}K$, of $\mathrm{N}^{(2}\mathrm{P}^{(2}K$, defined by the $p \times q$ configuration of squares in $\mathrm{P}^{(2}K$

$$\left(\begin{tikzcd} \epsilon^r_i x & \epsilon^r_j \arrow[l, "\epsilon^r_{i,j}x"'] \\ \epsilon^s_i x \arrow[u, "\epsilon^{r,s}_i x"] \arrow[ur, phantom, "\epsilon^{r,s}_{i,j}x" description] & \epsilon^s_j x \arrow[l, "\epsilon^s_{i,j}x"] \arrow[u, "\epsilon^{r,s}_j x"'] \end{tikzcd}\right)_{\substack{0 \leq i \leq j \leq p \\ 0 \leq r \leq s \leq q}},$$

where

$$\begin{aligned}
\epsilon^{r,s}_{i,j}x &= [[xd^p_{\mathrm{h}} \cdots d^{j+1}_{\mathrm{h}} d^{j-1}_{\mathrm{h}} \cdots d^{i+1}_{\mathrm{h}} d^{i-1}_{\mathrm{h}} \cdots d^0_{\mathrm{h}} d^q_{\mathrm{v}} \cdots d^{s+1}_{\mathrm{v}} d^{s-1}_{\mathrm{v}} \cdots d^{r+1}_{\mathrm{v}} d^{r-1}_{\mathrm{v}} \cdots d^0_{\mathrm{v}}]],\\
\epsilon^{r,s}_{j}x &= [xd^p_{\mathrm{h}} \cdots d^{j+1}_{\mathrm{h}} d^{j-1}_{\mathrm{h}} \cdots d^0_{\mathrm{h}} d^q_{\mathrm{v}} \cdots d^{s+1}_{\mathrm{v}} d^{s-1}_{\mathrm{v}} \cdots d^{r+1}_{\mathrm{v}} d^{r-1}_{\mathrm{v}} \cdots d^0_{\mathrm{v}}]_{\mathrm{v}},\\
\epsilon^{r}_{i,j}x &= [xd^p_{\mathrm{h}} \cdots d^{j+1}_{\mathrm{h}} d^{j-1}_{\mathrm{h}} \cdots d^{i+1}_{\mathrm{h}} d^{i-1}_{\mathrm{h}} \cdots d^0_{\mathrm{h}} d^q_{\mathrm{v}} \cdots d^{r+1}_{\mathrm{v}} d^{r-1}_{\mathrm{v}} \cdots d^0_{\mathrm{v}}]_{\mathrm{h}},\\
\epsilon^{r}_{i}x &= xd^p_{\mathrm{h}} \cdots d^{i+1}_{\mathrm{h}} d^{i-1}_{\mathrm{h}} \cdots d^0_{\mathrm{h}} d^q_{\mathrm{v}} \cdots d^{r+1}_{\mathrm{v}} d^{r-1}_{\mathrm{v}} \cdots d^0_{\mathrm{v}}.
\end{aligned}$$

Since a straightforward verification shows that $\mathrm{P}^{(2}\epsilon(K)$ is the identity map on $\mathrm{P}^{(2}K$, for any K, and $\epsilon(\mathrm{N}^{(2}\mathcal{G})$ is the identity map on $\mathrm{N}^{(2}\mathcal{G}$, for any double groupoid $\mathcal{G}$, it follows that $\mathrm{N}^{(2}$ is right adjoint to $\mathrm{P}^{(2}$, with ϵ and the identity being the unit and the counit of the adjunction respectively. □

With the next theorem we show that the double groupoid $\mathrm{P}^{(2}K$ represents the same homotopy 2-type as the bisimplicial set K.

Theorem 1.7 *Let K be any bisimplicial set satisfying the extension condition and such that, for all base vertices a, $\pi_2(K_{0,*}, a) = 0$ and $\pi_2(K_{*,0}, a) = 0$. Then, the induced map by unit of the adjunction $|\epsilon| : |K| \to |\mathrm{N}^{(2}\mathrm{P}^{(2}K| = \mathrm{BP}^{(2}K$ is a weak homotopy 2-equivalence.*

Proof: By Facts 1.5 (1) and (3) and Theorem 1.3, the map $|\epsilon| : |K| \to |\mathrm{N}^{(2}\mathrm{P}^{(2}K|$ is, up to natural homotopy equivalences, induced by the simplicial map $\overline{W}\epsilon : \overline{W}K \to \overline{W}\mathrm{N}^{(2}\mathrm{P}^{(2}K$, where both $\overline{W}K$ and $\overline{W}\mathrm{N}^{(2}\mathrm{P}^{(2}K$ are Kan-complexes.

At dimension 0, we have the equalities $\overline{W}K_0 = K_{0,0} = \overline{W}\mathrm{N}^{(2}\mathrm{P}^{(2}K_0$, and the map $\overline{W}\epsilon$ is the identity on 0-simplices. At dimension 1, the map

$$\overline{W}\epsilon : (x_{0,1}, x_{1,0}) \mapsto ([x_{0,1}]_{\mathrm{v}}, [x_{1,0}]_{\mathrm{h}}),$$

is clearly surjective, whence we conclude that the induced

$$\pi_0\overline{W}\epsilon : \pi_0\overline{W}K \to \pi_0\overline{W}\mathrm{N}^{(2}\mathrm{P}^{(2}K \overset{(1.18)}{\cong} \pi_0\mathrm{P}^{(2}K$$

is a bijection and also that, for any vertex $a \in K_{0,0}$, that induced on the π_1-groups

$$\pi_1\overline{W}\epsilon : \pi_1(\overline{W}K, a) \to \pi_1(\overline{W}\mathrm{N}^{(2}\mathrm{P}^{(2}K, a) \overset{(1.18)}{\cong} \pi_1(\mathrm{P}^{(2}K, a)$$

is surjective. To see that $\pi_1\overline{W}\epsilon$ is actually an isomorphism, suppose that $(x_{0,1}, x_{1,0}) \in \overline{W}K_1$, with $x_{0,1}d^1_{\mathrm{v}} = a = x_{1,0}d^0_{\mathrm{h}}$, represents an element in the kernel of $\pi_1\overline{W}\epsilon$. This implies the existence of a bisimplex $x : \Delta[1,1] \to K$ whose bihomotopy class is a square in $\mathrm{P}^{(2}K$ with boundary as in

$$\begin{tikzcd}
a \arrow[r, equal, "{[as^0_{\mathrm{h}}]_{\mathrm{h}}}"] & a \arrow[d, equal, "{[as^0_{\mathrm{v}}]_{\mathrm{v}}}"] \\
\cdot \arrow[u, "{[x_{0,1}]_{\mathrm{v}}}"] \arrow[ur, phantom, "{[[x]]}"] & a \arrow[l, "{[x_{1,0}]_{\mathrm{h}}}"]
\end{tikzcd}$$

Using Lemma 1.9 twice (one in each direction), we can find a bisimplex $x_{1,1} : \Delta[1,1] \to K$, such that $[[x_{1,1}]] = [[x]]$, $x_{1,1}d^1_{\mathrm{v}} = as^0_{\mathrm{h}}$, and $x_{1,1}d^0_{\mathrm{h}} = as^0_{\mathrm{v}}$. Moreover, since $[x_{1,1}d^0_{\mathrm{v}}]_{\mathrm{h}} = [x_{1,0}]_{\mathrm{h}}$ and $[x_{1,1}d^1_{\mathrm{h}}]_{\mathrm{v}} = [x_{0,1}]_{\mathrm{v}}$, there are bisimplices $x_{2,0} : \Delta[2,0] \to K$ and $x_{0,2} : \Delta[0,2] \to K$, with faces as in the picture

$$\begin{tikzcd}
a & a \arrow[l, "as^0_{\mathrm{v}}"'] & a \arrow[l, "as^0_{\mathrm{h}}"'] \\
 & \cdot \arrow[ul, "x_{0,1}"] \arrow[u, "x_{0,2}"] \arrow[ur, phantom, "x_{1,1}"] & a \arrow[l] \arrow[u, "as^0_{\mathrm{v}}"'] \\
 & & a \arrow[ul, "x_{1,0}", "x_{2,0}"'] \arrow[u, "as^0_{\mathrm{h}}"']
\end{tikzcd}$$

This amounts to saying that the triplet $(x_{0,2}, x_{1,1}, x_{2,0})$ is a 2-simplex of $\overline{W}K$ which is a homotopy from $(x_{0,1}, x_{1,0})$ to $(as^0_{\mathrm{v}}, as^0_{\mathrm{h}})$. Then, $(x_{0,1}, x_{1,0})$ represents the identity element of the group $\pi_1(\overline{W}K, a)$. This proves that $\pi_1\overline{W}\epsilon$ is an isomorphism.

Let us now analyze the homomorphism

$$\pi_2\overline{W}\epsilon : \pi_2(\overline{W}K, a) \to \pi_2(\overline{W}\mathrm{N}^{(2}\mathrm{P}^{(2}K, a) \overset{(1.18)}{\cong} \pi_2(\mathrm{P}^{(2}K, a).$$

An element of $\pi_2(\mathrm{P}^{(2}K, a)$ is a square in $\mathrm{P}^{(2}K$ of the form

$$\begin{tikzcd}
a & a \arrow[l, "{[as^0_{\mathrm{h}}]_{\mathrm{h}}}"'] \\
a \arrow[u, "{[as^0_{\mathrm{v}}]_{\mathrm{v}}}"] \arrow[ur, phantom, "{[[x]]}"] & a \arrow[l, "{[as^0_{\mathrm{h}}]_{\mathrm{h}}}"] \arrow[u, "{[as^0_{\mathrm{v}}]_{\mathrm{v}}}"']
\end{tikzcd}$$

and the homomorphism $\pi_2\overline{W}\epsilon$ is induced by the mapping

$$\begin{tikzcd}
a & a \arrow[l, "as^0_{\mathrm{v}}"'] & a \arrow[l, "as^0_{\mathrm{h}}"'] & \\
 & a \arrow[ul, "as^0_{\mathrm{v}}"] \arrow[u, "x_{0,2}"] \arrow[ur, phantom, "x_{1,1}"] & a \arrow[l] \arrow[u, "as^0_{\mathrm{v}}"'] & \longmapsto [[x_{1,1}]]. \\
 & & a \arrow[ul, "as^0_{\mathrm{h}}"] \arrow[u, "x_{2,0}"] \arrow[u, "as^0_{\mathrm{h}}"'] &
\end{tikzcd}$$

That $\pi_2\overline{W}\epsilon$ is surjective is proven using a parallel argument to that given previously for proving that $\pi_1\overline{W}\epsilon$ is injective (given $[[x]]$, using Lemma 1.9 twice, we can find $x_{1,1} : \Delta[1,1] \to K$, etc.). To prove that $\pi_2\overline{W}\epsilon$ is also injective, suppose $(x_{0,2}, x_{1,1}, x_{2,0})$ as above, representing an element of $\pi_2(\overline{W}K, a)$ into the kernel of $\pi_2\overline{W}\epsilon$, that is, such that $[[x_{1,1}]] = [[as^0_{\mathrm{h}}s^0_{\mathrm{v}}]]$. Then, there is a bisimplex $y : \Delta[1,1] \to K$ such that $[x_{1,1}]_{\mathrm{v}} = [y]_{\mathrm{v}}$ and $[y]_{\mathrm{h}} = [as^0_{\mathrm{h}}s^0_{\mathrm{v}}]_{\mathrm{h}}$, whence we can find bisimplices $\alpha' : \Delta[1,2] \to K$ and $\beta' : \Delta[2,1] \to K$ such that

$$\alpha' d^0_{\mathrm{v}} = y d^0_{\mathrm{v}} s^0_{\mathrm{v}}, \quad \alpha' d^1_{\mathrm{v}} = y, \quad \alpha' d^2_{\mathrm{v}} = x_{1,1}, \quad \beta' d^0_{\mathrm{h}} = as^0_{\mathrm{h}}s^0_{\mathrm{v}}, \quad \beta' d^1_{\mathrm{h}} = as^0_{\mathrm{h}}s^0_{\mathrm{v}}, \quad \beta' d^2_{\mathrm{h}} = y.$$

Let us now choose $\theta : \Delta[2,2] \to K$ and $\theta' : \Delta[1,3] \to K$ as respective solutions to the following extension problems

$$\begin{tikzcd}
\Lambda^{2,0}[2,2] \arrow[r, "{(as^0_{\mathrm{h}}s^0_{\mathrm{v}}s^0_{\mathrm{v}}, as^0_{\mathrm{h}}s^0_{\mathrm{v}}s^0_{\mathrm{v}}, -; -, \beta' d^1_{\mathrm{v}} s^0_{\mathrm{v}}, \beta')}"] \arrow[d, hook] & K & \Delta[1]\widetilde{\times}\Lambda^2[3] \arrow[r, "{(\theta d^2_{\mathrm{h}} d^0_{\mathrm{v}} s^0_{\mathrm{v}}, \theta d^2_{\mathrm{h}}, -, \alpha')}"] \arrow[d, hook] & K \\
\Delta[2,2] \arrow[ur, dotted, "\theta"'] & & \Delta[1,3] \arrow[ur, dotted, "\theta'"'] &
\end{tikzcd}$$

Then, for $\alpha = \theta' d^2_{\mathrm{v}} : \Delta[1,2] \to K$ and $\beta = \theta d^0_{\mathrm{v}} : \Delta[2,1] \to K$, we have the equalities

$$\alpha d^0_{\mathrm{v}} = \beta d^2_{\mathrm{h}}, \quad \alpha d^1_{\mathrm{v}} = as^0_{\mathrm{h}}s^0_{\mathrm{v}}, \quad \alpha d^2_{\mathrm{v}} = x_{1,1}, \quad \beta d^0_{\mathrm{h}} = as^0_{\mathrm{h}}s^0_{\mathrm{v}}, \quad \beta d^1_{\mathrm{h}} = as^0_{\mathrm{h}}s^0_{\mathrm{v}}. \tag{1.25}$$

By Lemma 1.2, as the 2nd homotopy groups of $K_{0,*}$ vanish and both bisimplices αd^0_{h} and $as^0_{\mathrm{v}}s^0_{\mathrm{v}}$ have the same vertical faces, there is a vertical homotopy $\omega : \Delta[0,3] \to K$ from $as^0_{\mathrm{v}}s^0_{\mathrm{v}}$ to αd^0_{h}. And similarly, since βd^1_{v} and $x_{2,0}$ have the same horizontal faces and the 2nd homotopy groups of $K_{*,0}$ are all trivial, there is a horizontal homotopy, say $\omega' : \Delta[3,0] \to K$, from βd^1_{v} to $x_{2,0}$. Now, let $\Gamma : \Delta[1,3] \to K$ and $\Gamma' : \Delta[3,1] \to K$ be bisimplices solving, respectively, the extension problems

$$\begin{tikzcd}
\Lambda^{1,2}[1,3] \arrow[r, "{(\omega, -; \alpha d^0_{\mathrm{v}} s^1_{\mathrm{v}}, as^0_{\mathrm{v}}s^0_{\mathrm{v}}s^0_{\mathrm{h}}, -, \alpha)}"] \arrow[d, hook] & K & \Lambda^{3,0}[3,1] \arrow[r, "{(as^0_{\mathrm{v}}s^0_{\mathrm{h}}s^0_{\mathrm{h}}, as^0_{\mathrm{v}}s^0_{\mathrm{h}}s^0_{\mathrm{h}}, \beta, -; -, \omega')}"] \arrow[d, hook] & K \\
\Delta[1,3] \arrow[ur, dotted, "\Gamma"'] & & \Delta[3,1] \arrow[ur, dotted, "\Gamma'"'] &
\end{tikzcd}$$

and take $x_{1,2} = \Gamma d^2_{\mathrm{v}} : \Delta[1,2] \to K$ and $x_{2,1} = \Gamma' d^3_{\mathrm{h}} : \Delta[2,1] \to K$. Then, the same equalities as in (1.25) hold for $x_{1,2}$ instead of α and $x_{2,1}$ instead of β, and moreover $x_{1,2}d_{\rm h}^0 = as_{\rm v}^0 s_{\rm v}^0$ and $x_{2,1}d_{\rm v}^1 = x_{2,0}$. Finally, by taking $x_{0,3}:\Delta[0,3] \to K$ any bisimplex with $x_{0,3}d_{\rm v}^0 = x_{1,2}d_{\rm h}^1$, $x_{0,3}d_{\rm v}^1 = as_{\rm v}^0 s_{\rm v}^0$, $x_{0,3}d_{\rm v}^2 = as_{\rm v}^0 s_{\rm v}^0$ and $x_{0,3}d_{\rm v}^3 = x_{0,2}$, and $x_{3,0}:\Delta[3,0] \to K$ any horizontal homotopy from $as_{\rm h}^0 s_{\rm h}^0$ to $x_{2,1}d_{\rm v}^0$ (which exist thanks to Lemma 1.2), we have the 3-simplex $(x_{0,3}, x_{1,2}, x_{2,1}, x_{3,0})$ of $\overline{W}K$, which is easily recognized as a homotopy from $(as_{\rm v}^0 s_{\rm v}^0, as_{\rm h}^0 s_{\rm v}^0, as_{\rm h}^0 s_{\rm h}^0)$ to $(x_{0,2}, x_{1,1}, x_{2,0})$. Consequently, $(x_{0,2}, x_{1,1}, x_{2,0})$ represents the identity of the group $\pi_2(\overline{W}K, a)$. Therefore, $\pi_2\overline{W}\epsilon$ is an isomorphism, and the proof is complete. □

1.7 The equivalence of homotopy categories.

Recall that the category of weak homotopy types is defined to be the localization of the category of topological spaces with respect to the class of weak equivalences, and the *category of homotopy* 2*-types*, hereafter denoted by Ho(**2-types**), is its full subcategory given by those spaces X with $\pi_i(X,a) = 0$ for any integer $i > 2$ and any base point a.

We now define the *homotopy category of double groupoids satisfying the filling condition*, denoted by $\mathrm{Ho}(\mathbf{DG}_{\rm fc})$, to be the localization of the category $\mathbf{DG}_{\rm fc}$, of these double groupoids, with respect to the class of weak equivalences, as defined in Subsection 1.3.4.

By Corollaries 1.2 and 1.1, both the geometric realization functor $\mathcal{G} \mapsto \mathrm{B}\mathcal{G}$ and the homotopy double groupoid funtor $X \mapsto \Pi^{(2}X$ induce equally denoted functors

$$\mathrm{B} : \mathrm{Ho}(\mathbf{DG}_{\rm fc}) \to \mathrm{Ho}(\textbf{2-types}), \tag{1.26}$$

$$\Pi^{(2} : \mathrm{Ho}(\textbf{2-types}) \to \mathrm{Ho}(\mathbf{DG}_{\rm fc}). \tag{1.27}$$

One of the main goals in this section is to prove the following:

Theorem 1.8 *The induced functors* (1.26) *and* (1.27) *are mutually quasi-inverse, establishing an equivalence of categories*

$$\mathrm{Ho}(\mathbf{DG}_{\rm fc}) \simeq \mathrm{Ho}(\textbf{2-types}).$$

The proof of this Theorem 1.8 is somewhat indirect. Previously, we shall establish the following result, where **KC** is the category of Kan complexes and

$$\mathrm{Ho}(L \in \mathbf{KC} \mid \pi_i L = 0,\ i > 2)$$

is the full subcategory of the homotopy category of Kan complexes given by those L such that $\pi_i(L,a) = 0$ for all $i > 2$ and base vertex $a \in L_0$:

Theorem 1.9 *There are adjoint functors,* $\overline{W}\mathrm{N}^{(2} : \mathbf{DG}_{\rm fc} \to \mathbf{KC}$*, the right adjoint, and* $\mathrm{P}^{(2}\mathrm{Dec} : \mathbf{KC} \to \mathbf{DG}_{\rm fc}$*, the left adjoint, that induce an equivalence of categories*

$$\mathrm{Ho}(\mathbf{DG}_{\rm fc}) \simeq \mathrm{Ho}(L \in \mathbf{KC} \mid \pi_i L = 0,\ i > 2).$$

Proof: The pair of adjoint functors $\mathrm{P}^{(2}\mathrm{Dec} \dashv \overline{W}\mathrm{N}^{(2}$ is obtained by composition of the pair of adjoint functors $\mathrm{Dec} \dashv \overline{W}$, recalled in (1.10), with the pair of adjoint functors $\mathrm{P}^{(2} \dashv \mathrm{N}^{(2}$, stated in Theorem 1.6. For any double groupoid $\mathcal{G} \in \mathbf{DG}_{\mathrm{fc}}$, its double nerve $\mathrm{N}^{(2}\mathcal{G}$ satisfies the extension condition, by Theorem 1.3, and therefore, by Fact 1.5 (4), the simplicial set $\overline{W}\mathrm{N}^{(2}\mathcal{G}$ is a Kan complex. Conversely, if L is any Kan complex, then the bisimplicial set $\mathrm{Dec}L$ satisfies the extension condition by Fact 1.5 (5) and, moreover, $\pi_2(\mathrm{Dec}L_{*,0}, a) = 0 = \pi_2(\mathrm{Dec}L_{0,*}, a)$ for all vertices a, since both augmented simplicial sets $\mathrm{Dec}L_{*,0} \overset{d_0}{\to} L_0$ and $\mathrm{Dec}L_{0,*} \overset{d_1}{\to} L_0$ have simplicial contractions, given respectively by the families of degeneracies $(s_p : L_p \to L_{p+1})_{p\geq 0}$ and $(s_0 : L_q \to L_{q+1})_{q\geq 0}$. Therefore, in accordance with Theorem 1.6, the composite functor $L \mapsto \mathrm{P}^{(2}\mathrm{Dec}L$ is well defined on Kan complexes.

By Fact 1.4 (3), the homotopy equivalences in Fact 1.5 (1), and Corollary 1.2, it follows that a double functor $F : \mathcal{G} \to \mathcal{G}'$, in $\mathbf{DG}_{\mathrm{fc}}$, is a weak equivalence if and only if the induced simplicial map $\overline{W}\mathrm{N}^{(2}F : \overline{W}\mathrm{N}^{(2}\mathcal{G} \to \overline{W}\mathrm{N}^{(2}\mathcal{G}'$ is a homotopy equivalence.

By Facts 1.4 (3) and 1.5 (2), Theorem 1.7, and Corollary 1.2, if $f : L \to L'$ is any simplicial map between Kan complexes L, L' such that $\pi_i(L, a) = 0 = \pi_i(L', a')$ for all $i \geq 3$ and base vertices $a \in L_0$, $a' \in L'_0$, then f is a homotopy equivalence if and only if the induced $\mathrm{P}^{(2}\mathrm{Dec}f : \mathrm{P}^{(2}\mathrm{Dec}L \to \mathrm{P}^{(2}\mathrm{Dec}L'$ is a weak equivalence of double groupoids.

If L is any Kan complex such that $\pi_i(L, a) = 0$ for all $i \geq 3$ and all base vertices $a \in L_0$, then the unit of the adjunction $L \to \overline{W}\mathrm{N}^{(2}\mathrm{P}^{(2}\mathrm{Dec}L$ is a homotopy equivalence since it is the composition of the simplicial maps

$$L \xrightarrow{\mathrm{u}} \overline{W}\mathrm{Dec}L \xrightarrow{\overline{W}\epsilon(\mathrm{Dec}L)} \overline{W}\mathrm{N}^{(2}\mathrm{P}^{(2}\mathrm{Dec}\, L,$$

where u is a homotopy equivalence by Fact 1.5(3) and Fact 1.4 (3), and then $\overline{W}\epsilon(\mathrm{Dec}L)$ is also a homotopy equivalence by Theorem 1.7 and Fact 1.4 (3).

Finally, the counit $\mathrm{P}^{(2}\mathrm{v}(\mathrm{N}^{(2}\mathcal{G}) : \mathrm{P}^{(2}\mathrm{Dec}\overline{W}\mathrm{N}^{(2}\mathcal{G} \to \mathrm{P}^{(2}\mathrm{N}^{(2}\mathcal{G} = \mathcal{G}$, at any double groupoid $\mathcal{G}$, is a weak equivalence, thanks to Fact 1.5(3), Theorem 1.7, and Corollary 1.2. This makes the proof complete. $\square$

Since, by Facts 1.4, the adjoint pair of functors $|\ | \dashv \mathrm{S} : \mathbf{Top} \leftrightarrows \mathbf{KC}$ induces mutually quasi-inverse equivalences of categories

$$\mathrm{Ho}(\mathbf{2\text{-}types}) \simeq \mathrm{Ho}(L \in \mathbf{KC} \mid \pi_i L = 0,\, i > 2),$$

the following follows from Theorem 1.9 above, and Fact 1.5 (1):

Theorem 1.10 *The induced functor* (1.26), $\mathrm{B} : \mathrm{Ho}(\mathbf{DG}_{\mathrm{fc}}) \to \mathrm{Ho}(\mathbf{2\text{-}types})$, *is an equivalence of categories with a quasi-inverse induced by the functor* $X \mapsto \mathrm{P}^{(2}\mathrm{Dec}\,\mathrm{S}X$.

Theorem 1.10 gives half of Theorem 1.8. The remaining part, that is, that the induced functor (1.27) is a quasi-inverse equivalence of (1.26), follows from the proposition below.

Theorem 1.11 *The two induced functors* $\Pi^{(2}, \mathrm{P}^{(2}\mathrm{Dec\,S} : \mathrm{Ho}(\mathbf{2\text{-}types}) \to \mathrm{Ho}(\mathbf{DG}_{\mathrm{fc}})$ *are naturally equivalent.*

Proof: The proof consists in displaying a natural double functor

$$\eta : \mathrm{P}^{(2}\mathrm{DecS}X \to \Pi^{(2}X,$$

which is a weak equivalence for any topological space X. This is as follows:

On objects of $\mathrm{P}^{(2}\mathrm{DecS}X$, the double functor η carries a continuous map $u : \Delta_1 \to X$ to the path $\eta_u : I \to X$ given by

$$\eta_u(x) = u(1-x, x),$$

that is, the obtained from u through the homeomorphism

$$I \cong \Delta_1 : \quad x \mapsto (1-x, x).$$

On horizontal morphisms of $\mathrm{P}^{(2}\mathrm{DecS}X$, η acts by

$$(gd^0 \xrightarrow{[g]_{\mathrm{h}}} gd^1) \ \overset{\eta}{\mapsto}\ (\eta_{gd^0} \to \eta_{gd^1}),$$

for any continuous map $g : \Delta_2 \to X$. That is, η carries the horizontal homotopy class of g in $\mathrm{DecS}X$ to the unique horizontal morphism in $\Pi^{(2}X$ from the path η_{gd^0} to the path η_{gd^1}. This correspondence is well defined since

$$\eta_{gd^0}(1) = gd^0(0,1) = g(0,0,1) = gd^1(0,1) = \eta_{gd^1}(1),$$

and, moreover, if $[g]_{\mathrm{h}} = [g']_{\mathrm{h}}$ in $\mathrm{DecS}X$, then $gd^i = g'd^i$ for $i = 0, 1$.

Similarly, on vertical morphisms, η is given by

$$(gd^1 \xrightarrow{[g]_{\mathrm{v}}} gd^2) \ \overset{\eta}{\mapsto}\ (\eta_{gd^1} \to \eta_{gd^2}).$$

On squares in $\mathrm{P}^{(2}\mathrm{DecS}X$, η is defined through the embedding $I \times I \hookrightarrow \Delta_3$

$$\begin{tikzcd}
 & & & & (0,0,1,0) & & \\
(0,1) & (1,1) \arrow[l] \arrow[d] & {} \arrow[r, hook] & {} & & & (1,0,0,0) \arrow[llu, "d^1d^2"'] \arrow[ld, "d^1d^1"] \\
(0,0) \arrow[u] \arrow[r] & (1,0) & & (0,1,0,0) \arrow[ruu, "d^0d^2"] \arrow[rr, "d^0d^1"'] \arrow[rrru, dotted] & & (0,0,0,1) \arrow[luu, dotted] &
\end{tikzcd}$$

given by: $(x, y) \mapsto \big(xy, (1-x)(1-y), (1-x)y, x(1-y)\big)$. That is, for any continuous map $\alpha : \Delta_3 \to X$, η is defined by

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "{[\alpha d^3]_{\mathrm{h}}}"'] & & \eta_{\alpha d^1 d^2} & \eta_{\alpha d^0 d^2} \arrow[l] \\
\cdot \arrow[u, "{[\alpha d^1]_{\mathrm{v}}}"] \arrow[ur, phantom, "{[[\alpha]]}"] & \cdot \arrow[l, "{[\alpha d^2]_{\mathrm{h}}}"] \arrow[u, "{[\alpha d^0]_{\mathrm{v}}}"'] & \overset{\eta}{\mapsto} & \eta_{\alpha d^1 d^1} \arrow[u] \arrow[ur, phantom, "{[\eta_\alpha]}"] & \eta_{\alpha d^0 d^1} \arrow[l] \arrow[u]
\end{tikzcd}$$

where $\eta_\alpha : I \times I \to X$ is the square in X given by the formula

$$\eta_\alpha(x, y) = \alpha\big(xy, (1-x)(1-y), (1-x)y, x(1-y)\big).$$

To see that η is well defined on squares in $\mathrm{P}^{(2}\mathrm{DecS}X$, suppose $[[\alpha_1]] = [[\alpha_2]]$. This means that $[\alpha_1]_{\mathrm{h}} = [\alpha]_{\mathrm{h}}$ and $[\alpha_2]_{\mathrm{v}} = [\alpha]_{\mathrm{v}}$, for some $\alpha : \Delta_3 \to X$, in the bisimplicial set $\mathrm{DecS}X$. Then, there are maps $\beta, \gamma : \Delta_4 \to X$ such that the following equalities hold:

$$\beta d^0 = \alpha_1 d^0 s^0,\ \beta d^1 = \alpha_1,\ \beta d^2 = \alpha = \gamma d^3,\ \gamma d^4 = \alpha_2,\ \gamma d^2 = \alpha_2 d^2 s^2;$$

whence the equalities of squares in $\Pi^{(2}X$, $[\eta_{\alpha_1}] = [\eta_\alpha] = [\eta_{\alpha_2}]$, follow from the relative homotopies $F_1 : \eta_{\alpha_1} \to \eta_\alpha$ and $F_2 : \eta_\alpha \to \eta_{\alpha_2}$, respectively given by the formulas

$$F_1(x, y, t) = \beta\big(xy, t(1-x)(1-y), (1-t)(1-x)(1-y), (1-x)y, x(1-y)\big),$$
$$F_2(x, y, t) = \gamma\big(xy, (1-x)(1-y), (1-x)y, tx(1-y), (1-t)x(1-y)\big).$$

Most of the details to confirm η is actually a double functor are routine and easily verifiable. We leave them to the reader since the only ones with any difficulty are those a)- d) proven below.

a) For $\omega : \Delta_4 \to X$, $[\eta_{\omega d^1}] = [\eta_{\omega d^2}] \circ_{\mathrm{h}} [\eta_{\omega d^0}]$,

b) For $\omega : \Delta_4 \to X$, $[\eta_{\omega d^3}] = [\eta_{\omega d^4}] \circ_{\mathrm{v}} [\eta_{\omega d^2}]$,

c) For $g : \Delta_2 \to X$, $[\eta_{g s^2}] = 1^{\mathrm{v}}_{(\eta_{g d^1}, \eta_{g d^0})}$,

d) For $g : \Delta_2 \to X$, $[\eta_{g s^0}] = 1^{\mathrm{h}}_{(\eta_{g d^2}, \eta_{g d^1})}$.

To prove a), let $H : I^2 \times I \to X$ be the continuous map defined by

$$H(x,y,t) = \begin{cases} \omega\Big((1\text{-}t)xy, 2tx(x+y), (1\text{-}x)(1\text{-}y)+tx(2x\text{-}2+y), \\ \qquad y(1\text{-}x)+tx(1\text{-}2x\text{-}y), x(1\text{-}y)+tx(1\text{-}2x\text{-}y)\Big) & \text{if } x+y \le 1, x \le y, \\ \omega\Big((1\text{-}t)xy, 2ty(x+y), (1\text{-}x)(1\text{-}y)+ty(2y\text{-}2+x), \\ \qquad y(1\text{-}x)+ty(1\text{-}x\text{-}2y), x(1\text{-}y)+ty(1\text{-}x\text{-}2y)\Big) & \text{if } x+y \le 1, x \ge y, \\ \omega\Big(xy+t(1\text{-}y)(1\text{-}x\text{-}2y), 2t(1\text{-}y)(2\text{-}x\text{-}y), (1\text{-}t)(1\text{-}x)(1\text{-}y), \\ \qquad y(1\text{-}x)\text{-}t(1\text{-}y)(2\text{-}x\text{-}2y), (1\text{-}y)(x\text{-}t(2\text{-}x\text{-}2y)\Big) & \text{if } x+y \ge 1, x \le y, \\ \omega\Big(xy+t(1\text{-}x)(1\text{-}2x\text{-}y), 2t(1\text{-}x)(2\text{-}x\text{-}y), (1\text{-}t)(1\text{-}x)(1\text{-}y), \\ \qquad (1\text{-}x)(y\text{-}t(2\text{-}2x\text{-}y)), x(1\text{-}y)\text{-}t(1\text{-}x)(2\text{-}2x\text{-}y)\Big) & \text{if } x+y \ge 1, x \ge y. \end{cases}$$

When $t = 0$, then

$$H(x,y,0) = \omega\big(xy, 0, (1-x)(1-y), (1-x)y, x(1-y)\big) = \eta_{\omega d^1}(x,y),$$

while, if $t = 1$, then
$H(x,y,1) =$

- $\omega\big((0,2x(x+y),(1\text{-}x)(1\text{-}y)+x(2x\text{-}2+y),y(1\text{-}x)+x(1\text{-}2x\text{-}y),x(1\text{-}y)+tx(1\text{-}2x\text{-}y)\big)$ if $x+y\leq 1, x\leq y,$
- $\omega\big((0,2y(x+y),(1\text{-}x)(1\text{-}y)+y(2y\text{-}2+x),y(1\text{-}x)+ty(1\text{-}x\text{-}2y),x(1\text{-}y)+y(1\text{-}x\text{-}2y)\big)$ if $x+y\leq 1, x\geq y,$
- $\omega\big(xy+(1\text{-}y)(1\text{-}x\text{-}2y),2(1\text{-}y)(2\text{-}x\text{-}y),0,y(1\text{-}x)\text{-}(1\text{-}y)(2\text{-}x\text{-}2y),(1\text{-}y)(x\text{-}(2\text{-}x\text{-}2y)\big)$ if $x+y\geq 1, x\leq y,$
- $\omega\big(xy+(1\text{-}x)(1\text{-}2x\text{-}y),2(1\text{-}x)(2\text{-}x\text{-}y),0,(1\text{-}x)(y\text{-}(2\text{-}2x\text{-}y)),x(1\text{-}y)\text{-}(1\text{-}x)(2\text{-}2x\text{-}y)\big)$ if $x+y\geq 1, x\geq y.$

$$\overset{(1.15)}{=} (\eta_{\omega d^2} \circ_{\mathrm{h}} \eta_{\omega d^0})(x,y).$$

Hence, since the following equalities hold:

$$\begin{aligned}
H(x,0,t) &= \omega(0,0,1-x,0,x) = \omega d^1 d^0 d^1(1-x,x) = \eta_{\omega d^1 d^0 d^1}(x),\\
H(0,y,t) &= \omega(0,0,1-y,y,0) = \omega d^1 d^0 d^2(1-y,y) = \eta_{\omega d^1 d^0 d^2}(y),\\
H(x,1,t) &= \omega(x,0,0,1-x,0) = \omega d^1 d^1 d^2(x,1-x) = \eta_{\omega d^1 d^1 d^2}(1-x),\\
H(1,y,t) &= \omega(y,0,0,0,1-y) = \omega d^1 d^1 d^1(y,1-y) = \eta_{\omega d^1 d^1 d^1}(1-y),
\end{aligned}$$

we conclude that H is actually a relative homotopy

$$\begin{tikzcd}
\cdot & & \cdot \arrow[ll] \arrow[dd] & & & \cdot & & \cdot \arrow[ll] \arrow[dd] \\
& \eta_{\omega d^1} & & {} \arrow[r, "H"] & {} & & \cdot \arrow[ul, "\eta_{\omega d^2}"] \arrow[dr, "\eta_{\omega d^0}"'] & \\
\cdot \arrow[uu] \arrow[rr] & & \cdot & & & \cdot \arrow[uu] \arrow[rr] & & \cdot
\end{tikzcd}$$

between $\eta_{\omega d^1}$ and $\eta_{\omega d^2} \circ_{\mathrm{h}} \eta_{\omega d^0}$, as required for proving a). The proof of b) is completely similar.

Now we attend to the case c). For, let $H : I^2 \times I \to X$ be the continuous map defined by

$$H(x,y,t) = \begin{cases} g\big((1\text{-}t)xy,(1\text{-}x)(1\text{-}y)-txy,x+y+2xy(t\text{-}1)\big) & \text{if } x+y \leq 1,\\ g\big(xy+t(x+y\text{-}1\text{-}xy),(1\text{-}t)(1\text{-}x)(1\text{-}y),x+y\text{-}2xy+2t(1\text{-}x)(1\text{-}y)\big) & \text{if } x+y \geq 1. \end{cases}$$

When $t = 0$, then

$$\begin{aligned}
H(x,y,0) &= g\big(xy, (1-x)(1-y), x+y-2xy\big)\\
&= gs^2\big(xy, (1-x)(1-y), (1-x)y, x(1-y)\big) = \eta_{gs^2}(x,y),
\end{aligned}$$

while, if $t = 1$, then

$$\begin{aligned}
H(x,y,1) &= \begin{cases} g(0, 1-x-y, x+y) & \text{if } x+y \leq 1,\\ g(x+y-1, 0, 2-x-y) & \text{if } x+y \geq 1. \end{cases}\\
&= \begin{cases} gd^0(1-x-y, x+y) & \text{if } x+y \leq 1,\\ gd^1(x+y-1, 2-x-y) & \text{it } x+y \geq 1. \end{cases}\\
&\overset{(1.16)}{=} e^{\mathrm{v}}(gd^1, gd^0)(x,y).
\end{aligned}$$

Moreover,

$$\begin{aligned}
H(x,0,t) &= g(0,1-x,x) = gd^0(1-x,x) = \eta_{gd^0}(x),\\
H(0,y,t) &= g(0,1-y,y) = gd^0(1-y,y) = \eta_{gd^0}(y),\\
H(x,1,t) &= g(x,0,1-x) = gd^1(x,1-x) = \eta_{gd^1}(1-x),\\
H(1,y,t) &= g(y,0,1-y) = gd^1(y,1-y) = \eta_{gd^1}(1-y),
\end{aligned}$$

and therefore we conclude that H is actually a relative homotopy

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, "gd^1"'] \arrow[d, "gd^1"] & & \cdot & \cdot \arrow[l, "gd^1"'] \arrow[d, "gd^1"] \\
\cdot \arrow[u, "gd^0"] \arrow[r, "gd^0"'] & \cdot & & \cdot \arrow[u, "gd^0"] \arrow[r, "gd^0"'] & \cdot
\arrow[from=1-1, to=2-2, phantom, "\eta_{gs^2}"]
\arrow[from=1-4, to=2-5, phantom, "e^{\mathrm{v}}"]
\arrow[from=1-2, to=2-4, phantom, "\xrightarrow{H}"]
\end{tikzcd},$$

between η_{gs^2} and $e^{\mathrm{v}}(gd^1, gd^0)$. This proves c), and the proof of d) is parallel.

Hence, $\eta : \mathrm{P}^{(2}\mathrm{DecS}X \to \Pi^{(2}X$ is a double functor, which is clearly natural on the topological space X. Moreover, for any X, it is actually a weak equivalence since, for any 1-simplex $u : \Delta_1 \to X$ and integer $i \geq 0$, the induced map $\pi_i\eta : \pi_i(\mathrm{P}^{(2}\mathrm{DecS}X, u) \to \pi_i(\Pi^{(2}X, \eta_u)$ occurs in this commutative diagram

$$\begin{tikzcd}
\pi_i(\mathrm{P}^{(2}\mathrm{DecS}X, u) \arrow[ddd, "\pi_i\eta"'] & \pi_i(\mathrm{BP}^{(2}\mathrm{DecS}X, u) \arrow[l, "\mathrm{Th.1.4}"', "\cong"] & \pi_i(|\mathrm{DecS}X|, u) \arrow[l, "\mathrm{Th.1.7}"', "\cong"] \arrow[d, "\cong", "\mathrm{Fact1.5(1)}"'] \\
 & & \pi_i(|\overline{W}\mathrm{DecS}X|, u) \arrow[dd, "\cong", "\mathrm{Fact1.5(3)}"'] \\
 & & \\
\pi_i(\Pi^{(2}X, \eta_u) \arrow[r, "\mathrm{Th.1.2}", "\cong"'] & \pi_i(X, u(1,0)) & \pi_i(|\mathrm{S}X|, u(1,0)) \arrow[l, "\mathrm{Fact1.4(6)}"', "\cong"]
\end{tikzcd}$$

in which all other maps are bijections (group isomorphisms for $i \geq 1$) by the references in the labels. □

Chapter 2

Comparing geometric realizations of tricategories

2.1 Introduction and summary

As we mention in Chapter 1, the process of taking classifying spaces of categorical structures has shown its relevance as a tool in algebraic topology and algebraic K-theory, and one of the main reasons is that the classifying space constructions transport categorical coherence to homotopic coherence. We can easily stress the historical relevance of the construction of classifying spaces by recalling that Quillen [109] defines a higher algebraic K-theory by taking homotopy groups of the classifying spaces of certain categories. Joyal and Tierney [89] have shown that Gray-groupoids are a suitable framework for studying homotopy 3-types. Monoidal categories were shown by Stasheff [114] to be algebraic models for loop spaces, and work by May [105] and Segal [112] showed that classifying spaces of symmetric monoidal categories provide the most noteworthy examples of spaces with the extra structure required to define an Ω-spectrum, a fact exploited with great success in algebraic K-theory.

This chapter contributes to the study of classifying spaces for (small) tricategories, introduced by Gordon, Power and Street in [69]. The homotopy theory of higher categorical structures has demonstrated relevance as a tool for the treatment of an extensive list of subjects of recognized mathematical interest in several mathematical contexts beyond homotopy theory, such as algebraic geometry, geometric structures on low-dimensional manifolds, string theory, or topological quantum field theory and conformal field theory.

We explore the relationship amongst three different 'nerves' that might reasonably be associated to any tricategory $\mathcal{T}$. These are the pseudo-simplicial bicategory called the *Grothendieck nerve* $\mathrm{N}\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Bicat}$, the simplicial bicategory termed the *Segal nerve* $\mathrm{S}\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Hom}$, and the simplicial set called the *geometric nerve* of the tricategory $\Delta\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$. Since, as we prove, these three nerve constructions lead to homotopy equivalent spaces, any one of these spaces could therefore be taken as *the* classifying space $\mathrm{B}_3\mathcal{T}$ of the tricategory[1]. Many properties of the classifying space construction for tricategories, $\mathcal{T} \mapsto \mathrm{B}_3\mathcal{T}$, may be easier to establish depending on the nerve used for realizations. Here, both for historical reasons and for theoretical interest, it is appropriate to start with the Grothendieck nerve construction to introduce $\mathrm{B}_3\mathcal{T}$. Let us briefly recall that it was Grothendieck who first associated a simplicial set $\mathrm{N}\mathcal{C}$ to a small category $\mathcal{C}$, calling it its nerve, whose p-simplices are composable p-tuples $x_0 \to \cdots \to x_p$ of morphisms in $\mathcal{C}$. Geometric realization of its nerve is the classifying space of the category, $\mathrm{B}\mathcal{C} = |\mathrm{N}\mathcal{C}|$. A first result in this chapter shows how the Grothendieck nerve construction for categories rises to tricategories. Thus, we prove

> **Theorem 2.1** *Any tricategory $\mathcal{T}$ defines a pseudo-simplicial bicategory, that is, a trihomomorphism $\mathrm{N}\mathcal{T} = (\mathrm{N}\mathcal{T}, \chi, \omega)\colon \Delta^{\mathrm{op}} \to$ **Bicat**, whose bicategory of p-simplices*
>
> $$\mathrm{N}\mathcal{T}_p = \bigsqcup_{(x_0,\dots,x_p)\in \mathrm{Ob}\mathcal{T}^{p+1}} \mathcal{T}(x_{p-1},x_p)\times\mathcal{T}(x_{p-2},x_{p-1})\times\cdots\times\mathcal{T}(x_0,x_1)$$
>
> *consists of p-tuples of horizontally composable cells.*

Then, heavily dependent on the results by Carrasco, Cegarra and Garzón in [42], where an analysis of classifying spaces is performed for lax diagrams of bicategories following the way Segal [112] and Thomason [120] analyzed lax diagrams of categories, we introduce *the classifying space* $\mathrm{B}_3\mathcal{T}$, of a tricategory $\mathcal{T}$, to be the classifying space of its Grothendieck nerve $\mathrm{N}\mathcal{T}$. Briefly, say that the so-called Grothendieck construction [42, §3.1] on the pseudo-simplicial bicategory $\mathrm{N}\mathcal{T}$ produces a bicategory $\int_\Delta \mathrm{N}\mathcal{T}$. Again, the Grothendieck nerve construction on this bicategory $\int_\Delta \mathrm{N}\mathcal{T}$ now gives rise to a normal pseudo-simplicial category $\mathrm{N}(\int_\Delta \mathrm{N}\mathcal{T})\colon \Delta^{\mathrm{op}} \to$ **Cat**, on which once more the Grothendieck construction leads to a category, $\int_\Delta \mathrm{N}(\int_\Delta \mathrm{N}\mathcal{T})$, whose classifying space is, by definition, the classifying space of the tricategory, that is:

$$\mathrm{B}_3\mathcal{T} = |\mathrm{N}(\textstyle\int_\Delta \mathrm{N}(\int_\Delta \mathrm{N}\mathcal{T}))|$$

or, in other words, $\mathrm{B}_3\mathcal{T} = \mathrm{B}_2 \int_\Delta \mathrm{N}\mathcal{T} = \mathrm{B}\int_\Delta \mathrm{N}(\int_\Delta \mathrm{N}\mathcal{T})$, where $\mathrm{B}_2\mathcal{B}$ denotes the classifying space of any bicategory $\mathcal{B}$ as defined by Carrasco, Cegarra and Garzón [41, Definition 3.1]. The behavior of this classifying space construction, $\mathcal{T} \mapsto \mathrm{B}_3\mathcal{T}$, can be summarized as follows (see Propositions 2.1 and 2.4 and Corollary 2.3):

> - *Any trihomomorphism $F\colon \mathcal{T} \to \mathcal{T}'$ induces a continuous map* $\mathrm{B}_3F\colon \mathrm{B}_3\mathcal{T} \to \mathrm{B}_3\mathcal{T}'$.

---

[1] Throughout this chapter we indicate the different classifying spaces of categories, bicategories and tricategories by B, B_2 and B_3 respectively to avoid confusion. In the rest of the thesis we will omit the index since it will be clear in each case by context.

- *For any composable trihomomorphisms* $F\colon \mathcal{T} \to \mathcal{T}'$ *and* $F'\colon \mathcal{T}' \to \mathcal{T}''$, *there is a homotopy* $\mathrm{B}_3F'\,\mathrm{B}_3F \simeq \mathrm{B}_3(F'F)\colon \mathrm{B}_3\mathcal{T} \to \mathrm{B}_3\mathcal{T}''$ *and, for any tricategory* $\mathcal{T}$, *there is a homotopy* $\mathrm{B}_3 1_{\mathcal{T}} \simeq 1_{\mathrm{B}_3\mathcal{T}}$.
- *If* $F, G\colon \mathcal{T} \to \mathcal{T}'$ *are two trihomomorphisms, then any tritransformation,* $F \Rightarrow G$, *canonically defines a homotopy* $\mathrm{B}_3F \simeq \mathrm{B}_3G\colon \mathrm{B}_3\mathcal{T} \to \mathrm{B}_3\mathcal{T}'$ *between the induced maps on classifying spaces.*
- *Any triequivalence of tricategories* $\mathcal{T} \to \mathcal{T}'$ *induces a homotopy equivalence on classifying spaces* $\mathrm{B}_3\mathcal{T} \simeq \mathrm{B}_3\mathcal{T}'$.

For instance, for every tricategory $\mathcal{T}$ there is a Gray-category $\mathrm{G}(\mathcal{T})$ with a triequivalence $\mathcal{T} \to \mathrm{G}(\mathcal{T})$, thanks to the coherence theorem for tricategories by Gordon, Power and Street [69, Theorem 8.1]. Then, it is a consequence of the properties above that

- *There is an induced homotopy equivalence* $\mathrm{B}_3\mathcal{T} \simeq \mathrm{B}_3\mathrm{G}(\mathcal{T})$.

To deal with the delooping properties of certain classifying spaces, for any tricategory $\mathcal{T}$, we introduce its Segal nerve $\mathrm{S}\mathcal{T}$. This is a simplicial bicategory whose bicategory of p-simplices, $\mathrm{S}\mathcal{T}_p$, is *the bicategory of unitary trihomomorphisms* of the ordinal category $[p]$ into $\mathcal{T}$. Each $\mathrm{S}\mathcal{T}$ is a *special* simplicial bicategory, in the sense that the Segal projection homomorphisms on it are biequivalences of bicategories, and thus it is a weak 3-category from the standpoint of Tamsamani [119] and Simpson [113]. When $\mathcal{T}$ is a reduced tricategory (i.e., with only one object), then the simplicial space $\mathrm{B}_2\mathrm{S}\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Top}$, obtained by replacing the bicategories $\mathrm{S}\mathcal{T}_p$ by their classifying spaces $\mathrm{B}_2(\mathrm{S}\mathcal{T}_p)$, is a special simplicial space. Therefore, according to Segal [112, Proposition 1.5], under favorable circumstances, $\Omega|\mathrm{B}_2\mathrm{S}\mathcal{T}|$ is homotopy equivalent to $\mathrm{B}_2(\mathrm{S}\mathcal{T}_1)$. In our development here, the relevant result is

Theorem 2.2 *For any tricategory* $\mathcal{T}$, *there is a homotopy equivalence* $\mathrm{B}_3\mathcal{T} \simeq |\mathrm{B}_2\mathrm{S}\mathcal{T}|$.

Any monoidal bicategory $(\mathcal{B}, \otimes)$ gives rise to a one-object tricategory $\Sigma(\mathcal{B}, \otimes)$, its '*suspension*' tricategory following Street's terminology [117, §9, Example 2] (or '*delooping*' in the terminology of Kapranov-Voevodsky [90] or Berger [17]). Defining the classifying space of a monoidal bicategory $(\mathcal{B}, \otimes)$ to be the classifying space of its suspension tricategory, that is, $\mathrm{B}_3(\mathcal{B}, \otimes) = \mathrm{B}_3\Sigma(\mathcal{B}, \otimes)$, we prove the following extension to bicategories of the aforementioned fact by Stasheff on monoidal categories.

Theorem 2.3 *Let* $(\mathcal{B}, \otimes)$ *be a monoidal bicategory such that, for any object* $x \in \mathcal{B}$, *the homomorphism* $x \otimes -\colon \mathcal{B} \to \mathcal{B}$ *induces a homotopy autoequivalence on the classifying space* $\mathrm{B}_2\mathcal{B}$ *of* $\mathcal{B}$. *Then, there is a homotopy equivalence*

$$\mathrm{B}_2\mathcal{B} \simeq \Omega\mathrm{B}_3(\mathcal{B}, \otimes),$$

between the classifying space of the underlying bicategory and the loop space of the classifying space of the monoidal bicategory.

If $(\mathcal{C}, \otimes, \boldsymbol{c})$ is any braided monoidal category, then, thanks to the braiding, the suspension of the underlying monoidal category $\Sigma(\mathcal{C}, \otimes)$, which is actually a bicategory, has a structure of monoidal bicategory. Hence, the double suspension tricategory $\Sigma^2(\mathcal{C}, \otimes, \boldsymbol{c})$ is defined. According to Jardine [86, Setion 3], the classifying space of the monoidal category $\mathrm{B}_2(\mathcal{C}, \otimes)$ is the classifying space of its suspension bicategory, and, following Carrasco, Cegarra and Garzón [42, Definition 6.1], the classifying space of the braided monoidal category $\mathrm{B}_3(\mathcal{C}, \otimes, \boldsymbol{c})$ is the classifying space of its double suspension tricategory. Hence, from the above result we get the following.

Corollary 2.2 (*i*) *For any braided monoidal category* $(\mathcal{C}, \otimes, \boldsymbol{c})$ *there is a homotopy equivalence*

$$\mathrm{B}_2(\mathcal{C}, \otimes) \simeq \Omega \mathrm{B}_3(\mathcal{C}, \otimes, \boldsymbol{c}).$$

(*ii*) *Let* $(\mathcal{C}, \otimes, \boldsymbol{c})$ *be a braided monoidal category such that, for any object* $x \in \mathcal{C}$, *the functor* $x \otimes -\colon \mathcal{C} \to \mathcal{C}$ *induces a homotopy auto-equivalence on the classifying space of* $\mathcal{C}$. *Then, there is a homotopy equivalence*

$$\mathrm{B}\mathcal{C} \simeq \Omega^2 \mathrm{B}_3(\mathcal{C}, \otimes, \boldsymbol{c}).$$

Thus, under natural hypothesis, the double suspension tricategory $\Sigma^2(\mathcal{C}, \otimes, \boldsymbol{c})$ is a categorical model for the double deelooping space of the classifying space of the underlying category $\mathcal{C}$, a fact already proved in [42, Theorem 6.10] (cf. Berger [17, Proposition 2.11] and Balteanu, Fiedorowicz, Schwänzl and Vogt [12, Theorem 2.2]).

The process followed for defining the classifying space of a tricategory $\mathcal{T}$, by means of its Grothendieck nerve $\mathrm{N}\mathcal{T}$, is quite indirect and the CW-complex $\mathrm{B}_3\mathcal{T}$ thus obtained has little apparent intuitive connection with the cells of the original tricategory. However, when $\mathcal{T}$ is a (strict) 3-category, then the space $|\mathrm{diagNNN}\mathcal{T}|$, the geometric realization of the simplicial set diagonal of the 3-simplicial set 3-fold nerve of $\mathcal{T}$, has usually been taken as the 'correct' classifying space of the 3-category. In Example 2.4, we state that, for a 3-category $\mathcal{T}$, there is homotopy equivalence

$$\mathrm{B}_3\mathcal{T} \simeq |\mathrm{diagNNN}\mathcal{T}|.$$

The construction of the simplicial set $\mathrm{diagNNN}\mathcal{T}$ for 3-categories does not work in the non-strict case since the compositions in arbitrary tricategories are not associative and not unitary, which is crucial for the 3-simplicial structure of the triple nerve $\mathrm{NNN}\mathcal{T}$, but only up to coherent equivalences or isomorphisms. There is, however, another convincing way of associating a simplicial set to a 3-category $\mathcal{T}$ through its *geometric nerve* $\Delta\mathcal{T}$, thanks to Street [116]. He extends each ordinal $[p] = \{0 < 1 < \cdots < p\}$ to a p-category $\mathcal{O}_p$, the p^{th}-*oriental*, such that the p-simplices of $\Delta\mathcal{T}$ are just the p-functors $\mathcal{O}_p \to \mathcal{T}$. Thus, $\Delta\mathcal{T}$ is a simplicial set whose 0-simplices are the objects (0-cells) $F0$ of $\mathcal{T}$, whose 1-simplices are the 1-cells $F_{0,1}\colon F0 \to F1$, whose 2-simplices

$$\begin{tikzcd}[column sep=small] & F0 \arrow[dl, "F_{0,1}"'] \arrow[dr, "F_{0,2}"] \arrow[d, phantom, "\overset{F_{0,1,2}}{\Rightarrow}"] & \\ F1 \arrow[rr, "F_{1,2}"'] & {} & F2, \end{tikzcd}$$

consist of two composable 1-cells and a 2-cell $F_{0,1,2}$: $F_{1,2} \otimes F_{0,1} \Rightarrow F_{0,2}$, and so on. In fact, the geometric nerve construction $\Delta\mathcal{T}$ even works for arbitrary tricategories $\mathcal{T}$, as Duskin [58] and Street [118] pointed out, and we discuss here in detail. The geometric nerve $\Delta\mathcal{T}$ is defined to be the simplicial set whose p-simplices are *unitary lax functors* of the ordinal category $[p]$ in the tricategory $\mathcal{T}$. This is a simplicial set which completely encodes all the structure of the tricategory and, furthermore, the cells of its geometric realization $|\Delta\mathcal{T}|$ have a pleasing geometrical description in terms of the cells of $\mathcal{T}$. As a main result in the chapter, we state and prove that

Theorem 2.4 *For any tricategory* $\mathcal{T}$*, there is a homotopy equivalence* $\mathrm{B}_3\mathcal{T} \simeq |\Delta\mathcal{T}|$.

If $(\mathcal{B}, \otimes)$ is any monoidal bicategory, then its geometric nerve, $\Delta(\mathcal{B}, \otimes)$, is defined to be the geometric nerve of its suspension tricategory $\Sigma(\mathcal{B}, \otimes)$. Then, in Example 2.10, we obtain that there is a homotopy equivalence

$$\mathrm{B}_3(\mathcal{B}, \otimes) \simeq |\Delta(\mathcal{B}, \otimes)|.$$

For instance, since the geometric nerve of a braided monoidal category $(\mathcal{C}, \otimes, \boldsymbol{c})$ is the geometric nerve of its double suspension tricategory, that is, $\Delta(\mathcal{C}, \otimes, \boldsymbol{c}) = \Delta\Sigma^2(\mathcal{C}, \otimes, \boldsymbol{c})$, the existence of a homotopy equivalence $\mathrm{B}_3(\mathcal{C}, \otimes, \boldsymbol{c}) \simeq |\Delta(\mathcal{C}, \otimes, \boldsymbol{c})|$ follows, a fact proved by Carrasco, Cegarra and Garzón [42, Theorem 6.9].

The geometric nerve $\Delta(\mathcal{B}, \otimes)$, of any given monoidal bicategory $(\mathcal{B}, \otimes)$, is a Kan complex if and only if $(\mathcal{B}, \otimes)$ is a *bicategorical group*, that is, a monoidal bicategory whose 2-cells are isomorphisms, whose 1-cells are equivalences, and each object x has a quasi-inverse with respect to the tensor product. In other words, a bicategorical group is a monoidal bicategory whose suspension tricategory $\Sigma(\mathcal{B}, \otimes)$ is a *trigroupoid* (or *Azumaya tricategory* in the terminology of Gordon, Power and Street [69]). The geometric nerve of any bicategorical group $(\mathcal{B}, \otimes)$ is then a Kan complex, whose classifying space $\mathrm{B}_3(\mathcal{B}, \otimes)$ is a path-connected homotopy 3-type. In fact, every connected homotopy 3-type can be realized in this way from a bicategorical group, as suggested by the unpublished but widely publicized result of Joyal and Tierney [89] that Gray-groups (called semistrict 3-groups by Baez and Neuchl [9]) model connected homotopy 3-types (see also Berger [17], Lack [94], or Leroy [97]). Recall that, by the coherence theorem for tricategories, every bicategorical group is monoidal biequivalent to a Gray-group. In the last Subsection 2.5.1, we outline in some detail the proof of the following statement.

Proposition 2.5 *For any path-connected pointed CW-complex* X*, there is a bicategorical group* $(\mathcal{B}(X), \otimes)$ *with a homotopy equivalence* $\mathrm{B}_3(\mathcal{B}(X), \otimes) \simeq X$*, if and only if* $\pi_i X = 0$ *for* $i \geq 4$.

The bicategorical group $(\mathcal{B}(X), \otimes)$ we build, associated to any space X as above, might be recognized as a skeleton of Gurski's *monoidal fundamental bigroupoid* of the loop space of X, $(\Pi_2(\Omega X),\otimes)$, [75, Theorem 1.4]. In the particular case when, in addition, $\pi_3 X = 0$, then the resulting bicategorical group $(\mathcal{B}(X),\otimes)$ has all its 2-cells identities, so it is actually a *categorical group* in the sense of Joyal and Street [88, Definition 3.1]. While in the particular case where $\pi_1 X = 0$, the bicategorical group $(\mathcal{B}(X),\otimes)$ has only one object, so that it is the suspension of a *braided categorical group*, see Cheng and Gurski [53, §2]. Hence, our proof implicitly covers two relevant particular cases, already well-known from Joyal and Tierney [89] (see also Carrasco and Cegarra [39, Theorems 2.6 and 2.10]), one stating that categorical groups are a convenient algebraic model for connected homotopy 2-types, and the other that braided categorical groups are algebraic models for connected, simply-connected homotopy 3-types.

2.1.1 The organization of the chapter

The plan of this chapter is, briefly, as follows. After this introductory Section 2.1, the chapter is organized in five sections. Section 2.2 is quite technical, but crucial to our discussions. It is dedicated to establishing some needed results concerning the notion of lax functor from a category into a tricategory, which is at the heart of the several constructions of nerves for tricategories used in the chapter. In Section 2.3, we mainly include the construction of the Grothendieck nerve $\mathrm{N}\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Bicat}$, for any tricategory $\mathcal{T}$, and the study of the basic properties concerning the behavior of the Grothendieck nerve construction, $\mathcal{T} \mapsto \mathrm{N}\mathcal{T}$, with respect to trihomomorphisms of tricategories. Section 2.4 contains the definition of classifying space $\mathrm{B}_3\mathcal{T}$, for any tricategory $\mathcal{T}$. The main facts concerning the classifying space construction $\mathcal{T} \mapsto \mathrm{B}_3\mathcal{T}$ are established here. In this section we also study the relationship between $\mathrm{B}_3\mathcal{T}$ and the space realization of the Segal nerve of a tricategory, $\mathrm{S}\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Hom}$, which, for instance, we apply to show how the classifying space of any monoidal bicategory realizes a delooping space. Section 2.5 is mainly dedicated to describing the geometric nerve $\Delta\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$, of any tricategory $\mathcal{T}$, and to proving the existence of homotopy equivalences $\mathrm{B}_3\mathcal{T} \simeq |\Delta\mathcal{T}|$. Also, by means of the geometric nerve construction for monoidal bicategories, we show here that bicategorical groups are a convenient algebraic model for connected homotopy 3-types. And finally, Section 2.6 collects the expression of various coherence conditions used throughout the chapter and the proofs of lemmas in the preparatory Section 2.2.

2.1.2 Notations

We refer the reader to the papers by Bénabou [15], Street [117], Gordon-Power-Street [69], Gurski [77], and Leinster [96], for the background on bicategories. The bicategorical conventions and the notations that we use along the thesis are the same as in [42, §2.1] and [41, §2.4]. Thus, given any bicategory $\mathcal{B}$, the composition in each hom-category $\mathcal{B}(x,y)$, that is, the vertical composition of 2-cells, is denoted by $\beta\cdot\alpha$, while the symbol $\circ$ is used to denote the horizontal composition functors

$\mathcal{B}(y,z) \times \mathcal{B}(x,y) \xrightarrow{\circ} \mathcal{B}(x,z)$. Identities are denoted as $1_a : a \Rightarrow a$, for any 1-cell $a : x \to y$, and $1_x : x \to x$, for any object $x \in \mathrm{Ob}\mathcal{B}$. The associativity, right unit, and left unit constraints of the bicategory are respectively denoted by the letters $\boldsymbol{a}$, $\boldsymbol{r}$, and $\boldsymbol{l}$.

A *lax functor* $F : \mathcal{A} \to \mathcal{B}$ will have structure constraints

$$Fa \circ Fb \Rightarrow F(a \circ b), \quad 1_{Fx} \Rightarrow F1_x.$$

The lax functor is termed a *pseudo functor* or *homomorphism* whenever all these structure constraints are invertible. If the unit constraints are all identities, then the lax functor is qualified as (strictly) *unitary* or *normal* and if, moreover, the composition constraints are also identities, then F is called a 2-*functor*.

If $F, G : \mathcal{A} \to \mathcal{B}$ are lax functors, then we follow the convention of [69] in what is meant by a *lax transformation* $\alpha : F \Rightarrow G$. Thus, α consists of morphisms $\alpha x : Fx \to Gx$, $x \in \mathrm{Ob}\mathcal{A}$, and of 2-cells $\alpha_a : \alpha y \circ Fa \Rightarrow Ga \circ \alpha x$, subject to the usual axioms. When all the 2-cells α_a are invertible, we say that $\alpha : F \Rightarrow G$ is a *pseudo transformation.*

In accordance with the orientation of the naturality 2-cells chosen, if $\alpha, \beta : F \Rightarrow G$ are two lax transformations, then a *modification* $\sigma : \alpha \Rrightarrow \beta$ will consist of 2-cells $\sigma x : \alpha x \Rightarrow \beta x$, $x \in \mathrm{Ob}\mathcal{A}$, subject to the commutativity condition, for any morphism $a : x \to y$ of $\mathcal{A}$:

$$\begin{tikzcd} Fx \arrow[r, "\beta x"] \arrow[r, bend right, "\alpha x"', ""{name=A}] \arrow[d, "Fa"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & Gx \arrow[d, "Ga"] \\ Fy \arrow[r, "\alpha y"'] & Gy \end{tikzcd} \Uparrow\sigma \quad = \quad \begin{tikzcd} Fx \arrow[r, "\beta x"] \arrow[d, "Fa"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & Gx \arrow[d, "Ga"] \\ Fy \arrow[r, "\alpha y"'] \arrow[r, bend left, "\beta y"] & Gy. \end{tikzcd} \Uparrow\sigma$$

Bicat denotes the tricategory of of bicategories, homomorphisms, pseudo transformations, and modifications, while **Hom** will denote the category of bicategories and homomorphisms. Thus we follow the notations by Gordon, Power and Street [69, Notation 4.9 and §5] and Gurski [77, §5.1]. In the structure of **Bicat** we use, the composition of pseudo transformations is taken to be

$$\Big(\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{C} \end{tikzcd} \Big) \Big(\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd} \Big) = \Big(\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "GF"] \arrow[r, bend right, "G'F'"'] \arrow[r, phantom, "\Downarrow\beta\alpha"] & \mathcal{C} \end{tikzcd} \Big),$$

where $\beta\alpha = \beta F' \circ G\alpha : \big(GF \xRightarrow{G\alpha} GF' \xRightarrow{\beta F'} G'F' \big)$, but note the existence of the useful invertible modification

$$\begin{tikzcd} GF \arrow[r, Rightarrow, "\beta F"] \arrow[d, Rightarrow, "G\alpha"'] \arrow[dr, phantom, "\Rrightarrow"] & G'F \arrow[d, Rightarrow, "G'\alpha"] \\ GF' \arrow[r, Rightarrow, "\beta F'"] & G'F' \end{tikzcd} \tag{2.1}$$

whose component at an object x of $\mathcal{A}$, is $\beta_{\alpha x}$, the component of β at the morphism αx. The following fact will be also very useful.

Fact 2.1 *Let* $\alpha\colon F \Rightarrow F'\colon \mathcal{A} \to \mathcal{B}$ *be a lax transformation between homomorphisms of bicategories. Then, for any* 2*-cell in* $\mathcal{A}$

$$\begin{tikzcd}
 & x_1 \arrow[r] & \cdots \arrow[r] & x_n \arrow[dr, "a_n"] & \\
x \arrow[ur, "a_0"] \arrow[dr, "b_0"'] & & \Downarrow u & & x', \\
 & x'_1 \arrow[r] & \cdots \arrow[r] & x'_m \arrow[ur, "b_m"'] &
\end{tikzcd}$$

the following equality holds:

$$\begin{tikzcd}
 & Fx_1 \arrow[r] & \cdots \arrow[r] & Fx_n \arrow[dr, "Fa_n"] & \\
Fx \arrow[ur, "Fa_0"] \arrow[dr, "Fb_0"'] \arrow[dd, "\alpha x"'] & & \Downarrow Fu & & Fx' \arrow[dd, "\alpha x'"] \\
 & Fx'_1 \arrow[r] \arrow[dd] & \cdots \arrow[r] & Fx'_m \arrow[ur, "Fb_m"'] \arrow[dd] & \\
F'x \arrow[dr, "F'b_0"'] & \Downarrow\alpha_{b_0} & \cdots & \Downarrow\alpha_{b_m} & F'x' \\
 & F'x'_1 \arrow[r] & \cdots \arrow[r] & F'x'_m \arrow[ur, "F'b_m"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & Fx_1 \arrow[r] \arrow[dd] & \cdots \arrow[r] & Fx_n \arrow[dr, "Fa_n"] \arrow[dd] & \\
Fx \arrow[ur, "Fa_0"] \arrow[dd, "\alpha x"'] & \Downarrow\alpha_{a_0} & \cdots & \Downarrow\alpha_{a_n} & Fx' \arrow[dd, "\alpha x'"] \\
 & F'x_1 \arrow[r] & \cdots \arrow[r] & F'x_n \arrow[dr, "F'a_n"] & \\
F'x \arrow[ur, "F'a_0"] \arrow[dr, "F'b_0"'] & & \Downarrow F'u & & F'x'. \\
 & F'x'_1 \arrow[r] & \cdots \arrow[r] & F'x'_m \arrow[ur, "F'b_m"'] &
\end{tikzcd}$$

In this thesis we use the notion of tricategory $\mathcal{T} = (\mathcal{T}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \pi, \mu, \lambda, \rho)$ as it was introduced by Gordon, Power and Street in [69], but with a minor alteration: we require that the homomorphisms of bicategories picking out units are normalized, and then written simply as $1_t \in \mathcal{T}(t,t)$. This restriction is not substantive, see Gurski [77, Theorem 7.24], but it does slightly reduce the amount of coherence data we have to deal with. For any object t of the tricategory $\mathcal{T}$, the arrow $\boldsymbol{r}1_t\colon 1_t \to 1_t \otimes 1_t$ is an equivalence in the hom-bicategory $\mathcal{T}(t,t)$, with the arrow $\boldsymbol{l}1_t\colon 1_t \otimes 1_t \to 1_t$ an adjoint quasi-inverse, see [77, Lemma 7.7]. Hereafter, we suppose the adjoint quasi-inverse of $\boldsymbol{r}$, $\boldsymbol{r}^\bullet \dashv \boldsymbol{r}$, has been chosen such that $\boldsymbol{r}^\bullet 1_t = \boldsymbol{l}1_t$, with the isomorphism $\boldsymbol{r}^\bullet 1_t \cong \boldsymbol{l}1_t$ being an identity. We will extensively use the coherence results in [77, Corollaries 10.6 and 10.15], particulary the following facts, easily deduced from them.

Fact 2.2 *Any two pasting diagrams in a tricategory* $\mathcal{T}$ *with the same source and target constructed only out of constraints* 2*-cells and* 3*-cells of* $\mathcal{T}$ *are equal*[2]*.*

Fact 2.3 *Given a trihomomorphism* $H\colon \mathcal{T} \to \mathcal{T}'$*, any two pasting diagrams in* $\mathcal{T}'$ *with the same source and target constructed only out of constraints* 2*-cells and* 3*-cells of* $\mathcal{T}$*,* $\mathcal{T}'$*, and* H*, are equal.*

For the general background on simplicial sets, we mainly refer to the book by Goerss and Jardine [68]. The *simplicial category*[3] is denoted by Δ, and its objects,

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[2] Actually, as stated these facts aren't exactly true. To be more precise, we should ask that the diagrams appear as diagrams in a free tricategory. This condition is easy to check for all the diagrams in which these facts have being applied.

[3] Notice that here we change the directions for the arrows of $[n]$ with respect to Chapter 1, we will keep this convention for the rest of the thesis.

that is, the ordered sets $[n] = \{0, 1, \dots, n\}$, are usually considered as categories with only one morphism $(i, j)\colon i \to j$ when $0 \leq i \leq j \leq n$. Then, a non-decreasing map $[n] \to [m]$ is the same as a functor, so that we see Δ, the simplicial category of finite ordinal numbers, as a full subcategory of **Cat**, the category (actually the 2-category) of small categories. Throughout the chapter, Segal's geometric realization [111] of a simplicial (compactly generated topological) space $X\colon \Delta^{\mathrm{op}} \to \mathbf{Top}$ is denoted by $|X|$. By regarding a set as a discrete space, the (Milnor) geometric realization of a simplicial set $X\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ is $|X|$. Following Quillen [109], the classifying space of a category $\mathcal{C}$ is denoted by $\mathrm{B}\mathcal{C}$.

2.2 Lax functors from categories into tricategories

As we will show in this chapter, the classifying space of any tricategory can be realized up to homotopy by a simplicial set $\Delta\mathcal{T}$, whose p-simplices $\Delta[p] \to \Delta\mathcal{T}$ are lax functors $[p] \to \mathcal{T}$, where $[p]$ is regarded as a tricategory in which the 2-cells and 3-cells are all identities, satisfying various requirements of normality. To be more precise, we recall the following.

A *lax functor* $F\colon \mathcal{I} \to \mathcal{T}$, of a category $\mathcal{I}$ in a tricategory $\mathcal{T}$, is a system of data consisting of

- for each object i in $\mathcal{I}$, an object $Fi \in \mathrm{Ob}\mathcal{T}$,
- for each arrow $a\colon i \to j$ in $\mathcal{I}$, a 1-cell $Fa\colon Fi \to Fj$,
- for each pair of composable arrows $i \xrightarrow{b} j \xrightarrow{a} k$ in $\mathcal{I}$, a 2-cell $F_{a,b}\colon Fa \otimes Fb \Rightarrow F(ab)$,
- for each object $i \in \mathrm{Ob}\mathcal{I}$, a 2-cell $F_i\colon 1_{Fi} \Rightarrow F1_i$,
- for any three composable arrows $i \xrightarrow{c} j \xrightarrow{b} k \xrightarrow{a} l$ in $\mathcal{I}$, a 3-cell

$$\begin{tikzcd}
(Fa\otimes Fb)\otimes Fc \arrow[rr, Rightarrow, "a"] \arrow[d, Rightarrow, "F_{a,b}\otimes 1"'] & \overset{F_{a,b,c}}{\Rrightarrow} & Fa\otimes(Fb\otimes Fc) \arrow[d, Rightarrow, "1\otimes F_{b,c}"] \\
F(ab)\otimes Fc \arrow[r, Rightarrow, "F_{ab,c}"'] & F(abc) & Fa\otimes F(bc), \arrow[l, Rightarrow, "F_{a,bc}"]
\end{tikzcd}$$

- for any arrow $i \xrightarrow{a} j$ in the index category $\mathcal{I}$, two 3-cells

$$\begin{tikzcd}
& 1_{Fj}\otimes Fa \arrow[dl, Rightarrow, "F_j\otimes 1"'] \arrow[dr, Rightarrow, "l"] & & & Fa\otimes 1_{Fi} \arrow[dl, Rightarrow, "1\otimes F_i"'] \arrow[dr, Rightarrow, "r^\bullet"] & \\
F1_j\otimes Fa \arrow[rr, Rightarrow, "F_{1,a}"', "\overset{\widehat{F}_a}{\Rrightarrow}" above=12pt] & & Fa, & Fa\otimes F1_i \arrow[rr, Rightarrow, "F_{a,1}"', "\overset{\widetilde{F}_a}{\Rrightarrow}" above=12pt] & & Fa.
\end{tikzcd}$$

These data are required to satisfy the coherence conditions **(CR1)**, **(CR2)**, and **(CR3)** as stated in the **Appendix**, §2.6.1.

Notice that we use a weaker notion of lax functor from that by Garner and Gurski in [65], where it is required for the structure 3-cells to be invertible. Furthermore, here the hom-functors $\mathcal{I}(i, j) \to \mathcal{T}(Fi, Fj)$ are normal.

The set of lax functors from a small category $\mathcal{I}$ to a small tricategory $\mathcal{T}$ is denoted by

$$\mathrm{Lax}(\mathcal{I},\mathcal{T}).$$

A lax functor $F\colon \mathcal{I}\to\mathcal{T}$ is termed *unitary* or *normal* whenever the following conditions hold: for each object i of $\mathcal{I}$, $F1_i = 1_{Fi}$ and $F_i = 1_{1_{Fi}}$; for each arrow $a\colon i\to j$ of $\mathcal{I}$, $F_{a,1_i} = \boldsymbol{r}^{\bullet}\colon Fa\otimes 1 \Rightarrow Fa$, $F_{1_j,a} = \boldsymbol{l}\colon 1\otimes Fa\Rightarrow Fa$, and the 3-cells $F_{1,b,c}$, $F_{a,1,c}$, $F_{a,b,1}$, $\widehat{F}_a$, and $\widetilde{F}_a$ are the unique coherence isomorphisms. Furthermore, a lax functor $F\colon \mathcal{I}\to\mathcal{T}$ whose structure 2-cells $F_{a,b}$ are all equivalences (in the corresponding hom-bicategories of $\mathcal{T}$ where they lie) and whose structure 3-cells $F_{a,b,c}$, $\widehat{F}_a$, and $\widetilde{F}_a$, are all invertible is called *homomorphic* (or, trihomomorphism). The subsets of $\mathrm{Lax}(\mathcal{I},\mathcal{T})$ whose elements are the unitary, homomorphic, and unitary homomorphic lax functors, are denoted respectively by

$$\mathrm{Lax}_{\mathrm{u}}(\mathcal{I},\mathcal{T}),\ \mathrm{Lax}_{\mathrm{h}}(\mathcal{I},\mathcal{T}),\ \mathrm{Lax}_{\mathrm{uh}}(\mathcal{I},\mathcal{T}). \tag{2.2}$$

Example 2.1 Let A be an abelian group, and let $\Sigma^2 A$ denote the tricategory (actually a 3-groupoid) having only one i-cell for $0\leq i\leq 2$ and whose 3-cells are the elements of A, with all the compositions given by addition in A. Then, for any small category $\mathcal{I}$ (e.g., a group G or a monoid M), a unitary lax functor $F\colon \mathcal{I}\to\Sigma^2 A$ is the same as a function $F\colon \mathrm{N}\mathcal{I}_3\to A$ satisfying the equations

$$F(b,c,d)+F(a,bc,d)+F(a,b,c) = F(ab,c,d)+F(a,b,cd),$$

and such that $F(a,b,c)=0$ whenever any of the arrows a, b, or c is an identity. Thus $\mathrm{Lax}_{\mathrm{u}}(\mathcal{I},\Sigma^2A) = Z^3(I,A)$, the set of normalized 3-cocycles of (the nerve $\mathrm{N}\mathcal{I}$ of) the category $\mathcal{I}$ with coefficients in the abelian group A.

2.2.1 The bicategories $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$, $\mathbb{L}\mathrm{ax}_{\mathrm{u}}(\mathcal{I},\mathcal{T})$, $\mathbb{L}\mathrm{ax}_{\mathrm{h}}(\mathcal{I},\mathcal{T})$, $\mathbb{L}\mathrm{ax}_{\mathrm{uh}}(\mathcal{I},\mathcal{T})$

For any category $\mathcal{I}$ and any tricategory $\mathcal{T}$, the set $\mathrm{Lax}(\mathcal{I},\mathcal{T})$ of lax functors from $\mathcal{I}$ into $\mathcal{T}$ is the set of objects of a bicategory whose 1-cells are a kind of degenerated lax transformations between lax functors that agree on objects, called *oplax icons* by Garner and Gurski in [65]. When $\mathcal{T}=\mathcal{B}$ is a bicategory, that is, when the 3-cells are all identities, these transformations have been considered by Bullejos and Cegarra in [35] under the name of *relative to objects lax transformations*, whereas they are termed *icons* by Lack in [93]. This bicategory, denoted by $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$, is as follows:

- *The cells of* $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$. As we said above, lax functors $F\colon\mathcal{I}\to\mathcal{T}$ are the 0-cells of this bicategory. For any two lax functors $F,G\colon\mathcal{I}\to\mathcal{T}$, a 1-cell $\Phi\colon F\Rightarrow G$ may exists only if F and G agree on objects, and is then given by specifying

- for every arrow $a\colon i\to j$ in $\mathcal{I}$, a 2-cell $\Phi a\colon Fa\Rightarrow Ga$ of $\mathcal{T}$,

- for each pair of composable arrows $i \xrightarrow{b} j \xrightarrow{a} k$ in $\mathcal{I}$, a 3-cell

$$\begin{tikzcd}
Fa \otimes Fb \arrow[r, Rightarrow, "F_{a,b}"] \arrow[d, Rightarrow, "\Phi a \otimes \Phi b"'] \arrow[dr, phantom, "\overset{\Phi_{a,b}}{\Rrightarrow}"] & F(ab) \arrow[d, Rightarrow, "\Phi(ab)"] \\
Ga \otimes Gb \arrow[r, Rightarrow, "G_{a,b}"'] & G(ab),
\end{tikzcd} \tag{2.3}$$

- for each object i of the category $\mathcal{I}$, a 3-cell

$$\begin{tikzcd}
& 1_{Fi=Gi} \arrow[dl, Rightarrow, "F_i"'] \arrow[dr, Rightarrow, "G_i"] \arrow[d, phantom, "\overset{\Phi_i}{\Lleftarrow}"] & \\
F1_i \arrow[rr, Rightarrow, "\Phi 1_i"'] & & G1_i,
\end{tikzcd} \tag{2.4}$$

This data are subject to the axioms **(CR4)** and **(CR5)** as stated in the **Appendix**, §2.6.1.

A 2-cell $M\colon \Phi \Rrightarrow \Psi$, for $\Phi, \Psi\colon F \Rightarrow G$ two 1-cells in $\mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{T})$, is an *icon modification* in the sense of [65], so it consists of a family of 3-cells in $\mathcal{T}$, $Ma\colon \Phi a \Rrightarrow \Psi a$, one for each arrow $a\colon i \to j$ in $\mathcal{I}$, subject to the coherence condition **(CR6)**.

• *Compositions in* $\mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{T})$. The vertical composition of a 2-cell $M\colon \Phi \Rrightarrow \Psi$ with a 2-cell $N\colon \Psi \Rrightarrow \Gamma$, for $\Phi, \Psi, \Gamma\colon F \Rightarrow G$, yields the 2-cell $N \cdot M\colon \Phi \Rrightarrow \Gamma$ which is defined using pointwise vertical composition in the hom-bicategories of $\mathcal{T}$; that is, for each $a\colon i \to j$ in $\mathcal{I}$, $(N \cdot M)a = (Na) \cdot (Ma)\colon \Phi a \Rrightarrow \Gamma a\colon Fa \Rightarrow Ga$. The horizontal composition of 1-cells $\Phi\colon F \Rightarrow G$ and $\Psi\colon G \Rightarrow H$, for $F, G, H\colon \mathcal{I} \to \mathcal{T}$ lax functors, is $\Psi \circ \Phi\colon F \Rightarrow H$, where $(\Psi \circ \Phi)a = \Psi a \circ \Phi a\colon Fa \Rightarrow Ha$, for each arrow $a\colon i \to j$ in $\mathcal{I}$. Its component

$$(\Psi \circ \Phi)_{a,b}\colon H_{a,b} \circ ((\Psi \circ \Phi)a \otimes (\Psi \circ \Phi)b) \Rrightarrow (\Psi(ab) \circ \Phi(ab)) \circ F_{a,b},$$

attached at a pair of composable arrows $i \xrightarrow{b} j \xrightarrow{a} k$ of the category $\mathcal{I}$, is given by pasting in the bicategory $\mathcal{T}(Fi, Fk)$ the diagram

$$\begin{tikzcd}
Fa \otimes Fb \arrow[rr, Rightarrow, "F_{a,b}"] \arrow[dr, Rightarrow, "\Phi a \otimes \Phi b"] \arrow[dd, Rightarrow, "(\Psi a \circ \Phi a) \otimes (\Psi b \circ \Phi b)"'] & & F(ab) \arrow[dr, Rightarrow, "\Phi(ab)"] & \\
\cong & Ga \otimes Gb \arrow[rr, Rightarrow, "G_{a,b}"] \arrow[dl, Rightarrow, "\Psi a \otimes \Psi b"] & \overset{\Phi_{a,b}}{\Rrightarrow} & G(ab) \arrow[dl, Rightarrow, "\Psi(ab)"] \\
Ha \otimes Fb \arrow[rr, Rightarrow, "H_{a,b}"'] & & H(ab), & \\
& & \overset{\Psi_{a,b}}{\Rrightarrow} \arrow[u, phantom] &
\end{tikzcd}$$

and its component $(\Psi \circ \Phi)_i\colon H_i \Rrightarrow (\Psi \circ \Phi)1_i \circ F_i$, attached at any object i of $\mathcal{I}$, is given by pasting in $\mathcal{T}(Fi, Fi)$ the diagram

$$\begin{tikzcd}
& 1_{Fi=Gi=Hi} \arrow[dl, Rightarrow, "F_i"'] \arrow[d, Rightarrow, "G_i"] \arrow[dr, Rightarrow, "H_i"] & \\
F1_i \arrow[r, Rightarrow, "\Phi 1_i"'] \arrow[ur, phantom, "\overset{\Phi_i}{\Lleftarrow}"] & G1_i \arrow[r, Rightarrow, "\Psi 1_i"'] \arrow[ur, phantom, "\overset{\Psi_i}{\Lleftarrow}"] & H1_i.
\end{tikzcd}$$

The horizontal composition of 2-cells $M\colon \Phi \Rrightarrow \Psi\colon F \Rightarrow G$ and $N\colon \Gamma \Rrightarrow \Theta\colon G \Rightarrow H$ in $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$ is $N \circ M\colon \Gamma \circ \Phi \Rrightarrow \Theta \circ \Psi$, which, at each $a\colon i \to j$ in $\mathcal{I}$, is given by the formula $(N \circ M)a = Na \circ Ma$.

- *Identities in* $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$. The identity 1-cell of a lax functor $F\colon \mathcal{I} \to \mathcal{T}$ is $1_F\colon F \Rightarrow F$, where $(1_F)a = 1_{Fa}$, the identity of Fa in the bicategory $\mathcal{T}(Fi,Fi)$, for each $a\colon i \to j$ in $\mathcal{I}$. Its structure 3-cell $(1_F)_{a,b}\colon F_{a,b} \circ (1_{Fa} \otimes 1_{Fa}) \Rrightarrow 1_{F(ab)} \otimes F_{a,b}$, attached at each pair of composable arrows $i \overset{b}{\to} j \overset{a}{\to} k$, is the canonical one obtained from the identity constraints of the bicategory $\mathcal{T}(Fk,Fi)$ by pasting the diagram

$$\begin{tikzcd}
Fa \otimes Fb \arrow[r, Rightarrow, "F_{a,b}"] \arrow[d, Rightarrow, "1\otimes 1"'] \arrow[d, Rightarrow, bend left=60, "1"] \arrow[d, phantom, "\cong", xshift=2ex] \arrow[dr, phantom, "\cong"] & F(ab) \arrow[d, Rightarrow, "1"] \\
Fa \otimes Fb \arrow[r, Rightarrow, "F_{a,b}"'] & F(ab),
\end{tikzcd}$$

and its component attached at any object i of $\mathcal{I}$ is obtained from the left unit constraints of the bicategory $\mathcal{T}(Fi,Fi)$ at $F_i\colon 1_{F_i} \Rightarrow F1_i$, that is, $(1_F)_i = \boldsymbol{l}^{-1}\colon F_i \cong 1_{F1_i} \circ F_i$. The identity 2-cell 1_Φ, of a 1-cell $\Phi\colon F \Rightarrow G$, is defined at any arrow $a\colon i \to j$ of I by the simple formula $(1_\Phi)a = 1_{\Phi a}\colon \Phi a \Rrightarrow \Phi a$.

- *The structure constraints in* $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$. For any three composable 1-cells $F \overset{\Phi}{\Rightarrow} G \overset{\Psi}{\Rightarrow} H \overset{\Theta}{\Rightarrow} K$ in $\mathbb{L}\mathrm{ax}(\mathcal{I},T)$, the component of the structure associativity isomorphism $(\Theta \circ \Psi) \circ \Phi \cong \Theta \circ (\Psi \circ \Phi)$, at any arrow $i \overset{a}{\to} j$ of the category $\mathcal{I}$, is provided by the associativity constraint $(\Theta a \circ \Psi a) \circ \Phi a \cong \Theta a \circ (\Psi a \circ \Phi a)$ of the hom-bicategory $\mathcal{T}(Fi,Fj)$. And similarly, the components of structure left and right identity isomorphisms $1_G \circ \Phi \cong \Phi$ and $\Phi \circ 1_F \cong \Phi$, at any arrow $a\colon i \to j$ as above, are provided by the identity constraints $1_{Ga} \circ \Phi a \cong \Phi a$, and $\Phi a \circ 1_{Fa} \cong \Phi a$, of the bicategory $\mathcal{T}(Fi,Fj)$, respectively.

The bicategory $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$, contains three sub-bicategories that are of interest in our development: The *bicategory of unitary lax functors*, denoted by

$$\mathbb{L}\mathrm{ax}_{\mathrm{u}}(\mathcal{I},\mathcal{T}), \tag{2.5}$$

whose 1-cells are those $\Phi\colon F \Rightarrow G$ in $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$ that are unitary, in the sense that $\Phi 1_i = 1_{1_{Fi}}$, the 3-cells $\Phi_{1,a}$, $\Phi_{a,1}$ in (2.3), and Φ_i in (2.4) are those given by the constraints of the tricategory, and it is full on 2-cells between such normalized 1-cells.

The *bicategory of homomorphic lax functors* (i.e., of trihomomorphisms), denoted by

$$\mathbb{L}\mathrm{ax}_{\mathrm{h}}(\mathcal{I},\mathcal{T}),$$

whose 1-cells are those $\Phi\colon F \Rightarrow G$ in $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$ such that the structure 3-cells $\Phi_{a,b}$ and Φ_i are all invertible, and it is full on 2-cells $M\colon \Phi \Rrightarrow \Psi$ between such 1-cells.

The *bicategory of unitary homomorphic lax functors*, denoted by $\mathbb{L}\mathrm{ax}_{\mathrm{uh}}(\mathcal{I},\mathcal{T})$, which is defined to be the intersection of the above two, that is,

$$\mathbb{L}\mathrm{ax}_{\mathrm{uh}}(\mathcal{I},\mathcal{T}) = \mathbb{L}\mathrm{ax}_{\mathrm{u}}(\mathcal{I},\mathcal{T}) \cap \mathbb{L}\mathrm{ax}_{\mathrm{h}}(\mathcal{I},\mathcal{T}). \tag{2.6}$$

Example 2.2 Let $\Sigma^2 A$ be the strict tricategory defined by an abelian group A as in Example 2.1 and let $\mathcal{I}$ be any category. Then, the bicategory $\mathbb{L}\mathrm{ax}_{\mathrm{u}}(\mathcal{I}, \Sigma^2 A)$ is actually a 2-groupoid whose objects are normalized 3-cocycles of $\mathcal{I}$ with coefficients in A. If F, G: $\mathrm{N}\mathcal{I}_3 \to A$ are two such 3-cocycles, then a 1-cell Φ: $F \Rightarrow G$ is a normalized 2-cochain Φ: $\mathrm{N}\mathcal{I}_2 \to A$ satisfying

$$G(a,b,c) + \Phi(b,c) + \Phi(a,bc) = F(a,b,c) + \Phi(ab,c) + \Phi(a,b),$$

that is, $G = F + \partial\Phi$. And for any two 1-cells Φ, Ψ: $F \Rightarrow G$ as above, a 2-cell M: $\Phi \Rrightarrow \Psi$ consists of a normalized 1-cochain M: $\mathrm{N}\mathcal{I}_1 \to A$ such that $\Psi = \Phi + \partial M$, that is, such that $\Psi(a,b) + M(a) + M(b) = M(ab) + \Phi(a,b)$.

2.2.2 Functorial properties of $\mathbb{L}\mathrm{ax}(\mathcal{I}, -)$

For any given tricategory $\mathcal{T}$, any functor α: $\mathcal{I} \to \mathcal{J}$ induces a strict functor

$$\alpha^*\colon \mathbb{L}\mathrm{ax}(\mathcal{J}, \mathcal{T}) \to \mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{T})$$

given on cells in the following way: For any F: $\mathcal{J} \to \mathcal{T}$, $\alpha^* F$: $\mathcal{I} \to \mathcal{T}$ is the lax functor acting both on objects and arrows as the composite $F\alpha$, and whose structure cells are simply given by the rules

$$\begin{array}{lll} (\alpha^*F)_{a,b} = F_{\alpha a,\alpha b}, & (\alpha^*F)_i = F_{\alpha i}, & \\ (\alpha^*F)_{a,b,c} = F_{\alpha a,\alpha b,\alpha c}, & \widehat{(\alpha^*F)}_a = \widehat{F}_{\alpha a}, & \widetilde{(\alpha^*F)}_a = \widetilde{F}_{\alpha a}. \end{array}$$

Notice that $\alpha^* F$ is slightly different from the composite lax functor $F\alpha$, which will give structure 2-cells $(F\alpha)_{a,b} = 1_{F(\alpha a \alpha b)} \circ F_{\alpha a,\alpha b}$.

For Φ: $F \Rightarrow G$ a 1-cell of $\mathbb{L}\mathrm{ax}(\mathcal{J}, \mathcal{T})$, $\alpha^*\Phi$: $\alpha^* F \Rightarrow \alpha^* G$ is the 1-cell of $\mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{T})$ with

$$(\alpha^*\Phi)a = \Phi(\alpha a),\ (\alpha^*\Phi)_{a,b} = \Phi_{\alpha a,\alpha b},\ (\alpha^*\Phi)_i = \Phi_{\alpha i}.$$

Similarly, for any 2-cell M: $\Phi \Rrightarrow \Psi$ in $\mathbb{L}\mathrm{ax}(\mathcal{J}, \mathcal{T})$, $\alpha^* M$: $\alpha^*\Phi \Rrightarrow \alpha^*\Psi$ is the 2-cell of $\mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{T})$ with $(\alpha^* M)a = M(\alpha a)$.

Using the definition above, the construction $\mathcal{I} \mapsto \mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{T})$ is functorial on the category $\mathcal{I}$. For a trihomomorphism of tricategories $H = (H, \chi, \iota, \omega, \gamma, \delta)$: $\mathcal{T} \to \mathcal{T}'$, as defined by Gordon, Power and Street in [69, Definition 3.1], we have the following result.

Lemma 2.1 *Let $\mathcal{I}$ be any given small category.*

(i) Every trihomomorphism H: $\mathcal{T} \to \mathcal{T}'$ gives rise to a homomorphism

$$H_*\colon \mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{T}) \to \mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{T}'),$$

which is natural on $\mathcal{I}$, that is, for any functor α: $\mathcal{I} \to \mathcal{J}$,

$$H_*\alpha^* = \alpha^* H_*\colon \mathbb{L}\mathrm{ax}(\mathcal{J}, \mathcal{T}) \to \mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{T}').$$

(ii) If $H\colon \mathcal{T} \to \mathcal{T}'$ *and* $H'\colon \mathcal{T}' \to \mathcal{T}''$ *are any two composable trihomomorphisms, then there is a pseudo-equivalence* $m\colon H'_* H_* \Rightarrow (H'H)_*$*, such that, for any functor* $\alpha\colon \mathcal{I} \to \mathcal{J}$*, the equality* $m\alpha^* = \alpha^* m$ *holds.*

(iii) For any tricategory $\mathcal{T}$*, there is a pseudo-equivalence* $m\colon (1_{\mathcal{T}})_* \Rightarrow 1$*, such that, for any functor* $\alpha\colon \mathcal{I} \to \mathcal{J}$*, the equality* $m\alpha^* = \alpha^* m$ *holds.*

Proof: This is given in the **Appendix**, §2.6.2. ☐

2.2.3 Lax functors from free categories

Let us now replace the category $\mathcal{I}$ above by a (directed) graph $\mathcal{G}$. For any tricategory $\mathcal{T}$, there is a bicategory

$$\mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{T}),$$

where a 0-cell $f\colon \mathcal{G} \to \mathcal{T}$ consists of a pair of maps that assigns an object fi to each vertex $i \in \mathcal{G}$ and a 1-cell $fa\colon fi \to fj$ to each edge $a\colon i \to j$ in $\mathcal{G}$, respectively. A 1-cell $\phi\colon f \Rightarrow g$ may exist only if f and g agree on vertices, that is, $fi = gi$ for all $i \in \mathcal{G}$; and then it consists of a map that assigns to each edge $a\colon i \to j$ in the graph a 2-cell $\phi a\colon fa \Rightarrow ga$ of $\mathcal{T}$. And a 2-cell $m\colon \phi \Rrightarrow \psi$, for $\phi, \psi\colon f \Rightarrow g$ two 1-cells as above, consists of a family of 3-cells in $\mathcal{T}$, $ma\colon \phi a \Rrightarrow \psi a$, one for each arrow $a\colon i \to j$ in I. Compositions in $\mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{T})$ are defined in the natural way by the same rules as those stated above for the bicategory of lax functors from a category into a tricategory.

Suppose now that $\mathcal{I}(\mathcal{G})$ is the free category generated by the graph $\mathcal{G}$. Then, restriction to the basic graph gives a strict functor

$$R\colon \mathbb{L}\mathrm{ax}(\mathcal{I}(\mathcal{G}),\mathcal{T}) \to \mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{T}),$$

and we shall state the following auxiliary statement to be used later.

Lemma 2.2 *Let* $\mathcal{I} = \mathcal{I}(\mathcal{G})$ *be the free category generated by a graph* $\mathcal{G}$*. Then, for any tricategory* $\mathcal{T}$*, there is a homomorphism*

$$L\colon \mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{T}) \to \mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T}),$$

and a lax transformation $\mathrm{v}\colon LR \Rightarrow 1_{\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})}$*, such that the following facts hold:*

(a) $RL = 1_{\mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{T})}$*,* $\mathrm{v}L = 1_L$*,* $R\mathrm{v} = 1_R$*.*

(b) *The image of* L *is contained in the sub-bicategory* $\mathbb{L}\mathrm{ax}_{\mathrm{uh}}(\mathcal{I},\mathcal{T}) \subseteq \mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$*.*

(c) *The restricted homomorphisms of* L *and* R *establish biadjoint biequivalences*

$$\begin{tikzcd} \mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{T}) \arrow[r, shift left, "L"] \arrow[r, phantom, "\sim"] & \mathbb{L}\mathrm{ax}_{\mathrm{h}}(\mathcal{I},\mathcal{T}) \arrow[l, shift left, "R"] \end{tikzcd}, \tag{2.7}$$

$$\begin{tikzcd} \mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{T}) \arrow[r, shift left, "L"] \arrow[r, phantom, "\sim"] & \mathbb{L}\mathrm{ax}_{\mathrm{uh}}(I,\mathcal{T}) \arrow[l, shift left, "R"] \end{tikzcd}, \tag{2.8}$$

whose respective unit is the identity $1\colon 1 \Rightarrow RL$*, the counit is given by the corresponding restriction of* $\mathrm{v}\colon LR \Rightarrow 1$*, and whose triangulators are the canonical modifications* $1 \cong 1 \circ 1 = \mathrm{v}L \circ L1$ *and* $R\mathrm{v} \circ 1R = 1 \circ 1 \cong 1$*, respectively.*

Proof: This is given in the **Appendix**, §2.6.3. □

2.3 The Grothendieck nerve of a tricategory

Let us briefly recall that it was Grothendieck who first associated a simplicial set

$$\mathrm{N}\mathcal{C}\colon \Delta^{\mathrm{op}} \to \mathbf{Set} \tag{2.9}$$

to a small category $\mathcal{C}$, calling it its *nerve*. The set of p-simplices

$$\mathrm{N}\mathcal{C}_p = \bigsqcup_{(c_0,\dots,c_p)} \mathcal{C}(c_{p-1},c_p)\times\mathcal{C}(c_{p-2},c_{p-1})\times\cdots\times\mathcal{C}(c_0,c_1)$$

consists of length p sequences of composable morphisms in $\mathcal{C}$. Geometric realization of its nerve is the *classifying space* of the category, $\mathrm{B}\mathcal{C}$. The main result here shows how the Grothendieck nerve construction for categories rises to tricategories.

When a tricategory $\mathcal{T}$ is strict, that is, a 3-category, then the nerve construction (2.9) actually works by giving a simplicial 2-category (see Example 2.4). However, for an arbitrary tricategory, the device is more complicated since the compositions of cells in a tricategory is in general not associative and not unitary (which is crucial for the simplicial structure in the construction of $\mathrm{N}\mathcal{T}$ as above), but it is only so up to coherent isomorphisms. This 'defect' has the effect of forcing one to deal with the classifying space of a nerve of $\mathcal{T}$ in which the simplicial identities are replaced by coherent isomorphisms, that is, a *pseudo-simplicial bicategory* as stated in the theorem below. Pseudo-simplicial bicategories, and the tricategory they form (whose 1-cells are pseudo-simplicial homomorphisms, 2-cells pseudo-simplicial transformations, and 3-cells pseudo-simplicial modifications) were studied by Carrasco, Cegarra and Garzón in [42], to which we refer the reader[4].

Theorem 2.1 *Any tricategory $\mathcal{T}$ defines a normal pseudo-simplicial bicategory, that is, a unitary trihomomorphism from the simplicial category Δ^{op} into the tricategory of bicategories,*

$$\mathrm{N}\mathcal{T} = (\mathrm{N}\mathcal{T},\chi,\omega)\colon \Delta^{\mathrm{op}} \to \mathbf{Bicat}, \tag{2.10}$$

called the nerve of the tricategory*, whose bicategory of p-simplices, for $p \geq 1$, is*

$$\mathrm{N}\mathcal{T}_p = \bigsqcup_{(t_0,\dots,t_p)\in\ \mathrm{Ob}\mathcal{T}^{p+1}} \mathcal{T}(t_{p-1},t_p)\times\mathcal{T}(t_{p-2},t_{p-1})\times\cdots\times\mathcal{T}(t_0,t_1),$$

---

[4] See also Subsection 3.2.1 and Example 3.1 in Chapter 3.

and $\mathrm{N}\mathcal{T}_0 = \mathrm{Ob}\mathcal{T}$*, as a discrete bicategory. The face and degeneracy homomorphisms are defined on* 0*-cells,* 1*-cells and* 2*-cells of* $\mathrm{N}\mathcal{T}_p$ *by the ordinary formulas*

$$\begin{aligned} d_i(x_p,\dots,x_1) &= \begin{cases} (x_p,\dots,x_2) & \text{if} \quad i=0,\\ (x_p,\dots,x_{i+1}\otimes x_i,\dots,x_1) & \text{if} \quad 0<i<p,\\ (x_{p-1},\dots,x_1) & \text{if} \quad i=p, \end{cases} \\ s_i(x_p,\dots,x_1) &= (x_p,\dots,x_{i+1},1,x_i,\dots,x_1). \end{aligned} \tag{2.11}$$

Indeed, if $a\colon [q]\to[p]$ *is any map in the simplicial category* Δ*, then the associated homomorphism*

$$\mathrm{N}\mathcal{T}_a\colon \mathrm{N}\mathcal{T}_p \to \mathrm{N}\mathcal{T}_q$$

is induced by the composition $\mathcal{T}(t',t)\times\mathcal{T}(t'',t')\xrightarrow{\otimes}\mathcal{T}(t'',t)$ *and unit* $1_t\colon 1\to\mathcal{T}(t,t)$ *homomorphisms. The structure pseudo-equivalences*

$$\begin{tikzcd} \mathrm{N}\mathcal{T}_p \arrow[r, bend left, "\mathrm{N}\mathcal{T}_b\ \mathrm{N}\mathcal{T}_a"] \arrow[r, bend right, "\mathrm{N}\mathcal{T}_{ab}"'] \arrow[r, phantom, "\Downarrow\chi_{a,b}"] & \mathrm{N}\mathcal{T}_n, \end{tikzcd} \tag{2.12}$$

for each pair of composable maps $[n]\xrightarrow{b}[q]\xrightarrow{a}[p]$ *in* Δ*, and the invertible modifications*

$$\begin{tikzcd} \mathrm{N}\mathcal{T}_c\ \mathrm{N}\mathcal{T}_b\ \mathrm{N}\mathcal{T}_a \arrow[r, Rightarrow, "\mathrm{N}\mathcal{T}_c\,\chi_{a,b}"] \arrow[d, Rightarrow, "\chi_{b,c}\mathrm{N}\mathcal{T}_a"'] \arrow[dr, phantom, "\omega_{a,b,c}\Rrightarrow"] & \mathrm{N}\mathcal{T}_c\ \mathrm{N}\mathcal{T}_{ab} \arrow[d, Rightarrow, "\chi_{ab,c}"] \\ \mathrm{N}\mathcal{T}_{bc}\ \mathrm{N}\mathcal{T}_a \arrow[r, Rightarrow, "\chi_{a,bc}"'] & \mathrm{N}\mathcal{T}_{abc}, \end{tikzcd} \tag{2.13}$$

respectively associated to triplets of composable arrows $[m]\xrightarrow{c}[n]\xrightarrow{b}[q]\xrightarrow{a}[p]$*, canonically arise all from the structure pseudo equivalences and modifications data of the tricategory.*

We shall prove Theorem 2.1 simultaneously with the Proposition 2.1 below, which states the basic properties concerning the behavior of the Grothendieck nerve construction, $\mathcal{T}\mapsto\mathrm{N}\mathcal{T}$, with respect to trihomomorphisms of tricategories.

Proposition 2.1 *(i) Any trihomomorphism between tricategories* $H\colon\mathcal{T}\to\mathcal{T}'$ *induces a normal pseudo-simplicial homomorphism*

$$\mathrm{N}H = (\mathrm{N}H,\theta,\Pi)\colon \mathrm{N}\mathcal{T}\to\mathrm{N}\mathcal{T}',$$

which, at any integer $p\geq 0$*, is the evident homomorphism* $\mathrm{N}H_p\colon\mathrm{N}\mathcal{T}_p\to\mathrm{N}\mathcal{T}'_p$ *defined on any cell* $(x_p,\dots,x_1)$ *of* $\mathrm{N}\mathcal{T}_p$ *by*

$$\mathrm{N}H_p(x_p,\dots,x_1) = (Hx_p,\dots,Hx_1).$$

The structure pseudo-equivalence

$$\begin{tikzcd}
\mathrm{N}\mathcal{T}_p \arrow[r, "\mathrm{N}H_p"] \arrow[d, "\mathrm{N}\mathcal{T}_a"'] \arrow[dr, phantom, "\theta_a \Rightarrow"] & \mathrm{N}\mathcal{T}'_p \arrow[d, "\mathrm{N}\mathcal{T}'_a"] \\
\mathrm{N}\mathcal{T}_q \arrow[r, "\mathrm{N}H_q"'] & \mathrm{N}\mathcal{T}'_q,
\end{tikzcd} \tag{2.14}$$

for each map a: $[q] \to [p]$ in Δ, and the invertible modifications

$$\begin{tikzcd}
\mathrm{N}H_n\ \mathrm{N}\mathcal{T}_b\ \mathrm{N}\mathcal{T}_a \arrow[rr, Rightarrow, "\mathrm{N}H_n\chi"] \arrow[d, Rightarrow, "\theta\mathrm{N}\mathcal{T}_a"'] & \Pi_{a,b} \Rrightarrow & \mathrm{N}H_n\ \mathrm{N}\mathcal{T}_{ab} \arrow[d, Rightarrow, "\theta"] \\
\mathrm{N}\mathcal{T}'_b\ \mathrm{N}H_q\ \mathrm{N}\mathcal{T}_a \arrow[r, Rightarrow, "\mathrm{N}\mathcal{T}'_b\theta"'] & \mathrm{N}\mathcal{T}'_b\ \mathrm{N}\mathcal{T}'_a\ \mathrm{N}H_p \arrow[r, Rightarrow, "\chi\mathrm{N}H_p"'] & \mathrm{N}\mathcal{T}'_{ab}\ \mathrm{N}H_p,
\end{tikzcd} \tag{2.15}$$

respectively associated to pairs of composable arrows $[n] \xrightarrow{b} [q] \xrightarrow{a} [p]$, canonically arise all from the structure pseudo equivalences and modifications data of the trihomomorphism and the involved tricategories.

(ii) For any pair of composable trihomomorphisms H: $\mathcal{T} \to \mathcal{T}'$ and H': $\mathcal{T}' \to \mathcal{T}''$, there is a pseudo-simplicial pseudo-equivalence

$$\mathrm{N}H'\ \mathrm{N}H \Rightarrow \mathrm{N}(H'H). \tag{2.16}$$

(iii) For any tricategory $\mathcal{T}$, there is a pseudo-simplicial pseudo-equivalence

$$\mathrm{N}1_{\mathcal{T}} \Rightarrow 1_{\mathrm{N}\mathcal{T}}. \tag{2.17}$$

Proof:[Proof of Theorem 2.1 and Proposition 2.1.] Let us note that, for any $p \geq 0$, the category $[p]$ is free on the graph $\mathcal{G}_p = (0 \to 1 \to \cdots \to p)$, and that $\mathrm{N}\mathcal{T}_p = \mathbb{L}\mathrm{ax}(\mathcal{G}_p, \mathcal{T})$. Hence, the existence of a biadjoint biequivalence

$$L_p \dashv R_p\colon \mathrm{N}\mathcal{T}_p \rightleftarrows \mathbb{L}\mathrm{ax}_{\mathrm{h}}([p], \mathcal{T})$$

follows from Lemma 2.2, where R_p is the strict functor defined by restricting to the basic graph $\mathcal{G}_p$ of the category $[p]$, such that $R_pL_p = 1$, whose unity is the identity, and whose counit v_p: $L_pR_p \Rightarrow 1$ is a pseudo-equivalence satisfying the equalities $\mathrm{v}_pL_p = 1$ and $R_p\mathrm{v}_p = 1$. Then, if a: $[q] \to [p]$ is any map in the simplicial category, the associated homomorphism $\mathrm{N}\mathcal{T}_a$: $\mathrm{N}\mathcal{T}_p \to \mathrm{N}\mathcal{T}_q$, is defined to be the composite

$$\begin{tikzcd}
\mathrm{N}\mathcal{T}_p \arrow[r, dotted, "\mathrm{N}\mathcal{T}_a"] \arrow[d, "L_p"'] & \mathrm{N}\mathcal{T}_q \\
\mathbb{L}\mathrm{ax}_{\mathrm{h}}([p], \mathcal{T}) \arrow[r, "a^*"] & \mathbb{L}\mathrm{ax}_{\mathrm{h}}([q], \mathcal{T}). \arrow[u, "R_q"']
\end{tikzcd}$$

Observe that, thus defined, the homomorphism $\mathrm{N}\mathcal{T}_a$ maps the component bicategory of $\mathrm{N}\mathcal{T}_p$ at $(t_p,\dots,t_0)$ into the component at $(t_{a(q)},\dots,t_{a(0)})$ of $\mathrm{N}\mathcal{T}_q$, and it acts on 0-cells, 1-cells, and 2-cells of $\mathrm{N}\mathcal{T}_p$ by the formula $\mathrm{N}\mathcal{T}_a(x_p,\dots,x_1)=(y_q,\dots,y_1)$, where, for $0\leq k<q$, (see (2.43) for the definition of $\overset{\mathrm{or}}{\otimes}$)

$$y_{k+1}=\begin{cases}\overset{\mathrm{or}}{\otimes}(x_{a(k+1)},\dots,x_{a(k)+1}) & \textit{if}\ \ a(k)<a(k+1),\\ 1 & \textit{if}\ \ a(k)=a(k+1).\end{cases}$$

Thus we get, in particular, the formulas for the face and degeneracy homomorphisms.

The pseudo natural equivalence (2.12) is

$$\mathrm{N}\mathcal{T}_b\ \mathrm{N}\mathcal{T}_a=R_nb^*L_qR_qa^*L_p\xRightarrow{\chi_{a,b}=R_nb^*\mathrm{v}_qa^*L_p}R_nb^*a^*L_p=R_n(ab)^*L_p=\mathrm{N}\mathcal{T}_{ab},$$

and the invertible modification (2.13) is

$$\omega_{a,b,c}=R_mc^*\omega'_ba^*L_p,$$

where ω'_b is the canonical modification below.

$$\begin{tikzcd}[column sep=large] & L_nR_nb^*L_qR_q \arrow[r, Rightarrow, "L_nR_nb^*\mathrm{v}_q"] \arrow[d, Rightarrow, "\mathrm{v}_nb^*L_qR_q"'] \arrow[dr, phantom, "\overset{(2.1)}{\Rightarrow}"] & L_nR_nb^* \arrow[d, Rightarrow, "\mathrm{v}_nb^*"] \\ \omega'_b\colon & b^*L_qR_q \arrow[r, Rightarrow, "b^*\mathrm{v}_q"'] & b^* \end{tikzcd} \tag{2.18}$$

Thus defined, $\mathrm{N}\mathcal{T}$ is actually a normal pseudo-simplicial bicategory. Both coherence conditions for $\mathrm{N}\mathcal{T}$, that is, conditions (**CC1**) and (**CC2**) in [42], follow from the equalities $R_pL_p=1$, $\mathrm{v}_pL_p=1$, and $R_p\mathrm{v}_p=1$, and the coherence in the tricategory **Bicat**. This proves Theorem 2.1.

And when it comes to Proposition 2.1, first, let us note that the homomorphisms $\mathrm{N}H_p\colon\mathrm{N}\mathcal{T}_p\to\mathrm{N}\mathcal{T}'_p$, $p\geq 0$, make commutative the diagrams

$$\begin{tikzcd} \mathrm{N}\mathcal{T}_p \arrow[r, "\mathrm{N}H_p"] \arrow[d, "L=L_p^{\mathcal{T}}"'] & \mathrm{N}\mathcal{T}'_p \\ \mathbb{L}\mathrm{ax}_{\mathrm{h}}([p],\mathcal{T}) \arrow[r, "H_*"] & \mathbb{L}\mathrm{ax}_{\mathrm{h}}([p],\mathcal{T}'), \arrow[u, "R=R_p^{\mathcal{T}'}"'] \end{tikzcd}$$

where H_* is the induced homomorphism by the trihomomorphism $H\colon\mathcal{T}\to\mathcal{T}'$, as stated in Lemma 2.1 (i). Then, the pseudo-equivalence (2.14), θ_a, is provided by the pseudo-equivalences $\mathrm{v}\colon LR\Rightarrow 1$ and their adjoint quasi-inverses $\mathrm{v}^\bullet\colon 1\Rightarrow LR$ (which we can choose such that $R\mathrm{v}^\bullet=1$ and $\mathrm{v}^\bullet L=1$); that is, $\theta_a=Ra^*\mathrm{v}^\bullet H_*L\circ RH_*LRa^*L$,

$$\begin{tikzcd}[column sep=small] & RH_*a^*L=Ra^*H_*L \arrow[dr, Rightarrow, "Ra^*\mathrm{v}^\bullet H_*L"] & \\ \mathrm{N}H_q\ \mathrm{N}\mathcal{T}_a=RH_*LRa^*L \arrow[ur, Rightarrow, "RH_*\mathrm{v}a^*L"] \arrow[rr, dotted, "\theta_a"] & & Ra^*LRH_*L=\mathrm{N}\mathcal{T}'_a\ \mathrm{N}H_p. \end{tikzcd}$$

And, for $[n]\xrightarrow{b}[q]\xrightarrow{a}[p]$, any two composable arrows of Δ, the structure invertible modification (2.15), $\Pi_{a,b}$, is the modification obtained by pasting the diagram

$$\begin{tikzcd}
RH_*LRb^*LRa^*L \arrow[rr, Rightarrow, "RH_*LRb^*\mathrm{v}a^*L"] \arrow[d, Rightarrow, "RH_*\mathrm{v}b^*LRa^*L"'] \arrow[drr, phantom, "\overset{(2.1)}{\Rrightarrow}"] & & RH_*RLb^*a^*L \arrow[d, Rightarrow, "RH_*\mathrm{v}b^*a^*L"] \\
RH_*b^*LRa^*L \arrow[rr, Rightarrow, "Rb^*H_*\mathrm{v}a^*L"] \arrow[d, Rightarrow, "Rb^*\mathrm{v}^\bullet H_*LRa^*L"'] \arrow[dr, phantom, "\overset{(2.1)}{\Rrightarrow}"] & & RH_*b^*a^*L \arrow[d, Rightarrow, "Rb^*a^*\mathrm{v}^\bullet H_*L"] \arrow[dll, Rightarrow, bend right=15, "Rb^*\mathrm{v}^\bullet H_*a^*L"'] \arrow[dl, Rightarrow, "1"] \arrow[d, phantom, shift right=4ex, "\cong"] \\
Rb^*LRH_*LRa^*L \arrow[d, Rightarrow, "Rb^*LRH_*\mathrm{v}a^*L"'] \arrow[r, phantom, "\overset{(A)}{\cong}"] & Rb^*a^*H_*L \arrow[r, Rightarrow, "Rb^*a^*\mathrm{v}^\bullet H_*L"'] \arrow[dr, phantom, "\overset{(2.1)}{\Rrightarrow}"] & Rb^*a^*LRH_*L \\
Rb^*LRH_*a^*L \arrow[ur, Rightarrow, "Rb^*\mathrm{v}a^*H_*L"'] \arrow[rr, Rightarrow, "Rb^*LRa^*\mathrm{v}^\bullet H_*L"] & & Rb^*LRa^*LRH_*L \arrow[u, Rightarrow, "Rb^*\mathrm{v}a^*LRH_*L"']
\end{tikzcd}$$

where the isomorphism labelled (A) is given by the adjunction invertible modification $\mathrm{v}\circ\mathrm{v}^\bullet\cong 1$. The coherence conditions for $\mathrm{N}H\colon \mathrm{N}\mathcal{T}\to\mathrm{N}\mathcal{T}'$, that is, conditions (**CC3**) and (**CC4**) in [42], are easily verified again from coherence in **Bicat**.

Suppose now that $\mathcal{T}\overset{H}{\to}\mathcal{T}'\overset{H'}{\to}\mathcal{T}''$ are two composable trihomomorphisms. Then, the pseudo-simplicial pseudo-equivalence (2.16), $\alpha\colon \mathrm{N}H'\ \mathrm{N}H\Rightarrow\mathrm{N}(H'H)$, is, at any integer $p\geq 0$, given by $\alpha_p=RmL\circ RH'_*\mathrm{v}H_*L$,

$$\begin{tikzcd}
 & RH'_*H_*L \arrow[dr, Rightarrow, "RmL"] & \\
\mathrm{N}H'_p\ \mathrm{N}H_p = RH'_*LRH_*L \arrow[ur, Rightarrow, "RH'_*\mathrm{v}H_*L"] \arrow[rr, dotted, "\alpha_p"] & & R(H'H)_*L = \mathrm{N}(H'H)_p,
\end{tikzcd}$$

where the pseudo-equivalence $m\colon H'_*H_*\Rightarrow(H'H)_*\colon \mathbb{L}\mathrm{ax}_\mathrm{h}([p],\mathcal{T})\to\mathbb{L}\mathrm{ax}_\mathrm{h}([p],\mathcal{T}'')$ is that given in Lemma 2.1 (*ii*). The naturality component of α at any map $a\colon[q]\to[p]$,

$$\begin{tikzcd}
\mathrm{N}H'_q\ \mathrm{N}H_q\ \mathrm{N}\mathcal{T}_a \arrow[rr, Rightarrow, "\alpha_q\mathrm{N}\mathcal{T}_a"] \arrow[d, Rightarrow, "\mathrm{N}H'_q\,\theta_a"'] \arrow[drr, phantom, "\Rrightarrow"] & & \mathrm{N}(H'H)_q\ \mathrm{N}\mathcal{T}_a \arrow[d, Rightarrow, "\theta_a"] \\
\mathrm{N}H'_q\ \mathrm{N}\mathcal{T}'_a\ \mathrm{N}H_p \arrow[r, Rightarrow, "\theta_a\mathrm{N}H_p"'] & \mathrm{N}\mathcal{T}''_a\ \mathrm{N}H'_p\ \mathrm{N}H_p \arrow[r, Rightarrow, "\mathrm{N}\mathcal{T}''_a\alpha_p"'] & \mathrm{N}\mathcal{T}''_a\ \mathrm{N}(H'H)_p,
\end{tikzcd}$$

is provided by the invertible modification obtained by pasting in

$$\begin{tikzcd}
H'_*LRH_*LRa^* \arrow[rr, Rightarrow, "H'_*\mathrm{v}LRa^*"] \arrow[d, Rightarrow, "H'_*LRH_*\mathrm{v}a^*"'] \arrow[drr, phantom, "\overset{(2.1)}{\Rrightarrow}"] & & H'_*H_*LRa^* \arrow[r, Rightarrow, "mLRa^*"] \arrow[d, Rightarrow, "H'_*H_*\mathrm{v}a^*"'] \arrow[dr, phantom, "\overset{(2.1)}{\Rrightarrow}"] & (H'H)_*LRa^* \arrow[d, Rightarrow, "(H'H)_*\mathrm{v}a^*"] \\
H'_*LRH_*a^* \arrow[rr, Rightarrow, "H'_*\mathrm{v}H_*a^*"] \arrow[d, Rightarrow, "H'_*LRa^*\mathrm{v}^\bullet H_*"'] \arrow[dr, Rightarrow, "1"] \arrow[d, phantom, shift left=6ex, "\overset{(A)}{\Rrightarrow}"] & \cong & H'_*H_*a^* \arrow[r, Rightarrow, "ma^*=a^*m"] \arrow[dd, Rightarrow, "a^*\mathrm{v}^\bullet H'_*H_*"'] \arrow[ddr, phantom, "\overset{(2.1)}{\Rrightarrow}"] & (H'H)_*a^* \arrow[dd, Rightarrow, "a^*\mathrm{v}^\bullet(H'H)_*"] \\
H'_*LRa^*LRH_* \arrow[r, Rightarrow, "H'_*LRa^*\mathrm{v}H_*"'] \arrow[d, Rightarrow, "H'_*\mathrm{v}a^*LRH_*"'] \arrow[dr, phantom, "\overset{(2.1)}{\Rrightarrow}"] & H'_*LRa^*H_* \arrow[ur, Rightarrow, "H'_*\mathrm{v}H_*a^*"] & \arrow[d, phantom, "\overset{(2.1)}{\Rrightarrow}"] & \\
H'_*a^*LRH_* \arrow[r, Rightarrow, "a^*\mathrm{v}^\bullet H'_*LRH_*"'] \arrow[uurr, Rightarrow, bend left=20, "H'_*a^*\mathrm{v}H_*"] & a^*LRH'_*LRH_* \arrow[r, Rightarrow, "a^*LRH'_*\mathrm{v}H_*"'] & a^*LRH'_*H_* \arrow[r, Rightarrow, "a^*LRm"'] & a^*LR(H'H)_*,
\end{tikzcd}$$

where the isomorphism (A) is given by the adjunction invertible modification $\mathrm{v}\circ\mathrm{v}^\bullet \cong 1$.

Finally, the pseudo-simplicial pseudo-equivalence (2.17), β: $\mathrm{N}1_{\mathcal{T}} \Rightarrow 1_{\mathrm{N}\mathcal{T}}$, is defined by the family of pseudo-equivalences

$$\mathrm{N}(1_{\mathcal{T}})_p = R(1_{\mathcal{T}})_*L \xRightarrow{\beta_p=RmL} RL = 1_{\mathrm{N}\mathcal{T}_p},$$

where m: $(1_{\mathcal{T}})_* \Rightarrow 1$: $\mathbb{L}\mathrm{ax}_{\mathrm{h}}([p],\mathcal{T}) \to \mathbb{L}\mathrm{ax}_{\mathrm{h}}([p],\mathcal{T})$ is the pseudo-equivalence in Lemma 2.1 (*iii*). The naturality invertible modification attached at any map a: $[q] \to [p]$,

$$\begin{tikzcd}
\mathrm{N}(1_{\mathcal{T}})_p\ \mathrm{N}\mathcal{T}_a \arrow[r, Rightarrow, "\beta_p\mathrm{N}\mathcal{T}_a"] \arrow[d, Rightarrow, "\theta_a"'] \arrow[dr, phantom, "\Rrightarrow"] & \mathrm{N}\mathcal{T}_a \arrow[d, Rightarrow, "1"] \\
\mathrm{N}\mathcal{T}_a\ \mathrm{N}(1_{\mathcal{T}})_q \arrow[r, Rightarrow, "\mathrm{N}\mathcal{T}_a\beta_q"'] & \mathrm{N}\mathcal{T}_a,
\end{tikzcd}$$

is that obtained by pasting the diagram

$$\begin{tikzcd}
R(1_{\mathcal{T}})_*LRa^*L \arrow[rr, Rightarrow, "RmLRa^*L"] \arrow[d, Rightarrow, "R(1_{\mathcal{T}})_*\mathrm{v}a^*L"'] \arrow[dr, phantom, "\overset{(2.1)}{\Rrightarrow}"] & & RLRa^*L = Ra^*L \arrow[dl, Rightarrow, "1=R\mathrm{v}a^*L"'] \arrow[dd, Rightarrow, "1"] \\
R(1_{\mathcal{T}})_*a^*L \arrow[r, Rightarrow, "Rma^*L"] \arrow[d, Rightarrow, "Ra^*\mathrm{v}^\bullet(1_{\mathcal{T}})_*L"'] \arrow[dr, phantom, "\overset{(2.1)}{\Rrightarrow}"] & Ra^*L \arrow[dr, Rightarrow, "1=Ra^*\mathrm{v}^\bullet L"'] \arrow[r, phantom, "\cong"] & \\
Ra^*LR(1_{\mathcal{T}})_*L \arrow[rr, Rightarrow, "Ra^*LRmL"'] & & Ra^*LRL = Ra^*L.
\end{tikzcd}$$

The conditions (**CC5**) and (**CC6**) in [42], for both α and β, are plainly verified. □

2.4 The classifying space of a tricategory

2.4.1 Preliminaries on classifying spaces of bicategories

When a bicategory $\mathcal{B}$ is regarded as a tricategory all of whose 3-cells are identities, the nerve construction on it actually produces a normal pseudo-simplicial category $\mathrm{N}\mathcal{B} = (\mathrm{N}\mathcal{B}, \chi)\colon \Delta^{\mathrm{op}} \to \mathbf{Cat}$, which is called by Carrasco, Cegarra and Garzón [41, §3] the *pseudo-simplicial nerve of the bicategory*. The *classifying space of the bicategory*, denoted here by $\mathrm{B}_2\mathcal{B}$, is then defined to be the ordinary classifying space of the category obtained by the Grothendieck construction [73] on the nerve of the bicategory, that is,

$$\mathrm{B}_2\mathcal{B} = \mathrm{B}\textstyle\int_{\Delta}\mathrm{N}\mathcal{B}\,.$$

The following facts are proved in [41].

Fact 2.4 *Each homomorphism between bicategories $F\colon \mathcal{B} \to \mathcal{C}$ induces a continuous (cellular) map $\mathrm{B}_2F\colon \mathrm{B}_2\mathcal{B} \to \mathrm{B}_2\mathcal{C}$. Thus, the classifying space construction, $\mathcal{B} \mapsto \mathrm{B}_2\mathcal{B}$, defines a functor from the category* ***Hom*** *of bicategories to CW-complexes.*

Fact 2.5 *If $F, F'\colon \mathcal{B} \to \mathcal{C}$ are two homomorphisms between bicategories, then any lax (or oplax) transformation, $F \Rightarrow F'$, canonically defines a homotopy between the induced maps on classifying spaces, $\mathrm{B}_2F \simeq \mathrm{B}_2F'\colon \mathrm{B}_2\mathcal{B} \to \mathrm{B}_2\mathcal{C}$.*

Fact 2.6 *If a homomorphism of bicategories has a left or right biadjoint, the map induced on classifying spaces is a homotopy equivalence. In particular, any biequivalence of bicategories induces a homotopy equivalence on classifying spaces.*

Furthermore, we should recall that the classifying space of any pseudo-simplicial bicategory $\mathcal{F}\colon \Delta^{\mathrm{op}} \to \mathbf{Bicat}$ is defined by Carrasco, Cegarra and Garzón in [42, Definition 5.4] to be the classifying space of its *bicategory of simplices* $\int_{\Delta}\mathcal{F}$, also called *the Grothendieck construction on* $\mathcal{F}$ [42, §3.1][5]. That is, the bicategory whose objects are the pairs (x, p), where $p \geq 0$ is an integer and x is an object of the bicategory $\mathcal{F}_p$, and whose hom-categories are

$$\textstyle\int_{\Delta}\mathcal{F}\big((y,q),(x,p)\big) = \displaystyle\bigsqcup_{[q]\xrightarrow{a}[p]} \mathcal{F}_q(y, a^*x),$$

where the disjoint union is over all arrows $a\colon [q] \to [p]$ in the simplicial category Δ; compositions, identities, and structure constraints are defined in the natural way. We refer the reader to [42, §3] for details about the bicategorical Grothendieck construction trihomomorphism

$$\textstyle\int_{\Delta}-\colon \mathbf{Bicat}^{\Delta^{\mathrm{op}}} \to \mathbf{Bicat},$$

from the tricategory of pseudo-simplicial bicategories to the tricategory of bicategories. The following facts are proved in [42].

---

[5] See also Section 3.3.

Fact 2.7 *(i) If* $\mathcal{F}, \mathcal{G}\colon \Delta^{\mathrm{op}} \to \mathbf{Bicat}$ *are pseudo-simplicial bicategories, then each pseudo-simplicial homomorphism* $F\colon \mathcal{F} \to \mathcal{G}$ *induces a continuous map*

$$\mathrm{B}_2\textstyle\int_\Delta F\colon \mathrm{B}_2\int_\Delta \mathcal{F} \to \mathrm{B}_2\int_\Delta \mathcal{G}.$$

(ii) For any pseudo-simplicial bicategory $\mathcal{F}\colon \Delta^{\mathrm{op}} \to \mathbf{Bicat}$*, there is a homotopy*

$$\mathrm{B}_2\textstyle\int_\Delta 1_{\mathcal{F}} \simeq 1_{\mathrm{B}_2\int_\Delta \mathcal{F}}\colon \mathrm{B}_2\int_\Delta \mathcal{F} \to \mathrm{B}_2\int_\Delta \mathcal{F}.$$

(iii) For any pair of composable pseudo-simplicial homomorphisms $F\colon \mathcal{F} \to \mathcal{G}$, $G\colon \mathcal{G} \to \mathcal{H}$*, there is a homotopy*

$$\mathrm{B}_2\textstyle\int_\Delta G\ \ \mathrm{B}_2\int_\Delta F \simeq \mathrm{B}_2\int_\Delta (GF)\colon \mathrm{B}_2\int_\Delta \mathcal{F} \to \mathrm{B}_2\int_\Delta \mathcal{H}.$$

Fact 2.8 *Any pseudo-simplicial transformation* $F \Rightarrow G\colon \mathcal{F} \to \mathcal{G}$ *induces a homotopy*

$$\mathrm{B}_2\textstyle\int_\Delta F \simeq \mathrm{B}_2\int_\Delta G\colon \mathrm{B}_2\int_\Delta \mathcal{F} \to \mathrm{B}_2\int_\Delta \mathcal{G}.$$

Fact 2.9 *If* $F\colon \mathcal{F} \to \mathcal{G}$ *is a pseudo simplicial homomorphism, between pseudo simplicial bicategories* $\mathcal{F}, \mathcal{G}\colon \Delta^{\mathrm{op}} \to \mathbf{Bicat}$*, such that the induced map* $\mathrm{B}_2 F_p\colon \mathrm{B}_2\mathcal{F}_p \to \mathrm{B}_2\mathcal{G}_p$ *is a homotopy equivalence for all* $p \geq 0$*, then the induced map* $\mathrm{B}_2\int_\Delta F\colon \mathrm{B}_2\int_\Delta \mathcal{F} \to \mathrm{B}_2\int_\Delta \mathcal{G}$ *is a homotopy equivalence.*

Fact 2.10 *If* $\mathcal{F}\colon \Delta^{\mathrm{op}} \to \mathbf{Hom} \subset \mathbf{Bicat}$ *is a simplicial bicategory, then there is a natural homotopy equivalence*

$$\mathrm{B}_2\textstyle\int_\Delta \mathcal{F} \simeq |\mathrm{B}_2\mathcal{F}| = |[p] \mapsto \mathrm{B}_2\mathcal{F}_p|,$$

where the latter is the geometric realization of the simplicial space $\mathrm{B}_2\mathcal{F}\colon \Delta^{\mathrm{op}} \to \mathbf{Top}$*, obtained by composing* $\mathcal{F}$ *with the classifying space functor* $\mathrm{B}_2\colon \mathbf{Hom} \to \mathbf{Top}$.

2.4.2 The classifying space construction for tricategories

We are now ready to make the following definition, which recovers the more traditional way through which a classifying space is assigned in the literature to certain specific kinds of tricategories, such as 3-categories, bicategories, monoidal categories, or braided monoidal categories (see Examples 2.4, 2.5, and 2.6 below).

Definition 2.1 *The classifying space* $\mathrm{B}_3\mathcal{T}$*, of a tricategory* $\mathcal{T}$*, is the classifying space of its bicategorical pseudo-simplicial Grothendieck nerve,* $\mathrm{N}\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Bicat}$*, that is,*

$$\mathrm{B}_3\mathcal{T} = \mathrm{B}_2\textstyle\int_\Delta \mathrm{N}\mathcal{T}\ .$$

Let us remark that the classifying space of a tricategory $\mathcal{T}$ is then realized as the classifying space of a category canonically associated to it, namely, as

$$\mathrm{B}_3\mathcal{T} = \mathrm{B}{\textstyle\int_\Delta}\mathrm{N}\big({\textstyle\int_\Delta}\mathrm{N}\mathcal{T}\big) = |\mathrm{N}\big({\textstyle\int_\Delta}\mathrm{N}({\textstyle\int_\Delta}\mathrm{N}\mathcal{T})\big)|.$$

Example 2.3 (Classifying spaces of categories and bicategories) When a bicategory $\mathcal{B}$ is viewed as a tricategory whose 3-cells are all identities, then its classifying space is homotopy equivalent to the classifying space of the bicategory $\mathrm{B}_2\mathcal{B}$ as defined by Carrasco, Cegarra and Garzón in [41, Definition 3.1], that is

$$\mathrm{B}_3\mathcal{B} \simeq \mathrm{B}_2\mathcal{B}.$$

To see that, let us recall that, for any simplicial set $X\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$, there is a natural homotopy equivalence $\mathrm{B}\int_\Delta X \simeq |X|$, between the classifying space of its category of simplices and its geometric realization (see Illusie [85, Theorem 3.3]). As, for any bicategory $\mathcal{B}$, $\mathrm{N}\big(\int_\Delta \mathrm{N}\mathcal{B}\big)$ is actually a simplicial set, we have a homotopy equivalence

$$\mathrm{B}_3\mathcal{B} = \mathrm{B}\,{\textstyle\int_\Delta}\mathrm{N}\big({\textstyle\int_\Delta}\mathrm{N}\mathcal{B}\big)| \simeq |\mathrm{N}\big({\textstyle\int_\Delta}\mathrm{N}\mathcal{B}\big)| = \mathrm{B}\,{\textstyle\int_\Delta}\mathrm{N}\mathcal{B} = \mathrm{B}_2\mathcal{B}.$$

Similarly, for $\mathcal{C}$ any category regarded as a tricategory whose 2- and 3-cells are all identities, we have homotopy equivalences

$$\mathrm{B}_3\mathcal{C} \simeq \mathrm{B}_2\mathcal{C} = \mathrm{B}\,{\textstyle\int_\Delta}\mathrm{N}\mathcal{C} \simeq |\mathrm{N}\mathcal{C}| = \mathrm{B}\mathcal{C},$$

since, in this case, $\mathrm{N}\mathcal{C}$ is a simplicial set.

The next proposition deals with some properties concerning the homotopy behavior of the classifying space construction, $\mathcal{T} \mapsto \mathrm{B}_3\mathcal{T}$, with respect to trihomomorphisms of tricategories.

Proposition 2.2 *(i) Any trihomomorphism between tricategories $H\colon \mathcal{T} \to \mathcal{T}'$ induces a continuous (cellular) map $\mathrm{B}_3H\colon \mathrm{B}_3\mathcal{T} \to \mathrm{B}_3\mathcal{T}'$.*

(ii) For any pair of composable trihomomorphisms $H\colon \mathcal{T} \to \mathcal{T}'$ and $H'\colon \mathcal{T}' \to \mathcal{T}''$, there is a homotopy

$$\mathrm{B}_3H'\ \mathrm{B}_3H \simeq \mathrm{B}_3(H'H)\colon \mathrm{B}_3\mathcal{T} \to \mathrm{B}_3\mathcal{T}''.$$

(iii) For any tricategory $\mathcal{T}$, there is a homotopy $\mathrm{B}_3 1_{\mathcal{T}} \simeq 1_{\mathrm{B}_3\mathcal{T}}\colon \mathrm{B}_3\mathcal{T} \to \mathrm{B}_3\mathcal{T}$.

Proof: (i) By Proposition 2.1 (i), any trihomomorphism $H\colon \mathcal{T} \to \mathcal{T}'$ gives rise to a pseudo-simplicial homomorphism on the corresponding Grothendieck nerves $\mathrm{N}H\colon \mathrm{N}\mathcal{T} \to \mathrm{N}\mathcal{T}'$, which, by Fact 2.7 ($i$), produces the claimed continuous map $\mathrm{B}_3H = \mathrm{B}_2\int_\Delta \mathrm{N}H\colon \mathrm{B}_3\mathcal{T} \to \mathrm{B}_3\mathcal{T}'$.

(*ii*) Suppose that $\mathcal{T} \overset{H}{\to} \mathcal{T}' \overset{H'}{\to} \mathcal{T}''$ are trihomomorphisms. By Proposition 2.1 (*ii*), there is a pseudo-simplicial pseudo-equivalence $\mathrm{N}H'\,\mathrm{N}H \Rightarrow \mathrm{N}(H'H)$, which, by Fact 2.8, induces a homotopy

$$\mathrm{B}_2\textstyle\int_\Delta(\mathrm{N}H'\,\mathrm{N}H) \simeq \mathrm{B}_2\textstyle\int_\Delta\mathrm{N}(H'H) = \mathrm{B}_3(H'H).$$

Then, the result follows since, by Fact 2.7 (*iii*), there is a homotopy

$$\mathrm{B}_2\textstyle\int_\Delta(\mathrm{N}H'\,\mathrm{N}H) \simeq \mathrm{B}_2\textstyle\int_\Delta\mathrm{N}H'\;\mathrm{B}_2\textstyle\int_\Delta\mathrm{N}H = \mathrm{B}_3H'\,\mathrm{B}_3H.$$

(*iii*) By Proposition 2.1 (*iii*), there is a pseudo-simplicial pseudo-equivalence

$$\mathrm{N}1_{\mathcal{T}} \Rightarrow 1_{\mathrm{N}\mathcal{T}},$$

which, by Fact 2.8, induces a homotopy $\mathrm{B}_3 1_{\mathcal{T}} = \mathrm{B}_2\int_\Delta\mathrm{N}1_{\mathcal{T}} \simeq \mathrm{B}_2\int_\Delta 1_{\mathrm{N}\mathcal{T}}$. Since, by Fact 2.7 (*ii*), there is a homotopy $\mathrm{B}_2\int_\Delta 1_{\mathrm{N}\mathcal{T}} \simeq \mathrm{B}_2 1_{\int_\Delta\mathrm{N}\mathcal{T}} = 1_{\mathrm{B}_3\mathcal{T}}$, the result follows. ☐

Example 2.4 (Classifying spaces of 3-categories) In [111], Segal observed that, if $\mathbb{C}$ is a topological category, then its Grothendieck nerve (2.9) is, in a natural way, a simplicial space, that is, $\mathrm{N}\mathbb{C}\colon \Delta^{\mathrm{op}} \to \mathbf{Top}$. Then, he defines the classifying space of a topological category $\mathbb{C}$ to be $|\mathrm{N}\mathbb{C}|$, the geometric realization of this simplicial space. This notion given by Segal provides, for instance, the usual definition of classifying spaces of strict bicategories, or 2-categories, and strict tricategories, or 3-categories. Thus, the *classifying space of a 2-category* $\mathcal{B}$ is, by definition, the classifying space of the topological category whose space of objects is the discrete space of objects of $\mathcal{B}$, and whose hom spaces are obtained by replacing the hom categories $\mathcal{B}(b, b')$ by their classifying spaces. In other words, the classifying space of the 2-category is the geometric realization of the simplicial space obtained by composing $\mathrm{N}\mathcal{B}\colon \Delta^{\mathrm{op}} \to \mathbf{Cat}$ with the classifying space functor $\mathrm{B}\colon \mathbf{Cat} \to \mathbf{Top}$, that is, the space

$$|[p] \mapsto \mathrm{B}(\mathrm{N}\mathcal{B}_p)| = |[p] \mapsto |\mathrm{N}(\mathrm{N}\mathcal{B}_p)|| = |[p] \mapsto |[q] \mapsto \mathrm{N}(\mathrm{N}\mathcal{B}_p)_q|| \cong |\mathrm{diagNN}\mathcal{B}|$$

where $\mathrm{diagNN}\mathcal{B}$ is the diagonal simplicial set of the bisimplicial set $\mathrm{NN}\mathcal{B}$ double nerve of $\mathcal{B}$, $([p],[q]) \mapsto \mathrm{N}(\mathrm{N}\mathcal{B}_p)_q$, and the last homeomorphism above is a consequence of Quillen's Lemma [109, page 86]. Similarly, as a 3-category $\mathcal{T}$ is just a category enriched in the category of 2-categories and 2-functors, that is, a category $\mathcal{T}$ endowed with 2-categorical hom-sets $\mathcal{T}(t', t)$, in such a way that the compositions $\mathcal{T}(t', t) \times \mathcal{T}(t'', t') \to \mathcal{T}(t'', t)$ are 2-functors, by replacing the hom 2-categories $\mathcal{T}(t', t)$ by their classifying spaces as above, we obtain a topological category, whose classifying space is the *classifying space of the* 3*-category* $\mathcal{T}$. That is, the space

$$|[p] \mapsto |\mathrm{diagNN}(\mathrm{N}\mathcal{T}_p)|| = |[p] \mapsto |[q] \mapsto \mathrm{N}(\mathrm{N}(\mathrm{N}\mathcal{T}_p)_q)_q| \cong |\mathrm{diagNNN}\mathcal{T}|.$$

For $\mathcal{B}$ any 2-category, Thomason's Homotopy Colimit Theorem [120, Theorem 1.2] and a result by Bousfield and Kan [20, XII, 4.3] give the existence of natural homotopy equivalences

$$\mathrm{B}_2\mathcal{B} = \mathrm{B}\textstyle\int_{\Delta}\mathrm{N}\mathcal{B} \simeq |\mathrm{hocolim}_{\Delta}\mathrm{NN}\mathcal{B}| \simeq |\mathrm{diagNN}\mathcal{B}|.$$

For $\mathcal{T}$ any 3-category, by Fact 2.10, there is a homotopy equivalence $\mathrm{B}_3\mathcal{T} = \mathrm{B}_2\int_{\Delta}\mathrm{N}\mathcal{T} \simeq |[p] \mapsto \mathrm{B}_2(\mathrm{N}\mathcal{T}_p)|$, and therefore we have an induced homotopy equivalence

$$\mathrm{B}_3\mathcal{T} \simeq |[p] \mapsto |\mathrm{diagNN}(\mathrm{N}\mathcal{T}_p)|| \cong |\mathrm{diagNNN}\mathcal{T}|.$$

2.4.3 The Segal nerve of a tricategory

Several theoretical interests suggest dealing with the so-called Segal nerve construction for tricategories. This associates to any tricategory $\mathcal{T}$ a simplicial bicategory, denoted by $\mathrm{S}\mathcal{T}$, which can be thought of as a 'rectification' of the pseudo-simplicial Grothendieck nerve of the tricategory $\mathrm{N}\mathcal{T}$, since both are biequivalent in the tricategory of pseudo-simplicial bicategories and therefore model the same homotopy type. Furthermore, $\mathrm{S}\mathcal{T}$ is a weak 3-category under the point of view of Tamsamani [119] and Simpson [113] (see Proposition 2.3 below), in the sense that it is a *special simplicial bicategory*, that is, a simplicial bicategory S: $\Delta^{\mathrm{op}} \to \mathbf{Hom}$ satisfying the following two conditions:

(i) S_0 is discrete (i.e., all its 1- and 2-cells are identities).

(ii) for $n \geq 2$, the Segal projection homomorphisms below are biequivalences.

$$p_n = \prod_{k=1}^{n} d_n \cdots d_{k+1} d_{k-2} \cdots d_0 \colon \mathrm{S}_n \longrightarrow \mathrm{S}_1 {}_{d_1}\!\times_{d_0} \mathrm{S}_1 {}_{d_1}\!\times_{d_0} \cdots {}_{d_1}\!\times_{d_0} \mathrm{S}_1. \tag{2.19}$$

For a given tricategory $\mathcal{T}$, the construction of the bicategory of unitary homomorphic lax functors from any small category I in the tricategory $\mathcal{T}$, $I \mapsto \mathbb{L}\mathrm{ax}_{\mathrm{uh}}(I, \mathcal{T})$, given in (2.6), is functorial on the category I, and this leads to the definition below.

Definition 2.2 *The* Segal nerve *of a tricategory $\mathcal{T}$ is the simplicial bicategory*

$$\mathrm{S}\mathcal{T} \colon \Delta^{\mathrm{op}} \to \mathbf{Hom} \subset \mathbf{Bicat}, \quad [p] \mapsto \mathrm{S}\mathcal{T}_p = \mathbb{L}\mathrm{ax}_{\mathrm{uh}}([p], \mathcal{T}). \tag{2.20}$$

We should remark that, when $\mathcal{T} = \mathcal{B}$ is a bicategory, that is, when its 3-cells are all identities, then the Segal nerve $\mathrm{S}\mathcal{B}$ was introduced by Carrasco, Cegarra and Garzón in [41, Definition 5.2], although it was first studied by Lack and Paoli in [95] under the name of '2-nerve of $\mathcal{B}$'.

Proposition 2.3 *Let $\mathcal{T}$ be a tricategory. Then, the following statements hold:*

(i) There is a normal pseudo-simplicial homomorphism

$$L \colon \mathrm{N}\mathcal{T} \to \mathrm{S}\mathcal{T}, \tag{2.21}$$

such that, for any $p \geq 0$*, the homomorphism* $L_p\colon \mathrm{N}\mathcal{T}_p \to \mathrm{S}\mathcal{T}_p$ *is a biequivalence.*

(ii) *The simplicial bicategory* $\mathrm{S}\mathcal{T}$ *is special.*

Proof: We keep the notations established in the construction of $\mathrm{N}\mathcal{T} = (\mathrm{N}\mathcal{T}, \chi, \omega)$ given in the proof of Theorem 2.1, and recall from Lemma 2.2 (c) that we have a biadjoint biequivalence

$$L_p \dashv R_p\colon \mathrm{N}\mathcal{T}_p \rightleftarrows \mathrm{S}\mathcal{T}_p.$$

The normal pseudo-simplicial homomorphism $L = (L, \theta, \Pi)\colon \mathrm{N}\mathcal{T} \to \mathrm{S}\mathcal{T}$ is then defined by the homomorphisms $L_p\colon \mathrm{N}\mathcal{T}_p \to \mathrm{S}\mathcal{T}_p$, $p \geq 0$. For any $a\colon [q] \to [p]$, the structure pseudo-equivalence

$$\begin{tikzcd}
\mathrm{N}\mathcal{T}_p \arrow[r, "L_p"] \arrow[d, "\mathrm{N}\mathcal{T}_a"'] \arrow[dr, phantom, "\overset{\theta_a}{\Rightarrow}"] & \mathrm{S}\mathcal{T}_p \arrow[d, "a^*"] \\
\mathrm{N}\mathcal{T}_q \arrow[r, "L_q"'] & \mathrm{S}\mathcal{T}_q,
\end{tikzcd}$$

is provided by the counit pseudo-equivalence $\mathrm{v}_q\colon L_q R_q \Rightarrow 1_{\mathrm{S}\mathcal{T}_q}$; that is,

$$L_q\ \mathrm{N}\mathcal{T}_a = L_q R_q a^* L_p \xRightarrow{\theta_a = \mathrm{v}_q a^* L_p} a^*\, L_p.$$

For $[n] \xrightarrow{b} [q] \xrightarrow{a} [p]$, any two composable arrows of Δ, the structure invertible modification

$$\begin{tikzcd}[column sep=small]
L_n\, \mathrm{N}\mathcal{T}_b\, \mathrm{N}\mathcal{T}_a \arrow[rr, Rightarrow, "L_n\chi"] \arrow[d, Rightarrow, "\theta \mathrm{N}\mathcal{T}_a"'] \arrow[drr, phantom, "\overset{\Pi_{a,b}}{\Rightarrow}"] & & L_n\, \mathrm{N}\mathcal{T}_{ab} \arrow[d, Rightarrow, "\theta"] \\
b^* L_q\, \mathrm{N}\mathcal{T}_a \arrow[r, Rightarrow, "b^*\theta"'] & b^* a^* L_p \arrow[r, equal, "1"'] & (ab)^* L_p
\end{tikzcd}$$

is directly provided by the canonical modification (2.18), $\Pi_{a,b} = \omega'_b a^* L_p$.

The coherence conditions for L (i.e., conditions (**CC3**) and (**CC4**) in [42], with the modifications Γ the coherence isomorphisms $1 \circ 1 \cong 1$), are easily verified by using Fact 2.1. This complete the proof of part (i).

And when it comes to part (ii), that is, that $\mathrm{S}\mathcal{T}$ is a special simplicial bicategory, we have the following isomorphisms between bicategories:

$$\mathrm{S}\mathcal{T}_0 = \mathbb{L}\mathrm{ax}_{\mathrm{uh}}([0], \mathcal{T}) \cong \mathrm{Ob}\mathcal{T} = \mathrm{N}\mathcal{T}_0, \tag{2.22}$$

which identifies a normal trihomomorphism $F\colon [0] \to \mathcal{T}$ with the object $F0$,

$$\mathrm{S}\mathcal{T}_1 = \mathbb{L}\mathrm{ax}_{\mathrm{uh}}([1], \mathcal{T}) \cong \bigsqcup_{(t_0, t_1)} \mathcal{T}(t_0, t_1) = \mathrm{N}\mathcal{T}_1, \tag{2.23}$$

which carries a normal trihomomorphism $F\colon [1] \to \mathcal{T}$ to the 1-cell $F_{0,1}\colon F0 \to F1$, and, for any integer $p \geq 2$,

$$\begin{aligned}
&\mathrm{S}\mathcal{T}_{1\,d_1}\!\times_{d_0}\cdots{}_{d_1}\!\times_{d_0}\mathrm{S}\mathcal{T}_1\\
&\quad\cong \bigsqcup_{(t_{p-1},t_p)}\mathcal{T}(t_{p-1},t_p)\times\bigsqcup_{(t_{p-2},t_{p-1})}\mathcal{T}(t_{p-2},t_{p-1})\times\cdots\times\bigsqcup_{(t_0,t_1)}\mathcal{T}(t_0,t_1)\\
&\quad\cong \bigsqcup_{(t_0,\dots,t_p)}\mathcal{T}(t_{p-1},t_p)\times\mathcal{T}(t_{p-2},t_{p-1})\times\cdots\times\mathcal{T}(t_0,t_1)=\mathrm{N}\mathcal{T}_p.
\end{aligned}$$

Through these isomorphisms we see that, for any integer $p\geq 2$, the Segal projection homomorphism (2.19) is precisely the biequivalence R_p: $\mathrm{S}\mathcal{T}_p\to\mathrm{N}\mathcal{T}_p$ which, recall, is defined by restricting it to the basic graph of the category $[p]$. Whence the simplicial bicategory $\mathrm{S}\mathcal{T}$ is special. □

The following theorem states that the classifying space of a tricategory $\mathcal{T}$ can be realized, up to homotopy equivalence, by its Segal nerve $\mathrm{S}\mathcal{T}$. This fact will be relevant for our later discussions on loop spaces. Let

$$\mathrm{B}_2\mathrm{S}\mathcal{T}\colon\Delta^{\mathrm{op}}\to\mathbf{Top},\quad [p]\mapsto\mathrm{B}_2(\mathrm{S}\mathcal{T}_p),$$

be the simplicial space obtained by composing $\mathrm{S}\mathcal{T}\colon\Delta^{\mathrm{op}}\to\mathbf{Hom}\subset\mathbf{Bicat}$ with the classifying functor B_2: $\mathbf{Hom}\to\mathbf{Top}$ (recall Fact 2.4).

Theorem 2.2 *For any tricategory $\mathcal{T}$, there is a homotopy equivalence*

$$\mathrm{B}_3\mathcal{T}\simeq|\mathrm{B}_2\mathrm{S}\mathcal{T}|.$$

Proof: Let us consider the pseudo-simplicial homomorphism (2.21), L: $\mathrm{N}\mathcal{T}\to\mathrm{S}\mathcal{T}$. Since, for every integer $p\geq 0$, the homomorphism L_p: $\mathrm{N}\mathcal{T}_p\to\mathrm{S}\mathcal{T}_p$ is a biequivalence, it follows from Fact 2.6 that the induced cellular map B_2L_p: $\mathrm{B}_2(\mathrm{N}\mathcal{T}_p)\to\mathrm{B}_2(\mathrm{S}\mathcal{T}_p)$ is a homotopy equivalence. Then, by Fact 2.9, the induced map $\mathrm{B}_2\int_\Delta L$: $\mathrm{B}_2\int_\Delta\mathrm{N}\mathcal{T}\to\mathrm{B}_2\int_\Delta\mathrm{S}\mathcal{T}$ is a homotopy equivalence. Since, by definition, $\mathrm{B}_3\mathcal{T}=\mathrm{B}_2\int_\Delta\mathrm{N}\mathcal{T}$, whereas, by Fact 2.10, there is a homotopy equivalence $\mathrm{B}_2\int_\Delta\mathrm{S}\mathcal{T}\simeq|\mathrm{B}_2\mathrm{S}\mathcal{T}|$, the claimed homotopy equivalence follows. □

Example 2.5 (Classifying spaces of monoidal bicategories) Any monoidal bicategory $(\mathcal{B},\otimes)=(\mathcal{B},\otimes,\mathrm{I},\boldsymbol{a},\boldsymbol{l},\boldsymbol{r},\pi,\mu,\lambda,\rho)$ can be viewed as a tricategory

$$\Sigma(\mathcal{B},\otimes)\tag{2.24}$$

with only one object, say $*$, whose hom-bicategory is the underlying bicategory. Thus, $\Sigma(\mathcal{B},\otimes)(*,*)=\mathcal{B}$, and its composition given by the tensor functor $\otimes$: $\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ and the identity at the object is $1_*=\mathrm{I}$, the unit object of the monoidal bicategory. The structure pseudo-equivalences and modifications $\boldsymbol{a}$, $\boldsymbol{l}$, $\boldsymbol{r}$, π, μ, λ, and ρ for $\Sigma(\mathcal{B},\otimes)$ are just those of the monoidal bicategory, respectively (see the paper by Cheng and Gurski [53, §3] for details). Call this tricategory the *suspension*, or *delooping, tricategory of the bicategory $\mathcal{B}$ induced by the monoidal structure given on it*, and call its corresponding Grothendieck nerve the *nerve of the monoidal bicategory*, hereafter denoted by $\mathrm{N}(\mathcal{B},\otimes)$. Thus,

$$\mathrm{N}(\mathcal{B},\otimes)=\mathrm{N}\Sigma(\mathcal{B},\otimes)\colon \Delta^{\mathrm{op}} \to \mathbf{Bicat}, \quad [p] \mapsto \mathcal{B}^p,$$

is a normal pseudo-simplicial bicategory, whose bicategory of p-simplices is the p-fold power of the underlying bicategory $\mathcal{B}$, with face and degeneracy homomorphisms induced by the tensor homomorphism $\otimes\colon \mathcal{B}\times\mathcal{B}\to\mathcal{B}$ and unit object I, following the familiar formulas (2.11), in analogy with those of the reduced bar construction on a topological monoid, and with structure pseudo-equivalences and modifications canonically arising from the data of the monoidal structure on $\mathcal{B}$. The general Definition 2.1 for classifying spaces of tricategories leads to the following.

Definition 2.3 *The* classifying space *of the monoidal bicategory, denoted by* $\mathrm{B}_3(\mathcal{B},\otimes)$*, is defined to be the classifying space of its delooping tricategory* $\Sigma(\mathcal{B},\otimes)$*. Thus,*

$$\mathrm{B}_3(\mathcal{B},\otimes) = \mathrm{B}_3\Sigma(\mathcal{B},\otimes) = \mathrm{B}_2\textstyle\int_\Delta \mathrm{N}(\mathcal{B},\otimes)\ .$$

The next theorem extends a well-known result by Mac Lane and Stasheff on monoidal categories to monoidal bicategories.

Theorem 2.3 *Let* $(\mathcal{B},\otimes)$ *be a monoidal bicategory such that, for any object* $x\in\mathcal{B}$*, the homomorphism* $x\otimes -\colon \mathcal{B}\to\mathcal{B}$ *induces a homotopy auto-equivalence on the classifying space of* $\mathcal{B}$*,* $\mathrm{B}_2(x\otimes -)\colon \mathrm{B}_2\mathcal{B}\simeq\mathrm{B}_2\mathcal{B}$*. Then, there is a homotopy equivalence*

$$\mathrm{B}_2\mathcal{B}\simeq\Omega\mathrm{B}_3(\mathcal{B},\otimes),$$

between the classifying space of the underlying bicategory and the loop space of the classifying space of the monoidal bicategory.

Proof: By Theorem 2.2, $\mathrm{B}_3(\mathcal{B},\otimes)$ is homotopy equivalent to $|X_\bullet|$, the geometric realization of the the simplicial space $X_\bullet = \mathrm{B}_2\mathrm{S}\Sigma(\mathcal{B},\otimes)\colon \Delta^{\mathrm{op}} \to \mathbf{Top}$ obtained by taking classifying spaces on the simplicial bicategory $\mathrm{S}\Sigma(\mathcal{B},\otimes)$, the Segal nerve of the suspension tricategory of the monoidal bicategory. By Proposition 2.3, $\mathrm{S}\Sigma(\mathcal{B},\otimes)$ is a special simplicial bicategory. Furthermore, since the tricategory $\Sigma(\mathcal{B},\otimes)$ has only one object, the simplicial bicategory $\mathrm{S}\Sigma(\mathcal{B},\otimes)$ is reduced (see (2.22)), that is, $\mathrm{S}\Sigma(\mathcal{B},\otimes)_0 = *$ the one-object discrete bicategory. Hence, the simplicial space $\mathrm{B}_2\mathrm{S}\Sigma(\mathcal{B},\otimes)$ satisfies hypothesis (*i*) and (*ii*) of Segal's Proposition 1.5 in [112]: $X_0 = \mathrm{B}_2(\mathrm{S}\Sigma(\mathcal{B},\otimes)_0)$ is contractible, and the Segal projections maps

$$\mathrm{B}_2p_n\colon X_n = \mathrm{B}_2(\mathrm{S}\Sigma(\mathcal{B},\otimes)_n) \to X_1^n = \mathrm{B}_2(\mathrm{S}\Sigma(\mathcal{B},\otimes)_1^n) \overset{(2.23)}{\cong} \mathrm{B}_2\mathcal{B}^n$$

are homotopy equivalences. Since the H-space structure on $X_1 = \mathrm{B}_2\mathcal{B}$ is just induced by tensor homomorphism $\otimes\colon \mathcal{B}\times\mathcal{B}\to\mathcal{B}$, we have, by hypothesis, that X_1 has homotopy inverses. Therefore, from [112, Proposition 1.5 (b)], we can conclude that the canonical map $X_1\to\Omega|X_\bullet|$ is a homotopy equivalence, whence the homotopy equivalence $\mathrm{B}_2\mathcal{B}\simeq\Omega\mathrm{B}_3(\mathcal{B},\otimes)$ follows. □

Example 2.6 (Classifying spaces of braided monoidal categories) If $(\mathcal{C},\otimes)$ is a monoidal category , then $\Sigma(\mathcal{C},\otimes)$ is the bicategory called by Kapranov and Voevodsky in [90, 2.10] the *delooping bicategory of the category induced by its monoidal structure.* The nerve of $\Sigma(\mathcal{C},\otimes)$ then becomes the pseudo-simplicial category used by Jardine in [86, §3] to define the classifying space of the monoidal category just as above: $\mathrm{B}_2(\mathcal{C},\otimes) = \mathrm{B}\int_\Delta \mathrm{N}(\mathcal{C},\otimes)$ (see also Bullejos and Cegarra [36], or Balteanu et al. [12]). Thus,

$$\mathrm{B}_2(\mathcal{C},\otimes) = \mathrm{B}_2\Sigma(\mathcal{C},\otimes). \tag{2.25}$$

Now, let $(\mathcal{C},\otimes,\boldsymbol{c})$ be a braided monoidal category as defined by Joyal and Street in [88, Definition 3.1]. Thanks to the braidings $\boldsymbol{c}\colon x\otimes y \to y\otimes x$, the given tensor product on $\mathcal{C}$ defines in a natural way a tensor product homomorphism on the suspension bicategory of the underlying monoidal category, $\otimes\colon \Sigma(\mathcal{C},\otimes)\times\Sigma(\mathcal{C},\otimes) \to \Sigma(\mathcal{C},\otimes)$. Thus $(\Sigma(\mathcal{C},\otimes),\otimes)$ is a monoidal bicategory. The corresponding suspension tricategory,

$$\Sigma^2(\mathcal{C},\otimes,\boldsymbol{c}) = \Sigma(\Sigma(\mathcal{C},\otimes),\otimes)$$

is called the *double suspension*, or *double delooping, of the underlying category* $\mathcal{C}$ *associated to the given braided monoidal structure on it* (see Berger [17, 4,2.5], Kapranov and Voevodsky [90, 4.2] or Gordon, Power and Street [69, 7.9]). We refer the reader to Cheng and Gurski [53, §2] for the (nontrivial details) of this construction and, briefly, recall that this is a tricategory with only one object, say $*$, only one arrow $* = 1_*\colon * \to *$, the objects of $\mathcal{C}$ are the 2-cells, and the morphisms of $\mathcal{C}$ are the 3-cells. The hom-bicategory is $\Sigma^2(\mathcal{C},\otimes,\boldsymbol{c})(*,*) = \Sigma(\mathcal{C},\otimes)$, the suspension bicategory of the underlying monoidal category $(\mathcal{C},\otimes)$, the composition is also (as the horizontal one in $\Sigma(\mathcal{C},\otimes)$) given by the tensor functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and the interchange 3-cell between the two different composites of 2-cells is given by the braiding.

The most striking instance is for $(\mathcal{C},\otimes,\boldsymbol{c}) = (A,+,0)$, the strict braided monoidal category with only one object defined by an abelian group A, where both composition and tensor product are given by the addition $+$ in A; in this case, the double suspension tricategory $\Sigma^2 A$ is precisely the 3-category treated in Examples 2.1 and 2.2.

For any braided monoidal category $(\mathcal{C},\otimes,\boldsymbol{c})$, the Grothendieck nerve of the double suspension tricategory $\Sigma^2(\mathcal{C},\otimes,\boldsymbol{c})$ coincides with the pseudo-simplicial bicategory called by Carrasco, Cegarra and Garzón in [42] the *nerve of the braided monoidal category*, and denoted by $\mathrm{N}(\mathcal{C},\otimes,\boldsymbol{c})$. Thus,

$$\mathrm{N}(\mathcal{C},\otimes,\boldsymbol{c}) = \mathrm{N}\Sigma^2(\mathcal{C},\otimes,\boldsymbol{c})\colon \Delta^{\mathrm{op}} \to \mathbf{Bicat}, \quad [p] \mapsto (\Sigma(\mathcal{C},\otimes))^p = \Sigma(\mathcal{C}^p,\otimes),$$

is a normal pseudo-simplicial one-object bicategory whose bicategory of p-simplicies is the suspension bicategory of the monoidal category p-fold power of $(\mathcal{C},\otimes)$. Since the *classifying space of the braided monoidal category* [42, Definition 6.1], $\mathrm{B}_3(\mathcal{C},\otimes,\boldsymbol{c})$, is just given by

$$\mathrm{B}_3(\mathcal{C},\otimes,\boldsymbol{c}) = \mathrm{B}_2\textstyle\int_\Delta \mathrm{N}(\mathcal{C},\otimes,\boldsymbol{c}),$$

we have the following.

Corollary 2.1 *The classifying space of a braided monoidal category is the classifying space of its double suspension tricategory, that is,* $\mathrm{B}_3(\mathcal{C},\otimes,\boldsymbol{c}) = \mathrm{B}_3\Sigma^2(\mathcal{C},\otimes,\boldsymbol{c})$.

It is known that the group completion of the classifying space $\mathrm{B}\mathcal{C}$ of a braided monoidal category $(\mathcal{C},\otimes,\boldsymbol{c})$ is a double loop space. This fact was first noticed by J. D. Stasheff in [114], but originally proven by Z. Fiedorowicz in [64] (other proofs can be found in Balteanu et al. [12] or Berger [17]). We shall show below that under favorable circumstances $\mathrm{B}_3(\mathcal{C},\otimes,\boldsymbol{c})$ is a model for such a double delooping of $\mathrm{B}\mathcal{C}$.

Corollary 2.2 (i) *For any braided monoidal category* $(\mathcal{C},\otimes,\boldsymbol{c})$ *there is a homotopy equivalence*

$$\mathrm{B}_2(\mathcal{C},\otimes) \simeq \Omega\mathrm{B}_3(\mathcal{C},\otimes,\boldsymbol{c}).$$

(ii) *Let* $(\mathcal{C},\otimes,\boldsymbol{c})$ *be a braided monoidal category such that, for any object* $x\in\mathcal{C}$, *the functor* $x\otimes - \colon \mathcal{C}\to\mathcal{C}$ *induces a homotopy auto-equivalence on the classifying space of* $\mathcal{C}$, $\mathrm{B}(x\otimes -)\colon \mathrm{B}\mathcal{C}\simeq\mathrm{B}\mathcal{C}$. *Then, there is a homotopy equivalence*

$$\mathrm{B}\mathcal{C} \simeq \Omega^2\mathrm{B}_3(\mathcal{C},\otimes,\boldsymbol{c}),$$

between the classifying space of the underlying category and the double loop space of the classifying space of the underlying category.

Proof: (i) By Corollary 2.1, the classifying space of any braided monoidal category $(\mathcal{C},\otimes,\boldsymbol{c})$ is the same as the classifying space of the monoidal bicategory $\Sigma(\Sigma(\mathcal{C},\otimes),\otimes)$. Therefore, $\Omega\mathrm{B}_3(\mathcal{C},\otimes,\boldsymbol{c}) = \Omega\mathrm{B}_3\Sigma(\Sigma(\mathcal{C},\otimes),\otimes)$. Since $\Sigma(\mathcal{C},\otimes)$ has only one object, it is obvious that its monoid of connected components $\pi_0\Sigma(\mathcal{C},\otimes) = 1$, the trivial group. Then, by Theorem 2.3, there is a homotopy equivalence $\mathrm{B}_2\Sigma(\mathcal{C},\otimes) \simeq \Omega\mathrm{B}_3\Sigma(\Sigma(\mathcal{C},\otimes),\otimes)$. Since, by (2.25), $\mathrm{B}_2(\mathcal{C},\otimes) = \mathrm{B}_2\Sigma(\mathcal{C},\otimes)$, the result follows.

(ii) From the discussion in Example 2.3, we have homotopy equivalences $\mathrm{B}\mathcal{C}\simeq\mathrm{B}_2\mathcal{C}$ and $\mathrm{B}_2(\mathcal{C},\otimes)\simeq\mathrm{B}_3(\mathcal{C},\otimes)$. Then, by Theorem 2.3, there is a homotopy equivalence $\mathrm{B}\mathcal{C}\simeq\Omega\mathrm{B}_2(\mathcal{C},\otimes)$. By the already proved in (i), there is a homotopy equivalence $\Omega\mathrm{B}_2(\mathcal{C},\otimes)\simeq\Omega^2\mathrm{B}_3(\mathcal{C},\otimes,\boldsymbol{c})$, whence the result. □

2.5 The geometric nerve of a tricategory

With the notion of the classifying space of a tricategory $\mathcal{T}$ given above, the resulting CW-complex $\mathrm{B}_3\mathcal{T}$ thus obtained has many cells with little apparent intuitive connection with the cells of the original tricategory, and they do not enjoy any proper geometric meaning. This leads one to search for any simplicial set realizing the space $\mathrm{B}_3\mathcal{T}$ and whose cells give a logical geometric meaning to the data of the tricategory. With the definition below, we give a natural candidate for such a simplicial set, which, up to minor changes affecting the direction conventions on 2- and 3-cells, is essentially due to Street [116, 118].

For any given tricategory $\mathcal{T}$, the construction $\mathcal{I} \mapsto \mathrm{Lax_u}(\mathcal{I}, \mathcal{T})$ given in (2.2), which carries each category $\mathcal{I}$ to the set of unitary lax functors from $\mathcal{I}$ into $\mathcal{T}$, is clearly functorial on the small category $\mathcal{I}$, whence we have the following simplicial set.

Definition 2.4 *The* geometric nerve *of a tricategory $\mathcal{T}$ is the simplicial set*

$$\Delta\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Set}, \quad [p] \mapsto \mathrm{Lax_u}([p], \mathcal{T}).$$

The simplicial set $\Delta\mathcal{T}$ encodes the entire tricategorical structure of $\mathcal{T}$ and, as we will prove below, represents the classifying space of the tricategory $\mathcal{T}$, up to homotopy. We shall stress here that the simplices of the geometric nerve $\Delta\mathcal{T}$ have the following pleasing geometric description, where we have taken into account the coherence theorem for tricategories in order to interpret correctly the pasting diagrams (i.e., by thinking of $\mathcal{T}$ as a Gray-category). The vertices of $\Delta\mathcal{T}$ are points labelled with the objects $F0$ of $\mathcal{T}$. The 1-simplices are paths labelled with the 1-cells

$$F_{0,1}\colon F0 \to F1. \tag{2.26}$$

The 2-simplices are oriented triangles

$$\begin{tikzcd}
& F0 \arrow[dl, "F_{0,1}"'] \arrow[dr, "F_{0,2}"] & \\
F1 \arrow[rr, "F_{1,2}"'] \arrow[rr, phantom, bend left=20, "\overset{F_{0,1,2}}{\Rightarrow}"] & & F2,
\end{tikzcd} \tag{2.27}$$

with objects Fi placed on the vertices, 1-cells $F_{i,j}\colon Fi \to Fj$ on the edges, and labelling the interior as a 2-cell $F_{0,1,2}\colon F_{1,2} \otimes F_{0,1} \Rightarrow F_{0,2}$. For $p \geq 3$, a p-simplex of $\Delta\mathcal{T}$ is geometrically represented by a diagram in $\mathcal{T}$ with the shape of the 3-skeleton of an oriented standard p-simplex whose 3-faces are oriented tetrahedrons

$$\begin{tikzcd}
& Fi \arrow[dl] \arrow[dr] \arrow[dd] & \\
Fj \arrow[rr, dotted] \arrow[dr] & & Fl, \\
& Fk \arrow[ur] &
\end{tikzcd} \tag{2.28}$$

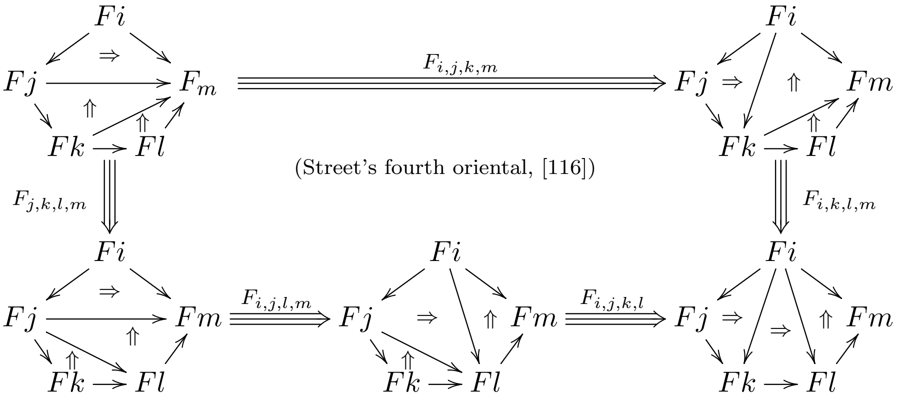
one for each $0 \leq i < j < k < l \leq p$, whose faces

$$\begin{tikzcd}[column sep=small]
& Fj \arrow[dl, "F_{j,k}"'] \arrow[dr, "F_{j,l}"] & & & Fi \arrow[dl, "F_{i,k}"'] \arrow[dr, "F_{i,l}"] & & & Fi \arrow[dl, "F_{i,j}"'] \arrow[dr, "F_{i,l}"] & & & F_i \arrow[dl, "F_{i,j}"'] \arrow[dr, "F_{i,k}"] & \\
Fk \arrow[rr, "F_{k,l}"'] \arrow[rr, phantom, bend left=20, "\overset{F_{j,k,l}}{\Rightarrow}"] & & Fl, & Fk \arrow[rr, "F_{k,l}"'] \arrow[rr, phantom, bend left=20, "\overset{F_{i,k,l}}{\Rightarrow}"] & & Fl, & Fj \arrow[rr, "F_{j,l}"'] \arrow[rr, phantom, bend left=20, "\overset{F_{i,j,l}}{\Rightarrow}"] & & Fl, & Fj \arrow[rr, "F_{j,k}"'] \arrow[rr, phantom, bend left=20, "\overset{F_{i,j,k}}{\Rightarrow}"] & & Fk,
\end{tikzcd} \tag{2.29}$$

are geometric 2-simplices as above, and

$$\begin{tikzcd}
& Fi \arrow[dl] \arrow[dr] & & & & Fi \arrow[dl] \arrow[dr] \arrow[dd] & \\
Fj \arrow[rr] \arrow[dr] \arrow[rr, phantom, bend left=25, "\Rightarrow"] \arrow[rr, phantom, bend right=25, "\Uparrow"] & & Fl & \overset{F_{i,j,k,l}}{\Rrightarrow} & Fj \arrow[dr] \arrow[r, phantom, "\Rightarrow"] & {} \arrow[r, phantom, "\Uparrow"] & Fl \\
& Fk \arrow[ur] & & & & Fk \arrow[ur] &
\end{tikzcd} \tag{2.30}$$

is a 3-cell of the tricategory that labels the interior of the tetrahedron. For $p \geq 4$, these data are required to satisfy the coherence condition (**CR1**), as stated in the Appendix, §2.6.1; that is, for each $0 \leq i < j < k < l < m \leq p$, the following diagram commutes:



The simplicial set $\Delta\mathcal{T}$ is coskeletal in dimensions greater than 4. More precisely, for $p \geq 4$, a p-simplex $F\colon \Delta[p] \to \mathcal{T}$ of $\Delta\mathcal{T}$ is determined uniquely by its boundary $\partial F = (Fd^0, \ldots, Fd^p)$

$$\begin{tikzcd} \partial\Delta[p] \arrow[r, "\partial F"] \arrow[d, hook] & \Delta\mathcal{T}, \\ \Delta[p] \arrow[ur, "F"'] & \end{tikzcd}$$

and, for $p \geq 5$, every possible boundary of a p-simplex, $\partial\Delta[p] \to \Delta\mathcal{T}$, is actually the boundary ∂F of a geometric p-simplex F of the tricategory $\mathcal{T}$.

Example 2.7 (Geometric nerves of bicategories) When a bicategory $\mathcal{B}$ is regarded as a tricategory, all of whose 3-cells are identities, then the simplicial set $\Delta\mathcal{B}$ is precisely the unitary *geometric nerve of the bicategory*, as it is called by Carrasco, Cegarra and Garzón in [41] (but denoted by $\Delta^{\mathrm{u}}\mathcal{B}$). The construction of the geometric nerve for a bicategory was first given in the late eighties by J. Duskin and R. Street (see [117, pag. 573]). In [58], Duskin gave a characterization of the unitary geometric nerve of a bicategory $\mathcal{B}$ in terms of its simplicial structure. The result states that a simplicial set is isomorphic to the geometric nerve of a bicategory if and only if it satisfies the coskeletal conditions above as well as supporting appropriate sets of 'abstractly invertible' 1- and 2-simplices (see Gurski [74], for an interesting new approach to this subject). In [41, Theorem 6.1], the following fact is proved.

Fact 2.11 *For any bicategory $\mathcal{B}$, there is a homotopy equivalence* $\mathrm{B}_2\mathcal{B} \simeq |\Delta\mathcal{B}|$.

We now state a main result of this chapter.

Theorem 2.4 *For any tricategory $\mathcal{T}$, there is a homotopy equivalence* $\mathrm{B}_3\mathcal{T} \simeq |\Delta\mathcal{T}|$.

Proof: Let us consider, for any given tricategory $\mathcal{T}$, the simplicial bicategory

$$\underline{\Delta}\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Hom} \subset \mathbf{Bicat}, \quad [q] \mapsto \mathbb{L}\mathrm{ax}_{\mathrm{u}}([q],\mathcal{T}),$$

whose bicategories of q-simplices are the bicategories of unitary lax functors (2.5) of $[q]$ into $\mathcal{T}$. In this simplicial bicategory, the homomorphism induced by any map $a\colon [q] \to [p]$, $a^*\colon \underline{\Delta}\mathcal{T}_p \to \underline{\Delta}\mathcal{T}_q$, is actually a strict functor. Hence, the bisimplicial set

$$\Delta\underline{\Delta}\mathcal{T}\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}, \quad ([p],[q]) \mapsto \Delta(\underline{\Delta}\mathcal{T}_q)_p = \mathrm{Lax}_{\mathrm{u}}([p],\mathbb{L}\mathrm{ax}_{\mathrm{u}}([q],\mathcal{T})),$$

is well defined, since the geometric nerve construction Δ is functorial on unitary homomorphisms between bicategories. The plan is to prove the existence of homotopy equivalences

$$\mathrm{B}_3\mathcal{T} \simeq |\mathrm{diag}\Delta\underline{\Delta}\mathcal{T}|, \tag{2.31}$$

$$|\Delta\mathcal{T}| \simeq |\mathrm{diag}\Delta\underline{\Delta}\mathcal{T}|, \tag{2.32}$$

whence the theorem follows.

• *The homotopy equivalence* (2.31): The Segal nerve of the tricategory (2.20) is a simplicial sub-bicategory of $\underline{\Delta}\mathcal{T}$. Let $L\colon \mathrm{N}\mathcal{T} \to \underline{\Delta}\mathcal{T}$ be the pseudo-simplicial homomorphism obtained by composing the pseudo simplicial homomorphism (2.21), equally denoted by $L\colon \mathrm{N}\mathcal{T} \to \mathrm{S}\mathcal{T}$, with the simplicial inclusion $\mathrm{S}\mathcal{T} \subseteq \underline{\Delta}\mathcal{T}$. At any degree $p \geq 0$, the homomorphism $L_p\colon \mathrm{N}\mathcal{T}_p \to \underline{\Delta}\mathcal{T}_p$ is precisely the homomorphism in Lemma 2.2, $\mathbb{L}\mathrm{ax}(\mathcal{G}_p,\mathcal{T}) \to \mathbb{L}\mathrm{ax}_{\mathrm{u}}([p],\mathcal{T})$, corresponding with the basic graph of the category $[p]$. Then, we have a homomorphism $R_p\colon \underline{\Delta}\mathcal{T}_p \to \mathrm{N}\mathcal{T}_p$ such that $R_pL_p = 1_{\mathrm{N}\mathcal{T}_p}$, and a lax transformation $\mathrm{v}_p\colon L_pR_p \Rightarrow 1_{\underline{\Delta}\mathcal{T}_p}$. It follows from Fact 2.5 that every induced map $\mathrm{B}_2L_p\colon \mathrm{B}_2(\mathrm{N}\mathcal{T}_p) \to \mathrm{B}_2(\underline{\Delta}\mathcal{T}_p)$ is a homotopy equivalence. Then, by Fact 2.9, the induced map $\mathrm{B}_2\int_\Delta L\colon \mathrm{B}_2\int_\Delta \mathrm{N}\mathcal{T} \to \mathrm{B}_2\int_\Delta \underline{\Delta}\mathcal{T}$ is a homotopy equivalence. Let

$$\mathrm{B}_2\underline{\Delta}\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Top}, \quad [p] \mapsto \mathrm{B}_2(\underline{\Delta}\mathcal{T}_p),$$

be the simplicial space obtained by composing $\underline{\Delta}\mathcal{T}$ with the classifying space functor $\mathrm{B}_2\colon \mathbf{Hom} \to \mathbf{Top}$ (see Fact 2.4). Since, by definition, $\mathrm{B}_3\mathcal{T} = \mathrm{B}_2\int_\Delta \mathrm{N}\mathcal{T}$, whereas, by Fact 2.10, there is a homotopy equivalence $\mathrm{B}_2\int_\Delta \underline{\Delta}\mathcal{T} \simeq |\mathrm{B}_2\underline{\Delta}\mathcal{T}|$, we have a homotopy equivalence $\mathrm{B}_3\mathcal{T} \simeq |\mathrm{B}_2\underline{\Delta}\mathcal{T}|$. Furthermore, by Fact 2.11, we have a homotopy equivalence

$$|\mathrm{B}_2\underline{\Delta}\mathcal{T}| = |[q] \mapsto \mathrm{B}_2(\underline{\Delta}\mathcal{T}_q)| \simeq |[q] \mapsto |\Delta(\underline{\Delta}\mathcal{T}_q)|| \cong |\mathrm{diag}\Delta\underline{\Delta}\mathcal{T}|,$$

where for the last homeomorphism we refer to Quillen's Lemma in [109, page 86]. Thus, $\mathrm{B}\mathcal{T} \simeq |\mathrm{diag}\Delta\underline{\Delta}\mathcal{T}|$, as claimed.

• *The homotopy equivalence* (2.32): Note that the geometric nerve $\Delta\mathcal{T}$ is the simplicial set of objects of the simplicial bicategory $\underline{\Delta}\mathcal{T}$, that is, $\Delta\mathcal{T} = \Delta(\underline{\Delta}\mathcal{T})_0$. Therefore, if we regard $\Delta\mathcal{T}$ as a simplicial discrete bicategory (i.e., all 1-cells and 2-cells are identities), then $\Delta\Delta\mathcal{T}$ becomes a bisimplicial set that is constant in the horizontal direction, and there is a natural bisimplicial map $\Delta\Delta\mathcal{T} \hookrightarrow \Delta\underline{\Delta}\mathcal{T}$, which is, at each horizontal level $p \geq 1$, the composite simplicial map

$$\Delta\mathcal{T} = \Delta(\underline{\Delta}\mathcal{T})_0 \overset{s_0^{\mathrm{h}}}{\hookrightarrow} \Delta(\underline{\Delta}\mathcal{T})_1 \hookrightarrow \cdots \hookrightarrow \Delta(\underline{\Delta}\mathcal{T})_{p-1} \overset{s_{p-1}^{\mathrm{h}}}{\hookrightarrow} \Delta(\underline{\Delta}\mathcal{T})_p. \tag{2.33}$$

Next, we prove that the simplicial map $\Delta\mathcal{T} \to \mathrm{diag}\Delta\underline{\Delta}\mathcal{T}$, induced on diagonals, is a weak homotopy equivalence, whence the announced homotopy equivalence in (2.32). It suffices to prove that every one of the simplicial maps in (2.33) is a weak homotopy equivalence and, in fact, we will prove more: *Every simplicial map s_{p-1}^{h}, $p \geq 1$, embeds the simplicial set $\Delta(\underline{\Delta}\mathcal{T})_{p-1}$ into $\Delta(\underline{\Delta}\mathcal{T})_p$ as a simplicial deformation retract.* Since $d_p^{\mathrm{h}} s_{p-1}^{\mathrm{h}} = 1$, it is enough to exhibit a simplicial homotopy h: $1 \Rightarrow s_{p-1}^{\mathrm{h}} d_p^{\mathrm{h}}$: $\Delta(\underline{\Delta}\mathcal{T})_p \to \Delta(\underline{\Delta}\mathcal{T})_p$.

To do so, we shall use the following notation for the bisimplices in $\Delta\underline{\Delta}\mathcal{T}$. Since such a bisimplex of bidegree (p, q), say $F \in \Delta(\underline{\Delta}\mathcal{T}_q)_p$, is a unitary lax functor of the category $[p]$ in the bicategory of unitary lax functors $\mathbb{L}\mathrm{ax}_{\mathrm{u}}([q], \mathcal{T})$, it consists of

- unitary lax functors F^u: $[q] \to \mathcal{T}$ for $0 \leq u \leq p$,
- 1-cells $F^{u,v}$: $F^u \Rightarrow F^v$ for $0 \leq u < v \leq p$,
- 2-cells $F^{u,v,w}$: $F^{v,w} \circ F^{u,v} \Rrightarrow F^{u,w}$ for $0 \leq u < v < w \leq p$,

such that the diagrams

$$\begin{tikzcd}
(F^{w,t} \circ F^{v,w}) \circ F^{u,v} \arrow[rr, Rightarrow, "a"] \arrow[d, Rightarrow, "F^{v,w,t} \circ 1"'] & & F^{w,t} \circ (F^{v,w} \circ F^{u,v}) \arrow[d, Rightarrow, "1 \circ F^{u,v,w}"] \\
F^{v,t} \circ F^{u,v} \arrow[r, Rightarrow, "F^{u,v,t}"] & F^{u,t} & F^{w,t} \circ F^{u,w} \arrow[l, Rightarrow, "F^{u,w,t}"']
\end{tikzcd}$$

commute for $u < v < w < t$. Hence, such a (p, q)-simplex is described by a list of cells of the tricategory $\mathcal{T}$

$$F = \big(Fi, F_{i,j}^u, F_{i,j,k}^u, F_{i,j,k,l}^u, F_{i,j}^{u,v}, F_{i,j,k}^{u,v}, F_{i,j}^{u,v,w}\big), \tag{2.34}$$

with $0 \leq i < j < k < l \leq q$, where

- Fi $(= F^0 i)$ is an object of $\mathcal{T}$,
- $F_{i,j}^u$: $Fi \to Fj$ are 1-cells in $\mathcal{T}$,
- $F_{i,j,k}^u$: $F_{j,k}^u \otimes F_{i,j}^u \Rightarrow F_{i,k}^u$ are 2-cells in $\mathcal{T}$,
- $F_{i,j}^{u,v}$: $F_{i,j}^u \Rightarrow F_{i,j}^v$ are 2-cells in $\mathcal{T}$,

- $F^u_{i,j,k,l}$, $F^{u,v}_{i,j,k}$ and $F^{u,v,w}_{i,j}$ are 3-cells in $\mathcal{T}$ of the form

$$\begin{tikzcd}
(F^u_{k,l}\otimes F^u_{j,k})\otimes F^u_{i,j} \arrow[rr, Rightarrow, "\boldsymbol{a}"] \arrow[d, Rightarrow, "F^u_{j,k,l}\otimes 1"'] & {} \arrow[d, phantom, "\overset{F^u_{i,j,k,l}}{\Rrightarrow}"] & F^u_{k,l}\otimes(F^u_{j,k}\otimes F^u_{i,j}) \arrow[d, Rightarrow, "1\otimes F^u_{i,j,k}"] \\
F^u_{j,l}\otimes F^u_{i,j} \arrow[r, Rightarrow, "F^u_{i,j,l}"'] & F^u_{i,l} & F^u_{k,l}\otimes F^u_{i,k}, \arrow[l, Rightarrow, "F^u_{i,k,l}"]
\end{tikzcd}$$

$$\begin{tikzcd}
F^u_{j,k}\otimes F^u_{i,j} \arrow[r, Rightarrow, "F^u_{i,j,k}"] \arrow[d, Rightarrow, "F^{u,v}_{j,k}\otimes F^{u,v}_{i,j}"'] \arrow[dr, phantom, "\overset{F^{u,v}_{i,j,k}}{\Rrightarrow}"] & F^u_{i,k} \arrow[d, Rightarrow, "F^{u,v}_{i,k}"] \\
F^v_{j,k}\otimes F^v_{i,j} \arrow[r, Rightarrow, "F^v_{i,j,k}"'] & F^v_{i,k},
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F^u_{i,j} \arrow[dl, Rightarrow, "F^{u,v}_{i,j}"'] \arrow[dr, Rightarrow, "F^{u,w}_{i,j}"] \arrow[d, phantom, "\overset{F^{u,v,w}_{i,j}}{\Rrightarrow}"] & \\
F^v_{i,j} \arrow[rr, Rightarrow, "F^{v,w}_{i,j}"'] & {} & F^w_{i,j},
\end{tikzcd}$$

satisfying the various conditions.

The horizontal faces and degeneracies of such a bisimplex (2.34) are given by the simple rules $d^{\mathrm{h}}_m F = (Fi, F^{d^m u}_{i,j}, \dots)$ and $s^{\mathrm{h}}_m F = (Fi, F^{s^m u}_{i,j}, \dots)$, whereas the vertical ones are given by $d^{\mathrm{v}}_m F = (Fd^m i, F^u_{d^m i, d^m j}, \dots)$ and $s^{\mathrm{v}}_m F = (Fs^m i, F^u_{s^m i, s^m j}, \dots)$.

We have the following simplicial homotopy h: $1 \Rightarrow s^{\mathrm{h}}_{p-1} d^{\mathrm{h}}_p$: $\Delta(\underline{\Delta}\mathcal{T})_p \to \Delta(\underline{\Delta}\mathcal{T})_p$. For each $0 \leq m \leq q$, the map h_m: $\Delta(\underline{\Delta}\mathcal{T}_q)_p \to \Delta(\underline{\Delta}\mathcal{T}_{q+1})_p$ takes a (p,q)-simplex F as in (2.34) of $\Delta\underline{\Delta}\mathcal{T}$ to the $(p, q+1)$-simplex $h_m F$ consisting of:

◇ The lax functors $h_m F^u = (s^m)^* F^u$: $[q+1] \to \mathcal{T}$ for $0 \leq u < p$.

◇ The lax functors $h_m F^p$: $[q+1] \to \mathcal{T}$, with

- $(h_m F)^p i = F s^m i$ for $0 \leq i \leq q+1$,
- $(h_m F)^p_{i,j} = \begin{cases} F^{p-1}_{s^m i, s^m j} & \text{if } j \leq m, \\ F^p_{s^m i, j-1} & \text{if } m < j, \end{cases}$
- $(h_m F)^p_{i,j,k} = \begin{cases} F^{p-1}_{s^m i, s^m j, s^m k} & \text{if } k \leq m, \\ F^p_{j,k-1}\otimes F^{p-1}_{i,j} \xRightarrow{1\otimes F^{p-1,p}_{i,j}} F^p_{j,k-1}\otimes F^p_{i,j} \xRightarrow{F^p_{i,j,k-1}} F^p_{i,k-1} & \text{if } j \leq m < k, \\ F^p_{s^m i, j-1, k-1} & \text{if } m < j, \end{cases}$
- $(h_m F)^p_{i,j,k,l} = \begin{cases} F^{p-1}_{s^m i, s^m j, s^m k, s^m l} & \text{if } l \leq m, \\ \text{the 3-cell given by the pasting diagram (2.35)} & \text{if } k \leq m < l, \\ \text{the 3-cell given by the pasting diagram (2.36)} & \text{if } j \leq m < k, \\ F^p_{s^m i, j-1, k-1, l-1} & \text{if } m < j. \end{cases}$

$$\begin{tikzcd}
 & (F^p_{k,l-1}\otimes F^{p-1}_{j,k})\otimes F^{p-1}_{i,j} \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[dl, Rightarrow, "(1\otimes F^{p-1,p}_{j,k})\otimes 1"'] \arrow[dd, Rightarrow, "(1\otimes F^{p-1,p}_{j,k})\otimes F^{p-1,p}_{i,j}"] \arrow[ddr, phantom, "\cong"] & F^p_{k,l}\otimes(F^{p-1}_{j,k}\otimes F^{p-1}_{i,j}) \arrow[dr, Rightarrow, "1\otimes F^{p-1}_{i,j,k}"] \arrow[dd, Rightarrow, "1\otimes(F^{p-1,p}_{j,k}\otimes F^{p-1,p}_{i,j})"'] \arrow[dddr, phantom, "\overset{1\otimes F^{p-1,p}_{i,j,k}}{\Rrightarrow}"] & \\
(F^p_{k,l-1}\otimes F^p_{j,k})\otimes F^{p-1}_{i,j} \arrow[dd, Rightarrow, "F^p_{j,k,l-1}\otimes 1"'] \arrow[ddr, phantom, "\cong"] & & & F^p_{k,l-1}\otimes F^{p-1}_{i,k} \arrow[dd, Rightarrow, "1\otimes F^{p-1,p}_{i,k}"] \\
 & (F^p_{k,l-1}\otimes F^p_{j,k})\otimes F^p_{i,j} \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[dd, Rightarrow, "F^p_{j,k,l-1}\otimes 1"'] \arrow[ddr, phantom, "\overset{F^p_{i,j,k,l-1}}{\Rrightarrow}"] & F^p_{k,l-1}\otimes(F^p_{j,k}\otimes F^p_{i,j}) \arrow[dr, Rightarrow, "1\otimes F^p_{i,j,k}"] & \\
F^p_{j,l-1}\otimes F^p_{i,j} \arrow[dr, Rightarrow, "1\otimes F^{p-1,p}_{i,j}"'] & & & F^p_{k,l-1}\otimes F^p_{i,k} \arrow[dl, Rightarrow, "F^p_{i,k,l-1}"] \\
 & F^p_{j,l-1}\otimes F^p_{i,j} \arrow[r, Rightarrow, "F^p_{i,j,l-1}"] & F^p_{i,l-1} &
\end{tikzcd} \tag{2.35}$$

$$\begin{tikzcd}
 & (F^p_{k-1,l-1}\otimes F^p_{j,k-1})\otimes F^{p-1}_{i,j} \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[dl, Rightarrow, "F^p_{j,k-1,l-1}\otimes 1"'] \arrow[d, Rightarrow, "(1\otimes 1)\otimes F^{p-1,p}_{i,j}"] \arrow[dr, phantom, "\cong"] & F^p_{k-1,l-1}\otimes(F^p_{j,k-1}\otimes F^{p-1}_{i,j}) \arrow[d, Rightarrow, "1\otimes(1\otimes F^{p-1,p}_{i,j})"] \\
F^p_{j,l-1}\otimes F^{p-1}_{i,j} \arrow[r, phantom, "\cong"] \arrow[d, Rightarrow, "1\otimes F^{p-1,p}_{i,j}"'] & (F^p_{k-1,l-1}\otimes F^p_{j,k-1})\otimes F^p_{i,j} \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[dl, Rightarrow, "F^p_{j,k-1,l-1}\otimes 1"'] \arrow[dr, phantom, "\overset{F^p_{i,j,k-1,l-1}}{\Rrightarrow}"] & F^p_{k-1,l-1}\otimes(F^p_{j,k-1}\otimes F^p_{i,j}) \arrow[d, Rightarrow, "1\otimes F^p_{i,j,k-1}"] \\
F^p_{j,l-1}\otimes F^p_{i,j} \arrow[r, Rightarrow, "F^p_{i,j,l-1}"] & F^p_{i,l-1} & F^p_{k-1,l-1}\otimes F^p_{i,k-1} \arrow[l, Rightarrow, "F^p_{i,k-1,l-1}"']
\end{tikzcd} \tag{2.36}$$

◇ The 1-cells $h_mF^{u,v} = (s^m)^*F^{u,v}$: $h_mF^u \Rightarrow h_mF^v$ for $0 \leq u < v < p$.

◇ The 1-cells $h_mF^{u,p}$: $h_mF^u \Rightarrow h_mF^p$ with

$$\bullet\, (h_mF)^{u,p}_{i,j} = \begin{cases} F^{u,p-1}_{s^mi,s^mj} & \text{if } j \leq m, \\ F^{u,p}_{s^mi,j-1} & \text{if } j > m, \end{cases}$$

$$\bullet\, (h_mF)^{u,p}_{i,j,k} = \begin{cases} F^{u,p-1}_{s^mi,s^mj,s^mk} & \text{if } k \leq m, \\ \text{the 3-cell given by the pasting diagram (2.37)} & \text{if } j \leq m < k, \\ F^{u,p}_{s^mi,j-1,k-1} & \text{if } m < j, \end{cases}$$

$$\begin{tikzcd}
 & F^u_{j,k-1}\otimes F^u_{i,j} \arrow[r, Rightarrow, "F^u_{i,j,k-1}"] \arrow[dl, Rightarrow, "F^{u,p}_{j,k-1}\otimes F^{u,p-1}_{i,j}"'] \arrow[dd, Rightarrow, "F^{u,p}_{j,k-1}\otimes F^{u,p}_{i,j}" description] \arrow[ddr, phantom, "\overset{F^{u,p}_{i,j,k-1}}{\Rrightarrow}"] & F^u_{i,k-1} \arrow[dd, Rightarrow, "F^{u,p}_{i,k-1}"] \\
F^p_{j,k-1}\otimes F^{p-1}_{i,j} \arrow[dr, Rightarrow, "1\otimes F^{p-1,p}_{i,j}"'] \arrow[r, phantom, "\overset{F^{u,p-1,p}_{i,j}}{\Rrightarrow}"] & & \\
 & F^p_{j,k-1}\otimes F^p_{i,j} \arrow[r, Rightarrow, "F^p_{i,j,k-1}"] & F^p_{i,k-1},
\end{tikzcd} \tag{2.37}$$

◇ The 2-cells $(h_mF)^{u,v,w} = (s^m)^*F^{u,v,w}$ for $w < p$.

◇ The 2-cells $(h_mF)^{u,v,p}$ are given by $(h_mF)^{u,v,p}_{i,j} = \begin{cases} F^{u,v,p-1}_{s^m i, s^m j} & \text{if } j \leq m, \\ F^{u,v,p}_{s^m i, j-1} & \text{if } m < j. \end{cases}$

So defined, to show that $h\colon 1 \Rightarrow s^{\mathrm{h}}_{p-1} d^{\mathrm{h}}_p$ is actually a simplicial homotopy, is a straightforward (though quite tedious) verification. □

As an application of the theorem above, we shall prove that tritransformations produce homotopies. To do so, the following lemma is the key.

Lemma 2.3 *Suppose* $\theta = (\theta, \Pi, \mathrm{M})\colon F \Rightarrow G\colon \mathcal{T} \to \mathcal{T}'$ *is a tritransformation. There is a trihomomorphism* $H\colon \mathcal{T} \times [1] \to \mathcal{T}'$ *making the diagram commutative*

$$\begin{tikzcd}
\mathcal{T}\times[0]\cong\mathcal{T} \arrow[d, "1\times\delta_1"'] \arrow[rrd, "F"] & & \\
\mathcal{T}\times[1] \arrow[rr, "H"] & & \mathcal{T}'. \\
\mathcal{T}\times[0]\cong\mathcal{T} \arrow[u, "1\times\delta_0"] \arrow[rru, "G"'] & &
\end{tikzcd} \qquad (2.38)$$

Proof: For any objects x, y of $\mathcal{T}$, $H\colon (\mathcal{T}\times[1])((x,0),(y,1)) \to \mathcal{T}'(Fx, Gy)$ is the homomorphism composite of

$$\mathcal{T}(x,y)\times\{(0,1)\} \cong \mathcal{T}(x,y) \overset{G}{\to} \mathcal{T}'(Gx,Gy) \xrightarrow{\mathcal{T}'(\theta x,1)} \mathcal{T}'(Fx,Gy).$$

For objects x, y, z of $\mathcal{T}$, the pseudo-equivalence

$$\begin{tikzcd}
(\mathcal{T}\times[1])((y,1),(z,1)) \times (\mathcal{T}\times[1])((x,0),(y,1)) \arrow[r, "H\times H"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow\chi^H"] & \mathcal{T}'(Gy,Gz) \times \mathcal{T}'(Fx,Gy) \arrow[d, "\otimes"] \\
(\mathcal{T}\times[1])((x,0),(z,1)) \arrow[r, "H"] & \mathcal{T}'(Fx,Gz)
\end{tikzcd}$$

is obtained by pasting the diagram

$$\begin{tikzcd}
\mathcal{T}(q,r)\times\mathcal{T}(x,y) \arrow[r, "G\times G"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow\chi^G"] & \mathcal{T}'(Gy,Gz)\times\mathcal{T}'(Gx,Gy) \arrow[r, "1\times\mathcal{T}'(\theta x,1)"] \arrow[d, "\otimes"] \arrow[dr, phantom, "\cong"] & \mathcal{T}'(Gy,Gz)\times\mathcal{T}'(Fx,Gy) \arrow[d, "\otimes"] \\
\mathcal{T}(x,z) \arrow[r, "G"] & \mathcal{T}'(Gx,Gz) \arrow[r, "\mathcal{T}'(\theta x,1)"] & \mathcal{T}'(Fx,Gz),
\end{tikzcd}$$

and the pseudo-equivalence

$$\begin{tikzcd}
(\mathcal{T}\times[1])((y,0),(z,1)) \times (\mathcal{T}\times[1])((x,0),(y,0)) \arrow[r, "H\times H"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow\chi^H"] & \mathcal{T}'(Fy,Gz) \times \mathcal{T}'(Fx,Fy) \arrow[d, "\otimes"] \\
(\mathcal{T}\times[1])((x,0),(z,1)) \arrow[r, "H"] & \mathcal{T}'(Fx,Gz)
\end{tikzcd}$$

by pasting in

$$\begin{tikzcd}
\mathcal{T}(y,z)\times\mathcal{T}(x,y) \arrow[rr, "G\times F"] \arrow[drr, "F\times F"'] \arrow[ddd, "\otimes"'] & & \mathcal{T}'(Gy,Gz)\times\mathcal{T}'(Fx,Fy) \arrow[rr, "\mathcal{T}'(\theta y{,}1)\times 1"] \arrow[d, phantom, "\Downarrow\theta^{\bullet}\times 1"] & & \mathcal{T}'(Fy,Gz)\times\mathcal{T}'(Fx,Fy) \arrow[ddd, "\otimes"] \\
 & \Downarrow\chi^F & \mathcal{T}'(Fy,Fz)\times\mathcal{T}'(Fx,Fy) \arrow[urr, "\mathcal{T}'(1{,}\theta z)\times 1"'] \arrow[d, "\otimes"] & \cong & \\
 & & \mathcal{T}'(Fx,Fz) \arrow[drr, "\mathcal{T}'(1{,}\theta z)"] \arrow[d, phantom, "\Downarrow\theta"] & & \\
\mathcal{T}(x,z) \arrow[urr, "F"] \arrow[rr, "G"] & & \mathcal{T}'(Gx,Gz) \arrow[rr, "\mathcal{T}'(\theta x{,}1)"] & & \mathcal{T}'(Fx,Gz).
\end{tikzcd}$$

For x,y,z,t any objects of $\mathcal{T}$, the components of the invertible modification ω^H at the triples of composable 1-cells of $\mathcal{T}\times[1]$

$$(x,0)\xrightarrow{(a,(0,1))}(y,1)\xrightarrow{(b,1_1)}(z,1)\xrightarrow{(c,1_1)}(t,1),$$
$$(x,0)\xrightarrow{(a,1_0))}(y,0)\xrightarrow{(b,(0,1))}(z,1)\xrightarrow{(c,1_1)}(t,1),$$
$$(x,0)\xrightarrow{(a,1_0))}(y,0)\xrightarrow{(b,1_0))}(z,0)\xrightarrow{(c,(0,1))}(t,1),$$

are canonically provided by the 3-cells (2.39), (2.40) and (2.41) below.

$$\begin{tikzcd}
((Gc\otimes Gb)\otimes Ga)\otimes\theta x \arrow[rr, Rightarrow, "(\chi^G\otimes 1)\otimes 1"] \arrow[dd, Rightarrow, "\boldsymbol{a}\otimes 1"'] & & (G(c\otimes b)\otimes Ga)\otimes\theta x \arrow[d, Rightarrow, "\chi^G\otimes 1"] \\
 & \overset{\omega^G\otimes 1}{\Leftarrow} & G((c\otimes b)\otimes a)\otimes\theta x \arrow[d, Rightarrow, "G\boldsymbol{a}\otimes 1"] \\
(Gc\otimes(Gb\otimes Ga))\otimes\theta x \arrow[r, Rightarrow, "(1\otimes\chi^G)\otimes 1"] & (Gc\otimes(G(b\otimes a))\otimes\theta x \arrow[r, Rightarrow, "\chi^G\otimes 1"] & G(c\otimes(b\otimes a))\otimes\theta x.
\end{tikzcd} \tag{2.39}$$

$$\begin{tikzcd}
(Gc\otimes Gb)\otimes(\theta y\otimes Fa) \arrow[r, Rightarrow, "\chi^G\otimes 1"] \arrow[d, Rightarrow, "1\otimes\theta"'] \arrow[dr, phantom, "\cong"] & G(c\otimes b)\otimes(\theta y\otimes Fa) \arrow[d, Rightarrow, "1\otimes\theta"] \\
(Gc\otimes Gb)\otimes(Ga\otimes\theta x) \arrow[r, Rightarrow, "\chi^G\otimes 1"] & G(c\otimes b)\otimes(Ga\otimes\theta x)
\end{tikzcd} \tag{2.40}$$

$$\begin{tikzcd}
Gc\otimes((\theta z\otimes Fb)\otimes Fa) \arrow[r, Rightarrow, "1\otimes(\theta\otimes 1)"] \arrow[d, Rightarrow, "1\otimes\boldsymbol{a}"'] & Gc\otimes((Gb\otimes\theta y)\otimes Fa) \arrow[r, Rightarrow, "1\otimes\boldsymbol{a}"] & Gc\otimes(Gb\otimes(\theta y\otimes Fa)) \arrow[d, Rightarrow, "1\otimes(1\otimes\theta)"] \\
Gc\otimes(\theta z\otimes(Fb\otimes Fa)) \arrow[d, Rightarrow, "1\otimes(1\otimes\chi^F)"'] & \overset{1\otimes\Pi}{\Leftarrow} & Gc\otimes(Gb\otimes(Ga\otimes\theta x)) \arrow[d, Rightarrow, "1\otimes\boldsymbol{a}^{\bullet}"] \\
Gc\otimes(\theta z\otimes F(b\otimes a)) \arrow[r, Rightarrow, "1\otimes\theta"] & Gc\otimes(G(b\otimes c)\otimes\theta x) & Gc\otimes((Gb\otimes Ga)\otimes\theta x) \arrow[l, Rightarrow, "1\otimes(\chi^G\otimes 1)"']
\end{tikzcd} \tag{2.41}$$

To finish the description of the homomorphism H, say that the component of the invertible modification δ^H at any morphism $(a,(0,1))\colon (x,0)\to(y,1)$ is canonically obtained from the 3-cells $1 \otimes \mathrm{M}$ and $\delta^G \otimes 1$ below, while the component of γ^H is provided by 3-cell $\gamma^G \otimes 1$.

$$\begin{tikzcd}
Ga\otimes\theta x \arrow[r, Rightarrow, "1\otimes\boldsymbol{r}"] \arrow[d, Rightarrow, "1\otimes\boldsymbol{l}^\bullet"'] & Ga\otimes(\theta x\otimes 1) \arrow[r, Rightarrow, "1\otimes(1\otimes\iota^F)"] \arrow[d, phantom, "\overset{1\otimes\mathrm{M}}{\Leftarrow}"] & Ga\otimes(\theta x\otimes F1) \arrow[d, Rightarrow, "1\otimes\theta"] \\
Ga\otimes(1\otimes\theta x) \arrow[rr, Rightarrow, "1\otimes(\iota^G\otimes 1)"] & & Ga\otimes(G1\otimes\theta y),
\end{tikzcd}$$

$$\begin{tikzcd}
Ga\otimes\theta x \arrow[r, Rightarrow, "\boldsymbol{r}\otimes 1"] \arrow[d, Rightarrow, "G\boldsymbol{r}\otimes 1"'] \arrow[dr, phantom, "\overset{\delta^G\otimes 1}{\Rrightarrow}"] & (Gx\otimes 1)\otimes\theta_p \arrow[d, Rightarrow, "(1\otimes\iota^G)\otimes 1"] & (1\otimes Ga)\otimes\theta x \arrow[r, Rightarrow, "(\iota^G\otimes 1)\otimes 1"] \arrow[d, Rightarrow, "\boldsymbol{l}\otimes 1"'] \arrow[dr, phantom, "\overset{\gamma^G\otimes 1}{\Lleftarrow}"] & (G1\otimes Ga)\otimes\theta x \arrow[d, Rightarrow, "\chi^G\otimes 1"] \\
G(a\otimes 1)\otimes\theta x & (Ga\otimes G1)\otimes\theta x, \arrow[l, Rightarrow, "\chi^G\otimes 1"] & Ga\otimes\theta x & G(1\otimes a)\otimes\theta x. \arrow[l, Rightarrow, "G\boldsymbol{l}\otimes 1"]
\end{tikzcd}$$

□

Proposition 2.4 *If $F, G\colon \mathcal{T} \to \mathcal{T}'$ are two trihomomorphisms between tricategories, then any tritransformation, $F \Rightarrow G$, defines a homotopy between the induced maps on classifying spaces*

$$\mathrm{B}_3F \simeq \mathrm{B}_3G\colon \mathrm{B}_3\mathcal{T} \to \mathrm{B}_3\mathcal{T}'.$$

Proof: Taking geometric nerves in the diagram (2.38), we obtain a commutative diagram of simplicial maps

$$\begin{tikzcd}
\Delta\mathcal{T}\times\Delta[0]\cong\Delta\mathcal{T} \arrow[d, "1\times\delta_1"'] \arrow[dr, "\Delta F"] & \\
\Delta\mathcal{T}\times\Delta[1] \arrow[r, "\Delta H"] & \Delta\mathcal{T}', \\
\Delta\mathcal{T}\times\Delta[0]\cong\Delta\mathcal{T} \arrow[u, "1\times\delta_0"] \arrow[ur, "\Delta G"'] &
\end{tikzcd}$$

Since $|\Delta[1]| \cong [0, 1]$, the unity interval, the result follows by Theorem 2.4. □

As a consequence for triequivalences between tricategories, we have the following.

Corollary 2.3 (i) *If $F\colon \mathcal{T} \to \mathcal{T}'$ is any trihomomorphism such that there is a trihomomorphism $G\colon \mathcal{T}' \to \mathcal{T}$ and tritransformations $FG \Rightarrow 1_{\mathcal{T}'}$ and $1_{\mathcal{T}} \Rightarrow GF$, then the induced map $\mathrm{B}_3F\colon \mathrm{B}_3\mathcal{T} \to \mathrm{B}_3\mathcal{T}'$ is a homotopy equivalence.*

(ii) *Any triequivalence of tricategories induces a homotopy equivalence on classifying spaces.*

Proof: (i) Given any trihomomorphism $F\colon \mathcal{T} \to \mathcal{T}'$ in the hypothesis, by Proposition 2.2 (ii), there is a homotopy $\mathrm{B}_3F\ \mathrm{B}_3G \simeq \mathrm{B}_3(FG)$. By Proposition 2.4, the existence of a homotopy $\mathrm{B}_3(FG) \simeq \mathrm{B}_3 1_{\mathcal{T}'}$ follows. Since, by Proposition 2.2 (iii), there is a homotopy $\mathrm{B}_3 1_{\mathcal{T}'} \simeq 1_{\mathrm{B}_3\mathcal{T}'}$, we conclude the existence of a homotopy $\mathrm{B}_3 F\ \mathrm{B}_3 G \simeq 1_{\mathrm{B}_3\mathcal{T}'}$. Analogously, we can prove that $1_{\mathrm{B}_3\mathcal{T}} \simeq \mathrm{B}_3 G\ \mathrm{B}_3 F$, which completes the proof of this part. Part (*ii*) clearly follows from part (*i*). □

Example 2.8 (Geometric nerves of 3-categories) In [116], Street gave a precise notion of nerve for n-categories. He extended each graph $\mathcal{G}_p = (0 \to 1 \to \cdots \to p)$ to a "free" ω-category $\mathcal{O}_p$ (called the p^{th}-*oriental*) such that, for any n-category $\mathcal{X}$, the p-simplices of its nerve, are just n-functors $\mathcal{O}_p \to \mathcal{X}$, from the underlying n-category of the p^{th}-oriental to $\mathcal{X}$. In the case when $n = 3$, Street's nerve construction on a 3-category $\mathcal{T}$ just produces, up to some directional changes, its geometric nerve $\Delta\mathcal{T}$, as stated in Definition 2.4. After the discussion in Example 2.4, from Theorem 2.4 we get, for any 3-category $\mathcal{T}$, homotopy equivalences

$$|\mathrm{diagNNN}\mathcal{T}| \simeq \mathrm{B}_3\mathcal{T} \simeq |\Delta\mathcal{T}|.$$

Example 2.9 (Geometric nerves of braided monoidal categories) If A is any abelian group, then the braided monoidal category with only one object it defines, $(A, +, 0)$, has as double suspension the tricategory $\Sigma^2 A$, treated in Examples 2.1 and 2.2. For any integer $p \geq 0$, we have

$$\mathrm{Lax}_{\mathrm{u}}([p], \Sigma^2 A) = Z^3([p], A) = Z^3(\Delta[p], A) = K(A,3)_p,$$

whence $\Delta\Sigma^2 A = K(A,3)$, the minimal Eilenberg-Mac Lane complex. Hence, from Theorem 2.4 and Corollary 2.1, it follows that $\mathrm{B}_3(A,+,0) = |K(A,3)|$.

If $(\mathcal{C}, \otimes, \boldsymbol{c})$ is any braided monoidal category, then a unitary lax functor of a category $\mathcal{I}$ in the double suspension tricategory, $\mathcal{I} \to \Sigma^2(\mathcal{C}, \otimes, \boldsymbol{c})$, is what was called by Carrasco, Cegarra and Garzón in [42, Definition 6.6] and by Cegarra and Khmaladze in [48, §4] a (normal) 3-*cocycle* of $\mathcal{I}$ with coefficients in the braided monoidal category. Therefore, the geometric nerve $\Delta\Sigma^2(\mathcal{C}, \otimes, \boldsymbol{c})$ coincides with the simplicial set [42, Definition 6.7]

$$Z^3(\mathcal{C}, \otimes, \boldsymbol{c})\colon \Delta^{\mathrm{op}} \ \to\ \mathbf{Set}, \quad [p] \mapsto Z^3([p], (\mathcal{C}, \otimes, \boldsymbol{c})),$$

whose p-simplices are the 3-cocycles of $[p]$ in the braided monoidal category. The geometric nerve $Z^3(\mathcal{C}, \otimes, \boldsymbol{c})$ is then a 4-coskeletal 1-reduced (one vertex, one 1-simplex) simplicial set, whose 2-simplices are the objects $F_{0,1,2}$ of $\mathcal{C}$, and whose p-simplices, for $p \geq 3$, are families of morphisms of the form

$$F_{i,j,k,l}\colon F_{i,j,l} \otimes F_{j,k,l} \to F_{i,k,l} \otimes F_{i,j,k}, \quad 0 \leq i < j < k < l \leq p,$$

making commutative, for $0 \leq i < j < k < l < m \leq p$, the diagrams

$$\begin{tikzcd}
F_{i,j,m} \otimes (F_{j,k,m} \otimes F_{k,l,m}) \arrow[r, "\boldsymbol{a}(F_{i,j,k,m}\otimes 1)\boldsymbol{a}^{-1}"] \arrow[d, "1\otimes F_{j,k,l,m}"'] & F_{i,k,m} \otimes (F_{i,j,k} \otimes F_{k,l,m}) \arrow[d, "1\otimes \boldsymbol{c}"] \\
F_{i,j,m} \otimes (F_{j,l,m} \otimes F_{j,k,l}) \arrow[d, "(F_{i,j,l,m}\otimes 1)\boldsymbol{a}^{-1}"'] & F_{i,k,m} \otimes (F_{k,l,m} \otimes F_{i,,j,k}) \arrow[d, "(F_{i,k,l,m}\otimes 1)\boldsymbol{a}^{-1}"] \\
(F_{i,l,m} \otimes F_{i,j,l}) \otimes F_{j,k,l} \arrow[r, "\boldsymbol{a}^{-1}(1\otimes F_{i,j,k,l})\boldsymbol{a}"] & (F_{i,l,m} \otimes F_{i,k,l}) \otimes F_{i,j,k}.
\end{tikzcd}$$

From Theorem 2.4 and Corollary 2.1, we obtain the following known result.

Corollary 2.4 ([42, Theorem 6.11]) *For any braided monoidal category* $(\mathcal{C}, \otimes, \boldsymbol{c})$*, there is a homotopy equivalence* $\mathrm{B}_3(\mathcal{C}, \otimes, \boldsymbol{c}) \simeq |Z^3(\mathcal{C}, \otimes, \boldsymbol{c})|$.

Example 2.10 (Geometric nerves of monoidal bicategories) If $(\mathcal{B}, \otimes)$ is any monoidal bicategory, then we define its *geometric nerve*, denoted by $\Delta(\mathcal{B}, \otimes)$, as the geometric nerve of its suspension 3-category $\Sigma(\mathcal{B}, \otimes)$, (2.24). That is,

$$\Delta(\mathcal{B}, \otimes) \colon \Delta^{\mathrm{op}} \to \mathbf{Simpl.Set}, \quad [p] \mapsto \mathrm{Lax}_{\mathrm{u}}([p], \Sigma(\mathcal{B}, \otimes)).$$

Then, Theorem 2.4 specializes to monoidal bicategories giving a homotopy equivalence

$$\mathrm{B}_3(\mathcal{B}, \otimes) \simeq |\Delta(\mathcal{B}, \otimes)|.$$

2.5.1 Bicategorical groups and homotopy 3-types

Recall that a bigroupoid is a bicategory $\mathcal{B}$ in which every 2-cell is invertible, that is, all the hom-categories $\mathcal{B}(x, y)$ are groupoids, and every 1-cell $u \colon x \to y$ is an equivalence, that is, there exist a morphism $u' \colon y \to x$ and 2-cells $u \circ u' \Rightarrow 1_y$ and $1_x \Rightarrow u' \circ u$. By a *bicategorical group* we shall mean a monoidal bigroupoid $(\mathcal{B}, \otimes)$ in which for every object x there is an object x' with 1-cells $1 \to x \otimes x'$ and $x' \otimes x \to 1$. Bicategorical groups correspond to those *Picard* 2*-categories*, in the sense of Gurski [76, §6], whose underlying bicategory is a bigroupoid.

In any bicategorical group $(\mathcal{B}, \otimes)$, the homomorphisms $x \otimes - \colon \mathcal{B} \to \mathcal{B}$ and $- \otimes x \colon \mathcal{B} \to \mathcal{B}$ are biequivalences, for any object $x \in \mathcal{B}$. Hence, by Theorem 2.3, there is a homotopy equivalence

$$\mathrm{B}_2\mathcal{B} \simeq \Omega \mathrm{B}_3(\mathcal{B}, \otimes),$$

between the classifying space of the underlying bigroupoid and the loop space of the classifying space of the bicategorical group.

If $(\mathcal{B}, \otimes)$ is any monoidal bicategory, then its geometric nerve $\Delta(\mathcal{B}, \otimes)$ is a 4-coskeletal reduced (one vertex) simplicial set, which satisfies the Kan extension condition if and only if $(\mathcal{B}, \otimes)$ is a bicategorical group. In such a case, the homotopy groups of its geometric realization

$$\pi_i \mathrm{B}_3(\mathcal{B}, \otimes) \cong \pi_i \Delta(\mathcal{B}, \otimes) \cong \pi_{i-1} \Delta \mathcal{B}$$

are plainly recognized to be

- $\pi_i \mathrm{B}_3(\mathcal{B}, \otimes) = 0, \quad \text{if } i \neq 1, 2, 3.$
- $\pi_1 \mathrm{B}_3(\mathcal{B}, \otimes) = \mathrm{Ob}\mathcal{B}/\!\sim,$ the group of equivalence classes of objects in $\mathcal{B}$ where multiplication is induced by the tensor product.
- $\pi_2 \mathrm{B}_3(\mathcal{B}, \otimes) = \mathrm{Aut}_{\mathcal{B}}(1)/\!\cong,$ the group of isomorphism classes of autoequivalences of the unit object where the operation is induced by the horizontal composition in $\mathcal{B}$.
- $\pi_3 \mathrm{B}_3(\mathcal{B}, \otimes) = \mathrm{Aut}_{\mathcal{B}}(1_1)$, the group of automorphisms of the identity 1-cell of the unit object where the operation is vertical composition in $\mathcal{B}$.

Thus, bicategorical groups arise as algebraic path-connected homotopy 3-types, a fact that supports the *Homotopy Hypothesis* of Baez [5]. Indeed, every path-connected homotopy 3-type can be realized in this way from a bicategorical group, as we show below (cf. Berger [17], Joyal and Tierney [89], Lack [94], or Leroy [97], for alternative approaches to this issue).

Proposition 2.5 *For any path-connected pointed CW-complex X for which $\pi_i X = 0$ for $i \geq 4$, there is a bicategorical group $(\mathcal{B}(X), \otimes)$ whose classifying space* $\mathrm{B}_3(\mathcal{B}(X), \otimes)$ *is homotopy equivalent to X.*

Proof: Given X as above, let $M(X) \subseteq S(X)$ be a minimal subcomplex that is a deformation retract of the total singular complex of X, so that $|M(X)| \simeq X$. Taking into account the Postnikov k-invariants, this minimal complex M can be described (see Goerss and Jardine [68, VI. Corollary 5.13]), up to isomorphism,

$$M(X) = K(B,3) \times_t (K(A,2) \times_h K(G,1)), \tag{2.42}$$

by means of the group $G = \pi_1 X$, the G-modules $A = \pi_2 X$ and $B = \pi_3 X$, and two maps,

$$h\colon G^3 \to A, \quad t\colon A^6 \times G^4 \to B,$$

defining normalized cocycles $h \in Z^3(G, A)$ and $t \in Z^4(K(A,2) \times_h K(G,1), B)$. That is, $M(X)$ is the 4^{th} coskeleton of the truncated simplicial set

$$\begin{tikzcd}
\mathrm{tr}_4 M(X) = B^4 \times A^6 \times G^4 \arrow[r, shift left=2, "d_0"] \arrow[r, phantom, "\cdots"] \arrow[r, shift right=2, "d_4"'] & B \times A^3 \times G^3 \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=70, "s_3"', "\cdots"] \arrow[r, shift left=2, "d_0"] \arrow[r, phantom, "\cdots"] \arrow[r, shift right=2, "d_3"'] & A \times G^2 \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=60] \arrow[l, bend right=70, "s_2"'] \arrow[r, shift left=2, "d_0"] \arrow[r] \arrow[r, shift right=2, "d_2"'] & G \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=70, "s_1"'] \arrow[r, shift left=1, "d_0"] \arrow[r, shift right=1, "d_1"'] & 1, \arrow[l, bend right=40, "s_0"']
\end{tikzcd}$$

whose face and degeneracy operators are given by ($\sigma_i \in G$, $x_j \in A$, $u_k \in B$)

$$d_i(x_1, \sigma_1, \sigma_2) = \begin{cases} \sigma_2 & i = 0, \\ \sigma_1\sigma_2 & i = 1, \\ \sigma_1 & i = 2. \end{cases}$$

$$d_i(u_1,x_1,x_2,x_3,\sigma_1,\sigma_2,\sigma_3)=\begin{cases}({}^{\sigma_1^{-1}}x_3,\sigma_2,\sigma_3) & i=0,\\ (x_2+x_3,\sigma_1\sigma_2,\sigma_3) & i=1,\\ (x_1+x_2,\sigma_1,\sigma_2\sigma_3) & i=2,\\ (x_1-h(\sigma_1,\sigma_2,\sigma_3),\sigma_1,\sigma_2) & i=3.\end{cases}$$

$$d_i(u_1,u_2,u_3,u_4,x_1,x_2,x_3,x_4,x_5,x_6,\sigma_1,\sigma_2,\sigma_3,\sigma_4)=$$

$$\begin{cases}({}^{\sigma_1^{-1}}u_4,{}^{\sigma_1^{-1}}x_4,{}^{\sigma_1^{-1}}x_5,{}^{\sigma_1^{-1}}x_6,\sigma_2,\sigma_3,\sigma_4) & i=0,\\ (u_3+u_4,x_2+x_4,x_3+x_5,x_6,\sigma_1\sigma_2,\sigma_3,\sigma_4) & i=1,\\ (u_2+u_3,x_1+x_2,x_3,x_5+x_6,\sigma_1,\sigma_2\sigma_3,\sigma_4) & i=2,\\ (u_1+u_2,x_1,x_2+x_3,x_4+x_5,\sigma_1,\sigma_2,\sigma_3\sigma_4) & i=3\\ (\bar{u}_1,\bar{x}_1,\bar{x}_2,\bar{x}_3,\sigma_1,\sigma_2,\sigma_3) & i=4,\end{cases}$$

where $\bar{u}_1=u_1-t(x_1,x_2,x_3,x_4,x_5,x_6,\sigma_1,\sigma_2,\sigma_3,\sigma_4)$, $\bar{x}_1=x_1-h(\sigma_1,\sigma_2,\sigma_3\sigma_4)+h(\sigma_1,\sigma_2,\sigma_3)$, $\bar{x}_2=x_2-h(\sigma_1\sigma_2,\sigma_3,\sigma_4)+{}^{\sigma_1^{-1}}h(\sigma_2,\sigma_3,\sigma_4)$, and $\bar{x}_3=x_4-{}^{\sigma_1^{-1}}h(\sigma_2,\sigma_3,\sigma_4)$.

Then, a bicategorical group $(\mathcal{B}(X),\otimes)$ with a simplicial isomorphism $\Delta(\mathcal{B}(X),\otimes)\cong M(X)$ is defined as follows:

- A 0-cell of $\mathcal{B}=\mathcal{B}(X)$ is an element $\sigma\in G$. If $\sigma\neq\tau$ are different elements of G, then $\mathcal{B}(\sigma,\tau)=\emptyset$, that is, there is no 1-cell between them, whereas if $\sigma=\tau$, then a 1-cell $x\colon\sigma\to\sigma$ is an element $x\in A$. Similarly, there is no 2-cell in $\mathcal{B}$ between two 1-cells $x,y\colon\sigma\to\sigma$ if $x\neq y$, whereas, when $x=y$, a 2-cell $u\colon x\Rightarrow x$ is an element $u\in B$.
- The vertical composition of 2-cells is given by addition in B, that is,

$$(x\overset{u}{\Longrightarrow}x)\cdot(x\overset{v}{\Longrightarrow}x)=(x\overset{u+v}{\Longrightarrow}x).$$

- The horizontal composition of 1-cells and 2-cells is given by addition in A and B respectively, that is,

$$\begin{tikzcd}(\sigma \arrow[r, bend left, "x"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "\Downarrow u"] & \sigma)\circ(\sigma \arrow[r, bend left, "y"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow v"] & \sigma) = (\sigma \arrow[r, bend left, "x+y"] \arrow[r, bend right, "x+y"'] \arrow[r, phantom, "\Downarrow u+v"] & \sigma).\end{tikzcd}$$

- The associativity isomorphism is

$$\begin{tikzcd}\sigma \arrow[r, bend left, "(x+y)+z"] \arrow[r, bend right, "x+(y+z)"'] \arrow[r, phantom, "\Downarrow \boldsymbol{a}"] & \sigma,\end{tikzcd} \qquad \boldsymbol{a}=t(x,y,z,0,0,0,\sigma,1,1,1),$$

and the 0 of A gives the (strict) unit on each σ, that is, $1_\sigma=0\colon\sigma\to\sigma$.

- The (strictly unitary) tensor homomorphism $\otimes\colon\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ is given on cells of $\mathcal{B}$ by

$$\begin{tikzcd}(\sigma \arrow[r, bend left, "x"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "\Downarrow u"] & \sigma)\otimes(\tau \arrow[r, bend left, "y"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow v"] & \tau) = (\sigma\tau \arrow[r, bend left, "x+{}^{\sigma}y"] \arrow[r, bend right, "x+{}^{\sigma}y"'] \arrow[r, phantom, "\Downarrow u+{}^{\sigma}v"] & \sigma\tau).\end{tikzcd}$$

- The structure interchange isomorphism, for any 1-cells $\sigma \xrightarrow{x'} \sigma \xrightarrow{x} \sigma$ and $\tau \xrightarrow{y'} \tau \xrightarrow{y} \tau$,

$$\begin{tikzcd}
\sigma\tau \arrow[r, bend left, "(x+{}^{\sigma}y)+(x'+{}^{\sigma}y')"] \arrow[r, bend right, "(x+x')+{}^{\sigma}(y+y')"'] \arrow[r, phantom, "\Downarrow"] & \sigma\tau,
\end{tikzcd}$$

is that obtained by pasting in the bigroupoid $\mathcal{B}$ the diagram

$$\begin{tikzcd}
\sigma\tau \arrow[r, "{}^{\sigma}y'"] \arrow[dr, "{}^{\sigma}(y+y')"'] & \sigma\tau \arrow[r, "x'"] \arrow[d, "{}^{\sigma}y"] \arrow[dl, phantom, "\Leftarrow\chi"] & \sigma\tau \arrow[r, "{}^{\sigma}y"] \arrow[d, phantom, "\Downarrow \boldsymbol{c}"] & \sigma\tau \arrow[d, "x"] \arrow[dl, phantom, "\Downarrow\bar{\chi}"] \\
& \sigma\tau \arrow[urr, "x'"] \arrow[rr, "x+x'"'] & & \sigma\tau
\end{tikzcd}$$

$\chi = -t(0,0,0,{}^{\sigma}y,{}^{\sigma}y',0,\sigma,\tau,1,1),$

$\boldsymbol{c} = t(0,x,0,0,{}^{\sigma}y,0,\sigma,\tau,1,1) - t(0,0,x,{}^{\sigma}y,0,0,\sigma,\tau,1,1) - t(x,0,0,0,0,{}^{\sigma}y,\sigma,1,\tau,1),$

$\bar{\chi} = -t(0,x,x',0,0,0,\sigma,\tau,1,1)+t(x,0,x',0,0,0,\sigma,1,\tau,1)-t(x,x',0,0,0,0,\sigma,1,1,\tau).$

- The associativity pseudo-equivalence $(-\otimes-)\otimes- \overset{\boldsymbol{a}}{\Rightarrow} -\otimes(-\otimes-)\colon \mathcal{B}^3 \to \mathcal{B}$ is defined by the 1-cells

$$h(\sigma,\tau,\gamma)\colon (\sigma\tau)\gamma \to \sigma(\tau\gamma).$$

The naturality component of $\boldsymbol{a}$, at any 1-cells $\sigma \xrightarrow{x} \sigma$, $\tau \xrightarrow{y} \tau$ and $\gamma \xrightarrow{z} \gamma$,

$$\begin{tikzcd}
(\sigma\tau)\gamma \arrow[r, "h=h(\sigma,\tau,\gamma)"] \arrow[d, "(x+{}^{\sigma}y)+{}^{\sigma\tau}z"'] \arrow[dr, phantom, "\Rightarrow"] & \sigma(\tau\gamma) \arrow[d, "x+{}^{\sigma}(y+{}^{\tau}z)"] \\
(\sigma\tau)\gamma \arrow[r, "h(\sigma,\tau,\gamma)"'] & \sigma(\tau\gamma)
\end{tikzcd}$$

is given by pasting in $\mathcal{B}$ the diagram

$$\begin{tikzcd}
(\sigma\tau)\gamma \arrow[r, "h"] \arrow[d, "x"'] \arrow[dr, phantom, "\Downarrow\Omega"] & \sigma(\tau\gamma) \arrow[r, "x"] & \sigma(\tau\gamma) \arrow[r, "{}^{\sigma}y"] & \sigma(\tau\gamma) \arrow[d, "{}^{\sigma\tau}z"] \\
(\sigma\tau)\gamma \arrow[r, "{}^{\sigma}y"'] \arrow[urr, "h"] & (\sigma\tau)\gamma \arrow[r, "{}^{\sigma\tau}z"'] \arrow[urr, "h"] \arrow[ur, phantom, "\Downarrow\Psi"] & (\sigma\tau)\gamma \arrow[r, "h"'] \arrow[ur, phantom, "\Downarrow\Phi"] & \sigma(\tau\gamma)
\end{tikzcd}$$

$$\begin{aligned}
\Phi = {} & t(0,h,0,0,{}^{\sigma\tau}z,0,\sigma,\tau\gamma,1,1) - t(h,0,0,0,0,{}^{\sigma\tau}z,\sigma,\tau,\gamma,1) \\
& -t(0,0,h,{}^{\sigma\tau}z,0,0,\sigma,\tau\gamma,1,1), \\
\Psi = {} & t(h,0,0,{}^{\sigma}y,0,0,\sigma,\tau,1,\gamma) - t(h,0,0,0,{}^{\sigma}y,0,\sigma,\tau,\gamma,1) \\
& +t(0,h,0,0,{}^{\sigma}y,0,\sigma,\tau\gamma,1,1) - t(0,0,h,{}^{\sigma}y,0,0,\sigma,\tau\gamma,1,1), \\
\Omega = {} & -t(x,h,0,0,0,0,\sigma,1,\tau,\gamma) + t(h,x,0,0,0,0,\sigma,\tau,1,\gamma) \\
& -t(h,0,x,0,0,0,\sigma,\tau,\gamma,1) + t(x,0,h,0,0,0,\sigma,1,\tau\gamma,1) \\
& +t(0,h,x,0,0,0,\sigma,\tau\gamma,1,1) - t(0,x,h,0,0,0,\sigma,\tau\gamma,1,1),
\end{aligned}$$

- The structure modification π, at any objects $\sigma,\tau,\gamma,\delta\in G$, is

$$\begin{tikzcd}
((\sigma\tau)\gamma)\delta \arrow[rr, "h_1=h(\sigma\tau,\gamma,\delta)"] \arrow[d, "h_4=h(\sigma,\tau,\gamma)"'] & \overset{\pi}{\Rightarrow} & (\sigma\tau)(\gamma\delta) \arrow[d, "h_3=h(\sigma,\tau,\gamma\delta)"] \\
(\sigma(\tau\gamma))\delta \arrow[r, "h_2=h(\sigma,\tau\gamma,\delta)"] & \sigma((\tau\gamma)\delta) \arrow[r, "h_0={}^{\sigma}h(\tau,\gamma,\delta)"] & \sigma(\tau(\gamma\delta)),
\end{tikzcd}$$

$$\begin{aligned}\pi = {} & t(h_3,h_1-h_0,0,h_0,0,0,\sigma,\tau,\gamma,\delta)-t(h_2,h_4,0,0,0,0,\sigma,\tau\gamma,1,\delta)\\ &+t(h_2,0,h_4,0,0,0,\sigma,\tau\gamma,\delta,1)-t(h_3,0,h_1-h_0,0,h_0,0,\sigma,\tau,\gamma\delta,1)\\ &+t(0,h_3,h_1-h_0,0,h_0,0,\sigma,\tau\gamma\delta,1,1)-t(0,0,h_2+h_4,h_0,0,0,\sigma,\tau\gamma\delta,1,1)\\ &-t(0,h_2,h_4,0,0,0,\sigma,\tau\gamma\delta,1,1).\end{aligned}$$

This completes the description of the bicategorical group $(\mathcal{B},\otimes)=(\mathcal{B}(X),\otimes)$, whose geometric nerve is recognized to be isomorphic to the minimal complex $M(X)$ in (2.42) by means of the simplicial map φ: $\Delta(\mathcal{B},\otimes)\to M$ which, in dimensions ≤ 4,

$$\begin{tikzcd}
\Delta(\mathcal{B},\otimes)_4 \arrow[r, shift left, "\cdots"] \arrow[r, shift right] \arrow[d, "\varphi"] & \Delta(\mathcal{B},\otimes)_3 \arrow[r, shift left, "\cdots"] \arrow[r, shift right] \arrow[d, "\varphi"] & \Delta(\mathcal{B},\otimes)_2 \arrow[r, shift left=1.5] \arrow[r] \arrow[r, shift right=1.5] \arrow[d, "\varphi"] & \Delta(\mathcal{B},\otimes)_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\varphi"] & 1, \arrow[d] \\
B^4\times A^6\times G^4 \arrow[r, shift left, "\cdots"] \arrow[r, shift right] & B\times A^3\times G^3 \arrow[r, shift left, "\cdots"] \arrow[r, shift right] & A\times G^2 \arrow[r, shift left=1.5] \arrow[r] \arrow[r, shift right=1.5] & G \arrow[r, shift left] \arrow[r, shift right] & 1,
\end{tikzcd}$$

carries (keeping the notations in (2.26) – (2.30))

- a unitary lax functor F: $[1]\to\Sigma(\mathcal{B},\otimes)$ to $\varphi(F)=F_{0,1}$,
- a unitary lax functor F: $[2]\to\Sigma(\mathcal{B},\otimes)$ to $\varphi(F)=(-F_{0,1,2},F_{0,1},F_{1,2})$,
- a unitary lax functor F: $[3]\to\Sigma(\mathcal{B},\otimes)$ to

$$\varphi(F)= \\ (-F_{0,1,2,3},-{}^{F_{0,1}}F_{1,2,3}+F_{0,2,3}-F_{0,1,3},{}^{F_{0,1}}F_{1,2,3}-F_{0,2,3},-{}^{F_{0,1}}F_{1,2,3},F_{0,1},F_{1,2},F_{2,3}),$$

- a F: $[4]\to\Sigma(\mathcal{B},\otimes)$ to

$$\varphi(F)=(u_1,u_2,u_3,u_4,x_1,x_2,x_3,x_4,x_5,x_6,F_{0,1},F_{1,2},F_{2,3},F_{3,4}),$$

where

$$\begin{array}{ll}
u_1={}^{F_{0,1}}F_{1,2,3,4}-F_{0,1,2,4}+F_{0,1,3,4}-F_{0,2,3,4}, & x_2={}^{F_{0,1}}F_{1,2,4}-{}^{F_{0,1}}F_{1,3,4}+F_{0,3,4}-F_{0,2,4},\\
u_2=F_{0,2,3,4}-F_{0,1,3,4}-{}^{F_{0,1}}F_{1,2,3,4}, & x_3={}^{F_{0,1}}F_{1,3,4}-F_{0,3,4},\\
u_3={}^{F_{0,1}}F_{1,2,3,4}-F_{0,2,3,4}, & x_4={}^{F_{0,1}}F_{1,3,4}-{}^{F_{0,1}}F_{1,2,4}-{}^{F_{0,2}}F_{2,3,4},\\
u_4=-{}^{F_{0,1}}F_{1,2,3,4}, & x_5={}^{F_{0,2}}F_{2,3,4}-{}^{F_{0,1}}F_{1,3,4},\\
x_1=F_{0,2,4}-{}^{F_{0,1}}F_{1,2,4}-F_{0,1,4}, & x_6=-{}^{F_{0,2}}F_{2,3,4}.
\end{array}$$

□

Remark 2.1 The *fundamental bigroupoid* $\Pi_2(X)$ of a space X was independently described by Hardie, Kamps and Kieboom in [80] and by Stevenson in [115]. The objects of $\Pi_2(X)$ are the points $x \in X$, the 1-cells $f\colon x \to y$ are paths $f\colon I = [0,1] \to X$ with $f(0) = x$ and $f(1) = y$, and the 2-cells $[\alpha]\colon f \Rightarrow g$ are relative homotopy classes of homotopies between paths $\alpha\colon I \times I \to X$ with $\alpha|_{I\times 0} = f$ and $\alpha|_{I\times 1} = g$. In [75, Theorem 1.4], Gurski proves that when the space X is endowed with a structure of algebra for the little cubes operad $\mathcal{C}_1$, then $\Pi_2(X)$ has the structure of a monoidal bicategory, say $(\Pi_2(X), \otimes)$. In this way, any given pointed topological space X has associated a bicategorical group $(\Pi_2(\Omega X), \otimes)$, Gurski's monoidal bicategory of the loop space ΩX. Although we will not give details here, we want to point out that, for X a path-connected pointed space X for which $\pi_i X = 0$ for $i \geq 4$, the bicategorical group $(\mathcal{B}(X), \otimes)$ in Proposition 2.5 is a skeleton of the bicategorical group $(\Pi_2(\Omega X), \otimes)$. That is, there is a monoidal biequivalence $(\mathcal{B}(X), \otimes) \simeq (\Pi_2(\Omega X), \otimes)$, and the bigroupoid $\mathcal{B}(X)$ is skeletal, in the sense that any two isomorphic 1-cells are equal and any two equivalent objects are equal.

To finish, we shall remark on two particular relevant cases of the demonstrated relationship between monoidal bicategories and path-connected homotopy 3-types. Since *categorical groups*, in the sense of Joyal and Street [88, §3], are the same thing as bicategorical groups in which all 2-cells are identities, then categorical groups are algebraic models for path-connected homotopy 2-types, see Carrasco and Cegarra [39, §2.1] and Cegarra and Garzón [44, §5]. This fact goes back to Whitehead (1949) [124] and Mac Lane and Whitehead (1950) [101] since every categorical group is equivalent to a strict one, and strict categorical groups are the same as crossed modules. On the other hand, if $(\mathcal{C}, \otimes, \boldsymbol{c})$ is any *braided categorical group* [88], then its classifying space $\mathrm{B}_3(\mathcal{C}, \otimes, \boldsymbol{c})$ is the classifying space of its suspension bicategorical group $(\Sigma(\mathcal{C}, \otimes), \otimes)$ (see Examples 2.6 and 2.9), which is precisely a one-object bicategorical group. Therefore, we conclude from the above discussion that braided categorical groups are algebraic models for path-connected simply connected homotopy 3-types, a fact due to Joyal and Tierney [89], but also proved by Carrasco and Cegarra in [39, §2.2] (cf. [40, Theorem 3.8]) and, implicitly, by Joyal and Street in [88, Theorem 3.3].

2.6 Appendix

2.6.1 Coherence conditions

(CR1): for any four composable arrows in $\mathcal{I}$, $i \overset{d}{\to} j \overset{c}{\to} k \overset{b}{\to} l \overset{a}{\to} m$, the equation $A = A'$ on 3-cells in $\mathcal{T}$ holds, where:

$$A= \begin{tikzcd}
((Fa\otimes Fb)\otimes Fc)\otimes Fd \arrow[rr, Rightarrow, "\boldsymbol{a}"] \arrow[dd, Rightarrow, "(F_{a,b}\otimes 1)\otimes 1"'] \arrow[dr, Rightarrow, "\boldsymbol{a}\otimes 1"] & & (Fa\otimes Fb)\otimes(Fc\otimes Fd) \arrow[rr, Rightarrow, "\boldsymbol{a}"] \arrow[d, phantom, "\overset{\pi}{\cong}"] & & Fa\otimes(Fb\otimes(Fc\otimes Fd)) \arrow[dd, Rightarrow, "1\otimes(1\otimes F_{c,d})"] \\
& (Fa\otimes(Fb\otimes Fc))\otimes Fd \arrow[rr, Rightarrow, "\boldsymbol{a}"] \arrow[dd, Rightarrow, "(1\otimes F_{b,c})\otimes 1"] & & Fa\otimes((Fb\otimes Fc)\otimes Fd) \arrow[ur, Rightarrow, "1\otimes\boldsymbol{a}"] \arrow[dd, Rightarrow, "1\otimes(F_{b,c}\otimes 1)"] & \\
(F(ab)\otimes Fc)\otimes Fd \arrow[dd, Rightarrow, "F_{ab,c}\otimes 1"'] & \overset{F_{a,b,c}\otimes 1}{\Rrightarrow} & \cong & \overset{1\otimes F_{b,c,d}}{\Rrightarrow} & Fa\otimes(Fb\otimes F(cd)) \arrow[dd, Rightarrow, "1\otimes F_{b,cd}"] \\
& (Fa\otimes F(bc))\otimes Fd \arrow[rr, Rightarrow, "\boldsymbol{a}"] \arrow[dl, Rightarrow, "F_{a,bc}\otimes 1"] & \overset{F_{a,bc,d}}{\Rrightarrow} & Fa\otimes(F(bc)\otimes Fd) \arrow[dr, Rightarrow, "1\otimes F_{bc,d}"] & \\
F(abc)\otimes Fd \arrow[rr, Rightarrow, "F_{abc,d}"] & & F(abcd) & & Fa\otimes F(bcd) \arrow[ll, Rightarrow, "F_{a,bcd}"']
\end{tikzcd}$$

$$A'= \begin{tikzcd}
((Fa\otimes Fb)\otimes Fc)\otimes Fd \arrow[rr, Rightarrow, "\boldsymbol{a}"] \arrow[dd, Rightarrow, "(F_{a,b}\otimes 1)\otimes 1"'] \arrow[dr, phantom, "\cong"] & & (Fa\otimes Fb)\otimes(Fc\otimes Fd) \arrow[rr, Rightarrow, "\boldsymbol{a}"] \arrow[dl, Rightarrow, "F_{a,b}\otimes(1\otimes 1)"'] \arrow[dr, Rightarrow, "(1\otimes 1)\otimes F_{c,d}"] & & Fa\otimes(Fb\otimes(Fc\otimes Fd)) \arrow[dd, Rightarrow, "1\otimes(1\otimes F_{c,d})"] \arrow[dl, phantom, "\cong"] \\
& F(ab)\otimes(Fc\otimes Fd) \arrow[dr, Rightarrow, "1\otimes F_{c,d}"'] & \cong & (Fa\otimes Fb)\otimes F(cd) \arrow[dl, Rightarrow, "F_{a,b}\otimes 1"] \arrow[dr, Rightarrow, "\boldsymbol{a}"] & \\
(F(ab)\otimes Fc)\otimes Fd \arrow[ur, Rightarrow, "\boldsymbol{a}"] \arrow[dd, Rightarrow, "F_{ab,c}\otimes 1"'] & & F(ab)\otimes F(cd) \arrow[dd, Rightarrow, "F_{ab,cd}"] & & Fa\otimes(Fb\otimes F(cd)) \arrow[dd, Rightarrow, "1\otimes F_{b,cd}"] \\
& \overset{F_{ab,c,d}}{\Rrightarrow} & & \overset{F_{a,b,cd}}{\Rrightarrow} & \\
F(abc)\otimes Fd \arrow[rr, Rightarrow, "F_{abc,d}"] & & F(abcd) & & Fa\otimes F(bcd) \arrow[ll, Rightarrow, "F_{a,bcd}"']
\end{tikzcd}$$

(CR2): for any two composable arrows $i \overset{b}{\to} j \overset{a}{\to} k$ in $\mathcal{I}$, the equations $B = B'$, $C = C'$, and $D = D'$, on 3-cells in $\mathcal{T}$ hold, where:

$$B= \begin{tikzcd}
& & (Fa\otimes 1_{Fj})\otimes Fb \arrow[rr, Rightarrow, "(1\otimes F_j)\otimes 1"] \arrow[dll, Rightarrow, "\boldsymbol{a}"'] \arrow[dd, Rightarrow, "\boldsymbol{r}^{\bullet}\otimes 1"] & & (Fa\otimes F1_j)\otimes Fb \arrow[ddll, Rightarrow, "F_{a,1}\otimes 1"] \arrow[dr, Rightarrow, "F_{a,1}\otimes 1"] & \\
Fa\otimes(1_{Fj}\otimes Fb) \arrow[drr, Rightarrow, "1\otimes\boldsymbol{l}"'] & \overset{\mu}{\cong} & & \overset{\widetilde{F}_a\otimes 1}{\Lleftarrow} & = & Fa\otimes Fb \arrow[dl, Rightarrow, "F_{a,b}"] \\
& & Fa\otimes Fb \arrow[rr, Rightarrow, "F_{a,b}"] & & F(ab) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & (Fa\otimes 1_{Fj})\otimes Fb \arrow[rr, Rightarrow, "(1\otimes F_j)\otimes 1"] \arrow[dl, Rightarrow, "\boldsymbol{a}"'] \arrow[dr, phantom, "\cong"] & & (Fa\otimes F1_j)\otimes Fb \arrow[dl, Rightarrow, "\boldsymbol{a}"] \arrow[dr, Rightarrow, "F_{a,1}\otimes 1"] & \\
B' = & Fa\otimes(1_{Fj}\otimes Fb) \arrow[rr, Rightarrow, "1\otimes(F_j\otimes 1)"] \arrow[dr, Rightarrow, "1\otimes\boldsymbol{l}"'] & \arrow[d, phantom, "\overset{1\otimes\widehat{F}_a}{\Leftarrow}"] & Fa\otimes(F1_j\otimes Fb) \arrow[dl, Rightarrow, "1\otimes F_{1,b}"] & \overset{F_{a,1,b}}{\Leftarrow} & Fa\otimes Fb \arrow[dl, Rightarrow, "F_{a,b}"] \\
 & & Fa\otimes Fb \arrow[rr, Rightarrow, "F_{a,b}"] & & F(ab) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & (1_{Fk}\otimes Fa)\otimes Fb \arrow[dl, Rightarrow, "\boldsymbol{a}"'] \arrow[dr, Rightarrow, "(F_k\otimes 1)\otimes 1"] \arrow[ddr, Rightarrow, "\boldsymbol{l}\otimes 1"'] & \\
 & 1_{Fk}\otimes(Fa\otimes Fb) \arrow[d, Rightarrow, "1\otimes F_{a,b}"'] \arrow[drr, Rightarrow, "\boldsymbol{l}"] & \overset{\lambda}{\cong} & (F1_k\otimes Fa)\otimes Fb \arrow[d, Rightarrow, "F_{1,a}\otimes 1"] \arrow[dl, phantom, "\overset{\widehat{F}_a\otimes 1}{\Leftarrow}"] \\
C = & 1_{Fk}\otimes F(ab) \arrow[dr, Rightarrow, "\boldsymbol{l}"'] & \cong & Fa\otimes Fb \arrow[dl, Rightarrow, "F_{a,b}"] \\
 & & F(ab) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & & (1_{Fk}\otimes Fa)\otimes Fb \arrow[dll, Rightarrow, "\boldsymbol{a}"'] \arrow[drr, Rightarrow, "(F_k\otimes 1)\otimes 1"] & & \\
 & 1_{Fk}\otimes(Fa\otimes Fb) \arrow[ddd, Rightarrow, "1\otimes F_{a,b}"'] \arrow[drr, Rightarrow, "F_k\otimes 1"] & & \cong & & (F1_k\otimes Fa)\otimes Fb \arrow[dll, Rightarrow, "\boldsymbol{a}"'] \arrow[ddd, Rightarrow, "F_{1,a}\otimes 1"] \\
 & & & F1_k\otimes(Fa\otimes Fb) \arrow[d, Rightarrow, "1\otimes F_{a,b}"] & & \\
C' = & & \cong & F1_k\otimes F(ab) \arrow[dd, Rightarrow, "F_{1,ab}"] & \overset{F_{1,a,b}}{\Leftarrow} & \\
 & 1_{Fk}\otimes F(ab) \arrow[urr, Rightarrow, "F_k\otimes 1"] \arrow[drr, Rightarrow, "\boldsymbol{l}"'] & \overset{\widehat{F}_{ab}}{\Leftarrow} & & & Fa\otimes Fb \arrow[dll, Rightarrow, "F_{a,b}"] \\
 & & & F(ab) & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & & (Fa\otimes Fb)\otimes 1_{Fi} \arrow[dll, Rightarrow, "\boldsymbol{a}"'] \arrow[drr, Rightarrow, "F_{a,b}\otimes 1"] \arrow[ddll, Rightarrow, "\boldsymbol{r}^{\bullet}"] & & \\
D = & Fa\otimes(Fb\otimes 1_{Fi}) \arrow[d, Rightarrow, "1\otimes\boldsymbol{r}^{\bullet}"'] \arrow[ur, phantom, "\overset{\rho}{\cong}"] & & & & F(ab)\otimes 1_{Fi} \arrow[d, Rightarrow, "1\otimes F_i"] \arrow[ddll, Rightarrow, "\boldsymbol{r}^{\bullet}"'] \arrow[d, phantom, "\overset{\widetilde{F}_{ab}}{\Leftarrow}"', shift right=6] \\
 & Fa\otimes Fb \arrow[drr, Rightarrow, "F_{a,b}"'] & & \cong & & F(ab)\otimes F1_i \arrow[dll, Rightarrow, "F_{ab,1}"] \\
 & & & F(ab) & &
\end{tikzcd}$$

$$D' = \begin{tikzcd}
 & & & (Fa\otimes Fb)\otimes 1_{Fi} \arrow[dlll, Rightarrow, "\boldsymbol{a}"'] \arrow[drrr, Rightarrow, "F_{a,b}\otimes 1"] \arrow[ddrr, Rightarrow, "1\otimes F_i"] & & & \\
Fa\otimes(Fb\otimes 1_{Fi}) \arrow[ddd, Rightarrow, "1\otimes \boldsymbol{r}^\bullet"'] \arrow[ddr, Rightarrow, "1\otimes(1\otimes F_i)"] & & & \cong & & \cong & F(ab)\otimes 1_{Fi} \arrow[ddd, Rightarrow, "1\otimes F_i"] \\
 & \overset{1\otimes\widetilde{F}_b}{\Leftarrow} & & & & (Fa\otimes Fb)\otimes F1_i \arrow[dllll, Rightarrow, "\boldsymbol{a}"] \arrow[ddr, Rightarrow, "F_{a,b}\otimes 1"'] & \\
 & Fa\otimes(Fb\otimes F1_i) \arrow[dl, Rightarrow, "1\otimes F_{b,1}"] & & \overset{F_{a,b,1}}{\Leftarrow} & & & \\
Fa\otimes Fb \arrow[drrr, Rightarrow, "F_{a,b}"'] & & & & & & F(ab)\otimes F1_i \arrow[dlll, Rightarrow, "F_{ab,1}"] \\
 & & & F(ab) & & &
\end{tikzcd}$$

(CR3): for any object $i \in \mathcal{I}$,the following equation holds:

$$\begin{tikzcd}
 & 1_{Fi}\otimes 1_{Fi} \arrow[dl, Rightarrow, "F_i\otimes 1"'] \arrow[dr, Rightarrow, "\boldsymbol{r}^\bullet"] & \\
F1_i\otimes 1_{Fi} \arrow[d, Rightarrow, "1\otimes F_i"'] \arrow[drr, Rightarrow, "\boldsymbol{r}^\bullet"] \arrow[dr, phantom, "\overset{\widetilde{F}_1}{\Rightarrow}"] & \cong & 1_{Fi} \arrow[d, Rightarrow, "F_i"] \\
F1_i\otimes F1_i \arrow[rr, Rightarrow, "F_{1,1}"] & & F1_i
\end{tikzcd}
=
\begin{tikzcd}
 & & 1_{Fi}\otimes 1_{Fi} \arrow[dll, Rightarrow, "F_i\otimes 1"'] \arrow[d, Rightarrow, "1\otimes F_i"] \arrow[drr, Rightarrow, "\boldsymbol{r}^\bullet=\boldsymbol{l}"] & & \\
F1_i\otimes 1_{Fi} \arrow[d, Rightarrow, "1\otimes F_i"'] & \cong & 1_{Fi}\otimes F1_i \arrow[dll, Rightarrow, "F_i\otimes 1"'] \arrow[drr, Rightarrow, "\boldsymbol{l}"] \arrow[d, phantom, "\overset{\widehat{F}_1}{\Rightarrow}"] & \cong & 1_{Fi} \arrow[d, Rightarrow, "F_i"] \\
F1_i\otimes F1_i \arrow[rrrr, Rightarrow, "F_{1,1}"] & & \, & & F1_i
\end{tikzcd}$$

(CR4): for any triplet of composable morphisms of $\mathcal{I}$, $i \overset{c}{\to} j \overset{b}{\to} k \overset{a}{\to} l$, the equation $E = E'$ on 3-cells in $\mathcal{T}$ holds, where:

$$E = \begin{tikzcd}
(Fa\otimes Fb)\otimes Fc \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[d, Rightarrow, "(\Phi a\otimes\Phi b)\otimes\Phi c"'] \arrow[dr, Rightarrow, "F_{a,b}\otimes 1"] & Fa\otimes(Fb\otimes Fc) \arrow[r, Rightarrow, "1\otimes F_{b,c}"] \arrow[dr, phantom, "\overset{F_{a,b,c}}{\Rightarrow}"] & Fa\otimes F(bc) \arrow[d, Rightarrow, "F_{a,bc}"] \\
(Ga\otimes Gb)\otimes Gc \arrow[d, Rightarrow, "G_{a,b}\otimes 1"'] \arrow[r, phantom, "\overset{\Phi_{a,b}\otimes 1}{\Rightarrow}"] & F(ab)\otimes Fc \arrow[r, Rightarrow, "F_{ab,c}"] \arrow[dl, Rightarrow, "\Phi(ab)\otimes\Phi c"] \arrow[dr, phantom, "\overset{\Phi_{ab,c}}{\Rightarrow}"] & F(abc) \arrow[d, Rightarrow, "\Phi(abc)"] \\
G(ab)\otimes Gc \arrow[rr, Rightarrow, "G_{ab,c}"] & & G(abc),
\end{tikzcd}$$

$$E' = \begin{tikzcd}
(Fa\otimes Fb)\otimes Fc \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[d, Rightarrow, "(\Phi a\otimes\Phi b)\otimes\Phi c"'] \arrow[dr, phantom, "\cong"] & Fa\otimes(Fb\otimes Fc) \arrow[rr, Rightarrow, "1\otimes F_{b,c}"] \arrow[d, Rightarrow, "\Phi a\otimes(\Phi b\otimes\Phi c)"] \arrow[dr, phantom, "\overset{1\otimes\Phi_{b,c}}{\Rightarrow}"] & & Fa\otimes F(bc) \arrow[dl, Rightarrow, "\Phi a\otimes\Phi(bc)"'] \arrow[d, Rightarrow, "F_{a,bc}"] \\
(Ga\otimes Gb)\otimes Gc \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[d, Rightarrow, "G_{a,b}\otimes 1"'] \arrow[dr, phantom, "\overset{G_{a,b,c}}{\Rightarrow}"] & Ga\otimes(Gb\otimes Gc) \arrow[r, Rightarrow, "1\otimes G_{b,c}"'] & Ga\otimes G(bc) \arrow[drr, Rightarrow, "G_{a,bc}"'] \arrow[r, phantom, "\overset{\Phi_{a,bc}}{\Rightarrow}"] & F(abc) \arrow[d, Rightarrow, "\Phi(abc)"] \\
G(ab)\otimes Gc \arrow[rrr, Rightarrow, "G_{ab,c}"] & & & G(abc).
\end{tikzcd}$$

(CR5): for any morphism of $\mathcal{I}$, $i \overset{a}{\to} j$, the following two pasting equalities hold:

$$\begin{tikzcd}
 & 1\otimes Fa \arrow[dl, Rightarrow, "1\otimes\Phi a"'] \arrow[d, Rightarrow, "F_j\otimes 1"] \arrow[dr, Rightarrow, "\boldsymbol{l}"] \arrow[dr, phantom, "\overset{\widehat{F}_a}{\Rrightarrow}"', near start] & & & & 1\otimes Fa \arrow[dl, Rightarrow, "1\otimes\Phi a"'] \arrow[dr, Rightarrow, "\boldsymbol{l}"] & \\
1\otimes Ga \arrow[d, Rightarrow, "G_j\otimes 1"'] \arrow[r, phantom, "\overset{\Phi_j\otimes 1}{\Rrightarrow}"] & F1\otimes Fa \arrow[r, Rightarrow, "F_{1,a}"'] \arrow[dl, Rightarrow, "\Phi 1\otimes\Phi a"] \arrow[dr, phantom, "\overset{\Phi_{1,a}}{\Rrightarrow}"] & Fa \arrow[d, Rightarrow, "\Phi a"] & = & 1\otimes Ga \arrow[d, Rightarrow, "G_j\otimes 1"'] \arrow[drr, Rightarrow, "\boldsymbol{l}"] \arrow[dr, phantom, "\overset{\widehat{G}_a}{\Rrightarrow}"'] & \cong & Fa \arrow[d, Rightarrow, "\Phi a"] \\
G1\otimes Ga \arrow[rr, Rightarrow, "G_{1,a}"'] & & Ga & & G1\otimes Ga \arrow[rr, Rightarrow, "G_{1,a}"'] & & Ga,
\end{tikzcd}$$

$$\begin{tikzcd}
 & Fa\otimes 1 \arrow[dl, Rightarrow, "\Phi a\otimes 1"'] \arrow[d, Rightarrow, "1\otimes F_i"] \arrow[dr, Rightarrow, "\boldsymbol{r}^\bullet"] \arrow[dr, phantom, "\overset{\widetilde{F}_a}{\Rrightarrow}"', near start] & & & & Fa\otimes 1 \arrow[dl, Rightarrow, "\Phi a\otimes 1"'] \arrow[dr, Rightarrow, "\boldsymbol{r}^\bullet"] & \\
Ga\otimes 1 \arrow[d, Rightarrow, "1\otimes G_i"'] \arrow[r, phantom, "\overset{1\otimes\Phi_i}{\Rrightarrow}"] & Fa\otimes F1 \arrow[r, Rightarrow, "F_{a,1}"'] \arrow[dl, Rightarrow, "\Phi a\otimes\Phi 1"] \arrow[dr, phantom, "\overset{\Phi_{a,1}}{\Rrightarrow}"] & Fa \arrow[d, Rightarrow, "\Phi a"] & = & Ga\otimes 1 \arrow[d, Rightarrow, "1\otimes G_i"'] \arrow[drr, Rightarrow, "\boldsymbol{r}^\bullet"] \arrow[dr, phantom, "\overset{\widetilde{G}_a}{\Rrightarrow}"'] & \cong & Fa \arrow[d, Rightarrow, "\Phi a"] \\
Ga\otimes G1 \arrow[rr, Rightarrow, "G_{a,1}"'] & & Ga & & Ga\otimes G1 \arrow[rr, Rightarrow, "G_{a,1}"'] & & Ga,
\end{tikzcd}$$

(CR6): for any object i and each two composable arrows $i \xrightarrow{b} j \xrightarrow{a} k$ of $\mathcal{I}$, the diagrams of 3-cells below commute.

$$\begin{tikzcd}
\Phi 1_i\circ F_i \arrow[rr, Rrightarrow, "M1_i\circ 1"] & & \Psi 1_i\circ F_i & & G_{a,b}\circ(\Phi a\otimes\Phi b) \arrow[r, Rrightarrow, "\Phi_{a,b}"] \arrow[d, Rrightarrow, "1\circ(Ma\otimes Mb)"'] & \Phi(ab)\circ F_{a,b} \arrow[d, Rrightarrow, "M(ab)\otimes 1"] \\
 & G_i \arrow[ul, Rrightarrow, "\Phi_i"] \arrow[ur, Rrightarrow, "\Psi_i"'] & & & G_{a,b}\circ(\Psi a\otimes\Psi b) \arrow[r, Rrightarrow, "\Psi_{a,b}"] & \Psi(ab)\circ F_{a,b}
\end{tikzcd}$$

2.6.2 Proof of Lemma 2.1

(i) The homomorphism H_* is defined as follows: It carries a lax functor $F\colon \mathcal{I}\to\mathcal{T}$ to the lax functor $H_*F\colon \mathcal{I}\to\mathcal{T}'$, which is defined on objects i of $\mathcal{I}$ by $(H_*F)i = HFi$, and on arrows $a\colon i\to j$ by $(H_*F)a = HFa\colon HFi\to HFj$. The 2-cell $(H_*F)_{a,b}\colon (H_*F)a\otimes (H_*F)b \Rightarrow (H_*F)(ab)$, for each pair of composable arrows $i\xrightarrow{b} j\xrightarrow{a} k$, is the composition $HFa\otimes HFb \overset{\chi}{\Longrightarrow} H(Fa\otimes Fb) \overset{HF_{a,b}}{\Longrightarrow} HF(ab)$. For each object i, the 2-cell $(H_*F)_i\colon 1_{(H_*F)i}\Rightarrow (H_*F)1_i$ is the composite of $1_{HFi}\overset{\iota}{\Longrightarrow} H1_{Fi}\overset{HF_i}{\Longrightarrow} HF1_i$. The structure 3-cell of $H_*F\colon \mathcal{I}\to\mathcal{T}'$ associated to any three composable arrows $i\xrightarrow{c} j\xrightarrow{b} k\xrightarrow{a} l$, is that obtained by pasting the diagram

$$\begin{tikzcd}
& (HFa\otimes HFb)\otimes HFc \arrow[rr, Rightarrow, "\boldsymbol{a}"] \arrow[dl, Rightarrow, "\chi\otimes 1"'] & & HFa\otimes(HFb\otimes HFc) \arrow[dr, Rightarrow, "1\otimes\chi"] & \\
H(Fa\otimes Fb)\otimes HFc \arrow[dr, Rightarrow, "\chi"] \arrow[dd, Rightarrow, "HF_{a,b}\otimes 1"'] & & \overset{\omega}{\Rrightarrow} & & HFa\otimes H(Fb\otimes Fc) \arrow[dl, Rightarrow, "\chi"'] \arrow[dd, Rightarrow, "1\otimes HF_{b,c}"] \\
& H((Fa\otimes Fb)\otimes Fc) \arrow[rr, Rightarrow, "H\boldsymbol{a}"] \arrow[dd, Rightarrow, "H(F_{a,b}\otimes 1)"] & & H(Fa\otimes(Fb\otimes Fc)) \arrow[dd, Rightarrow, "H(1\otimes F_{b,c})"] & \\
HF(ab)\otimes HFc \arrow[dr, Rightarrow, "\chi"'] \arrow[ur, phantom, "\cong"] & & \overset{F_{a,b,c}}{\Rrightarrow} & & HFa\otimes HF(bc) \arrow[dl, Rightarrow, "\chi"] \arrow[ul, phantom, "\cong"] \\
& H(F(ab)\otimes Fc) \arrow[dr, Rightarrow, "HF_{ab,c}"'] & & H(Fa\otimes F(bc)) \arrow[dl, Rightarrow, "HF_{a,bc}"] & \\
& & HF(abc) & &
\end{tikzcd}$$

whereas the structure 3-cells of the lax functor H_*F attached to an arrow a: $i \to j$ of the category $\mathcal{I}$, are respectively those obtained by pasting the diagrams below.

$$\begin{tikzcd}
& H1_{Fj}\otimes HFa \arrow[dl, Rightarrow, "HF_j\otimes 1"'] \arrow[dr, Rightarrow, "\chi"'] & & 1_{HFj}\otimes HFa \arrow[ll, Rightarrow, "\iota\otimes 1"'] \arrow[dd, Rightarrow, "\boldsymbol{l}"] \\
\cong & & H(1_{Fj}\otimes Fa) \arrow[dl, Rightarrow, "H(F_j\otimes 1)"'] \arrow[dr, Rightarrow, "H\boldsymbol{l}"] & \overset{\gamma}{\Rrightarrow} \\
HF1_j\otimes HFa \arrow[r, Rightarrow, "\chi"] & H(F1_j\otimes Fa) \arrow[rr, Rightarrow, "HF_{1,a}"] & \overset{H\widehat{F}_a}{\Rrightarrow} & HFa,
\end{tikzcd}$$

$$\begin{tikzcd}
& HFa\otimes H1_{Fi} \arrow[dl, Rightarrow, "1\otimes HF_i"'] \arrow[dr, Rightarrow, "\chi"'] & & HFa\otimes 1_{HFi} \arrow[ll, Rightarrow, "1\otimes\iota"'] \arrow[dd, Rightarrow, "\boldsymbol{r}^\bullet"] \\
\cong & & H(Fa\otimes 1_{Fi}) \arrow[dl, Rightarrow, "H(1\otimes F_i)"'] \arrow[dr, Rightarrow, "H\boldsymbol{r}^\bullet"] & \overset{\delta^{-1}}{\Rrightarrow} \\
HFa\otimes HF1_i \arrow[r, Rightarrow, "\chi"] & H(Fa\otimes F1_i) \arrow[rr, Rightarrow, "HF_{a,1}"] & \overset{H\widetilde{F}_a}{\Rrightarrow} & HFa.
\end{tikzcd}$$

If Φ: $F \Rightarrow G$ is any 1-cell in the bicategory $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$, then $H_*\Phi$: $H_*F \Rightarrow H_*G$ is the 1-cell in $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T}')$ whose component at an arrow a: $i \to j$ of $\mathcal{I}$ is the 2-cell of $\mathcal{T}'$ defined by $(H_*\Phi)a = H\Phi a$: $HFa \Rightarrow HGa$. For any pair of composable arrows $i \xrightarrow{b} j \xrightarrow{a} k$ and any object i of $\mathcal{I}$, the corresponding structure 3-cells (2.3) and (2.4), $(H_*\Phi)_{a,b}$ and $(H_*\Phi)_i$, are respectively given by pasting in

$$\begin{tikzcd}
HFa\otimes HFb \arrow[r, Rightarrow, "\chi"] \arrow[d, Rightarrow, "H\Phi a\otimes H\Phi b"'] & H(Fa\otimes Fb) \arrow[r, Rightarrow, "HF_{a,b}"] \arrow[d, Rightarrow, "H(\Phi a\otimes\Phi b)"] \arrow[dl, phantom, "\cong"] & HF(ab) \arrow[d, Rightarrow, "H\Phi(ab)"] \arrow[dl, phantom, "\overset{H\Phi_{a,b}}{\Rrightarrow}"] \\
HGa\otimes HGb \arrow[r, Rightarrow, "\chi"'] & H(Ga\otimes Gb) \arrow[r, Rightarrow, "HG_{a,b}"'] & HG(ab),
\end{tikzcd}$$

$$\begin{tikzcd}
& 1_{HFi=HGi} \arrow[d, Rightarrow, "\iota"] & \\
& H1_{Fi} \arrow[dl, Rightarrow, "HF_i"'] \arrow[dr, Rightarrow, "HG_i"] & \\
HF1_i \arrow[rr, Rightarrow, "H\Phi_i"'] & \overset{H\Phi_i}{\Lleftarrow} & HG1_i.
\end{tikzcd}$$

And a 2-cell M: $\Phi \Rrightarrow \Psi$ of $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$ is applied by the homomorphism H_* to the 2-cell H_*M: $H_*\Phi \Rrightarrow H_*\Psi$ of $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T}')$, such that $(H_*M)a = HMa$: $H\Phi a \Rrightarrow H\Psi a$ for any arrow a: $i \to j$ of the category $\mathcal{I}$.

Finally, if $\Phi\colon F \Rightarrow G$ and $\Psi\colon G \Rightarrow H$ are any two composable 1-cells in $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$, and $F\colon \mathcal{I} \to \mathcal{T}$ is any lax functor, then the constraints $(H_*\Psi) \circ (H_*\Phi) \cong H_*(\Psi \circ \Phi)$ and $1_{H_*F} \cong H_*1_F$ are, at any arrow $a\colon i \to j$ of $\mathcal{I}$, the structure isomorphisms $H\Psi a \circ H\Phi a \cong H(\Psi a \circ \Phi a)$ and $1_{HFa} \cong H1_{Fa}$ of the homomorphism $H\colon \mathcal{T}(Fi, Fj) \to \mathcal{T}'(HFi, HFj)$, respectively.

If $\alpha\colon \mathcal{I} \to \mathcal{J}$ is a functor then, recalling the definition of α^* given at the beginning of Subsection 2.2.2, one easily checks the equality $H_*\alpha^* = \alpha^* H_*$.

(*ii*) For any lax functor $F\colon \mathcal{I} \to \mathcal{T}$, the 2-cell attached by

$$m = m_F\colon H'_*(H_*F) \Rightarrow (H'H)_*F$$

at any arrow $a\colon i \to j$ of $\mathcal{I}$ is the identity, that is, $ma = 1_{H'HFa}$. For any pair of composable arrows $i \xrightarrow{b} j \xrightarrow{a} k$ and any object i of $\mathcal{I}$, the corresponding invertible structure 3-cells (2.3) and (2.4),

$$\begin{aligned} &m_{a,b}\colon (H'_*(H_*F))_{a,b} \circ (ma \otimes mb) \Rrightarrow m(ab) \circ ((H'H)_*F)_{a,b},\\ &m_i\colon m1_i \circ (H'_*(H_*F))_i \Rrightarrow ((H'H)_*F)_i, \end{aligned}$$

are, respectively, given by pasting in the diagrams below.

$$\begin{tikzcd}
H'HFa \otimes H'HFb \arrow[r, Rightarrow, "\chi"] \arrow[d, Rightarrow, "1\otimes 1"'] \arrow[dr, phantom, "\cong"] & H'(HFa \otimes HFb) \arrow[rr, Rightarrow, "H'(HF_{a,b}\circ\chi)"] \arrow[d, Rightarrow, "1"'] \arrow[dr, Rightarrow, "H'\chi"] \arrow[d, phantom, shift left=4, "\cong"] & \arrow[d, phantom, "\cong"] & H'HF(ab) \arrow[d, Rightarrow, "1"] \arrow[dl, phantom, shift left=2, "\cong"] \\
H'HFa \otimes H'HFb \arrow[r, Rightarrow, "\chi"'] & H'(HFa \otimes HFb) \arrow[r, Rightarrow, "H'\chi"'] & H'H(Fa \otimes Fb) \arrow[r, Rightarrow, "H'HF_{a,b}"'] \arrow[ur, Rightarrow, "H'HF_{a,b}"] & H'HF(ab),
\end{tikzcd}$$

$$\begin{tikzcd}
1_{H'HFi} \arrow[r, Rightarrow, "\iota"] \arrow[dr, Rightarrow, "\iota"'] \arrow[r, phantom, shift right=4, "\cong"] & H'1_{HFi} \arrow[rr, Rightarrow, "H'(HF_i\circ\iota)"] \arrow[d, Rightarrow, "1"'] \arrow[dr, Rightarrow, "H'\iota"] \arrow[d, phantom, shift left=4, "\cong"] & \arrow[d, phantom, "\cong"] & H'HF1_i \arrow[d, Rightarrow, "1"] \arrow[dl, phantom, shift left=2, "\cong"] \\
 & H'1_{HFi} \arrow[r, Rightarrow, "H'\iota"'] & H'H1_{Fi} \arrow[r, Rightarrow, "H'HF_i"'] \arrow[ur, Rightarrow, "H'HF_i"] & H'HF1_i.
\end{tikzcd}$$

If $\Phi\colon F \Rightarrow G$ any 1-cell in $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$, then the invertible naturality 2-cell $m_\Phi\colon m_G \circ (H'_*(H_*\Phi)) \Rrightarrow (H'H)_*\Phi \circ m_F$, at any arrow $a\colon i \to j$ of $\mathcal{I}$, is the canonical isomorphism $1 \circ H'H\Phi a \cong H'H\Phi a \circ 1$ in the bicategory $\mathcal{T}''(H'HFi, H'HFj)$.

For a functor $\alpha\colon \mathcal{I} \to \mathcal{J}$, it is easy to see that $m\alpha^* = \alpha^* m$.

(*iii*) For any lax functor $F\colon \mathcal{I} \to \mathcal{T}$, the 2-cell attached by

$$m = m_F\colon (1_{\mathcal{T}})_*F \Rightarrow F$$

at any arrow $a\colon i \to j$ of $\mathcal{I}$ is the identity, that is, $ma = 1_{Fa}$. For any pair of composable arrows $i \xrightarrow{b} j \xrightarrow{a} k$ and any object i of $\mathcal{I}$, the corresponding invertible structure 3-cells (2.3) and (2.4),

$$m_{a,b}\colon F_{a,b} \circ (ma \otimes mb) \Rrightarrow m(ab) \circ ((1_{\mathcal{T}})_*F)_{a,b}, \quad m_i\colon m1_i \circ ((1_{\mathcal{T}})_*F)_i \Rrightarrow F_i,$$

are, respectively, the canonical isomorphisms in the diagrams below.

$$\begin{tikzcd}
Fa\otimes Fb \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1\otimes 1"'] & Fa\otimes Fb \arrow[r, Rightarrow, "F_{a,b}"] \arrow[d, phantom, "\cong"] & F(ab) \arrow[d, Rightarrow, "1"] \\
Fa\otimes Fb \arrow[rr, Rightarrow, "F_{a,b}"] & & F(ab),
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1_{Fi} \arrow[dl, Rightarrow, "1"'] \arrow[dr, Rightarrow, "F_i"] \arrow[d, phantom, "\cong"] & \\
1_{Fi} \arrow[r, Rightarrow, "F_i"] & F1_i \arrow[r, Rightarrow, "1"] & F1_i.
\end{tikzcd}$$

If $\Phi\colon F\Rrightarrow G$ any 1-cell in $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{T})$, then the invertible naturality 2-cell $m_\Phi\colon m_G\circ((1_{\mathcal{T}})_*\Phi))\Rrightarrow \Phi\circ m_F$, at any arrow $a\colon i\to j$ of I, is provided by the canonical isomorphism $1\circ\Phi a\cong\Phi a\circ 1$ in $\mathcal{T}(Fi,Fj)$.

Again, for $\alpha\colon\mathcal{I}\to\mathcal{J}$ a functor, it is straightforward to check that the equality $m\alpha^*=\alpha^*m$ holds. □

2.6.3 Proof of Lemma 2.2

To describe the homomorphism L, we shall use the following useful construction: For any list $(t_0,\dots,t_p)$ of objects in the tricategory $\mathcal{T}$, let

$$\overset{\text{or}}{\otimes}\colon \mathcal{T}(t_{p-1},t_p)\times\mathcal{T}(t_{p-2},t_{p-1})\times\cdots\times\mathcal{T}(t_0,t_1)\longrightarrow\mathcal{T}(t_0,t_p) \tag{2.43}$$

denote the homomorphism recursively defined as the composite

$$\mathcal{T}(t_{p-1},t_p)\times\prod_{i=1}^{p-1}\mathcal{T}(t_{i-1},t_i)\xrightarrow{1\times\overset{\text{or}}{\otimes}}\mathcal{T}(t_{p-1},t_p)\times\mathcal{T}(t_0,t_{p-1})\xrightarrow{\otimes}\mathcal{T}(t_0,t_p).$$

That is, $\overset{\text{or}}{\otimes}$ is the homomorphism obtained by iterating composition in the tricategory, which acts on 0-cells, 1-cells and 2-cells of the product bicategory $\prod_{i=1}^{p}\mathcal{T}(t_{i-1},t_i)$ by the recursive formula

$$\overset{\text{or}}{\otimes}(x_p,\dots,x_1)=\begin{cases} x_1 & \text{if}\quad p=1,\\ x_p\otimes\big(\overset{\text{or}}{\otimes}(x_{p-1},\dots,x_1)\big) & \text{if}\quad p\geq 2.\end{cases}$$

- *The definition of* L *on* 0*-cells* . The homomorphism L takes a lax functor of the graph in the tricategory, say $f\colon\mathcal{G}\to\mathcal{T}$, to the unitary homomorphic lax functor from the free category

$$L(f)=F\colon\mathcal{I}\to\mathcal{T},$$

such that $Fi=fi$, for any vertex i of $\mathcal{G}$ (= objects of $\mathcal{I}$), and associates to strings $a\colon a(0)\overset{a_1}{\to}\cdots\overset{a_p}{\to}a(p)$ in $\mathcal{G}$ the 1-cells $Fa=\overset{\text{or}}{\otimes}(fa_p,\dots,fa_1)\colon fa(0)\to fa(p)$. The structure 2-cells $F_{a,b}\colon Fa\otimes Fb\Rightarrow F(ab)$, for any pair of strings in the graph, $a=a_p\cdots a_1$ as above and $b=b_q\cdots b_1$ with $b(q)=a(0)$, are canonically obtained from the associativity constraint in the tricategory: first by taking $F_{a_1,b}=1_{F(a_1b)}$ and then, recursively for $p>1$, defining $F_{a,b}$ as the composite

$$F_{a,b}\colon\ Fa\otimes Fb\xRightarrow{\boldsymbol{a}}Fa_p\otimes(Fa'\otimes Fb)\xRightarrow{1\otimes F_{a',b}}F(ab),$$

where $a' = a_{p-1} \cdots a_1$ (whence $Fa = Fa_p \otimes Fa'$). And the structure 3-cells $F_{a,b,c}$, for any three strings in the graph a, b and c as above with $a(0) = b(q)$ and $b(0) = c(r)$, are the unique isomorphisms constructed from the tricategory coherence 3-cells π. For a particular construction of these isomorphisms, we can first take each $F_{a_1,b,c}$ to be the canonical isomorphism

$$\begin{tikzcd}
 & (Fa_1 \otimes Fb) \otimes Fc \arrow[rrr, Rightarrow, "\boldsymbol{a}"] \arrow[d, Rightarrow, "1\otimes 1"'] & & & Fa_1 \otimes (Fb \otimes Fc) \arrow[d, Rightarrow, "1\otimes F_{b,c}"] \\
F_{a_1,b,c}\colon & F(a_1b) \otimes Fc \arrow[r, Rightarrow, "\boldsymbol{a}"] & Fa_1 \otimes (Fb \otimes Fc) \arrow[r, Rightarrow, "1\otimes F_{b,c}"] \arrow[u, phantom, "\cong"] & F(a_1bc) & Fa_1 \otimes F(bc), \arrow[l, Rightarrow, "1"']
\end{tikzcd}$$

and then, recursively for $p > 1$, take $F_{a,b,c}$ to be the 3-cell canonically obtained from $F_{a_{p-1}\cdots a_1,b,c}$ by pasting the diagram bellow, where, as above, we write a' for $a_{p-1} \cdots a_1$.

$$\begin{tikzcd}[column sep=tiny]
 & & & (Fa\otimes Fb)\otimes Fc \arrow[dll, Rightarrow, "\boldsymbol{a}\otimes 1"'] \arrow[drrr, Rightarrow, "\boldsymbol{a}"] & & & & \\
 & (Fa_p\otimes(Fa'\otimes Fb))\otimes Fc \arrow[drr, Rightarrow, "\boldsymbol{a}"] \arrow[ddd, Rightarrow, "(1\otimes F_{a',b})\otimes 1"'] & & & \overset{\pi}{\Rrightarrow} & & Fa\otimes(Fb\otimes Fc) \arrow[dl, Rightarrow, "\boldsymbol{a}"'] \arrow[dr, Rightarrow, "1\otimes F_{b,c}"] & \\
 & & & Fa_p\otimes((Fa'\otimes Fb)\otimes Fc) \arrow[dr, Rightarrow, "1\otimes\boldsymbol{a}"] \arrow[ddd, Rightarrow, "1\otimes(F_{a',b}\otimes 1)"] & & & & \\
F_{a,b,c}\colon & & \cong & & & Fa_p\otimes(Fa'\otimes(Fb\otimes Fc)) \arrow[ddr, Rightarrow, "1\otimes(1\otimes F_{b,c})"] & \cong & Fa\otimes F(bc) \arrow[ddl, Rightarrow, "\boldsymbol{a}"] \\
 & F(ab)\otimes Fc \arrow[drr, Rightarrow, "\boldsymbol{a}"'] & & & \overset{1\otimes F_{a',b,c}}{\Rrightarrow} & & & \\
 & & & Fa_p\otimes(F(a'b)\otimes Fc) \arrow[drr, Rightarrow, "1\otimes F_{a'b,c}"'] & & & Fa_p\otimes(Fa'\otimes F(bc)) \arrow[dl, Rightarrow, "1\otimes F_{a',bc}"] & \\
 & & & & & F(abc) & &
\end{tikzcd}$$

The conditions **(CR1)**, **(CR2)**, and **(CR3)**, are verified thanks to Fact 2.2, since we are only using constraints 2-cells and 3-cells. Note that, since all structure 2-cells $F_{a,b}$ are equivalences in the corresponding hom-bicategories of $\mathcal{T}$ in which they lie, as well as all the structure 3-cells $F_{a,b,c}$ are invertible, the so defined unitary lax functor $F\colon \mathcal{I} \to \mathcal{T}$ is actually a homomorphic one; that is, $L(f) = F \in \mathbb{L}\mathrm{ax}_{\mathrm{uh}}(\mathcal{I}, \mathcal{T}) \subseteq \mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{T})$.

- *The definition of L on 1-cells* . Any 1-cell $\phi\colon f \Rightarrow g$ of $\mathbb{L}\mathrm{ax}(\mathcal{G}, \mathcal{T})$, is taken by L to the 1-cell in $\mathbb{L}\mathrm{ax}_{\mathrm{uh}}(\mathcal{I}, \mathcal{T})$

$$L(\phi) = \Phi\colon F \Rightarrow G,$$

consisting of the 2-cells in the tricategory $\Phi a = \overset{\mathrm{or}}{\otimes}(\phi a_p, \ldots, \phi a_1)\colon Fa \Rightarrow Ga$, attached to the strings of adjacent edges in the graph $a = a_p \cdots a_1$. The structure (actually invertible) 3-cells $\Phi_{a,b}$, for any pair of strings in the graph, a and b with $b(q) = a(0)$ as above, are defined by induction on the length of a as follows: each $\Phi_{a_1,b}$ is the canonical isomorphism

$$\begin{tikzcd}
 & Fa_1 \otimes Fb \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "\Phi a_1\otimes\Phi b"'] \arrow[dr, phantom, "\cong"] & F(a_1b) \arrow[d, Rightarrow, "\Phi(a_1b)=\Phi a_1\otimes\Phi b"] \\
\Phi_{a_1,b}\colon & Ga_1 \otimes Gb \arrow[r, Rightarrow, "1"] & G(a_1b),
\end{tikzcd}$$

and, for $p > 1$, each $\Phi_{a,b}$ is obtained from $\Phi_{a',b}$, where $a' = a_{p-1}\cdots a_1$, by pasting

$$\Phi_{a,b}:\quad \begin{tikzcd}
Fa\otimes Fb \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[d, Rightarrow, "\Phi a\otimes\Phi b"'] \arrow[dr, phantom, "\cong"] & Fa_p\otimes(Fa'\otimes Fb) \arrow[r, Rightarrow, "1\otimes F_{a',b}"] \arrow[d, Rightarrow, "\Phi a_p\otimes(\Phi a'\otimes\Phi b)"'] \arrow[dr, phantom, "\overset{1\otimes\Phi_{a',b}}{\Rrightarrow}"] & F(ab) \arrow[d, Rightarrow, "\Phi(ab)=\Phi a_p\otimes(\Phi a'\otimes\Phi b)"] \\
Ga\otimes Gb \arrow[r, Rightarrow, "\boldsymbol{a}"'] & Ga_p\otimes(Ga'\otimes Gb) \arrow[r, Rightarrow, "1\otimes G_{a',b}"'] & G(ab).
\end{tikzcd}$$

Again, Fact 2.2 ensures that conditions **(CR4)** and **(CR5)** are satisfied. Note that, since all structure 3-cells $\Phi_{a,b}$ are invertible, the thus defined unitary 1-cell $L(\phi) = \Phi$ of $\mathbb{L}\mathrm{ax}_{\mathrm{u}}(\mathcal{I},\mathcal{T})$ is actually a 1-cell of $\mathbb{L}\mathrm{ax}_{\mathrm{uh}}(\mathcal{I},\mathcal{T})$.

- *The definition of L on 2-cells* . For ϕ, ψ: $f \Rightarrow g$, any two 1-cells in $\mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{T})$, the homomorphism L on a 2-cell m: $\phi \Rrightarrow \psi$ gives the 2-cell of $\mathbb{L}\mathrm{ax}_{\mathrm{uh}}(\mathcal{I},\mathcal{T})$

$$L(m) = M\colon \Phi \Rrightarrow \Psi,$$

consisting of the 3-cells in the tricategory $Ma = \overset{\text{or}}{\otimes}(ma_p,\dots,ma_1)\colon \Phi a \Rightarrow \Psi a$, for the strings $a = a_p\cdots a_1$ of adjacent edges in the graph $\mathcal{G}$.

- *The structure constraints of L.* If ϕ: $f \Rightarrow g$ and ψ: $g \Rightarrow h$ are 1-cells in $\mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{T})$, then the structure isomorphism in $\mathbb{L}\mathrm{ax}_{\mathrm{u}}(I,\mathcal{T})$

$$L_{\psi,\phi}\colon L(\psi)\circ L(\phi) \cong L(\psi\circ\phi),$$

at each string $a = a_p\cdots a_1$ as above, is recursively defined as the identity 3-cell on $\psi a_1 \circ \phi a_1$ if $p = 1$, while, for $p > 1$, $L_{\psi,\phi}\, a$: $L(\psi)a \circ L(\phi)a \Rrightarrow L(\psi\circ\phi)a$ is obtained from $L_{\psi,\phi}\, a'$, where $a' = a_{p-1}\cdots a_1$, as the composite

$$\begin{aligned} L(\psi)a\circ L(\phi)a &= (\psi a_p \otimes L(\psi)a')\circ(\phi a_p\otimes L(\phi)a') \\ &\cong (\psi a_p\circ\phi a_p)\otimes(L(\psi)a'\circ L(\phi)a') \\ &\overset{1\otimes L_{\psi,\phi}a'}{\Rrightarrow} (\psi a_p\circ\phi a_p)\otimes L(\psi\circ\phi)a' = L(\psi\circ\phi)a. \end{aligned}$$

Similarly, the structure isomorphism L_f: $1_{L(f)} \cong L(1_f)$ consists of the 3-cells $L_f\, a$: $1_{L(f)a} \Rrightarrow L(1_f)a$, where $L_f a_1 = 1$: $1_{fa_1} \Rrightarrow 1_{fa_1}$ and, for $p > 1$, $L_f a$ is recursively obtained from $L_f a'$, $a' = a_{p-1}\cdots a_1$, as the composite

$$1_{L(f)a} = 1_{fa_p\otimes L(f)a'} \cong 1_{fa_p}\otimes 1_{L(f)a'} \overset{1\otimes L_f a'}{\Rrightarrow} 1_{fa_p}\otimes L(1_f)a' = L(1_f)a.$$

This completes the description of the homomorphism L.

- *The definition of the lax transformation* v. The component of this lax transformation at a lax functor F: $I \to \mathcal{T}$, $\mathrm{v} = \mathrm{v}(F)$: $LR(F) \Rightarrow F$, is defined on identities by

$$\mathrm{v}1_i = F_i\colon 1_{Fi} \Rightarrow F1_i,$$

for any vertex i of $\mathcal{G}$, and it associates to each string of adjacent edges in the graph $a = a_p \cdots a_1$ the 2-cell

$$\mathrm{v}a\colon \overset{\mathrm{or}}{\otimes}(Fa_p, \dots, Fa_1) \Rightarrow Fa, \tag{2.44}$$

which is given by taking $\mathrm{v}a_1 = 1_{Fa_1}$ if $p = 1$, and then, recursively for $p > 1$, by taking $\mathrm{v}a$ as the composite $\mathrm{v}a\colon\ \overset{\mathrm{or}}{\otimes}(Fa_p, \dots, Fa_1) \xRightarrow{1\otimes \mathrm{v}a'} Fa_p \otimes Fa' \xRightarrow{F_{a_p,a'}} Fa$, where $a' = a_{p-1} \cdots a_1$.

The structure 3-cell

$$\mathrm{v}_{a,b}\colon F_{a,b} \circ (\mathrm{v}a \otimes \mathrm{v}b) \Rrightarrow \mathrm{v}(ab) \circ LR(F)_{a,b}, \tag{2.45}$$

for any pair of composable morphisms in $\mathcal{I}$, is defined as follows: when $a = 1_j$ or $b = 1_i$ are identities, then $\mathrm{v}_{1_j,b}$ and $\mathrm{v}_{a,1_i}$ are respectively given by pasting the diagrams

$$\begin{tikzcd}
\mathrm{v}_{1_j,b}: & 1_{Fj}\otimes LR(F)b \arrow[rr, Rightarrow, "\boldsymbol{l}"] \arrow[dd, Rightarrow, "F_j\otimes \mathrm{v}b"'] \arrow[dr, Rightarrow, "1\otimes \mathrm{v}b"] & & LR(F)b \arrow[dd, Rightarrow, "\mathrm{v}b"] \\
& \cong & 1_{Fj}\otimes Fb \arrow[dl, Rightarrow, "F_j\otimes 1"'] \arrow[dr, Rightarrow, "\boldsymbol{l}"] \arrow[d, phantom, "\overset{\widehat{F}_b}{\Rrightarrow}"] & \cong \\
& F1_j\otimes Fb \arrow[rr, Rightarrow, "F_{1_j,b}"'] & & Fb,
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{v}_{a,1_i}: & LR(F)a\otimes 1_{Fi} \arrow[rr, Rightarrow, "\boldsymbol{r}^\bullet"] \arrow[dd, Rightarrow, "\mathrm{v}a\otimes F_i"'] \arrow[dr, Rightarrow, "\mathrm{v}a\otimes 1"] & & LR(F)a \arrow[dd, Rightarrow, "\mathrm{v}a"] \\
& \cong & Fa\otimes 1_{Fi} \arrow[dl, Rightarrow, "1\otimes F_i"'] \arrow[dr, Rightarrow, "\boldsymbol{r}^\bullet"] \arrow[d, phantom, "\overset{\widetilde{F}_a}{\Rrightarrow}"] & \cong \\
& Fa\otimes F1_i \arrow[rr, Rightarrow, "F_{a,1_i}"'] & & Fa,
\end{tikzcd}$$

and, for strings a and b in the graph with $b(q) = a(0)$, $\mathrm{v}_{a,b}$ is defined by induction on the length of a by taking $\mathrm{v}_{a_1,b}$ to be the canonical isomorphism

$$\begin{tikzcd}
\mathrm{v}_{a_1,b}: & Fa_1\otimes LR(F)b \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1\otimes \mathrm{v}b"'] \arrow[dr, phantom, "\cong"] & LR(F)(a_1b) \arrow[d, Rightarrow, "\mathrm{v}(a_1b)=F_{a_1,b}\circ(1\otimes \mathrm{v}b)"] \\
& Fa_1\otimes Fb \arrow[r, Rightarrow, "F_{a_1,b}"'] & F(a_1b),
\end{tikzcd}$$

and then, for $p > 1$, $\mathrm{v}_{a,b}$ is obtained from $\mathrm{v}_{a',b}$, where $a' = a_{p-1} \cdots a_1$, by pasting

$$\begin{tikzcd}
& LR(F)a\otimes LR(F)b \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[d, Rightarrow, "(1\otimes \mathrm{v}a')\otimes \mathrm{v}b"'] \arrow[dr, phantom, "\cong"] & Fa_p\otimes(LR(F)a'\otimes LR(F)b) \arrow[r, Rightarrow, "1\otimes LR(F)_{a',b}"] \arrow[d, Rightarrow, "1\otimes(\mathrm{v}a'\otimes \mathrm{v}b)"] \arrow[dr, phantom, "\overset{1\otimes \mathrm{v}_{a',b}}{\Rrightarrow}"] & LR(F)(ab) \arrow[d, Rightarrow, "1\otimes \mathrm{v}(a'b)"] \\
\mathrm{v}_{a,b}: & (Fa_p\otimes Fa')\otimes Fb \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[d, Rightarrow, "F_{a_p,a'}\otimes 1"'] & Fa_p\otimes(Fa'\otimes Fb) \arrow[r, Rightarrow, "1\otimes F_{a',b}"'] \arrow[d, phantom, "\overset{F_{a_p,a',b}}{\Rrightarrow}"] & Fa_p\otimes F(a'b) \arrow[d, Rightarrow, "F_{a_p,a'b}"] \\
& Fa\otimes Fb \arrow[rr, Rightarrow, "F_{a,b}"] & & F(ab).
\end{tikzcd}$$

And the structure 3-cell

$$\mathrm{v}_i\colon \mathrm{v}1_i \circ LR(F)_i \Rrightarrow F_i, \tag{2.46}$$

for any vertex i of the graph, is the canonical isomorphism $F_i \circ 1 \cong F_i$. Conditions **(CR4)** and **(CR5)** are verified by using conditions **(CR1)**, **(CR2)**, and **(CR3)** for F, and Facts 2.2, 2.3, and 2.1.

The naturality component of v at a 1-cell $\Phi\colon F \Rightarrow G$ in $\mathbb{L}\mathrm{ax}_{\mathrm{u}}(\mathcal{I}, \mathcal{T})$,

$$\mathrm{v}_{\Phi}\colon \mathrm{v}(G) \circ LR(\Phi) \Rrightarrow \Phi \circ \mathrm{v}(F), \tag{2.47}$$

is given on identities by

$$\begin{tikzcd}
\mathrm{v}_{\Phi}1_i : & 1_{Fi} \arrow[r, Rightarrow, "F_i"] \arrow[d, Rightarrow, "1"'] \arrow[dr, Rightarrow, "G_i"' near start] \arrow[d, phantom, shift left=4ex, "\cong"] & F1_i \arrow[d, Rightarrow, "\Phi 1_i"] \arrow[dl, phantom, shift right=3ex, "\overset{\Phi_i}{\Rrightarrow}"] \\
& 1_{Fi} \arrow[r, Rightarrow, "G_i"'] & G1_i,
\end{tikzcd}$$

and it is recursively defined at each string in the graph $a = a_p \cdots a_1$, by the 3-cells $\mathrm{v}_{\Phi}a$, where

$$\begin{tikzcd}
\mathrm{v}_{\Phi}a_1 : & Fa_1 \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "\Phi a_1"'] \arrow[dr, phantom, "\cong"] & Fa_1 \arrow[d, Rightarrow, "\Phi a_1"] \\
& Ga_1 \arrow[r, Rightarrow, "1"] & Ga_1,
\end{tikzcd}$$

and then, when $p > 1$, $\mathrm{v}_{\Phi}a$ is obtained from $\mathrm{v}_{\Phi}a'$, where $a' = a_{p-1} \cdots a_1$, by pasting

$$\begin{tikzcd}
\mathrm{v}_{\Phi}a : & Fa_p \otimes LR(F)a' \arrow[r, Rightarrow, "1\otimes \mathrm{v}(F)a'"] \arrow[d, Rightarrow, "\Phi a_p \otimes LR(\Phi)a'"'] \arrow[dr, phantom, "\overset{1\otimes \mathrm{v}_{\Phi}a'}{\Rrightarrow}"] & Fa_p \otimes Fa' \arrow[r, Rightarrow, "F_{a_p,a'}"] \arrow[d, Rightarrow, "\Phi a_p \otimes \Phi a'"] \arrow[dr, phantom, "\overset{\Phi_{a_p,a'}}{\Rrightarrow}"] & Fa \arrow[d, Rightarrow, "\Phi a"] \\
& Ga_p \otimes LR(G)a' \arrow[r, Rightarrow, "1\otimes \mathrm{v}(G)a'"] & Ga_p \otimes Ga' \arrow[r, Rightarrow, "G_{a_p,a}"] & Ga.
\end{tikzcd}$$

Condition **(CR6)** for this 2-cell is verified using conditions **(CR4)** and **(CR5)** for the 1-cell Φ, together with Facts 2.2, 2.3, and 2.1.

We are now ready to complete the proof. That the equalities $RL = 1$, $\mathrm{v}L = 1$, and $R\mathrm{v} = 1$ hold only requires a straightforward verification, and then (a) follows. Moreover, (b) has already been shown by construction of the homomorphism L.

- *The proof of* (c). Suppose that $F\colon \mathcal{I} \to \mathcal{T}$ is any homomorphic lax functor. This means that all structure 2-cells $F_{a,b}$ and F_i are equivalences, and 3-cells $F_{a,b,c}$, $\widehat{F}_a$, and $\widetilde{F}_a$ are isomorphisms in the hom-bicategories of $\mathcal{T}$ in which they lie. Then, directly from the construction given, it easily follows that all the 2-cells $\mathrm{v}(F)a$ in (2.44) are equivalences in the corresponding hom-bicategories, and that all the 3-cells $\mathrm{v}(F)_{a,b}$ in (2.45), and v_i in (2.46) are invertible. Hence, each $\mathrm{v}(F)\colon LR(F) \Rightarrow F$, for $F\colon \mathcal{I} \to \mathcal{T}$ any homomorphic lax functor, is an equivalence in the bicategory $\mathbb{L}\mathrm{ax}_{\mathrm{h}}(\mathcal{I}, \mathcal{T})$. Moreover, if $\Phi\colon F \Rightarrow G$ is any 1-cell in $\mathbb{L}\mathrm{ax}_{\mathrm{h}}(\mathcal{I}, \mathcal{T})$, so that every 3-cell $\Phi_{a,b}$ and Φ_i is an isomorphism, then we see that the component (2.47) of v at Φ consists only of invertible 3-cells $\mathrm{v}_{\Phi}a$, whence v_{Φ} is invertible itself. Therefore, when v is restricted to $\mathbb{L}\mathrm{ax}_{\mathrm{h}}(\mathcal{I}, \mathcal{T})$, it actually gives a pseudo-equivalence between LR and 1, the identity homomorphism on the bicategory $\mathbb{L}\mathrm{ax}_{\mathrm{h}}(\mathcal{I}, \mathcal{T})$. The claimed biadjoint biequivalence (2.7) is now an easy consequence of all the already parts proved. Finally, it is clear that the biadjoint biequivalence (2.7) gives by restriction the biadjoint biequivalence (2.8). □

Chapter 3

Bicategorical homotopy fiber sequences

3.1 Introduction and summary

The process of taking classifying spaces of bicategories reveals a way to transport categorical coherence to homotopical coherence since the construction $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$ preserves products, any lax or oplax functor between bicategories, $F : \mathcal{A} \to \mathcal{B}$, induces a continuous map on classifying spaces $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$, any lax or oplax transformation between these, $\alpha : F \Rightarrow F'$, induces a homotopy between the corresponding induced maps $\mathrm{B}\alpha : \mathrm{B}F \Rightarrow \mathrm{B}F'$, and any modification between these, $\varphi : \alpha \Rrightarrow \beta$, a homotopy $\mathrm{B}\varphi : \mathrm{B}\alpha \Rrightarrow \mathrm{B}\beta$ between them. Thus, if $\mathcal{A}$ and $\mathcal{B}$ are biequivalent bicategories or if a homomorphism $\mathcal{A} \to \mathcal{B}$ has a biadjoint, then their associated classifying spaces are homotopy equivalent.

In this chapter we show the subtlety of this theory by analyzing the homotopy fibers of the map $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$, which is induced by a lax functor between small bicategories $F : \mathcal{A} \to \mathcal{B}$, such as Quillen did in [109] where he stated his celebrated Theorems A and B for the classifying spaces of small categories. Every object $b \in \mathrm{Ob}\mathcal{B}$ has an associated *homotopy fiber bicategory* $F{\downarrow}_b$ whose objects are the 1-cells $f : Fa \to b$ in $\mathcal{B}$, with a an object of $\mathcal{A}$; the 1-cells consist of all triangles

$$\begin{tikzcd}
Fa \arrow[rr, "Fu"] \arrow[dr, "f"'] & \overset{\beta}{\Rightarrow} & Fa' \arrow[dl, "f'"] \\
& b &
\end{tikzcd}$$

with $u : a \to a'$ a 1-cell in $\mathcal{A}$ and $\beta : f \Rightarrow f' \circ Fu$ a 2-cell in $\mathcal{B}$, and the 2-cells of this bicategory are commutative diagrams of 2-cells in $\mathcal{B}$ of the form

$$\begin{tikzcd}
& f \arrow[dl, Rightarrow, "\beta"'] \arrow[dr, Rightarrow, "\beta'"] & \\
f' \circ Fu \arrow[rr, Rightarrow, "1_{f'} \circ F\alpha"] & & f' \circ Fu'
\end{tikzcd}$$

with $\alpha : u \Rightarrow u'$ a 2-cell in $\mathcal{A}$. Compositions, identities, and the structure associativity and unit constraints in $F{\downarrow}_b$ are canonically provided by those of the involved bicategories and the structure 2-cells of the lax functor (see Section 3.5 for details). For the case $F = 1_{\mathcal{B}}$, we have the *comma* bicategory $\mathcal{B}{\downarrow}_b$. Then, we prove (see Theorem 3.2):

> *"For every object b of the bicategory $\mathcal{B}$, the induced square*
>
> $$\begin{tikzcd}
> \mathrm{B}(F{\downarrow}_b) \arrow[r] \arrow[d] & \mathrm{B}(\mathcal{B}{\downarrow}_b) \arrow[d] \\
> \mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B}
> \end{tikzcd}$$
>
> *is homotopy cartesian if and only if all the maps $\mathrm{B}p : \mathrm{B}(F{\downarrow}_b) \to \mathrm{B}(F{\downarrow}_{b'})$, induced by the 1-cells $p : b \to b'$ of $\mathcal{B}$, are homotopy equivalences."*

Since the spaces $\mathrm{B}(\mathcal{B}{\downarrow}_b)$ are contractible (Lemma 3.7), the result above tells us that, under the minimum necessary conditions, the classifying space of the homotopy fiber bicategory $F{\downarrow}_b$ is homotopy equivalent to the homotopy fiber of $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ at its 0-cell $\mathrm{B}b \in \mathrm{B}\mathcal{B}$. Thus, the name 'homotopy fiber bicategory' is well chosen. Furthermore, as a corollary, we obtain (see Theorem 3.3):

> *"If all the spaces $\mathrm{B}(F{\downarrow}_b)$ are contractible, then the map $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ is a homotopy equivalence."*

When the bicategories $\mathcal{A}$ and $\mathcal{B}$ involved in the results above are actually categories, then they are reduced to the well-known Theorems A and B by Quillen [109]. Indeed, the methods used in the proof of Theorem 3.2 we give follow similar lines to those used by Quillen in his proof of Theorem B. However, the situation with bicategories is more complicated than with categories. Let us stress the two main differences between both situations: On one hand, every 2-cell $\sigma : p \Rightarrow q : b \to b'$ in $\mathcal{B}$ gives rise to a homotopy

$$\mathrm{B}\sigma : \mathrm{B}p \simeq \mathrm{B}q : \mathrm{B}(F{\downarrow}_b) \to \mathrm{B}(F{\downarrow}_{b'})$$

that must be taken into account. On the other hand, for $p : b \to b'$ and $p' : b' \to b''$ any two composable 1-cells in $\mathcal{B}$, we have a homotopy

$$\mathrm{B}p' \circ \mathrm{B}p \simeq \mathrm{B}(p' \circ p) : \mathrm{B}(F{\downarrow}_b) \to \mathrm{B}(F{\downarrow}_{b''}),$$

rather than the identity $\mathrm{B}p' \circ \mathrm{B}p = \mathrm{B}(p' \circ p)$, as it happens in the category case. This unfortunate behavior is due to the fact that neither is the horizontal composition of 1-cells in the bicategories involved (strictly) associative nor does the lax functor preserve (strictly) that composition. Therefore, in the process of taking homotopy fiber bicategories, $F{\downarrow}: b \mapsto F{\downarrow}_b$, we are forced to deal with *lax bidiagrams of bicategories*

$$\mathfrak{F} : \mathcal{B} \to \mathbf{Bicat}, \quad b \mapsto \mathfrak{F}_b,$$

which are a type of lax functors in the sense of Gordon, Power and Street [69] from the bicategory $\mathcal{B}$ to the tricategory of small bicategories, rather than ordinary diagrams of small categories, that is, functors $\mathfrak{F} : \mathcal{B} \to \mathbf{Cat}$, as it happens when both $\mathcal{A}$ and $\mathcal{B}$ are categories.

After this introductory Section 3.1, the chapter is organized in four sections. Section 3.2 is an attempt to make the chapter as self-contained as possible; hence, at the same time as we set notations and terminology, we define and describe in detail the kind of lax functors $\mathfrak{F} : \mathcal{B} \to \mathbf{Bicat}$ we are going to work with. Section 3.3 is very technical but crucial to our discussions. It is mainly dedicated to describing in detail a *bicategorical Grothendieck construction*, which assembles any lax bidiagram of bicategories $\mathfrak{F} : \mathcal{B} \to \mathbf{Bicat}$ into a bicategory $\int_{\mathcal{B}} \mathfrak{F}$. This is similar to what the ordinary construction, due to Grothendieck [72, 73], Giraud [66, 67], and Thomason [120] on lax diagrams of categories with the shape of any given category. By means of this higher Grothendieck construction, in Section 3.4 we establish the third relevant result of the chapter, namely (see Theorem 3.1):

> "*If* $\mathfrak{F} : \mathcal{B} \to \mathbf{Bicat}$ *is a lax bidiagram of bicategories such that each 1-cell* $p : b \to b'$ *in the bicategory* $\mathcal{B}$ *induces a homotopy equivalence* $\mathrm{B}\mathfrak{F}_b \simeq \mathrm{B}\mathfrak{F}_{b'}$*, then, for every object* $b \in \mathrm{Ob}\mathcal{B}$*, there is an induced homotopy cartesian square*
>
> $$\begin{tikzcd} \mathrm{B}\mathfrak{F}_b \arrow[r] \arrow[d] & \mathrm{B}\int_{\mathcal{B}} \mathfrak{F} \arrow[d] \\ pt \arrow[r, "\mathrm{B}b"] & \mathrm{B}\mathcal{B}. \end{tikzcd}$$
>
> *That is, the classifying space* $\mathrm{B}\mathfrak{F}_b$ *is homotopy equivalent to the homotopy fiber of the map induced on classifying spaces by the projection homomorphism* $\int_{\mathcal{B}} \mathfrak{F} \to \mathcal{B}$ *at the 0-cell corresponding to the object* b*.*"

Thanks to Thomason's Homotopy Colimit Theorem [120], when $\mathcal{B}$ is a small category and $\mathfrak{F}$ take values in $\mathbf{Cat}$, the result above is equivalent to the relevant lemma used by Quillen in his proof of Theorem B. Similarly here, the proof of the bicategorical Theorem B, given in the last Section 3.5, essentially consists of two steps: First, to apply that key result above to the lax bidiagram of homotopy fiber bicategories, $F{\downarrow} : \mathcal{B} \to \mathbf{Bicat}$, of a lax functor $F : \mathcal{A} \to \mathcal{B}$. Second, to prove that there is a homomorphism $\int_{\mathcal{B}} F{\downarrow} \to \mathcal{A}$ inducing a homotopy equivalence $\mathrm{B}(\int_{\mathcal{B}} F{\downarrow}) \simeq \mathrm{B}\mathcal{A}$, so that the bicategory $\int_{\mathcal{B}} F{\downarrow}$ may be thought of as the "total bicategory" of the lax functor F. Section 5 also includes some applications to classifying spaces of monoidal categories. For instance, we find a new proof of the well-known result by Mac Lane [99] and Stasheff [114]:

> "*Let* $(\mathcal{M}, \otimes) = (\mathcal{M}, \otimes, I, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$ *be a monoidal category. If multiplication for each object* $x \in \mathrm{Ob}\mathcal{M}$*,* $y \mapsto y \otimes x$*, induces a homotopy autoequivalence*

on $\mathrm{B}\mathcal{M}$*, then there is a homotopy equivalence*

$$\mathrm{B}\mathcal{M} \simeq \Omega\mathrm{B}(\mathcal{M},\otimes),$$

between the classifying space of the underlying category and the loop space of the classifying space of the monoidal category."

3.2 Bicategorical preliminaries: Lax bidiagrams of bicategories

In this chapter we shall work with small bicategories following the notations in Subsection 2.1.2, with some minor changes. Namely, for a lax functor $F:\mathcal{A}\to\mathcal{B}$ we will name its constraint 2-cells by

$$\widehat{F}_{g,f}: Fg\circ Ff \Rightarrow F(g\circ f), \quad \widehat{F}_a: 1_{Fa}\Rightarrow F1_a.$$

while for a lax transformation $\alpha: F\to G$ we will use $\widehat{\alpha}_f:\alpha b\circ Ff\Rightarrow Gf\circ\alpha a$ for the constraint 2-cell at a 1-cell $f:a\to b$ in $\mathcal{A}$.

We will use the fact that the commutative triangles

$$\begin{tikzcd}
(1\circ g)\circ f \arrow[rr, Rightarrow, "\boldsymbol{a}"] \arrow[dr, Rightarrow, "\boldsymbol{l}\circ 1"'] & & 1\circ(g\circ f) \arrow[dl, Rightarrow, "\boldsymbol{l}"] & (g\circ f)\circ 1 \arrow[rr, Rightarrow, "\boldsymbol{a}"] \arrow[dr, Rightarrow, "\boldsymbol{r}"'] & & g\circ(f\circ 1) \arrow[dl, Rightarrow, "1\circ\boldsymbol{r}"] \\
 & g\circ f & & & g\circ f &
\end{tikzcd} \tag{3.1}$$

and the equality

$$\boldsymbol{r}_1 = \boldsymbol{l}_1 : 1\circ 1 \cong 1 \tag{3.2}$$

are consequences of the other axioms (this is not obvious, but a proof can be done paralleling the one given, for monoidal categories, by Kelly in [91] or Joyal and Street in [88, Proposition 1.1]).

3.2.1 Lax bidiagrams of bicategories

The next concept of fibered bicategory in bicategories is the basis of most of our subsequent discussions. Let $\mathcal{B}$ be a bicategory. Regarding $\mathcal{B}$ as a tricategory in which the 3-cells are all identities, we define a *lax bidiagram of bicategories*

$$\mathfrak{F} = (\mathfrak{F},\chi,\xi,\omega,\gamma,\delta):\mathcal{B}^{\mathrm{op}}\to\mathbf{Bicat} \tag{3.3}$$

to be a contravariant[1] lax functor of tricategories from $\mathcal{B}$ to **Bicat**, all of whose coherence modifications are invertible, they are called *lax homomorphisms* by Garner

---

[1] $\mathcal{B}^{\mathrm{op}}$ means that we are inverting the direction of the 1-cells but not of the 2-cells, that is $\mathcal{B}^{\mathrm{op}}(a,b)=\mathcal{B}(b,a)$.

and Gurski in [65]. More explicitly, a lax bidiagram of bicategories $\mathfrak{F}$ as above consists of the following data:

(**D1**) for each object b in $\mathcal{B}$, a bicategory $\mathfrak{F}_b$;

(**D2**) for each 1-cell $f : a \to b$ of $\mathcal{B}$, a homomorphism $f^* : \mathfrak{F}_b \to \mathfrak{F}_a$;

(**D3**) for each 2-cell $\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \end{tikzcd}$ of $\mathcal{B}$, a pseudo transformation $\alpha^* : f^* \Rightarrow g^*$;

$$\begin{tikzcd} \mathfrak{F}_b \arrow[r, bend left, "f^*"] \arrow[r, bend right, "g^*"'] \arrow[r, phantom, "\Downarrow\alpha^*"] & \mathfrak{F}_a \end{tikzcd}$$

(**D4**) for each two composable 1-cells $a \xrightarrow{f} b \xrightarrow{g} c$ in the bicategory $\mathcal{B}$, a pseudo transformation $\chi_{g,f} : f^*g^* \Rightarrow (g \circ f)^*$;

$$\begin{tikzcd} & \mathfrak{F}_c \arrow[dl, "g^*"'] \arrow[dr, "(g\circ f)^*"] \arrow[d, phantom, "\overset{\chi}{\Rightarrow}"] & \\ \mathfrak{F}_b \arrow[rr, "f^*"'] & & \mathfrak{F}_a \end{tikzcd}$$

(**D5**) for each object b of $\mathcal{B}$, a pseudo transformation $\chi_b : 1_{\mathfrak{F}_b} \Rightarrow 1_b^*$;

$$\begin{tikzcd} \mathfrak{F}_b \arrow[r, bend left, "1_{\mathfrak{F}_b}"] \arrow[r, bend right, "1_b^*"'] \arrow[r, phantom, "\Downarrow\chi"] & \mathfrak{F}_b \end{tikzcd}$$

(**D6**) for any two vertically composable 2-cells $\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \end{tikzcd}$ and $\begin{tikzcd} a \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta"] & b \end{tikzcd}$ in $\mathcal{B}$, an invertible modification $\xi_{\beta,\alpha} : \beta^* \circ \alpha^* \Rrightarrow (\beta \cdot \alpha)^*$;

$$\begin{tikzcd} & f^* \arrow[dl, Rightarrow, "\alpha^*"'] \arrow[dr, Rightarrow, "(\beta\cdot\alpha)^*"] \arrow[d, phantom, "\overset{\xi}{\Rrightarrow}"] & \\ g^* \arrow[rr, Rightarrow, "\beta^*"'] & & h^* \end{tikzcd}$$

(**D7**) for each 1-cell $f : a \to b$ of $\mathcal{B}$, an invertible modification $\xi_f : 1_{f^*} \Rrightarrow 1_f^*$;

$$\begin{tikzcd} f^* \arrow[d, Rightarrow, bend right, "1_{f^*}"'] \arrow[d, Rightarrow, bend left, "1_f^*"] \arrow[d, phantom, "\overset{\xi}{\Rrightarrow}"] \\ f^* \end{tikzcd}$$

(**D8**) for every two horizontally composable 2-cells $\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, bend left, "g"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & c \end{tikzcd}$, an invertible modification $\chi_{\beta,\alpha} : (\beta \circ \alpha)^* \circ \chi_{g,f} \Rrightarrow \chi_{k,h} \circ (\alpha^* \beta^*)$;

$$\begin{tikzcd}
f^*\, g^* \arrow[r, Rightarrow, "\alpha^*\beta^*"] \arrow[d, Rightarrow, "\chi"'] \arrow[dr, phantom, "\overset{\chi}{\Rrightarrow}"] & h^*\, k^* \arrow[d, Rightarrow, "\chi"] \\
(g \circ f)^* \arrow[r, Rightarrow, "(\beta\circ\alpha)^*"'] & (k \circ h)^*
\end{tikzcd}$$

(**D9**) for every three composable 1-cells $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$ in $\mathcal{B}$, an invertible modification $\omega_{h,g,f} : \boldsymbol{a}^* \circ (\chi_{h\circ g,f} \circ f^*\chi_{h,g}) \Rrightarrow \chi_{h,g\circ f} \circ \chi_{g,f} h^*$;

$$\begin{tikzcd}
f^*g^*h^* \arrow[rr, Rightarrow, "\chi h^*"] \arrow[d, Rightarrow, "f^*\chi"'] & \overset{\omega}{\Rrightarrow} & (g \circ f)^* h^* \arrow[d, Rightarrow, "\chi"] \\
f^*(h \circ g)^* \arrow[r, Rightarrow, "\chi"] & ((h \circ g) \circ f)^* \arrow[r, Rightarrow, "\boldsymbol{a}^*"] & (h \circ (g \circ f))^*
\end{tikzcd}$$

(**D10**) for any 1-cell $f : a \to b$ of $\mathcal{B}$, two invertible modifications

$$\gamma_f : \boldsymbol{l}_f^* \circ (\chi_{1_b,f} \circ f^*\chi_b) \Rrightarrow 1_{f^*}, \quad \delta_f : \boldsymbol{r}_f^* \circ (\chi_{f,1_a} \circ \chi_a f^*) \Rrightarrow 1_{f^*}.$$

$$\begin{tikzcd}
f^* 1_b^* \arrow[d, Rightarrow, "\chi"'] \arrow[dr, phantom, "\overset{\gamma}{\Rrightarrow}"] & f^* \arrow[l, Rightarrow, "f^*\chi"'] \arrow[r, Rightarrow, "\chi f^*"] \arrow[d, Rightarrow, "1_{f^*}"] \arrow[dr, phantom, "\overset{\delta}{\Lleftarrow}"] & 1_a^* f^* \arrow[d, Rightarrow, "\chi"] \\
(1_b \circ f)^* \arrow[r, Rightarrow, "\boldsymbol{l}^*"'] & f^* & (f \circ 1_a)^* \arrow[l, Rightarrow, "\boldsymbol{r}^*"]
\end{tikzcd}$$

These data must satisfy the following coherence conditions:

(**C1**) for any three composable 2-cells $f \overset{\alpha}{\Rightarrow} g \overset{\beta}{\Rightarrow} h \overset{\zeta}{\Rightarrow} k : a \to b$ in $\mathcal{B}$, the equation on modifications below holds;

$$\begin{tikzcd}
g^* \arrow[d, Rightarrow, "\beta^*"'] & f^* \arrow[l, Rightarrow, "\alpha^*"'] \arrow[dl, Rightarrow, "(\beta\cdot\alpha)^*" description] \arrow[d, Rightarrow, "(\zeta\cdot\beta\cdot\alpha)^*"] \\
h^* \arrow[r, Rightarrow, "\zeta^*"'] & k^*
\end{tikzcd}
\;\; = \;\;
\begin{tikzcd}
g^* \arrow[d, Rightarrow, "\beta^*"'] \arrow[dr, Rightarrow, "(\zeta\cdot\beta)^*" description] & k^* \arrow[l, Rightarrow, "\alpha^*"'] \arrow[d, Rightarrow, "(\zeta\cdot\beta\cdot\alpha)^*"] \\
h^* \arrow[r, Rightarrow, "\zeta^*"'] & k^*
\end{tikzcd}$$

with $\overset{\xi}{\Rrightarrow}$ in each triangle.

(**C2**) for any 2-cell $f \overset{\alpha}{\Rightarrow} g : a \to b$ of $\mathcal{B}$,

$$\begin{tikzcd}
& f^* \arrow[dl, Rightarrow, bend right=40, "1_{f^*}"'] \arrow[dl, Rightarrow, bend left=20, "1_f^*"] \arrow[dr, Rightarrow, "\alpha^*"] & \\
f^* \arrow[rr, Rightarrow, "\alpha^*"'] & & g^*
\end{tikzcd}
= \boldsymbol{r}_{\alpha^*}\,, \qquad
\begin{tikzcd}
& g^* & \\
g^* \arrow[ur, Rightarrow, bend left=40, "1_{g^*}"] \arrow[ur, Rightarrow, bend right=20, "1_g^*"'] & & f^* \arrow[ul, Rightarrow, "\alpha^*"'] \arrow[ll, Rightarrow, "\alpha^*"]
\end{tikzcd}
= \boldsymbol{l}_{\alpha^*};$$

with $\overset{\xi}{\Rrightarrow}$ in each region.

Notation 3.1 Thanks to conditions (**C1**) and (**C2**), for each objects $a, b \in \mathrm{Ob}\mathcal{B}$, we have a homomorphism $\mathcal{B}(a,b) \to \mathbf{Bicat}(\mathfrak{F}_b, \mathfrak{F}_a)$ such that

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, phantom, "\mapsto"] & \mathfrak{F}_b \arrow[r, bend left, "f^*"] \arrow[r, bend right, "g^*"'] \arrow[r, phantom, "\Downarrow\alpha^*"] & \mathfrak{F}_a, \end{tikzcd}$$

and whose structure constraints are the deformations ξ in (**D6**) and (**D7**). Then, whenever it is given a commutative diagram in the category $\mathcal{B}(a,b)$ of the form

$$\begin{tikzcd} f \arrow[r, Rightarrow, "\beta_0"] \arrow[d, Rightarrow, "\alpha_0"'] & g_1 \arrow[r, Rightarrow, "\beta_1"] & \cdots \arrow[r, Rightarrow] & g_n \arrow[d, Rightarrow, "\beta_n"] \\ f_1 \arrow[r, Rightarrow, "\alpha_1"] & \cdots \arrow[r, Rightarrow] & f_m \arrow[r, Rightarrow, "\alpha_m"] & g, \end{tikzcd} \tag{3.4}$$

we will denote by

$$\begin{tikzcd} f^* \arrow[r, Rightarrow, "\beta_0^*"] \arrow[d, Rightarrow, "\alpha_0^*"'] & g_1^* \arrow[r, Rightarrow, "\beta_1^*"] \arrow[d, phantom, "\overset{\xi}{\cong}"] & \cdots \arrow[r, Rightarrow] & g_n^* \arrow[d, Rightarrow, "\beta_n^*"] \\ f_1^* \arrow[r, Rightarrow, "\alpha_1^*"] & \cdots \arrow[r, Rightarrow] & f_m^* \arrow[r, Rightarrow, "\alpha_m^*"] & g^* \end{tikzcd} \tag{3.5}$$

the invertible modification obtained by an (any) appropriate composition of the modifications ξ and their inverses ξ^{-1}, once any particular bracketing in the strings $\alpha_0^*, \ldots, \alpha_m^*$ and $\beta_0^*, \ldots, \beta_n^*$ has been chosen. That diagram (3.5) is well defined from diagram (3.4) is a consequence of the coherence theorem for homomorphisms of bicategories [69, Theorem 1.6].

Furthermore, for any diagram $\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\alpha"] & c \arrow[r, "h"] & d \end{tikzcd}$ in $\mathcal{B}$, we shall denote by

$$\chi_{\alpha,f} : (\alpha \circ 1_f)^* \circ \chi_{g,f} \Rrightarrow \chi_{g',f} \circ f^*\alpha^*, \quad \chi_{h,\alpha} : (1_h \circ \alpha)^* \circ \chi_{h,g} \Rrightarrow \chi_{h,g'} \circ \alpha^* h^*,$$

$$\begin{tikzcd} f^*g^* \arrow[r, Rightarrow, "f^*\alpha^*"] \arrow[d, Rightarrow, "\chi"'] \arrow[dr, phantom, "\overset{\chi}{\Rrightarrow}"] & f^*g'^* \arrow[d, Rightarrow, "\chi"] & g^*h^* \arrow[r, Rightarrow, "\alpha^*h^*"] \arrow[d, Rightarrow, "\chi"'] \arrow[dr, phantom, "\overset{\chi}{\Rrightarrow}"] & g'^*h^* \arrow[d, Rightarrow, "\chi"] \\ (g \circ f)^* \arrow[r, Rightarrow, "(\alpha \circ 1_f)^*"'] & (g' \circ f)^* & (h \circ g)^* \arrow[r, Rightarrow, "(1_h \circ \alpha)^*"'] & (h \circ g')^* \end{tikzcd}$$

the modifications obtained, respectively, by pasting the diagrams in **Bicat** below.

$$\begin{tikzcd}[column sep=huge, row sep=huge] f^*g^* \arrow[r, Rightarrow, bend left=40, "f^*\alpha^*"] \arrow[r, Rightarrow, no head, "1_{f^*}\alpha^*" near end] \arrow[r, phantom, bend left=20, "\cong"] \arrow[r, Rightarrow, bend right=40, "1_f^*\alpha^*"'] \arrow[r, phantom, bend right=20, "\overset{\xi 1_{\alpha^*}}{\cong}"] \arrow[d, Rightarrow, "\chi"'] \arrow[dr, phantom, "\overset{\chi}{\Rrightarrow}"] & f^*g'^* \arrow[d, Rightarrow, "\chi"] & g^*h^* \arrow[r, Rightarrow, bend left=40, "\alpha^*h^*"] \arrow[r, Rightarrow, "\alpha^*1_{h^*}" near end] \arrow[r, phantom, bend left=20, "\cong"] \arrow[r, Rightarrow, bend right=40, "\alpha^*1_h^*"'] \arrow[r, phantom, bend right=20, "\overset{1_{\alpha^*}\xi}{\cong}"] \arrow[d, Rightarrow, "\chi"'] \arrow[dr, phantom, "\overset{\chi}{\Rrightarrow}"] & g'^*h^* \arrow[d, Rightarrow, "\chi"] \\ (g \circ f)^* \arrow[r, Rightarrow, "(\alpha \circ 1_f)^*"'] & (g' \circ f)^* & (h \circ g)^* \arrow[r, Rightarrow, "(1_h \circ \alpha)^*"'] & (h \circ g')^* \end{tikzcd}$$

(**C3**) for every diagram of 2-cells $\begin{tikzcd} a \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=2.2ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.2ex, "\Downarrow\alpha'"] & b \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=2.2ex, "\Downarrow\beta"] \arrow[r, phantom, shift right=2.2ex, "\Downarrow\beta'"] & c \end{tikzcd}$ in $\mathcal{B}$,

$$\begin{tikzcd}
(g\circ f)^* \arrow[d, Rightarrow, "(\beta\circ\alpha)^*"'] & \overset{\chi}{\Rrightarrow} & f^*g^* \arrow[ll, Rightarrow, "\chi"'] \arrow[dl, Rightarrow, "\alpha^*\beta^*"'] \arrow[dd, Rightarrow, bend right=30, "(\alpha'^*\circ\alpha^*)(\beta'^*\circ\beta^*)" description] \arrow[dd, Rightarrow, bend left=40, "(\alpha'\cdot\alpha)^*(\beta'\cdot\beta)^*"] & & (g\circ f)^* \arrow[d, Rightarrow, "(\beta\circ\alpha)^*"'] \arrow[dd, Rightarrow, bend left=50, "((\beta'\cdot\beta)\circ(\alpha'\cdot\alpha))^*"] & & f^*g^* \arrow[ll, Rightarrow, "\chi"'] \arrow[dd, Rightarrow, bend left=30, "(\alpha'\cdot\alpha)^*(\beta'\cdot\beta)^*"] \\
(g'\circ f')^* \arrow[d, Rightarrow, "(\beta'\circ\alpha')^*"'] & f'^*g'^* \arrow[l, Rightarrow, "\chi"'] \arrow[dr, Rightarrow, "\alpha'^*\beta'^*"'] & \overset{(2.1)}{\cong} \quad \overset{\xi\xi}{\cong} & = & (g'\circ f')^* \arrow[d, Rightarrow, "(\beta'\circ\alpha')^*"'] \arrow[r, phantom, "\overset{\xi}{\cong}"] & \overset{\chi}{\Rrightarrow} & \\
(g''\circ f'')^* & \overset{\chi}{\Rrightarrow} & f''^*g''^* \arrow[ll, Rightarrow, "\chi"] & & (g''\circ f'')^* & & f''^*g''^* \arrow[ll, Rightarrow, "\chi"]
\end{tikzcd}$$

(**C4**) for every pair of composable 1-cells $a \xrightarrow{f} b \xrightarrow{g} c$,

$$\begin{tikzcd}
(g\circ f)^* \arrow[d, Rightarrow, bend right=50, "1_{(g\circ f)^*}"'] \arrow[d, Rightarrow, bend left=50, "1^*_{g\circ f}"] \arrow[d, phantom, "\overset{\xi}{\Rrightarrow}"] & f^*g^* \arrow[l, Rightarrow, "\chi"'] \arrow[d, Rightarrow, bend left=50, "1^*_f1^*_g"] \arrow[l, phantom, shift right=4ex, "\overset{\chi}{\Rrightarrow}"] & = & (g\circ f)^* \arrow[d, Rightarrow, bend right=50, "1_{(g\circ f)^*}"'] \arrow[d, phantom, shift left=3ex, "\cong"] & f^*g^* \arrow[l, Rightarrow, "\chi"'] \arrow[d, Rightarrow, bend right=50, "1_{f^*}1_{g^*}"'] \arrow[d, Rightarrow, bend left=50, "1^*_f1^*_g"] \arrow[d, phantom, "\overset{\xi\xi}{\Rrightarrow}"] \\
(g\circ f)^* & f^*g^* \arrow[l, Rightarrow, "\chi"'] & & (g\circ f)^* & f^*g^* \arrow[l, Rightarrow, "\chi"']
\end{tikzcd}$$

(**C5**) for every 2-cells $\begin{tikzcd} a \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & c \arrow[r, bend left=50, "h"] \arrow[r, bend right=50, "h'"'] \arrow[r, phantom, "\Downarrow\zeta"] & d \end{tikzcd}$, the equation $A = A'$ holds, where

$$A = \begin{tikzcd}
& ((h\circ g)\circ f)^* \arrow[dl, Rightarrow, "\boldsymbol{a}^*"'] \arrow[d, Rightarrow, "((\zeta\circ\beta)\circ\alpha)^*" description] \arrow[r, phantom, shift right=5ex, "\overset{\chi}{\Rrightarrow}"] & f^*(h\circ g)^* \arrow[l, Rightarrow, "\chi"'] \arrow[d, Rightarrow, "\alpha^*(\zeta\circ\beta)^*" description] \arrow[r, phantom, shift right=3ex, "\overset{\alpha^*\chi}{\Rrightarrow}"] & f^*g^*h^* \arrow[l, Rightarrow, "f^*\chi"'] \arrow[d, Rightarrow, bend right=40, "\alpha^*(\beta^*\zeta^*)"'] \arrow[d, Rightarrow, "(\alpha^*\beta^*)\zeta^*"] \arrow[d, phantom, shift right=1.5ex, "\cong"] \\
(h\circ(g\circ f))^* \arrow[r, phantom, "\overset{\xi}{\cong}"] \arrow[dr, Rightarrow, "(\zeta\circ(\beta\circ\alpha))^*"'] & ((h'\circ g')\circ f')^* \arrow[d, Rightarrow, "\boldsymbol{a}^*"] & f'^*(h'\circ g')^* \arrow[l, Rightarrow, "\chi"] \arrow[dl, phantom, "\overset{\omega}{\Rrightarrow}"] & f'^*g'^*h'^* \arrow[l, Rightarrow, "f'^*\chi"] \arrow[dl, Rightarrow, "\chi h'^*"] \\
& (h'\circ(g'\circ f'))^* & (g'\circ f')^*h'^* \arrow[l, Rightarrow, "\chi"'] &
\end{tikzcd}$$

$$A' = \begin{tikzcd}
& ((h\circ g)\circ f)^* \arrow[dl, Rightarrow, "\boldsymbol{a}^*"'] & f^*(h\circ g)^* \arrow[l, Rightarrow, "\chi"'] \arrow[dl, phantom, "\overset{\omega}{\Rrightarrow}"] & f^*g^*h^* \arrow[l, Rightarrow, "f^*\chi"'] \arrow[dl, Rightarrow, "\chi h^*"'] \arrow[d, Rightarrow, "(\alpha^*\beta^*)\zeta^*"] \\
(h\circ(g\circ f))^* \arrow[dr, Rightarrow, "(\zeta\circ(\beta\circ\alpha))^*"'] & \overset{\chi}{\Rrightarrow} & (g\circ f)^*h^* \arrow[ll, Rightarrow, "\chi"'] \arrow[d, Rightarrow, "(\beta\circ\alpha)^*\zeta^*"] \arrow[r, phantom, "\overset{\chi\zeta^*}{\Rrightarrow}"] & f'^*g'^*h'^* \arrow[dl, Rightarrow, "\chi h'^*"] \\
& (h'\circ(g'\circ f'))^* & (g'\circ f')^*h'^* \arrow[l, Rightarrow, "\chi"'] &
\end{tikzcd}$$

(**C6**) for every four composable 1-cells $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d \xrightarrow{k} e$, the equation $B = B'$ holds, where

$$B = \begin{tikzcd}
 & & f^*g^*h^*k^* \arrow[dll, Rightarrow, "f^*g^*\chi"'] \arrow[drr, Rightarrow, "\chi h^*k^*"] \arrow[d, phantom, "\overset{(2.1)}{\cong}"] & & \\
f^*g^*(k\circ h)^* \arrow[rr, Rightarrow, "\chi(k\circ h)^*"] \arrow[d, Rightarrow, "f^*\chi"'] & & (g\circ f)^*(k\circ h)^* \arrow[dd, Rightarrow, "\chi"] & & (g\circ f)^*h^*k^* \arrow[ll, Rightarrow, "(g\circ f)^*\chi"'] \arrow[d, Rightarrow, "\chi k^*"] \\
f^*((k\circ h)\circ g)^* \arrow[d, Rightarrow, "\chi"'] & \overset{\omega}{\Rrightarrow} & & \overset{\omega}{\Rrightarrow} & (h\circ(g\circ f))^*k^* \arrow[d, Rightarrow, "\chi"] \\
(((k\circ h)\circ g)\circ f)^* \arrow[rr, Rightarrow, "\boldsymbol{a}^*"] \arrow[dr, Rightarrow, "(\boldsymbol{a}\circ 1_f)^*"'] & & ((k\circ h)\circ(g\circ f))^* \arrow[rr, Rightarrow, "\boldsymbol{a}^*"] \arrow[d, phantom, "\overset{\xi}{\cong}"] & & (k\circ(h\circ(g\circ f)))^* \\
 & ((k\circ(h\circ g))\circ f)^* \arrow[rr, Rightarrow, "\boldsymbol{a}^*"'] & & (k\circ((h\circ g)\circ f))^* \arrow[ur, Rightarrow, "(1_k\circ\boldsymbol{a})^*"'] &
\end{tikzcd}$$

$$B' = \begin{tikzcd}
 & & f^*g^*h^*k^* \arrow[dll, Rightarrow, "f^*g^*\chi"'] \arrow[drr, Rightarrow, "\chi h^*k^*"] \arrow[d, Rightarrow, "f^*\chi\, k^*"] & & \\
f^*g^*(k\circ h)^* \arrow[d, Rightarrow, "f^*\chi"'] & \overset{f^*\omega}{\Rrightarrow} & f^*(h\circ g)^*k^* \arrow[dl, Rightarrow, "f^*\chi"'] \arrow[dr, Rightarrow, "\chi k^*"] & \overset{\omega k^*}{\Rrightarrow} & (g\circ f)^*h^*k^* \arrow[d, Rightarrow, "\chi k^*"] \\
f^*((k\circ h)\circ g)^* \arrow[r, Rightarrow, "f^*\boldsymbol{a}^*"] \arrow[d, Rightarrow, "\chi"'] & f^*(k\circ(h\circ g))^* \arrow[dd, Rightarrow, "\chi"] & & ((h\circ g)\circ f)^*k^* \arrow[r, Rightarrow, "\boldsymbol{a}^*k^*"] \arrow[dd, Rightarrow, "\chi"'] & (h\circ(g\circ f))^*k^* \arrow[d, Rightarrow, "\chi"] \\
(((k\circ h)\circ g)\circ f)^* \arrow[r, phantom, "\overset{\chi}{\Rrightarrow}"] \arrow[dr, Rightarrow, "(\boldsymbol{a}\circ 1_f)^*"'] & & \overset{\omega}{\Rrightarrow} & & (k\circ(h\circ(g\circ f)))^* \arrow[l, phantom, "\overset{\chi}{\Rrightarrow}"] \\
 & ((k\circ(h\circ g))\circ f)^* \arrow[rr, Rightarrow, "\boldsymbol{a}^*"'] & & (k\circ((h\circ g)\circ f))^* \arrow[ur, Rightarrow, "(1_k\circ\boldsymbol{a})^*"'] &
\end{tikzcd}$$

(**C7**) for every 2-cell $f \overset{\alpha}{\Longrightarrow} g : a \to b$, the following two equations on modifications hold:

$$\begin{tikzcd}
 & 1_a^*f^* \arrow[dl, Rightarrow, "\chi"'] \arrow[d, phantom, "\overset{\delta}{\Rrightarrow}"] & f^* \arrow[l, Rightarrow, "\chi f^*"'] \arrow[dl, Rightarrow, "1_{f^*}"'] \arrow[d, Rightarrow, "\alpha^*"] & & & 1_a^*f^* \arrow[dl, Rightarrow, "\chi"'] \arrow[d, Rightarrow, "1_a^*\alpha^*"] \arrow[dr, phantom, "\overset{(2.1)}{\cong}"] & f^* \arrow[l, Rightarrow, "\chi f^*"'] \arrow[d, Rightarrow, "\alpha^*"] \\
(f\circ 1_a)^* \arrow[r, Rightarrow, "\boldsymbol{r}^*"] \arrow[d, Rightarrow, "(\alpha\circ 1)^*"'] \arrow[dr, phantom, "\overset{\xi}{\cong}"] & f^* \arrow[d, Rightarrow, "\alpha^*"] \arrow[r, phantom, "\cong"] & g^* \arrow[dl, Rightarrow, "1_{g^*}"] & = & (f\circ 1_a)^* \arrow[r, phantom, "\overset{\chi}{\Rrightarrow}"] \arrow[d, Rightarrow, "(\alpha\circ 1)^*"'] & 1_a^*g^* \arrow[dl, Rightarrow, "\chi"'] \arrow[d, phantom, "\overset{\delta}{\Rrightarrow}"] & g^* \arrow[l, Rightarrow, "\chi g^*"'] \arrow[dl, Rightarrow, "1_{g^*}"] \\
(g\circ 1_a)^* \arrow[r, Rightarrow, "\boldsymbol{r}^*"] & g^* & & & (g\circ 1_a)^* \arrow[r, Rightarrow, "\boldsymbol{r}^*"] & g^* &
\end{tikzcd}$$

$$\begin{tikzcd}
 & f^*1_b^* \arrow[dl, Rightarrow, "\chi"'] \arrow[d, phantom, "\overset{\gamma}{\Rrightarrow}"] & f^* \arrow[l, Rightarrow, "f^*\chi"'] \arrow[dl, Rightarrow, "1_{f^*}"'] \arrow[d, Rightarrow, "\alpha^*"] \\
(1_b\circ f)^* \arrow[r, Rightarrow, "\boldsymbol{l}^*"] \arrow[d, Rightarrow, "(1\circ\alpha)^*"'] \arrow[dr, phantom, "\overset{\xi}{\cong}"] & f^* \arrow[d, Rightarrow, "\alpha^*"] \arrow[r, phantom, "\cong"] & g^* \arrow[dl, Rightarrow, "1_{g^*}"] \\
(1_b\circ g)^* \arrow[r, Rightarrow, "\boldsymbol{l}^*"] & g^* &
\end{tikzcd}
=
\begin{tikzcd}
 & f^*1_b^* \arrow[dl, Rightarrow, "\chi"'] \arrow[d, Rightarrow, "\alpha^*1_b^*"] \arrow[dr, phantom, "\overset{(2.1)}{\cong}"] & f^* \arrow[l, Rightarrow, "f^*\chi"'] \arrow[d, Rightarrow, "\alpha^*"] \\
(1_b\circ f)^* \arrow[r, phantom, "\overset{\chi}{\Rrightarrow}"] \arrow[d, Rightarrow, "(1\circ\alpha)^*"'] & g^*1_b^* \arrow[dl, Rightarrow, "\chi"'] \arrow[d, phantom, "\overset{\gamma}{\Rrightarrow}"] & g^* \arrow[l, Rightarrow, "g^*\chi"'] \arrow[dl, Rightarrow, "1_{g^*}"] \\
(1_b\circ g)^* \arrow[r, Rightarrow, "\boldsymbol{l}^*"] & g^* &
\end{tikzcd}$$

(**C8**) for every pair of composable 1-cells $a \xrightarrow{f} b \xrightarrow{g} c$, the following equation holds:

$$\begin{tikzcd}
f^*1_b^*g^* \arrow[d, Rightarrow, "f^*\chi"'] \arrow[dr, Rightarrow, "\chi g^*"] & \arrow[d, phantom, "\overset{\gamma g^*}{\Rrightarrow}"] & f^*g^* \arrow[ll, Rightarrow, "f^*\chi g^*"'] \arrow[d, Rightarrow, "1_{f^*g^*}"] \\
f^*(g\circ 1_b)^* \arrow[ddd, Rightarrow, "\chi"'] \arrow[ddr, phantom, "\overset{\omega}{\Rrightarrow}"] & (1_b\circ f)^*g^* \arrow[r, Rightarrow, "\boldsymbol{l}^*g^*"] \arrow[dd, Rightarrow, "\chi"] & f^*g^* \arrow[ddd, Rightarrow, "\chi"] \arrow[ddl, phantom, "\overset{\chi}{\Rrightarrow}"] \\
 & & \\
 & (g\circ(1_b\circ f))^* \arrow[dr, Rightarrow, "(1_g\circ\boldsymbol{l})^*"] \arrow[d, phantom, "\overset{\xi}{\cong}"] & \\
((g\circ 1_b)\circ f)^* \arrow[ur, Rightarrow, "\boldsymbol{a}^*"] \arrow[rr, Rightarrow, "(\boldsymbol{r}\circ 1_f)^*"'] & & (g\circ f)^*
\end{tikzcd}
=
\begin{tikzcd}
f^*1_b^*g^* \arrow[d, Rightarrow, "f^*\chi"'] \arrow[dr, phantom, "\overset{f^*\delta}{\Rrightarrow}"] & f^*g^* \arrow[l, Rightarrow, "f^*\chi g^*"'] \arrow[d, Rightarrow, bend right, "f^*1_{g^*}"', ""{name=A}] \arrow[d, Rightarrow, "1_{f^*g^*}", ""'{name=B}] \arrow[from=A, to=B, phantom, "\cong"] \\
f^*(g\circ 1_b)^* \arrow[r, Rightarrow, "f^*\boldsymbol{r}^*"] \arrow[d, Rightarrow, "\chi"'] \arrow[dr, phantom, "\overset{\chi}{\Rrightarrow}"] & f^*g^* \arrow[d, Rightarrow, "\chi"] \\
((g\circ 1_b)\circ f)^* \arrow[r, Rightarrow, "(\boldsymbol{r}\circ 1_f)^*"'] & (g\circ f)^*
\end{tikzcd}$$

A lax bidiagram of bicategories $\mathfrak{F} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Bicat}$ is called a *pseudo bidiagram of bicategories* whenever each of the pseudo transformations χ, in (**D4**) and (**D5**), is a pseudo equivalence; that is, regarding $\mathcal{B}$ as a tricategory whose 3-cells are all identities, a trihomomorphism $\mathfrak{F} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Bicat}$ in the sense of Gordon-Power-Street [69, Definition 3.1].

Example 3.1 If $\mathcal{C}$ is any small category viewed as a bicategory, then a lax bidiagram of bicategories over $\mathcal{C}$, as above, in which the deformations ξ in (**D6**) and (**D7**), and χ in (**D8**), are all identities is the same thing as a *lax diagram of bicategories* $\mathfrak{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Bicat}$ as in [42, §2.2].

For instance, let X be any topological space and let $\mathcal{C}(X)$ denote its poset of open subsets, regarded as a category. Then a *fibered bicategory in bigroupoids above* X is a lax diagram of bicategories

$$\mathfrak{F} : \mathcal{C}(X)^{\mathrm{op}} \to \mathbf{Bicat},$$

such that all the bicategories $\mathfrak{F}_U$ are bigroupoids, that is, bicategories whose 1-cells are invertible up to a 2-cell, and whose 2-cells are strictly invertible. In particular, when all the bigroupoids $\mathfrak{F}_U$ are strict, that is, 2-categories, and all the homomorphisms $f^* : \mathfrak{F}_U \to \mathfrak{F}_V$ associated to the inclusions of open sets $f : V \hookrightarrow U$ are 2-functors, we have the notion of *fibered* 2*-category in* 2*-groupoids above the space* X. Thus, 2-stacks and 2-gerbes on spaces are relevant examples of lax diagrams of bicategories (see e.g. Breen [22, Definitions 6.1, 6.2, and 6.3]).

For another example, if $\mathcal{T}$ is any small tricategory, then its *Grothendieck nerve* (2.10)

$$\mathrm{N}\mathcal{T} : \Delta^{\mathrm{op}} \to \mathbf{Bicat},$$

gives a striking example of a pseudo diagram of bicategories.

Example 3.2 For any bicategory $\mathcal{B}$, a *lax bidiagram of categories* over $\mathcal{B}$, that is, a lax bidiagram $\mathfrak{F} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Bicat}$ in which every bicategory $\mathfrak{F}_a$, $a \in \mathrm{Ob}\mathcal{B}$, is a category (i.e., a bicategory where all the 2-cells are identities) is the same thing as a contravariant lax functor $\mathfrak{F} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$ to the 2-category $\mathbf{Cat}$ of small categories, functors, and natural transformations, since the condition of all $\mathfrak{F}_a$ being categories forces all the modifications in (**D6**) – (**D10**) to be identities.

For example, any object b of a bicategory $\mathcal{B}$ defines a pseudo bidiagram of categories [117, Example 10]

$$\mathcal{B}(-, b) : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat},$$

which carries an object $x \in \mathrm{Ob}\mathcal{B}$ to the hom-category $\mathcal{B}(x, b)$, a 1-cell $g : x \to y$ to the functor $g^* : \mathcal{B}(y, b) \to \mathcal{B}(x, b)$ defined by

$$\begin{tikzcd} y \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\beta"] & b \arrow[r, mapsto, "g^*"] & x \arrow[r, bend left, "f\circ g"] \arrow[r, bend right, "f'\circ g"'] \arrow[r, phantom, "\Downarrow\beta\circ 1_g"] & b\ , \end{tikzcd}$$

and a 2-cell $\alpha : g \Rightarrow g'$ is carried to the natural transformation $\alpha^* : g^* \Rightarrow g'^*$ that assigns to each 1-cell $f : y \to b$ in $\mathcal{B}$ the 2-cell $1_f \circ \alpha : f \circ g \Rightarrow f \circ g'$. For $x \xrightarrow{g} y \xrightarrow{h} z$ any two composable 1-cells of $\mathcal{B}$, the structure natural equivalence $\chi : g^* h^* \cong (h \circ g)^*$, at any $f : z \to b$, is provided by the associativity constraint $\boldsymbol{a} : (f \circ h) \circ g \cong f \circ (h \circ g)$, whereas for any $x \in \mathrm{Ob}\mathcal{B}$, the structure natural equivalence $\chi : 1_{\mathcal{B}(x,b)} \cong 1_x^*$, at any $f : x \to b$, is the right unit isomorphism $\boldsymbol{r}^{-1} : f \cong f \circ 1_x$.

3.3 The Grothendieck construction on lax bidiagrams of bicategories

The well-known 'Grothendieck construction', due to Grothendieck [72, 73] and Giraud [66, 67], on pseudo diagrams $(\mathfrak{F}, \chi) : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$ of small categories with the shape of any given small category, was implicitly used in the proof given by Quillen of his famous Theorems A and B for the classifying spaces of small categories [109]. Subsequently, since Thomason established his celebrate Homotopy Colimit Theorem [120], the Grothendieck construction has become an essential tool in homotopy theory of classifying spaces.

In this section, our work is dedicated to extending the Grothendieck construction to lax bidiagrams of bicategories $\mathfrak{F} = (\mathfrak{F}, \chi, \xi, \omega, \gamma, \delta) : \mathcal{B}^{\mathrm{op}} \to \mathbf{Bicat}$, where $\mathcal{B}$ is any bicategory, since its use is a key for proving our main results in this chapter. But we are not claiming here much originality, since extensions of the ubiquitous

Grothendieck construction have been developed in many general frameworks. In particular, we should mention here three recent approaches to our construction: In [42], Carrasco, Cegarra, and Garzón study the bicategorical Grothendieck construction on lax diagrams of bicategories, as in Example 3.1. In [10, 11], Baković performs the Grothendieck construction on normal pseudo bidiagrams of bicategories, that is, lax bidiagrams $\mathfrak{F}$ whose modifications χ_b in (**D5**) and ξ_f in (**D7**) are identities, and whose pseudo transformations $\chi_{g,f}$ in (**D4**) are pseudo equivalences. Buckley, in [33], presents the more general case of pseudo bidiagrams, that is, when all the pseudo transformations $\chi_{g,f}$ and χ_b in (**D4**) and (**D5**) are pseudo equivalences.

The *Grothendieck construction on a lax bidiagram of bicategories* $\mathfrak{F} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Bicat}$, as in (3.3), assembles it into a bicategory, denoted by

$$\textstyle\int_{\mathcal{B}}\mathfrak{F}\ ,$$

which is defined as follows:

The objects are pairs (x, a), where $a \in \mathrm{Ob}\mathcal{B}$ and $x \in \mathrm{Ob}\mathfrak{F}_a$.

The 1-cells are pairs $(u, f) : (x, a) \to (y, b)$, where $f : a \to b$ is a 1-cell in $\mathcal{B}$ and $u : x \to f^*y$ is a 1-cell in $\mathfrak{F}_a$.

The 2-cells are pairs $\begin{tikzcd}(x,a) \arrow[r, bend left, "{(u,f)}"] \arrow[r, bend right, "{(v,g)}"'] \arrow[r, phantom, "\Downarrow(\phi,\alpha)"] & (y,b)\end{tikzcd}$, consisting of a 2-cell $\begin{tikzcd}a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b\end{tikzcd}$ of $\mathcal{B}$ together with a 2-cell $\phi : \alpha^*y \circ u \Rightarrow v$ in $\mathfrak{F}_a$,

$$\begin{tikzcd}
 & f^*y \arrow[dr, "\alpha^*y"] \arrow[d, phantom, "\Downarrow\phi"] & \\
x \arrow[ur, "u"] \arrow[rr, "v"'] & {} & g^*y.
\end{tikzcd}$$

The vertical composition of 2-cells in $\int_{\mathcal{B}}\mathfrak{F}$, $\begin{tikzcd}(x,a) \arrow[r, bend left, "{(u,f)}"] \arrow[r, bend right, "{(v,g)}"'] \arrow[r, phantom, "\Downarrow(\phi,\alpha)"] & (y,b)\end{tikzcd}$ and $\begin{tikzcd}(x,a) \arrow[r, bend left, "{(v,g)}"] \arrow[r, bend right, "{(w,h)}"'] \arrow[r, phantom, "\Downarrow(\psi,\beta)"] & (y,b)\end{tikzcd}$,

is the 2-cell

$$\begin{tikzcd}(x,a) \arrow[r, bend left, "{(u,f)}"] \arrow[r, bend right, "{(w,h)}"'] \arrow[r, phantom, "\Downarrow(\psi\odot\phi,\beta\cdot\alpha)"] & (y,b),\end{tikzcd}$$

where $\beta \cdot \alpha$ is the vertical composition of β with α in $\mathcal{B}$, and $\psi \odot \phi : (\beta \cdot \alpha)^*y \circ u \Rightarrow w$ is the 2-cell of $\mathfrak{F}_a$ obtained by pasting the diagram below.

$$\psi \odot \phi : \quad \begin{tikzcd}
 & f^*y \arrow[d, "\alpha^*y"] \arrow[ddr, "(\beta\cdot\alpha)^*y"] & \\
\Downarrow\phi & g^*y \arrow[dr, "\beta^*y"'] \arrow[r, phantom, "\overset{\xi}{\cong}"] & {} \\
x \arrow[uur, "u"] \arrow[ur, "v"'] \arrow[rr, "w"'] & \Downarrow\psi & h^*y
\end{tikzcd}$$

The vertical composition of 2-cells so defined is associative and unitary thanks to the coherence conditions (**C1**) and (**C2**). The identity 2-cell, for each 1-cell $(u,f):(x,a)\to(y,b)$, is

$$1_{(u,f)} = (\dot{1}_{(u,f)}, 1_f) : (u,f) \Rightarrow (u,f).$$

$$\dot{1}_{(u,f)} = \big(\, 1_f^* y \circ u \xRightarrow{\xi^{-1}\circ 1} 1_{f^*y} \circ u \overset{\boldsymbol{l}}{\Rightarrow} u \,\big)$$

Hence, we have defined the hom-category $\int_{\mathcal{B}} \mathfrak{F}\big((x,a),(y,b)\big)$, for any two objects (x,a) and (y,b) of $\int_{\mathcal{B}} \mathfrak{F}$. Before continuing the description of this bicategory, we shall do the following useful observation:

Lemma 3.1 *A 2-cell $(\phi,\alpha):(u,f)\Rightarrow(v,g)$ in $\int_{\mathcal{B}} \mathfrak{F}\big((x,a),(y,b)\big)$ is an isomorphism if and only if both $\alpha : f \Rightarrow g$, in $\mathcal{B}(a,b)$, and $\phi : \alpha^* y \circ u \Rightarrow v$, in $\mathfrak{F}_a(x, g^*y)$, are isomorphisms.*

Proof: It is quite straightforward, and we leave it to the reader. □

We return now to the description of the bicategory $\int_{\mathcal{B}} \mathfrak{F}$.

The horizontal composition of two 1-cells $(x,a) \xrightarrow{(u,f)} (y,b) \xrightarrow{(u',f')} (z,c)$ is the 1-cell

$$(u',f') \circ (u,f) = (u' \circledcirc u, f' \circ f) : (x,a) \longrightarrow (z,c),$$

where $f' \circ f : a \to c$ is the composite in $\mathcal{B}$ of the 1-cells f and f', while

$$u' \circledcirc u = \chi z \circ (f^* u' \circ u) : x \longrightarrow (f' \circ f)^* z$$

is the composite in $\mathfrak{F}_a$ of $x \xrightarrow{u} f^*y \xrightarrow{f^*u'} f^*f'^*z \xrightarrow{\chi z} (f'\circ f)^*z$.

The horizontal composition of 2-cells is defined by

$$\begin{tikzcd}
(x,a) \arrow[r, bend left, "{(u,f)}"] \arrow[r, bend right, "{(v,g)}"'] \arrow[r, phantom, "\Downarrow(\phi{,}\alpha)"] & (y,b) \arrow[r, bend left, "{(u',f')}"] \arrow[r, bend right, "{(v',g')}"'] \arrow[r, phantom, "\Downarrow(\phi'{,}\alpha')"] & (z,c) & \overset{\circ}{\mapsto} & (x,a) \arrow[r, bend left, "{(u'\circledcirc u,f'\circ f)}"] \arrow[r, bend right, "{(v'\circledcirc v,g'\circ g)}"'] \arrow[r, phantom, "\Downarrow(\phi'\circledcirc\phi{,}\alpha'\circ\alpha)"] & (z,c),
\end{tikzcd}$$

where $\alpha' \circ \alpha$ is the horizontal composition in $\mathcal{B}$ of α' with α, and $\phi' \circledcirc \phi$ is the 2-cell in $\mathfrak{F}_a$ canonically obtained by pasting the diagram below.

$$\phi' \circledcirc \phi : \quad \begin{tikzcd}
 & & f^*y \arrow[rr, "f^*u'"] \arrow[rrdd, "f^*v'"'] \arrow[dddd, "\alpha^*y"] & & f^*f'^*z \arrow[rr, "\chi z"] \arrow[dd, "f^*\alpha'^*z"] & & (f'\circ f)^*z \arrow[dddd, "(\alpha'\circ\alpha)^*z"] \\
 & & & \Downarrow f^*\phi' & & & \\
x \arrow[uurr, "u"] \arrow[ddrr, "v"'] & \Downarrow\phi & & & f^*g'^*z \arrow[dd, "\alpha^*g'^*z"] & \overset{\chi z}{\cong} & \\
 & & & \overset{\widehat{\alpha^*}}{\cong} & & & \\
 & & g^*y \arrow[rr, "g^*v'"'] & & g^*g'^*z \arrow[rr, "\chi z"'] & & (g'\circ g)^*z
\end{tikzcd}$$

Owing to the coherence conditions (**C3**) and (**C4**), the horizontal composition so defined truly gives, for any three objects $(x,a),(y,b),(z,c)$ of $\int_{\mathcal{B}}\mathfrak{F}$, a functor

$$\textstyle\int_{\mathcal{B}}\mathfrak{F}((y,b),(z,c)) \times \int_{\mathcal{B}}\mathfrak{F}((x,a),(y,b)) \xrightarrow{\ \circ\ } \int_{\mathcal{B}}\mathfrak{F}((x,a),(z,c)).$$

The structure associativity isomorphism, for any three composable morphisms

$$(x,a) \overset{(u,f)}{\longrightarrow} (y,b) \overset{(v,g)}{\longrightarrow} (z,c) \overset{(w,h)}{\longrightarrow} (t,d),$$

$$(\mathring{\boldsymbol{a}},\boldsymbol{a}) : \big((w \circledcirc v) \circledcirc u, (h \circ g) \circ f\big) \cong \big(w \circledcirc (v \circledcirc u), h \circ (g \circ f)\big),$$

is provided by the associativity constraint $\boldsymbol{a} : (h \circ g) \circ f \cong h \circ (g \circ f)$ of the bicategory $\mathcal{B}$, together with the isomorphism in the bicategory $\mathfrak{F}_a$

$$\mathring{\boldsymbol{a}} : \boldsymbol{a}^*t \circ ((w \circledcirc v) \circledcirc u) \cong w \circledcirc (v \circledcirc u),$$

canonically obtained from the 2-cell pasted of the diagram

$$\begin{tikzcd}
 & & & f^*(h\circ g)^*t \arrow[rr, "\chi t"] & & ((h\circ g)\circ f)^*t \arrow[dd, "\boldsymbol{a}^*t"] \\
x \arrow[r, "u"] \arrow[drr, "v\circledcirc u"'] & f^*y \arrow[r, "f^*v"] \arrow[urr, "f^*(w\circledcirc v)"] \arrow[d, phantom, "\cong"] & f^*g^*z \arrow[r, "f^*g^*w"] \arrow[d, "\chi z"'] \arrow[u, phantom, "\cong"] \arrow[dr, phantom, "\overset{\widehat{\chi}}{\cong}"] & f^*g^*h^*t \arrow[u, "f^*\chi t"'] \arrow[d, "\chi h^*t"] & \overset{\omega t}{\cong} & \\
 & & (g\circ f)^*z \arrow[r, "(g\circ f)^*w"'] & (g\circ f)^*h^*t \arrow[rr, "\chi t"'] & & (h\circ(g\circ f))^*t
\end{tikzcd}$$

By Lemma 3.1, these associativity 2-cells are actually isomorphisms in $\int_{\mathcal{B}}\mathfrak{F}$. Furthermore, they are natural thanks to the coherence condition (**C5**), while the pentagon axiom for them holds because of condition (**C6**).

The identity 1-cell, for each object (x,a) in $\int_{\mathcal{B}}\mathfrak{F}$, is provided by the pseudo transformation $\chi_a : 1_{\mathfrak{F}_a} \Rightarrow 1_a^*$, by

$$1_{(x,a)} = (\chi x, 1_a) : (x,a) \to (x,a).$$

The left and right unit constraints, for each morphism $(u,f) : (x,a) \to (y,b)$ in $\int_{\mathcal{B}}\mathfrak{F}$,

$$(\mathring{\boldsymbol{l}},\boldsymbol{l}) : 1_{(y,b)} \circ (u,f) \cong (u,f), \quad (\mathring{\boldsymbol{r}},\boldsymbol{r}) : (u,f) \circ 1_{(x,a)} \cong (u,f),$$

are respectively given by the 2-cells $\boldsymbol{l} : 1_b \circ f \Rightarrow f$ and $\boldsymbol{r} : f \circ 1_a \Rightarrow f$ of $\mathcal{B}$, together with the 2-cells in $\mathfrak{F}_a$ obtained by pasting the diagrams below.

$$\mathring{\boldsymbol{l}} : \begin{tikzcd}
f^*y \arrow[r, "f^*\chi y"] \arrow[drr, "1_{f^*y}"'] & f^*1_b^*y \arrow[r, "\chi y"] \arrow[d, phantom, "\overset{\gamma y}{\cong}"] & (1_b\circ f)^*y \arrow[d, "\boldsymbol{l}^*y"] \\
x \arrow[u, "u"] \arrow[rr, "u"'] \arrow[ur, phantom, "\overset{\boldsymbol{l}}{\cong}"] & & f^*y
\end{tikzcd}
\qquad
\mathring{\boldsymbol{r}} : \begin{tikzcd}
1_a^*x \arrow[r, "1_a^*u"] \arrow[dr, phantom, "\overset{\widehat{\chi}}{\cong}"] & 1_a^*f^*y \arrow[r, "\chi y"] \arrow[dr, phantom, "\overset{\delta y}{\cong}"] & (f\circ 1_a)^*y \arrow[dd, "\boldsymbol{r}^*y"] \\
 & f^*y \arrow[u, "\chi f^*y"'] \arrow[dr, "1_{f^*y}"] \arrow[d, phantom, "\overset{\boldsymbol{l}}{\cong}"] & \\
x \arrow[uu, "\chi x"] \arrow[ur, "u"] \arrow[rr, "u"'] & & f^*y
\end{tikzcd}$$

These unit constraints in $\int_{\mathcal{B}}\mathfrak{F}$ are isomorphisms by Lemma 3.1, natural due to coherence condition (**C7**), and the coherence triangle for them follows from condition (**C8**). Hence, $\int_{\mathcal{B}}\mathfrak{F}$ is actually a bicategory.

As a consequence of the above construction we obtain the following equalities on lax bidiagram of bicategories, which is used many times along the chapter for several proofs:

Lemma 3.2 *Let $\mathfrak{F} = (\mathfrak{F}, \chi, \xi, \omega, \gamma, \delta) : \mathcal{B}^{\mathrm{op}} \to$ **Bicat** be a lax bidiagram of bicategories. The equations on modifications below hold.*

(i) For any object a of $\mathcal{B}$,

$$\begin{tikzcd}
1_a^* \arrow[d, Rightarrow, "1_a^*\chi"'] \arrow[ddrr, Rightarrow, "1"] & & 1_{\mathcal{F}_a} \arrow[ll, Rightarrow, "\chi"'] \arrow[dd, Rightarrow, "\chi"] \\
1_a^*1_a^* \arrow[d, Rightarrow, "\chi"'] & \overset{\gamma}{\Rrightarrow} & \cong \\
(1_a \circ 1_a)^* \arrow[rr, Rightarrow, "l_1^*=r_1^*"'] & & 1_a^*
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
1_a^* \arrow[d, Rightarrow, "1_a^*\chi"'] & \overset{(2.1)}{\cong} & 1_{\mathcal{F}_a} \arrow[ll, Rightarrow, "\chi"'] \arrow[dl, Rightarrow, "\chi"'] \arrow[dd, Rightarrow, "\chi"] \\
1_a^*1_a^* \arrow[d, Rightarrow, "\chi"'] & 1_a^* \arrow[l, Rightarrow, "\chi 1_a^*"'] \arrow[dr, Rightarrow, "1"] \arrow[r, phantom, "\cong"] & {} \\
(1_a \circ 1_a)^* \arrow[rr, Rightarrow, "l_1^*=r_1^*"'] & \overset{\delta}{\Rrightarrow} & 1_a^*
\end{tikzcd}$$

(ii) for every pair of composable 1-cells $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathcal{B}$,

$$\begin{tikzcd}
f^*g^*1_c^* \arrow[d, Rightarrow, "f^*\chi"'] \arrow[dr, Rightarrow, "\chi 1_c^*"] & \overset{(2.1)}{\cong} & f^*g^* \arrow[ll, Rightarrow, "f^*g^*\chi"'] \arrow[d, Rightarrow, "\chi"] \\
f^*(1_c\circ g)^* \arrow[dd, Rightarrow, "\chi"'] & (g\circ f)^*1_c^* \arrow[d, Rightarrow, "\chi"] & (g\circ f)^* \arrow[l, Rightarrow, "(g\circ f)^*\chi"'] \arrow[dd, Rightarrow, "1_{(g\circ f)^*}"] \\
\overset{\omega}{\Rrightarrow} & (1_c\circ(g\circ f))^* \arrow[dr, Rightarrow, "l^*"] & \overset{\gamma}{\Rrightarrow} \\
((1_c\circ g)\circ f)^* \arrow[ur, Rightarrow, "a^*"] \arrow[rr, Rightarrow, "(l\circ 1_f)^*"'] & \overset{\xi}{\cong} & (g\circ f)^*
\end{tikzcd}
=
\begin{tikzcd}
f^*g^*1_c^* \arrow[d, Rightarrow, "f^*\chi"'] & \overset{f^*\gamma}{\Rrightarrow} & f^*g^* \arrow[ll, Rightarrow, "f^*g^*\chi"'] \arrow[dl, Rightarrow, "f^*1_{g^*}"'] \arrow[d, Rightarrow, "\chi"] \\
f^*(1_c\circ g)^* \arrow[r, Rightarrow, "f^*l^*"] \arrow[d, Rightarrow, "\chi"'] & f^*g^* \arrow[r, phantom, "\cong"] \arrow[dr, Rightarrow, "\chi"] & (g\circ f)^* \arrow[d, Rightarrow, "1_{(g\circ f)^*}"] \\
((1_c\circ g)\circ f)^* \arrow[rr, Rightarrow, "(l\circ 1_f)^*"'] & \overset{\chi}{\Rrightarrow} & (g\circ f)^*
\end{tikzcd}$$

$$\begin{tikzcd}
1_a^*f^*g^* \arrow[d, Rightarrow, "1_a^*\chi"'] \arrow[dr, Rightarrow, "\chi g^*"'] & \overset{\delta g^*}{\Rrightarrow} & f^*g^* \arrow[ll, Rightarrow, "\chi f^*g^*"'] \arrow[d, Rightarrow, "1_{f^*g^*}"] \\
1_a^*(g\circ f)^* \arrow[dd, Rightarrow, "\chi"'] & (f\circ 1_a)^*g^* \arrow[r, Rightarrow, "r^*g^*"] \arrow[d, Rightarrow, "\chi"] & f^*g^* \arrow[dd, Rightarrow, "\chi"] \\
\overset{\omega}{\Rrightarrow} & (g\circ(f\circ 1_a))^* \arrow[dr, Rightarrow, "(1_g\circ r)^*"] & \overset{\chi}{\Rrightarrow} \\
((g\circ f)\circ 1_a)^* \arrow[ur, Rightarrow, "a^*"] \arrow[rr, Rightarrow, "r^*"'] & \overset{\xi}{\cong} & (g\circ f)^*
\end{tikzcd}
=
\begin{tikzcd}
1_a^*f^*g^* \arrow[d, Rightarrow, "1_a^*\chi"'] & \overset{(2.1)}{\cong} & f^*g^* \arrow[ll, Rightarrow, "\chi f^*g^*"'] \arrow[dl, Rightarrow, "\chi"'] \arrow[d, Rightarrow, "1_{f^*g^*}"] \\
1_a^*(g\circ f)^* \arrow[d, Rightarrow, "\chi"'] & (g\circ f)^* \arrow[l, Rightarrow, "\chi(g\circ f)^*"'] \arrow[r, phantom, "\cong"] \arrow[dr, Rightarrow, "1_{(g\circ f)^*}"'] & f^*g^* \arrow[d, Rightarrow, "\chi"] \\
((g\circ f)\circ 1_a)^* \arrow[rr, Rightarrow, "r^*"'] & \overset{\delta}{\Rrightarrow} & (g\circ f)^*
\end{tikzcd}$$

Proof: (i) follows from the equality (3.2) in the bicategory $\int_{\mathcal{B}}\mathfrak{F}$, that is, $\boldsymbol{r}_{1_{(x,a)}} = \boldsymbol{l}_{1_{(x,a)}}$, for any $x \in \mathfrak{F}_a$. Similarly, (ii) is consequence of the commutativity of triangles (3.1) in $\int_{\mathcal{B}}\mathfrak{F}$, for any pair of composable 1-cells of the form

$$(f^*g^*x, a) \xrightarrow{(1,f)} (g^*x, b) \xrightarrow{(1,g)} (x, c)\ ,$$

for any $x \in \mathrm{Ob}\mathfrak{F}_c$. $\square$

3.3.1 A cartesian square

Let $\mathfrak{F} = (\mathfrak{F}, \chi, \xi, \omega, \gamma, \delta) : \mathcal{B}^{\mathrm{op}} \to \mathbf{Bicat}$ be any given lax bidiagram of bicategories. For any bicategory $\mathcal{A}$ and any lax functor $F : \mathcal{A} \to \mathcal{B}$, we shall denote by

$$\mathfrak{F}F = (\mathfrak{F}F, \chi_F, \xi_F, \omega_F, \gamma_F, \delta_F) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Bicat} \tag{3.6}$$

the lax bidiagram of bicategories obtained by composing, in the natural sense, $\mathfrak{F}$ with F; that is, the lax bidiagram consisting of the following data:

(**D1**) for each object a in $\mathcal{A}$, the bicategory $\mathfrak{F}_{Fa}$;

(**D2**) for each 1-cell $f : a \to b$ of $\mathcal{A}$, the homomorphism $(Ff)^* : \mathfrak{F}_{Fb} \to \mathfrak{F}_{Fa}$;

(**D3**) for each 2-cell $a \underset{g}{\overset{f}{\underset{\Downarrow\alpha}{\rightrightarrows}}} b$ of $\mathcal{A}$, the pseudo transformation $(F\alpha)^* : (Ff)^* \Rightarrow (Fg)^*$;

(**D4**) for each two composable 1-cells $a \xrightarrow{f} b \xrightarrow{g} c$ in the bicategory $\mathcal{A}$, the pseudo transformation $\chi_{F_{g,f}} : (Ff)^*(Fg)^* \Rightarrow F(g \circ f)^*$ obtained by pasting

$$\begin{tikzcd}
 & \mathfrak{F}_{Fc} \arrow[dl, "(Fg)^*"'] \arrow[dr, "F(g\circ f)^*"] \arrow[dr, bend right=20, "(Fg\circ Ff)^*"'] & \\
\mathfrak{F}_{Fb} \arrow[rr, "(Ff)^*"'] \arrow[r, phantom, "\overset{\chi}{\Rightarrow}"] & \arrow[r, phantom, "\overset{\widehat{F}^*}{\Rightarrow}"] & \mathfrak{F}_{Fa};
\end{tikzcd}$$

(**D5**) for each object a of $\mathcal{A}$, the pseudo transformation

$$\chi_{F_a} = \big(1_{\mathfrak{F}_{Fa}} \overset{\chi_{Fa}}{\Longrightarrow} 1^*_{Fa} \overset{\widehat{F}^*_a}{\Longrightarrow} F(1_a)^*\big);$$

(**D6**) for any two vertically composable 2-cells $f \overset{\alpha}{\Rightarrow} g \overset{\beta}{\Rightarrow} h$ in $\mathcal{A}$, the invertible modification $\xi_{F_{\beta,\alpha}} = \xi_{F\beta,F\alpha} : F(\beta)^* \circ F(\alpha)^* \Rrightarrow F(\beta \cdot \alpha)^*$;

(**D7**) for each 1-cell $f : a \to b$ of $\mathcal{A}$, the invertible modification $\xi_{F_f} = \xi_{Ff} : 1_{F(f)^*} \Rrightarrow 1^*_{Ff}$;

(**D8**) for every two horizontally composable 2-cells $a \underset{h}{\overset{f}{\underset{\rightarrow}{\overset{\rightarrow}{\Downarrow\alpha}}}} b \underset{k}{\overset{g}{\underset{\rightarrow}{\overset{\rightarrow}{\Downarrow\beta}}}} c$ in $\mathcal{A}$,

$$\chi_{F_{\beta,\alpha}} : F(\beta \circ \alpha)^* \circ \chi_{F_{g,f}} \Rrightarrow \chi_{F_{k,h}} \circ (F(\alpha)^* F(\beta)^*)$$

is the invertible modification obtained by pasting the diagram below;

$$\begin{tikzcd}
F(f)^*F(g)^* \arrow[r, Rightarrow, "F(\alpha)^*F(\beta)^*"] \arrow[d, Rightarrow, "\chi"'] & F(h)^*F(k)^* \arrow[r, Rightarrow, "\chi"] \arrow[d, phantom, "\overset{\chi}{\Rrightarrow}"] & (Fk \circ Fh)^* \arrow[d, Rightarrow, "\widehat{F}^*"] \\
(Fg \circ Ff)^* \arrow[r, Rightarrow, "\widehat{F}^*"'] \arrow[urr, Rightarrow, "(F\beta \circ F\alpha)^*"' near end] & F(g \circ f)^* \arrow[r, Rightarrow, "F(\beta\circ\alpha)^*"'] \arrow[ur, phantom, "\overset{\xi}{\Rrightarrow}"] & F(k \circ h)^*
\end{tikzcd}$$

(**D9**) for every three composable 1-cells $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$ in $\mathcal{A}$, the invertible modification

$$\omega_{F_{h,g,f}} : F(\boldsymbol{a})^* \circ (\chi_{F_{h\circ g,f}} \circ F(f)^* \chi_{F_{h,g}}) \Rrightarrow \chi_{F_{h,g\circ f}} \circ \chi_{F_{g,f}} F(h)^*$$

is obtained from the modification pasted of the diagram below;

$$\begin{tikzcd}
F(f)^*F(g)^*F(h)^* \arrow[r, Rightarrow, "\chi F(h)^*"] \arrow[d, Rightarrow, "F(f)^*\chi"'] \arrow[dr, phantom, "\overset{\omega}{\cong}"] & (Fg \circ Ff)^*F(h)^* \arrow[r, Rightarrow, "\widehat{F}^* F(h)^*"] \arrow[dr, Rightarrow, "\chi"] & F(g \circ f)^*F(h)^* \arrow[dr, Rightarrow, "\chi"] \arrow[d, phantom, "\overset{\chi}{\cong}"] & \\
F(f)^*(Fh \circ Fg)^* \arrow[r, Rightarrow, "\chi"] \arrow[d, Rightarrow, "F(f)^*\widehat{F}^*"'] \arrow[dr, phantom, "\overset{\chi}{\cong}"] & ((Fh \circ Fg) \circ Ff)^* \arrow[r, Rightarrow, "\boldsymbol{a}^*"] \arrow[d, Rightarrow, "(\widehat{F}\circ 1)^*"'] & (Fh \circ (Fg \circ Ff))^* \arrow[r, Rightarrow, "(1\circ\widehat{F})^*"] \arrow[d, phantom, "\overset{\xi}{\cong}"] & (Fh \circ F(g \circ f))^* \arrow[d, Rightarrow, "\widehat{F}^*"] \\
F(f)^*F(h \circ g)^* \arrow[r, Rightarrow, "\chi"'] & (F(h \circ g) \circ Ff)^* \arrow[r, Rightarrow, "\widehat{F}^*"'] & F((h \circ g) \circ f)^* \arrow[r, Rightarrow, "F(\boldsymbol{a})^*"'] & F(h \circ (g \circ f))^*
\end{tikzcd}$$

(**D10**) for any 1-cell $f : a \to b$ of $\mathcal{A}$, the invertible modifications

$$\gamma_{F_f} : F(\boldsymbol{l}_f)^* \circ (\chi_{F_{1,f}} \circ F(f)^* \chi_{F_b}) \Rrightarrow 1_{F(f)^*},$$
$$\delta_{F_f} : F(\boldsymbol{r}_f)^* \circ (\chi_{F_{f,1}} \circ \chi_{F_a} F(f)^*) \Rrightarrow 1_{F(f)^*},$$

are, respectively, canonically obtained from the modification pasted of the diagrams below.

$$\begin{tikzcd}
 & F(f)^*F(1_b)^* \arrow[d, Rightarrow, "\chi"'] \arrow[dr, phantom, "\overset{\chi}{\cong}"] & F(f)^*1^*_{Fb} \arrow[l, Rightarrow, "F(f)^*\widehat{F}^*"'] \arrow[d, Rightarrow, "\chi"] \arrow[dr, phantom, "\overset{\gamma}{\cong}"] & F(f)^* \arrow[l, Rightarrow, "F(f)^*\chi"'] \arrow[dd, Rightarrow, "1"] \\
\gamma_F : & (F1_b \circ Ff)^* \arrow[d, Rightarrow, "\widehat{F}^*"'] & (1_{Fb} \circ Ff)^* \arrow[l, Rightarrow, "(\widehat{F}\circ 1)^*"] \arrow[dr, Rightarrow, "\boldsymbol{l}^*"] \arrow[dl, phantom, "\overset{\xi}{\cong}"] & \\
 & F(1_b \circ f)^* \arrow[rr, Rightarrow, "F(\boldsymbol{l})^*"'] & & F(f)^*
\end{tikzcd}$$

$$\begin{tikzcd}
 & F(1_a)^*F(f)^* \arrow[d, Rightarrow, "\chi"'] \arrow[dr, phantom, "\overset{\chi}{\cong}"] & 1^*_{Fa}F(f)^* \arrow[l, Rightarrow, "\widehat{F}^*F(f)^*"'] \arrow[d, Rightarrow, "\chi"] \arrow[dr, phantom, "\overset{\delta}{\cong}"] & F(f)^* \arrow[l, Rightarrow, "\chi F(f)^*"'] \arrow[dd, Rightarrow, "1"] \\
\delta_F : & (Ff\circ F1_a)^* \arrow[d, Rightarrow, "\widehat{F}^*"'] \arrow[dr, phantom, "\overset{\xi}{\cong}"] & (Ff\circ 1_{Fa})^* \arrow[l, Rightarrow, "(1\circ\widehat{F})^*"] \arrow[dr, Rightarrow, "\boldsymbol{r}^*"] & \\
 & F(f\circ 1_a)^* \arrow[rr, Rightarrow, "F(\boldsymbol{r})^*"'] & & F(f)^*
\end{tikzcd}$$

There is an induced lax funtor

$$\bar{F} : \textstyle\int_{\mathcal{A}}\mathfrak{F}F \to \int_{\mathcal{B}}\mathfrak{F} \tag{3.7}$$

given on cells by

$$\begin{tikzcd}
(x,a) \arrow[r, bend left, "{(u,f)}"] \arrow[r, bend right, "{(v,g)}"'] \arrow[r, phantom, "\Downarrow(\phi,\alpha)"] & (y,b) \arrow[r, phantom, "\overset{\bar{F}}{\mapsto}"] & (x,Fa) \arrow[r, bend left, "{(u,Ff)}"] \arrow[r, bend right, "{(v,Fg)}"'] \arrow[r, phantom, "\Downarrow(\phi,F\alpha)"] & (y,Fb),
\end{tikzcd}$$

and whose structure constraints are canonically given by those of F, namely: For every two composable 1-cells $(x,a) \xrightarrow{(u,f)} (y,b) \xrightarrow{(v,g)} (z,c)$ in $\int_{\mathcal{A}}\mathfrak{F}F$, the corresponding structure 2-cell of $\bar{F}$ for their composition is

$$(\boldsymbol{a}^{-1},\widehat{F}) : \bar{F}(v,g)\circ\bar{F}(u,f) \cong \bar{F}((v,g)\circ(u,f)),$$

where $\widehat{F} = \widehat{F}_{g,f} : Fg\circ Ff \Rightarrow F(g\circ f)$ is the structure 2-cell of F, and

$$\boldsymbol{a}^{-1} : \widehat{F}^*_{g,f}\circ(\chi_{Fg,Ff}z\circ(F(f)^*(v)\circ u)) \cong (\widehat{F}^*_{g,f}\circ\chi_{Fg,Ff}z)\circ(F(f)^*(v)\circ u)$$

is the associativity isomorphism in the bicategory $\mathfrak{F}_{Fa}$. For (x,a) any object of the bicategory $\int_{\mathcal{A}}\mathfrak{F}F$, the corresponding structure 2-cell of $\bar{F}$ for its identity is

$$(1,\widehat{F}) : 1_{\bar{F}(x,a)} \Rightarrow \bar{F}1_{(x,a)},$$

where $\widehat{F} = \widehat{F}_a : 1_{Fa} \Rightarrow F(1_a)$ is the structure 2-cell of F, and 1 is the is the identity 2-cell of the 1-cell $\widehat{F}^*_a x\circ\chi_{Fa}x : x \to F(1_a)^*x$ in the bicategory $\mathfrak{F}_{Fa}$.

Then, although the category of bicategories and lax functors has no pullbacks in general, if, for any lax bidiagram of bicategories $\mathfrak{F} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Bicat}$ as above, we denote by

$$P : \textstyle\int_{\mathcal{B}}\mathfrak{F} \to \mathcal{B} \tag{3.8}$$

the canonical projection 2-functor, which is defined by

$$\begin{tikzcd}
(x,a) \arrow[r, bend left, "{(u,f)}"] \arrow[r, bend right, "{(v,g)}"'] \arrow[r, phantom, "\Downarrow(\phi,\alpha)"] & (y,b) \arrow[r, phantom, "\overset{P}{\mapsto}"] & a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b,
\end{tikzcd}$$

the following fact holds:

Lemma 3.3 *Let $\mathfrak{F} : \mathcal{B}^{\mathrm{op}} \to$* **Bicat** *be a lax bidiagram of bicategories. For any lax functor $F : \mathcal{A} \to \mathcal{B}$, the induced square*

$$\begin{tikzcd}
\int_{\mathcal{A}} \mathfrak{F}F \arrow[r, "\bar{F}"] \arrow[d, "P"'] & \int_{\mathcal{B}} \mathfrak{F} \arrow[d, "P"] \\
\mathcal{A} \arrow[r, "F"] & \mathcal{B}
\end{tikzcd}$$

is cartesian in the category of bicategories and lax functors.

Proof: Any pair of lax functors, say $L : \mathcal{C} \to \mathcal{A}$ and $M : \mathcal{C} \to \int_{\mathcal{B}} \mathfrak{F}$, such that $FL = PM$ determines a unique lax functor $N : \mathcal{C} \to \int_{\mathcal{A}} \mathfrak{F}F$

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, dotted, "N"] \arrow[drr, "M"] \arrow[ddr, "L"'] & & \\
& \int_{\mathcal{A}} \mathfrak{F}F \arrow[r, "\bar{F}"] \arrow[d, "P"] & \int_{\mathcal{B}} \mathfrak{F} \arrow[d, "P"] \\
& \mathcal{A} \arrow[r, "F"] & \mathcal{B}
\end{tikzcd}$$

such that $PN = L$ and $\bar{F}N = M$, which is defined as follows: Observe that the lax functor M carries any object $a \in \mathrm{Ob}\mathcal{C}$ to an object of $\int_{\mathcal{B}} \mathfrak{F}$ which is necessarily written in the form $Ma = (Da, FLa)$ for some object Da of the bicategory $\mathfrak{F}_{FLa}$. Similarly, for any 1-cell $f : a \to b$ in $\mathcal{C}$, we have $Mf = (Df, FLf)$, for some 1-cell $Df : Da \to FL(f)^* Db$ in $\mathfrak{F}_{FLa}$, and, for any 2-cell $\alpha : f \Rightarrow g \in \mathcal{C}(a, b)$, we have $M\alpha = (D\alpha, FL\alpha)$, for $D\alpha : FL(\alpha)^* Db \circ Df \Rightarrow Dg$ a 2-cell in $\mathfrak{F}_{FLa}$. Also, for any pair of composable 1-cells $a \xrightarrow{f} b \xrightarrow{g} c$ and any object a in $\mathcal{C}$, the structure 2-cells of M can be respectively written in a similar form as

$$\widehat{M}_{g,f} = (\widehat{D}_{g,f}, F\widehat{L}_{g,f} \circ \widehat{F}_{Lg,Lf}) : (Dg, FLg) \circ (Df, FLf) \Rightarrow (D(g \circ f), FL(g \circ f))$$

$$\widehat{M}_a = (\widehat{D}_a, F(\widehat{L}_a) \circ \widehat{F}_{La}) : 1_{(Da,FLa)} \Rightarrow (D1_a, FL1_a),$$

for some 2-cells $\widehat{D}_{g,f}$ and $\widehat{D}_a$ of the bicategory $\mathfrak{F}_{FLa}$. Then, the claimed $N : \mathcal{C} \to \int_{\mathcal{A}} \mathfrak{F}F$ is the lax functor which acts on cells by

$$\begin{tikzcd}
a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, phantom, "\overset{N}{\mapsto}"] & (Da, La) \arrow[r, bend left, "{(Df,Lf)}"] \arrow[r, bend right, "{(Dg,Lg)}"'] \arrow[r, phantom, "\Downarrow(D\alpha,L\alpha)"] & (Db, Lb)
\end{tikzcd}$$

and whose respective structure 2-cells, for any pair of composable 1-cells $a \xrightarrow{f} b \xrightarrow{g} c$ and any object a in $\mathcal{C}$, are

$$\widehat{N}_{g,f} = (\widehat{D}_{g,f}, \widehat{L}_{g,f}) : (Dg, Lg) \circ (Df, Lf) \Rightarrow (D(g \circ f), L(g \circ f)),$$

$$\widehat{N}_a = (\widehat{D}_a, \widehat{L}_a) : 1_{(Da,La)} \Rightarrow (D1_a, L1_a).$$

□

Remark 3.1 There exist different other 'dual' notions of *bidiagrams of bicategories*, depending on the covariant or contravariant choices for (**D2**) and (**D3**), and the direction of the pseudo transformations χ in (**D4**) and (**D5**), but the results we present about lax bidiagrams are similarly proved for the different cases. For example, in a *covariant oplax bidiagram of bicategories* $\mathfrak{F} : \mathcal{B} \to \mathbf{Bicat}$ the data in (**D2**) are specified with homomorphisms $f_* : \mathfrak{F}_a \to \mathfrak{F}_b$ for the 1-cells $f : a \to b$ of $\mathcal{B}$, while in (**D4**), the pseudo transformations are of the form $\chi_{g,f} : (g \circ f)_* \Rightarrow g_* f_*$. The corresponding data in (**D5**), (**D8**), (**D9**) and (**D10**) change in a natural way. The Grothendieck construction on such a bidiagram, has now 1-cells $(u, f) : (x, a) \to (y, b)$ given by $f : a \to b$ a 1-cell in $\mathcal{B}$ and $u : f_*x \to y$ a 1-cell in $\mathfrak{F}_b$. The 2-cells $(\phi, \alpha) : (u, f) \Rightarrow (v, g)$ are now given by a 2-cell $\alpha : f \Rightarrow g$ in $\mathcal{B}$ and a 2-cell $\phi : u \Rightarrow v \circ \alpha_* x$. The compositions and constraints of this bicategory are defined in the same way as in the contravariant lax case.

3.4 The homotopy cartesian square induced by a lax bidiagram

For the general background on simplicial sets we mainly refer to the book by Goerss and Jardine [68]. In particular, we will use the following result, which can be easily proved from the discussion made in [68, IV, 5.1] and Quillen's Lemma [109, Lemma in page 14] (or [68, IV, Lemma 5.7])[2]:

Lemma 3.4 *Let $p : E \to B$ be an arbitrary simplicial map. For any n-simplex $x \in B_n$, let $p^{-1}(x)$ be the simplicial set defined by the pullback diagram*

$$\begin{tikzcd} p^{-1}(x) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "\Delta x"] & B, \end{tikzcd}$$

where $\Delta[n] = \Delta(-, [n])$ is the standard simplicial n-simplex, whose m-simplices are the maps $[m] \to [n]$ in the simplicial category Δ, and $\Delta x : \Delta[n] \to B$ denotes the simplicial map such that $\Delta x(1_{[n]}) = x$.

---

[2] This is also a consequence of Lemma 4.1.

Suppose that, for every n-simplex $x \in B_n$, and for any map $\sigma : [m] \to [n]$ in the simplicial category, the induced simplicial map $p^{-1}(\sigma^*x) \to p^{-1}(x)$

$$\begin{tikzcd}
p^{-1}(\sigma^*x) \arrow[dd] \arrow[dr, dotted] \arrow[drr] & & \\
& p^{-1}(x) \arrow[r] \arrow[dd] & E \arrow[dd, "p"] \\
\Delta[m] \arrow[dr, "\Delta\sigma"'] \arrow[drr, "\Delta(\sigma^*x)"] & & \\
& \Delta[n] \arrow[r, "\Delta x"'] & B
\end{tikzcd}$$

gives a homotopy equivalence on geometric realizations $|p^{-1}(\sigma^*x)| \simeq |p^{-1}(x)|$. Then, for each vertex $v \in B_0$, the induced square of spaces

$$\begin{tikzcd}
{|p^{-1}(v)|} \arrow[r] \arrow[d] & {|E|} \arrow[d, "|p|"] \\
pt \arrow[r, "|v|"] & {|B|}
\end{tikzcd}$$

is homotopy cartesian, that is, $|p^{-1}(v)|$ is homotopy equivalent to the homotopy fiber of the map $|p| : |E| \to |B|$ over the 0-cell $|v|$ of $|B|$.

Like categories, bicategories are closely related to spaces through the classifying space construction. We shall recall briefly from [41, Theorem 6.1] that the *classifying space* of a (small) bicategory[3]can be defined by means of several, but always homotopy equivalent, simplicial and pseudo simplicial objects that have been characteristically associated to it. For instance, the classifying space $\mathrm{B}\mathcal{B}$ of the bicategory $\mathcal{B}$ may be thought of as

$$\mathrm{B}\mathcal{B} = |\Delta\mathcal{B}|,$$

the geometric realization of its (non-unitary) *geometric nerve* [41, Definition 4.3]; that is, the simplicial set

$$\Delta\mathcal{B} : \Delta^{\mathrm{op}} \to \ \mathbf{Set}, \qquad [n] \mapsto \mathrm{Lax}([n], \mathcal{B}),$$

whose n-simplices are all lax functors $\mathbf{z} : [n] \to \mathcal{B}$. Here, the ordered sets $[n] = \{0, \ldots, n\}$ are considered as categories with only one morphism $(i, j) : i \to j$ when $0 \leq i \leq j \leq n$, so that a non-decreasing map $[m] \to [n]$ is the same as a functor. Hence, a geometric n-simplex of $\mathcal{B}$ is a list of cells of the bicategory

$$\mathbf{z} = (\mathbf{z}_i, \mathbf{z}_{i,j}, \widehat{\mathbf{z}}_{i,j,k}, \widehat{\mathbf{z}}_i)$$

---

[3] See also Section 2.4 in Chapter 2 or Appendix 4.6.

which is geometrically represented by a diagram in $\mathcal{B}$ with the shape of the 2-skeleton of an oriented standard n-simplex, whose faces are triangles

$$\begin{tikzcd}
& \mathbf{z}_j \arrow[dr, "\mathbf{z}_{j,k}"] & \\
\mathbf{z}_i \arrow[ur, "\mathbf{z}_{i,j}"] \arrow[rr, "\mathbf{z}_{i,k}"'] & \Downarrow \widehat{\mathbf{z}}_{i,j,k} & \mathbf{z}_k
\end{tikzcd}$$

with objects $\mathbf{z}_i$ placed on the vertices, 1-cells $\mathbf{z}_{i,j} : \mathbf{z}_i \to \mathbf{z}_j$ on the edges, and 2-cells $\widehat{\mathbf{z}}_{i,j,k} : \mathbf{z}_{j,k} \circ \mathbf{z}_{i,j} \Rightarrow \mathbf{z}_{i,k}$ in the inner, together with 2-cells $\widehat{\mathbf{z}}_i : 1_{\mathbf{z}_i} \Rightarrow \mathbf{z}_{i,i}$. These data are required to satisfy the condition that, for $0 \leq i \leq j \leq k \leq l \leq n$, each tetrahedron is commutative in the sense that

$$\begin{tikzcd}
\mathbf{z}_i \arrow[r, "\mathbf{z}_{i,l}"] \arrow[d, "\mathbf{z}_{i,j}"'] \arrow[dr, "\mathbf{z}_{i,k}" description] \arrow[dr, phantom, "\widehat{\mathbf{z}}\Uparrow", shift left=4ex] \arrow[dr, phantom, "\overset{\widehat{\mathbf{z}}}{\Rightarrow}"', shift right=4ex] & \mathbf{z}_l \\
\mathbf{z}_j \arrow[r, "\mathbf{z}_{j,k}"'] & \mathbf{z}_k \arrow[u, "\mathbf{z}_{k,l}"']
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{z}_i \arrow[r, "\mathbf{z}_{i,l}"] \arrow[d, "\mathbf{z}_{i,j}"'] & \mathbf{z}_l \\
\mathbf{z}_j \arrow[r, "\mathbf{z}_{j,k}"'] \arrow[ur, "\mathbf{z}_{j,l}" description] \arrow[ur, phantom, "\widehat{\mathbf{z}}\Uparrow", shift left=4ex] \arrow[ur, phantom, "\overset{\widehat{\mathbf{z}}}{\Leftarrow}"', shift right=4ex] & \mathbf{z}_k \arrow[u, "\mathbf{z}_{k,l}"']
\end{tikzcd}$$

and, moreover,

$$\begin{tikzcd}
& \mathbf{z}_i \arrow[dl, bend right=50, "1_{\mathbf{z}_i}"'] \arrow[dl, bend left=20, "\mathbf{z}_{i,i}"] \arrow[dl, phantom, "\overset{\widehat{\mathbf{z}}}{\Rightarrow}"', shift right=2ex] \arrow[dr, "\mathbf{z}_{i,j}"] & \\
\mathbf{z}_i \arrow[rr, "\mathbf{z}_{i,j}"'] & \overset{\widehat{\mathbf{z}}}{\Rightarrow} & \mathbf{z}_j
\end{tikzcd}
= \boldsymbol{r}_{\mathbf{z}_{i,j}} \, ,
\qquad
\begin{tikzcd}
& \mathbf{z}_j & \\
\mathbf{z}_j \arrow[ur, bend left=50, "1_{\mathbf{z}_j}"] \arrow[ur, bend right=20, "\mathbf{z}_{j,j}"'] \arrow[ur, phantom, "\overset{\widehat{\mathbf{z}}}{\Rightarrow}", shift left=2ex] & \overset{\widehat{\mathbf{z}}}{\Rightarrow} & \mathbf{z}_i \arrow[ll, "\mathbf{z}_{i,j}"] \arrow[ul, "\mathbf{z}_{i,j}"']
\end{tikzcd}
= \boldsymbol{l}_{\mathbf{z}_{i,j}} .$$

If $\sigma : [m] \to [n]$ is any map in Δ, that is, a functor, the induced $\sigma^* : \Delta\mathcal{B}_n \to \Delta\mathcal{B}_m$ carries any $\mathbf{z} : [n] \to \mathcal{B}$ to $\sigma^*\mathbf{z} = \mathbf{z}\sigma : [m] \to \mathcal{B}$, the composite lax functor of $\mathbf{z}$ with σ.

On a small category $\mathcal{C}$, viewed as a bicategory in which all 2-cells are identities, the geometric nerve construction $\Delta\mathcal{C}$ gives the usual Grothendieck nerve of the category [72], since, for any integer $n \geq 0$, we have $\mathrm{Lax}([n], \mathcal{C}) = \mathrm{Func}([n], \mathcal{C})$. Hence, the space $\mathrm{B}\mathcal{C} = |\Delta\mathcal{C}|$ of a category $\mathcal{C}$, is the usual classifying space of the category, as considered by Quillen in [109]. In particular, the geometric nerve of the category $[n]$ is precisely $\Delta[n]$, the standard simplicial n-simplex, so the notation is not confusing. Furthermore, for any bicategory $\mathcal{B}$, the simplicial map $\Delta\mathbf{z} : \Delta[n] \to \Delta\mathcal{B}$ defined by a n-simplex $\mathbf{z} \in \Delta\mathcal{B}_n$, that is, such that $\Delta\mathbf{z}(1_{[n]}) = \mathbf{z}$, is precisely the simplicial map obtained by taking geometric nerves on the lax functor $\mathbf{z} : [n] \to \mathcal{B}$. Thus, if $\sigma : [m] \to [n]$ is any map in Δ, then

$$\Delta(\sigma^*\mathbf{z}) = \Delta(\mathbf{z}\sigma) = \Delta\mathbf{z}\Delta\sigma : \Delta[m] \to \Delta\mathcal{B}.$$

The following fact[4], which is proved in [41, Proposition 7.1], will be used repeatedly in our subsequent discussions:

---

[4] It also appears as Fact 2.5 in Chapter 2

Lemma 3.5 *If $F, G : \mathcal{A} \to \mathcal{B}$ are two lax functors between bicategories, then any lax or oplax transformation, $\varepsilon : F \Rightarrow G$, canonically defines a homotopy $\mathrm{B}\varepsilon : \mathrm{B}F \simeq \mathrm{B}G$ between the induced maps on classifying spaces $\mathrm{B}F, \mathrm{B}G : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$.*

Suppose that $\mathfrak{F} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$ is a functor, where $\mathcal{B}$ is any small category, such that for every morphism $f : b \to c$ of $\mathcal{B}$ the induced map $\mathrm{B}f^* : \mathrm{B}\mathfrak{F}_c \to \mathrm{B}\mathfrak{F}_b$ is a homotopy equivalence. Then, by Quillen's Lemma [109, Lemma in page 14], the induced commutative square of spaces

$$\begin{tikzcd}
\mathrm{B}\mathfrak{F}_a \arrow[r] \arrow[d] & \mathrm{hocolim}_{\mathcal{B}}\mathrm{B}\mathfrak{F} \arrow[d] \\
pt \arrow[r] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

is homotopy cartesian. By Thomason's Homotopy Colimit Theorem [120], there is a natural homotopy equivalence $\mathrm{hocolim}_{\mathcal{B}}\mathrm{B}\mathfrak{F} \simeq \mathrm{B}\int_{\mathcal{B}}\mathfrak{F}$. Therefore, there is a homotopy cartesian square

$$\begin{tikzcd}
\mathrm{B}\mathfrak{F}_a \arrow[r] \arrow[d] & \mathrm{B}\int_{\mathcal{B}}\mathfrak{F} \arrow[d] \\
pt \arrow[r] & \mathrm{B}\mathcal{B}.
\end{tikzcd}$$

We are now ready to state and prove the following important result in this chapter, which generalizes the result above, as well as the results in [41, Theorem 7.4] and [43, Theorem 4.3]:

Theorem 3.1 *Let $\mathfrak{F} = (\mathfrak{F}, \chi, \xi, \omega, \gamma, \delta) : \mathcal{B}^{\mathrm{op}} \to \mathbf{Bicat}$ be any given lax bidiagram of bicategories. For any object $a \in \mathrm{Ob}\mathcal{B}$, there is a commutative square in* **Bicat**

$$\begin{tikzcd}
\mathfrak{F}_a \arrow[r, "J"] \arrow[d] & \int_{\mathcal{B}}\mathfrak{F} \arrow[d, "P"] \\
{[0]} \arrow[r, "a"] & \mathcal{B}
\end{tikzcd} \tag{3.9}$$

where P is the projection 2-functor (3.8)*, a denotes the normal lax functor carrying* 0 *to a, and J is the natural embedding homomorphism* (3.11) *described below, such that, whenever each 1-cell $f : b \to c$ in $\mathcal{B}$ induces a homotopy equivalence $\mathrm{B}f^* : \mathrm{B}\mathfrak{F}_c \simeq \mathrm{B}\mathfrak{F}_b$, then the square of spaces induced on classifying spaces below is homotopy cartesian.*

$$\begin{tikzcd}
\mathrm{B}\mathfrak{F}_a \arrow[r, "\mathrm{B}J"] \arrow[d] & \mathrm{B}\int_{\mathcal{B}}\mathfrak{F} \arrow[d, "\mathrm{B}P"] \\
pt \arrow[r, "\mathrm{B}a"] & \mathrm{B}\mathcal{B}
\end{tikzcd} \tag{3.10}$$

Proof: This is divided into four parts.

Part 1. Here we exhibit the embedding homomorphism in the square (3.9)

$$J = J(\mathfrak{F}, a) :\ \mathfrak{F}_a \longrightarrow \textstyle\int_{\mathcal{B}} \mathfrak{F}. \tag{3.11}$$

It is defined on cells of $\mathfrak{F}_a$ by

$$\begin{tikzcd}
x \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\phi"] & y \arrow[r, phantom, "\overset{J}{\mapsto}"] & (x,a) \arrow[r, bend left, "{(\chi y\circ u,1_a)}"] \arrow[r, bend right, "{(\chi y\circ v,1_a)}"'] \arrow[r, phantom, "{\Downarrow(\tilde{\phi},1)}"] & (y,a)
\end{tikzcd}$$

where $\chi y \circ u$ is the 1-cell of $\mathfrak{F}_a$ composite of $x \xrightarrow{u} y \xrightarrow{\chi_a y} 1_a^* y$, $1 = 1_{1_a}$, the identity 2-cell in $\mathcal{B}$ of the identity 1-cell of a, and $\tilde{\phi}$ is the 2-cell given by the pasting in the diagram below.

$$\begin{tikzcd}
 & & & 1_a^*y \arrow[dd, bend right, "1"'] \arrow[dd, bend left, "1^*_{1_a}y"] \arrow[dd, phantom, "\overset{\xi}{\cong}"] \\
\tilde{\phi} : & x \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\phi"] & y \arrow[ur, "\chi y"] \arrow[dr, "\chi y"'] \arrow[r, phantom, "\overset{\boldsymbol{l}}{\cong}"] & \\
 & & & 1_a^*y
\end{tikzcd}$$

For $x \xrightarrow{u} y \xrightarrow{v} z$, two composable 1-cells in $\mathfrak{F}_a$, the corresponding constraint 2-cell for their composition is $(\widehat{J}, \boldsymbol{l}) : Jv \circ Ju \cong J(v \circ u)$, where $\boldsymbol{l} = \boldsymbol{l}_{1_a} : 1_a \circ 1_a \cong 1_a$, while $\widehat{J} = \widehat{J}_{v,u}$ is the 2-cell of $\mathfrak{F}_a$ given by pasting the diagram

$$\begin{tikzcd}
 & x \arrow[r, "u"] \arrow[ddr, "v\circ u"'] & y \arrow[r, "\chi y"] \arrow[dd, "v"] \arrow[ddl, phantom, "="] & 1_a^*y \arrow[rr, "1_a^*(\chi z\circ v)"] \arrow[dr, "1_a^*v"'] \arrow[dd, phantom, "\overset{\widehat{\chi}}{\cong}"] & & 1_a^*1_a^*z \arrow[r, "\chi z"] \arrow[dd, phantom, "\overset{\gamma}{\cong}"] & (1_a\circ 1_a)^*z \arrow[dd, "\boldsymbol{l}^*z"] \\
\widehat{J}_{v,u} : & & & \arrow[r, phantom, "\cong"] & 1_a^*z \arrow[ur, "1_a^*\chi z"'] \arrow[drr, "1"] \arrow[d, phantom, "\overset{\boldsymbol{l}}{\cong}"] & & \\
 & & z \arrow[urr, "\chi z"] \arrow[rrrr, "\chi z"] & & {} & & 1_a^*z,
\end{tikzcd}$$

and, for any object x of $\mathfrak{F}_a$, the structure isomorphism for its identity is $(\widehat{J}, 1) : 1_{Jx} \cong J(1_x)$, where $1 = 1_{1_a}$, and $\widehat{J} = \widehat{J}_x$ is provided by pasting the diagram in $\mathfrak{F}_a$ below.

$$\begin{tikzcd}
 & x \arrow[r, "\chi x"] \arrow[d, "1_x"'] \arrow[dr, phantom, "\cong"] & 1_a^*x \arrow[d, bend right, "1"'] \arrow[d, "1^*_{1_a}x"] \arrow[d, phantom, "\overset{\xi}{\cong}"] \\
\widehat{J}_x : & x \arrow[r, "\chi x"] & 1_a^*x
\end{tikzcd}$$

So defined, it is straightforward to verify that J is functorial on vertical composition of 2-cells in $\mathfrak{F}_a$. The naturality of the structure 2-cells $Jv \circ Ju \cong J(v \circ u)$ follows from the coherence conditions in (**C1**) and (**C2**), whereas the hexagon coherence condition for them is verified thanks to conditions (**C1**), (**C2**), and (**C7**), and the result in Lemma 3.2(ii) relating γ with ω. As for the other two coherence conditions, one amounts to the equality in Lemma 3.2(i), and the other is easily checked.

Part 2. Let $\mathbf{z} : [n] \to \mathcal{B}$ be any given geometric n-simplex of the bicategory, $n \geq 0$. Then, as in (3.6), we have a composite lax bidiagram of bicategories $\mathfrak{F}\mathbf{z} : [n] \to \mathbf{Bicat}$. In this part of the proof, we show that the homomorphism

$$J = J(\mathfrak{F}\mathbf{z}, 0) \colon\ \mathfrak{F}_{\mathbf{z}_0} \longrightarrow \textstyle\int_{[n]} \mathfrak{F}\mathbf{z}$$

induces a homotopy equivalence on classifying spaces:

$$\mathrm{B}J \colon \mathrm{B}\mathfrak{F}_{\mathbf{z}_0} \ \simeq\ \mathrm{B}\textstyle\int_{[n]} \mathfrak{F}\mathbf{z}. \tag{3.12}$$

This is a direct consequence of the following general observation:

Lemma 3.6 *Suppose $\mathcal{C}$ is a small category with an initial object 0, and let us regard $\mathcal{C}$ as a bicategory whose 2-cells are all identities. Then, for any lax bidiagram of bicategories $\mathfrak{L} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Bicat}$, the homomorphism $J = J(\mathfrak{L}, 0) : \mathfrak{L}_0 \to \int_{\mathcal{C}} \mathfrak{L}$ induces a homotopy equivalence on classifying spaces, $\mathrm{B}J \colon \mathrm{B}\mathfrak{L}_0 \simeq \mathrm{B}\int_{\mathcal{C}} \mathfrak{L}$.*

Proof: For any object $a \in \mathrm{Ob}\mathcal{C}$, let $0_a : 0 \to a$ denote the unique morphism in $\mathcal{C}$ from the initial object to a. There is a homomorphism

$$K = K(\mathfrak{L}, 0) : \textstyle\int_{\mathcal{C}} \mathfrak{L} \to \mathfrak{L}_0,$$

which carries any object (x, a) to $K(x, a) = 0_a^* x$, the image of x by the homomorphism $0_a^* : \mathfrak{L}_a \to \mathfrak{L}_0$, a 1-cell $(u, f) : (x, a) \to (y, b)$ to

$$K(u, f) = \big(\, 0_a^* x \xrightarrow{\ 0_a^* u\ } 0_a^* f^* y \xrightarrow{\ \chi_{f, 0_a} y\ } (f \circ 0_a)^* y = 0_b^* y \,\big),$$

and a 2-cell $(x, a) \underset{(v,f)}{\overset{(u,f)}{\Downarrow(\phi,1)}} (y, b)$ to the 2-cell $K(\phi, 1) : K(u, f) \Rightarrow K(v, f)$ obtained by pasting the diagram below, where $(A) = \big(\, 1_{0_a^* f^* y} \overset{\widehat{0_a^*}}{\Longrightarrow} 0_a^* 1_{f^* y} \overset{0_a^* \xi}{\Longrightarrow} 0_a^* 1_f^* y \,\big)$.

$$K(\phi, 1) : \quad \begin{tikzcd}
 & & 0_a^* f^* y \arrow[dd, "0_a^* 1_f^* y"'] \arrow[dd, bend left, "1"] \arrow[drr, "\chi y"] & & \\
0_a^* x \arrow[urr, "0_a^* u"] \arrow[drr, "0_a^* v"'] & \Downarrow 0_a^* \phi & \underset{\cong}{(A)} & \overset{\boldsymbol{r}}{\cong} & 0_b^* y \\
 & & 0_a^* f^* y \arrow[urr, "\chi y"'] & &
\end{tikzcd}$$

For each object (x,a) of $\int_{\mathcal{C}}\mathfrak{L}$, the structure isomorphism $\widehat{K}: 1_{K(x,a)} \cong K1_{(x,a)}$ is

$$\widehat{K}: \quad \begin{tikzcd}
0_a^*x \arrow[rr, "0_a^*\chi x"] \arrow[dd, "1"'] & & 0_a^*1_a^*x \arrow[dl, "\chi x"'] \arrow[ddll, bend left=50, "\chi x"] \arrow[dl, phantom, shift left=6ex, "\overset{\boldsymbol{l}}{\cong}"] \\
 & 0_a^*x \arrow[dl, bend right, "1^*_{0_a}x"'] \arrow[dl, bend left, "1"] \arrow[ul, phantom, "\overset{\gamma}{\cong}"] \arrow[dl, phantom, "\cong\xi"] & \\
0_a^*x & &
\end{tikzcd}$$

while the constraint $\widehat{K}: K(v,g)\circ K(u,f) \cong K((v,g)\circ(u,f))$, for each pair of composable 1-cells $(x,a) \overset{(u,f)}{\to} (y,b) \overset{(v,g)}{\to} (z,c)$ of $\int_{\mathcal{C}}\mathfrak{L}$, is given by pasting in $\mathfrak{L}_0$ the diagram below.

$$\widehat{K}: \quad \begin{tikzcd}
0_a^*f^*y \arrow[r, "\chi y"] & 0_b^*y \arrow[r, "0_b^*v"] & 0_b^*g^*z \arrow[rr, "\chi z"] & & 0_c^*z \\
 & 0_a^*f^*y \arrow[u, "\chi y"] \arrow[r, "0_a^*f^*v"'] \arrow[ul, phantom, "="] & 0_a^*f^*g^*z \arrow[u, "\chi g^*z"'] \arrow[ul, phantom, "\overset{\widehat{\chi}}{\cong}"] \arrow[ur, phantom, "\overset{\omega}{\cong}"] & 0_c^*z \arrow[ur, bend left, "1^*_{0_c}z"] \arrow[ur, bend right, "1"'] \arrow[ur, phantom, "\overset{\xi}{\cong}"] & \\
0_a^*x \arrow[uu, "0_a^*u"] \arrow[ur, "0_a^*u"'] \arrow[rrr, "0_a^*(\chi z\circ(f^*v\circ u))"] \arrow[urr, phantom, "\cong"] & & & 0_a^*(g\circ f)^*z \arrow[u, "\chi z"'] \arrow[uur, bend right, "\chi z"'] \arrow[ur, phantom, "\overset{\boldsymbol{l}}{\cong}"] &
\arrow[from=2-3, to=3-4, "0_a^*\chi z"']
\end{tikzcd}$$

There are also two pseudo transformations

$$\varepsilon: JK \Rightarrow 1_{\int_{\mathcal{C}}\mathfrak{L}}, \quad \eta: 1_{\mathfrak{L}_0} \Rightarrow KJ,$$

which are defined as follows: The component of ε at an object (x,a) of $\int_{\mathcal{C}}\mathfrak{L}$ is

$$\varepsilon(x,a) = (1_{0_a^*x}, 0_a) : (0_a^*x, 0) \to (x,a),$$

and its naturality component at a morphism $(u,f):(x,a)\to(y,b)$ is

$$(\widehat{\varepsilon},1): \varepsilon(y,b)\circ JK(u,f) \cong (u,f)\circ\varepsilon(x,a),$$

where $\widehat{\varepsilon}$ is the 2-cell of $\mathfrak{L}_0$ pasted of the diagram below.

$$\begin{tikzcd}
0_a^*x \arrow[r, "0_a^*u"] \arrow[dd, "1"'] & 0_a^*f^*y \arrow[r, "\chi y"] & 0_b^*y \arrow[r, "\chi 0_b^*y"] \arrow[ddrr, "1"'] \arrow[dl, phantom, "\cong"] & 1_0^*0_b^*y \arrow[r, "1_0^*1_{0_b^*y}"] \arrow[r, bend right, "1"'] \arrow[r, phantom, shift right=1ex, "\cong"] \arrow[dr, "\chi y"'] \arrow[d, phantom, "\overset{\delta}{\cong}"] & 1_0^*0_b^*y \arrow[d, "\chi y"] \arrow[dl, phantom, "\overset{\boldsymbol{r}}{\cong}"] \\
 & & & & 0_b^*y \arrow[d, "1^*_{0_b}y"] \\
0_a^*x \arrow[r, "0_a^*u"] & 0_a^*f^*y \arrow[rrr, "\chi y"] & & & 0_b^*y
\end{tikzcd}$$

The pseudo transformation $\eta : 1 \Rightarrow KJ$ assigns to each object x of the bicategory $\mathfrak{L}_0$ the 1-cell $\eta x = \chi x : x \to 1_0^* x$, while its naturality isomorphism at any 1-cell $u : x \to y$,

$$\widehat{\eta} : \eta y \circ u \cong KJ(u) \circ \eta x,$$

is obtained by pasting the diagram below.

$$\begin{tikzcd}
x \arrow[rrrr, "u"] \arrow[ddd, "\chi x"'] \arrow[dr, phantom, "\overset{\widehat{\chi}}{\cong}"] & & & & y \arrow[dlll, "\chi y"'] \arrow[ddd, "\chi y"] \arrow[dl, phantom, "\overset{\boldsymbol{l}}{\cong}"] \\
& 1_0^*y \arrow[dd, "1_0^*\chi y"] \arrow[dddrrr, bend left, "1"] \arrow[dr, phantom, "\overset{\gamma}{\cong}"] & & & \\
& & & 1_0^*y \arrow[dr, bend left, "1_0^*y"] \arrow[dr, bend right, "1"'] \arrow[dr, phantom, "\overset{\xi}{\cong}"] \arrow[d, phantom, "\cong"] & \\
1_0^*x \arrow[uur, "1_0^*u"] \arrow[r, "1_0^*(\chi y \circ u)"'] \arrow[ur, phantom, "\cong"] & 1_0^*1_0^*y \arrow[urr, "\chi y"] \arrow[rrr, "\chi y"'] & & & 1_0^*y
\end{tikzcd}$$

Hence, by Lemma 3.5, there are induced homotopies $\mathrm{B}\varepsilon : \mathrm{B}J\,\mathrm{B}K = \mathrm{B}(JK) \simeq \mathrm{B}1_{\int_{\mathcal{C}}\mathfrak{L}} = 1_{\mathrm{B}\int_{\mathcal{C}}\mathfrak{L}}$ and $\mathrm{B}\eta : 1_{\mathrm{B}\mathfrak{L}_0} = \mathrm{B}1_{\mathfrak{L}_0} \simeq \mathrm{B}(KJ) = \mathrm{B}K\,\mathrm{B}J$, and it follows that both maps $\mathrm{B}J$ and $\mathrm{B}K$ are actually homotopy equivalences. $\square$

Part 3. Let $\sigma : [m] \to [n]$ be a map in the simplicial category. By Lemma 3.3, for any geometric n-simplex $\mathbf{z} : [n] \to \mathcal{B}$ of the bicategory $\mathcal{B}$, we have the square

$$\begin{tikzcd}
\int_{[m]} \mathfrak{F}\mathbf{z}\sigma \arrow[r, "\bar{\sigma}"] \arrow[d, "P"'] & \int_{[n]} \mathfrak{F}\mathbf{z} \arrow[d, "P"] \\
{[m]} \arrow[r, "\sigma"] & {[n]}
\end{tikzcd}$$

which is cartesian in the category of bicategories and lax functors. This part has the goal of proving that the lax functor $\bar{\sigma}$ induces a homotopy equivalence on classifying spaces:

$$\mathrm{B}\bar{\sigma}\colon\ \mathrm{B}\textstyle\int_{[m]} \mathfrak{F}\mathbf{z}\sigma \simeq \mathrm{B}\int_{[n]} \mathfrak{F}\mathbf{z}. \tag{3.13}$$

To do that, let us consider the square of lax functors

$$\begin{tikzcd}
\mathfrak{F}_{\mathbf{z}_{\sigma 0}} \arrow[r, "J=J(\mathfrak{F}\mathbf{z}\sigma{,}0)"] \arrow[d, "\mathbf{z}^*_{0,\sigma 0}"'] & \int_{[m]} \mathfrak{F}\mathbf{z}\sigma \arrow[d, "\bar{\sigma}"] \\
\mathfrak{F}_{\mathbf{z}_0} \arrow[r, "J=J(\mathfrak{F}\mathbf{z}{,}0)"] & \int_{[n]} \mathfrak{F}\mathbf{z},
\end{tikzcd}$$

where $\mathbf{z}^*_{0,\sigma 0}$ is the homomorphism attached by the lax diagram $\mathfrak{F} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Bicat}$ to the 1-cell $\mathbf{z}_{0,\sigma 0} : \mathbf{z}_0 \to \mathbf{z}_{\sigma 0}$ of $\mathcal{B}$, and the homomorphisms J are defined as in (3.11). This square is not commutative, but there is a pseudo transformation $\theta : J\mathbf{z}^*_{0,\sigma 0} \Rightarrow \bar{\sigma}J$, whose component at any object x of $\mathfrak{F}_{\mathbf{z}\sigma 0}$ is the 1-cell of $\int_{[n]}\mathfrak{F}\mathbf{z}$

$$\theta x = (1_{\mathbf{z}^*_{0,\sigma 0}x}, (0, \sigma 0)) : (\mathbf{z}^*_{0,\sigma 0}x, 0) \to (x, \sigma 0),$$

and whose naturality isomorphism, at any 1-cell $u : x \to y$ in $\mathfrak{F}_{\mathbf{z}\sigma 0}$, is

$$\widehat{\theta}_u = (\tilde{\theta}, 1_{\mathbf{z}_{0,\sigma 0}}) : \theta y \circ J\mathbf{z}^*_{0,\sigma 0}u \cong \bar{\sigma}Ju \circ \theta x,$$

where $\tilde{\theta}$ is given by pasting in $\mathfrak{F}_{\mathbf{z}_0}$ the diagram below.

$$\begin{tikzcd}
\mathbf{z}^*_{0,\sigma 0}x \arrow[r, "\mathbf{z}^*_{0,\sigma 0}u"] \arrow[ddd, "1"'] \arrow[dddr, phantom, "\cong"] & \mathbf{z}^*_{0,\sigma 0}y \arrow[rr, "\chi_{\mathbf{z}}\mathbf{z}^*_{0,\sigma 0}y"] \arrow[ddd, "\mathbf{z}^*_{0,\sigma 0}\chi_{\mathbf{z}}y"] \arrow[dddrrr, bend left, "1"] \arrow[ddrr, phantom, "\overset{\gamma}{\cong}"] & & \mathbf{z}^*_{0,0}\mathbf{z}^*_{0,\sigma 0}y \arrow[r, "\mathbf{z}^*_{0,0}1"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\cong"] \arrow[dr, "\chi_{\mathbf{z}}y"'] \arrow[dd, phantom, "\overset{\delta}{\cong}"] & \mathbf{z}^*_{0,0}\mathbf{z}^*_{0,\sigma 0}y \arrow[d, "\chi_{\mathbf{z}}y"] \arrow[d, phantom, "\cong"'] \\
& & & & \mathbf{z}^*_{0,\sigma 0}y \arrow[dd, "1^*_{\mathbf{z}_{0,\sigma 0}}y"] \\
& \arrow[d, phantom, "\cong"] & & \mathbf{z}^*_{0,\sigma 0}y \arrow[dr, "1^*_{\mathbf{z}_{0,\sigma 0}}y"] \arrow[dr, bend right, "1"'] \arrow[dr, phantom, "\overset{\xi}{\cong}"] \arrow[d, phantom, "\cong"] & \\
\mathbf{z}^*_{0,\sigma 0}x \arrow[uuur, "\mathbf{z}^*_{0,\sigma 0}u"] \arrow[r, "\mathbf{z}^*_{0,\sigma 0}(\chi_{\mathbf{z}}y\circ u)"] & \mathbf{z}^*_{0,\sigma 0}\mathbf{z}^*_{\sigma 0,\sigma 0}y \arrow[urr, "\chi_{\mathbf{z}}y"] \arrow[rrr, "\chi_{\mathbf{z}}y"] & & & \mathbf{z}^*_{0,\sigma 0}y
\end{tikzcd}$$

Hence, by Lemma 3.5, the induced square on classifying spaces

$$\begin{tikzcd}
\mathrm{B}\mathfrak{F}_{\mathbf{z}_{\sigma 0}} \arrow[r, "\mathrm{B}J"] \arrow[d, "\mathrm{B}\mathbf{z}^*_{0,\sigma 0}"'] & \mathrm{B}\int_{[m]}\mathfrak{F}\mathbf{z}\sigma \arrow[d, "\mathrm{B}\bar{\sigma}"] \\
\mathrm{B}\mathfrak{F}_{\mathbf{z}_0} \arrow[r, "\mathrm{B}J"] & \mathrm{B}\int_{[n]}\mathfrak{F}\mathbf{z}
\end{tikzcd}$$

is homotopy commutative. Moreover, both maps BJ in the square are homotopy equivalences, as we showed in the proof of Lemma 3.6 above. Since, by hypothesis, the map $\mathrm{B}\mathbf{z}^*_{0,\sigma 0} : \mathrm{B}\mathfrak{F}_{\mathbf{z}_{\sigma 0}} \to \mathrm{B}\mathfrak{F}_{\mathbf{z}_0}$ is also a homotopy equivalence, it follows that the remaining map in the square have the same property, that is, the map $\mathrm{B}\bar{\sigma} : \mathrm{B}\int_{[m]}\mathfrak{F}\mathbf{z}\sigma \longrightarrow \mathrm{B}\int_{[n]}\mathfrak{F}\mathbf{z}$ is a homotopy equivalence.

Part 4. Finally, we are ready to complete here the proof of the theorem as follows: Let us consider the induced simplicial map on geometric nerves $\Delta P : \Delta\int_{\mathcal{B}}\mathfrak{F} \to \Delta\mathcal{B}$. This verifies the hypothesis in Lemma 3.4. In effect, thanks to Lemma 3.3, for any geometric n-simplex of $\mathcal{B}$, $\mathbf{z} : [n] \to \mathcal{B}$, the square

$$\begin{tikzcd}
\int_{[n]}\mathfrak{F}\mathbf{z} \arrow[r, "\bar{\mathbf{z}}"] \arrow[d, "P"'] & \int_{\mathcal{B}}\mathfrak{F} \arrow[d, "P"] \\
{[n]} \arrow[r, "\mathbf{z}"] & \mathcal{B}
\end{tikzcd}$$

is a pullback in the category of bicategories and lax functors, whence the square induced by taking geometric nerves

$$\begin{tikzcd}
\Delta\int_{[n]}\mathfrak{F}\mathbf{z} \arrow[r, "\Delta\bar{\mathbf{z}}"] \arrow[d, "\Delta P"'] & \Delta\int_{\mathcal{B}}\mathfrak{F} \arrow[d, "\Delta P"] \\
\Delta[n] \arrow[r, "\Delta\mathbf{z}"] & \Delta\mathcal{B}
\end{tikzcd}$$

is a pullback in the category of simplicial sets. Thus, $\Delta P^{-1}(\Delta\mathbf{z}) = \Delta\int_{[n]}\mathfrak{F}\mathbf{z}$.

Furthermore, for any map $\sigma : [m] \to [n]$ in the simplicial category, since the diagram of lax functors

$$\begin{tikzcd}
\int_{[m]}\mathfrak{F}\mathbf{z}\sigma \arrow[dr, "\bar{\sigma}"'] \arrow[drr, "\overline{\mathbf{z}\sigma}"] \arrow[dd, "P"'] & & \\
& \int_{[n]}\mathfrak{F}\mathbf{z} \arrow[r, "\bar{\mathbf{z}}"'] \arrow[dd, "P"] & \int_{\mathcal{B}}\mathfrak{F} \arrow[dd, "P"] \\
{[m]} \arrow[dr, "\sigma"'] \arrow[drr, "\mathbf{z}\sigma"] & & \\
& {[n]} \arrow[r, "\mathbf{z}"'] & \mathcal{B}
\end{tikzcd}$$

is commutative, the induced diagram of simplicial maps

$$\begin{tikzcd}
\Delta\int_{[m]}\mathfrak{F}\mathbf{z}\sigma \arrow[dr, "\Delta\bar{\sigma}"'] \arrow[drr, "\Delta\overline{\mathbf{z}\sigma}"] \arrow[dd, "\Delta P"'] & & \\
& \Delta\int_{[n]}\mathfrak{F}\mathbf{z} \arrow[r, "\Delta\bar{\mathbf{z}}"'] \arrow[dd, "\Delta P"] & \Delta\int_{\mathcal{B}}\mathfrak{F} \arrow[dd, "\Delta P"] \\
\Delta[m] \arrow[dr, "\Delta\sigma"'] \arrow[drr, "\Delta(\mathbf{z}\sigma)"] & & \\
& \Delta[n] \arrow[r, "\Delta\mathbf{z}"'] & \Delta\mathcal{B}
\end{tikzcd}$$

is also commutative. Then, as $\sigma^*\mathbf{z} = \mathbf{z}\sigma$, the induced simplicial map $\Delta P^{-1}(\sigma^*\mathbf{z}) \to \Delta P^{-1}(\mathbf{z})$ is precisely the map $\Delta\bar{\sigma} : \Delta\int_{[m]}\mathfrak{F}\mathbf{z}\sigma \to \Delta\int_{[n]}\mathfrak{F}\mathbf{z}$, whose induced map on geometric realizations is the homotopy equivalence (3.13), $\mathrm{B}\bar{\sigma}\colon\ \mathrm{B}\int_{[m]}\mathfrak{F}\mathbf{z}\sigma \simeq \mathrm{B}\int_{[n]}\mathfrak{F}\mathbf{z}$.

Hence, by Lemma 3.4, for each object $a \in \mathrm{Ob}\mathcal{B}$, the square

$$\begin{tikzcd}
{|\Delta P^{-1}(a)|} \arrow[r] \arrow[d] & {|\Delta\int_{\mathcal{B}}\mathfrak{F}|} \arrow[d, "|\Delta P|"] & & \mathrm{B}\int_{[0]}\mathfrak{F}a \arrow[r, "\mathrm{B}\bar{a}"] \arrow[d] & \mathrm{B}\int_{\mathcal{B}}\mathfrak{F} \arrow[d, "\mathrm{B}P"] \\
pt \arrow[r, "|\Delta a|"] & {|\Delta\mathcal{B}|} & = & \mathrm{B}\Delta[0] \arrow[r, "\mathrm{B}a"] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

is homotopy cartesian. Furthermore, since the diagram of lax functors

$$\begin{tikzcd}
\mathfrak{F}a \arrow[dr, "J(\mathfrak{F}a{,}0)"'] \arrow[drr, "J(\mathfrak{F}{,}a)"] \arrow[dd] & & \\
& \int_{[0]}\mathfrak{F}a \arrow[r, "\bar{a}"'] \arrow[dd] & \int_{\mathcal{B}}\mathfrak{F} \arrow[dd, "P"] \\
{[0]} \arrow[dr] \arrow[drr, "a"] & & \\
& {[0]} \arrow[r, "a"'] & \mathcal{B}
\end{tikzcd}$$

commutes, it follows that the square (3.10) is homotopy cartesian square (3.10) as it is the composite of the squares

$$\begin{tikzcd}
\mathrm{B}\mathfrak{F}_a \arrow[r, "\mathrm{B}J(\mathfrak{F}a{,}0)"] \arrow[d] & \mathrm{B}\int_{[0]}\mathfrak{F}a \arrow[r, "\mathrm{B}\bar{a}"] \arrow[d] & \mathrm{B}\int_{\mathcal{B}}\mathfrak{F} \arrow[d, "\mathrm{B}P"] \\
pt \arrow[r] & pt \arrow[r, "\mathrm{B}a"] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

where the map $\mathrm{B}J(\mathfrak{F}a, 0) \colon \mathrm{B}\mathfrak{F}_a \simeq \mathrm{B}\int_{[0]}\mathfrak{F}a$ in the left square is one of the homotopy equivalences (3.12), while the square on the right is homotopy cartesian. ☐

3.5 The homotopy cartesian square induced by a lax functor

In this section we prove the main theorem of this chapter, that is, a generalization to lax functors (monoidal functors, for instance) of the well-known Quillen's Theorem B [109]. We shall first extend Gray's construction [71, Section 3.1] of homotopy fiber 2-categories to *homotopy fiber bicategories* of an arbitrary lax functor between bicategories, so we can state the corresponding 'Theorem B' in terms of them.

Let $F : \mathcal{A} \to \mathcal{B}$ be any given lax functor between bicategories. As in Example 3.2, each object b of $\mathcal{B}$ gives rise to a pseudo bidiagram of categories

$$\mathcal{B}(-, b) : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat},$$

which carries an object $x \in \mathrm{Ob}\mathcal{B}$ to the hom-category $\mathcal{B}(x, b)$, and then also to the lax bidiagram of categories

$$\mathcal{B}(-, b)F \colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}, \tag{3.14}$$

obtained, as in (3.6), by composing $\mathcal{B}(-, b)$ with F. The Grothendieck construction on these lax bidiagrams leads to the notions of *homotopy fiber* and *comma bicategories*:

Definition 3.1 *The* homotopy fiber, $F\downarrow_b$, *of a lax functor between bicategories* $F:\mathcal{A}\to\mathcal{B}$ *over an object* $b\in\mathrm{Ob}\mathcal{B}$, *is the bicategory obtained as the Grothendieck construction on the lax bidiagram* (3.14), *that is,*

$$F{\downarrow_b}=\textstyle\int_{\mathcal{A}}\mathcal{B}(-,b)F.$$

In particular, when $F=1_{\mathcal{B}}$ *is the identity functor on* $\mathcal{B}$,

$$\mathcal{B}{\downarrow_b}=\textstyle\int_{\mathcal{B}}\mathcal{B}(-,b)$$

is the comma bicategory *of objects over* b *of the bicategory* $\mathcal{B}$.

It will be useful to develop here the Grothendieck construction, exposed in Section 3.3, in this particular case. Its objects are pairs

$$(f\colon Fa\to b,a)\tag{3.15}$$

with a a 0-cell of $\mathcal{A}$ and f a 1-cell of $\mathcal{B}$ whose source is Fa and target the fixed object b. The 1-cells

$$(\beta,u):(f,a)\to(f',a')\tag{3.16}$$

consist of a 1-cell $u:a\to a'$ in $\mathcal{A}$, together with a 2-cell $\beta:f\Rightarrow f'\circ Fu$ in the bicategory $\mathcal{B}$,

$$\begin{tikzcd}
Fa \arrow[rr, "Fu"] \arrow[dr, "f"'] & \overset{\beta}{\Rightarrow} & Fa' \arrow[dl, "f'"] \\
& b &
\end{tikzcd}$$

A 2-cell in $F{\downarrow_b}$,

$$\begin{tikzcd}
(f,a) \arrow[r, bend left, "{(\beta,u)}"] \arrow[r, bend right, "{(\beta',u')}"'] \arrow[r, phantom, "\Downarrow\alpha"] & (f',a')
\end{tikzcd}\ ,\tag{3.17}$$

is a 2-cell $\alpha:u\Rightarrow u'$ in $\mathcal{A}$, such that the equation below holds in the category $\mathcal{B}(Fa,b)$.

$$\begin{tikzcd}
Fa \arrow[rr, "Fu'"] \arrow[rr, bend right, "Fu"'] \arrow[rr, phantom, bend right=15, "\Uparrow F\alpha"] \arrow[dr, "f"'] & \overset{\beta}{\Rightarrow} & Fa' \arrow[dl, "f'"] \\
& b &
\end{tikzcd}
=
\begin{tikzcd}
Fa \arrow[rr, "Fu'"] \arrow[dr, "f"'] & \overset{\beta'}{\Rightarrow} & Fa' \arrow[dl, "f'"] \\
& b &
\end{tikzcd}\tag{3.18}$$

Compositions, identities, and the structure associativity and unit constraints in $F{\downarrow_b}$ are as follows: For any given objects (f,a) and (f',a') as in (3.15), the vertical composition of 2-cells

$$\begin{tikzcd}
(f,a) \arrow[r, bend left=50, "{(\beta,u)}"] \arrow[r, "{(\beta',u')}" description] \arrow[r, bend right=50, "{(\beta'',u'')}"'] \arrow[r, phantom, bend left=25, "\Downarrow\alpha"] \arrow[r, phantom, bend right=25, "\Downarrow\alpha'"] & (f',a')
\end{tikzcd}
\ \overset{\cdot}{\mapsto}\
\begin{tikzcd}
(f,a) \arrow[r, bend left, "{(\beta,u)}"] \arrow[r, bend right, "{(\beta'',u'')}"'] \arrow[r, phantom, "\Downarrow\alpha'\cdot\alpha"] & (f',a')
\end{tikzcd}$$

is given by the vertical composition $\alpha' \cdot \alpha$ of 2-cells in $\mathcal{A}$. The horizontal composition of two 1-cells in $F{\downarrow}_b$,

$$(f,a) \xrightarrow{(\beta,u)} (f',a') \xrightarrow{(\gamma,v)} (f'',a'')$$

is the 1-cell

$$(\gamma,v)\circ(\beta,u) = (\gamma \circledcirc \beta, v\circ u) : (f,a) \to (f'',a''),$$

where the second component is the horizontal composition $v \circ u$ in $\mathcal{A}$, while the first one is the 2-cell in $\mathcal{B}$ obtained by pasting the diagram below.

$$\gamma \circledcirc \beta : \quad \begin{tikzcd} Fa \arrow[r, "Fu"] \arrow[rr, bend left, "F(v\circ u)"] \arrow[rr, phantom, bend left=15, "\widehat{F}\Uparrow"] \arrow[dr, "f"'] & Fa' \arrow[r, "Fv"] \arrow[d, "f'"] \arrow[dl, phantom, near start, "\overset{\beta}{\Rightarrow}"] \arrow[dr, phantom, near start, "\overset{\gamma}{\Rightarrow}"] & Fa'' \arrow[dl, "f''"] \\ & b & \end{tikzcd} \tag{3.19}$$

The horizontal composition of 2-cells is simply given by the horizontal composition of 2-cells in $\mathcal{B}$,

$$\begin{tikzcd}[column sep=large] (f,a) \arrow[r, bend left, "(\beta,u)"] \arrow[r, bend right, "(\beta',u')"'] \arrow[r, phantom, "\Downarrow\alpha"] & (f',a') \arrow[r, bend left, "(\gamma,v)"] \arrow[r, bend right, "(\gamma',v')"'] \arrow[r, phantom, "\Downarrow\alpha'"] & (f'',a'') \arrow[r, phantom, "\mapsto"] & (f,a) \arrow[r, bend left, "(\gamma\circledcirc\beta,v\circ u)"] \arrow[r, bend right, "(\gamma'\circledcirc\beta',v'\circ u')"'] \arrow[r, phantom, "\Downarrow\alpha'\circ\alpha"] & (f'',a'') \end{tikzcd} ,$$

and the identity 1-cell of each 0-cell $(f : Fa \to b, a)$ is

$$1_{(f,a)} = (\overset{\circ}{1}_{(f,a)}, 1_a) : (f,a) \to (f,a),$$
$$\overset{\circ}{1}_{(f,a)} = \big(\, f \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} f \circ 1_{Fa} \overset{1_f \circ \widehat{F}}{\Longrightarrow} f \circ F(1_a) \,\big)$$

Finally, the associativity, left and right unit constraints are obtained from those of $\mathcal{A}$ by the formulas

$$\boldsymbol{a}_{(\beta'',u''),(\beta',u'),(\beta,u)} = \boldsymbol{a}_{u'',u',u}, \quad \boldsymbol{r}_{(\beta,u)} = \boldsymbol{r}_u, \quad \boldsymbol{l}_{(\beta,u)} = \boldsymbol{l}_u.$$

We shall prove below that, under reasonable necessary conditions, the classifying spaces of the homotopy fiber bicategories $\mathrm{B}(F{\downarrow}_b)$, of a lax functor $F : \mathcal{A} \to \mathcal{B}$, realize the homotopy fibers of the induced map on classifying spaces, $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$. This fact will justify the name of 'homotopy fiber bicategories' for them. As a first step to do it, we state the following particular case, when $F = 1_{\mathcal{B}}$ is the identity homomorphism:

Lemma 3.7 *For any object b of a bicategory $\mathcal{B}$, the classifying space of the comma bicategory $\mathcal{B}{\downarrow}_b$ is contractible, that is,* $\mathrm{B}(\mathcal{B}{\downarrow}_b) \simeq pt$.

Proof: Let $[0] \to \mathcal{B}{\downarrow}_b$ denote the normal lax functor that carries 0 to the object $(1_b, b)$, and let $\mathrm{Ct} : \mathcal{B}{\downarrow}_b \to \mathcal{B}{\downarrow}_b$ be the composite of $\mathcal{B}{\downarrow}_b \to [0] \to \mathcal{B}{\downarrow}_b$. Then, the induced map on classifying spaces

$$\mathrm{B}(\mathcal{B}{\downarrow}_b) \xrightarrow{\mathrm{BCt}} \mathrm{B}(\mathcal{B}{\downarrow}_b) \;=\; \mathrm{B}(\mathcal{B}{\downarrow}_b) \longrightarrow \mathrm{B}[0] = pt \longrightarrow \mathrm{B}(\mathcal{B}{\downarrow}_b)$$

is a constant map. Now, let us observe that there is a canonical oplax transformation $1_{\mathcal{B}\downarrow_b} \Rightarrow \mathrm{Ct}$, whose component at any object $(f : a \to b, a)$ is the 1-cell $(\boldsymbol{l}_f^{-1}, f) : (f, a) \to (1_b, b)$, and whose naturality component at a 1-cell $(\beta, u) : (f, a) \to (f', a')$ is

$$\begin{tikzcd}
(f,a) \arrow[r, "(\boldsymbol{l}^{-1}{,}f)"] \arrow[d, "(\beta{,}u)"'] \arrow[dr, phantom, "\overset{\beta\cdot\boldsymbol{l}}{\Leftarrow}"] & (1_b,b) \arrow[d, "(\boldsymbol{r}^{-1}{,}1_b)"] \\
(f',a') \arrow[r, "(\boldsymbol{l}^{-1}{,}f')"'] & (1_b,b).
\end{tikzcd}$$

This oplax transformation gives, thanks to Lemma 3.5, a homotopy between $\mathrm{B}(1_{\mathcal{B}\downarrow_b}) = 1_{\mathrm{B}(\mathcal{B}\downarrow_b)}$ and the constant map BCt, and so we obtain the result. □

Example 3.3 Let $\mathcal{B}$ be a bicategory, and suppose $b \in \mathrm{Ob}\mathcal{B}$ is an object such that the induced maps $\mathrm{B}p^* : \mathrm{B}\mathcal{B}(y, b) \to \mathrm{B}\mathcal{B}(x, b)$ are homotopy equivalences for the different morphisms $p : x \to y$ in $\mathcal{B}$ (for instance, any object of a bigroupoid). By Theorem 3.1, we have the fiber sequence

$$\mathrm{B}\mathcal{B}(b, b) \to \mathrm{B}\mathcal{B}{\downarrow}_b \to \mathrm{B}\mathcal{B}$$

in which the space $\mathrm{B}\mathcal{B}{\downarrow}_b$ is contractible by Lemma 3.7. Hence, we conclude the existence of a homotopy equivalence

$$\Omega(\mathrm{B}\mathcal{B}, \mathrm{B}b) \simeq \mathrm{B}(\mathcal{B}(b, b)) \tag{3.20}$$

between the loop space of the classifying space of the bicategory with base point Bb and the classifying space of the category of endomorphisms of b in $\mathcal{B}$.

The homotopy equivalence above is already known when the bicategory is strict, that is, when $\mathcal{B}$ is a 2-category. It appears as a main result in the paper by del Hoyo [84, Theorem 8.5], and it was also stated at the same time by Cegarra in [43, Example 4.4]. Indeed, that homotopy equivalence (3.20), for the case when $\mathcal{B}$ is a 2-category, can be deduced from a result by Tillmann about simplicial categories in [121, Lemma 3.3].

Returning to an arbitrary lax functor $F : \mathcal{A} \to \mathcal{B}$, we shall now pay attention to two constructions with fiber homotopy bicategories. First, we have that any 1-cell $p : b \to b'$ in $\mathcal{B}$ determines a 2-functor

$$p_*\colon F{\downarrow}_b \to F{\downarrow}_{b'} \tag{3.21}$$

whose function on objects is defined by

$$p_*(Fa \xrightarrow{f} b, a) = (Fa \xrightarrow{p\circ f} b', a).$$

A 1-cell $(\beta, u) : (f, a) \to (f', a')$ of $\mathcal{B}{\downarrow_b}$, as in (3.16) , is carried to the 1-cell of $\mathcal{B}{\downarrow_{b'}}$

$$p_*(\beta, u) = (p \circledcirc \beta, u) : (p \circ f, a) \to (p \circ f', a'),$$
$$p \circledcirc \beta = \big(\, p \circ f \overset{1_p\circ\beta}{\Longrightarrow} p \circ (f' \circ Fu) \overset{\boldsymbol{a}^{-1}}{\Longrightarrow} (p \circ f') \circ Fu \,\big)$$

while, for $\alpha : (\beta, u) \Rightarrow (\beta', u')$ any 2-cell in $\mathcal{B}{\downarrow_b}$ as in (3.17),

$$p_*(\alpha) = \alpha : (p \circledcirc \beta, u) \Rightarrow (p \circledcirc \beta', u').$$

Secondly, by Lemma 3.3, we have a pullback square in the category of bicategories and lax functors for any $b \in \mathrm{Ob}\mathcal{B}$

$$\begin{tikzcd}
F{\downarrow_b} \arrow[r, "\bar{F}"] \arrow[d, "P"'] & \mathcal{B}{\downarrow_b} \arrow[d, "P"] & = & \int_{\mathcal{A}} \mathcal{B}(-,b)F \arrow[r, "\bar{F}"] \arrow[d, "P"'] & \int_{\mathcal{B}} \mathcal{B}(-,b) \arrow[d, "P"] \\
\mathcal{A} \arrow[r, "F"] & \mathcal{B} & & \mathcal{A} \arrow[r, "F"] & \mathcal{B}
\end{tikzcd} \tag{3.22}$$

where, recall, the 2-functors P are the canonical projections (3.8), and $\bar{F}$ is the induced lax functor (3.7), which acts on cells by

$$\begin{tikzcd}
(f,a) \arrow[r, bend left, "{(\beta,u)}"] \arrow[r, bend right, "{(\beta',u')}"'] \arrow[r, phantom, "\Downarrow\alpha"] & (f',a') & \overset{\bar{F}}{\mapsto} & (f,Fa) \arrow[r, bend left, "{(\beta,Fu)}"] \arrow[r, bend right, "{(\beta',Fu')}"'] \arrow[r, phantom, "\Downarrow F\alpha"] & (f',Fa'),
\end{tikzcd}$$

and whose structure constraints are canonically given by those of F.

We are now ready to state and prove the following theorem, which is just the well-known Quillen's Theorem B [109] when the lax functor F in the hypothesis is an ordinary functor between categories. The result therein also generalizes a similar result by Cegarra [43, Theorem 3.2], which was stated for the case when F is a 2-functor between 2-categories, but the extension to arbitrary lax functors between bicategories is highly nontrivial and the proof we give here uses different tools.

Theorem 3.2 *Let $F : \mathcal{A} \to \mathcal{B}$ be a lax functor between bicategories. The following statements are equivalent:*

(i) For every 1-cell $p : b \to b'$ in $\mathcal{B}$, the induced map $\mathrm{B}p_* : \mathrm{B}(F{\downarrow_b}) \to \mathrm{B}(F{\downarrow_{b'}})$ is a homotopy equivalence.

(ii) For every object b of $\mathcal{B}$, the induced square by (3.22) on classifying spaces

$$\begin{tikzcd}
\mathrm{B}(F{\downarrow_b}) \arrow[r, "\mathrm{B}\bar{F}"] \arrow[d, "\mathrm{B}P"'] & \mathrm{B}(\mathcal{B}{\downarrow_b}) \arrow[d, "\mathrm{B}P"] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B}
\end{tikzcd} \tag{3.23}$$

is homotopy cartesian.

Therefore, in such a case, for each object $a \in \mathrm{Ob}\mathcal{A}$ *such that* $Fa = b$, *there is a homotopy fiber sequence*

$$\mathrm{B}(F{\downarrow}_b) \to \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B},$$

relative to the base 0-cells $\mathrm{B}a$ *of* $\mathrm{B}\mathcal{A}$, $\mathrm{B}b$ *of* $\mathrm{B}\mathcal{B}$ *and* $\mathrm{B}(1_b, a)$ *of* $\mathrm{B}(F{\downarrow}_b)$, *that induces a long exact sequence on homotopy groups*

$$\cdots \to \pi_{n+1}\mathrm{B}\mathcal{B} \to \pi_n\mathrm{B}(F{\downarrow}_b) \to \pi_n\mathrm{B}\mathcal{A} \to \pi_n\mathrm{B}\mathcal{B} \to \cdots .$$

Proof: $(ii) \Rightarrow (i)$ Suppose that $p : b \to b'$ is any 1-cell of $\mathcal{B}$. Then, taking $\mathbf{z} : [1] \to \mathcal{B}$ the normal lax functor such that $\mathbf{z}_{0,1} = p$, we have the path $\mathrm{B}\mathbf{z} : \mathrm{B}[1] = I \to \mathrm{B}\mathcal{B}$, whose origen is the point $\mathrm{B}a$ and whose end is $\mathrm{B}b$ (actually, $\mathrm{B}\mathcal{B}$ is a CW-complex and $\mathrm{B}\mathbf{z}$ is one of its 1-cells). Since the homotopy fibers of a continuous map whose over points are connected by a path are homotopy equivalent, the result follows.

$(i) \Rightarrow (ii)$ This is divided into three parts.

Part 1. We begin here by noting that the bicategorical homotopy fiber construction is actually the function on objects of a covariant oplax bidiagram of bicategories

$$F{\downarrow} = (F{\downarrow}\,, \chi, \xi, \omega, \gamma, \delta) : \mathcal{B} \to \mathbf{Bicat}$$

consisting of the following data:

(**D1**) for each object b in $\mathcal{B}$, the homotopy fiber bicategory $F{\downarrow}_b$;

(**D2**) for each 1-cell $p : b \to b'$ of $\mathcal{B}$, the 2-functor $p_*\colon F{\downarrow}_b \to F{\downarrow}_{b'}$ in (3.21);

(**D3**) for each 2-cell $\begin{tikzcd} b \arrow[r, bend left, "p"] \arrow[r, bend right, "p'"'] \arrow[r, phantom, "\Downarrow\sigma"] & b' \end{tikzcd}$ of $\mathcal{B}$, the pseudo transformation $\sigma_* : p_* \Rightarrow p'_*$, whose component at an object (f, a) of $F{\downarrow}_b$, is the 1-cell

$$\sigma_*(f, a) = (\sigma \circledcirc f, 1_a) : (p \circ f, a) \to (p' \circ f, a),$$
$$\sigma \circledcirc f = \big(\, p \circ f \overset{\sigma\circ 1}{\Longrightarrow} p' \circ f \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} (p' \circ f) \circ 1_{Fa} \overset{1\circ\widehat{F}}{\Longrightarrow} (p' \circ f) \circ F1_a \,\big)$$

and whose naturality component at any 1-cell $(\beta, u) : (f, a) \to (f', a')$, as in (3.16), is the canonical isomorphism $\boldsymbol{r}^{-1} \cdot \boldsymbol{l} : 1_{a'} \circ u \cong u \circ 1_a$;

$$\begin{tikzcd}
(p \circ f, a) \arrow[r, "(\sigma\circledcirc f{,}1_a)"] \arrow[d, "(p\circledcirc\beta{,}u)"'] \arrow[dr, phantom, "\boldsymbol{r}^{-1}_{\cong}\cdot\boldsymbol{l}"] & (p' \circ f, a) \arrow[d, "(p'\circledcirc\beta{,}u)"] \\
(p \circ f', a') \arrow[r, "(\sigma\circledcirc f'{,}1_{a'})"'] & (p' \circ f', a')
\end{tikzcd}$$

(**D4**) for each two composable 1-cells $b \xrightarrow{p} b' \xrightarrow{p'} b''$ in the bicategory $\mathcal{B}$, the pseudo transformation $\chi_{p',p} : (p' \circ p)_* \Rightarrow p'_* p_*$ has component, at an object (f, a) of $F{\downarrow}_b$, the

1-cell

$$(\boldsymbol{r}\mathrm{a}, 1_a) : ((p' \circ p) \circ f, a) \to (p' \circ (p \circ f), a),$$
$$\boldsymbol{r}\mathrm{a} = \big(\ (p' \circ p) \circ f \overset{\boldsymbol{a}}{\Longrightarrow} p' \circ (p \circ f) \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} (p' \circ (p \circ f)) \circ 1_{Fa} \overset{1 \circ \widehat{F}}{\Longrightarrow} (p' \circ (p \circ f)) \circ F1_a\ \big)$$

and whose naturality component at a 1-cell $(\beta, u) : (f, a) \to (f', a')$, is

$$\begin{tikzcd}
((p' \circ p) \circ f, a) \arrow[r, "(\boldsymbol{r}\mathrm{a}{,}1)"] \arrow[d, "((p'\circ p)\circledcirc\beta){,}u)"'] \arrow[dr, phantom, "\boldsymbol{r}^{-1}_{\cong}\cdot \boldsymbol{l}"] & (p' \circ (p \circ f), a) \arrow[d, "(p'\circledcirc(p\circledcirc\beta){,}u)"] \\
((p' \circ p) \circ f', a') \arrow[r, "(\boldsymbol{r}\mathrm{a}{,}1)"'] & (p' \circ (p \circ f'), a');
\end{tikzcd}$$

(**D5**) for each object b of $\mathcal{B}$, $\chi_b : 1_{b*} \Rightarrow 1_{F\downarrow_b}$ is the pseudo transformation whose component at any object (f, a) is the 1-cell

$$(\mathring{1} \cdot \boldsymbol{l}, 1_a) : (1_b \circ f, a) \to (f, a),$$
$$\mathring{1} \cdot \boldsymbol{l} = \big(\ 1_b \circ f \overset{\boldsymbol{l}}{\Longrightarrow} f \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} f \circ 1_{Fa} \overset{1 \circ \widehat{F}}{\Longrightarrow} f \circ F1_a\ \big)$$

and whose naturality component, at a 1-cell $(\beta, u) : (f, a) \to (f', a')$, is

$$\begin{tikzcd}
(1_b \circ f, a) \arrow[r, "(\mathring{1}{,}1)"] \arrow[d, "(1_b\circledcirc\beta{,}u)"'] \arrow[dr, phantom, "\boldsymbol{r}^{-1}_{\cong}\cdot \boldsymbol{l}"] & (f, a) \arrow[d, "(\beta{,}u)"] \\
(1_b \circ f', a') \arrow[r, "(\mathring{1}{,}1)"'] & (f', a');
\end{tikzcd}$$

(**D6**) for any two vertically composable 2-cells $p \overset{\sigma}{\Rightarrow} p' \overset{\tau}{\Rightarrow} p''$ in $\mathcal{B}$, the invertible modification $\xi_{\tau,\sigma} : \tau_* \circ \sigma_* \Rrightarrow (\tau \cdot \sigma)_*$ has component, at any object (f, a), the canonical isomorphism $\boldsymbol{l} : 1_a \circ 1_a \cong 1_a$

$$\begin{tikzcd}
& (p \circ f, a) \arrow[dl, "(\sigma\circledcirc f{,}1_a)"'] \arrow[dr, "((\tau\cdot\sigma)\circledcirc f{,}1_a)"] \arrow[d, phantom, "\overset{\boldsymbol{l}}{\cong}"] & \\
(p' \circ f, a) \arrow[rr, "(\tau\circledcirc f{,}1_a)"'] & & (p'' \circ f, a);
\end{tikzcd}$$

(**D7**) for each 1-cell $p : b \to b'$ of $\mathcal{B}$, $(1_p)_* = 1_{p*}$, and ξ_p is the identity modification;

(**D8**) for every two horizontally composable 2-cells $\begin{tikzcd} b \arrow[r, bend left, "p"] \arrow[r, bend right, "q"'] \arrow[r, phantom, "\Downarrow\sigma"] & b' \arrow[r, bend left, "p'"] \arrow[r, bend right, "q'"'] \arrow[r, phantom, "\Downarrow\tau"] & b'' \end{tikzcd}$ in $\mathcal{B}$, the equality $(\tau_* \sigma_*) \circ \chi_{p',p} = \chi_{q',q} \circ (\tau \circ \sigma)_*$ holds and the modification $\chi_{\tau,\sigma}$ is the identity;

(**D9**) for every three composable 1-cells $b \xrightarrow{p} b' \xrightarrow{p'} b'' \xrightarrow{p''} b'''$ in $\mathcal{B}$, the invertible modification $\omega_{p'',p',p}$, at any object (f,a), is the canonical isomorphism $\boldsymbol{r} : (1_a \circ 1_a) \circ 1_a \cong 1_a \circ 1_a$,

$$\begin{tikzcd}
(((p'' \circ p') \circ p) \circ f, a) \arrow[rr, "(\boldsymbol{a} \circledcirc f{,}1_a)"] \arrow[d, "(\boldsymbol{r}\mathrm{a}{,}1_a)"'] & \overset{\boldsymbol{r}}{\cong} & ((p'' \circ (p' \circ p)) \circ f, a) \arrow[d, "(\boldsymbol{r}\mathrm{a}{,}1_a)"] \\
((p'' \circ p') \circ (p \circ f), a) \arrow[r, "(\boldsymbol{r}\mathrm{a}{,}1_a)"] & (p'' \circ (p' \circ (p \circ f)), a) & (p'' \circ ((p' \circ p) \circ f), a); \arrow[l, "(p'' \circledcirc \boldsymbol{r}\mathrm{a}{,}1_a)"']
\end{tikzcd}$$

(**D10**) for any 1-cell $p : b \to b'$ of $\mathcal{B}$, the invertible modifications γ_p and δ_p, at any object (f,a) are given by the canonical isomorphism $1_a \circ (1_a \circ 1_a) \cong 1_a$,

$$\begin{tikzcd}
(1_{b'} \circ (p \circ f), a) \arrow[r, "(\mathring{1} \cdot \boldsymbol{l}{,}1_a)"] \arrow[rd, phantom, "\overset{\boldsymbol{r} \cdot \boldsymbol{r}}{\cong}"] & (p \circ f, a) \arrow[d, "(\mathring{1}{,}1_a)"] & (p \circ (1_{b'} \circ f), a) \arrow[r, "(p \circledcirc (\mathring{1} \cdot \boldsymbol{l}){,}1_a)"] \arrow[rd, phantom, "\overset{\boldsymbol{r} \cdot \boldsymbol{r}}{\cong}"] & (p \circ f, a) \arrow[d, "(\mathring{1}{,}1_a)"] \\
((1_{b'} \circ p) \circ f, a) \arrow[u, "(\boldsymbol{r}\mathrm{a}{,}1_a)"] \arrow[r, "(\boldsymbol{l} \circledcirc f{,}1_a)"'] & (p \circ f, a) & ((p \circ 1_{b'}) \circ f, a) \arrow[u, "(\boldsymbol{r}\mathrm{a}{,}1_a)"] \arrow[r, "(\boldsymbol{r} \circledcirc f{,}1_a)"'] & (p \circ f, a).
\end{tikzcd}$$

Observe that all the 2-cells given above are well defined since all the data is obtained from the constraints of the bicategories involved and the lax functor F. Then the coherence conditions of these give us the equality (3.18) in each case. For the same reason the axioms (**C1**) – (**C8**) hold.

Part 2. In this part, we consider the Grothendieck construction on the oplax bidiagram of homotopy fibers $F{\downarrow} : \mathcal{B} \to \mathbf{Bicat}$, and we shall prove the following:

Lemma 3.8 *There is a homomorphism*

$$Q : \textstyle\int_{\mathcal{B}} F{\downarrow} \to \mathcal{A}, \tag{3.24}$$

inducing a homotopy equivalence on classifying spaces, $\mathrm{B}Q : \mathrm{B}\textstyle\int_{\mathcal{B}} F{\downarrow} \simeq \mathrm{B}\mathcal{A}$.

Before starting the proof of the lemma, we shall briefly describe the bicategory $\int_{\mathcal{B}} F{\downarrow}$. It has objects the triplets (f,a,b), with $a \in \mathrm{Ob}\mathcal{A}$, $b \in \mathrm{Ob}\mathcal{B}$, and $f : Fa \to b$ a 1-cell of $\mathcal{B}$. Its 1-cells

$$(\beta, u, p) : (f, a, b) \to (f', a', b'),$$

consist of a 1-cell $p : b \to b'$ in $\mathcal{B}$, together with a 1-cell $(\beta, u) : p_*(f,a) = (p \circ f, a) \to (f', a')$ in $F{\downarrow}_{b'}$, that is, a 1-cell $u : a \to a'$ in $\mathcal{A}$ and a 2-cell $\beta : p \circ f \Rightarrow f' \circ Fu$ in $\mathcal{B}$

$$\begin{tikzcd}
Fa \arrow[r, "Fu"] \arrow[d, "f"'] \arrow[rd, phantom, "\overset{\beta}{\Rightarrow}"] & Fa' \arrow[d, "f'"] \\
b \arrow[r, "p"] & b'.
\end{tikzcd}$$

A 2-cell in $\int_{\mathcal{B}} F{\downarrow}$,

$$\begin{tikzcd}
(f,a,b) \arrow[r, bend left, "{(\beta,u,p)}"] \arrow[r, bend right, "{(\beta',u',p')}"'] \arrow[r, phantom, "\Downarrow(\alpha,\sigma)"] & (f',a',b')
\end{tikzcd}\ ,$$

consists of a 2-cell $\sigma : p \Rightarrow p'$ in $\mathcal{B}$, together with a 2-cell $\alpha : (\beta,u) \Rightarrow (\beta',u') \circ \sigma_*(f,a)$ in $F{\downarrow}_{b'}$, that is, (after some work using coherence equations) a 2-cell $\alpha : u \Rightarrow u' \circ 1_a$ in $\mathcal{A}$, such that the equation below holds.

$$\begin{tikzcd}
Fa \arrow[r, "Fu'"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\beta'}{\Rightarrow}"] & Fa' \arrow[d, "f'"] & & Fa \arrow[r, "Fu'"] \arrow[r, bend right, "Fu"', "\Uparrow F(\boldsymbol{r}\cdot\alpha)"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & Fa' \arrow[d, "f'"] \\
b \arrow[r, "p"'] \arrow[r, bend left, "p'", "\Uparrow\sigma"'] & b' & = & b \arrow[r, "p"'] & b'
\end{tikzcd}$$

We shall look carefully at the vertical composition of 2-cells and the horizontal composition of 1-cells in $\int_{\mathcal{B}} F{\downarrow}$ since we will use them later: Given two vertically composable 2-cells, say (α,σ) as above and $(\alpha',\sigma') : (\beta',u',p') \Rightarrow (\beta'',u'',p'')$, their vertical composition is given by the formula

$$(\alpha',\sigma') \cdot (\alpha,\sigma) = (\alpha' \cdot \boldsymbol{r} \cdot \alpha, \sigma' \cdot \sigma) : (\beta,u,p) \Rightarrow (\beta'',u'',p'').$$

Given two composable 1-cells, say (β,u,p) as above and $(\beta',u',p') : (f',a',b') \to (f'',a'',b'')$, their horizontal composition is

$$(\beta',u',p') \circ (\beta,u,p) = (F\boldsymbol{r}^{-1} \cdot (\beta' \circledcirc (1_{p'} \circ \beta)), (u' \circ u) \circ 1_a, p' \circ p) : (f,a,b) \to (f'',a'',b''),$$

where $\beta' \circledcirc (1_{p'} \circ \beta)$ is as in (3.19), thus

$$F\boldsymbol{r}^{-1} \cdot (\beta' \circledcirc (1_{p'} \circ \beta)) : \quad \begin{tikzcd}
Fa \arrow[r, "Fu"] \arrow[rr, bend left, "F((u'\circ u)\circ 1_a)", "\Uparrow F\boldsymbol{r}^{-1}\cdot\widehat{F}"'] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & Fa' \arrow[r, "Fu'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\overset{\beta'}{\Rightarrow}"] & Fa'' \arrow[d, "f''"] \\
b \arrow[r, "p"] & b' \arrow[r, "p'"] & b''.
\end{tikzcd}$$

The identity 1-cell at an object (f,a,b) is

$$1_{(f,a,b)} = (\mathring{1}_{(f,a)} \cdot \boldsymbol{l}, 1_a, 1_b) : (f,a,b) \to (f,a,b).$$

$$\mathring{1}_{(f,a)} \cdot \boldsymbol{l} = \Big(\ 1_b \circ f \overset{\boldsymbol{l}}{\Rightarrow} f \overset{\boldsymbol{r}^{-1}}{\Rightarrow} f \circ 1_{Fa} \overset{1_f \circ \widehat{F}}{\Rightarrow} f \circ F(1_a)\ \Big)$$

Proof of Lemma 3.8. The homomorphism Q in (3.24) is defined on cells by

$$\begin{tikzcd}
(f,a,b) \arrow[r, bend left, "{(\beta,u,p)}"] \arrow[r, bend right, "{(\beta',u',p')}"'] \arrow[r, phantom, "\Downarrow(\alpha,\sigma)"] & (f',a',b') & \overset{Q}{\mapsto} & a \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow\boldsymbol{r}\cdot\alpha"] & a',
\end{tikzcd}$$

$$\boldsymbol{r} \cdot \alpha = \big(u \overset{\alpha}{\Rightarrow} u' \circ 1_a \overset{\boldsymbol{r}}{\Rightarrow} u'\big)$$

This homomorphism Q is strictly unitary, and its structure isomorphism at any two composable 1-cells, say (β,u,p) as above and $(\beta',u',p'):(f',a',b')\to(f'',a'',b'')$, is

$$\widehat{Q}=\boldsymbol{r}_{u'\circ u}:Q((\beta',u',p')\circ(\beta,u,p))\cong Q(\beta',u',p')\circ Q(\beta,u,p).$$

To prove that this homomorphism Q induces a homotopy equivalence on classifying spaces, let us observe that there is also a lax functor $L:\mathcal{A}\to\int_{\mathcal{B}}F\!\downarrow$, such that $Q\,L=1_{\mathcal{A}}$. This is defined on cells of $\mathcal{A}$ by

$$\begin{tikzcd}
a \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow\alpha"] & a' \arrow[r, phantom, "\overset{L}{\mapsto}"] & (1_{Fa},a,Fa) \arrow[r, bend left, "(\boldsymbol{l}^{-1}\cdot\boldsymbol{r},u,Fu)"] \arrow[r, bend right, "(\boldsymbol{l}^{-1}\cdot\boldsymbol{r},u',Fu')"'] \arrow[r, phantom, "\Downarrow(\boldsymbol{r}^{-1}\cdot\alpha,F\alpha)"] & (1_{Fa'},a',Fa'),
\end{tikzcd}$$

$$\boldsymbol{r}^{-1}\cdot\alpha=\big(u\overset{\alpha}{\Rightarrow}u'\overset{\boldsymbol{r}^{-1}}{\Rightarrow}u'\circ 1_a\big)$$

where the first component of $(\boldsymbol{l}^{-1}\cdot\boldsymbol{r},u,Fu)$ is the canonical isomorphism $Fu\circ 1_{Fa}\cong 1_{Fa'}\circ Fu$. Its structure 2-cells, at any pair of composable 1-cells $a\overset{u}{\to}a'\overset{u'}{\to}a''$ and at any object a of $\mathcal{A}$, are respectively defined by

$$\begin{aligned}
&\widehat{L}_{u',u}=(1_{(u'\circ u)\circ 1_a},\widehat{F}_{u',u}):\ Lu'\circ Lu\Rightarrow L(u'\circ u),\\
&\widehat{L}_a=(\boldsymbol{r}_{1_a}^{-1},\widehat{F}_a):1_{La}\Rightarrow L1_a.
\end{aligned}$$

The equality $QL=1_{\mathcal{A}}$ is easily checked. Furthermore, there is an oplax transformation $\iota:LQ\Rightarrow 1_{\int_{\mathcal{B}}F\downarrow}$ assigning to each object (f,a,b) of the bicategory $\int_{\mathcal{B}}F\!\downarrow$ the 1-cell

$$\iota(f,a,b)=(1_f\circ\widehat{F}_a,1_a,f):(1_{Fa},a,Fa)\to(f,a,b),$$

and whose naturality component at any 1-cell $(\beta,u,p):(f,a,b)\to(f',a',b')$ is the 2-cell

$$\begin{tikzcd}
(1_{Fa},a,Fa) \arrow[r, "(\boldsymbol{l}^{-1}\cdot\boldsymbol{r},u,Fu)"] \arrow[d, "(1_f\circ\widehat{F}_a,1_a,f)"'] \arrow[dr, phantom, "\overset{\widehat{\iota}=((\boldsymbol{l}^{-1}\circ 1)\circ 1,\beta)}{\Rightarrow}"] & (1_{Fa'},a',Fa') \arrow[d, "(1_{f'}\circ\widehat{F}_{a'},1_{a'},f')"] \\
(f,a,b) \arrow[r, "(\beta,u,p)"'] & (f',a',b').
\end{tikzcd}$$

Therefore, by taking classifying spaces, we have $\mathrm{B}Q\,\mathrm{B}L=1_{\mathrm{B}\mathcal{A}}$ and, by Lemma 3.5, $\mathrm{B}L\,\mathrm{B}Q\simeq 1_{\mathrm{B}\int_{\mathcal{B}}F\downarrow}$, whence $\mathrm{B}Q$ is actually a homotopy equivalence. $\square$

Part 3. We complete here the proof of the theorem as follows: There is a canonical homomorphism

$$\bar{F}:\textstyle\int_{\mathcal{B}}F\!\downarrow\longrightarrow\int_{\mathcal{B}}\mathcal{B}\!\downarrow \tag{3.25}$$

making commutative, for any object $b \in \mathrm{Ob}\mathcal{B}$, the diagrams

$$(A): \quad \begin{tikzcd} F{\downarrow}_b \arrow[r, "J"] \arrow[rr, bend left, "P"] \arrow[d, "\bar{F}"'] & \int_{\mathcal{B}} F{\downarrow} \arrow[r, "Q"] \arrow[d, dashed, "\bar{F}"] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{B}{\downarrow}_b \arrow[r, "J"] \arrow[rr, bend right, "P"'] & \int_{\mathcal{B}} \mathcal{B}{\downarrow} \arrow[r, "Q"] & \mathcal{B} \end{tikzcd} \qquad (B): \quad \begin{tikzcd} F{\downarrow}_b \arrow[r, "F"] \arrow[rr, bend left] \arrow[d, "J"'] & \mathcal{B}{\downarrow}_b \arrow[r] \arrow[d, "J"] & {[0]} \arrow[d, "b"] \\ \int_{\mathcal{B}} F{\downarrow} \arrow[r, dashed, "\bar{F}"] \arrow[rr, bend right, "P"'] & \int_{\mathcal{B}} \mathcal{B}{\downarrow} \arrow[r, "P"] & \mathcal{B} \end{tikzcd}$$

in which $Q : \int_{\mathcal{B}} F{\downarrow} \to \mathcal{A}$ is the homomorphism in (3.24) and $Q : \int_{\mathcal{B}} \mathcal{B}{\downarrow} \to \mathcal{B}$ is the corresponding one for $F = 1_{\mathcal{B}}$, all the 2-functors P are the canonical projections (3.8), and the embedding homomorphisms J are the corresponding ones defined as in (3.11). This homomorphism (3.25) is defined on cells by

$$\begin{tikzcd} (f,a,b) \arrow[r, bend left, "{(\beta,u,p)}"] \arrow[r, bend right, "{(\beta',u',p')}"'] \arrow[r, phantom, "\Downarrow(\alpha,\sigma)"] & (f',a',b') \arrow[r, phantom, "\overset{\bar{F}}{\mapsto}"] & (f,Fa,b) \arrow[r, bend left, "{(\beta,Fu,p)}"] \arrow[r, bend right, "{(\beta',Fu',p')}"'] \arrow[r, phantom, "\Downarrow(\boldsymbol{r}^{-1}\cdot F\boldsymbol{r}\cdot F\alpha,\sigma)"] & (f',Fa',b'). \end{tikzcd}$$

$$\boldsymbol{r}^{-1}\cdot F\boldsymbol{r}\cdot F\alpha = \Big(Fu \overset{F\alpha}{\Rightarrow} F(u'\circ 1_a) \overset{F\boldsymbol{r}}{\Rightarrow} Fu' \overset{\boldsymbol{r}^{-1}}{\Rightarrow} Fu'\circ 1_{Fa} \Big)$$

Its composition constraint at a pair of composable 1-cells, say (β,u,p) as above and $(\beta',u',p') : (f',a',b') \to (f'',a'',b'')$, is the 2-cell

$$(\widetilde{F}, 1_{p'\circ p}) : \bar{F}(\beta',u',p') \circ \bar{F}(\beta,u,p) \Rightarrow \bar{F}((\beta',u',p')\circ(\beta,u,p)),$$
$$\widetilde{F} = \Big((Fu'\circ Fu)\circ 1_{Fa} \overset{\widehat{F}\circ 1}{\Longrightarrow} F(u'\circ u)\circ 1_{Fa} \overset{F(\boldsymbol{r}^{-1})\circ 1}{\Longrightarrow} F((u'\circ u)\circ 1_a)\circ 1_{Fa} \Big)$$

while its unit constraint at an object (f,a,b) is

$$(\widetilde{F}, 1_{1_b}) : 1_{\bar{F}(f,a,b)} \Rightarrow \bar{F}(1_{(f,a,b)}).$$
$$\widetilde{F} = \Big(1_{Fa} \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} 1_{Fa}\circ 1_{Fa} \overset{\widehat{F}\circ 1}{\Longrightarrow} F(1_a)\circ 1_{Fa} \Big)$$

Let us now observe that (the covariant and oplax version of) Theorem 3.1 applies both to the bidiagram of homotopy fibers $F{\downarrow}$, by hypothesis, and to the bidiagram of comma bicategories $\mathcal{B}{\downarrow}$, since the spaces $\mathrm{B}\mathcal{B}{\downarrow}_b$ are contractible by Lemma 3.7 and therefore any 1-cell $p : b \to b'$ in $\mathcal{B}$ obviously induces a homotopy equivalence $\mathrm{B}p_* : \mathrm{B}\mathcal{B}{\downarrow}_b \simeq \mathrm{B}\mathcal{B}{\downarrow}_{b'}$. Hence, the squares

$$\begin{tikzcd} F{\downarrow}_b \arrow[r, "J"] \arrow[d] & \int_{\mathcal{B}} F{\downarrow} \arrow[d, "P"] \\ {[0]} \arrow[r, "b"] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}{\downarrow}_b \arrow[r, "J"] \arrow[d] & \int_{\mathcal{B}} \mathcal{B}{\downarrow} \arrow[d, "P"] \\ {[0]} \arrow[r, "b"] & \mathcal{B} \end{tikzcd} \tag{3.26}$$

induce homotopy cartesian squares on classifying spaces

$$\begin{tikzcd}
\mathrm{B}F{\downarrow}_b \arrow[r, "\mathrm{B}J"] \arrow[d] & \mathrm{B}\int_{\mathcal{B}} F{\downarrow} \arrow[d, "\mathrm{B}P"] & \mathrm{B}\mathcal{B}{\downarrow}_b \arrow[r, "\mathrm{B}J"] \arrow[d] & \mathrm{B}\int_{\mathcal{B}} \mathcal{B}{\downarrow} \arrow[d, "\mathrm{B}P"] \\
pt \arrow[r, "\mathrm{B}b"] & \mathrm{B}\mathcal{B}, & pt \arrow[r, "\mathrm{B}b"] & \mathrm{B}\mathcal{B}.
\end{tikzcd}$$

By [68, II, Lemma 8.22 (2)(b)], it follows from the commutativity of diagram (B) above that the induced square

$$\begin{tikzcd}
\mathrm{B}F{\downarrow}_b \arrow[r, "\mathrm{B}\bar{F}"] \arrow[d, "\mathrm{B}J"'] & \mathrm{B}\mathcal{B}{\downarrow}_b \arrow[d, "\mathrm{B}J"] \\
\mathrm{B}\int_{\mathcal{B}} F{\downarrow} \arrow[r, "\mathrm{B}\bar{F}"] & \mathrm{B}\int_{\mathcal{B}} \mathcal{B}{\downarrow}
\end{tikzcd}$$

is homotopy cartesian. Then, by [68, II, Lemma 8.22 (1), (2)(a)], the theorem follows from the commutativity of diagram (A), since, by Lemma 3.8, in the induced square

$$\begin{tikzcd}
\mathrm{B}\int_{\mathcal{B}} F{\downarrow} \arrow[r, "\mathrm{B}\bar{F}"] \arrow[d, "\mathrm{B}Q"'] & \mathrm{B}\int_{\mathcal{B}} \mathcal{B}{\downarrow} \arrow[d, "\mathrm{B}Q"] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

both maps $\mathrm{B}Q$ are homotopy equivalences and therefore it is homotopy cartesian. □

The following corollary generalizes Quillen's Theorem A in [109]:

Theorem 3.3 *Let $F : \mathcal{A} \to \mathcal{B}$ be a lax functor between bicategories. The induced map on classifying spaces $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ is a homotopy equivalence whenever the classifying spaces of the homotopy fiber bicategories $\mathrm{B}F\downarrow_b$ are contractible for all objects b of $\mathcal{B}$.*

Particular cases of the result above have been also stated in [35, Theorem 1.2], for the case when $F : \mathcal{A} \to \mathcal{B}$ is any 2-functor between 2-categories, and in [84, Theorem 6.4], for the case when F is a lax functor from a category $\mathcal{A}$ to a 2-category $\mathcal{B}$. In [54, Théorème 6.5], it is stated a relative Theorem A for lax functors between 2-categories, which also implies the particular case of Theorem 3.3 when F is any lax functor between 2-categories.

Example 3.4 Let $(\mathcal{M}, \otimes) = (\mathcal{M}, \otimes, I, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$ be a monoidal category (see e.g. [100]), and let $\Sigma(\mathcal{M}, \otimes)$ denote its *suspension* or *delooping* bicategory. That is, $\Sigma(\mathcal{M}, \otimes)$ is the bicategory with only one object, say $\star$, whose hom-category is $\mathcal{M}$, and whose horizontal composition is given by the tensor functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. The identity

1-cell on the object is the unit object I of the monoidal category, and the constraints $\boldsymbol{a}$, $\boldsymbol{l}$, and $\boldsymbol{r}$ for $\Sigma(\mathcal{M},\otimes)$ are just those of the monoidal category. By [35, Theorem 1],

$$\mathrm{B}(\mathcal{M},\otimes)=\mathrm{B}\Sigma(\mathcal{M},\otimes),$$

that is, the classifying space of the monoidal category is the classifying space of its suspension bicategory. Then, Theorem 3.2 is applicable to monoidal functors between monoidal categories. However, we should stress that the homotopy fiber bicategory of the homomorphism between the suspension bicategories that a monoidal functor $F:(\mathcal{M},\otimes)\to(\mathcal{M}',\otimes)$ defines, $\Sigma F:\Sigma(\mathcal{M},\otimes)\to\Sigma(\mathcal{M}',\otimes)$, at the unique object of $\Sigma(\mathcal{M}',\otimes)$, is not a monoidal category but a genuine bicategory: The 0-cells of $\Sigma F{\downarrow_\star}$ are the objects $x'\in\mathcal{M}'$, its 1-cells $(u',x):x'\to y'$ are pairs with x an object in $\mathcal{M}$ and $u':x'\to y'\otimes F(x)$ a morphism in $\mathcal{M}'$, and its 2-cells

$$\begin{tikzcd} x' \arrow[r, bend left, "{(u',x)}"{name=U}] \arrow[r, bend right, "{(v',y)}"'{name=D}] & y' \arrow[from=U, to=D, phantom, "\Downarrow u"] \end{tikzcd}$$

are those morphisms $u:x\to y$ in $\mathcal{M}$ making commutative the triangle

$$\begin{tikzcd} x' \arrow[r, "u'"] \arrow[dr, "v'"'] & y'\otimes Fx \arrow[d, "y'\otimes Fu"] \\ & y'\otimes Fy. \end{tikzcd}$$

The vertical composition of 2-cells is given by the composition of arrows in $\mathcal{M}$. The horizontal composition of two 1-cells $x'\xrightarrow{(u',x)}y'\xrightarrow{(v',y)}z'$ is the 1-cell $(v'\circledcirc u',y\otimes x):x'\to z'$,

$$v'\circledcirc u'=\Big(x'\xrightarrow{u'}y'\otimes Fx\xrightarrow{v'\otimes Fx}(z'\otimes Fy)\otimes Fx\ \cong z'\otimes(Fy\otimes Fx)\cong z'\otimes F(y\otimes x)\Big)$$

and the horizontal composition of 2-cells is given by tensor product of arrows in $\mathcal{M}$. The identity 1-cell of any 0-cell x is $(\mathring{1}_x,I):x\to x$, where $\mathring{1}_x=(x'\cong x'\otimes I'\cong x'\otimes FI)$. The associativity, left and right constraints are obtained from those of $(\mathcal{M},\otimes)$ by the formulas

$$\boldsymbol{a}_{(w',z),(v',y),(u',x)}=\boldsymbol{a}_{z,y,x},\quad \boldsymbol{r}_{(u',x)}=\boldsymbol{r}_x,\quad \boldsymbol{l}_{(u',x)}=\boldsymbol{l}_x.$$

Following the terminology of [36, page 228], we shall call this bicategory $\Sigma F{\downarrow_\star}$ the *homotopy fiber bicategory* of the monoidal functor $F:(\mathcal{M},\otimes)\to(\mathcal{M}',\otimes)$, and write it by $\mathcal{K}_F$. This bicategories have been studied by Vitale in [123] for the case of a monoidal functor between categorical groups, where he calls them the *cokernel* of the functor[5]. Every object z' of $\mathcal{M}'$, determines a 2-endofunctor $z'\otimes -:\mathcal{K}_F\to\mathcal{K}_F$, which

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[5] We would like to thank Niles Johnson for pointing this out.

is defined on cells by

$$\begin{tikzcd} x' \arrow[r, bend left, "{(u',x)}"] \arrow[r, bend right, "{(v',y)}"'] \arrow[r, phantom, "\Downarrow u"] & y' \end{tikzcd} \mapsto \begin{tikzcd} z'\otimes x' \arrow[r, bend left, "{(z'\circledcirc u',x)}"] \arrow[r, bend right, "{(z'\circledcirc v',y)}"'] \arrow[r, phantom, "\Downarrow u"] & z'\otimes y' \end{tikzcd},$$

where $z'\circledcirc u' = \Big(\, z'\otimes x' \xrightarrow{z'\otimes u'} z'\otimes(y'\otimes Fx)\, \cong (z'\otimes y')\otimes Fx\Big)$, and from Theorems 3.2 and 3.3, we get the following:

Theorem 3.4 *For any monoidal functor $F:(\mathcal{M},\otimes)\to(\mathcal{M}',\otimes)$, the following statements hold: (i) There is an induced homotopy fiber sequence*

$$\mathrm{B}\mathcal{K}_F \to \mathrm{B}(\mathcal{M},\otimes) \xrightarrow{\mathrm{B}F} \mathrm{B}(\mathcal{M}',\otimes),$$

whenever the induced maps $\mathrm{B}(z'\otimes -):\mathrm{B}\mathcal{K}_F\to\mathrm{B}\mathcal{K}_F$ are homotopy autoequivalences, for all $z'\in Ob\mathcal{M}'$. (ii) The induced map $\mathrm{B}F:\mathrm{B}(\mathcal{M},\otimes)\to\mathrm{B}(\mathcal{M}',\otimes)$ is a homotopy equivalence if the space $\mathrm{B}\mathcal{K}_F$ is contractible.

For any monoidal category $(\mathcal{M},\otimes)$, pseudo bidiagrams of categories over its suspension bicategory,

$$\mathcal{N}=(\mathcal{N},\chi):\Sigma(\mathcal{M},\otimes)^{\mathrm{op}}\to\mathbf{Cat},$$

are interesting to consider, since they can be regarded as a category $\mathcal{N}$ (the one associated to the unique object of the suspension bicategory) endowed with a coherente right pseudo action of the monoidal category $(\mathcal{M},\otimes)$ (see e.g. [86, §1]). Namely, by the functor $\otimes:\mathcal{N}\times\mathcal{M}\to\mathcal{N}$, which is defined on objects by $a\otimes x = x^*a$ and on morphism by

$$(a\xrightarrow{f} b)\otimes(x\xrightarrow{u} y) = \big(x^*a\xrightarrow{x^*f} x^*b\xrightarrow{u^*b} y^*b\big) \;=\; \big(x^*a\xrightarrow{u^*a} y^*a\xrightarrow{y^*f} y^*b\big),$$

together with the coherent natural isomorphisms

$$\begin{aligned} &(a\otimes x)\otimes y = y^*x^*a \xrightarrow{\chi_{x,y}a} (x\otimes y)^*a = a\otimes(x\otimes y)\\ &a\xrightarrow{\chi_I a} I^*a = a\otimes I. \end{aligned}$$

For each such $(\mathcal{M},\otimes)$-category $\mathcal{N}$, the cells of the bicategory $\int_{\Sigma(\mathcal{M},\otimes)}\mathcal{N}$ has the following easy description: Its objects are the same as the objects of the category $\mathcal{N}$. A 1-cell $(f,x):a\to b$ is a pair with x an object of $\mathcal{M}$ and $f:a\to b\otimes x$ a morphism in $\mathcal{N}$, and a 2-cell

$$\begin{tikzcd} a \arrow[r, bend left, "{(f,x)}"] \arrow[r, bend right, "{(g,y)}"'] \arrow[r, phantom, "\Downarrow u"] & b \end{tikzcd}$$

is a morphism $u : x \to y$ in $\mathcal{M}$ such that the triangle

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[dr, "g"] & \\ b \otimes x \arrow[rr, "b\otimes u"] & & b \otimes y. \end{tikzcd}$$

is commutative. Many of the homotopy theoretical properties of the classifying space of the monoidal category, $\mathrm{B}(\mathcal{M}, \otimes)$, can actually be more easily reviewed by using Grothendieck bicategories $\int_{\Sigma(\mathcal{M},\otimes)} \mathcal{N}$, instead of the Borel pseudo simplicial categories

$$E_{(\mathcal{M},\otimes)}\mathcal{N} : \Delta^{\mathrm{op}} \to \mathbf{Cat}, \quad [p] \mapsto \mathcal{N} \times \mathcal{M}^p$$

as, for example, Jardine did in [86] for $(\mathcal{M}, \otimes)$-categories $\mathcal{N}$. Thus, one sees, for example, that if the action is such that multiplication by each object x of $\mathcal{M}$, that is, the endofunctor $- \otimes x : \mathcal{N} \to \mathcal{N}$, induces a homotopy equivalence $\mathrm{B}\mathcal{N} \simeq \mathrm{B}\mathcal{N}$, then, by Theorem 3.1, one has an induced homotopy fiber sequence (cf. [86, Proposition 3.5])

$$\mathrm{B}\mathcal{N} \to \mathrm{B} \textstyle\int_{\Sigma(\mathcal{M},\otimes)} \mathcal{N} \xrightarrow{\mathrm{B}P} \mathrm{B}(\mathcal{M}, \otimes).$$

In particular, the right action of $(\mathcal{M}, \otimes)$ on the underlying category $\mathcal{M}$ leads to the bicategory

$$\textstyle\int_{\Sigma(\mathcal{M},\otimes)} \mathcal{M} = \Sigma(\mathcal{M}, \otimes)\!\downarrow_{\star}\,,$$

the comma bicategory of the suspension bicategory over its unique object, whose classifying space is contractible by Lemma 3.7 (cf. [86, Proposition 3.8]). Then, it follows the well-known result by Mac Lane [99] and Stasheff [114] that there is a homotopy equivalence

$$\mathrm{B}\mathcal{M} \simeq \Omega\mathrm{B}(\mathcal{M}, \otimes),$$

between the classifying space of the underlying category and the loop space of the classifying space of the monoidal category, whenever multiplication by each object $x \in \mathrm{Ob}\mathcal{M}$, $y \mapsto y \otimes x$, induces a homotopy autoequivalence on $\mathrm{B}\mathcal{M}$ (cf. Example 3.3).

Chapter 4

Bicategorical homotopy pullbacks

4.1 Introduction and summary

If $A \overset{\phi}{\to} B \overset{\phi'}{\leftarrow} A'$ are continuous maps between topological spaces, its *homotopy-fiber product* $A\times^{\mathrm{h}}_B A'$ is the subspace of the product $A\times B^I\times A'$, where $I=[0,1]$ and B^I is taken with the compact-open topology, whose points are triples (a,γ,a') with $a\in A$, $a'\in A'$, and $\gamma:\phi a\to\phi' a'$ a path in B joining ϕa and $\phi' a'$, that is $\gamma: I\to B$ is a path starting at $\gamma 0=\phi a$ and ending at $\gamma 1=\phi' a'$. In particular, the *homotopy-fiber* of a continuous map $\phi: A\to B$ over a base point $b\in B$ is $\mathrm{Fib}(\phi,b)=A\times^{\mathrm{h}}_B\{b\}$, the homotopy-fiber product of ϕ and the constant inclusion map $\{b\}\hookrightarrow B$. That is, $\mathrm{Fib}(\phi,b)$ is the space of pairs (a,γ), where $a\in A$, and $\gamma:\phi a\to b$ is a path in B joining ϕa with the base point b.

If $\mathcal{A}\overset{F}{\to}\mathcal{B}\overset{F'}{\leftarrow}\mathcal{A}'$ are now functors between (small) categories, its *homotopy-fiber product category* is the comma category $F\downarrow F'$ consisting of triples (a,f,a') with $f: Fa\to F'a'$ a morphism in $\mathcal{B}$, in which a morphism from (a_0,f_0,a'_0) to (a_1,f_1,a'_1) is a pair of morphisms $u: a_0\to a_1$ in $\mathcal{A}$ and $u': a'_0\to a'_1$ in A' such that $F'u'\circ f_0=f_1\circ Fu$. In particular, the *homotopy-fiber category* $F\downarrow b$ of a functor $F:\mathcal{A}\to\mathcal{B}$, relative to an object $b\in\mathrm{Ob}\mathcal{B}$, is the homotopy-fiber product category of F and the constant functor $\{b\}\hookrightarrow\mathcal{B}$. These naive categorical emulations of the topological constructions are, however, subtle. Let $\mathrm{B}:\mathbf{Cat}\to\mathbf{Top}$ be the classifying space functor. The homotopy-fiber product category $F\downarrow F'$ comes with a canonical map from its classifying space to the homotopy-fiber product space of the induced maps $\mathrm{B}F:\mathrm{B}\mathcal{A}\to\mathrm{B}\mathcal{B}$ and $\mathrm{B}F':\mathrm{B}\mathcal{A}'\to\mathrm{B}\mathcal{B}$, and Barwick and Kan [13, Theorem 3.5] [14, Theorem 8.2] have proven that *this canonical map* $\mathrm{B}(F\downarrow F')\to\mathrm{B}\mathcal{A}\times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}}\mathrm{B}\mathcal{A}'$ *is a homotopy equivalence whenever the maps* $\mathrm{B}(F\downarrow b_0)\to\mathrm{B}(F\downarrow b_1)$*, induced by the different morphisms* $b_0\to b_1$ *of* $\mathcal{B}$*, are homotopy equivalences.* This result extends the well-known Quillen's Theorem B, which asserts that *under such an hypothesis, the canonical maps* $\mathrm{B}(F\downarrow b)\to\mathrm{Fib}(\mathrm{B}F,\mathrm{B}b)$ *are homotopy equivalences.* Actually, the result by Barwick and Kan is a consequence of a Theorem B by Cisinski [55, Théorèm 6.4.15][1]. Let us stress again that Theorem B and its consequent Theorem A have been fundamental for higher algebraic K-theory since the early 1970s, when Quillen [109] published his seminal paper, and they are now two of the most important theorems in the foundation of homotopy theory.

Similar categorical lax limit constructions have been used to describe homotopy pullbacks in many settings of enriched categories, where a homotopy theory has been established (see Grandis [70], for instance). Here, we focus on the bicategorical case. Recall again that like categories, small Bénabou bicategories [15] and, in particular, 2-categories and Mac Lane's monoidal categories, are closely related to topological spaces through the classifying space construction, as shown by Carrasco, Cegarra, and Garzón in [41]. This assigns to each bicategory $\mathcal{B}$ a CW-complex $\mathrm{B}\mathcal{B}$, whose cells give a natural geometric meaning to the cells of the bicategory. Further, we should mention that the category of (strict) 2-categories and 2-functors has a Thomason-type Quillen model structure, as was first announced by Worytkiewicz, Hess, Parent and Tonks in [125, Theorem 4.5.1] and fully proved by Ara and Maltsiniotis in [2, Théorème 6.27], such that the classifying space functor $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$ is an equivalence of homotopy theories between 2-categories and topological spaces.

In the preparatory Section 4.2 of this chapter, for any diagram $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$, where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A}'$ are bicategories, F is a lax functor, and F' is an oplax functor (for instance, if F and F' are both homomorphisms), we present a *homotopy-fiber product bicategory* $F{\downarrow}F'$, whose 0-cells, or objects, are triples (a, f, a') with $f : Fa \to F'a'$ a 1-cell in $\mathcal{B}$ as in the case when F and F' are functors between categories. But now, a 1-cell from (a_0, f_0, a'_0) to (a_1, f_1, a'_1) is a triple (u, β, u') consisting of 1-cells $u : a_0 \to a_1$ in $\mathcal{A}$ and $u' : a'_0 \to a'_1$ in $\mathcal{A}'$, together with a 2-cell $\beta : F'u' \circ f_0 \Rightarrow f_1 \circ Fu$ in $\mathcal{B}$. And $F{\downarrow}F'$ has 2-cells $(\alpha, \alpha') : (u, \beta, u') \Rightarrow (v, \gamma, v')$, which are given by 2-cells $\alpha : u \Rightarrow v$ in $\mathcal{A}$ and $\alpha' : u' \Rightarrow v'$ in $\mathcal{A}'$ such that $(1_{f_1} \circ F\alpha) \cdot \beta = (\gamma \circ F'\alpha') \circ 1_{f_0}$. In particular, for any object $b \in \mathcal{B}$, we have the *homotopy-fiber bicategories*[2] $F{\downarrow}b$ and $b{\downarrow}F'$, in terms of which we state and prove our main results of the chapter. These are exposed in Section 4.3, and they can be summarized as follows (see Theorem 4.1 and Corollary 4.2):

- *For any diagram of bicategories* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$, *where* F *is a lax functor and* F' *is an oplax functor, there is a canonical map* $\mathrm{B}(F \downarrow F') \to \mathrm{B}\mathcal{A} \times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$, *from the classifying space of the homotopy-fiber product bicategory to the homotopy-fiber product space of the induced maps* $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ *and* $\mathrm{B}F' : \mathrm{B}\mathcal{A}' \to \mathrm{B}\mathcal{B}$.

- *For a given lax functor* $F : \mathcal{A} \to \mathcal{B}$, *the following properties are equivalent:*

- For any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$, *the map* $\mathrm{B}(F \downarrow F') \to \mathrm{B}\mathcal{A} \times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$ *is a homotopy equivalence.*

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[1] We thank the referee for pointing out this fact.

[2] These bicategories are isomorphic to the bicategories $F{\downarrow}_b$ of Chapter 3, but they are defined in a slightly different way

- For any 1-cell $b_0 \to b_1$ *of* $\mathcal{B}$*, the map* $\mathrm{B}(F \downarrow b_0) \to \mathrm{B}(F \downarrow b_1)$ *is a homotopy equivalence.*

- For any 0-cell b *of* $\mathcal{B}$*, the map* $\mathrm{B}(F \downarrow b) \to \mathrm{Fib}(\mathrm{B}F, \mathrm{B}b)$ *is a homotopy equivalence.*

- *For a given oplax functor* $F' : \mathcal{A}' \to \mathcal{B}$*, the following properties are equivalent:*

-For any lax functor $F : \mathcal{A} \to \mathcal{B}$*, the map* $\mathrm{B}(F \downarrow F') \to \mathrm{B}\mathcal{A} \times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$ *is a homotopy equivalence.*

- For any 1-cell $b_0 \to b_1$ *of* $\mathcal{B}$*, the map* $\mathrm{B}(b_1 \downarrow F') \to \mathrm{B}(b_0 \downarrow F)$ *is a homotopy equivalence.*

- For any 0-cell b *of* $\mathcal{B}$*, the map* $\mathrm{B}(b \downarrow F') \to \mathrm{Fib}(\mathrm{B}F', \mathrm{B}b)$ *is a homotopy equivalence.*

Let us remark that, if the map $\mathrm{B}(F \downarrow F') \to \mathrm{B}\mathcal{A} \times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$ is a homotopy equivalence, then, by Dyer and Roitberg [61], there are Mayer-Vietoris type long exact sequences on homotopy groups

$$\cdots \to \pi_{n+1}\mathrm{B}\mathcal{B} \to \pi_n\mathrm{B}(F \downarrow F') \to \pi_n\mathrm{B}\mathcal{A} \times \pi_n\mathrm{B}\mathcal{A}' \to \pi_n\mathrm{B}\mathcal{B} \to \cdots .$$

The above results include the aforementioned results by Barwick and Kan, but also the extension of Quillen's Theorems A and B to lax functors between bicategories stated in Chapter 3.

We also study conditions on a bicategory $\mathcal{B}$ in order to ensure that the space $\mathrm{B}(F \downarrow F')$ is always homotopy equivalent to the homotopy-fiber product of the induced maps $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ and $\mathrm{B}F' : \mathrm{B}\mathcal{A}' \to \mathrm{B}\mathcal{B}$. Thus, in Theorem 4.2, we prove

- *For a bicategory* $\mathcal{B}$*, the following properties are equivalent:*

- For any diagram $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$ *, where* F *is a lax functor and* F' *is an oplax functor, the map* $\mathrm{B}(F \downarrow F') \to \mathrm{B}\mathcal{A} \times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$ *is a homotopy equivalence*

- For any object b *and* 1*-cell* $b_0 \to b_1$ *in* $\mathcal{B}$*, the induced map* $\mathrm{B}\mathcal{B}(b, b_0) \to \mathrm{B}\mathcal{B}(b, b_1)$ *is a homotopy equivalence.*

- For any object b *and* 1*-cell* $b_0 \to b_1$ *in* $\mathcal{B}$*, the induced map* $\mathrm{B}\mathcal{B}(b_1, b) \to \mathrm{B}\mathcal{B}(b_0, b)$ *is a homotopy equivalence.*

- For any two objects $b, b' \in \mathcal{B}$*, the canonical map*

$$\mathrm{B}\mathcal{B}(b, b') \to \{\gamma : I \to \mathrm{B}\mathcal{B} \mid \gamma(0) = \mathrm{B}b, \gamma(1) = \mathrm{B}b'\} \subseteq \mathrm{B}\mathcal{B}^I$$

is a homotopy equivalence.

For a bicategory $\mathcal{B}$ satisfying the conditions above[3], we conclude the existence of a canonical homotopy equivalence

$$\mathrm{B}\mathcal{B}(b, b) \simeq \Omega(\mathrm{B}\mathcal{B}, \mathrm{B}b)$$

between the loop space of the classifying space of the bicategory with base point $\mathrm{B}b$ and the classifying space of the category of endomorphisms of b in $\mathcal{B}$ (see Corollary

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[3] See also Example 3.3.

4.4). This result for $\mathcal{B}$ a 2-category should be attributed to Tillmann [121, Lemma 3.3], but it has been independently proven by both Cegarra [43, Example 4.4] and by del Hoyo [84, Theorem 8.5].

Since any monoidal category can be regarded as a bicategory with only one 0-cell, our results are applicable to them. Thus, any diagram of monoidal functors and monoidal categories, $(\mathcal{N},\otimes)\overset{F}{\to}(\mathcal{M},\otimes)\overset{F'}{\leftarrow}(\mathcal{N}',\otimes)$, gives rise to a *homotopy-fiber product bicategory* $F\overset{\otimes}{\downarrow}F'$, whose 0-cells are the objects $m\in\mathcal{M}$, whose 1-cells $(n,f,n'):m_0\to m_1$ consist of objects $n\in\mathcal{N}$ and $n'\in\mathcal{N}'$, and a morphism $f: F'n'\otimes m_0\to m_1\otimes Fn$ in $\mathcal{M}$, and whose 2-cells $(u,u'):(n,f,n')\Rightarrow(\bar{n},\bar{f},\bar{n}')$ are given by a pair of morphisms, $u:n\to\bar{n}$ in $\mathcal{N}$ and $u':n'\to\bar{n}'$ in $\mathcal{N}'$, such that $(1\otimes Fu)\cdot f=\bar{f}\cdot(F'u'\otimes 1)$. In particular, for any monoidal functor F as above, we have the *homotopy-fiber bicategory* $F\overset{\otimes}{\downarrow}\mathrm{I}$, where $\mathrm{I}:([0],\otimes)\to(\mathcal{M},\otimes)$ denotes the monoidal functor from the trivial (one-arrow) monoidal category $[0]$ to $\mathcal{M}$ that carries its unique object 0 to the unit object I of the monoidal category $\mathcal{M}$. Then, our main conclusions concerning monoidal categories, which are presented throughout Section 4.4, are summarized as follows (see Theorems 4.4, 4.5, and 4.6).

• *The following properties on a monoidal functor $F:(\mathcal{N},\otimes)\to(\mathcal{M},\otimes)$ are equivalent:*

- For any monoidal functor $F':(\mathcal{N}',\otimes)\to(\mathcal{M},\otimes)$, the canonical map

$$\mathrm{B}(F\overset{\otimes}{\downarrow}F')\to\mathrm{B}(\mathcal{N},\otimes)\times^{\mathrm{h}}_{\mathrm{B}(\mathcal{M},\otimes)}\mathrm{B}(\mathcal{N}',\otimes)$$

is a homotopy equivalence.

- For any object $m\in\mathcal{M}$, the homomorphism $m\otimes-:F\overset{\otimes}{\downarrow}\mathrm{I}\to F\overset{\otimes}{\downarrow}\mathrm{I}$ induces a homotopy autoequivalence on $\mathrm{B}(F\overset{\otimes}{\downarrow}\mathrm{I})$.

- The canonical map $\mathrm{B}(F\overset{\otimes}{\downarrow}\mathrm{I})\to\mathrm{Fib}(\mathrm{B}F,\mathrm{BI})$ is a homotopy equivalence.

• *The following properties on a monoidal category $(\mathcal{M},\otimes)$ are equivalent:*

- For any diagram of monoidal functors $(\mathcal{N},\otimes)\xrightarrow{F}(\mathcal{M},\otimes)\xleftarrow{F'}(\mathcal{N}',\otimes)$, the canonical map $\mathrm{B}(F\overset{\otimes}{\downarrow}F')\to\mathrm{B}(\mathcal{N},\otimes)\times^{\mathrm{h}}_{\mathrm{B}(\mathcal{M},\otimes)}\mathrm{B}(\mathcal{N}',\otimes)$ is a homotopy equivalence.

- For any object $m\in\mathcal{M}$, the functor $m\otimes-:\mathcal{M}\to\mathcal{M}$ induces a homotopy autoequivalence on the classifying space $\mathrm{B}\mathcal{M}$.

- For any object $m\in\mathcal{M}$, the functor $-\otimes m:\mathcal{M}\to\mathcal{M}$ induces a homotopy autoequivalence on the classifying space $\mathrm{B}\mathcal{M}$.

- The canonical map from the classifying space of the underlying category into the loop space of the classifying space of the monoidal category is a homotopy equivalence, $\mathrm{B}\mathcal{M}\simeq\Omega\mathrm{B}(\mathcal{M},\otimes)$.

The equivalence between the two last statements in the first result above might be considered as a version of Quillen's Theorem B for monoidal functors. A monoidal version of Theorem A follows: *If the homotopy-fiber bicategory of a monoidal functor*

$F : (\mathcal{N},\otimes) \to (\mathcal{M},\otimes)$ *is contractible, that is,* $\mathrm{B}(F \overset{\otimes}{\downarrow} \mathrm{I}) \simeq \mathrm{pt}$*, then the induced map* $\mathrm{B}F : \mathrm{B}(\mathcal{N},\otimes) \to \mathrm{B}(\mathcal{M},\otimes)$ *is a homotopy equivalence.* The equivalence of the three last statements in the second one are essentially due to Stasheff [114].

Thanks to the equivalence between the category of crossed modules and the category of 2-groupoids, by Brown and Higgins [28, Theorem 4.1], our results on bicategories also find application in the setting of crossed modules, what we do in Section 4.5. Briefly, for any diagram of crossed modules $(\mathcal{G},\mathcal{P},\partial) \xrightarrow{(\varphi,F)} (\mathcal{H},\mathcal{Q},\partial) \xleftarrow{(\varphi',F')} (\mathcal{G}',\mathcal{P}',\partial)$, we construct its *homotopy-fiber product crossed module* $(\varphi,F)\!\downarrow\!(\varphi',F')$, and we prove as the main result here (see Theorem 4.7) the following:

- *There is a canonical homotopy equivalence*

$$\mathrm{B}\big((\varphi,F)\!\downarrow\!(\varphi',F')\big) \simeq \mathrm{B}(\mathcal{G},\mathcal{P},\partial) \times^{\mathrm{h}}_{\mathrm{B}(\mathcal{H},\mathcal{Q},\partial)} \mathrm{B}(\mathcal{G}',\mathcal{P}',\partial)$$

between the classifying space of the homotopy-fiber product crossed module and the homotopy-fiber product space of the induced maps $\mathrm{B}(\varphi,F) : \mathrm{B}(\mathcal{G},\mathcal{P},\partial) \to \mathrm{B}(\mathcal{H},\mathcal{Q},\partial)$ *and* $\mathrm{B}(\varphi',F') : \mathrm{B}(\mathcal{G}',\mathcal{P}',\partial) \to \mathrm{B}(\mathcal{H},\mathcal{Q},\partial)$.

(Here, $(\mathcal{G},\mathcal{P},\partial) \mapsto \mathrm{B}(\mathcal{G},\mathcal{P},\partial)$ denotes the classifying space of crossed modules functor by Brown and Higgins [29].) Recalling that the category of crossed complexes has a closed model structure, as shown by Brown and Golasinki in [26], we also prove that the constructed homotopy-fiber product crossed module $(\varphi,F)\!\downarrow\!(\varphi',F')$ is compatible with the construction of homotopy pullbacks in this model category. More precisely, in Theorem 4.8, we prove that

- *If one of the morphisms* (φ,F) *or* (φ',F') *is a fibration, then the canonical morphism*

$$(\mathcal{G},\mathcal{P},\partial) \times_{(\mathcal{H},\mathcal{Q},\partial)} (\mathcal{G}',\mathcal{P}',\partial) \to (\varphi,F)\!\downarrow\!(\varphi',F'),$$

from the pullback crossed module to the homotopy-fiber product crossed module induces a homotopy equivalence on classifying spaces.

The chapter also includes some new results concerning classifying spaces of bicategories, which are needed here to obtain the aforementioned results on homotopy-fiber products. On the one hand, although in [41, §4] it was proven that the classifying space construction is a functor from the category of bicategories and homomorphisms to the category **Top** of spaces, in this chapter we need to extend that fact as given below (see Lemma 4.3).

- *The assignment* $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$ *is the function on objects of two functors*

$$\mathbf{Lax} \xrightarrow{\mathrm{B}} \mathbf{Top} \xleftarrow{\mathrm{B}} \mathbf{opLax},$$

where **Lax** *is the category of bicategories and lax functors, and* **opLax** *the category of bicategories and oplax functors.*

On the other hand, we also need to work with Duskin and Street's *geometric nerves* of bicategories [58, 117]. That is, with the simplicial sets $\Delta^{\mathrm{u}}\mathcal{B}$, $\Delta\mathcal{B}$, $\nabla_{\mathrm{u}}\mathcal{B}$, and $\nabla\mathcal{B}$, whose respective p-simplices are the normal lax, lax, normal oplax, and oplax functors from the category $[p] = \{0 < \cdots < p\}$ into the bicategory $\mathcal{B}$. Although in [41, Theorem 6.1] the existence of homotopy equivalences

$$|\Delta^{\mathrm{u}}\mathcal{B}| \simeq |\Delta\mathcal{B}| \simeq \mathrm{B}\mathcal{B} \simeq |\nabla\mathcal{B}| \simeq |\nabla_{\mathrm{u}}\mathcal{B}|$$

was proved, their natural behaviour is not studied. Then, in Lemma 4.4 we state the following:

- *For any bicategory $\mathcal{B}$, the homotopy equivalence $|\Delta^{\mathrm{u}}\mathcal{B}| \simeq |\Delta\mathcal{B}|$ is natural on normal lax functors, the homotopy equivalence $|\Delta\mathcal{B}| \simeq \mathrm{B}\mathcal{B}$ is homotopy natural on lax functors, the homotopy equivalence $\mathrm{B}\mathcal{B} \simeq |\nabla\mathcal{B}|$ is homotopy natural on oplax functors, and the homotopy equivalence $|\nabla\mathcal{B}| \simeq |\nabla_{\mathrm{u}}\mathcal{B}|$ is natural on normal oplax functors.*

The proofs of these results are quite long and technical. Therefore, to avoid hampering the flow of the chapter, we have put most of them into an appendix, comprising Section 4.6.

4.2 Preparation: The constructions involved

This section aims to make this chapter as self-contained as possible; therefore, at the same time as fixing notations and terminology, we also review some necessary aspects and results about homotopy pullbacks of topological spaces, comma bicategories, and classifying spaces of small bicategories that are used throughout the chapter. However, some results, mainly those in Lemmas 4.1, 4.3, and 4.4, are actually new. For a detailed study of the definition of homotopy pullback of continuous maps we refer the reader to Mather's original paper [103] and to the more recent approach by Doeraene [57]. For a general background on simplicial sets and homotopy pullbacks in model categories, we recommend again the books by Goerss and Jardine [68] and Hirschhorn [81]. For a complete description of bicategories, lax functors, and lax transformations, we refer the reader to the papers by Bénabou [15, 16] and Street [117].

4.2.1 Homotopy pullbacks.

Throughout this chapter, *all topological spaces have the homotopy type of CW-complexes*, so that a continuous map is a homotopy equivalence if and only if it is a weak homotopy equivalence.

If $X \xrightarrow{f} B \xleftarrow{g} Y$ are continuous maps, recall that its *homotopy-fiber product* is the space

$$X \times^{\mathrm{h}}_{B} Y = X \times_B B^I \times_B Y$$

consisting of triples (x, γ, y) with x a point of X, y a point of Y, and $\gamma : I \to B$ a path of B joining $f(x)$ and $g(y)$. This space occurs in the so-called *standard homotopy* *pullback* of f and g, that is, the homotopy commutative square

$$\begin{tikzcd}
X \times^{\mathrm{h}}_B Y \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\overset{F}{\Rightarrow}"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & B
\end{tikzcd}$$

where f' and g' are the evident projection maps, and $F : (X \times^{\mathrm{h}}_B Y) \times I \to B$ is the homotopy from fg' to gf' given by $F(x, \gamma, y, t) = \gamma(t)$. In particular, for any continuous map $g : Y \to B$ and any point $b \in B$, we have the standard homotopy pullback

$$\begin{tikzcd}
\mathrm{Fib}(g, b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{F}{\Rightarrow}"] & Y \arrow[d, "g"] \\
\mathrm{pt} \arrow[r, "b"] & B,
\end{tikzcd}$$

where $\mathrm{Fib}(g, b) = \mathrm{pt} \times^{\mathrm{h}}_B Y$ is the *homotopy-fiber* of g over b. (We use pt to denote a one-point space.) For any $y \in g^{-1}(b)$, one has the exact homotopy sequence

$$\cdots \to \pi_{n+1}(B, b) \to \pi_n(\mathrm{Fib}(g, b), (\mathrm{Ct}_b, y)) \to \pi_n(Y, y) \to \pi_n(B, b) \to \cdots ,$$

from which g is a homotopy equivalence if and only if all its homotopy fibers are contractible.

More generally, following Mather's definition in [103], a homotopy commutative square

$$\begin{tikzcd}
Z \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\overset{H}{\Rightarrow}"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & B,
\end{tikzcd} \tag{4.1}$$

where $H : fg' \Rightarrow gf'$ is a homotopy, is called a *homotopy pullback* whenever the induced *whisker* map below is a homotopy equivalence.

$$w : Z \to X \times^{\mathrm{h}}_B Y, \quad z \mapsto (g'(z), H|_{z \times I}, f'(z)) \tag{4.2}$$

Throughout this thesis, we use only basic well-known properties of homotopy pullbacks. For instance, the *homotopy-fiber characterization of homotopy pullback squares*: The homotopy commutative square (4.1) is a homotopy pullback if and only if, for any point $x \in X$, the composite square

$$\begin{tikzcd}
\mathrm{Fib}(g', x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & Z \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\Rightarrow"] & Y \arrow[d, "g"] \\
\mathrm{pt} \arrow[r, "x"] & X \arrow[r, "f"] & B
\end{tikzcd}$$

is a homotopy pullback. That is, if and only if the induced whisker maps on homotopy fibers are homotopy equivalences, $w : \mathrm{Fib}(g', x) \overset{\sim}{\to} \mathrm{Fib}(g, f(x))$; or the *two out of three property* of homotopy pullbacks: Let

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \bullet \arrow[d] \\
X' \arrow[r] & X \arrow[r] & \bullet
\end{tikzcd}$$

be a diagram of homotopy commutative squares. If the right square is a homotopy pullback, then the left square is a homotopy pullback if and only if the composite square is as well. If $\pi_0 X' \to \pi_0 X$ is onto and the left and composite squares are homotopy pullbacks, then the right-hand square is a homotopy pullback.

Many other properties are easily deduced from the above ones. For example, the square (4.1) is a homotopy pullback whenever both maps g and g' are homotopy equivalences. If the square is a homotopy pullback and the map g is a homotopy equivalence, then so is g'. If the square is a homotopy pullback, g' is a homotopy equivalence, and the map $\pi_0 X \to \pi_0 B$ is surjective, then g is also a homotopy equivalence.

Hereafter, any (strictly) commutative square of spaces

$$\begin{tikzcd}
Z \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & B,
\end{tikzcd} \tag{4.3}$$

will be considered equipped with the static homotopy $Z \times I \to B$, $(z, t) \mapsto fg'(z) = gf'(z)$.

Remark 4.1 *A commutative square of spaces, as above, is a homotopy pullback if and only if it is a homotopy pullback in terms of the ordinary Quillen model structure for spaces.* To see that, simply observe that, given the commutative square (4.3), the whisker map (4.2) is the composite $Z \to X \times_B Y \overset{w}{\to} X \times^{\mathrm{h}}_B Y$, where $Z \to X \times_B Y$ is the canonical map $z \mapsto (g'(z), f'(z))$ into the topological fiber product. If f or g is a Serre fibration, the map $X \times_B Y \to X \times^{\mathrm{h}}_B Y$ is a homotopy equivalence, and therefore $Z \to X \times^{\mathrm{h}}_B Y$ is a homotopy equivalence if and only if $Z \to X \times_B Y$ is.

In [52, Proposition 5.4 and Corollary 5.5], Chachólski, Pitsch, and Scherer characterize continuous maps that always produce homotopy pullback squares when one pulls back with them. Along similar lines, we prove the needed lemma below for maps induced on geometric realizations by simplicial maps. More precisely, we characterize those simplicial maps $g : Y \to B$ such that, for any simplicial map $f : X \to B$, the pullback square of simplicial sets

$$\begin{tikzcd}
X \times_B Y \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & B
\end{tikzcd} \tag{4.4}$$

induces, by taking geometric realizations, a homotopy pullback square of spaces. To do so, recall the canonical homotopy colimit decomposition of a simplicial map, which allows the source of the map to be written as the homotopy colimit of its fibers over the simplices of the target: for a simplicial set B, we can consider its category of simplices $\Delta{\downarrow}B$ whose objects are the simplicial maps $\Delta[n] \to B$ and whose morphisms are the obvious commutative triangles. For a simplicial map $g : Y \to B$, we can then associate a functor from $\Delta{\downarrow}B$ to the category of spaces by mapping a simplex $x : \Delta[n] \to B$ to the geometric realization $|g^{-1}(x)|$ of the simplicial set $g^{-1}(x)$ defined by the pullback square

$$\begin{tikzcd}
g^{-1}(x) \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
\Delta[n] \arrow[r, "x"] & B.
\end{tikzcd}$$

By [68, Lemma IV.5.2], in the induced commutative diagram of spaces,

$$\begin{tikzcd}
\operatorname*{hocolim}_{x:\Delta[n]\to B} |g^{-1}(x)| \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "(a)"] & {|Y|} \arrow[d, "|g|"] \\
\operatorname*{hocolim}_{x:\Delta[n]\to B} |\Delta[n]| \arrow[r, "\sim"] & {|B|}
\end{tikzcd}$$

the horizontal maps are both homotopy equivalences.

Lemma 4.1 *For any given simplicial map $g : Y \to B$, the following statements are equivalent:*

(i) For any simplex of B, $x : \Delta[n] \to B$, and for any simplicial map $\sigma : \Delta[m] \to \Delta[n]$, the induced map $|g^{-1}(x\sigma)| \to |g^{-1}(x)|$ is a homotopy equivalence.

(ii) For any simplex $x : \Delta[n] \to B$, the induced pullback square of spaces

$$\begin{tikzcd}
{|g^{-1}(x)|} \arrow[r] \arrow[d] & {|Y|} \arrow[d, "|g|"] \\
{|\Delta[n]|} \arrow[r, "|x|"] & {|B|}
\end{tikzcd}$$

is a homotopy pullback.

(iii) For any simplicial map $f : X \to B$, the pullback square of spaces

$$\begin{tikzcd}
{|X \times_B Y|} \arrow[r, "|f'|"] \arrow[d, "|g'|"'] & {|Y|} \arrow[d, "|g|"] \\
{|X|} \arrow[r, "|f|"] & {|B|,}
\end{tikzcd}$$

induced by (4.4)*, is a homotopy pullback.*

Proof: $(i) \Rightarrow (ii)$: Let $x : \Delta[n] \to B$ be any simplex of B. We have the diagram

$$\begin{tikzcd}
{|g^{-1}(x)|} \arrow[r] \arrow[d] \arrow[dr, phantom, "(b)"] & \underset{x:\Delta[n]\to B}{\operatorname{hocolim}} |g^{-1}(x)| \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "(a)"] & {|Y|} \arrow[d, "|g|"] \\
{|\Delta[n]|} \arrow[r, "|x|"] \arrow[d, "\wr"'] \arrow[dr, phantom, "(c)"] & \underset{x:\Delta[n]\to B}{\operatorname{hocolim}} |\Delta[n]| \arrow[r, "\sim"] \arrow[d, "\wr"] & {|B|} \\
\mathrm{pt} \arrow[r, "x"] & \underset{x:\Delta[n]\to B}{\operatorname{hocolim}} \mathrm{pt},
\end{tikzcd}$$

where $\underset{x:\Delta[n]\to B}{\operatorname{hocolim}} \mathrm{pt} = \mathrm{B}(\Delta{\downarrow}B)$ is the classifying space of the simplex category. Since, by Quillen's Lemma [109, page 14], the composite square $(b)+(c)$ is a homotopy pullback, it follows that (b) is a homotopy pullback. Therefore, the composite $(b)+(a)$ is as well.

$(ii) \Rightarrow (i)$: For any simplicial map $\sigma : \Delta[m] \to \Delta[n]$ and any simplex $x : \Delta[n] \to B$, the right side and the large square in the diagram of spaces

$$\begin{tikzcd}
{|g^{-1}(x\sigma)|} \arrow[r] \arrow[d] & {|g^{-1}(x)|} \arrow[r] \arrow[d] & {|Y|} \arrow[d, "|g|"] \\
{|\Delta[m]|} \arrow[r, "|\sigma|"] & {|\Delta[n]|} \arrow[r, "|x|"] & {|B|}
\end{tikzcd}$$

are both homotopy pullback, and therefore so is the left-hand one. As $|\Delta[m]|$ and $|\Delta[n]|$ are both contractible, the map $|\sigma|$ is a homotopy equivalence, and therefore the map $|g^{-1}(x\sigma)| \to |g^{-1}(x)|$ is a homotopy equivalence.

$(i) \Rightarrow (iii)$: Suppose we have the pullback square of simplicial sets (4.4). Then, for any simplex $x : \Delta[n] \to X$ of X, we have a natural isomorphism of fibers $g'^{-1}(x) \cong g^{-1}(fx)$, and it follows that the map g' also satisfies the same condition (i) as g does. Then, by the already proven part $(i) \Leftrightarrow (ii)$, we know that, for any vertex $x : \Delta[0] \to X$, both the left side and the composite square in the diagram

$$\begin{tikzcd}
{|g'^{-1}(x)| \cong |g^{-1}(fx)|} \arrow[r] \arrow[d] & {|X \times_B Y|} \arrow[r] \arrow[d, "|g'|"'] & {|Y|} \arrow[d, "|g|"] \\
{\mathrm{pt} = |\Delta[0]|} \arrow[r, "|x|"] & {|X|} \arrow[r, "|f|"] & {|B|}
\end{tikzcd}$$

are homotopy pullbacks. Therefore, from the diagram on whisker maps

$$\begin{tikzcd}
{|g'^{-1}(x)|} \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathrm{Fib}(|g'|, |x|) \arrow[d, "w"] \\
{|g^{-1}(fx)|} \arrow[r, "\sim"] & \mathrm{Fib}(|g|, |fx|),
\end{tikzcd}$$

we conclude that the map $\mathrm{Fib}(|g'|,|x|)\to\mathrm{Fib}(|g|,|fx|)$ is a homotopy equivalence. Since the homotopy fibers of any map over points connected by a path are homotopy equivalent, and any point of $|X|$ is path-connected with a 0-cell $|x|$ defined by some 0-simplex $x:\Delta[0]\to X$ as above, the result follows from the homotopy fiber characterization.

(iii) ⇒ *(ii)*: This is obvious. □

4.2.2 Some bicategorical conventions.

We use the same notations of the previous chapters for bicategories, see Subsection 2.1.2 for reference.

Again, a *lax functor* is written as a pair $F=(F,\widehat{F}):\mathcal{B}\to\mathcal{C}$, and we will denote its structure constraints by

$$\widehat{F}_{f_2,f_1}:Ff_2\circ Ff_1\Rightarrow F(f_2\circ f_1),\quad \widehat{F}_b:1_{Fb}\Rightarrow F1_b,$$

for each pair of composable 1-cells, and each object of $\mathcal{B}$. Recall that the structure 2-cells $\widehat{F}_{f_2,f_1}$ are natural in $(f_2,f_1)\in\mathcal{B}(b_1,b_2)\times\mathcal{B}(b_0,b_1)$ and they satisfy the usual coherence conditions. Replacing the constraint 2-cells above by $\widehat{F}_{f_2,f_1}:F(f_2\circ f_1)\Rightarrow Ff_2\circ Ff_1$ and $\widehat{F}_b:F(1_b)\Rightarrow 1_{Fb}$, we have the notion of *oplax functor* $F=(F,\widehat{F}):\mathcal{B}\to\mathcal{C}$. Recall that, any lax or oplax functor F is termed a *pseudo-functor* or *homomorphism* whenever all the structure constraints $\widehat{F}_{f_2,f_1}$ and $\widehat{F}_b$ are invertible. When these 2-cells are all identities, then F is called a *2-functor.* If all the unit constraints $\widehat{F}_b$ are identities, then the lax or oplax functor is qualified as (strictly) *unitary* or *normal.*

We also recall that a *lax transformation* consists of morphisms $\alpha b:Fb\to F'b$, $b\in\mathrm{Ob}\mathcal{B}$, and 2-cells

$$\begin{tikzcd}
Fb_0 \arrow[r, "Ff"] \arrow[d, "\alpha b_0"'] \arrow[dr, phantom, "\overset{\widehat{\alpha}_f}{\Leftarrow}"] & Fb_1 \arrow[d, "\alpha b_1"] \\
F'b_0 \arrow[r, "F'f"'] & F'b_1
\end{tikzcd}$$

which are natural on the 1-cells $f:b_0\to b_1$ of $\mathcal{B}$, subject to the usual coherence axioms. Replacing the structure deformation above by $\widehat{\alpha}_f:F'f\circ\alpha b_0\Rightarrow\alpha b_1\circ Ff$, we have the notion of *oplax transformation* $\alpha:F\Rightarrow F'$. Any lax or oplax transformation α is termed a *pseudo-transformation* whenever all the naturality 2-cells $\widehat{\alpha}_f$ are invertible. Similarly, we have the notions of lax, oplax, and pseudo transformation between oplax functors.

4.2.3 Homotopy-fiber product bicategories.

We present a bicategorical comma construction in some detail, since it is fundamental for the results of this chapter. However, we are not claiming much originality since variations of the quite ubiquitous 'comma category' construction have been considered (just to define "homotopy pullbacks") in many general frameworks of enriched categories (where a homotopy theory has been established); see for instance Grandis [70].

Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$ be a diagram where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A}'$ are bicategories, F is a lax functor, and F' is an oplax functor. The "*homotopy-fiber product bicategory*"

$$F{\downarrow}F' \tag{4.5}$$

is defined as follows:

- *The* 0-*cells* of $F{\downarrow}F'$ are triples (a, f, a') with a a 0-cell of $\mathcal{A}$, a' a 0-cell of $\mathcal{A}'$, and $f : Fa \to F'a'$ a 1-cell in $\mathcal{B}$.

- *A* 1-*cell* $(u, \beta, u') : (a_0, f_0, a'_0) \to (a_1, f_1, a'_1)$ of $F{\downarrow}F'$ consists of a 1-cell $u : a_0 \to a_1$ in $\mathcal{A}$, a 1-cell $u' : a'_0 \to a'_1$ in $\mathcal{A}'$, and 2-cell $\beta : F'u' \circ f_0 \Rightarrow f_1 \circ Fu$ in $\mathcal{B}$,

$$\begin{tikzcd} Fa_0 \arrow[r, "Fu"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & Fa_1 \arrow[d, "f_1"] \\ F'a'_0 \arrow[r, "F'u'"] & F'a'_1. \end{tikzcd}$$

- *A* 2-*cell* in $F{\downarrow}F'$, $\begin{tikzcd} (a_0, f_0, a'_0) \arrow[r, bend left, "{(u,\beta,u')}"] \arrow[r, bend right, "{(\bar{u},\bar{\beta},\bar{u}')}"'] \arrow[r, phantom, "\Downarrow(\alpha,\alpha')"] & (a_1, f_1, a'_1), \end{tikzcd}$ is given by a 2-cell $\alpha : u \Rightarrow \bar{u}$ in $\mathcal{A}$ and a 2-cell $\alpha' : u' \Rightarrow \bar{u}'$ in $\mathcal{A}'$ such that the diagram below commutes.

$$\begin{tikzcd} F'u' \circ f_0 \arrow[r, Rightarrow, "F'\alpha' \circ 1"] \arrow[d, Rightarrow, "\beta"'] & F'\bar{u}' \circ f_0 \arrow[d, Rightarrow, "\bar{\beta}"] \\ f_1 \circ Fu \arrow[r, Rightarrow, "1 \circ F\alpha"] & f_1 \circ F\bar{u} \end{tikzcd}$$

- *The vertical composition of* 2-*cells* in $F{\downarrow}F'$ is induced by the vertical composition laws in $\mathcal{A}$ and $\mathcal{A}'$, thus $(\bar{\alpha}, \bar{\alpha}') \cdot (\alpha, \alpha') = (\bar{\alpha} \cdot \alpha, \bar{\alpha}' \cdot \alpha')$. The identity at a 1-cell is given by $1_{(u,\beta,u')} = (1_u, 1_{u'})$.

- *The horizontal composition of two* 1-*cells* in $F{\downarrow}F'$,

$$(a_0, f_0, a'_0) \xrightarrow{(u_1,\beta_1,u'_1)} (a_1, f_1, a'_1) \xrightarrow{(u_2,\beta_2,u'_2)} (a_2, f_2, a'_2)\ , \tag{4.6}$$

is the 1-cell $(u_2, \beta_2, u'_2) \circ (u_1, \beta_1, u'_1) = (u_2 \circ u_1, \beta_2 \circledcirc \beta_1, u'_2 \circ u'_1)$, where $\beta_2 \circledcirc \beta_1$ is the

2-cell pasted of the diagram in $\mathcal{B}$

$$\beta_2 \circledcirc \beta_1 = \begin{tikzcd} Fa_0 \arrow[r, "Fu_1"] \arrow[d, "f_0"'] \arrow[rr, bend left=40, "F(u_2\circ u_1)"] \arrow[rr, phantom, bend left=20, "\widehat{F}\Uparrow"] & Fa_1 \arrow[r, "Fu_2"] \arrow[d, "f_1"'] & Fa_2 \arrow[d, "f_2"] \\ F'a'_0 \arrow[r, "F'u'_1"] \arrow[rr, bend right=40, "F'(u'_2\circ u'_1)"'] \arrow[rr, phantom, bend right=20, "\widehat{F'}\Uparrow"] \arrow[ur, phantom, "\overset{\beta_1}{\Rightarrow}"] & F'a'_1 \arrow[r, "F'u'_2"] \arrow[ur, phantom, "\overset{\beta_2}{\Rightarrow}"] & F'a'_2, \end{tikzcd} \tag{4.7}$$

that is

$$\beta_2 \circledcirc \beta_1 = \Big(F'(u'_2 \circ u'_1) \circ f_0 \overset{\widehat{F'}\circ 1}{\Longrightarrow} (F'u'_2 \circ F'u'_1) \circ f_0 \overset{\boldsymbol{a}}{\Longrightarrow} F'u'_2 \circ (F'u'_1 \circ f_0) \overset{1\circ\beta_1}{\Longrightarrow}$$
$$F'u'_2 \circ (f_1 \circ Fu_1) \overset{\boldsymbol{a}^{-1}}{\Longrightarrow} (F'u'_2 \circ f_1) \circ Fu_1 \overset{\beta_2\circ 1}{\Longrightarrow} (f_2 \circ Fu_2) \circ Fu_1 \overset{\boldsymbol{a}}{\Longrightarrow}$$
$$f_2 \circ (Fu_2 \circ Fu_1) \overset{1\circ\widehat{F}}{\Longrightarrow} f_2 \circ F(u_2 \circ u_1)\Big).$$

- *The horizontal composition of* 2*-cells* in $F{\downarrow}F'$ is given by composing horizontally the 2-cells in $\mathcal{A}$ and $\mathcal{A}'$, thus $(\alpha_2, \alpha'_2) \circ (\alpha_1, \alpha'_1) = (\alpha_2 \circ \alpha_1, \alpha'_2 \circ \alpha'_1)$.
- *The identity* 1*-cell* in $F \downarrow F'$, at an object (a, f, a'), is $(1_a, \overset{\circ}{1}_{(a,f,a')}, 1_{a'})$, where $\overset{\circ}{1}_{(a,f,a')}$ is the 2-cell in $\mathcal{B}$ obtained by pasting the diagram

$$\overset{\circ}{1}_{(a,f,a')} = \begin{tikzcd} Fa \arrow[r, "1_{Fa}"] \arrow[d, "f"'] \arrow[r, bend left=50, "F1_a"] \arrow[r, phantom, bend left=25, "\widehat{F}\Uparrow"] \arrow[dr, phantom, "\overset{\boldsymbol{r}^{-1}\cdot\boldsymbol{l}}{\cong}"] & Fa \arrow[d, "f"] \\ F'a' \arrow[r, "1_{F'a'}"'] \arrow[r, bend right=50, "F'1_{a'}"'] \arrow[r, phantom, bend right=25, "\widehat{F'}\Uparrow"] & F'a', \end{tikzcd}$$

that is, $\overset{\circ}{1}_{(a,f,a')} = \Big(F'1_{a'} \circ f \overset{\widehat{F'}\circ 1}{\Longrightarrow} 1_{F'a'} \circ f \overset{\boldsymbol{l}}{\Longrightarrow} f \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} f \circ 1_{Fa} \overset{1\circ\widehat{F}}{\Longrightarrow} f \circ F1_a\Big)$.

- *The associativity, right and left unit constraints* of the bicategory $F \downarrow F'$ are provided by those of $\mathcal{A}$ and $\mathcal{A}'$ by the formulas

$$\boldsymbol{a}_{(u_3,\beta_3,u'_3),(u_2,\beta_2,u'_2),(u_1,\beta_1,u'_1)} = (\boldsymbol{a}_{u_3,u_2,u_1}, \boldsymbol{a}_{u'_3,u'_2,u'_1}), \ \ \boldsymbol{l}_{(u,\beta,u')} = (\boldsymbol{l}_u, \boldsymbol{l}_{u'}), \ \ \boldsymbol{r}_{(u,\beta,u')} = (\boldsymbol{r}_u, \boldsymbol{r}_{u'}).$$

The main square.

There is a (non-commutative!) square, which is of fundamental interest for the discussions below:

$$\begin{tikzcd} F{\downarrow}F' \arrow[r, "P'"] \arrow[d, "P"'] & \mathcal{A}' \arrow[d, "F'"] \\ \mathcal{A} \arrow[r, "F"] & \mathcal{B} \end{tikzcd} \tag{4.8}$$

where P and P' are projection 2-functors, which act on cells of $F{\downarrow}F'$ by

$$\begin{tikzcd} a_0 \arrow[r, bend left, "u"] \arrow[r, bend right, "\bar{u}"'] \arrow[r, phantom, "\Downarrow\alpha"] & a_1 & (a_0, f_0, a'_0) \arrow[l, mapsto, "P"'] \arrow[r, bend left, "{(u,\beta,u')}"] \arrow[r, bend right, "{(\bar{u},\bar{\beta},\bar{u}')}"'] \arrow[r, phantom, "\Downarrow(\alpha,\alpha')"] & (a_1, f_1, a'_1) \arrow[r, mapsto, "P'"] & a'_0 \arrow[r, bend left, "u'"] \arrow[r, bend right, "\bar{u}'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & a'_1. \end{tikzcd} \tag{4.9}$$

Two pullback squares.

We consider here three particular cases of the construction (4.5):

- For any lax functor $F : \mathcal{A} \to \mathcal{B}$, the bicategory $F{\downarrow}\mathcal{B} := F{\downarrow}1_{\mathcal{B}}$.
- For any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$, the bicategory $\mathcal{B}{\downarrow}F' := 1_{\mathcal{B}}{\downarrow}F'$.
- For any bicategory $\mathcal{B}$, the bicategory $\mathcal{B}{\downarrow}\mathcal{B} := 1_{\mathcal{B}}{\downarrow}1_{\mathcal{B}}$.

There are commutative squares

$$\begin{tikzcd} F{\downarrow}F' \arrow[r, "\bar{F}"] \arrow[d, "P"'] & \mathcal{B}{\downarrow}F' \arrow[d, "P"] & F{\downarrow}F' \arrow[r, "P'"] \arrow[d, "\bar{F}'"'] & \mathcal{A}' \arrow[d, "F'"] \\ \mathcal{A} \arrow[r, "F"] & \mathcal{B}, & F{\downarrow}\mathcal{B} \arrow[r, "P'"] & \mathcal{B}, \end{tikzcd} \tag{4.10}$$

where the first one is in the category of bicategories and lax functors, and the second one in the category of oplax functors. The lax functor $\bar{F} : F{\downarrow}F' \to \mathcal{B}{\downarrow}F'$ in the first square is given on cells by applying F to the first components

$$\begin{tikzcd} (a_0, f_0, a'_0) \arrow[r, bend left, "{(u,\beta,u')}"] \arrow[r, bend right, "{(\bar{u},\bar{\beta},\bar{u}')}"'] \arrow[r, phantom, "\Downarrow(\alpha,\alpha')"] & (a_1, f_1, a'_1) \arrow[r, mapsto, "\bar{F}"] & (Fa_0, f_0, a'_0) \arrow[r, bend left, "{(Fu,\beta,u')}"] \arrow[r, bend right, "{(F\bar{u},\bar{\beta},\bar{u}')}"'] \arrow[r, phantom, "\Downarrow(F\alpha,\alpha')"] & (Fa_1, f_1, a'_1), \end{tikzcd}$$

while the oplax functor $\bar{F}' : F{\downarrow}F' \to F{\downarrow}\mathcal{B}$ in the second one acts on cells through the application of F' to the last components

$$\begin{tikzcd} (a_0, f_0, a'_0) \arrow[r, bend left, "{(u,\beta,u')}"] \arrow[r, bend right, "{(\bar{u},\bar{\beta},\bar{u}')}"'] \arrow[r, phantom, "\Downarrow(\alpha,\alpha')"] & (a_1, f_1, a'_1) \arrow[r, mapsto, "\bar{F}'"] & (a_0, f_0, F'a'_0) \arrow[r, bend left, "{(u,\beta,F'u')}"] \arrow[r, bend right, "{(\bar{u},\bar{\beta},F'\bar{u}')}"'] \arrow[r, phantom, "\Downarrow(\alpha,F'\alpha')"] & (a_1, f_1, F'a'_1). \end{tikzcd}$$

At any pair of composable 1-cells in $F \downarrow F'$ as in (4.6), their respective structure constraints for the composition are the 2-cells

$$(\widehat{F}_{u_2,u_1}, 1_{u'_2\circ u'_1}) : \bar{F}(u_2,\beta_2,u'_2)\circ\bar{F}(u_1,\beta_1,u'_1) \Rightarrow \bar{F}\big((u_2,\beta_2,u'_2)\circ(u_1,\beta_1,u'_1)\big),$$
$$(1_{u_2\circ u_1}, \widehat{F}'_{u'_2,u'_1}) : \bar{F}'\big((u_2,\beta_2,u'_2)\circ(u_1,\beta_1,u'_1)\big) \Rightarrow \bar{F}'(u_2,\beta_2,u'_2)\circ\bar{F}'(u_1,\beta_1,u'_1),$$

and, at any object (a,f,a') of $F\!\downarrow\!F'$, their respective constraints for the identity are

$$(\widehat{F}_a, 1_{1_{a'}}) : 1_{\bar{F}(a,f,a')} \Rightarrow \bar{F}1_{(a,f,a')}, \quad (1_{1_a}, \widehat{F}'_{a'}) : \bar{F}'1_{(a,f,a')} \Rightarrow 1_{\bar{F}'(a,f,a')}.$$

Although neither the category of bicategories and lax functors nor the category of bicategories and oplax functors have pullbacks in general, the following fact holds.

Lemma 4.2 (*i*) *The first square in* (4.10) *is a pullback in the category of bicategories and lax functors.*

(*ii*) *The second square in* (4.10) *is a pullback in the category of bicategories and oplax functors.*

Proof: (*i*) Any pair of lax functors $L : \mathcal{D} \to \mathcal{A}$ and $M : \mathcal{D} \to \mathcal{B}\downarrow F'$ such that $FL = PM$ determines a unique oplax functor $N : \mathcal{D} \to F\!\downarrow\!F'$

$$\begin{tikzcd}
\mathcal{D} \arrow[dr, dotted, "N"] \arrow[drr, bend left, "M"] \arrow[ddr, bend right, "L"'] & & \\
& F\!\downarrow\!F' \arrow[r, "\bar{F}"] \arrow[d, "P"] & \mathcal{B}\!\downarrow\!F' \arrow[d, "P"] \\
& \mathcal{A} \arrow[r, "F"] & \mathcal{B}
\end{tikzcd}$$

such that $PN = L$ and $\bar{F}N = M$, which is defined as follows: The lax functor M carries any object $d \in \mathrm{Ob}\mathcal{D}$ to an object of $\mathcal{B}\!\downarrow\!F$ which is necessarily written in the form $M(d) = (FL(d), f(d), a'(d))$, with $a'(d)$ an object of $\mathcal{A}'$ and $f(d) : FL(d) \to F'a'(d)$ a 1-cell in $\mathcal{B}$. Similarly, for any 1-cell $h : d_0 \to d_1$ in $\mathcal{D}$, we have $M(h) = (FL(h), \beta(h), u'(h))$ for some 1-cell $u'(h) : a'(d_0) \to a'(d_1)$ in $\mathcal{A}'$ and some 2-cell

$$\begin{tikzcd}
FL(d_0) \arrow[r, "FL(h)"] \arrow[d, "f(d_0)"'] \arrow[dr, phantom, "\overset{\beta(h)}{\Rightarrow}"] & FL(d_1) \arrow[d, "f(d_1)"] \\
F'a'(d_0) \arrow[r, "F'u'(h)"'] & F'a'(d_1),
\end{tikzcd}$$

in $\mathcal{B}$, and for any 2-cell $\gamma : h_0 \Rightarrow h_1$ in $\mathcal{D}$, we have $M(\gamma) = (FL(\gamma), \alpha'(\gamma))$, for some 2-cell $\alpha'(\gamma) : u'(h_0) \Rightarrow u'(h_1)$ in $\mathcal{A}'$. Also, for any object d and any pair of composable

1-cells $h_1 : d_0 \to d_1$ and $h_2 : d_1 \to d_2$ in $\mathcal{D}$, the attached structure 2-cells of M can be respectively written in a similar form as

$$\begin{aligned}
&\widehat{M}_d = (F(\widehat{L}_d) \cdot \widehat{F}_{L(d)}, \widehat{\alpha}'_d) : 1_{M(d)} \Rightarrow M(1_d),\\
&\widehat{M}_{h_2,h_1} = (F(\widehat{L}_{h_2,h_1}) \cdot \widehat{F}_{L(h_2),L(h_1)}, \widehat{\alpha}'_{h_2,h_1}) : M(h_2) \circ M(h_1) \Rightarrow M(h_2 \circ h_1),
\end{aligned}$$

for some 2-cells $\widehat{\alpha}'_{h_2,h_1}$ and $\widehat{\alpha}'_d$ in $\mathcal{A}'$. Then, the claimed $N : \mathcal{D} \to F\downarrow F'$ is the lax functor which acts on cells by

$$\begin{tikzcd}
d_0 \arrow[r, bend left, "h"] \arrow[r, bend right, "\bar{h}"'] \arrow[r, phantom, "\Downarrow\gamma"] & d_1 & \overset{N}{\mapsto} & (L(d_0), f(d_0), a'(d_0)) \arrow[r, bend left, "{(L(h),\beta(h),u'(h))}"] \arrow[r, bend right, "{(L(\bar{h}),\beta(\bar{h}),u'(\bar{h}))}"'] \arrow[r, phantom, "{\Downarrow(L(\gamma),\alpha'(\gamma))}"] & (L(d_1), f(d_1), a'(d_1))
\end{tikzcd}$$

and its respective structure 2-cells, for any object d and any pair of composable 1-cells $h_1 : d_0 \to d_1$ and $h_2 : d_1 \to d_2$ in $\mathcal{D}$, are

$$\begin{aligned}
&\widehat{N}_d = (\widehat{L}_d, \widehat{\alpha}'_d) : 1_{N(d)} \Rightarrow N(1_d),\\
&\widehat{N}_{h_2,h_1} = (\widehat{L}_{h_2,h_1}, \widehat{\alpha}'_{h_2,h_1}) : N(h_2) \circ N(h_1) \Rightarrow N(h_2 \circ h_1).
\end{aligned}$$

The proof of (ii) is parallel to that given above for part (i), and it is left to the reader. ☐

4.2.4 The homotopy-fiber bicategories.

For any 0-cell $b \in \mathcal{B}$, we also denote by $b : [0] \to \mathcal{B}$ the normal homomorphism such that $b(0) = b$, and whose structure isomorphism is $\boldsymbol{l} : 1_b \otimes 1_b \cong 1_b$. Then, we have the bicategories[4]

- $F\downarrow b$, for any lax functor $F : \mathcal{A} \to \mathcal{B}$.
- $b\downarrow F'$, for any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$.
- $b\downarrow\mathcal{B} := b\downarrow 1_{\mathcal{B}}$, and $\mathcal{B}\downarrow b := 1_{\mathcal{B}}\downarrow b$.

Given F and F' as above, any 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$ determines 2-functors

$$p_* : F\downarrow b_0 \to F\downarrow b_1, \quad p^* : b_1\downarrow F' \to b_0\downarrow F', \tag{4.11}$$

respectively given on cells by

$$\begin{tikzcd}
(a_0, f_0) \arrow[r, bend left, "{(u,\beta)}"] \arrow[r, bend right, "{(\bar{u},\bar{\beta})}"'] \arrow[r, phantom, "\Downarrow\alpha"] & (a_1, f_1) & \overset{p_*}{\longmapsto} & (a_0, p \circ f_0) \arrow[r, bend left, "{(u,p\odot\beta)}"] \arrow[r, bend right, "{(\bar{u},p\odot\bar{\beta})}"'] \arrow[r, phantom, "\Downarrow\alpha"] & (a_1, p \circ f_1),
\end{tikzcd}$$

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[4] As we mentioned before these bicategories are isomorphic to the bicategories $F\downarrow_b$ defined in Chapter 3

$$\begin{tikzcd}
(f_0,a'_0) \arrow[r, bend left, "{(\beta,u')}"] \arrow[r, bend right, "{(\bar{\beta},\bar{u}')}"'] \arrow[r, phantom, "\Downarrow\alpha'"] & (f_1,a'_1) \arrow[r, mapsto, "p^*"] & (f_0\circ p,a'_0) \arrow[r, bend left, "{(\beta\odot p,u')}"] \arrow[r, bend right, "{(\bar{\beta}\odot p,\bar{u}')}"'] \arrow[r, phantom, "\Downarrow\alpha'"] & (f_1\circ p,a'_1),
\end{tikzcd}$$

where, for any $(u,\beta):(a_0,f_0)\to(a_1,f_1)$ in $F\!\downarrow\! b_0$ and $(\beta,u'):(f_0,a'_0)\to(f_1,a'_1)$ in $b_1\!\downarrow\! F'$, the 2-cells $p\odot\beta$ and $\beta\odot p$ are respectively obtained by pasting the diagrams

$$\begin{tikzcd}
 & Fa_0 \arrow[r, "Fu"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & Fa_1 \arrow[d, "f_1"] & & b_0 \arrow[r, "1"] \arrow[d, "p"'] \arrow[dr, phantom, "\cong"] & b_0 \arrow[d, "p"] \\
p\odot\beta: & b_0 \arrow[r, "1"] \arrow[d, "p"'] \arrow[dr, phantom, "\cong"] & b_0 \arrow[d, "p"] & \beta\odot p: & b_1 \arrow[r, "1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & b_1 \arrow[d, "f_1"] \\
 & b_1 \arrow[r, "1"] & b_1 & & F'a'_0 \arrow[r, "F'u'"] & F'a'_1
\end{tikzcd}$$

that is,

$$p\odot\beta=\Big(1\circ(p\circ f_0)\xRightarrow{\boldsymbol{l}}p\circ f_0\xRightarrow{1\circ\boldsymbol{l}^{-1}}p\circ(1\circ f_0)\xRightarrow{1\circ\beta}p\circ(f_1\circ Fu)\xRightarrow{\boldsymbol{a}^{-1}}(p\circ f_1)\circ Fu\Big),$$

$$\beta\odot p=\Big(F'u'\circ(f_0\circ p)\xRightarrow{\boldsymbol{a}^{-1}}(F'u'\circ f_0)\circ p\xRightarrow{\beta\circ 1}(f_1\circ 1)\circ p\xRightarrow{\boldsymbol{r}\circ 1}f_1\circ p\xRightarrow{\boldsymbol{r}^{-1}}(f_1\circ p)\circ 1\Big).$$

4.2.5 Classifying spaces of bicategories.

Briefly, let us recall[5] from [41, Definition 3.1] that the *classifying space* $\mathrm{B}\mathcal{B}$ of a (small) bicategory $\mathcal{B}$ is defined as the geometric realization of the *Grothendieck nerve* or *pseudo-simplicial nerve* of the bicategory, that is, the pseudo-functor from Δ^{op} to the 2-category **Cat** of small categories

$$\mathrm{N}\mathcal{B}:\Delta^{op}\to\mathbf{Cat},\quad [p]\mapsto\bigsqcup_{(b_0,\dots,b_p)}\mathcal{B}(b_{p-1},b_p)\times\mathcal{B}(b_{p-2},b_{p-1})\times\cdots\times\mathcal{B}(b_0,b_1),\tag{4.12}$$

whose face and degeneracy functors are defined in the standard way by using the horizontal composition and identity morphisms of the bicategory, and the natural isomorphisms $d_id_j\cong d_{j-1}d_i$, etc., being given from the associativity and unit constraints of the bicategory (see Theorem 4.9 in the Appendix, for more details). Thus,

$$\mathrm{B}\mathcal{B}=\mathrm{B}\textstyle\int_\Delta\mathrm{N}\mathcal{B}$$

is the classifying space of the category $\int_\Delta\mathrm{N}\mathcal{B}$ obtained by the Grothendieck construction[6] [73] on the pseudofunctor $\mathrm{N}\mathcal{B}$. In other words, $\mathrm{B}\mathcal{B}=|\mathrm{N}\int_\Delta\mathrm{N}\mathcal{B}|$ is the geometric realization of the simplicial set nerve of the category $\int_\Delta\mathrm{N}\mathcal{B}$. When $\mathcal{B}$ is a 2-category, then $\mathrm{B}\mathcal{B}$ is homotopy equivalent to Segal's classifying space [111] of the topological

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[5] Also see Chapter 2.

[6] See Section 3.3 in Chapter 3.

category obtained from $\mathcal{B}$ by replacing the hom-categories $\mathcal{B}(x,y)$ by their classifying spaces $\mathrm{B}\mathcal{B}(x,y)$, see [41, Remark 3.2].

In [41, §4], it is proven that the classifying space construction, $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$, is a functor $\mathrm{B}: \mathbf{Hom} \to \mathbf{Top}$, from the category of bicategories and homomorphisms to the category **Top** of spaces (actually of CW-complexes). In this chapter, we need the extension of this fact stated in part (*i*) of the lemma below.

Lemma 4.3 (*i*) *The assignment $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$ is the function on objects of two functors into the category of spaces*

$$\mathbf{Lax} \xrightarrow{\mathrm{B}} \mathbf{Top} \xleftarrow{\mathrm{B}} \mathbf{opLax}\,,$$

where **Lax** *(resp.* **opLax***) is the category of bicategories with lax (resp. oplax) functors between them as morphisms.*

(*ii*) *If $F, G : \mathcal{B} \to \mathcal{C}$ are two lax or oplax functors between bicategories, then any lax or oplax transformation between them $\alpha : F \Rightarrow G$ determines a homotopy, $\mathrm{B}\alpha : \mathrm{B}F \Rightarrow \mathrm{B}G : \mathrm{B}\mathcal{B} \to \mathrm{B}\mathcal{C}$, between the induced maps on classifying spaces.*

Proof: It is given in the Appendix, Corollaries 4.5 and 4.7. □

Other possibilities for defining $\mathrm{B}\mathcal{B}$ come from the *geometric nerves* of the bicategory, first defined by Street [117] and studied, among others, by Duskin [58], Gurski [74] and Carrasco, Cegarra, and Garzón [41]; that is, the simplicial sets

$$\begin{array}{lll} \Delta^{\mathrm{u}}\mathcal{B} : \Delta^{op} \to \mathbf{Set}, & [p] \mapsto \mathrm{NorLax}([p],\mathcal{B}), \\ \Delta\mathcal{B} : \Delta^{op} \to \mathbf{Set}, & [p] \mapsto \mathrm{Lax}([p],\mathcal{B}), \\ \nabla_{\mathrm{u}}\mathcal{B} : \Delta^{op} \to \mathbf{Set}, & [p] \mapsto \mathrm{NorOpLax}([p],\mathcal{B}), \\ \nabla\mathcal{B} : \Delta^{op} \to \mathbf{Set}, & [p] \mapsto \mathrm{OpLax}([p],\mathcal{B}), \end{array} \tag{4.13}$$

whose respective p-simplices are the normal lax, lax, normal oplax, and oplax functors from the category $[p]$ into the bicategory $\mathcal{B}$. In the Homotopy Invariance Theorem [41, Theorem 6.1] the existence of homotopy equivalences

$$|\Delta^{\mathrm{u}}\mathcal{B}| \simeq |\Delta\mathcal{B}| \simeq \mathrm{B}\mathcal{B} \simeq |\nabla\mathcal{B}| \simeq |\nabla_{\mathrm{u}}\mathcal{B}|, \tag{4.14}$$

it is proven, but their natural behaviour is not studied. Since, to establish the results in this chapter, we need to know that all the homotopy equivalences above are homotopy natural, we state the following

Lemma 4.4 *For any bicategory $\mathcal{B}$, the first homotopy equivalence in* (4.14) *is natural on normal lax functors, the second one is homotopy natural on lax functors, the third one is homotopy natural on oplax functors, and the fourth one is natural on normal oplax functors.*

Proof: By [41, Theorem 6.2], the homotopy equivalence $|\Delta^{\mathrm{u}}\mathcal{B}| \simeq |\Delta\mathcal{B}|$ is induced on geometric realizations by the inclusion map $\Delta^{\mathrm{u}}\mathcal{B} \hookrightarrow \Delta\mathcal{B}$. Therefore, it is clearly natural on normal lax functors between bicategories. Similarly, the homotopy equivalence $|\nabla_{\mathrm{u}}\mathcal{B}| \simeq |\nabla\mathcal{B}|$ is natural on normal oplax functors. The proof for the other two is more complicated and is given in the Appendix, Corollary 4.6. ☐

4.3 Inducing homotopy pullbacks on classifying spaces

Quillen's Theorem B [109] provides a sufficient condition on a functor between small categories $F : \mathcal{A} \to \mathcal{B}$ for the classifying space $\mathrm{B}(F\!\downarrow\! b)$ to be a homotopy-fiber over the 0-cell $\mathrm{B}b \in \mathrm{B}\mathcal{B}$ of the induced map $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$, for each object $b \in \mathrm{Ob}\mathcal{B}$. The condition is that the maps $\mathrm{B}p_* : \mathrm{B}(F\!\downarrow\! b) \to \mathrm{B}(F\!\downarrow\! b')$ are homotopy equivalences for every morphism $p : b \to b'$ in the category $\mathcal{B}$. That condition was referred to by Dwyer, Kan, and Smith in [60, §6] by saying that "the functor F has the property B" (see also Barwick, and Kan in [13, 14]), and by Cisinski in [55, 6.4.1] by saying that "the functor F is locally constant". To state our theorem below, we shall adapt that terminology to the bicategorical setting, and we will say that

(B_l) *a lax functor* between bicategories $F : \mathcal{A} \to \mathcal{B}$ *has the property* B_l if, for any 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the 2-functor $p_* : F\downarrow b_0 \to F\downarrow b_1$ in (4.11) induces a homotopy equivalence on classifying spaces, $\mathrm{B}(F\!\downarrow\! b_0) \simeq \mathrm{B}(F\!\downarrow\! b_1)$.

(B_o) *an oplax functor* between bicategories $F' : \mathcal{A}' \to \mathcal{B}$ *has the property* B_o if, for any 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the 2-functor $p^* : b_1\!\downarrow\! F' \to b_0\!\downarrow\! F'$ in (4.11) induces a homotopy equivalence on classifying spaces, $\mathrm{B}(b_1\!\downarrow\! F') \simeq \mathrm{B}(b_0\!\downarrow\! F')$.

The main result in this chapter can be summarized as follows:

Theorem 4.1 *Let* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$ *be a diagram of bicategories, where* F *is a lax functor and* F' *is an oplax functor (for instance, if* F *and* F' *are any two homomorphisms).*

(*i*) *There is a homotopy* $\mathrm{B}F\,\mathrm{B}P \Rightarrow \mathrm{B}F'\,\mathrm{B}P'$*, so that the square below, which is induced by* (4.8) *on classifying spaces, is homotopy commutative.*

$$\begin{tikzcd}
\mathrm{B}(F\!\downarrow\! F') \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}P"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{B}\mathcal{A}' \arrow[d, "\mathrm{B}F'"] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B}
\end{tikzcd} \tag{4.15}$$

(*ii*) *Suppose that* F *has the property* B_l *or* F' *has the property* B_o*. Then, the square* (4.15) *is a homotopy pullback.*

Therefore, by Dyer and Roitberg [61], for each $a \in \mathrm{Ob}\mathcal{A}$ *and* $a' \in \mathrm{Ob}\mathcal{A}'$ *such that* $Fa = F'a'$ *there is an induced Mayer-Vietoris type long exact sequence on homotopy* *groups based at the* 0*-cells* $\mathrm{B}a$ *of* $\mathrm{B}\mathcal{A}$*,* $\mathrm{B}Fa$ *of* $\mathrm{B}\mathcal{B}$*,* $\mathrm{B}a'$ *of* $\mathrm{B}\mathcal{A}'$*, and* $\mathrm{B}(a,1,a')$ *of* $\mathrm{B}(F\!\downarrow\!F')$*,*

$$\cdots \to \pi_{n+1}\mathrm{B}\mathcal{B} \to \pi_n\mathrm{B}(F\!\downarrow\!F') \to \pi_n\mathrm{B}\mathcal{A}\times\pi_n\mathrm{B}\mathcal{A}' \to \pi_n\mathrm{B}\mathcal{B} \to \cdots$$

$$\cdots \to \pi_1\mathrm{B}(F\!\downarrow\!F') \to \pi_1\mathrm{B}\mathcal{A}\times\pi_1\mathrm{B}\mathcal{A}' \to \pi_1\mathrm{B}\mathcal{B} \to \pi_0\mathrm{B}(F\!\downarrow\!F') \to \pi_0(\mathrm{B}\mathcal{A}\times\mathrm{B}\mathcal{A}').$$

(iii) If the square (4.15) *is a homotopy pullback for every* $F' = b : [0] \to \mathcal{B}$*,* $b \in \mathrm{Ob}\mathcal{B}$*, then* F *has the property* B_l*. Similarly, if the square* (4.15) *is a homotopy pullback for any* $F = b : [0] \to \mathcal{B}$*,* $b \in \mathrm{Ob}\mathcal{B}$*, then* F' *has the property* B_o*.*

The remainder of this section is devoted to the proof of this theorem. We shall start by recalling Lemma 3.7 from Chapter 3:

Lemma 4.5 *For any object* b *of a bicategory* $\mathcal{B}$*, the classifying spaces of the comma bicategories* $\mathcal{B}\!\downarrow\!b$ *and* $b\!\downarrow\!\mathcal{B}$ *are contractible, that is,* $\mathrm{B}(\mathcal{B}\!\downarrow\!b) \simeq \mathrm{pt} \simeq \mathrm{B}(b\!\downarrow\!\mathcal{B})$*.*

We also need the auxiliary result below. To state it, we use that, for any given diagram $F : \mathcal{A} \to \mathcal{B} \leftarrow \mathcal{A}' : F'$, with F a lax functor and F' an oplax functor, and for each objects a of $\mathcal{A}$ and a' of $\mathcal{A}'$, there are normal homomorphisms

$$Fa\!\downarrow\!F' \xrightarrow{\ J\ } F\!\downarrow\!F' \xleftarrow{\ J'\ } F\!\downarrow\!F'a', \tag{4.16}$$

where J acts on cells by

$$\begin{tikzcd}[column sep=large]
(f_0,a'_0) \arrow[r, bend left, "{(\beta,u')}"] \arrow[r, bend right, "{(\bar\beta,\bar u')}"'] \arrow[r, phantom, "\Downarrow\alpha'"] & (f_1,a'_1) \arrow[r, mapsto, "J"] & (a,f_0,a'_0) \arrow[r, bend left, "{(1_a,\imath(\beta,u'),u')}"] \arrow[r, bend right, "{(1_a,\imath(\bar\beta,\bar u'),\bar u')}"'] \arrow[r, phantom, "\Downarrow(1,\alpha')"] & (a,f_1,a'_1),
\end{tikzcd}$$

where, for any 1-cell $(\beta,u') : (f_0,a'_0) \to (f_1,a'_1)$ in $Fa\!\downarrow\!F'$, the 2-cell $\imath(\beta,u')$ is defined as the composite

$$\imath(\beta,u') = \big(F'u'\circ f_0 \overset{\beta}{\Longrightarrow} f_1\circ 1_{Fa} \overset{1\circ\widehat{F}_a}{\Longrightarrow} f_1\circ F1_a\big),$$

and whose constraints, at pairs of 1-cells $(f_0,a'_0)\xrightarrow{(\beta_1,u'_1)}(f_1,a'_1)\xrightarrow{(\beta_2,u'_2)}(f_2,a'_2)$ in $Fa\!\downarrow\!F'$, are the 2-cells of $F\!\downarrow\!F'$

$$(\boldsymbol{l}_{1_a}, 1_{u'_2\circ u'_1}) : (1_a\circ 1_a, \imath(\beta_2,u'_2)\circledcirc\imath(\beta_1,u'_1), u'_2\circ u'_1) \cong (1_a, \imath(\beta_2\circledcirc\beta_1, u'_2\circ u'_1), u'_2\circ u'_1).$$

Similarly, J' acts by

$$\begin{tikzcd}[column sep=large]
(a_0,f_0) \arrow[r, bend left, "{(u,\beta)}"] \arrow[r, bend right, "{(\bar u,\bar\beta)}"'] \arrow[r, phantom, "\Downarrow\alpha"] & (a_1,f_1) \arrow[r, mapsto, "J'"] & (a_0,f_0,a') \arrow[r, bend left, "{(u,\imath'(u,\beta),1_{a'})}"] \arrow[r, bend right, "{(\bar u,\imath'(\bar u,\bar\beta),1_{a'})}"'] \arrow[r, phantom, "\Downarrow(\alpha,1)"] & (a_1,f_1,a'),
\end{tikzcd}$$

where, for any 1-cell $(u,\beta):(a_0,f_0)\to(a_1,f_1)$ in $F\!\downarrow\!F'a'$, the 2-cell $\imath'(u,\beta)$ is defined as the composites

$$\imath'(u,\beta)=\big(F'1_{a'}\circ f_0 \overset{\widehat{F'}\circ 1}{\Longrightarrow} 1_{F'a'}\circ f_0 \overset{\beta}{\Longrightarrow} f_1\circ Fu\big),$$

and whose constraints, at pairs of 1-cells $(a_0,f_0)\xrightarrow{(u_1,\beta_1)}(a_1,f_1)\xrightarrow{(u_2,\beta_2)}(a_2,f_2)$ in $F\!\downarrow\!F'a'$, are the 2-cells of $F\!\downarrow\!F'$

$$(1_{u_2\circ u_1},\boldsymbol{l}_{1_{a'}}):(u_2\circ u_1,\imath'(u_2,\beta_2)\circledcirc\imath'(u_1,\beta_1),1_{a'}\circ 1_{a'})\cong(u_2\circ u_1,\imath'(u_2\circ u_1,\beta_2\circledcirc\beta_1),1_{a'}).$$

Lemma 4.6 *Let* $\mathcal{A}\xrightarrow{F}\mathcal{B}\xleftarrow{F'}\mathcal{A}'$ *be any diagram of bicategories, where* F *is a lax functor and* F' *is an oplax functor.*

(i) *If* $\mathcal{A}$ *is a category with an initial object* 0*, then the homomorphism* J *in* (4.16) *induces a homotopy equivalence on classifying spaces,* $\mathrm{B}(F0\!\downarrow\!F')\simeq\mathrm{B}(F\!\downarrow\!F')$.

(ii) *If* $\mathcal{A}'$ *is a category with a terminal object* 1*, then the homomorphism* J' *in* (4.16) *induces a homotopy equivalence on classifying spaces,* $\mathrm{B}(F\!\downarrow\!F'1)\simeq\mathrm{B}(F\!\downarrow\!F')$.

Proof: We only prove (i) since the proof of (ii) is parallel. Let $\langle a\rangle:0\to a$ be the unique morphism in $\mathcal{A}$ from the initial object to a. There is a 2-functor $L:F\!\downarrow\!F'\to F0\!\downarrow\!F'$ given on cells by

$$\begin{tikzcd}
(a_0,f_0,a'_0) \arrow[r, bend left, "{(u,\beta,u')}"] \arrow[r, bend right, "{(u,\bar\beta,\bar u')}"'] \arrow[r, phantom, "{\Downarrow(1,\alpha')}"] & (a_1,f_1,a'_1) \arrow[r, mapsto, "L"] & (f_0\circ F\langle a_0\rangle,a'_0) \arrow[r, bend left, "{(\ell(u,\beta,u'),u')}"] \arrow[r, bend right, "{(\ell(u,\bar\beta,\bar u'),\bar u')}"'] \arrow[r, phantom, "\Downarrow\alpha'"] & (f_1\circ F\langle a_1\rangle,a'_1),
\end{tikzcd}$$

where, for any 1-cell $(u,\beta,u'):(a_0,f_0,a'_0)\to(a_1,f_1,a'_1)$ of $F\downarrow F'$, $\ell(u,\beta,u')$ is the 2-cell of $\mathcal{B}$ obtained by pasting the diagram

$$\ell(u,\beta,u'):\quad\begin{tikzcd}
F0 \arrow[r, "1"] \arrow[d, "F\langle a_0\rangle"'] \arrow[dr, phantom, "\overset{\boldsymbol{r}^{-1}\cdot\widehat{F}}{\Rightarrow}"] & F0 \arrow[d, "F\langle a_1\rangle"] \\
Fa_0 \arrow[r, "Fu"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & Fa_1 \arrow[d, "f_1"] \\
F'a'_0 \arrow[r, "F'u'"'] & F'a'_1
\end{tikzcd}$$

that is,

$$\begin{aligned}\ell(u,\beta,u')=\Big(&F'u'\circ(f_0\circ F\langle a_0\rangle)\overset{\boldsymbol{a}^{-1}}{\Longrightarrow}(F'u'\circ f_0)\circ F\langle a_0\rangle\overset{\beta\circ 1}{\Longrightarrow}(f_1\circ Fu)\circ F\langle a_0\rangle\overset{\boldsymbol{a}}{\Longrightarrow}\\ &f_1\circ(Fu\circ F\langle a_0\rangle)\overset{1\circ\widehat{F}}{\Longrightarrow}f_1\circ F(u\circ\langle a_0\rangle)=f_1\circ F\langle a_1\rangle\overset{\boldsymbol{r}^{-1}}{\Longrightarrow}(f_1\circ F\langle a_1\rangle)\circ 1\Big).\end{aligned}$$

In addition, there are two pseudo-transformations

$$1_{F0\downarrow F'}\Rightarrow LJ,\quad JL\Rightarrow 1_{F\downarrow F'}.$$

The first one has as a component, at any object (f, a') of $F0{\downarrow}F'$, the 1-cell

$$(\eta(f,a'), 1_{a'}) : (f, a') \to (f \circ F1_0, a'),$$
$$\eta(f,a') = \big(F'1_{a'} \circ f \overset{\widehat{F'}\circ 1}{\Longrightarrow} 1_{F'a'} \circ f \overset{\boldsymbol{l}}{\Longrightarrow} f \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} f \circ 1_{F0} \overset{1\circ\widehat{F}}{\Longrightarrow} f \circ F1_0 \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} (f \circ F1_0) \circ 1_{F0}\big)$$

while its naturality component, at any 1-cell $(\beta, u') : (f_0, a'_0) \to (f_1, a'_1)$ of $F0{\downarrow}F'$, is given by the canonical isomorphism $\boldsymbol{l}^{-1} \cdot \boldsymbol{r} : u' \circ 1_{a'_0} \cong 1_{a'_1} \circ u'$,

$$\begin{tikzcd}
(f_0, a'_0) \arrow[r, "{(\beta,u')}"] \arrow[d, "{(\eta,1)}"'] \arrow[dr, phantom, "{\boldsymbol{l}^{-1}_{\cong}\cdot\boldsymbol{r}}"] & (f_1, a'_1) \arrow[d, "{(\eta,1)}"] \\
(f_0 \circ F1_0, a'_0) \arrow[r, "{(\ell(1_0,\imath(\beta,u'),u'),u')}"'] & (f_1 \circ F1_0, a'_1).
\end{tikzcd}$$

As for the pseudo-transformation $JL \Rightarrow 1_{F\downarrow F'}$, it associates to an object (a, f, a') in $F{\downarrow}F'$ the 1-cell

$$(\langle a\rangle, \epsilon(a,f,a'), 1_{a'}) : (0, f \circ F\langle a\rangle, a') \to (a, f, a')$$
$$\epsilon(a,f,a') = \big(F'1_{a'} \circ (f \circ F\langle a\rangle) \overset{\widehat{F'}\circ 1}{\Longrightarrow} 1_{F'a'} \circ (f \circ F\langle a\rangle) \overset{\boldsymbol{l}}{\Longrightarrow} f \circ F\langle a\rangle\big)$$

while its naturality component, at a 1-cell $(u, \beta, u') : (a_0, f_0, a'_0) \to (a_1, f_1, a'_1)$ of $F{\downarrow}F'$, is

$$\begin{tikzcd}
(0, f_0 \circ F\langle a_0\rangle, a'_0) \arrow[r, "{(1_0,\imath(\ell(u,\beta,u'),u'),u')}"] \arrow[d, "{(\langle a_0\rangle,\epsilon,1)}"'] \arrow[dr, phantom, "{(1,\boldsymbol{l}^{-1}_{\cong}\cdot\boldsymbol{r})}"] & (0, f_1 \circ F\langle a_1\rangle, a'_1) \arrow[d, "{(\langle a_1\rangle,\epsilon,1)}"] \\
(a_0, f_0, a'_0) \arrow[r, "{(u,\beta,u')}"'] & (a_1, f_1, a'_1).
\end{tikzcd}$$

Therefore, owing to Lemma 4.3, there are homotopies $\mathrm{B}J\,\mathrm{B}L \Rightarrow 1_{\mathrm{B}(F\downarrow F')}$ and $1_{\mathrm{B}(F0\downarrow F')} \Rightarrow \mathrm{B}L\,\mathrm{B}J$ making $\mathrm{B}J$ a homotopy equivalence. $\square$

As we will see below, the following result is the key for proving Theorem 4.1.

Lemma 4.7 (*i*) *If an oplax functor* $F' : \mathcal{A}' \to \mathcal{B}$ *has the property* B_o*, then, for any lax functor* $F : \mathcal{A} \to \mathcal{B}$*, the commutative square*

$$\begin{tikzcd}
\mathrm{B}(F{\downarrow}F') \arrow[r, "\mathrm{B}\bar{F}"] \arrow[d, "\mathrm{B}P"'] & \mathrm{B}(\mathcal{B}{\downarrow}F') \arrow[d, "\mathrm{B}P"] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B},
\end{tikzcd} \tag{4.17}$$

induced by the first square in (4.10) *on classifying spaces, is a homotopy pullback.*

(ii) If a lax functor $F : \mathcal{A} \to \mathcal{B}$ has the property B_l*, then, for any oplax functor* $F' : \mathcal{A}' \to \mathcal{B}$*, the commutative square*

$$\begin{tikzcd}
\mathrm{B}(F\!\downarrow\! F') \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}\bar{F}'"'] & \mathrm{B}\mathcal{A}' \arrow[d, "\mathrm{B}F'"] \\
\mathrm{B}(F\!\downarrow\!\mathcal{B}) \arrow[r, "\mathrm{B}P'"] & \mathrm{B}\mathcal{B},
\end{tikzcd} \tag{4.18}$$

induced by the second square in (4.10) *on classifying spaces, is a homotopy pullback.*

Proof: Suppose that $F' : \mathcal{A}' \to \mathcal{B}$ is any given oplax functor having the property B_o. We will prove that the simplicial map $\Delta P : \Delta(\mathcal{B}\!\downarrow\! F') \to \Delta\mathcal{B}$, induced on geometric nerves by the projection 2-functor $P : \mathcal{B}\!\downarrow\! F' \to \mathcal{B}$ in (4.9), satisfies the condition (*i*) of Lemma 4.1. To do so, let $\mathbf{x} : [n] \to \mathcal{B}$ be any geometric n-simplex of $\mathcal{B}$. Thanks to Lemma 4.2 (*i*), the square

$$\begin{tikzcd}
\mathbf{x}\!\downarrow\! F' \arrow[r, "\bar{\mathbf{x}}"] \arrow[d, "P"'] & \mathcal{B}\!\downarrow\! F' \arrow[d, "P"] \\
{[n]} \arrow[r, "\mathbf{x}"] & \mathcal{B}
\end{tikzcd}$$

is a pullback in the category of bicategories and lax functors, whence the square induced by taking geometric nerves

$$\begin{tikzcd}
\Delta(\mathbf{x}\!\downarrow\! F') \arrow[r, "\Delta\bar{\mathbf{x}}"] \arrow[d, "\Delta P"'] & \Delta(\mathcal{B}\!\downarrow\! F') \arrow[d, "\Delta P"] \\
\Delta[n] \arrow[r, "\Delta\mathbf{x}"] & \Delta\mathcal{B}
\end{tikzcd}$$

is a pullback in the category of simplicial sets. Therefore, $\Delta P^{-1}(\Delta\mathbf{x}) \cong \Delta(\mathbf{x}\!\downarrow\! F')$. Furthermore, for any map $\sigma : [m] \to [n]$ in the simplicial category, the diagram of lax functors

$$\begin{tikzcd}
\mathbf{x}\sigma\!\downarrow\! F' \arrow[dr, "\bar{\sigma}"'] \arrow[drr, "\overline{\mathbf{x}\sigma}"] \arrow[dd, "P"'] & & \\
& \mathbf{x}\!\downarrow\! F' \arrow[r, "\bar{\mathbf{x}}"'] \arrow[dd, "P"] & \mathcal{B}\!\downarrow\! F' \arrow[dd, "P"] \\
{[m]} \arrow[dr, "\sigma"'] \arrow[drr, "\mathbf{x}\sigma"] & & \\
& {[n]} \arrow[r, "\mathbf{x}"'] & \mathcal{B}
\end{tikzcd}$$

is commutative, whence the induced diagram of simplicial maps

$$\begin{tikzcd}
\Delta(\mathbf{x}\sigma\!\downarrow\! F') \arrow[dr, "\Delta\bar{\sigma}"'] \arrow[drr, "\Delta\overline{\mathbf{x}\sigma}"] \arrow[dd, "\Delta P"'] & & \\
& \Delta(\mathbf{x}\!\downarrow\! F') \arrow[r, "\Delta\bar{\mathbf{x}}"'] \arrow[dd, "\Delta P"] & \Delta(\mathcal{B}\!\downarrow\! F') \arrow[dd, "\Delta P"] \\
\Delta[m] \arrow[dr, "\Delta\sigma"'] \arrow[drr, "\Delta(\mathbf{x}\sigma)"] & & \\
& \Delta[n] \arrow[r, "\Delta\mathbf{x}"'] & \Delta\mathcal{B}
\end{tikzcd}$$

is also commutative. Consequently, the diagram below commutes.

$$\begin{tikzcd}
\Delta P^{-1}(\Delta\mathbf{x}\Delta\sigma) \arrow[r] \arrow[d, "\cong"'] & \Delta P^{-1}(\Delta\mathbf{x}) \arrow[d, "\cong"] \\
\Delta(\mathbf{x}\sigma{\downarrow}F') \arrow[r, "\Delta\bar{\sigma}"] & \Delta(\mathbf{x}{\downarrow}F')
\end{tikzcd}$$

Therefore, it suffices to prove that the lax functor $\bar{\sigma} : \mathbf{x}\sigma{\downarrow}F' \to \mathbf{x}{\downarrow}F'$ induces a homotopy equivalence on classifying spaces, $\mathrm{B}(\mathbf{x}\sigma{\downarrow}F') \simeq \mathrm{B}(\mathbf{x}{\downarrow}F')$. But note that we have the diagram

$$\begin{tikzcd}
\mathbf{x}\sigma0{\downarrow}F' \arrow[r, "J"] \arrow[d, "\mathbf{x}(0,\sigma0)^*"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & \mathbf{x}\sigma{\downarrow}F' \arrow[d, "\bar{\sigma}"] \\
\mathbf{x}0{\downarrow}F' \arrow[r, "J"] & \mathbf{x}{\downarrow}F'
\end{tikzcd}$$

where the homomorphisms J are given as in (4.16), and θ is the pseudo-transformation that assigns to every object (f, a') of $\mathbf{x}\sigma0{\downarrow}F'$ the 1-cell of $\mathbf{x}{\downarrow}F'$

$$\big((0,\sigma0), \theta(f,a'), 1_{a'}\big) : (0, f\circ\mathbf{x}(0,\sigma0), a') \to (\sigma0, f, a'),$$

where the 2-cell of $\mathcal{B}$

$$\begin{tikzcd}
\mathbf{x}0 \arrow[r, "\mathbf{x}(0,\sigma0)"] \arrow[d, "f\circ\mathbf{x}(0,\sigma0)"'] \arrow[dr, phantom, "\overset{\theta(f,a')}{\Rightarrow}"] & \mathbf{x}\sigma0 \arrow[d, "f"] \\
F'a' \arrow[r, "F'1_{a'}"'] & F'a'
\end{tikzcd}$$

is the composite

$$\theta(f,a') = \big(F'1_{a'} \circ (f\circ\mathbf{x}(0,\sigma0)) \overset{\widehat{F'}\circ 1}{\Longrightarrow} 1_{F'a'} \circ (f\circ\mathbf{x}(0,\sigma0)) \overset{\boldsymbol{l}}{\Longrightarrow} f\circ\mathbf{x}(0,\sigma0)\big),$$

and its naturality component at any 1-cell $(\beta, u') : (f_0, a'_0) \to (f_1, a'_1)$

$$\begin{tikzcd}
(0, f_0\circ\mathbf{x}(0,\sigma0), a'_0) \arrow[r, "{((0,0),\imath(\beta\circledcirc\mathbf{x}(0,\sigma0),u'),u')}"] \arrow[d, "{((0,\sigma0),\theta(f_0,a'_0),1_{a'_0})}"'] \arrow[dr, phantom, "{(1,\underset{\cong}{\boldsymbol{l}^{-1}}\cdot\boldsymbol{r})}"] & (0, f_1\circ\mathbf{x}(0,\sigma0), a'_1) \arrow[d, "{((0,\sigma0),\theta(f_1,a'_1),1_{a'_1})}"] \\
(\sigma0, f_0, a'_0) \arrow[r, "{((\sigma0,\sigma0),\imath(\beta,u'),u')}"'] & (\sigma0, f_1, a'_1)
\end{tikzcd}$$

is given by the canonical isomorphism $\boldsymbol{l}^{-1}\cdot\boldsymbol{r} : u'\circ 1_{a'_1} \cong 1_{a'_2}\circ u'$ in $\mathcal{A}'$. Therefore, by Lemma 4.3, the induced square on classifying spaces

$$\begin{tikzcd}
\mathrm{B}(\mathbf{x}\sigma0{\downarrow}F') \arrow[r, "\mathrm{B}J"] \arrow[d, "\mathrm{B}\mathbf{x}(0,\sigma0)^*"'] \arrow[dr, phantom, "\overset{\mathrm{B}\theta}{\Rightarrow}"] & \mathrm{B}(\mathbf{x}\sigma{\downarrow}F') \arrow[d, "\mathrm{B}\bar{\sigma}"] \\
\mathrm{B}(\mathbf{x}0{\downarrow}F') \arrow[r, "\mathrm{B}J"] & \mathrm{B}(\mathbf{x}{\downarrow}F')
\end{tikzcd}$$

is homotopy commutative. Moreover, by Lemma 4.6(i), both maps $\mathrm{B}J$ in the square are homotopy equivalences and, since the oplax functor F' has the property B_o, the map $\mathrm{B}\mathbf{x}(0,\sigma 0)^* : \mathrm{B}(\mathbf{x}\sigma 0 \downarrow F') \to \mathrm{B}(\mathbf{x}0 \downarrow F')$ is also a homotopy equivalence. It follows that the remaining map in the square has the same property, that is, the map $\mathrm{B}\bar{\sigma} : \mathrm{B}(\mathbf{x}\sigma\!\downarrow\! F') \simeq \mathrm{B}(\mathbf{x}\!\downarrow\! F')$ is a homotopy equivalence, as required.

Suppose now that $F : \mathcal{A} \to \mathcal{B}$ is any lax functor. Again, by Lemma 4.2(i), the first square in (4.10) is a pullback in the category of bicategories and lax functors, whence the square induced by taking geometric nerves

$$\begin{tikzcd}
\Delta(F\!\downarrow\! F') \arrow[r, "\Delta\bar{F}"] \arrow[d, "\Delta P"'] & \Delta(\mathcal{B}\!\downarrow\! F') \arrow[d, "\Delta P"] \\
\Delta\mathcal{A} \arrow[r, "\Delta F"] & \Delta\mathcal{B}
\end{tikzcd} \tag{4.19}$$

is a pullback in the category of simplicial sets. By what has been already proven above, it follows from Lemma 4.1 (iii) that the commutative square

$$\begin{tikzcd}
{|\Delta(F\!\downarrow\! F')|} \arrow[r, "|\Delta\bar{F}|"] \arrow[d, "|P|"'] & {|\Delta(\mathcal{B}\!\downarrow\! F')|} \arrow[d, "|P|"] \arrow[r, phantom, "\overset{(4.14)}{\simeq}"] & \mathrm{B}(F\!\downarrow\! F') \arrow[r, "\mathrm{B}\bar{F}"] \arrow[d, "\mathrm{B}P"'] & \mathrm{B}(\mathcal{B}\!\downarrow\! F') \arrow[d, "\mathrm{B}P"] \\
{|\Delta\mathcal{A}|} \arrow[r, "|\Delta F|"] & {|\Delta\mathcal{B}|} & \mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

is a homotopy pullback. This completes the proof of part (i) of the lemma.

The proof of part (ii) follows similar lines, but using the geometric nerve functor ∇ instead of Δ as above. Thus, for example, given any lax functor $F : \mathcal{A} \to \mathcal{B}$ having the property B_l, we start by proving that the simplicial map $\nabla P' : \nabla(F\!\downarrow\!\mathcal{B}) \to \nabla\mathcal{B}$ satisfies the condition (i) in Lemma 4.1, which we do by first getting natural simplicial isomorphisms $\nabla P'^{-1}(\nabla\mathbf{x}') \cong \nabla(F\!\downarrow\!\mathbf{x}')$, for the different oplax functors $\mathbf{x}' : [n] \to \mathcal{B}$ (i.e., the simplices of $\nabla\mathcal{B}$), and then by proving that any simplicial map $\sigma : [m] \to [n]$ induces a homotopy equivalence $\mathrm{B}(F\!\downarrow\!\mathbf{x}'\sigma) \simeq \mathrm{B}(F\!\downarrow\!\mathbf{x}')$. Here, we need to use the homomorphisms $J' : F\!\downarrow\!\mathbf{x}'n \to F\!\downarrow\!\mathbf{x}'$ in (4.16), which induce homotopy equivalences on classifying spaces by Lemma 4.6 (ii), and the existence of a pseudo-transformation $\theta' : \bar{\sigma}\, J' \Rightarrow J'\,\mathbf{x}'(\sigma m, n)_*$, which assigns to every object (a,f) of $F\!\downarrow\!\mathbf{x}'\sigma m$ the 1-cell $\big(1_a, \theta'(a,f), (\sigma m, n)\big) : (a, f, \sigma m) \to (a, \mathbf{x}'(\sigma m, n)\circ f, n)$, where

$$\theta'(a,f) = \big(\mathbf{x}'(\sigma m,n)\circ f \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} (\mathbf{x}'(\sigma m,n)\circ f)\circ 1_{Fa} \overset{1\circ\widehat{F}}{\Longrightarrow} (\mathbf{x}'(\sigma m,n)\circ f)\circ F1_a\big).$$

Using Lemma 4.2 (ii) therefore, we deduce that, for any lax functor $F' : \mathcal{A}' \to \mathcal{B}$, the square

$$\begin{tikzcd}
\nabla(F\!\downarrow\! F') \arrow[r, "\nabla P'"] \arrow[d, "\nabla\bar{F}'"'] & \nabla\mathcal{A}' \arrow[d, "\nabla F'"] \\
\nabla(F\!\downarrow\!\mathcal{B}) \arrow[r, "\nabla P'"] & \nabla\mathcal{B},
\end{tikzcd}$$

is a pullback in the category of simplicial sets which, by Lemma 4.1, induces a homotopy pullback square on geometric realizations. It follows that (4.18) is a homotopy pullback. □

With the corollary below we will be ready to complete the proof of Theorem 4.1.

Corollary 4.1 (*i*) *For any lax functor* $F : \mathcal{A} \to \mathcal{B}$*, the projection* 2*-functor* $P : F{\downarrow}\mathcal{B} \to \mathcal{A}$ *induces a homotopy equivalence on classifying spaces,* $\mathrm{B}(F{\downarrow}\mathcal{B}) \simeq \mathrm{B}\mathcal{A}$*.*

(*ii*) *For any oplax functor* $F' : \mathcal{A}' \to \mathcal{B}$*, the projection* 2*-functor* $P' : \mathcal{B}{\downarrow}F' \to \mathcal{A}'$ *induces a homotopy equivalence on classifying spaces,* $\mathrm{B}(\mathcal{B}{\downarrow}F') \simeq \mathrm{B}\mathcal{A}'$*.*

Proof: Once again we limit ourselves to proving (*i*). Let $F : \mathcal{A} \to \mathcal{B}$ be a lax functor.

The identity homomorphism $1_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ has the property B_o since, for any object $b \in \mathrm{Ob}\mathcal{B}$, the classifying space of the comma bicategory $b{\downarrow}\mathcal{B}$ is contractible, by Lemma 4.5. Therefore, Lemma 4.7 (*i*) applies to the case when $F' = 1_{\mathcal{B}}$, and tells us that the induced commutative square

$$\begin{tikzcd}
\mathrm{B}(F{\downarrow}\mathcal{B}) \arrow[r, "\mathrm{B}\bar{F}"] \arrow[d, "\mathrm{B}P"'] & \mathrm{B}(\mathcal{B}{\downarrow}\mathcal{B}) \arrow[d, "\mathrm{B}P"] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B},
\end{tikzcd}$$

is a homotopy pullback. So, it is enough to prove that the map $\mathrm{B}P : \mathrm{B}(\mathcal{B}{\downarrow}\mathcal{B}) \to \mathrm{B}\mathcal{B}$ is a homotopy equivalence. To do so, let b be any object of $\mathcal{B}$, and let us particularize the square above to the case where $F = b : [0] \to \mathcal{B}$. Then, we find the commutative homotopy pullback square

$$\begin{tikzcd}
\mathrm{B}(b{\downarrow}\mathcal{B}) \arrow[r, "\mathrm{B}\bar{b}"] \arrow[d, "\mathrm{B}P"'] & \mathrm{B}(\mathcal{B}{\downarrow}\mathcal{B}) \arrow[d, "\mathrm{B}P"] \\
\mathrm{pt} \arrow[r, "\mathrm{B}b"] & \mathrm{B}\mathcal{B},
\end{tikzcd}$$

where, by Lemma 4.5, the left vertical map is a homotopy equivalence. This tells us that the different homotopy fibers of the map $\mathrm{B}P : \mathrm{B}(\mathcal{B}{\downarrow}\mathcal{B}) \to \mathrm{B}\mathcal{B}$ over the 0-cells of $\mathrm{B}\mathcal{B}$ are all contractible, and consequently $\mathrm{B}P$ is actually a homotopy equivalence. □

We can now complete the proof of Theorem 4.1:

For any diagram $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$, where F is a lax functor and F' is an oplax functor, the square (4.15) occurs as the outside region in both of the following two diagrams:

$$\begin{tikzcd}
\mathrm{B}(F\!\downarrow\! F') \arrow[r, "\mathrm{B}\bar{F}"] \arrow[d, "\mathrm{B}P"'] \arrow[rr, bend left, "\mathrm{B}P'", "="'] \arrow[dr, phantom, "(4.17)"] & \mathrm{B}(\mathcal{B}\!\downarrow\! F') \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}P"'] \arrow[dr, phantom, "\overset{\mathrm{B}\omega}{\Rightarrow}"] & \mathrm{B}\mathcal{A}' \arrow[d, "\mathrm{B}F'"] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] \arrow[rr, bend right, "\mathrm{B}F"', "="] & \mathrm{B}\mathcal{B} \arrow[r, "\mathrm{B}1_{\mathcal{B}}"] & \mathrm{B}\mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{B}(F\!\downarrow\! F') \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}\bar{F}'"'] \arrow[dd, bend right=60, "\mathrm{B}P"', "="] \arrow[dr, phantom, "(4.18)"] & \mathrm{B}\mathcal{A}' \arrow[d, "\mathrm{B}F'"] \arrow[dd, bend left=60, "\mathrm{B}F'", "="'] \\
\mathrm{B}(F\!\downarrow\!\mathcal{B}) \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}P"'] \arrow[dr, phantom, "\overset{\mathrm{B}\omega'}{\Rightarrow}"] & \mathrm{B}\mathcal{B} \arrow[d, "\mathrm{B}1_{\mathcal{B}}"] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"'] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

(4.20)

where the inner squares with the homotopies labelled $\mathrm{B}\omega$ and $\mathrm{B}\omega'$ are the particular cases of the squares (4.15) obtained when $F = 1_{\mathcal{B}}$ and when $F' = 1_{\mathcal{B}}$, respectively. The homotopies are respectively induced, by Lemma 4.3, by the oplax transformations

$$\begin{tikzcd}
\mathcal{B}\!\downarrow\! F' \arrow[r, "P'"] \arrow[d, "P"'] \arrow[dr, phantom, "\overset{\omega}{\Rightarrow}"] & \mathcal{A}' \arrow[d, "F'"] \\
\mathcal{B} \arrow[r, "1_{\mathcal{B}}"] & \mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
F\!\downarrow\!\mathcal{B} \arrow[r, "P'"] \arrow[d, "P"'] \arrow[dr, phantom, "\overset{\omega'}{\Rightarrow}"] & \mathcal{B} \arrow[d, "1_{\mathcal{B}}"] \\
\mathcal{A} \arrow[r, "F"] & \mathcal{B}
\end{tikzcd}$$

which are defined as follows: The oplax transformation ω associates to any object (b, f, a') of $\mathcal{B}\!\downarrow\! F'$ the 1-cell $f : b \to F'a'$, and its naturality component at any 1-cell $(p, \beta, u') : (b_0, f_0, a'_0) \to (b_1, f_1, a'_1)$ is the 2-cell $\beta : F'u' \circ f_0 \Rightarrow f_1 \circ p$. Similarly, ω' associates to any object (a, f, b) of $F \downarrow \mathcal{B}$ the 1-cell $f : Fa \to b$, and its naturality component at any 1-cell $(u, \beta, p) : (a_0, f_0, b_0) \to (a_1, f_1, b_1)$ is $\beta : p \circ f_0 \Rightarrow f_1 \circ Fu$. Since, by Corollary 4.1, both maps $\mathrm{B}P' : \mathrm{B}(\mathcal{B}\!\downarrow\! F') \to \mathrm{B}\mathcal{A}'$ and $\mathrm{B}P : \mathrm{B}(F\!\downarrow\!\mathcal{B}) \to \mathrm{B}\mathcal{A}$ are homotopy equivalences, both squares are homotopy pullbacks. The other inner squares are those referred to therein.

The above implies the part (*i*) of Theorem 4.1 and, furthermore, it follows that the square (4.15) is a homotopy pullback whenever one of the inner squares (4.17) or (4.18) is a homotopy pullback. Therefore, Lemma 4.7 implies part (*ii*).

For proving part (*iii*), suppose a lax functor $F : \mathcal{A} \to \mathcal{B}$ is given such that the square (4.15) is a homotopy pullback for any $F' = b : [0] \to \mathcal{B}$, $b \in \mathrm{Ob}\mathcal{B}$. It follows from the diagram on the left in (4.20) that the inner square (4.17)

$$\begin{tikzcd}
\mathrm{B}(F\!\downarrow\! b) \arrow[r, "\mathrm{B}\bar{F}"] \arrow[d, "\mathrm{B}P"'] & \mathrm{B}(\mathcal{B}\!\downarrow\! b) \arrow[d, "\mathrm{B}P"] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

is a homotopy pullback for any object $b \in \mathcal{B}$. Then, if $p : b_0 \to b_1$ is any 1-cell of $\mathcal{B}$, since we have the commutative diagram

$$\begin{tikzcd}
\mathrm{B}(F\!\downarrow\! b_0) \arrow[rr, "\mathrm{B}\bar{F}"] \arrow[dr, "\mathrm{B}p_*"] \arrow[dd, "\mathrm{B}P"'] & & \mathrm{B}(\mathcal{B}\!\downarrow\! b_0) \arrow[dr, "\mathrm{B}p_*"] \arrow[dd, "\mathrm{B}P" near start] & \\
& \mathrm{B}(F\!\downarrow\! b_1) \arrow[rr, "\mathrm{B}\bar{F}" near start] \arrow[dl, "\mathrm{B}P"] & & \mathrm{B}(\mathcal{B}\!\downarrow\! b_1) \arrow[dl, "\mathrm{B}P"] \\
\mathrm{B}\mathcal{A} \arrow[rr, "\mathrm{B}F"] & & \mathrm{B}\mathcal{B} &
\end{tikzcd}$$

we deduce that the square

$$\begin{tikzcd}
\mathrm{B}(F\!\downarrow\! b_0) \arrow[r, "\mathrm{B}\bar{F}"] \arrow[d, "\mathrm{B}p_*"'] & \mathrm{B}(\mathcal{B}\!\downarrow\! b_0) \arrow[d, "\mathrm{B}p_*"] \\
\mathrm{B}(F\!\downarrow\! b_1) \arrow[r, "\mathrm{B}\bar{F}"] & \mathrm{B}(\mathcal{B}\!\downarrow\! b_1)
\end{tikzcd}$$

is also a homotopy pullback. Therefore, as $\mathrm{B}(\mathcal{B}\downarrow b_0) \simeq \mathrm{pt} \simeq \mathrm{B}(\mathcal{B}\downarrow b_1)$, by Lemma 4.5, we conclude that the induced map $\mathrm{B}p_* : \mathrm{B}(F\downarrow b_0) \simeq \mathrm{B}(F\downarrow b_1)$ is a homotopy equivalence. That is, the lax functor F has the property B_l. □

As a corollary, we obtain again Theorems 3.2 and 3.3:

Corollary 4.2 (*i*) *If a lax functor* $F : \mathcal{A} \to \mathcal{B}$ *has the property* B_l *then, for every object* $b \in \mathcal{B}$, *there is an induced homotopy fiber sequence*

$$\mathrm{B}(F\!\downarrow\! b) \xrightarrow{\mathrm{B}P} \mathrm{B}\mathcal{A} \xrightarrow{\mathrm{B}F} \mathrm{B}\mathcal{B}.$$

(*ii*) *If an oplax functor* $F' : \mathcal{A}' \to \mathcal{B}$ *has the property* B_o *then, for every object* $b \in \mathcal{B}$, *there is an induced homotopy fiber sequence*

$$\mathrm{B}(b\!\downarrow\! F') \xrightarrow{\mathrm{B}P'} \mathrm{B}\mathcal{A}' \xrightarrow{\mathrm{B}F'} \mathrm{B}\mathcal{B}.$$

Proof: It follows from Theorem 4.1, by taking $F' = b : [0] \to \mathcal{B}$ to obtain part (*i*) and $F = b : [0] \to \mathcal{B}$ for part (*ii*). □

Corollary 4.3 (*i*) *Let* $F : \mathcal{A} \to \mathcal{B}$ *be a lax functor such that the classifying spaces of its homotopy-fiber categories are contractible, that is,* $\mathrm{B}(F\downarrow b) \simeq \mathrm{pt}$ *for every object* $b \in \mathcal{B}$. *Then, the induced map on classifying spaces* $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ *is a homotopy equivalence.*

(*ii*) *Let* $F' : \mathcal{A}' \to \mathcal{B}$ *be an oplax functor such that the classifying spaces of its homotopy-fiber categories are contractible, that is,* $\mathrm{B}(b \downarrow F') \simeq \mathrm{pt}$ *for every object* $b \in \mathcal{B}$. *Then, the induced map on classifying spaces* $\mathrm{B}F' : \mathrm{B}\mathcal{A}' \to \mathrm{B}\mathcal{B}$ *is a homotopy equivalence.*

Next we study conditions on a bicategory $\mathcal{B}$ in order for the square (4.15) to always be a homotopy pullback. We use that, for any two objects b, b' of a bicategory $\mathcal{B}$, there is a diagram

$$\begin{tikzcd}
\mathcal{B}(b,b') \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & {[0]} \arrow[d, "b'"] \\
{[0]} \arrow[r, "b"] & \mathcal{B},
\end{tikzcd} \tag{4.21}$$

in which γ is the lax transformation defined by $\gamma f = f$, for any 1-cell $f : b \to b'$ in $\mathcal{B}$, and whose naturality component at a 2-cell $\beta : f_0 \Rightarrow f_1$, for any $f_0, f_1 : b \to b'$, is the composite 2-cell $\widehat{\gamma}_\beta = \big(1_{b'} \circ f_0 \overset{\boldsymbol{l}}{\cong} f_0 \overset{\beta}{\Rightarrow} f_1 \overset{\boldsymbol{r}^{-1}}{\cong} f_1 \circ 1_b\big)$.

Theorem 4.2 *The following properties of a bicategory $\mathcal{B}$ are equivalent:*

(i) *For any diagram of bicategories $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$, where F is a lax functor and F' is an oplax functor, the induced square* (4.15)

$$\begin{tikzcd}
\mathrm{B}(F{\downarrow}F') \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}P"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{B}\mathcal{A}' \arrow[d, "\mathrm{B}F'"] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

is a homotopy pullback.

(ii) *Any lax functor $F : \mathcal{A} \to \mathcal{B}$ has the property* B_l.

(iii) *Any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$ has the property* B_o.

(iv) *For any object b and* 1*-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the functor $p_* : \mathcal{B}(b, b_0) \to \mathcal{B}(b, b_1)$ induces a homotopy equivalence on classifying spaces,* $\mathrm{B}\mathcal{B}(b, b_0) \simeq \mathrm{B}\mathcal{B}(b, b_1)$.

(v) *For any object b and* 1*-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the functor $p^* : \mathcal{B}(b_1, b) \to \mathcal{B}(b_0, b)$ induces a homotopy equivalence on classifying spaces,* $\mathrm{B}\mathcal{B}(b_1, b) \simeq \mathrm{B}\mathcal{B}(b_0, b)$.

(vi) *For any two objects $b, b' \in \mathcal{B}$, the homotopy commutative square*

$$\begin{tikzcd}
\mathrm{B}\mathcal{B}(b,b') \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\mathrm{B}\gamma}{\Rightarrow}"] & \mathrm{pt} \arrow[d, "\mathrm{B}b'"] \\
\mathrm{pt} \arrow[r, "\mathrm{B}b"] & \mathrm{B}\mathcal{B},
\end{tikzcd}$$

induced by (4.21), *is a homotopy pullback. That is, the whisker map*

$$\mathrm{B}\mathcal{B}(b,b') \to \{\gamma : I \to \mathrm{B}\mathcal{B} \mid \gamma(0) = \mathrm{B}b, \gamma(1) = \mathrm{B}b'\} \subseteq \mathrm{B}\mathcal{B}^I$$

is a homotopy equivalence.

Proof: The implications $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are all direct consequences of Theorem 4.1. For the remaining implications, let us take into account that, for any objects $b, b' \in \mathcal{B}$ there is quite an obvious isomorphism of categories $b{\downarrow}b' \cong \mathcal{B}(b, b')$. With this identification in mind, we see that the homomorphism $b : [0] \to \mathcal{B}$ has the property B_l (resp. B_o) if and only if, for any 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the functor $p_* : \mathcal{B}(b, b_0) \to \mathcal{B}(b, b_1)$ (resp. $p^* : \mathcal{B}(b_1, b) \to \mathcal{B}(b_0, b)$) induces a homotopy equivalence on classifying spaces. Therefore, the implications $(ii) \Rightarrow (iv)$ and $(iii) \Rightarrow (v)$ are clear.

Furthermore, we see that the square in (vi) identifies the square

$$\begin{tikzcd}
\mathrm{B}(b{\downarrow}b') \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}P"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{B}[0] \arrow[d, "\mathrm{B}b'"] \\
\mathrm{B}[0] \arrow[r, "\mathrm{B}b"] & \mathrm{B}\mathcal{B}.
\end{tikzcd}$$

Then, for b fixed, it follows from Theorem 4.1 that the square in (vi) is a homotopy pullback for any b' if and only if $b : [0] \to \mathcal{B}$ has the property B_l, that is, the equivalence of statements $(vi) \Leftrightarrow (iv)$ holds.

Finally, to complete the proof, we are going to prove that $(iv) \Rightarrow (iii)$ and we shall leave it to the reader the proof that $(v) \Rightarrow (ii)$ since it is parallel. By hypothesis, for any object $b \in \mathrm{Ob}\mathcal{B}$, the normal homomorphism $b : [0] \to \mathcal{B}$ has the property B_l. Then, by Theorem 4.1 (ii), for any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$ the square

$$\begin{tikzcd}
\mathrm{B}(b{\downarrow}F') \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}P"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{B}\mathcal{A}' \arrow[d, "\mathrm{B}F'"] \\
\mathrm{B}[0] \arrow[r, "\mathrm{B}b"] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

is a homotopy pullback for any object $b \in \mathcal{B}$. Therefore, by Theorem 4.1 (iii), F' has the property B_o. □

We can state that

(B) *a bicategory $\mathcal{B}$ has the property* B if it has the properties in Theorem 4.2.

For example, *bigroupoids*, that is, bicategories whose 1-cells are invertible up to a 2-cell, and whose 2-cells are strictly invertible, have the property B: If $\mathcal{B}$ is any bigroupoid, for any object b and 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the functor $p^* : \mathcal{B}(b_1, b) \to \mathcal{B}(b_0, b)$ is actually an equivalence of categories and, therefore, induces a homotopy equivalence on classifying spaces $\mathrm{B}p^* : \mathrm{B}\mathcal{B}(b_1, b) \simeq \mathrm{B}\mathcal{B}(b_0, b)$. Recall that, by the correspondence $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$, bigroupoids correspond to homotopy 2-types, that is, CW-complexes whose n^{th} homotopy groups at any base point vanish for $n \geq 3$ (see Duskin [58, Theorem 8.6]).

Corollary 4.4 [7] *If a bicategory $\mathcal{B}$ has the property* B*, then, for any object $b \in \mathcal{B}$, there is a homotopy equivalence*

$$\Omega(\mathrm{B}\mathcal{B}, \mathrm{B}b) \simeq \mathrm{B}\mathcal{B}(b, b) \tag{4.22}$$

between the loop space of the classifying space of the bicategory with base point Bb *and the classifying space of the category of endomorphisms of b in $\mathcal{B}$.*

The above homotopy equivalence is already known when the bicategory is strict, that is, when $\mathcal{B}$ is a 2-category. It appears as a main result in the paper by Del Hoyo [84, Theorem 8.5], and it was also stated at the same time Cegarra in [43, Example 4.4]. Indeed, that homotopy equivalence (4.22), for the case when $\mathcal{B}$ is a 2-category, can be deduced from a result by Tillmann about simplicial categories in [121, Lemma 3.3].

4.3.1 Homotopy pullbacks of 2-categories.

As we recalled in the introduction, the category **2-Cat** of (strict) 2-categories and 2-functors has a Thomason-type Quillen model structure [2, Théorème 6.27], such that the classifying space functor is equivalence of homotopy theories between 2-categories and topological spaces [2, Corollaire 6.31]. Thus, in this model category, a 2-functor $F : \mathcal{A} \to \mathcal{B}$ is a weak equivalence if and only if the induced map on classifying spaces $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ is a homotopy equivalence, and a commutative square in **2-Cat**

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G'"] \arrow[d, "G"'] & \mathcal{A}' \arrow[d, "F'"] \\ \mathcal{A} \arrow[r, "F"] & \mathcal{B} \end{tikzcd} \tag{4.23}$$

is a homotopy pullback if and only if the induced on classifying spaces

$$\begin{tikzcd} \mathrm{B}\mathcal{C} \arrow[r, "\mathrm{B}G'"] \arrow[d, "\mathrm{B}G"'] & \mathrm{B}\mathcal{A}' \arrow[d, "\mathrm{B}F'"] \\ \mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B} \end{tikzcd} \tag{4.24}$$

is a homotopy pullback of spaces (see Remark 4.1). This is, for example, the case when F or F' in (4.23) is a fibration, $\mathcal{C} = \mathcal{A} \times_{\mathcal{B}} \mathcal{A}'$ is the pullback 2-category, and G and and G' are the respective projection 2-functors.

Our result in Theorem 4.1 has a natural interpretation in this setting as below. Observe that, when F and F' are 2-functors between 2-categories as above, the homotopy-fiber product bicategory $F \downarrow F'$ as well as the homotopy-fiber bicategories $F \downarrow b$ are actually 2-categories.

---

[7] This is also Example 3.3.

Proposition 4.1 *Let* (4.23) *be a commutative square in* **2-Cat**. *Suppose the* 2-*functor F has property B_l. Then the square is a homotopy pullback in the model category* **2-Cat** *(with the 'Thomason' model structure) if and only if the canonical* 2-*functor*

$$V : \mathcal{C} \to F{\downarrow}F'$$

$$\begin{tikzcd} c \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow\alpha"] & c' \arrow[r, mapsto, "V"] & (Gc,1,G'c) \arrow[r, bend left, "{(Gh,1,G'h)}"] \arrow[r, bend right, "{(Gh',1,G'h')}"'] \arrow[r, phantom, "{\Downarrow(G\alpha,G'\alpha)}"] & (Gc',1,G'c'), \end{tikzcd}$$

is a weak equivalence of 2-*categories.*

Proof: The square (4.24) is the composite of the induced squares

$$\begin{tikzcd} \mathrm{B}\mathcal{C} \arrow[r, "\mathrm{B}V"] \arrow[d, "\mathrm{B}G"'] & \mathrm{B}(F{\downarrow}F') \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}P"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{B}\mathcal{A}' \arrow[d, "\mathrm{B}F'"] \\ \mathrm{B}\mathcal{A} \arrow[r, equal] & \mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B}, \end{tikzcd} \tag{4.25}$$

where the right square is (4.15). In effect, one easily verifies that $PV = G$ and $P'V = G'$. Further, the homotopy $\mathrm{B}F\,\mathrm{B}P \Rightarrow \mathrm{B}F'\,\mathrm{B}P'$ is induced by the oplax transformation $\omega : FP \Rightarrow F'P'$ with $\omega(a,f,a') = f : Fa \to Fa'$, and whose naturality component at any 1-cell $(u,\beta,u') : (a_0,f_0,a_0') \to (a_1,f_1,a_1')$ is the 2-cell $\beta : F'u' \circ f_0 \Rightarrow f_1 \circ Fu$. The composite $\omega V : FPV = FG \Rightarrow F'G'V = F'G'$ is then the identity transformation, whence $\mathrm{B}\omega\mathrm{B}V$ is the static homotopy on $FG = F'G'$.

Suppose now that F has property B_l, so that the right square in (4.25) is a homotopy pullback of spaces, by Theorem 4.1. It follows that the composite square (4.24) is a homotopy pullback if and only if the left square in (4.25) is as well. As the later is a homotopy pullback if and only if the map $\mathrm{B}V$ is a homotopy equivalence, the proposition follows. □

4.3.2 The case when both functors are lax.

For a diagram $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{C}$, where both F and G are lax functors, the comma bicategory $F{\downarrow}G$ is not defined (unless G is a homomorphism). However, we can obtain a bicategorical model for the homotopy pullback of the maps induced on classifying spaces $\mathrm{B}\mathcal{A} \xrightarrow{\mathrm{B}F} \mathrm{B}\mathcal{B} \xleftarrow{\mathrm{B}G} \mathrm{B}\mathcal{C}$ as follows: Let

$$F{\downarrow_2}G := F{\downarrow}P'$$

be the comma bicategory defined as in (4.5) by the diagram $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{P'} G\downarrow\mathcal{B}$, where P' is the projection 2-functor (4.9) (the notation is taken from Dwyer, Kan, and Smith in [60] and Barwick and Kan in [13, 14]). Thus, $F\downarrow_2 G$ has 0-cells tuples (a,f,b,g,c), where $Fa \xrightarrow{f} b \xleftarrow{g} Gc$ are 1-cells of $\mathcal{B}$. A 1-cell

$$(u,\beta,p,\beta',v) : (a_0,f_0,b_0,g_0,c_0) \to (a_1,f_1,b_1,g_1,c_1)$$

in $F\downarrow_2 G$ consists of 1-cells $u : a_0 \to a_1$, $p : b_0 \to b_1$, and $v : c_0 \to c_1$, in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively, together with 2-cells β and β' of $\mathcal{B}$ as in the diagram

$$\begin{tikzcd}
Fa_0 \arrow[r, "f_0"] \arrow[d, "Fu"'] \arrow[dr, phantom, "\overset{\beta}{\Leftarrow}"] & b_0 \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\beta'}{\Rightarrow}"] & Gc_0 \arrow[l, "g_0"'] \arrow[d, "Gv"] \\
Fa_1 \arrow[r, "f_1"'] & b_1 & Gc_1, \arrow[l, "g_1"]
\end{tikzcd}$$

and a 2-cell

$$\begin{tikzcd}
(a_0,f_0,b_0,g_0,c_0) \arrow[r, bend left, "{(u,\beta,p,\beta',v)}"] \arrow[r, bend right, "{(\bar{u},\bar{\beta},\bar{p},\bar{\beta}',\bar{v})}"'] \arrow[r, phantom, "\Downarrow(\alpha,\delta,\rho)"] & (a_1,f_1,b_1,g_1,c_1),
\end{tikzcd}$$

is given by 2-cells $\alpha : u \Rightarrow \bar{u}$ in $\mathcal{A}$, $\delta : p \Rightarrow \bar{p}$ in $\mathcal{B}$, and $\rho : v \Rightarrow \bar{v}$ in $\mathcal{C}$, such that the diagrams below commute.

$$\begin{tikzcd}
p \circ f_0 \arrow[r, Rightarrow, "\delta\circ 1"] \arrow[d, Rightarrow, "\beta"'] & \bar{p} \circ f_0 \arrow[d, Rightarrow, "\bar{\beta}"] & p \circ g_0 \arrow[r, Rightarrow, "\delta\circ 1"] \arrow[d, Rightarrow, "\beta'"'] & \bar{p} \circ g_0 \arrow[d, Rightarrow, "\bar{\beta}'"] \\
f_1 \circ Fu \arrow[r, Rightarrow, "1\circ F\alpha"] & f_1 \circ F\bar{u} & g_1 \circ Gv \arrow[r, Rightarrow, "1\circ G\rho"] & g_1 \circ G\bar{v}
\end{tikzcd}$$

There is a (non-commutative) square

$$\begin{tikzcd}
F\downarrow_2 G \arrow[r, "Q"] \arrow[d, "P"'] & \mathcal{C} \arrow[d, "G"] \\
\mathcal{A} \arrow[r, "F"] & \mathcal{B}
\end{tikzcd} \tag{4.26}$$

where P and Q are projection 2-functors, which act on cells of $F\downarrow_2 G$ by

$$\begin{tikzcd}
a_0 \arrow[r, bend left, "u"] \arrow[r, bend right, "\bar{u}"'] \arrow[r, phantom, "\Downarrow\alpha"] & a_1 & (a_0,f_0,b_0,g_0,c_0) \arrow[l, mapsto, "P"'] \arrow[r, bend left, "{(u,\beta,p,\beta',v)}"] \arrow[r, bend right, "{(\bar{u},\bar{\beta},\bar{p},\bar{\beta}',\bar{v})}"'] \arrow[r, phantom, "\Downarrow(\alpha,\delta,\rho)"] & (a_1,f_1,b_1,g_1,c_1) \arrow[r, mapsto, "Q"] & c_0 \arrow[r, bend left, "v"] \arrow[r, bend right, "\bar{v}"'] \arrow[r, phantom, "\Downarrow\rho"] & c_1,
\end{tikzcd}$$

and we have the result given below.

Theorem 4.3 *Let* $\mathcal{A}\xrightarrow{F}\mathcal{B}\xleftarrow{G}\mathcal{C}$ *be a diagram where* F *and* G *are lax functors.*

(i) *There is a homotopy* $\mathrm{B}F\,\mathrm{B}P \Rightarrow \mathrm{B}G\,\mathrm{B}Q$ *so that the square below, which is induced by* (4.26) *on classifying spaces, is homotopy commutative.*

$$\begin{tikzcd}
\mathrm{B}(F\!\downarrow_2\!G) \arrow[r, "\mathrm{B}Q"] \arrow[d, "\mathrm{B}P"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{B}\mathcal{C} \arrow[d, "\mathrm{B}G"] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

(ii) *The square above is a homotopy pullback whenever* F *or* G *has property* B_l*.*

Proof: The part (i) follows from Theorem 4.1 (i) and the definition of $F\!\downarrow_2\!G$. For the part (ii), since $F\!\downarrow_2\!G \cong G\!\downarrow_2\!F$, it is enough, by symmetry, to prove the theorem when F has the property B_l. In this case, we have the homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{B}(F\!\downarrow_2\!G) \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}P"'] \arrow[rr, bend left=40, "\mathrm{B}Q"] \arrow[rr, phantom, bend left=15, "="] \arrow[dr, phantom, "(4.15)"] & \mathrm{B}(G\!\downarrow\!\mathcal{B}) \arrow[r, "\mathrm{B}P"] \arrow[d, "\mathrm{B}P'"'] \arrow[dr, phantom, "(4.15)"] & \mathrm{B}\mathcal{C} \arrow[d, "\mathrm{B}G"] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] \arrow[rr, bend right=40, "\mathrm{B}F"'] \arrow[rr, phantom, bend right=15, "="] & \mathrm{B}\mathcal{B} \arrow[r, "\mathrm{B}1_{\mathcal{B}}"] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

where, by Theorem 4.1, the inner squares (4.15) are both homotopy pullback. Then, the outside square is also a homotopy pullback, as claimed. □

4.4 Homotopy pullbacks of monoidal categories.

Recall [100, 110] that a *monoidal category* $(\mathcal{M},\otimes) = (\mathcal{M},\otimes,\mathrm{I},\boldsymbol{a},\boldsymbol{l},\boldsymbol{r})$ consists of a category $\mathcal{M}$ equipped with a tensor product $\otimes : \mathcal{M}\times\mathcal{M}\to\mathcal{M}$, a unit object I, and natural and coherent isomorphisms $\boldsymbol{a} : (m_3\otimes m_2)\otimes m_1 \cong m_3\otimes(m_2\otimes m_1)$, $\boldsymbol{l} : \mathrm{I}\otimes m \cong m$, and $\boldsymbol{r} : m\otimes \mathrm{I} \cong m$. Any monoidal category $(\mathcal{M},\otimes)$ can be viewed as a bicategory $\Sigma\mathcal{M}$ with only one object, say $*$, the objects m of $\mathcal{M}$ as 1-cells $m : * \to *$, and the morphisms of $\mathcal{M}$ as 2-cells. Thus, $\Sigma\mathcal{M}(*,*) = \mathcal{M}$, and the horizontal composition of cells is given by the tensor functor. The identity at the object is $1_* = \mathrm{I}$, the unit object of the monoidal category, and the associativity, left unit and right unit constraints for $\Sigma\mathcal{M}$ are precisely those of the monoidal category, that is, $\boldsymbol{a}$, $\boldsymbol{l}$, and $\boldsymbol{r}$, respectively. Furthermore, a monoidal functor $F = (F,\widehat{F}) : (\mathcal{N},\otimes)\to(\mathcal{M},\otimes)$ amounts precisely to a homomorphism $\Sigma F : \Sigma\mathcal{N}\to\Sigma\mathcal{M}$.

For any monoidal category $(\mathcal{M},\otimes)$, the Grothendieck nerve (4.12) of the bicategory $\Sigma\mathcal{M}$ is exactly the pseudo-simplicial category that the monoidal category defines by the reduced bar construction (see Jardine [86, Corollary 1.7]), whose category of

p-simplices is $\mathcal{M}^p$, the p-fold power of the underlying category $\mathcal{M}$. Therefore, the *classifying space of the monoidal category* $\mathrm{B}(\mathcal{M},\otimes)$ [86, §3] is the same as the classifying space $\mathrm{B}\Sigma\mathcal{M}$ of the one-object bicategory it defines [36], and thus the bicategorical results obtained above are applicable to monoidal functors between monoidal categories. This, briefly, can be done as follows:

Given any diagram $(\mathcal{N},\otimes) \xrightarrow{F} (\mathcal{M},\otimes) \xleftarrow{F'} (\mathcal{N}',\otimes)$, where F and F' are monoidal functors between monoidal categories, the "*homotopy- fiber product bicategory*"

$$F \overset{\otimes}{\downarrow} F' \tag{4.27}$$

(the notation $\overset{\otimes}{\downarrow}$ is to avoid confusion with the comma category $F{\downarrow}F'$ of the underlying functors) has as 0-cells the objects $m \in \mathcal{M}$. A 1-cell $(n,f,n') : m_0 \to m_1$ of $F \overset{\otimes}{\downarrow} F'$ consists of objects $n \in \mathcal{N}$ and $n' \in \mathcal{N}'$, and a morphism $f : F'n' \otimes m_0 \to m_1 \otimes Fn$ in $\mathcal{B}$. A 2-cell in $F \overset{\otimes}{\downarrow} F'$,

$$\begin{tikzcd} m_0 \arrow[r, bend left, "{(n,f,n')}"] \arrow[r, bend right, "{(\bar{n},\bar{f},\bar{n}')}"'] \arrow[r, phantom, "\Downarrow(u,u')"] & m_1, \end{tikzcd}$$

is given by a pair of morphisms, $u : n \to \bar{n}$ in $\mathcal{N}$ and $u' : n' \to \bar{n}'$ in $\mathcal{N}'$, such that the diagram below commutes.

$$\begin{tikzcd} F'n' \otimes m_0 \arrow[r, "F'u' \otimes 1"] \arrow[d, "f"'] & F'\bar{n}' \otimes m_0 \arrow[d, "\bar{f}"] \\ m_1 \otimes Fn \arrow[r, "1 \otimes Fu"] & m_1 \circ F\bar{n} \end{tikzcd}$$

The vertical composition of 2-cells is given by the composition of morphisms in $\mathcal{N}$ and $\mathcal{N}'$. The horizontal composition of the 1-cells $m_0 \xrightarrow{(n_1,f_1,n'_1)} m_1 \xrightarrow{(n_2,f_2,n'_2)} m_2$ is the 1-cell

$$(n_2 \otimes n_1, f_2 \circledcirc f_1, n'_2 \otimes n'_1) : m_0 \to m_2,$$

$$\begin{aligned} f_2 \circledcirc f_1 = \big(& F'(n'_2 \otimes n'_1) \otimes m_0 \overset{\widehat{F}'^{-1}\otimes 1}{\cong} (F'n'_2 \otimes F'n'_1) \otimes m_0 \overset{\boldsymbol{a}}{\cong} F'n'_2 \otimes (F'n'_1 \otimes m_0) \xrightarrow{1\otimes f_1} \\ & F'n'_2 \otimes (m_1 \otimes Fn_1) \overset{\boldsymbol{a}^{-1}}{\cong} (F'n'_2 \otimes m_1) \otimes Fn_1 \xrightarrow{f_2 \otimes 1} (m_2 \otimes Fn_2) \otimes Fn_1 \overset{\boldsymbol{a}}{\cong} \\ & m_2 \otimes (Fn_2 \circ Fn_1) \overset{1\otimes\widehat{F}}{\cong} m_2 \otimes F(n_2 \otimes n_1)\big), \end{aligned}$$

and the horizontal composition of 2-cells is given by the tensor product of morphisms in $\mathcal{N}$ and $\mathcal{N}'$. The identity 1-cell, at any 0-cell m, is $(\mathrm{I}, \mathring{1}_m, \mathrm{I}) : m \to m$, where

$$\mathring{1}_m = \big(F'\mathrm{I} \otimes m \overset{\widehat{F}'^{-1}\otimes 1}{\cong} \mathrm{I} \otimes m \overset{\boldsymbol{l}}{\cong} m \overset{\boldsymbol{r}^{-1}}{\cong} m \otimes \mathrm{I} \overset{1\otimes\widehat{F}}{\cong} m \otimes F\mathrm{I}\big).$$

The associativity, right, and left unit constraints of the bicategory $F\overset{\otimes}{\downarrow}F'$ are provided by those of $\mathcal{N}$ and $\mathcal{N}'$ by the formulas

$$\boldsymbol{a}_{(n_3,f_3,n'_3),(n_2,f_2,n'_2),(n_1,f_1,n'_1)}=(\boldsymbol{a}_{n_3,n_2,n_1},\boldsymbol{a}_{n'_3,n'_2,n'_1}),$$
$$\boldsymbol{l}_{(n,f,n')}=(\boldsymbol{l}_n,\boldsymbol{l}_{n'}),\ \ \boldsymbol{r}_{(n,f,n')}=(\boldsymbol{r}_n,\boldsymbol{r}_{n'}).$$

Remark 4.2 Let us stress that $F\overset{\otimes}{\downarrow}F'$ is not a monoidal category but a genuine bicategory, since it generally has more than one object.

In particular, for any monoidal functor $F:(\mathcal{N},\otimes)\to(\mathcal{M},\otimes)$, we have the *homotopy-fiber bicategories* (cf. [35])

$$F\overset{\otimes}{\downarrow}\mathrm{I},\quad \mathrm{I}\overset{\otimes}{\downarrow}F \tag{4.28}$$

where we denote by $\mathrm{I}:([0],\otimes)\to(\mathcal{M},\otimes)$ the monoidal functor that carries 0 to the unit object I, and whose structure isomorphism is $\boldsymbol{l}_\mathrm{I}=\boldsymbol{r}_\mathrm{I}:\mathrm{I}\otimes\mathrm{I}\cong\mathrm{I}$. Every object $m\in\mathcal{M}$ determines 2-endofunctors

$$m\otimes-:F\overset{\otimes}{\downarrow}\mathrm{I}\to F\overset{\otimes}{\downarrow}\mathrm{I},\quad -\otimes m:\mathrm{I}\overset{\otimes}{\downarrow}F\to\mathrm{I}\overset{\otimes}{\downarrow}F,$$

respectively given on cells by

$$\begin{tikzcd}
m_0 \arrow[r, bend left, "{(n,f)}"] \arrow[r, bend right, "{(\bar{n},\bar{f})}"'] \arrow[r, phantom, "\Downarrow u"] & m_1 \arrow[r, mapsto, "m\otimes-"] & m\otimes m_0 \arrow[r, bend left, "{(n,m\odot f)}"] \arrow[r, bend right, "{(\bar{n},m\odot\bar{f})}"'] \arrow[r, phantom, "\Downarrow u"] & m\otimes m_1, & m_0 \arrow[r, bend left, "{(g,n')}"] \arrow[r, bend right, "{(\bar{g},\bar{n}')}"'] \arrow[r, phantom, "\Downarrow u'"] & m_1 \arrow[r, mapsto, "-\otimes m"] & m_0\otimes m \arrow[r, bend left, "{(g\odot m,n')}"] \arrow[r, bend right, "{(\bar{g}\odot m,\bar{n}')}"'] \arrow[r, phantom, "\Downarrow u'"] & m_1\otimes m,
\end{tikzcd}$$

where, for any $(n,f):m_0\to m_1$ in $F\overset{\otimes}{\downarrow}\mathrm{I}$ and $(g,n):m_0\to m_1$ in $\mathrm{I}\overset{\otimes}{\downarrow}F$,

$$m\odot f=\big(\mathrm{I}\otimes(m\otimes m_0)\overset{\boldsymbol{l}}{\cong}m\otimes m_0\overset{1\otimes\boldsymbol{l}^{-1}}{\cong}m\otimes(\mathrm{I}\otimes m_0)\xrightarrow{1\otimes f}m\otimes(m_1\otimes Fn)\overset{\boldsymbol{a}^{-1}}{\cong}(m\otimes m_1)\otimes Fn\big),$$

$$g\odot m=\big(Fn\otimes(m_0\otimes m)\overset{\boldsymbol{a}^{-1}}{\cong}(Fn\otimes m_0)\otimes m\xrightarrow{g\otimes 1}(m_1\otimes\mathrm{I})\otimes m\overset{\boldsymbol{r}\otimes 1}{\cong}m_1\otimes m\overset{\boldsymbol{r}^{-1}}{\cong}(m_1\otimes m)\otimes\mathrm{I}\big).$$

We state that

(B_l) *the monoidal functor F has the property* B_l if, for any object $m\in\mathcal{M}$, the induced map $\mathrm{B}(m\otimes-):\mathrm{B}(F\overset{\otimes}{\downarrow}\mathrm{I})\to\mathrm{B}(F\overset{\otimes}{\downarrow}\mathrm{I})$ is a homotopy autoequivalence.

(B_o) *the monoidal functor F has the property* B_o if, for any object $m\in\mathcal{M}$, the induced map $\mathrm{B}(-\otimes m):\mathrm{B}(\mathrm{I}\overset{\otimes}{\downarrow}F)\to\mathrm{B}(\mathrm{I}\overset{\otimes}{\downarrow}F)$ is a homotopy autoequivalence.

Our main result here is a direct consequence of Theorem 4.1, after taking into account the identifications $\mathrm{B}(\mathcal{M},\otimes)=\mathrm{B}\Sigma\mathcal{M}$, $\mathrm{B}F=\mathrm{B}\Sigma F$, $F\overset{\otimes}{\downarrow}F'=\Sigma F\downarrow\Sigma F'$, $F\overset{\otimes}{\downarrow}\mathrm{I}=\Sigma F\downarrow *$, and $\mathrm{I}\overset{\otimes}{\downarrow}F=*\downarrow\Sigma F$, and the fact that a monoidal functor has the property B_l or B_o if and only if the homomorphism ΣF has that property. This result is as given below.

Theorem 4.4 (i) *Suppose* $(\mathcal{N},\otimes)\xrightarrow{F}(\mathcal{M},\otimes)\xleftarrow{F'}(\mathcal{N}',\otimes)$ *are monoidal functors between monoidal categories, such that* F *has the property* B_l *or* F' *has the property* B_o*. Then, there is an induced homotopy pullback square*

$$\begin{tikzcd}
\mathrm{B}(F\overset{\otimes}{\downarrow}F') \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}P"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{B}(\mathcal{N}',\otimes) \arrow[d, "\mathrm{B}F'"] \\
\mathrm{B}(\mathcal{N},\otimes) \arrow[r, "\mathrm{B}F"] & \mathrm{B}(\mathcal{M},\otimes).
\end{tikzcd} \tag{4.29}$$

Therefore, there is an induced Mayer-Vietoris type long exact sequence on homotopy groups, based at the 0*-cells* $\mathrm{B}*$ *of* $\mathrm{B}(\mathcal{M},\otimes)$*,* $\mathrm{B}(\mathcal{N},\otimes)$*, and* $\mathrm{B}(\mathcal{N}',\otimes)$ *respectively, and the* 0*-cell* $\mathrm{BI}\in\mathrm{B}(F\overset{\otimes}{\downarrow}F')$*,*

$$\cdots\to\pi_{n+1}\mathrm{B}(\mathcal{M},\otimes)\to\pi_n\mathrm{B}(F\overset{\otimes}{\downarrow}F')\to\pi_n\mathrm{B}(\mathcal{N},\otimes)\times\pi_n\mathrm{B}(\mathcal{N}',\otimes)\to\pi_n\mathrm{B}(\mathcal{M},\otimes)\to$$

$$\cdots\to\pi_1\mathrm{B}(F\overset{\otimes}{\downarrow}F')\to\pi_1\mathrm{B}(\mathcal{N},\otimes)\times\pi_1\mathrm{B}(\mathcal{N}',\otimes)\to\pi_1\mathrm{B}(\mathcal{M},\otimes)\to\pi_0\mathrm{B}(F\overset{\otimes}{\downarrow}F')\to 0.$$

(ii) *Given a monoidal functor* $F:(\mathcal{N},\otimes)\to(\mathcal{M},\otimes)$*, if the square* (4.29) *is a homotopy pullback for every monoidal functor* $F':(\mathcal{N}',\otimes)\to(\mathcal{M},\otimes)$*, then* F *has the property* B_l*. Similarly, if* F' *is a monoidal functor such that the square* (4.29) *is a homotopy pullback for any monoidal functor* F*, as above, then* F' *has the property* B_o*.*

Similarly, from Corollaries 4.2 and 4.3, we get the following extensions of Quillen's Theorems A and B to monoidal functors:

Theorem 4.5 *Let* $F:(\mathcal{N},\otimes)\to(\mathcal{M},\otimes)$ *be any monoidal functor.*

(i) *If* F *has the property* B_l*, then there is an induced homotopy fiber sequence*

$$\mathrm{B}(F\overset{\otimes}{\downarrow}\mathrm{I})\to\mathrm{B}(\mathcal{N},\otimes)\to\mathrm{B}(\mathcal{M},\otimes).$$

(ii) *If* F *has the property* B_o*, then there is an induced homotopy fiber sequence*

$$\mathrm{B}(\mathrm{I}\overset{\otimes}{\downarrow}F)\to\mathrm{B}(\mathcal{N},\otimes)\to\mathrm{B}(\mathcal{M},\otimes).$$

(iii) *If the classifying space of any of the two homotopy-fiber bicategories of* F *is contractible, that is, if* $\mathrm{B}(F\overset{\otimes}{\downarrow}\mathrm{I})\simeq\mathrm{pt}$ *or* $\mathrm{B}(\mathrm{I}\overset{\otimes}{\downarrow}F)\simeq\mathrm{pt}$*, then the induced map on classifying spaces* $\mathrm{B}F:\mathrm{B}(\mathcal{N},\otimes)\simeq\mathrm{B}(\mathcal{M},\otimes)$ *is a homotopy equivalence.*

For the last statement in the following theorem, let us note that there is a diagram of bicategories

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & {[0]} \arrow[d, "*"] \\
{[0]} \arrow[r, "*"] & \Sigma\mathcal{M}
\end{tikzcd} \tag{4.30}$$

in which γ is the lax transformation defined by $\gamma m = m : * \to *$, for any object $m \in \mathcal{M}$, and whose naturality component at a morphism $f : m_0 \to m_1$, is the composite 2-cell $\widehat{\gamma}_f = \big(\mathrm{I}\otimes m_0 \overset{\boldsymbol{l}}{\cong} m_0 \overset{f}{\Rightarrow} m_1 \overset{\boldsymbol{r}^{-1}}{\cong} m_1 \circ \mathrm{I}\big)$. Then, we have an induced homotopy commutative square on classifying spaces

$$\begin{tikzcd}
\mathrm{B}\mathcal{M} \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\mathrm{B}\gamma}{\Rightarrow}"] & \mathrm{pt} \arrow[d, "*"] \\
\mathrm{pt} \arrow[r, "*"] & \mathrm{B}(\mathcal{M},\otimes)
\end{tikzcd}$$

and a corresponding whisker map

$$\mathrm{B}\mathcal{M} \to \Omega(\mathrm{B}(\mathcal{M},\otimes),*). \tag{4.31}$$

Theorem 4.2 particularizes by giving

Theorem 4.6 *The following properties of a monoidal category* $(\mathcal{M},\otimes)$ *are equivalent:*

(i) *For any diagram of monoidal functors* $(\mathcal{N},\otimes) \xrightarrow{F} (\mathcal{M},\otimes) \xleftarrow{F'} (\mathcal{N}',\otimes)$ *, the induced square* (4.29)

$$\begin{tikzcd}
\mathrm{B}(F\overset{\otimes}{\downarrow}F') \arrow[r, "\mathrm{B}P'"] \arrow[d, "\mathrm{B}P"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{B}(\mathcal{N}',\otimes) \arrow[d, "\mathrm{B}F'"] \\
\mathrm{B}(\mathcal{N},\otimes) \arrow[r, "\mathrm{B}F"] & \mathrm{B}(\mathcal{M},\otimes).
\end{tikzcd}$$

is a homotopy pullback.

(ii) *Any monoidal functor* $F : (\mathcal{N},\otimes) \to (\mathcal{M},\otimes)$ *has property* B_l.

(iii) *Any monoidal functor* $F : (\mathcal{N},\otimes) \to (\mathcal{M},\otimes)$ *has property* B_o.

(iv) *For any object* $m \in \mathcal{M}$, *the functor* $m \otimes - : \mathcal{M} \to \mathcal{M}$ *induces a homotopy autoequivalence on the classifying space* $\mathrm{B}\mathcal{M}$.

(v) *For any object* $m \in \mathcal{M}$, *the functor* $- \otimes m : \mathcal{M} \to \mathcal{M}$ *induces a homotopy autoequivalence on the classifying space* $\mathrm{B}\mathcal{M}$.

(vi) *The whisker map* (4.31) *is a homotopy equivalence*

$$\mathrm{B}\mathcal{M} \simeq \Omega(\mathrm{B}(\mathcal{M},\otimes),*)$$

between the classifying space of the underlying category and the loop space of the classifying space of the monoidal category.

The implications $(iv) \Rightarrow (vi)$ and $(v) \Rightarrow (vi)$ in the above theorem are essentially due to Stasheff [114], but several other proofs can be found in the literature (see Jardine [86, Propositions 3.5 and 3.8], for example). When the equivalent properties in Theorem 4.6 hold, we say that *the monoidal category is homotopy regular.* For example, *regular monoidal categories* (as termed by Saavedra [110, Chap. I, (0.1.3)]), that is, monoidal categories $(\mathcal{M}, \otimes)$ where, for every object $m \in \mathcal{M}$, the functor $m \otimes - : \mathcal{M} \to \mathcal{M}$ is an autoequivalence of the underlying category $\mathcal{M}$, and, in particular, *categorical groups* (so named by Joyal and Street in [88, Definition 3.1] and also termed *Gr-categories* by Breen in [21, §2, 2.1]), that is, monoidal categories whose objects are invertible up to an isomorphism, and whose morphisms are all invertible, are homotopy regular.

4.5 Homotopy pullbacks of crossed modules

Thanks to the equivalence between the category of crossed modules and the category of 2-groupoids, the results in Section 4.3 can be applied to crossed modules. To do so in some detail, we shall start by briefly reviewing crossed modules and their classifying spaces.

Recall that, if $\mathcal{P}$ is any (small) groupoid, then the category of (left) *$\mathcal{P}$-groups* has objects the functors $\mathcal{P} \to \mathbf{Gp}$, from $\mathcal{P}$ into the category of groups, and its morphisms, called *$\mathcal{P}$-group homomorphisms*, are natural transformations. If $\mathcal{G}$ is a $\mathcal{P}$-group, then, for any arrow $p : a \to b$ in $\mathcal{P}$, we write the associated group homomorphism $\mathcal{G}(a) \to \mathcal{G}(b)$ by $g \mapsto {}^{p}g$, so that the equalities ${}^{1}g = g$,${}^{(q \circ p)}g = {}^{q}({}^{p}g)$, and ${}^{p}(g \cdot g') = {}^{p}g \cdot {}^{p}g'$ hold whenever they make sense. Here, the symbol $\circ$ denotes composition in the groupoid $\mathcal{P}$, whereas $\cdot$ denotes multiplication in $\mathcal{G}$. For instance, the assignment to each object of $\mathcal{P}$ its isotropy group, $a \mapsto \mathrm{Aut}_{\mathcal{P}}(a)$, is the function on objects of a $\mathcal{P}$-group $\mathrm{Aut}_{\mathcal{P}} : \mathcal{P} \to \mathbf{Gp}$ such that ${}^{p}q = p \circ q \circ p^{-1}$, for any $p : a \to b$ in $\mathcal{P}$ and $q \in \mathrm{Aut}_{\mathcal{P}}(a)$. Then, a *crossed module* (of groupoids) is a triplet

$$(\mathcal{G}, \mathcal{P}, \partial)$$

consisting of a groupoid $\mathcal{P}$, a $\mathcal{P}$-group $\mathcal{G}$, and a $\mathcal{P}$-group homomorphism $\partial : \mathcal{G} \to \mathrm{Aut}_{\mathcal{P}}$, called the *boundary map*, such that the Pfeiffer identity ${}^{\partial g}g' = g \cdot g' \cdot g^{-1}$ holds, for any $g, g' \in \mathcal{G}(a)$, $a \in \mathrm{Ob}\mathcal{P}$.

When a group P is regarded as a groupoid $\mathcal{P}$ with exactly one object, the above definition by Brown and Higgins [27] recovers the more classic notion of crossed module (G, P, ∂) due to Whitehead and Mac Lane [101, 124], now called *crossed modules of groups.* In fact, if $(\mathcal{G}, \mathcal{P}, \partial)$ is any crossed module, then, for any object a of $\mathcal{P}$, the triplet $(\mathcal{G}(a), \mathrm{Aut}_{\mathcal{P}}(a), \partial_a)$ is precisely a crossed module of groups.

Composition with any given functor $F : \mathcal{P} \to \mathcal{Q}$ defines a functor from the category of $\mathcal{Q}$-groups to the category of $\mathcal{P}$-groups: $(\varphi : \mathcal{G} \to \mathcal{H}) \mapsto (\varphi F : \mathcal{G}F \to \mathcal{H}F)$. For the particular case of the $\mathcal{Q}$-group of automorphisms $\mathrm{Aut}_{\mathcal{Q}}$, we have the $\mathcal{P}$-group homomorphism $F : \mathrm{Aut}_{\mathcal{P}} \to \mathrm{Aut}_{\mathcal{Q}} F$, which, at any $a \in \mathcal{P}$, is given by the map

$\mathrm{Aut}_{\mathcal{P}}(a) \to \mathrm{Aut}_{\mathcal{Q}}(Fa)$, $q \mapsto Fq$, defined by the functor F. Then, a *morphism of crossed modules*

$$(\varphi, F) : (\mathcal{G}, \mathcal{P}, \partial) \to (\mathcal{H}, \mathcal{Q}, \partial)$$

consists of a functor $F : \mathcal{P} \to \mathcal{Q}$ together with a $\mathcal{P}$-group homomorphism $\varphi : \mathcal{G} \to \mathcal{H}F$ such that the square below commutes.

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\partial"] \arrow[d, "\varphi"'] & \mathrm{Aut}_{\mathcal{P}} \arrow[d, "F"] \\
\mathcal{H}F \arrow[r, "\partial F"] & \mathrm{Aut}_{\mathcal{Q}}F.
\end{tikzcd}$$

The category of crossed modules, where compositions and identities are defined in the natural way, is denoted by **Xmod**. Let us now recall from Brown and Higgins [28, Theorem 4.1] that there is an equivalence between the category of crossed modules and the category of 2-groupoids

$$\beta : \mathbf{Xmod} \xrightarrow{\sim} 2\text{-}\mathbf{Gpd}, \tag{4.32}$$

which is as follows: Given any crossed module $(\mathcal{G}, \mathcal{P}, \partial)$, $\mathcal{P}$ is the underlying groupoid of the 2-groupoid $\beta(\mathcal{G}, \mathcal{P}, \partial)$, whose 2-cells

$$\begin{tikzcd}
a_0 \arrow[r, bend left, "p"] \arrow[r, bend right, "\bar{p}"'] \arrow[r, phantom, "\Downarrow g"] & a_1
\end{tikzcd}$$

are those elements $g \in \mathcal{G}(a_0)$ such that $\bar{p} \circ \partial g = p$. The vertical and horizontal composition of 2-cells are, respectively, given by

$$\begin{tikzcd}
a_0 \arrow[r, bend left=50, "p"] \arrow[r, "\bar{p}" description] \arrow[r, bend right=50, "\bar{\bar{p}}"'] \arrow[r, phantom, shift left=3, "\Downarrow g"] \arrow[r, phantom, shift right=3, "\Downarrow \bar{g}"] & a_1 & \overset{\cdot}{\mapsto} & a_0 \arrow[r, bend left=50, "p"] \arrow[r, bend right=50, "\bar{\bar{p}}"'] \arrow[r, phantom, "\Downarrow \bar{g}\cdot g"] & a_1\,, & a_0 \arrow[r, bend left, "p_1"] \arrow[r, bend right, "\bar{p}_1"'] \arrow[r, phantom, "\Downarrow g_1"] & a_1 \arrow[r, bend left, "p_2"] \arrow[r, bend right, "\bar{p}_2"'] \arrow[r, phantom, "\Downarrow g_2"] & a_2 & \overset{\circ}{\mapsto} & a_0 \arrow[r, bend left, "p_2\circ p_1"] \arrow[r, bend right, "\bar{p}_2\circ\bar{p}_1"'] \arrow[r, phantom, "\Downarrow {}^{\bar{p}_1^{-1}}g_2\cdot g_1"] & a_2.
\end{tikzcd}$$

A morphism of crossed modules $(\varphi, F) : (\mathcal{G}, \mathcal{P}, \partial) \to (\mathcal{H}, \mathcal{Q}, \partial)$ is carried by the equivalence to the 2-functor $\beta(\varphi, F) : \beta(\mathcal{G}, \mathcal{P}, \partial) \to \beta(\mathcal{H}, \mathcal{Q}, \partial)$ acting on cells by

$$\begin{tikzcd}
a_0 \arrow[r, bend left, "p"] \arrow[r, bend right, "\bar{p}"'] \arrow[r, phantom, "\Downarrow g"] & a_1 & \mapsto & Fa_0 \arrow[r, bend left, "Fp"] \arrow[r, bend right, "F\bar{p}"'] \arrow[r, phantom, "\Downarrow \varphi g"] & Fa_1.
\end{tikzcd}$$

Example 4.1 An example of crossed module is $\Pi(X, A, S) = (\pi_2(X, A), \pi(A, S), \partial)$, which comes associated to any triple (X, A, S), where X is any topological space, $A \subseteq X$ a subspace, and $S \subseteq A$ a set of (base) points. Here, $\pi(A, S)$ is the fundamental groupoid of homotopy classes of paths in A between points in S, $\pi_2(X, A) : \pi(A, S) \to$ **Gp** is the functor associating to each $a \in S$ the relative homotopy group $\pi_2(X, A, a)$, and, at any $a \in S$, the boundary map $\partial : \pi_2(X, A, a) \to \pi_1(A, a)$ is the usual boundary homomorphism in the exact sequence of homotopy groups based at a of the pair (X, A):

$$\left[\begin{tikzcd} a \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "g"] & a \arrow[d, equal] \\ a \arrow[r, equal] & a \end{tikzcd}\right] \overset{\partial}{\mapsto} \begin{tikzcd} a \arrow[r, "{[u]}"] & a. \end{tikzcd}$$

Furthermore, $\pi(A, S)$ is the underlying groupoid of the *Whitehead* 2*-groupoid* $W(X, A, S)$ presented by Moerdijk and Svensson [106], whose 2-cells

$$\left[\begin{tikzcd} a \arrow[r, "v"] \arrow[d, equal] \arrow[dr, phantom, "g"] & b \arrow[d, equal] \\ a \arrow[r, "w"'] & b \end{tikzcd}\right] : [v] \Rightarrow [w] : a \to b,$$

are equivalence classes of maps $g : I \times I \to X$, from the square $I \times I$ into X, which are constant along the vertical edges with values in S, and map the horizontal edges into A; two such maps are equivalent if they are homotopic by a homotopy that is constant along the vertical edges and deforms the horizontal edges within A.

Both constructions $\Pi(X, A, S)$ and $W(X, A, S)$ correspond to each other by the equivalence of categories (4.32). More precisely, there is a natural isomorphism

$$\beta\Pi(X, A, S) \cong W(X, A, S), \tag{4.33}$$

which is the identity on 0- and 1-cells, and carries a 2-cell $[g] : [v] \Rightarrow [w]$ of $\beta\Pi(X, A, S)$ to the 2-cell $1_{[w]} \circ [g] : [v] \Rightarrow [w]$ of $W(X, A, S)$:

$$\left(\left[\begin{tikzcd} a \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "g(s{,}t)"] & a \arrow[d, equal] \\ a \arrow[r, equal] & a \end{tikzcd}\right] : [v] \Rightarrow [w]\right) \mapsto \left(\left[\begin{tikzcd} a \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "g(2s{,}t)"] & a \arrow[r, "w"] \arrow[d, equal] \arrow[dr, phantom, "w(2s\text{-}1)"] & b \arrow[d, equal] \\ a \arrow[r, equal] & a \arrow[r, "w"'] & b \end{tikzcd}\right] : [v] \Rightarrow [w]\right).$$

For a simplicial set K, its *fundamental*, or *homotopy*, *crossed module* $\Pi(K)$ is defined as the crossed module

$$\Pi(K) = \Pi\big(|K|, |K^{(1)}|, |K^{(0)}|\big) \tag{4.34}$$

constructed in Example 4.1 (here, $K^{(n)}$ denotes the n-skeleton, as usual). The construction $K \mapsto \Pi(K)$ gives rise to a functor $\Pi : \mathbf{SimpSet} \to \mathbf{Xmod}$, from the category of simplicial sets to the category of crossed modules. To go in the other direction, we have the notion of *nerve of a crossed module*, which is actually a special case of the definition of nerve for crossed complexes by Brown and Higgins [29]. Thus, the *nerve* $N(\mathcal{G}, \mathcal{P}, \partial)$ of a crossed module $(\mathcal{G}, \mathcal{P}, \partial)$ is the simplicial set

$$N(\mathcal{G}, \mathcal{P}, \partial) : \Delta^{op} \longrightarrow \mathbf{Set}, \quad [n] \mapsto \mathbf{Xmod}\big(\Pi(\Delta[n]), (\mathcal{G}, \mathcal{P}, \partial)\big), \tag{4.35}$$

whose n-simplices are all morphisms of crossed modules $\Pi(\Delta[n]) \to (\mathcal{G}, \mathcal{P}, \partial)$.

The *classifying space* $\mathrm{B}(\mathcal{G},\mathcal{P},\partial)$ of a crossed module $(\mathcal{G},\mathcal{P},\partial)$ is the geometric realization of its nerve, that is,

$$\mathrm{B}(\mathcal{G},\mathcal{P},\partial) = |N(\mathcal{G},\mathcal{P},\partial)|. \tag{4.36}$$

By [29, Proposition 2.6], $\mathrm{B}(\mathcal{G},\mathcal{P},\partial)$ is a CW-complex whose 0-cells identify with the objects of the groupoid $\mathcal{P}$ and whose homotopy groups, at any $a \in \mathrm{Ob}\mathcal{P}$, can be algebraically computed as

$$\pi_i\big(\mathrm{B}(\mathcal{G},\mathcal{P},\partial),a\big) = \begin{cases} \text{the set of connected components of } \mathcal{P}, \text{ if } i=0, \\ \text{Coker } \partial : \mathcal{G}(a) \to \mathrm{Aut}_{\mathcal{P}}(a), \text{ if } i=1, \\ \text{Ker } \partial : \mathcal{G}(a) \to \mathrm{Aut}_{\mathcal{P}}(a), \text{ if } i=2, \\ 0, \text{ if } i \geq 3. \end{cases} \tag{4.37}$$

Therefore, classifying spaces of crossed modules are homotopy 2-types. Furthermore, it is a consequence of [29, Theorem 4.1] that, for any CW-complex X with $\pi_i(X,a)=0$ for all $i>2$ and base 0-cell a, there is a homotopy equivalence $X \simeq \mathrm{B}\Pi(X,X^{(1)},X^{(0)})$. Therefore, crossed modules are algebraic models for homotopy 2-types.

Lemma 4.8 *For any crossed module* $(\mathcal{G},\mathcal{P},\partial)$*, there is a homotopy natural homotopy equivalence*

$$\mathrm{B}(\mathcal{G},\mathcal{P},\partial) \simeq \mathrm{B}\beta(\mathcal{G},\mathcal{P},\partial). \tag{4.38}$$

Proof: By [29, Theorem 2.4], the functor $\Pi : \mathbf{SimpSet} \to \mathbf{Xmod}$ is left adjoint to the nerve functor $N : \mathbf{Xmod} \to \mathbf{SimpSet}$. Furthermore, in [106, Theorem 2.3] Moerdijk and Svensson show that the *Whitehead* 2*-groupoid functor* $W : \mathbf{SimpSet} \to 2\text{-}\mathbf{Gpd}$, $K \mapsto W(K) = W\big(|K|,|K^{(1)}|,|K^{(0)}|\big)$ (see Example 4.1) is left adjoint to the unitary geometric nerve functor $\Delta^{\mathrm{u}} : 2\text{-}\mathbf{Gpd} \to \mathbf{SimpSet}$. Since, owing to the isomorphisms (4.33), there is a natural isomorphism $\beta\,\Pi \cong W$, we conclude that $\Delta^{\mathrm{u}}\beta \cong N$. Therefore, for $(\mathcal{G},\mathcal{P},\partial)$ any crossed module, $\mathrm{B}(\mathcal{G},\mathcal{P},\partial) = |N(\mathcal{G},\mathcal{P},\partial)| \cong |\Delta^{\mathrm{u}}\beta(\mathcal{G},\mathcal{P},\partial)| \overset{(4.14)}{\simeq} \mathrm{B}\beta(\mathcal{G},\mathcal{P},\partial)$. □

Remark 4.3 For any crossed module $(\mathcal{G},\mathcal{P},\partial)$, the n-simplices of $\Delta^{\mathrm{u}}\beta(\mathcal{G},\mathcal{P},\partial)$, that is, the normal lax functors $[n] \to \beta(\mathcal{G},\mathcal{P},\partial)$, are precisely systems of data

$$(g,p,a) = \big(g_{i,j,k}, p_{i,j}, a_i\big)_{0\leq i\leq j\leq k\leq n}$$

consisting of objects a_i of $\mathcal{P}$, arrows $p_{i,j} : a_i \to a_j$ of $\mathcal{P}$, with $p_{i,i}=1$, and elements $g_{i,j,k} \in \mathcal{G}(a_i)$, with $g_{i,i,j} = g_{i,j,j} = 1$, such that the following conditions hold:

$$\begin{aligned} \partial(g_{i,j,k}) &= p_{i,k}^{-1} \circ p_{j,k} \circ p_{i,j} && \text{for } i \leq j \leq k, \\ g_{i,j,k}^{-1} \cdot g_{i,k,l}^{-1} \cdot g_{i,j,l} \cdot {}^{p_{i,j}^{-1}} g_{j,k,l} &= 1 && \text{for } i \leq j \leq k \leq l. \end{aligned}$$

Thus, the unitary geometric nerve $\Delta^{\mathrm{u}}\beta(\mathcal{G},\mathcal{P},\partial)$ coincides with the simplicial set called by Dakin [56, Chapter 5, §3] the *nerve of the crossed module* $(\mathcal{G},\mathcal{P},\partial)$ (cf. [29, page

99] and [4, Chapter 1, §11]). From the above explicit description, it is easily proven that the nerve of a crossed module is a Kan complex whose homotopy groups are given as in (4.37).

Thanks to Lemma 4.8, the bicategorical results obtained in Section 4.3 are transferable to the setting of crossed modules. To do so, if

$$(\mathcal{G},\mathcal{P},\partial)\xrightarrow{(\varphi,F)}(\mathcal{H},\mathcal{Q},\partial)\xleftarrow{(\varphi',F')}(\mathcal{G}',\mathcal{P}',\partial)$$

is any diagram in **Xmod**, then its *"homotopy-fiber product crossed module"*

$$(\varphi,F)\!\downarrow\!(\varphi',F')=\big(\mathcal{G}_{\varphi,F\downarrow\varphi',F'},\mathcal{P}_{\varphi,F\downarrow\varphi',F'},\partial\big) \tag{4.39}$$

is constructed as follows:

- *The groupoid* $\mathcal{P}_{\varphi,F\downarrow\varphi',F'}$ has objects the triples (a,q,a'), with $a\in\mathrm{Ob}\mathcal{P}$, $a'\in\mathrm{Ob}\mathcal{P}'$, and $q:Fa\to F'a'$ a morphism in $\mathcal{Q}$. A morphism $(p,h,p'):(a_0,q_0,a'_0)\to(a_1,q_1,a'_1)$ consists of a morphism $p:a_0\to a_1$ in $\mathcal{P}$, a morphism $p':a'_0\to a'_1$ in $\mathcal{P}'$, and an element $h\in\mathcal{H}(Fa_0)$, which measures the lack of commutativity of the square

$$\begin{tikzcd} Fa_0 \arrow[r, "q_0"] \arrow[d, "Fp"'] & F'a'_0 \arrow[d, "F'p'"] \\ Fa_1 \arrow[r, "q_1"] & F'a'_1 \end{tikzcd}$$

in the sense that the following equation holds: $\partial h=Fp^{-1}\circ q_1^{-1}\circ F'p'\circ q_0$. The composition of two morphisms $(a_0,q_0,a'_0)\xrightarrow{(p_1,h_1,p'_1)}(a_1,q_1,a'_1)\xrightarrow{(p_2,h_2,p'_2)}(a_2,q_2,a'_2)$ is given by the formula

$$(p_2,h_2,p'_2)\circ(p_1,h_1,p'_1)=(p_2\circ p_1,{}^{Fp_1^{-1}}h_2\cdot h_1,p'_2\circ p'_1).$$

For every object (a,q,a'), its identity is $1_{(a,q,a')}=(1_a,1,1_{a'})$, and the inverse of any morphism (p,h,p') as above is $(p,h,p')^{-1}=(p^{-1},{}^{Fp}h^{-1},p'^{-1})$.

- *The functor* $\mathcal{G}_{\varphi,F\downarrow\varphi',F'}:\mathcal{P}_{\varphi,F\downarrow\varphi',F'}\to\mathbf{Gp}$ is defined on objects by

$$\mathcal{G}_{\varphi,F\downarrow\varphi',F'}(a,q,a')=\mathcal{G}(a)\times\mathcal{G}'(a'),$$

and, for any morphism $(p,h,p'):(a_0,q_0,a'_0)\to(a_1,q_1,a'_1)$, the associated homomorphism is given by ${}^{(p,h,p')}(g,g')=({}^{p}g,{}^{p'}g')$.

- *The boundary map* $\partial:\mathcal{G}_{\varphi,F\downarrow\varphi',F'}\to\mathrm{Aut}_{\mathcal{P}_{\varphi,F\downarrow\varphi',F'}}$, at any object (a,q,a') of the groupoid $\mathcal{P}_{\varphi,F\downarrow\varphi',F'}$, is given by the formula

$$\partial(g,g')=(\partial g,\varphi g^{-1}\cdot{}^{q^{-1}}\varphi'g',\partial g').$$

For any crossed module $(\mathcal{H},\mathcal{Q},\partial)$, we identify any object $b\in\mathcal{Q}$ with the morphism from the trivial crossed module $b:(1,1,1)\to(\mathcal{H},\mathcal{Q},\partial)$ such that $b(1)=b$, so that, for any morphism of crossed modules $(\varphi,F):(\mathcal{G},\mathcal{P},\partial)\to(\mathcal{H},\mathcal{Q},\partial)$, we have defined the *"homotopy-fiber crossed module"*

$$(\varphi,F)\!\downarrow\! b.$$

Next, we summarize our results in this setting of crossed modules. The crossed module (4.39) comes with a (non-commutative) square

$$\begin{tikzcd}(\varphi,F)\!\downarrow\!(\varphi',F') \arrow[r, "(\pi',P')"] \arrow[d, "(\pi,P)"'] & (\mathcal{G}',\mathcal{P}',\partial) \arrow[d, "(\varphi',F')"] \\ (\mathcal{G},\mathcal{P},\partial) \arrow[r, "(\varphi,F)"] & (\mathcal{H},\mathcal{Q},\partial),\end{tikzcd} \tag{4.40}$$

where

$$(a_0\overset{p}{\to}a_1)\xleftarrow{\;P\;}\!\shortmid\big((a_0,q_0,a_0')\xrightarrow{(p,h,p')}(a_1,q_1,a_1')\big)\shortmid\!\xrightarrow{\;P'\;}(a_0'\overset{p'}{\to}a_1')$$

$$g\xleftarrow{\;\pi\;}\!\shortmid(g,g')\shortmid\!\xrightarrow{\;\pi'\;}g'$$

Theorem 4.7 *The following statements hold:*

(i) For any morphisms of crossed modules $(\mathcal{G},\mathcal{P},\partial)\xrightarrow{(\varphi,F)}(\mathcal{H},\mathcal{Q},\partial)\xleftarrow{(\varphi',F')}(\mathcal{G}',\mathcal{P}',\partial)$, *there is a homotopy* $\mathrm{B}(\varphi,F)\,\mathrm{B}(\pi,P)\Rightarrow\mathrm{B}(\varphi',F')\,\mathrm{B}(\pi',P')$ *making the homotopy commutative square*

$$\begin{tikzcd}\mathrm{B}((\varphi,F)\!\downarrow\!(\varphi',F')) \arrow[r, "\mathrm{B}(\pi',P')"] \arrow[d, "\mathrm{B}(\pi,P)"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{B}(\mathcal{G}',\mathcal{P}',\partial) \arrow[d, "\mathrm{B}(\varphi',F')"] \\ \mathrm{B}(\mathcal{G},\mathcal{P},\partial) \arrow[r, "\mathrm{B}(\varphi,F)"] & \mathrm{B}(\mathcal{H},\mathcal{Q},\partial),\end{tikzcd} \tag{4.41}$$

induced by (4.40) *on classifying spaces, a homotopy pullback square.*

(ii) For any morphism of crossed modules $(\varphi,F):(\mathcal{G},\mathcal{P},\partial)\to(\mathcal{H},\mathcal{Q},\partial)$ *and every object* $b\in\mathcal{Q}$*, there is an induced homotopy fiber sequence*

$$\mathrm{B}((\varphi,F)\!\downarrow\! b)\xrightarrow{\mathrm{B}(\pi,P)}\mathrm{B}(\mathcal{G},\mathcal{P},\partial)\xrightarrow{\mathrm{B}(\varphi,F)}\mathrm{B}(\mathcal{H},\mathcal{Q},\partial).$$

(iii) A morphism of crossed modules $(\varphi,F):(\mathcal{G},\mathcal{P},\partial)\to(\mathcal{H},\mathcal{Q},\partial)$ *induces a homotopy equivalence on classifying spaces,* $\mathrm{B}(\varphi,F):\mathrm{B}(\mathcal{G},\mathcal{P},\partial)\simeq\mathrm{B}(\mathcal{H},\mathcal{Q},\partial)$*, if and only if, for every object* $b\in\mathcal{Q}$*, the space* $\mathrm{B}((\varphi,F)\!\downarrow\! b)$ *is contractible.*

(iv) For any crossed module $(\mathcal{G},\mathcal{P},\partial)$ *and object* $a\in\mathcal{P}$*, there is a homotopy equivalence*

$$\mathrm{B}\big((\mathcal{G},\mathcal{P},\partial)(a)\big)\simeq\Omega(\mathrm{B}(\mathcal{G},\mathcal{P},\partial),a),$$

where $(\mathcal{G},\mathcal{P},\partial)(a)$ *is the groupoid whose objects are the automorphisms* $p:a\to a$ *in* $\mathcal{P}$*, and whose arrows* $g:p\to q$ *are those elements* $g\in\mathcal{G}(a)$ *such that* $p=q\circ\partial g$.

Proof: (*i*) Let us apply the equivalence of categories (4.32) to the square of crossed modules (4.40). Then, by direct comparison, we see that the equation between squares of 2-groupoids

$$\begin{tikzcd}
\beta\big((\varphi,F)\!\downarrow\!(\varphi',F')\big) \arrow[r, "\beta(\pi',P')"] \arrow[d, "\beta(\pi,P)"'] & \beta(\mathcal{G}',\mathcal{P}',\partial) \arrow[d, "\beta(\varphi',F')"] \\
\beta(\mathcal{G},\mathcal{P},\partial) \arrow[r, "\beta(\varphi,F)"] & \beta(\mathcal{H},\mathcal{Q},\partial)
\end{tikzcd}
=
\begin{tikzcd}
\big(\beta(\varphi,F)\!\downarrow\!\beta(\varphi',F')\big) \arrow[r, "P'"] \arrow[d, "P"'] & \beta(\mathcal{G}',\mathcal{P}',\partial) \arrow[d, "\beta(\varphi',F')"] \\
\beta(\mathcal{G},\mathcal{P},\partial) \arrow[r, "\beta(\varphi,F)"] & \beta(\mathcal{H},\mathcal{Q},\partial)
\end{tikzcd}$$

holds, where the square on the right is (4.8) for the 2-functors $\beta(\varphi,F)$ and $\beta(\varphi',F')$. As any 2-groupoid has property (*iv*) in Theorem 4.2 (see the comment before Corollary 4.4), that theorem gives a homotopy $\mathrm{B}\beta(\varphi,F)\,\mathrm{B}\beta(\pi,P) \Rightarrow \mathrm{B}\beta(\varphi',F')\,\mathrm{B}\beta(\pi',P')$ such that the induced square

$$\begin{tikzcd}
\mathrm{B}\beta\big((\varphi,F)\!\downarrow\!(\varphi',F')\big) \arrow[r, "\mathrm{B}\beta(\pi',P')"] \arrow[d, "\mathrm{B}\beta(\pi,P)"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{B}\beta(\mathcal{G}',\mathcal{P}',\partial) \arrow[d, "\mathrm{B}\beta(\varphi',F')"] \\
\mathrm{B}\beta(\mathcal{G},\mathcal{P},\partial) \arrow[r, "\mathrm{B}\beta(\varphi,F)"] & \mathrm{B}\beta(\mathcal{H},\mathcal{Q},\partial)
\end{tikzcd}$$

is a homotopy pullback. It follows that the square (4.41) is also a homotopy pullback since, by Lemma 4.8, it is homotopy equivalent to the square above.

The implications (*i*) ⇒ (*ii*) ⇒ (*iii*) are clear, and (*iv*) follows from Corollary 4.4, as $(\mathcal{G},\mathcal{P},\partial)(a) = \beta(\mathcal{G},\mathcal{P},\partial)(a,a)$ and $\mathrm{B}(\mathcal{G},\mathcal{P},\partial) \simeq \mathrm{B}\beta(\mathcal{G},\mathcal{P},\partial)$. $\square$

We can easily show how the construction $(\varphi,F)\!\downarrow\!(\varphi',F')$ works on basic examples (see below).

Example 4.2 (*i*) Let $P \xrightarrow{F} Q \xleftarrow{F'} P'$ be homomorphisms of groups. These induce homomorphisms of crossed modules of groups $(1,P,1) \xrightarrow{(1,F)} (1,Q,1) \xleftarrow{(1,F')} (1,P',1)$, whose homotopy-fiber product is $(1,F)\downarrow(1,F') = (1,F\downarrow F',1)$, where $F\downarrow F'$ is the groupoid having as objects the elements $q \in Q$ and as morphisms $(p,p') : q_0 \to q_1$ those pairs $(p,p') \in P \times P'$ such that $q_1 \cdot Fp = F'p' \cdot q_0$. Thus, (4.41) particularizes by giving a homotopy pullback square

$$\begin{tikzcd}
\mathrm{B}(F\!\downarrow\!F') \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & K(P',1) \arrow[d] \\
K(P,1) \arrow[r] & K(Q,1).
\end{tikzcd}$$

(*ii*) Let $A \xrightarrow{\varphi} B \xleftarrow{\varphi'} A'$ be homomorphisms of abelian groups. These induce homomorphisms of crossed modules of groups $(A,1,1) \xrightarrow{(\varphi,1)} (B,1,1) \xleftarrow{(\varphi',1)} (A',1,1)$, whose homotopy-fiber product is the abelian crossed module of groups $(\varphi,1)\!\downarrow\!(\varphi',1) =$

$(A\times A', B, \partial)$, where the coboundary map is given by $\partial(a,a') = \varphi'a' - \varphi a$. Thus, (4.41) particularizes by giving a homotopy pullback square

$$\begin{tikzcd}
\mathrm{B}(A\times A', B, \partial) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & K(A',2) \arrow[d] \\
K(A,2) \arrow[r] & K(B,2).
\end{tikzcd}$$

Let us stress that, as Example 4.2(*i*) shows, the homotopy-fiber product crossed module $(\varphi, F)\downarrow(\varphi', F')$ may be a genuine crossed module of groupoids even in the case when both (φ, F) and (φ', F') are morphisms between crossed modules of groups. The reader can find in this fact a good reason to be interested in the study of general crossed modules over groupoids.

To finish, recall that the category of crossed complexes has a closed model structure as described by Brown and Golasinski [26]. In this homotopy structure, a morphism of crossed modules $(\varphi, F) : (\mathcal{G}, \mathcal{P}, \partial) \to (\mathcal{H}, \mathcal{Q}, \partial)$ is a weak equivalence if the induced map on classifying spaces $\mathrm{B}(\varphi, F)$ is a homotopy equivalence, and it is a fibration (see Howie [83]) whenever the following conditions hold: (*i*) $F : \mathcal{P} \to \mathcal{Q}$ is a fibration of groupoids, that is, for every object $a \in \mathcal{P}$ and every morphism $q : Fa \to b$ in $\mathcal{Q}$, there is a morphism $p : a \to a'$ in $\mathcal{P}$ such that $Fp = q$, and (*ii*) for any object $a \in \mathcal{P}$, the homomorphism $\varphi : \mathcal{G}(a) \to \mathcal{H}(Fa)$ is surjective. Then, it is natural to ask whether the constructed homotopy-fiber product crossed module $(\varphi, F)\downarrow(\varphi', F')$ is compatible with the homotopy pullback in the model category of crossed complexes. The answer is positive as a consequence of the theorem below, and this fact implies that the classifying space functor $(\mathcal{G}, \mathcal{P}, \partial) \mapsto \mathrm{B}(\mathcal{G}, \mathcal{P}, \partial)$ preserves homotopy pullbacks.

Theorem 4.8 *Let* $(\mathcal{G}, \mathcal{P}, \partial) \xrightarrow{(\varphi,F)} (\mathcal{H}, \mathcal{Q}, \partial) \xleftarrow{(\varphi',F')} (\mathcal{G}', \mathcal{P}', \partial)$ *be a diagram of morphisms of crossed modules. If one of them is a fibration, then the canonical morphism*

$$(\mathcal{G}, \mathcal{P}, \partial) \times_{(\mathcal{H}, \mathcal{Q}, \partial)} (\mathcal{G}', \mathcal{P}', \partial) \to (\varphi, F)\downarrow(\varphi', F')$$

induces a homotopy equivalence

$$\mathrm{B}\big((\mathcal{G}, \mathcal{P}, \partial) \times_{(\mathcal{H}, \mathcal{Q}, \partial)} (\mathcal{G}', \mathcal{P}', \partial)\big) \simeq \mathrm{B}\big((\varphi, F)\downarrow(\varphi', F')\big).$$

Proof:

Let us observe that the pullback crossed module of (φ, F) and (φ', F') is

$$(\mathcal{G}, \mathcal{P}, \partial) \times_{(\mathcal{H}, \mathcal{Q}, \partial)} (\mathcal{G}', \mathcal{P}', \partial) = (\mathcal{G} \times_{\mathcal{H}F} \mathcal{G}', \mathcal{P} \times_{\mathcal{Q}} \mathcal{P}', \partial),$$

where $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$ is the pullback groupoid of $F : \mathcal{P} \to \mathcal{Q}$ and $F' : \mathcal{P}' \to \mathcal{Q}$. The functor $\mathcal{G} \times_{F} \mathcal{G}' : \mathcal{P} \times_{\mathcal{Q}} \mathcal{P}' \to \mathbf{Gp}$ is defined on objects by

$$(\mathcal{G} \times_{\mathcal{H}F} \mathcal{G}')(a,a') = \mathcal{G}(a) \times_{\mathcal{H}(Fa)} \mathcal{G}'(a') = \{(g,g') \in \mathcal{G}(a) \times \mathcal{G}'(a') \mid \varphi_a(g) = \varphi'_{a'}(g')\},$$

and the homomorphism associated to any morphism $(p,p') : (a,a') \to (b,b')$ in $\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$ is given by ${}^{(p,p')}(g,g') = ({}^{p}g, {}^{p'}g')$. The boundary map $\partial : \mathcal{G}\times_{\mathcal{H}F}\mathcal{G}' \to \mathrm{Aut}_{\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'}$, at any object of the groupoid $\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$, is given by the formula $\partial(g,g') = (\partial g, \partial g')$.

The canonical morphism

$$(\jmath, J) : (\mathcal{G}\times_{\mathcal{H}F}\mathcal{G}', \mathcal{P}\times_{\mathcal{Q}}\mathcal{P}', \partial) \to \big(\mathcal{G}_{\varphi,F\downarrow\varphi',F'}, \mathcal{P}_{\varphi,F\downarrow\varphi',F'}, \partial\big) \tag{4.42}$$

is as follows: The functor $J\colon \mathcal{P}\times_{\mathcal{Q}}\mathcal{P}' \to \mathcal{P}_{\varphi,F\downarrow\varphi',F'}$ sends a morphism $(p,p')\colon (a,a') \to (b,b')$ to the morphism $(p, 1_{\mathcal{H}(Fa)}, p') : (a, 1_{Fa}, a') \to (b, 1_{Fb}, b')$, and the $\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$-group homomorphism $\jmath : \mathcal{G}\times_{\mathcal{H}F}\mathcal{G}' \to \mathcal{P}_{\varphi,F\downarrow\varphi',F'}J$ is given at any object $(a,a') \in \mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$ by the inclusion map $\mathcal{G}(a)\times_{\mathcal{H}(Fa)}\mathcal{G}'(a') \hookrightarrow \mathcal{G}(a)\times\mathcal{G}'(a')$.

Next, we assume that (φ, F) is a fibration. Then, we verify that the canonical morphism (4.42) induces isomorphisms between the corresponding homotopy groups. Recall from (4.37) how to compute the homotopy groups of the classifying space of a crossed module.

- *The map $\pi_0(\jmath, J)$ is a bijection.*

Injectivity: Suppose objects $(a,a'), (b,b') \in \mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$, such that there is a morphism $(p,h,p') : (a,1_{Fa},a') \to (b,1_{Fb},b')$ in $\mathcal{P}_{\varphi,F\downarrow\varphi',F'}$. Then, as $\varphi : \mathcal{G}(a) \to \mathcal{H}(Fa)$ is surjective, there is $g \in \mathcal{G}(a)$ such that $\varphi(g) = h$, whence $(p\circ\partial g, p') : (a,a') \to (b,b')$ is a morphism in $\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$.

Surjectivity: Let (a,q,a') be an object of $\mathcal{P}_{\varphi,F\downarrow\varphi',F'}$. As $F : \mathcal{P} \to \mathcal{Q}$ is a fibration of groupoids, there is a morphism $p : a \to b$ in $\mathcal{P}$ such that $Fp = q$. Then, (b,a') is an object of the groupoid $\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$ with $J(b,a') = (b,1_{Fb},a')$ in the same connected component of (a,q,a'), since we have the morphism $(p, 1_{\mathcal{H}(Fb)}, 1_{a'}) : (a,q,a') \to (b,1_{Fb},a')$.

- *The homomorphisms $\pi_1(\jmath, J)$ are isomorphisms.* Let (a,a') be any object of $\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$.

Injectivity: Let $[(p,p')]$ be an element in the kernel of the homomorphism $\pi_1(\jmath,J)$ at (a,a'), that is, such that $[(p,1_{\mathcal{H}(Fa)},p')] = [(1_a, 1_{\mathcal{H}(Fa)}, 1_{a'})]$. This means that there is $(g,g') \in \mathcal{G}(a)\times\mathcal{G}'(a')$ with $\partial g = p$, $\partial(g') = p'$ and $\varphi(g)^{-1}\cdot\varphi'(g') = 1$. The last equation says that (g,g') is an element of $\mathcal{G}(a)\times_{\mathcal{H}(Fa)}\mathcal{G}'(a)$ which, by the first two, satisfies that $\partial(g,g') = (p,p')$. Hence, $[(p,p')] = [(1_a, 1_{a'})]$.

Surjectivity: Let $(p,h,p') : (a,1_{Fa},a') \to (a,1_{Fa},a')$ be an automorphism of $\mathcal{P}_{\varphi,F\downarrow\varphi',F'}$. As $\varphi : \mathcal{G}(a) \to \mathcal{H}(Fa)$ is surjective, there is a $g \in \mathcal{G}(a)$ such that $\varphi(g) = h$. Then, we have

$$\begin{aligned}(p,h,p')^{-1}\circ J(p\circ\partial g, p') &= (p,h,p')^{-1}\circ(p\circ\partial g, 1_{\mathcal{H}(Fa)}, p')\\ &= (p^{-1}\circ p\circ\partial g, h^{-1}, p'^{-1}\circ p')\\ &= (\partial g, \varphi(g)^{-1}\cdot 1_{\mathcal{H}(Fa)}, 1_{a'}) = \partial(g, 1_{\mathcal{G}'(a')})\end{aligned}$$

and therefore $[(p,h,p')] = [J(p\circ\partial g, p')]$.

- *The homomorphisms $\pi_2(\jmath, J)$ are isomorphisms.* At any object $(a,a') \in \mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$, the homomorphism $\pi_2(\jmath,J)$ is the restriction to the kernels of the boundary maps of the inclusion $\mathcal{G}(a)\times_{\mathcal{H}(Fa)}\mathcal{G}'(a') \hookrightarrow \mathcal{G}(a)\times\mathcal{G}'(a')$. Then, it is clearly injective. To see the surjectivity, let $(g,g') \in \mathcal{G}(a)\times\mathcal{G}'(a')$ with $\partial(g,g') = (1_a, 1_{\mathcal{H}(Fa)}, 1_{a'})$.

Then, we have $\partial g = 1_a$, $\partial g' = 1_{a'}$ and $\varphi(g)^{-1} \cdot \varphi'(g') = 1_{\mathcal{H}(Fa)}$. That is, that $(g, g') \in \mathcal{G}(a) \times_{\mathcal{H}(Fa)} \mathcal{G}'(a')$ and $(\partial g, \partial g') = (1_a, 1_{a'})$. $\square$

4.6 Appendix: Proofs of Lemmas 4.3 and 4.4

We shall only address lax functors below, but the discussions are easily dualized in order to obtain the corresponding results for oplax functors.

Our first goal is to accurately determine the functorial behaviour of the Grothendieck nerve construction $\mathcal{B} \mapsto \mathrm{N}\mathcal{B}$ (4.12) on lax functors between bicategories by means of the theorem below. The result in the first part of it is just the bicategorical version of Theorem 2.1. See also [41, §3, (21)], where a proof is given using Jardine's Supercoherence Theorem in [86]. The second part is a lax version of Proposition 2.1 for bicategories. The ideas in the proof are pretty similar to those used in Chapter 2.

Theorem 4.9 (*i*) *Any bicategory $\mathcal{B}$ defines a normal pseudo-simplicial category, that is, a unitary pseudo-functor from the simplicial category Δ^{op} into the 2-category of small categories,*

$$\mathrm{N}\mathcal{B} = (\mathrm{N}\mathcal{B}, \widehat{\mathrm{N}}\mathcal{B}) : \Delta^{\mathrm{op}} \to \mathbf{Cat},$$

which is called the Grothendieck *or* pseudo-simplicial nerve *of the bicategory, whose category of p-simplices, for $p \geq 0$, is*

$$\mathrm{N}\mathcal{B}_p := \bigsqcup_{(x_p,\ldots,x_0)\in \mathrm{Ob}\mathcal{B}^{p+1}} \mathcal{B}(x_{p-1}, x_p) \times \mathcal{B}(x_{p-2}, x_{p-1}) \times \cdots \times \mathcal{B}(x_0, x_1).$$

(*ii*) *Any lax functor between bicategories $F : \mathcal{B} \to \mathcal{B}'$ induces a lax transformation (i.e., a lax simplicial functor)*

$$\mathrm{N}F = (\mathrm{N}F, \widehat{\mathrm{N}}F) : \mathrm{N}\mathcal{B} \to \mathrm{N}\mathcal{B}'.$$

For any pair of composable lax functors $F : \mathcal{B} \to \mathcal{B}'$ and $F' : \mathcal{B}' \to \mathcal{B}''$, the equality $\mathrm{N}F'\,\mathrm{N}F = \mathrm{N}(F'F)$ holds, and, for any bicategory $\mathcal{B}$, $\mathrm{N}1_{\mathcal{B}} = 1_{\mathrm{N}\mathcal{B}}$.

Before starting with the proof, we shall describe some needed constructions and a few auxiliary facts. Given a category $\mathcal{I}$ and a bicategory $\mathcal{B}$, we denote by

$$\mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{B})$$

the category[8] whose objects are lax functors $F : \mathcal{I} \to \mathcal{B}$, and whose morphisms are *relative to object lax transformations*, as termed by Bullejos and Cegarra in [35], but also called *icons* by Lack in [93]. That is, for any two lax functors $F, G : \mathcal{I} \to \mathcal{B}$, a

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[8] This is the bicategorical version of the bicategories $\mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{T})$ defined in Subsection 2.2.1 of Chapter 2.

morphism $\Phi : F \Rightarrow G$ may exist only if F and G agree on objects, and it is then given by 2-cells in $\mathcal{B}$, $\Phi a : Fa \Rightarrow Ga$, for every arrow $a : i \to j$ in $\mathcal{I}$, such that the diagrams

$$\begin{tikzcd}
Fa \circ Fb \arrow[r, Rightarrow, "\widehat{F}_{a,b}"] \arrow[d, Rightarrow, "\Phi a \circ \Phi b"'] & F(ab) \arrow[d, Rightarrow, "\Phi(ab)"] \\
Ga \circ Gb \arrow[r, Rightarrow, "\widehat{G}_{a,b}"] & G(ab),
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1_{Fi=Gi} \arrow[dl, Rightarrow, "\widehat{F}_i"'] \arrow[dr, Rightarrow, "\widehat{G}_i"] & \\
F1_i \arrow[rr, Rightarrow, "\Phi 1_i"] & & G1_i,
\end{tikzcd}$$

commute for each pair of composable arrows $i \overset{b}{\to} j \overset{a}{\to} k$ and each object i. The composition of morphisms $\Phi : F \Rightarrow G$ and $\Psi : G \Rightarrow H$, for $F, G, H : \mathcal{I} \to \mathcal{B}$ lax functors, is $\Psi \cdot \Phi : F \Rightarrow H$, where $(\Psi \cdot \Phi)a = \Psi a \cdot \Phi a : Fa \Rightarrow Ha$, for each arrow $a : i \to j$ in $\mathcal{I}$. The identity morphism of a lax functor $F : \mathcal{I} \to \mathcal{B}$ is $1_F : F \Rightarrow F$, where $(1_F)a = 1_{Fa}$, the identity of Fa in the category $\mathcal{B}(Fi, Fj)$, for each $a : i \to j$ in $\mathcal{I}$.

Let us now replace the category $\mathcal{I}$ above by a (directed) graph $\mathcal{G}$. For any bicategory $\mathcal{B}$, there is a category

$$\mathbb{L}\mathrm{ax}(\mathcal{G}, \mathcal{B}),$$

where an object $f : \mathcal{G} \to \mathcal{B}$ consists of a pair of maps that assign an object fi to each vertex $i \in \mathcal{G}$ and a 1-cell $fa : fi \to fj$ to each edge $a : i \to j$ in $\mathcal{G}$, respectively. A morphism $\phi : f \Rightarrow g$ may exist only if f and g agree on vertices, that is, $fi = gi$ for all $i \in \mathcal{G}$; and then it consists of a map that assigns to each edge $a : i \to j$ in the graph a 2-cell $\phi a : fa \Rightarrow ga$ of $\mathcal{B}$. Compositions in $\mathbb{L}\mathrm{ax}(\mathcal{G}, \mathcal{B})$ are defined in the natural way by the same rules as those stated above for the category $\mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{B})$.

Lemma 4.9 *Let $\mathcal{I} = \mathcal{I}(\mathcal{G})$ be the free category generated by a graph $\mathcal{G}$, let $\mathcal{B}$ be a bicategory, and let*

$$R : \mathbb{L}\mathrm{ax}(\mathcal{I}(\mathcal{G}), \mathcal{B}) \to \mathbb{L}\mathrm{ax}(\mathcal{G}, \mathcal{B})$$

be the functor defined by restriction to the basic graph. Then, there is a functor

$$J : \mathbb{L}\mathrm{ax}(\mathcal{G}, \mathcal{B}) \to \mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{B}),$$

and a natural transformation

$$\mathrm{v} : JR \Rightarrow 1_{\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{B})}, \tag{4.43}$$

such that $RJ = 1_{\mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{B})}$, $\mathrm{v}J = 1_J$, $R\mathrm{v} = 1_R$. Thus, the functor R is right adjoint to the functor J.

Proof: To describe the functor J, we use the following useful construction: For any list $(x_0, \ldots, x_p)$ of objects in the bicategory $\mathcal{B}$, let

$$\overset{\mathrm{or}}{\circ} : \mathcal{B}(x_{p-1}, x_p) \times \mathcal{B}(x_{p-2}, x_{p-1}) \times \cdots \times \mathcal{B}(x_0, x_1) \longrightarrow \mathcal{B}(x_0, x_p)$$

denote the functor obtained by iterating horizontal composition in the bicategory, which acts on objects and arrows of the product category by the recursive formula

$$\overset{\mathrm{or}}{\circ}(u_p,\dots,u_1)=\begin{cases} u_1 & \text{if} \quad p=1,\\ u_p\circ\big(\overset{\mathrm{or}}{\circ}(u_{p-1},\dots,u_1)\big) & \text{if} \quad p\geq 2.\end{cases}$$

Then, the homomorphism J takes a graph map, say $f:\mathcal{G}\to\mathcal{B}$, to the unitary pseudo-functor from the free category

$$J(f)=F:\mathcal{I}\to\mathcal{B},$$

such that $Fi=fi$, for any vertex i of $\mathcal{G}$ (= objects of $\mathcal{I}$), and associates to strings $a: a(0)\overset{a_1}{\to}\cdots\overset{a_p}{\to}a(p)$ in $\mathcal{G}$ the 1-cells $Fa=\overset{\mathrm{or}}{\circ}(fa_p,\dots,fa_1):fa(0)\to fa(p)$. The structure 2-cells $\widehat{F}_{a,b}:Fa\circ Fb\Rightarrow F(ab)$, for any pair of strings in the graph, $a=a_p\cdots a_1$ as above and $b=b_q\cdots b_1$ with $b(q)=a(0)$, are canonically obtained from the associativity constraints in the bicategory: first by taking $\widehat{F}_{a_1,b}=1_{F(a_1b)}$ when $p=1$ and then, recursively for $p>1$, defining $\widehat{F}_{a,b}$ as the composite

$$\widehat{F}_{a,b}:\ Fa\circ Fb\xRightarrow{\boldsymbol{a}}Fa_p\circ(Fa'\circ Fb)\xRightarrow{1\circ\widehat{F}_{a',b}}F(ab),$$

where $a'=a_{p-1}\cdots a_1$ (whence $Fa=Fa_p\circ Fa'$). The coherence conditions for F are easily verified by using the coherence and naturality of the associativity constraint $\boldsymbol{a}$ of the bicategory.

Any morphism $\phi:f\Rightarrow g$ in $\mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{B})$ is taken by J to the morphism $J(\phi):F\Rightarrow G$ of $\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{B})$, consisting of the 2-cells in the bicategory $\overset{\mathrm{or}}{\circ}(\phi a_p,\dots,\phi a_1):Fa\Rightarrow Ga$, attached to the strings of adjacent edges in the graph $a=a_p\cdots a_1$. The coherence conditions of $J(\phi)$ are consequence of the naturality of the associativity constraint $\boldsymbol{a}$ of the bicategory. If $\phi:f\Rightarrow g$ and $\psi:g\Rightarrow h$ are 1-cells in $\mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{B})$, then $J(\psi)\cdot J(\phi)=J(\psi\cdot\phi)$ follows from the functoriality of the composition $\circ$, and so J is a functor.

The lax transformation v is defined as follows: The component of this lax transformation at a lax functor $F:\mathcal{I}\to\mathcal{B}$, $\mathrm{v}:JR(F)\Rightarrow F$, is defined on identities by $\mathrm{v}1_i=\widehat{F}_i:1_{Fi}\Rightarrow F1_i$, for any vertex i of $\mathcal{G}$, and it associates to each string of adjacent edges in the graph $a=a_p\cdots a_1$ the 2-cell $\mathrm{v}a:\overset{\mathrm{or}}{\circ}(Fa_p,\dots,Fa_1)\Rightarrow Fa$, which is given by taking $\mathrm{v}a_1=1_{Fa_1}$ if $p=1$, and then, recursively for $p>1$, by taking $\mathrm{v}a$ as the composite

$$\mathrm{v}a=\ \big(\overset{\mathrm{or}}{\circ}(Fa_p,\dots,Fa_1)\xRightarrow{1\circ\mathrm{v}a'}Fa_p\circ Fa'\xRightarrow{\widehat{F}_{a_p,a'}}Fa\big),$$

where $a'=a_{p-1}\cdots a_1$. The naturality condition $\widehat{F}_{a,b}\circ(\mathrm{v}a\circ\mathrm{v}b)=\mathrm{v}(ab)\circ\widehat{JR(F)}_{a,b}$, for any pair of composable morphisms in $\mathcal{I}$, can be checked as follows: when $a=1_i$

or $b = 1_j$ are identities, then it is a consequence of the commutativity of the diagrams

$$\begin{tikzcd}
1_{Fi} \circ JR(F)b \arrow[rr, Rightarrow, "\boldsymbol{l}"] \arrow[dr, Rightarrow, "1\circ \mathrm{v}b"] \arrow[dd, Rightarrow, "\widehat{F}_i\circ \mathrm{v}b"'] & & JR(F)b \arrow[dd, Rightarrow, "\mathrm{v}b"] \\
& 1_{Fi} \circ Fb \arrow[dl, Rightarrow, "\widehat{F}_i\circ 1"'] \arrow[dr, Rightarrow, "\boldsymbol{l}"] \arrow[l, phantom, "(A)"] \arrow[ur, phantom, "(B)"] \arrow[d, phantom, "(C)"] & \\
F1_i \circ Fb \arrow[rr, Rightarrow, "\widehat{F}_{1_i,b}"'] & {} & Fb,
\end{tikzcd}
\qquad
\begin{tikzcd}
JR(F)a \circ 1_{Fj} \arrow[rr, Rightarrow, "\boldsymbol{r}"] \arrow[dr, Rightarrow, "\mathrm{v}a\circ 1"] \arrow[dd, Rightarrow, "\mathrm{v}a\circ\widehat{F}_j"'] & & JR(F)a \arrow[dd, Rightarrow, "\mathrm{v}a"] \\
& Fa \circ 1_{Fj} \arrow[dl, Rightarrow, "1\circ\widehat{F}_j"'] \arrow[dr, Rightarrow, "\boldsymbol{r}"] \arrow[l, phantom, "(A)"] \arrow[ur, phantom, "(B)"] \arrow[d, phantom, "(C)"] & \\
Fa \circ F1_j \arrow[rr, Rightarrow, "\widehat{F}_{a,1_i}"'] & {} & Fa,
\end{tikzcd}$$

where the regions labelled with (A) commute by the functoriality of $\circ$, those with (B) by the naturality of $\boldsymbol{l}$ and $\boldsymbol{r}$, and those with (C) by the coherence of F. Now, for arbitrary strings a and b in the graph with $b(q) = a(0)$, we study the coherence recursively on the length of a. The case when $p = 1$ is the obvious commutative diagram

$$\begin{tikzcd}
Fa_1 \circ JR(F)b \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1\circ \mathrm{v}b"'] & JR(F)(a_1b) \arrow[d, Rightarrow, "\mathrm{v}(a_1b)=\widehat{F}_{a_1,b}\cdot(1\circ \mathrm{v}b)"] \\
Fa_1 \circ Fb \arrow[r, Rightarrow, "\widehat{F}_{a_1,b}"'] & F(a_1b),
\end{tikzcd}$$

and then, for $p > 1$, the result is a consequence of the diagram

$$\begin{tikzcd}
JR(F)a \circ JR(F)b \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[d, Rightarrow, "(1\circ \mathrm{v}a')\circ \mathrm{v}b"'] & Fa_p\circ(JR(F)a'\circ JR(F)b) \arrow[r, Rightarrow, "1\circ\widehat{JR(F)}_{a',b}"] \arrow[d, Rightarrow, "1\circ(\mathrm{v}a'\circ \mathrm{v}b)"] \arrow[dl, phantom, "(A)"] \arrow[dr, phantom, "(B)"] & JR(F)(ab) \arrow[d, Rightarrow, "1\circ \mathrm{v}(a'b)"] \\
(Fa_p\circ Fa')\circ Fb \arrow[r, Rightarrow, "\boldsymbol{a}"] \arrow[d, Rightarrow, "\widehat{F}_{a_p,a'}\circ 1"'] & Fa_p\circ(Fa'\circ Fb) \arrow[r, Rightarrow, "1\circ\widehat{F}_{a',b}"'] \arrow[d, phantom, "(C)"] & Fa_p \circ F(a'b) \arrow[d, Rightarrow, "\widehat{F}_{a_p,a'b}"] \\
Fa \circ Fb \arrow[rr, Rightarrow, "\widehat{F}_{a,b}"] & {} & F(ab)
\end{tikzcd}$$

where (A) commutes by the naturality of $\boldsymbol{a}$, (B) by induction, and (C) by the coherence of F.

To verify the equalities $RJ = 1$, $\mathrm{v}J = 1$, and $R\mathrm{v} = 1$ is straightforward. □

Let $\mathcal{I} = \mathcal{I}(\mathcal{G})$ again be the free category generated by a graph $\mathcal{G}$, as in Lemma 4.9 above, and suppose now that $F : \mathcal{B} \to \mathcal{B}'$ is a lax functor. Then, the square

$$\begin{tikzcd}
\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{B}) \arrow[r, "R"] \arrow[d, "F_*"'] & \mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{B}) \arrow[d, "F_*"] \\
\mathbb{L}\mathrm{ax}(\mathcal{I},\mathcal{B}') \arrow[r, "R'"] & \mathbb{L}\mathrm{ax}(\mathcal{G},\mathcal{B}')
\end{tikzcd} \tag{4.44}$$

commutes and, since $RJ = 1$, we have the equalities

$$R'F_*JR = F_*RJR = F_*R = R'F_*. \tag{4.45}$$

Furthermore, the naturality of $\mathrm{v} : JR \Rightarrow 1$ and $\mathrm{v}' : J'R' \Rightarrow 1$ means that the square

$$\begin{tikzcd}
J'R'F_*JR \arrow[r, Rightarrow, "\mathrm{v}'F_*JR"] \arrow[d, Rightarrow, "J'R'F_*\mathrm{v}"'] & F_*JR \arrow[d, Rightarrow, "F_*\mathrm{v}"] \\
J'R'F_* \arrow[r, Rightarrow, "\mathrm{v}'F_*"] & F_*
\end{tikzcd}$$

commutes. As $R\mathrm{v} = 1_R$ and then $J'R'F_*\mathrm{v} = J'F_*R\mathrm{v} = J'F_*1_R = 1_{J'R'F_*}$, we have the equality

$$F_*\mathrm{v} \circ \mathrm{v}'F_*JR = \mathrm{v}'F_*. \tag{4.46}$$

4.6.1 Proof of Theorem 4.9.

(i): Let us note that, for any integer $p \geq 0$, the category $[p]$ is free on the graph

$$\mathcal{G}_p = (0 \to 1 \cdots \to p).$$

Then, for any given bicategory $\mathcal{B}$, the existence of an adjunction

$$J_p \dashv R_p : \mathrm{N}\mathcal{B}_p = \mathbb{L}\mathrm{ax}(\mathcal{G}_p, \mathcal{B}) \rightleftarrows \mathbb{L}\mathrm{ax}([p], \mathcal{B}) \tag{4.47}$$

follows from Lemma 4.9, where R_p is the functor defined by restricting to the basic graph $\mathcal{G}_p$ of the category $[p]$, where $R_pJ_p = 1$, whose unity is the identity, and whose counit $\mathrm{v}_p : J_pR_p \Rightarrow 1$ satisfies the equalities $\mathrm{v}_pJ_p = 1$ and $R_p\mathrm{v}_p = 1$.

If $a : [q] \to [p]$ is any map in the simplicial category, then the associated functor $\mathrm{N}\mathcal{B}_a : \mathrm{N}\mathcal{B}_p \to \mathrm{N}\mathcal{B}_q$ is the composite

$$\begin{tikzcd}
\mathrm{N}\mathcal{B}_p \arrow[r, dotted, "\mathrm{N}\mathcal{B}_a"] \arrow[d, "J_p"'] & \mathrm{N}\mathcal{B}_q \\
\mathbb{L}\mathrm{ax}([p], \mathcal{B}) \arrow[r, "a^*"] & \mathbb{L}\mathrm{ax}([q], \mathcal{B}). \arrow[u, "R_q"']
\end{tikzcd}$$

Thus, $\mathrm{N}\mathcal{B}_a$ maps the component category of $\mathrm{N}\mathcal{B}_p$ at $(x_p, \ldots, x_0)$ into the component at $(x_{a(q)}, \ldots, x_{a(0)})$ of $\mathrm{N}\mathcal{B}_q$, and it acts both on objects and morphisms of $\mathrm{N}\mathcal{B}_p$ by the formula $\mathrm{N}\mathcal{B}_p(u_p, \ldots, u_1) = (v_q, \ldots, v_1)$, where, for $0 \leq k < q$,

$$v_{k+1} = \begin{cases} \overset{\mathrm{or}}{\circ}(u_{a(k+1)}, \ldots, u_{a(k)+1}) & \textit{if} \quad a(k) < a(k+1), \\ 1 & \textit{if} \quad a(k) = a(k+1), \end{cases}$$

whence, in particular, the usual formulas below for the face and degeneracy functors.

$$d_i(u_p, \ldots, u_1) = \begin{cases} (u_p, \ldots, u_2) & \text{if} \quad i = 0, \\ (u_p, \ldots, u_{i+1} \circ u_i, \ldots, u_1) & \text{if} \quad 0 < i < p, \\ (u_{p-1}, \ldots, u_1) & \text{if} \quad i = p, \end{cases}$$

$$s_i(u_p, \ldots, u_1) = (u_p, \ldots, u_{i+1}, 1, u_i, \ldots, u_0).$$

The structure natural transformation

$$\begin{tikzcd} \mathrm{N}\mathcal{B}_p \arrow[r, bend left, "\mathrm{N}\mathcal{B}_b\ \mathrm{N}\mathcal{B}_a"] \arrow[r, bend right, "\mathrm{N}\mathcal{B}_{ab}"'] \arrow[r, phantom, "\Downarrow \widehat{\mathrm{N}\mathcal{B}}_{a,b}"] & \mathrm{N}\mathcal{B}_n, \end{tikzcd} \tag{4.48}$$

for each pair of composable maps $[n] \xrightarrow{b} [q] \xrightarrow{a} [p]$ in Δ, is

$$\mathrm{N}\mathcal{B}_b\ \mathrm{N}\mathcal{B}_a = R_n b^* J_q R_q a^* J_p \xRightarrow{\widehat{\mathrm{N}\mathcal{B}}_{a,b}=R_n b^* \mathrm{v}_q a^* J_p} R_n b^* a^* J_p = R_n(ab)^* J_p = \mathrm{N}\mathcal{B}_{ab}.$$

Let us stress that, in spite of the natural transformation v in (4.43) not being invertible, the natural transformation $\widehat{\mathrm{N}\mathcal{B}}_{a,b}$ in (4.48) is invertible since, for any $\mathbf{x} \in \mathrm{N}\mathcal{B}_p$, the lax functor $a^* J_p\mathbf{x}$ is actually a homomorphism and therefore $\mathrm{v}_q a^* J_p \mathbf{x}$ is an isomorphism. Consequently, we only need to prove that these constraints $\widehat{\mathrm{N}\mathcal{B}}_{a,b}$ verify the coherence conditions for lax functors:

If $a = 1_{[p]}$, then $\widehat{\mathrm{N}\mathcal{B}}_{1,b} = R_n b^* \mathrm{v}_p J_p = R_n b^* 1_{J_p} = 1_{\mathrm{N}\mathcal{B}_b}$. Similarly, $\widehat{\mathrm{N}\mathcal{B}}_{a,1} = 1_{\mathrm{N}\mathcal{B}_a}$. Furthermore, for every triplet of composable arrows $[m] \xrightarrow{c} [n] \xrightarrow{b} [q] \xrightarrow{a} [p]$, the diagram

$$\begin{tikzcd} \mathrm{N}\mathcal{B}_c\ \mathrm{N}\mathcal{B}_b\ \mathrm{N}\mathcal{B}_a \arrow[r, Rightarrow, "\mathrm{N}\mathcal{B}_c\, \widehat{\mathrm{N}\mathcal{B}}_{a,b}"] \arrow[d, Rightarrow, "\widehat{\mathrm{N}\mathcal{B}}_{b,c}\, \mathrm{N}\mathcal{B}_a"'] & \mathrm{N}\mathcal{B}_c\ \mathrm{N}\mathcal{B}_{ab} \arrow[d, Rightarrow, "\widehat{\mathrm{N}\mathcal{B}}_{ab,c}"] \\ \mathrm{N}\mathcal{B}_{bc}\ \mathrm{N}\mathcal{B}_a \arrow[r, Rightarrow, "\widehat{\mathrm{N}\mathcal{B}}_{a,bc}"] & \mathrm{N}\mathcal{B}_{abc}, \end{tikzcd}$$

is commutative since it is obtained by applying the functors $R_m c^*$ on the left, and $a^* J_p$ on the right, to the diagram

$$\begin{tikzcd} J_n R_n b^* J_q R_q \arrow[r, Rightarrow, "J_n R_n b^* \mathrm{v}_q"] \arrow[d, Rightarrow, "\mathrm{v}_n b^* J_q R_q"'] & J_n R_n b^* \arrow[d, Rightarrow, "\mathrm{v}_n b^*"] \\ b^* J_q R_q \arrow[r, Rightarrow, "b^* \mathrm{v}_q"'] & b^*, \end{tikzcd} \tag{4.49}$$

which commutes by the naturality of v_n.

(ii): Suppose now that $F : \mathcal{B} \to \mathcal{B}'$ is a lax functor. Then, at any integer $p \geq 0$, the functor $\mathrm{N}F_p : \mathrm{N}\mathcal{B}_p \to \mathrm{N}\mathcal{B}'_p$ is the composite

$$\begin{tikzcd} \mathrm{N}\mathcal{B}_p \arrow[r, dotted, "\mathrm{N}F_p"] \arrow[d, "J_p"'] & \mathrm{N}\mathcal{B}'_p \\ \mathbb{L}\mathrm{ax}([p],\mathcal{B}) \arrow[r, "F_*"] & \mathbb{L}\mathrm{ax}([p],\mathcal{B}'), \arrow[u, "R'_p"'] \end{tikzcd}$$

which is explicitly given both on objects and arrows by the simple formula $\mathrm{N}F_p(u_p,\dots,u_1) = (Fu_p,\dots,Fu_1)$. The structure natural transformation

$$\begin{tikzcd}
\mathrm{N}\mathcal{B}_p \arrow[r, "\mathrm{N}\mathcal{B}_a"] \arrow[d, "\mathrm{N}F_p"'] \arrow[dr, phantom, "\overset{\widehat{\mathrm{N}}F_a}{\Rightarrow}"] & \mathrm{N}\mathcal{B}_q \arrow[d, "\mathrm{N}F_q"] \\
\mathrm{N}\mathcal{B}'_p \arrow[r, "\mathrm{N}\mathcal{B}'_a"'] & \mathrm{N}\mathcal{B}'_q,
\end{tikzcd}$$

at each map $a:[q]\to[p]$ in Δ, is

$$\begin{aligned}\mathrm{N}\mathcal{B}'_a\,\mathrm{N}F_p = R'_q a^* J'_p R'_p F_* J_p \xRightarrow{\widehat{\mathrm{N}}F_a = R'_q a^* \mathrm{v}'_p F_* J_p} & R'_q a^* F_* J_p = R'_q F_* a^* J_p \overset{(4.45)}{=} R'_q F_* J_q R_q a^* J_p \\ & = \mathrm{N}F_q\,\mathrm{N}\mathcal{B}_a.\end{aligned}$$

This family of natural transformations $\widehat{\mathrm{N}}F_a$ verifies the coherence conditions for lax transformations: If $a = 1_{[p]}$, then $\widehat{\mathrm{N}}F_1 = R'_p\mathrm{v}'_p F_* J_p = 1_{R'_p}F_*J_p = 1_{\widehat{\mathrm{N}}F_p}$. Suppose that $b:[n]\to[q]$ is any other map of Δ, then the coherence diagram

$$\begin{tikzcd}
\mathrm{N}\mathcal{B}'_b\,\mathrm{N}\mathcal{B}'_a\,\mathrm{N}F_p \arrow[r, Rightarrow, "\mathrm{N}\mathcal{B}'_b\,\widehat{\mathrm{N}}F_a"] \arrow[d, Rightarrow, "\widehat{\mathrm{N}}\mathcal{B}'_{a,b}\,\mathrm{N}F_p"'] & \mathrm{N}\mathcal{B}'_b\,\mathrm{N}F_q\,\mathrm{N}\mathcal{B}_a \arrow[r, Rightarrow, "\widehat{\mathrm{N}}F_b\,\mathrm{N}\mathcal{B}_a"] & \mathrm{N}F_n\,\mathrm{N}\mathcal{B}_b\,\mathrm{N}\mathcal{B}_a \arrow[d, Rightarrow, "\mathrm{N}F_n\,\widehat{\mathrm{N}}\mathcal{B}_{a,b}"] \\
\mathrm{N}\mathcal{B}'_{ab}\,\mathrm{N}F_p \arrow[rr, Rightarrow, "\widehat{\mathrm{N}}F_{ab}"] & & \mathrm{N}F_n\,\mathrm{N}\mathcal{B}_{ab}
\end{tikzcd}$$

commutes, since

$$\begin{aligned}
&(\mathrm{N}F_n\,\widehat{\mathrm{N}}\mathcal{B}_{a,b})\circ(\widehat{\mathrm{N}}F_b\,\mathrm{N}\mathcal{B}_a)\circ(\mathrm{N}\mathcal{B}'_b\,\widehat{\mathrm{N}}F_a) \\
&\qquad = (R'_nF_*J_nR_nb^*\mathrm{v}_qa^*J_p)\circ(R'_nb^*\mathrm{v}'_qF_*J_qR_qa^*J_p)\circ(R'_nb^*J'_qR'_qa^*\mathrm{v}'_pF_*J_p) \\
&\qquad \overset{(4.45)}{=} (R'_nb^*F_*\mathrm{v}_qa^*J_p)\circ(R'_nb^*\mathrm{v}'_qF_*J_qR_qa^*J_p)\circ(R'_nb^*J'_qR'_qa^*\mathrm{v}'_pF_*J_p) \\
&\qquad \overset{(4.46)}{=} (R'_nb^*\mathrm{v}'_qa^*F_*J_p)\circ(R'_nb^*J'_qR'_qa^*\mathrm{v}'_pF_*J_p) \\
&\qquad \overset{(4.49)}{=} (R'_nb^*a^*\mathrm{v}'_pF_*J_p)\circ(R'_nb^*\mathrm{v}'_qa^*J'_pR'_pF_*J_p) = \widehat{\mathrm{N}}F_{ab}\circ(\widehat{\mathrm{N}}\mathcal{B}'_{a,b}\,\mathrm{N}F_p).
\end{aligned}$$

To finish, let $F:\mathcal{B}\to\mathcal{B}'$ and $F':\mathcal{B}'\to\mathcal{B}''$ be lax functors. Then, $\mathrm{N}F'\,\mathrm{N}F = \mathrm{N}(F'F)$ and $\mathrm{N}1_{\mathcal{B}} = 1_{\mathrm{N}\mathcal{B}}$ since, at any $[p]$ and $a:[q]\to[p]$ in Δ, we have

$$\begin{aligned}
&\mathrm{N}F'_p\,\mathrm{N}F_p = R''F'_*J'_pR'_pF_*J_p \overset{(4.45)}{=} R''_pF'_*F_*J_p = R''_p(F'F)_*J_p = \mathrm{N}(F'F)_p, \\
&\widehat{\mathrm{N}F'\mathrm{N}}F_a = \mathrm{N}F'_q\,\widehat{\mathrm{N}}F_a\circ\widehat{\mathrm{N}}F'_a\,\mathrm{N}F_p = (R''_qF'_*J'_qR'_qa^*\mathrm{v}'_pF_*J_p)\circ(R''_qa^*\mathrm{v}''_pF'_*J'_pR'_pF_*J_p) \\
&\qquad \overset{(4.45)}{=} (R''_qa^*F'_*\mathrm{v}'_pF_*J_p)\circ(R''_qa^*\mathrm{v}''_pF'_*J'_pR'_pF_*J_p) \overset{(4.46)}{=} R''_qa^*\mathrm{v}''_pF'_*F_*J_p = \widehat{\mathrm{N}}(F'F)_a, \\
&\quad \mathrm{N}1_p = R_p1_*J_p \overset{(4.44)}{=} R_pJ_p = 1_{\mathrm{N}\mathcal{B}_p}, \\
&\quad \widehat{\mathrm{N}}1_a = R_qa^*\mathrm{v}_p1_*J_p = R_qa^*\mathrm{v}_pJ_p = R_qa^*1_{J_p} = 1_{R_qa^*J_p} = 1_{\mathrm{N}\mathcal{B}_a}.
\end{aligned}$$

This completes the proof of Theorem 4.9 and lets us prepare to prove the first part of Lemma 4.3.

Corollary 4.5 *The assignment* $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$ *is the function on objects of a functor*

$$\mathrm{B} : \mathbf{Lax} \to \mathbf{Top}.$$

Proof: By Theorem 4.9, any lax functor $F : \mathcal{B} \to \mathcal{B}'$ gives rise to a lax simplicial functor $\mathrm{N}F : \mathrm{N}\mathcal{B} \to \mathrm{N}\mathcal{B}'$, hence to a functor $\int_\Delta \mathrm{N}F : \int_\Delta \mathrm{N}\mathcal{B} \to \int_\Delta \mathrm{N}\mathcal{B}'$ and then to a cellular map $\mathrm{B}F : \mathrm{B}\mathcal{B} \to \mathrm{B}\mathcal{B}'$. For $F = 1_\mathcal{B}$, we have $\int_\Delta \mathrm{N}1_\mathcal{B} = \int_\Delta 1_{\mathrm{N}\mathcal{B}} = 1_{\int_\Delta \mathrm{N}\mathcal{B}}$, whence $\mathrm{B}1_\mathcal{B} = 1_{\mathrm{B}\mathcal{B}}$. For any other lax functor $F' : \mathcal{B}' \to \mathcal{B}''$, the equality $\mathrm{N}F'\,\mathrm{N}F = \mathrm{N}(F'F)$ gives that $\int_\Delta \mathrm{N}(F'F) = \int_\Delta \mathrm{N}F'\mathrm{N}F = \int_\Delta \mathrm{N}F' \int_\Delta \mathrm{N}F$, whence $\mathrm{B}(F'F) = \mathrm{B}F'\,\mathrm{B}F$. □

In [41, Definition 5.2], Carrasco, Cegarra, and Garzón defined the *categorical geometric nerve* of a bicategory $\mathcal{B}$ as the simplicial category

$$\underline{\Delta}\mathcal{B} : \Delta^{op} \to \mathbf{Cat}, \quad [p] \mapsto \mathbb{L}\mathrm{ax}([p], \mathcal{B}),$$

whose category of p-simplices is the category of lax functors $\mathbf{x} : [p] \to \mathcal{B}$, with relative to objects lax transformations (i.e., icons) between them as arrows. The proposition below shows how $\underline{\Delta}\mathcal{B}$ relates with the Grothendieck nerve $\mathrm{N}\mathcal{B}$.

Proposition 4.2 *For any bicategory* $\mathcal{B}$*, there is a lax simplicial functor*

$$R = (R, \widehat{R}) : \underline{\Delta}\mathcal{B} \to \mathrm{N}\mathcal{B} \tag{4.50}$$

inducing a homotopy equivalence

$$\mathrm{B}{\textstyle\int_\Delta} R : \ \mathrm{B}{\textstyle\int_\Delta}\underline{\Delta}\mathcal{B} \xrightarrow{\sim} \mathrm{B}{\textstyle\int_\Delta}\mathrm{N}\mathcal{B} = \mathrm{B}\mathcal{B}, \tag{4.51}$$

which is natural in $\mathcal{B}$ *on lax functors. That is, for any lax functor* $F : \mathcal{B} \to \mathcal{B}'$*, the square of spaces below commutes.*

$$\begin{tikzcd}
\mathrm{B}\int_\Delta\underline{\Delta}\mathcal{B} \arrow[r, "\mathrm{B}\int_\Delta R"] \arrow[d, "\mathrm{B}\int_\Delta\underline{\Delta}F"'] & \mathrm{B}\mathcal{B} \arrow[d, "\mathrm{B}F"] \\
\mathrm{B}\int_\Delta\underline{\Delta}\mathcal{B}' \arrow[r, "\mathrm{B}\int_\Delta R'"] & \mathrm{B}\mathcal{B}'
\end{tikzcd} \tag{4.52}$$

Proof: At any object $[p]$ of the simplicial category, R is given by the functor in (4.47)

$$R_p : \underline{\Delta}\mathcal{B}_p = \mathbb{L}\mathrm{ax}([p], \mathcal{B}) \longrightarrow \mathbb{L}\mathrm{ax}(\mathcal{G}_p, \mathcal{B}) = \mathrm{N}\mathcal{B}_p,$$

and, at any map $a : [q] \to [p]$, the natural transformation

$$\begin{tikzcd}
\underline{\Delta}\mathcal{B}_p \arrow[r, "a^*"] \arrow[d, "R_p"'] \arrow[dr, phantom, "\overset{\widehat{R}_a}{\Rightarrow}"] & \underline{\Delta}\mathcal{B}_q \arrow[d, "R_q"] \\
\mathrm{N}\mathcal{B}_p \arrow[r, "\mathrm{N}\mathcal{B}_a"'] & \mathrm{N}\mathcal{B}_q,
\end{tikzcd}$$

is defined by $\mathrm{N}\mathcal{B}_a R_p = R_q a^* J_p R_p \xRightarrow{\widehat{R}_a = R_q a^* \mathrm{v}_p} R_q a^*$. When $a = 1_{[p]}$, clearly $\widehat{R}_{1_{[p]}} = R_p \mathrm{v}_p = 1_{R_p}$ and, for any $b : [n] \to [q]$, the commutativity coherence condition

$$\begin{tikzcd}
\mathrm{N}\mathcal{B}_b\, \mathrm{N}\mathcal{B}_a\, R_p \arrow[r, Rightarrow, "\mathrm{N}\mathcal{B}_b \widehat{R}_a"] \arrow[d, Rightarrow, "\widehat{\mathrm{N}}\mathcal{B}_{a,b} R_p"'] & \mathrm{N}\mathcal{B}_b\, R_q\, a^* \arrow[d, Rightarrow, "\widehat{R}_b a^*"] \\
\mathrm{N}\mathcal{B}_{ab}\, R_p \arrow[r, Rightarrow, "\widehat{R}_{ab}"] & R_n b^* a^* = R_n (ab)^*,
\end{tikzcd}$$

holds since, by (4.49), $R_n b^* a^* \mathrm{v}_p \circ R_n b^* \mathrm{v}_q a^* J_p R_p = R_n b^* \mathrm{v}_q a^* \circ R_n b^* J_q R_q a^* \mathrm{v}_p$.

By [109, Corollary 1], every functor $R_p : \underline{\Delta}\mathcal{B}_p \to \mathrm{N}\mathcal{B}_p$ induces a homotopy equivalence on classifying spaces $\mathrm{B}R_p : \mathrm{B}\underline{\Delta}\mathcal{B}_p \xrightarrow{\sim} \mathrm{BN}\mathcal{B}_p$ since it has the functor J_p in (4.47) as a left adjoint. Then, the induced map in (4.51) is actually a homotopy equivalence by [120, Corollary 3.3.1].

Now let $F : \mathcal{B} \to \mathcal{B}'$ be any lax functor. Then, the square

$$\begin{tikzcd}
\underline{\Delta}\mathcal{B} \arrow[r, "R"] \arrow[d, "\underline{\Delta}F"'] & \mathrm{N}\mathcal{B} \arrow[d, "\mathrm{N}F"] \\
\underline{\Delta}\mathcal{B}' \arrow[r, "R'"] & \mathrm{N}\mathcal{B}'
\end{tikzcd}$$

commutes since, for any integer $p \geq 0$ and $a : [q] \to [p]$, we have

$$\begin{aligned}
&\mathrm{N}F_p\, R_p = R'_p F_* J_p R_p \overset{(4.45)}{=} R'_p F_* = R'_p\, \underline{\Delta}F_p,\\
&\widehat{\mathrm{N}F R}_a = \mathrm{N}F_q \widehat{R}_a \circ \widehat{\mathrm{N}}F_a R_p = R'_q F_* J_q R_q a^* \mathrm{v}_p \circ R'_q a^* \mathrm{v}'_p F_* J_p R_p\\
&\qquad \overset{(4.45)}{=} R'_q a^* F_* \mathrm{v}_p \circ R'_q a^* \mathrm{v}'_p F_* J_p R_p \overset{(4.46)}{=} R'_q a^* \mathrm{v}'_p F_* = \widehat{R}_a F_* = \widehat{R'\underline{\Delta}F}_a.
\end{aligned}$$

Hence, the commutativity of the square (4.52) follows:

$$\begin{aligned}
\mathrm{B}F\ \mathrm{B}{\textstyle\int_\Delta} R &= \mathrm{B}{\textstyle\int_\Delta}\mathrm{N}F\ \mathrm{B}{\textstyle\int_\Delta} R = \mathrm{B}({\textstyle\int_\Delta}\mathrm{N}F\ {\textstyle\int_\Delta} R) = \mathrm{B}\,{\textstyle\int_\Delta}(\mathrm{N}F\, R)\\
&= \mathrm{B}\,{\textstyle\int_\Delta}(R'\,\underline{\Delta}F) = \mathrm{B}({\textstyle\int_\Delta} R'\ {\textstyle\int_\Delta}\underline{\Delta}F) = \mathrm{B}{\textstyle\int_\Delta} R'\ \mathrm{B}{\textstyle\int_\Delta}\underline{\Delta}F.
\end{aligned}$$

□

We are now ready to complete the proof of Lemmas 4.3 and 4.4.

Corollary 4.6 *For any bicategory $\mathcal{B}$, there is a homotopy equivalence*

$$\kappa : |\Delta\mathcal{B}| \xrightarrow{\sim} \mathrm{B}\mathcal{B}, \tag{4.53}$$

which is homotopy natural on lax functors. That is, for any lax functor $F : \mathcal{B} \to \mathcal{B}'$, there is a homotopy $\kappa' \, |\Delta F| \Rightarrow \mathrm{B}F \, \kappa$,

$$\begin{tikzcd} |\Delta\mathcal{B}| \arrow[r, "\kappa"] \arrow[d, "|\Delta F|"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{B}\mathcal{B} \arrow[d, "\mathrm{B}F"] \\ |\Delta\mathcal{B}'| \arrow[r, "\kappa'"] & \mathrm{B}\mathcal{B}'. \end{tikzcd} \tag{4.54}$$

Proof: Let $\mathrm{N}\underline{\Delta}\mathcal{B} : \Delta^{op} \to \mathbf{SimplSet}$ be the bisimplicial set obtained from the simplicial category $\underline{\Delta}\mathcal{B} : \Delta^{op} \to \mathbf{Cat}$ with the nerve of categories functor $\mathrm{N} : \mathbf{Cat} \to \mathbf{SimplSet}$.

As $\Delta\mathcal{B}$ is the simplicial set of objects of the simplicial category $\underline{\Delta}\mathcal{B}$, if we regard $\Delta\mathcal{B}$ as a discrete simplicial category (i.e., with only identities as arrows), we have a simplicial category inclusion map $\Delta\mathcal{B} \hookrightarrow \underline{\Delta}\mathcal{B}$, whence a bisimplicial inclusion map $\mathrm{N}\Delta\mathcal{B} \hookrightarrow \mathrm{N}\underline{\Delta}\mathcal{B}$, where $\mathrm{N}\Delta\mathcal{B}$ is the bisimplicial set that is constant the simplicial set $\Delta\mathcal{B}$ in the vertical direction. Then, we have an induced simplicial set map on diagonals $i : \Delta\mathcal{B} \to \mathrm{diag}\,\mathrm{N}\underline{\Delta}\mathcal{B}$. This map is clearly natural in $\mathcal{B}$ on lax functors and, by [41, Theorem 6.2], it induces a homotopy equivalence on geometric realizations. Furthermore, a result by Bousfield and Kan [20, Chap. XII, 4.3] and Thomason's Homotopy Colimit Theorem [120] give us the existence of simplicial maps $\mu : \mathrm{hocolim}\,\mathrm{N}\underline{\Delta}\mathcal{B} \to \mathrm{diag}\,\mathrm{N}\underline{\Delta}\mathcal{B}$ and $\eta : \mathrm{hocolim}\,\mathrm{N}\underline{\Delta}\mathcal{B} \to \mathrm{N}\int_\Delta \underline{\Delta}\mathcal{B}$, which are natural on lax functors and both induce homotopy equivalences on geometric realizations.

We then have a chain of homotopy equivalences between spaces

$$|\Delta\mathcal{B}| \xrightarrow{|i|} |\mathrm{diag}\,\mathrm{N}\underline{\Delta}\mathcal{B}| \xleftarrow{|\mu|} |\mathrm{hocolim}\,\mathrm{N}\underline{\Delta}\mathcal{B}| \xrightarrow{|\eta|} \mathrm{B}\textstyle\int_\Delta \underline{\Delta}\mathcal{B} \xrightarrow{\mathrm{B}\int_\Delta R} \mathrm{B}\mathcal{B},$$

where the last one on the right is the homotopy equivalence (4.50), all of them natural on lax functors $F : \mathcal{B} \to \mathcal{B}'$. Therefore, taking $|\mu|^\bullet : |\mathrm{diag}\,\mathrm{N}\underline{\Delta}\mathcal{B}| \to |\mathrm{hocolim}\,\mathrm{N}\underline{\Delta}\mathcal{B}|$ to be any homotopy inverse map of $|\mu|$, we have a homotopy equivalence

$$\kappa = \mathrm{B}\textstyle\int_\Delta R \cdot |\eta| \cdot |\mu|^\bullet \cdot |i| : |\Delta\mathcal{B}| \xrightarrow{\sim} \mathrm{B}\mathcal{B},$$

which is homotopy natural on lax functors, as required. □

Corollary 4.7 *If $F, F' : \mathcal{B} \to \mathcal{B}'$ are two lax functors between bicategories, then any lax or oplax transformation between them $\alpha : F \Rightarrow F'$ determines a homotopy, $\mathrm{B}\alpha : \mathrm{B}F \Rightarrow \mathrm{B}F' : \mathrm{B}\mathcal{B} \to \mathrm{B}\mathcal{B}'$, between the induced maps on classifying spaces.*

Proof: In the proof of [41, Proposition 7.1 (ii)] it is proven that any $\alpha : F \Rightarrow G$ gives rise to a homotopy $H(\alpha) : |\Delta F| \Rightarrow |\Delta F'| : |\Delta\mathcal{B}| \to |\Delta\mathcal{B}'|$. Then, a homotopy $\mathrm{B}\alpha : \mathrm{B}F \Rightarrow \mathrm{B}F'$ is obtained as the composite of the homotopies

$$\mathrm{B}F \Longrightarrow \mathrm{B}F\kappa\kappa^\bullet \overset{(4.54)}{\Longrightarrow} \kappa'|\Delta F|\kappa^\bullet \overset{\kappa' H(\alpha)\kappa^\bullet}{\Longrightarrow} \kappa'|\Delta F'|\kappa^\bullet \overset{(4.54)}{\Longrightarrow} \mathrm{B}F'\kappa\kappa^\bullet \Longrightarrow \mathrm{B}F',$$

where $\kappa^\bullet$ is a homotopy inverse of the homotopy equivalence $\kappa : |\Delta\mathcal{B}| \to \mathrm{B}\mathcal{B}$ in (4.53).
$\square$

Resumen

Tras el reconocido artículo de Quillen [109] de 1973, la teoría de homotopía de estructuras categóricas se ha convertido en una parte importante de la maquinaria para el desarrollo de la topología algebraica y la K-teoría algebraica. Esta tesis contribuye al estudio de las relaciones existentes entre ciertas categorías superiores, los tipos de homotopía de sus espacios clasificadores (también llamados realizaciones geométricas) y algunas construcciones homotópicas clásicas aplicadas a dichos tipos de homotopía.

Las estructuras categóricas de dimensión superior son una herramienta poderosa para el estudio de diversas áreas de las matemáticas. Véase por ejemplo el libro recientemente publicado *Toward Higher Categories* [8], que sitúa adecuadamente el área en contexto. Además, dichas estructuras son aplicadas en otras áreas, tales como la física teórica o las ciencias de la computación, ya que aparecen en el estudio de las teorías de campos cuánticos topológicos (o TQFT por sus siglas en inglés, véase por ejemplo [6]), o más recientemente, sirven como punto de partida para el programa Univalent Foundations y su estudio de la teoría homotópica de tipos [122].

Esta tesis está formada por cuatro capítulos principales en los que se presentan los resultados obtenidos. Se ha intentado que los capítulos puedan ser leídos independiente, aunque comparten gran parte de la nomenclatura utilizada. Quitando algunos cambios en la notación, realizados con el objetivo de unificar la presentación de la tesis, el Capítulo 1 ha sido publicado en la revista *Applied Categorical Structures* (2012) como [46], el Capítulo 2 en la revista *Algebraic and Geometric Topology* (2014) como [45], el Capítulo 3 en la revista *Journal of Homotopy and Related Structures* (2014) como [38] y el Capítulo 4 en la revista *Theory and Applications of Categories* (2015) como [47].

En el Capítulo 1, nos concentramos en ciertas categorías dobles que modelan los 2-tipos de homotopía. Una *categoría doble* (definida por Ehresmann alrededor de 1963 [62, 63]) se puede interpretar como un conjunto de 'cuadrados' cuyos vértices son objetos, y cuyos lados son dos tipos de morfismos diferentes –uno vertical y otro horizontal– de la forma

$$\begin{tikzcd}
\cdot & \cdot \arrow[l] \\
\cdot \arrow[u] \arrow[ur, phantom, "\alpha"] & \cdot \arrow[l] \arrow[u]
\end{tikzcd},$$

junto con dos composiciones como en una categoría –una vertical y la otra horizontal– obedeciendo ciertas condiciones. Cualquier categoría doble (pequeña) $\mathcal{G}$ admite una construcción conocida como su doble nervio $\mathrm{N}^{(2}\mathcal{G}$, que puede ser transformada en un conjunto simplicial y por lo tanto, en un espacio topológico. Dicho espacio es llamado su *realización geométrica* o *espacio clasificador* y denotado por $\mathrm{B}\mathcal{G}$. El conjunto simplicial así obtenido no es un complejo de Kan, y por ello es complicado trabajar con él. A pesar de esto, una condición necesaria y suficiente para que una categoría doble produzca de esta forma un complejo de Kan es muy sencilla de formular: tiene que ser un *grupoide doble* que satisfaga una *condición de relleno.* Esta condición viene a decir que dados un par de morfismos, uno vertical y el otro horizontal, que tengan un vértice en común, podemos encontrar un cuadrado que tenga dichos morfismos en el borde:

$$\begin{tikzcd}
\cdot & \cdot \arrow[l] \\
{} \arrow[u, dotted] \arrow[ru, phantom, "\exists?"] & \cdot \arrow[l, dotted] \arrow[u]
\end{tikzcd}$$

Este resultado puede verse como una versión bidimensional del conocido hecho de que el nervio de una categoría es un complejo de Kan si y sólo si la categoría es un grupoide.

Un grupoide doble verificando la condición de relleno $\mathcal{G}$ tiene asociados una serie de *grupos de homotopía* $\pi_i(\mathcal{G}, a)$, que pueden ser definidos utilizando solamente su estructura algebraica y que son triviales cuando $i \geq 3$. Un primer resultado importante es el siguiente:

> *Para todo grupoide doble que verifica la condición de relleno $\mathcal{G}$, y para todo objeto a en él, existen isomorfismos naturales $\pi_i(\mathcal{G}, a) \cong \pi_i(\mathrm{B}\mathcal{G}, \mathrm{B}a), i \geq 0$.*

Así podemos hablar de la clase de *equivalencias débiles* entre tales grupoides dobles, es decir funtores dobles $F : \mathcal{G} \to \mathcal{G}'$ que inducen isomorfismos en los grupos de homotopía $\pi_i F : \pi_i(\mathcal{G}, a) \cong \pi_i(\mathcal{G}', Fa)$, y correspondientemente podemos definir su *categoría de homotopía* $\mathrm{Ho}(\mathbf{DG}_{\mathrm{fc}})$ como la localización de la categoría de dichos grupoides dobles con respecto a esta clase. Obtenemos de esta forma un funtor inducido

$$\mathrm{B} : \mathrm{Ho}(\mathbf{DG}_{\mathrm{fc}}) \to \mathrm{Ho}(\mathbf{Top}), \quad \mathcal{G} \mapsto \mathrm{B}\mathcal{G}\,,$$

donde $\mathrm{Ho}(\mathbf{Top})$ es la localización de la categoría de espacios topológicos con respecto a la clase de equivalencias homotópicas débiles. Además, describimos una construcción funtorial nueva para cualquier espacio topológico X, llamada su *grupoide doble de homotopía* $\Pi^{(2}X$ que induce un funtor

$$\mathrm{Ho}(\mathbf{Top}) \to \mathrm{Ho}(\mathbf{DG}_{\mathrm{fc}}), \quad X \mapsto \Pi^{(2}X.$$

El principal resultado de este capítulo establece lo siguiente

> *Los funtores $\mathcal{G} \mapsto \mathrm{B}\mathcal{G}$ y $X \mapsto \Pi^{(2}X$ inducen equivalencias mutuamente cuasi-inversas*
>
> $$\mathrm{Ho}(\mathbf{DG}_{\mathrm{fc}}) \simeq \mathrm{Ho}(\mathbf{2\text{-}types}),$$

donde Ho(**2-types**) es la subcategoría plena a la categoría de homotopía de los espacios topológicos, dada por aquellos espacios X cuyos grupos de homotopía $\pi_i(X,a)$ son triviales para $i > 2$, para todo punto a.

El Capítulo 2 se centra en el estudio de las tricategorías pequeñas, introducidas por Gordon, Power y Street en su artículo en AMS Memoir de 1995 [69]. En dicho artículo eran conscientes de que 3-grupoides estrictos no modelan 3-tipos de homotopía, y el objetivo de su trabajo era dar una definición explícita de una 3-categoría débil que no fuera equivalente (en el sentido tridimensional adecuado) a una 3-categoría estricta. Los resultados presentados en este capítulo se centran en el estudio de espacios clasificadores de tricategorías (pequeñas), con aplicaciones en la teoría de homotopía de categorías monoidales, bicategorías, categorías monoidales trenzadas y bicategorías monoidales. Cualquier tricategoría $\mathcal{T}$ tiene asociada varios objetos simpliciales o pseudo-simpliciales, y exploramos la relación entre tres de ellos: la bicategoría pseudo-simplicial llamada *nervio de Grothendieck* $\mathrm{N}\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Bicat}$, la bicategoría simplicial *nervio de Segal* $\mathrm{S}\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Hom}$, y el conjunto simplicial llamado *nervio geométrico de Street* $\Delta\mathcal{T}\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$. El principal resultado del capítulo se resume en que la realización geométrica de todos estos 'nervios de la tricategoría' son homotópicamente equivalentes, y por lo tanto podemos usar cualquiera de ellos como el espacio clasificador de la tricategoría $\mathrm{B}\mathcal{T}$. Estos nervios han sido usados recientemente por Buckley, Garner, Lack y Street en su trabajo sobre categorías skew-monoidales [34].

El nervio de Grothendieck $\mathrm{N}\mathcal{T}$ sirve como una generalización del triple nervio asociado a una 3-categoría estricta ya que asocia a cada número p la bicategoría

$$\mathrm{N}\mathcal{T}_p = \bigsqcup_{(x_0,\ldots,x_p)\in \mathrm{Ob}\mathcal{T}^{p+1}} \mathcal{T}(x_{p-1},x_p) \times \mathcal{T}(x_{p-2},x_{p-1}) \times \cdots \times \mathcal{T}(x_0,x_1).$$

Entonces, utilizando los resultados de Carrasco, Cegarra y Garzón en [42] introducimos el espacio clasificador de la tricategoría $\mathrm{B}\mathcal{T}$ de la siguiente manera: aplicamos la construcción de Grothendieck a la bicategoría pseudo-simplicial $\mathrm{N}\mathcal{T}$ para obtener un bicategoría $\int_\Delta \mathrm{N}\mathcal{T}$, usando de nuevo el nervio de Grothendieck obtenemos una categoría simplicial $\mathrm{N}(\int_\Delta \mathrm{N}\mathcal{T})$ cuya construcción de Grothendieck nos da una categoría $\int_\Delta \mathrm{N}(\int_\Delta \mathrm{N}\mathcal{T})$ y finalmente tomamos el espacio clasificador de la tricategoría como el espacio clasificador de esta categoría:

$$\mathrm{B}\mathcal{T} = |\mathrm{N}(\textstyle\int_\Delta \mathrm{N}(\int_\Delta \mathrm{N}\mathcal{T}))|$$

El comportamiento de esta construcción del espacio clasificador $\mathcal{T} \mapsto \mathrm{B}\mathcal{T}$ se puede resumir con los siguientes resultados:

> - *Cualquier trihomomorfismo* $F\colon \mathcal{T} \to \mathcal{T}'$ *induce una aplicación continua* $\mathrm{B}F\colon \mathrm{B}\mathcal{T} \to \mathrm{B}\mathcal{T}'$.
>
> - *Para cualquier pareja de trihomomorfismos componibles* $F\colon \mathcal{T} \to \mathcal{T}'$ *y* $F'\colon \mathcal{T}' \to \mathcal{T}''$*, existe una homotopía* $\mathrm{B}F'\mathrm{B}F \simeq \mathrm{B}(F'F)\colon \mathrm{B}\mathcal{T} \to \mathrm{B}\mathcal{T}''$*, y para cualquier tricategoría* $\mathcal{T}$*, existe una homotopía* $\mathrm{B}1_{\mathcal{T}} \simeq 1_{\mathrm{B}\mathcal{T}}$.

- Si $F, G\colon \mathcal{T} \to \mathcal{T}'$ son dos trihomomorfismos, entonces cualquier tritransformación $F \Rightarrow G$ induce una homotopía $\mathrm{B}F \simeq \mathrm{B}G\colon \mathrm{B}\mathcal{T} \to \mathrm{B}\mathcal{T}'$ entre las aplicaciones continuas inducidas en espacios clasificadores.

- Cualquier triequivalencia de tricategorías $\mathcal{T} \to \mathcal{T}'$ induce una equivalencia homotópica entre los espacios clasificadores $\mathrm{B}\mathcal{T} \simeq \mathrm{B}\mathcal{T}'$.

Cuando $\mathcal{T}$ es una 3-categoría estricta, el espacio $|\mathrm{diagNNN}\mathcal{T}|$ obtenido como la realización geométrica de su triple nervio es normalmente considerado como el espacio clasificador de la misma. Nosotros demostramos que existe una equivalencia homotópica

$$\mathrm{B}\mathcal{T} \simeq |\mathrm{diagNNN}\mathcal{T}|.$$

El nervio de Segal $\mathrm{S}\mathcal{T}$ asociada a cada número p la bicategoría $\mathrm{S}\mathcal{T}_p$ de trihomomorfismos unitarios desde la categoría $[p]$ a $\mathcal{T}$. Esta bicategoría simplicial es *especial* en el sentido de que las proyecciones de Segal son biequivalencias de bicategorías. Este hecho, viendo toda bicategoría monoidal $(\mathcal{B}, \otimes)$ como una tricategoría con un único objeto, nos permite demostrar lo siguiente:

Dada cualquier bicategoría monoidal $(\mathcal{B}, \times)$ tal que para todo objeto $x \in \mathcal{B}$, el homomorfismo $x \otimes -\colon \mathcal{B} \to \mathcal{B}$ induce una auto-equivalencia homotópica en el espacio clasificador $\mathrm{B}\mathcal{B}$ de la bicategoría, entonces existe una equivalencia homotópica

$$\mathrm{B}\mathcal{B} \simeq \Omega\mathrm{B}(\mathcal{B}, \otimes),$$

entre el espacio clasificador de la bicategoría subyacente y el espacio de lazos del espacio clasificador de la bicategoría monoidal, vista como una tricategoría con un sólo objeto.

De la misma forma, viendo una categoría monoidal trenzada $(\mathcal{C}, \otimes, \boldsymbol{c})$ como una tricategoría con un único objeto y una única 1-celda obtenemos el siguiente resultado conocido:

(i) Para cualquier categoría monoidal trenzada $(\mathcal{C}, \otimes, \boldsymbol{c})$, existe una equivalencia homotópica

$$\mathrm{B}(\mathcal{C}, \otimes) \simeq \Omega\mathrm{B}(\mathcal{C}, \otimes, \boldsymbol{c})$$

(ii) Si además para todo objeto $x \in \mathcal{C}$, el funtor $x \otimes -\colon \mathcal{C} \to \mathcal{C}$ induce una auto-equivalencia homotópica en el espacio clasificador de $\mathcal{C}$, entonces existe una equivalencia homotópica

$$\mathrm{B}\mathcal{C} \simeq \Omega^2\mathrm{B}(\mathcal{C}, \otimes, \boldsymbol{c})$$

El espacio clasificador $\mathrm{B}\mathcal{T}$ obtenido a través del nervio de Grothendieck es un CW-complejo cuyas celdas no tienen una conexión intuitiva con las celdas de la tricategoría. Para solucionar este problema, podemos usar el nervio geométrico de Street

$\Delta\mathcal{T}$. Dicho nervio es un conjunto simplicial, que tiene como p-símplices los funtores unitarios laxos desde la categoría $[p]$ a la tricategoría $\mathcal{T}$. Así pues, los 0-símplices no son más que los objetos de la tricategoría, los 1-símplices son las 1-celdas, mientras que los 2-símplices están dados por 2-celdas en la tricategoría de la forma

$$\begin{tikzcd}
 & F0 \arrow[dl, "F_{0,1}"'] \arrow[dr, "F_{0,2}"] \arrow[d, phantom, "\overset{F_{0,1,2}}{\Rightarrow}"] & \\
F1 \arrow[rr, "F_{1,2}"'] & & F2,
\end{tikzcd}$$

y así sucesivamente. Las celdas de la realización geométrica de este nervio $|\Delta\mathcal{T}|$ tienen así una descripción en términos de las celdas de la tricategoría, y podemos demostrar:

> *Para cualquier tricategoría $\mathcal{T}$ existe una equivalencia homotópica*
>
> $$\mathrm{B}\mathcal{T} \simeq |\Delta\mathcal{T}|.$$

Para finalizar el capítulo, damos una descripción de los 3-tipos de homotopía en función de los trigrupoides (también conocidos como tricategorías de Azumaya). Más concretamente, nos centramos en 3-tipos conexos por arcos y grupos bicategóricos, es decir, trigrupoides con un único objeto, y demostramos lo siguiente:

> *Para cualquier CW-complejo conexo por arcos X, existe un grupo bicategórico $(\mathcal{B}(X), \otimes)$ y una equivalencia homotópica $\mathrm{B}(\mathcal{B}(X), \otimes) \simeq X$ si y sólo si $\pi_i X = 0$ para $i > 3$.*

En los Capítulos 3 y 4 pasamos de modelar tipos de homotopía a utilizar estos modelos algebraicos en construcciones homotópicas que se suelen realizar con dichos tipos. Concretamente nos centraremos en cuadrados homotópicamente cartesianos. Los límites y colímites categóricos, incluyendo los cuadrados cartesianos, son una herramienta muy poderosa en teoría de categorías, con innumerables aplicaciones. Desgraciadamente, se comportan muy mal en términos de la teoría de homotopía, esto es, si reemplazamos nuestro diagrama original por uno homotópicamente equivalente a él, los límites (o colímites) correspondientes no son necesariamente homotópicamente equivalentes. Ése es el motivo por el que se estudian límites y colímites homotópicos.

Los Teoremas A y B (probados por Quillen [109]) son el punto de partida con los que Quillen dio una descripción homotópico-teórica de la K-teoría algebraica superior, y ahora son dos de los teoremas más importantes en los fundamentos de la teoría de homotopía. El Capítulo 3 de la tesis se centra en la generalización de dichos teoremas a funtores laxos entre bicategorías (definidas por Bénabou alrededor de 1967 [15]), que incluyen tanto a categorías monoidales como a 2-categorías. Dado un funtor laxo $F\colon \mathcal{A} \to \mathcal{B}$, y un objeto $b \in \mathcal{B}$ podemos asociarles una *bicategoría fibra homotópica* $F{\downarrow}_b$ cuyos objetos son las 1-celdas $f\colon Fa \to b$ en $\mathcal{B}$. En particular para $F = 1_{\mathcal{B}}$ tenemos las bicategorías coma $\mathcal{B}{\downarrow}_b$, entonces demostramos el Teorema B:

Dado cualquier objeto b de $\mathcal{B}$, el cuadrado inducido

$$\begin{tikzcd}
\mathrm{B}(F{\downarrow}_b) \arrow[r] \arrow[d] & \mathrm{B}(\mathcal{B}{\downarrow}_b) \arrow[d] \\
\mathrm{B}\mathcal{A} \arrow[r, "\mathrm{B}F"] & \mathrm{B}\mathcal{B}
\end{tikzcd}$$

es homotópicamente cartesiano si y sólo si todas las aplicaciones contínuas $\mathrm{B}p\colon \mathrm{B}(F{\downarrow}_b) \to \mathrm{B}(F{\downarrow}_{b'})$ *inducidas por una 1-celda* $p\colon b \to b'$ *en* $\mathcal{B}$ *son equivalencias homotópicas.*

Dado que los espacios $\mathrm{B}(\mathcal{B}{\downarrow}_b)$ son contráctiles, el resultado anterior nos dice que el espacio clasificador de la bicategoría $F{\downarrow}_b$ es homotópicamente equivalente a la fibra homotópica de la aplicación $\mathrm{B}F\colon \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ en el punto $\mathrm{B}b$. Además, como consecuencia obtenemos el Teorema A:

Si todos los espacios $\mathrm{B}(F{\downarrow}_b)$ *son contráctiles, la aplicación* $\mathrm{B}F\colon \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ *es una equivalencia homotópica.*

Es importante resaltar que el proceso de tomar bicategorías fibras homotópicas de un funtor laxo $F{\downarrow}\colon b \mapsto F{\downarrow}_b$ es más complicado que para funtores ordinarios entre categorías, ya que nos vemos obligados a trabajar con *bidiagramas laxos de bicategorías*

$$\mathfrak{F}\colon \mathcal{B} \to \mathbf{Bicat}, \quad b \mapsto \mathfrak{F}_b,$$

que son un tipo de trihomomorfismo desde la bicategoría $\mathcal{B}$ a la tricategoría de bicategorías. Una *construcción de Grothendieck* superior para tales bidiagramas $\int_{\mathcal{B}} \mathfrak{F}$ nos lleva a enunciar y demostrar una versión bicategórica del Lema Homotópico de Quillen [109] que, al igual que ocurre con funtores entre categorías, es un resultado clave en la demostración de los Teoremas A y B:

Si $\mathfrak{F}\colon \mathcal{B} \to \mathbf{Bicat}$ *es un bidiagrama laxo de bicategorías tal que cada 1-celda* $p\colon b \to b'$ *en* $\mathcal{B}$ *induce una equivalencia homotópica* $\mathrm{B}\mathfrak{F}_b \simeq \mathrm{B}\mathfrak{F}_{b'}$*, entonces para cada objeto* b *de* $\mathcal{B}$*, existe un cuadrado homotópicamente cartesiano*

$$\begin{tikzcd}
\mathrm{B}\mathfrak{F}_b \arrow[r] \arrow[d] & \mathrm{B}\int_{\mathcal{B}} \mathfrak{F} \arrow[d] \\
pt \arrow[r, "\mathrm{B}b"] & \mathrm{B}\mathcal{B}.
\end{tikzcd}$$

Es decir, el espacio clasificador $\mathrm{B}\mathfrak{F}_b$ *es homotópicamente equivalente a la fibra homotópica de la aplicación inducida en espacios clasificadores por la proyección* $\int_{\mathcal{B}} \mathfrak{F} \to \mathcal{B}$*.*

Yendo un paso más allá, en el Capítulo 4 tratamos cuadrados homotópicamente cartesianos en general. Dados un funtor laxo y un funtor oplaxo entre bicategorías con el mismo codominio

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}',$$

construimos una *bicategoría producto homotópicamente fibrado* $F{\downarrow}F'$ y obtenemos los siguientes resultados:

> — *Para cualquier diagrama de bicategorías* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$ *con* F *un funtor laxo y* F' *uno oplaxo, existe una aplicación inducida* $\mathrm{B}(F \downarrow F') \to \mathrm{B}\mathcal{A} \times^h_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$ *desde el espacio clasificador de la bicategoría producto homotópicamente fibrado en el espacio producto homotópicamente fibrado inducido por las aplicaciones* $\mathrm{B}F$ *y* $\mathrm{B}F'$*.*
>
> — *Dado un funtor laxo* $F\colon \mathcal{A} \to \mathcal{B}$*, las siguientes propiedades son equivalentes:*
>
> - *Para cualquier funtor oplaxo* $F'\colon \mathcal{A}' \to \mathcal{B}$*, la aplicación* $\mathrm{B}(F{\downarrow}F') \to \mathrm{B}\mathcal{A} \times^h_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$ *es una equivalencia homotópica.*
> - *Para cualquier 1-celda* $b \to b'$ *de* $\mathcal{B}$*, la aplicación* $\mathrm{B}(F{\downarrow}_b) \to \mathrm{B}(F{\downarrow}_{b'})$ *es una equivalencia homotópica*
> - *Para cualquier objeto* b *de* $\mathcal{B}$*, la aplicación* $\mathrm{B}(F{\downarrow}_b) \to \mathrm{Fib}(\mathrm{B}F, \mathrm{B}b)$ *a la fibra homotópica de* $\mathrm{B}F$ *en* $\mathrm{B}b$ *es una equivalencia homotópica.*

así como un resultado dual para funtores oplaxos $F'\colon \mathcal{A}' \to \mathcal{B}$.

Este resultado generaliza el Teorema B del Capítulo 3 y extiende de forma similar resultados recientes sobre diagramas de categorías debidos a Cisinski (2006) [55], y Barwick y Kan (2011) [13]. Además, la categoría de 2-categorías (estrictas) y 2-funtores tiene una estructura de modelos de tipo Thomason, tal y como anunciaron Worytkiewicz, Hess, Parent y Tonks (2007) en [125] y demostraron Ara y Maltsiniotis (2014) en [2], tal que el funtor espacio clasificador induce una equivalencia en las teorías de homotopía entre 2-categorías y espacios topológicos. Así, al restringir nuestros resultados a 2-categorías, encontramos una interpretación natural de nuestra construcción como un cuadrado homotópicamente cartesiano en la estructura de modelos de Thomason. De la misma forma, gracias a la equivalencia entre la categoría de módulos cruzados (sobre grupoides) y la categoría de 2-grupoides, podemos aplicarlos también en términos de la estructura de modelos para complejos cruzados definida por Brown y Golasinski (1989) [26]. Y también, dado que una categoría monoidal puede ser vista como una bicategoría con un único objeto, nuestros resultados se aplican también a categorías monoidales.

El Capítulo 4 también incluye algunos resultados nuevos relativos a espacios clasificadores de bicategorías, que son necesarios en este estudio para obtener los resultados principales del capítulo. El desarrollo del mismo es un gran ejemplo de cuán útil resulta establecer la relación entre los diferentes nervios, en este caso para bicategorías, para poder trabajar al mismo tiempo con funtores laxos y oplaxos. El nervio geométrico de Street de una bicategoría $\Delta\mathcal{B}$ es normalmente el más sencillo para trabajar, pero solamente es funtorial respecto a funtores laxos. También hay un nervio op-geométrico $\nabla\mathcal{B}$, que es funtorial respecto a funtores oplaxos. En el Apéndice del Capítulo 4 completamos el trabajo de Carrasco, Cegarra y Garzón [41] demostrando nuevos resultados de naturalidad para la comparación entre los nervios de bicategorías

$$|\Delta\mathcal{B}| \simeq \mathrm{B}\mathcal{B} \simeq |\nabla\mathcal{B}|$$

demostrando

> *Para cualquier bicategoría $\mathcal{B}$, la equivalencia homotópica $|\Delta\mathcal{B}| \simeq \mathrm{B}\mathcal{B}$ es homotópicamente natural con respecto a funtores laxos y la equivalencia $\mathrm{B}\mathcal{B} \simeq |\nabla\mathcal{B}|$ es homotópicamente natural con respecto a funtores oplaxos.*

Estos resultados siguen ideas similares a las usadas en el Capítulo 2 para el estudio de los nervios de tricategorías.

Posibles investigaciones futuras

El programa de hallar modelos para tipos de homotopía a través estructuras algebraicas está aún en pleno desarrollo. Algunos artículos recientemente publicados sobre el tema son [19, 78, 87]. En relación con los temas estudiados en esta tesis se encuentran los siguientes problemas abiertos:

- La conjetura de Brown: los grupoides dobles modelan 2-tipos de homotopía. Es decir, ¿es posible prescindir de la condición de relleno? Para ello habría que demostrar que todo grupoide doble es homotópicamente equivalente a un grupoide doble satisfaciendo la condición de relleno. Aún mejor sería la descripción de una estructura de modelos de Quillen en la categoría de grupoides dobles, en la que los objetos fibrados sean los que satisfacen dicha condición.

- Extensión de estos resultados a multigrupoides generales. En particular, ¿qué condición de relleno hay que exigir a un multigrupoide para que su nervio se convierta en un complejo de Kan?

- La construcción de espacios topológicos de bicategorías y tricategorías permite transformar coherencia categórica en coherencia homotópica. Es decir, las transformaciones entre ellas se transforman en homotopías entre sus espacios clasificadores. Esto apunta a la existencia de funtores

  $$\mathbf{Bicat} \to \infty\text{-}\mathbf{Top}$$

  $$\mathbf{Tricat} \to \infty\text{-}\mathbf{Top}$$

  cuyo codominio es la ∞-categoría de espacios topológicos. Una descripción de dichos funtores sería sin duda interesante.

- El estudio de cuadrados homotópicamente cartesianos entre tricategorías podría seguir los mismos derroteros llevados a cabo en el estudio del caso bicategórico. De la misma forma, usando métodos para el estudio de colímites homotópicos categóricos [120] se podría hacer un estudio de cuadrados homotópicamente cocartesianos entre tricategorías.
- Para ciertas definiciones de n-categorías débiles, existen diversas nociones no simpliciales de nervio [18]. Una comparación homotópica de dichos nervios podría ser útil para el estudio de la relación entre las diferentes definiciones de n-categorías débiles.

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