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Research Article

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Real Hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting shape operator

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Abstract: In this paper we prove a non-existence of real hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$, $m{\geq}3$, whose structure tensors $\{\phi_i\}_{i=1,2,3}$ commute with the shape operator.

Keywords: Real hypersurfaces, Complex hyperbolic two-plane Grassmannians, Commuting shape operator

MSC: 53C40, 53C15, 53C26

Introduction

It is one of the main topics in submanifold geometry to investigate an immersed real hypersurface in Hermitian symmetric spaces of rank 2 (HSS2) with certain geometric condition. Understanding and classifying real hypersurfaces in HSS2 is one of the important subjects in differential geometry. One of these spaces is complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. For indefinite complex Euclidean spaces, we give a definition of a complex hyperbolic two-plane Grassmannian, the set of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$ denoted by $SU_{2,m}/S(U_2{\cdot}U_m)$. This Riemannian symmetric space has a remarkable geometrical structure. It is the unique noncompact, Kähler, irreducible, quaternionic Kähler manifold with negative scalar curvature.

These are typical examples of HSS2. Characterizing typical model spaces of real hypersurfaces under certain geometric conditions has been one of our main interests in the classification theory in $G_2(\mathbb{C}^{m+2})$ (see [1]).

Now, thanks to Berndt and Suh [2], comparing to $G_2(\mathbb{C}^{m+2})$ with compact type, we have investigated geometry of submanifolds in $SU_{2,m}/S(U_2{\cdot}U_m)$. In the noncompact ambient space, we may find various types of hypersurfaces due to horospheres and an exceptional case.

Let M be a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, and let us denote by N a local unit normal vector field on M. Since $SU_{2,m}/S(U_2{\cdot}U_m)$ has the Kähler structure J, we may define a *Reeb vector field* $\xi = -JN$ and a 1-dimensional distribution $[\xi] = \text{Span}\{\xi\}$.

Let $\mathcal{C}$ be a distribution which stands for the orthogonal complement of $[\xi]$ in T_xM for any $x \in M$. It becomes the complex maximal subbundle of T_xM. Thus the tangent space of M consists of the direct sum of $\mathcal{C}$ and $\mathcal{C}^{\perp}(:= [\xi])$ as follows: $T_xM = \mathcal{C} \oplus \mathcal{C}^{\perp}$ for any $x \in M$. The real hypersurface M is said to be *Hopf* if $A\mathcal{C} \subset \mathcal{C}$, or equivalently, the Reeb vector field ξ is principal with principal curvature $\alpha = g(A\xi, \xi)$, where A denotes the shape operator of M with respect to N. In this case, the principal curvature $\alpha = g(A\xi, \xi)$ is said to be a *Reeb curvature* of M.

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From the quaternionic Kähler structure $\mathfrak{J} = \mathrm{Span}\{J_1, J_2, J_3\}$ of $SU_{2,m}/S(U_2{\cdot}U_m)$, there naturally exist *almost contact 3-structure* vector fields $\xi_i = -J_i N$, $i = 1,2,3$. Put $\mathcal{Q}^{\perp} = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$, which is a 3-dimensional distribution in the tangent vector space T_xM of M at $x \in M$. In addition, $\mathcal{Q}$ stands for the orthogonal complement of $\mathcal{Q}^{\perp}$ in T_xM. It becomes the quaternionic maximal subbundle of T_xM. Thus the tangent space of M consists of the direct sum of $\mathcal{Q}$ and $\mathcal{Q}^{\perp}$ as follows: $T_xM = \mathcal{Q} \oplus \mathcal{Q}^{\perp}$.

Thus we introduce the main two natural geometric conditions for real hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$, that the subbundles $\mathcal{C}$ and $\mathcal{Q}$ of TM are both invariant under the shape operator. By using these geometric conditions and the results in Eberlein [3], Berndt and Suh [2] proved the following:

Theorem A. *Let M be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 2$. Then the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M if and only if M is locally congruent to an open part of one of the following hypersurfaces:*

(A) *a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$;*

(B) *a tube around a totally geodesic $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$;*

(C) *a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular;*

or the following exceptional case holds:

(D) *The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$. Moreover, M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \sqrt{2}\,,\ \gamma = 0\,,\ \lambda = \frac{1}{\sqrt{2}},$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q})\,,\ T_\gamma = J(TM \ominus \mathcal{Q})\,,\ T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then we have $T_\mu \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}$, $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$.

Suh [7] has given a characterization of real hypersurfaces of type (A) when the shape operator A of M in $SU_{2,m}/S(U_2{\cdot}U_m)$ commutes with the structure tensor ϕ. The condition is said to be an isometric Reeb flow on M. Now in this paper we consider another commuting condition, that is, commuting shape operator which is defined by

$$A\phi_i X = \phi_i AX, \qquad i = 1,2,3, \tag{*}$$

where $\phi_i X$ denotes the tangential part of $J_i X$, $i = 1,2,3$ for the quaternionic Kähler structure $\mathfrak{J} = \mathrm{Span}\{J_1, J_2, J_3\}$ for $SU_{2,m}/S(U_2{\cdot}U_m)$.

Then we can assert the following without the assumption of Hopf.

Theorem 1. *There does not exist any real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, with commuting shape operator i.e., $A\phi_i = \phi_i A$, $i = 1,2,3$.*

On the other hand, let us consider a weaker condition than the above assumption, that is, $A\phi_i = \phi_i A$ on the distribution $\mathcal{C} = [\xi]^{\perp}$. Then with the assumption of Hopf, we can assert another theorem as follows:

Theorem 2. *There does not exist any Hopf hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, with commuting shape operator i.e., $A\phi_i = \phi_i A$, $i = 1,2,3$ on the distribution $\mathcal{C}$.*

Throughout this paper, we use some references [2], [7], [8], and [9] to recall the Riemannian geometry of $SU_{2,m}/S(U_2{\cdot}U_m)$ and some fundamental formulas including the Codazzi and Gauss equations for a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$.

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1 The complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$

In this section we summarize basic material about complex hyperbolic two-plane Grassmann manifolds $SU_{2,m}/S(U_2{\cdot}U_m)$, for details we refer to [1], [2], [4], [5], [7] and [8].

The Riemannian symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$, which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$, becomes a connected, simply connected, irreducible Riemannian symmetric space of noncompact type and with rank two. Let $G = SU_{2,m}$ and $K = S(U_2{\cdot}U_m)$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebra of the Lie group G and K respectively. Let B be the Killing form of $\mathfrak{g}$ and denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The Cartan involution $\theta \in \mathrm{A}ut(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m}AI_{2,m}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}$$

I_2 and I_m denotes the identity 2×2-matrix and $m \times m$-matrix respectively. Then $< X, Y >= -B(X, \theta Y)$ becomes a positive definite $\mathrm{A}d(K)$-invariant inner product on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ induces a metric g on $SU_{2,m}/S(U_2{\cdot}U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2{\cdot}U_m)$. Throughout this paper we consider $SU_{2,m}/S(U_2{\cdot}U_m)$ together with this particular Riemannian metric g.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where $\mathfrak{u}_1$ is the one-dimensional center of $\mathfrak{k}$. The adjoint action of $\mathfrak{su}_2$ on $\mathfrak{p}$ induces the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2{\cdot}U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2} I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2} I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure J on $SU_{2,m}/S(U_2{\cdot}U_m)$. By construction, J commutes with each almost Hermitian structure J_i in $\mathfrak{J}$ for $i = 1, 2, 3$. Recall that a canonical local basis $\{J_1, J_2, J_3\}$ of a quaternionic Kähler structure $\mathfrak{J}$ consists of three almost Hermitian structures J_1, J_2, J_3 in $\mathfrak{J}$ such that $J_i J_{i+1} = J_{i+2} = -J_{i+1} J_i$, where the index i is to be taken modulo 3. The tensor field JJ_i, which is locally defined on $SU_{2,m}/S(U_2{\cdot}U_m)$, is self-adjoint and satisfies $(JJ_i)^2 = I$ and $\mathrm{t}r(JJ_i) = 0$, where I is the identity transformation. For a nonzero tangent vector X we define $\mathbb{R}X = \{\lambda X | \lambda \in \mathbb{R}\}$, $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$, and $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$.

We identify the tangent space $T_o SU_{2,m}/S(U_2{\cdot}U_m)$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ at o with $\mathfrak{p}$ in the usual way. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Since $SU_{2,m}/S(U_2{\cdot}U_m)$ has rank two, the dimension of any such subspace is two. Every nonzero tangent vector $X \in T_o SU_{2,m}/S(U_2{\cdot}U_m) \cong \mathfrak{p}$ is contained in some maximal abelian subspace of $\mathfrak{p}$. Generically this subspace is uniquely determined by X, in which case X is called regular. If there exists more than one maximal abelian subspaces of $\mathfrak{p}$ containing X, then X is called singular. There is a simple and useful characterization of the singular tangent vectors: A nonzero tangent vector $X \in \mathfrak{p}$ is singular if and only if $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

Up to scaling there exists a unique $SU_{2,m}$-invariant Riemannian metric g on $SU_{2,m}/S(U_2{\cdot}U_m)$. Equipped with this metric $SU_{2,m}/S(U_2{\cdot}U_m)$ is a Riemannian symmetric space of rank two which is both Kähler and quaternionic Kähler. For computational reasons we normalize g such that the minimal sectional curvature of $(SU_{2,m}/S(U_2{\cdot}U_m), g)$ is -4. The sectional curvature K of the noncompact symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$ equipped with the Killing metric g is bounded by $-4 \le K \le 0$. The sectional curvature -4 is obtained for all 2-planes $\mathbb{C}X$ when X is a non-zero vector with $JX \in \mathfrak{J}X$.

When $m = 1$, $G_2^*(\mathbb{C}^3) = SU_{1,2}/S(U_1{\cdot}U_2)$ is isometric to the two-dimensional complex hyperbolic space $\mathbb{C}H^2$ with constant holomorphic sectional curvature -4.

When $m = 2$, we note that the isomorphism $SO(4, 2) \simeq SU_{2,2}$ yields an isometry between $G_2^*(\mathbb{C}^4) = SU_{2,2}/S(U_2{\cdot}U_2)$ and the indefinite real Grassmann manifold $G_2^*(\mathbb{R}_2^6)$ of oriented two-dimensional linear subspaces of an indefinite Euclidean space $\mathbb{R}_2^6$. For this reason we assume $m \ge 3$ from now on, although many of the subsequent results also hold for $m = 1, 2$.

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The Riemannian curvature tensor $\bar{R}$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z = -\frac{1}{2}\Big[& g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{i=1}^{3}\{g(J_iY,Z)J_iX - g(J_iX,Z)J_iY - 2g(J_iX,Y)J_iZ\} \\ & + \sum_{i=1}^{3}\{g(J_iJY,Z)J_iJX - g(J_iJX,Z)J_iJY\}\Big],\end{aligned} \tag{1}$$

where $\{J_1, J_2, J_3\}$ is any canonical local basis of $\mathfrak{J}$ (see [2]).

2 Fundamental formulas in $SU_{2,m}/S(U_2{\cdot}U_m)$

In this section we derive some basic formulas and the Codazzi equation for a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ (see [1], [2], [8], and [9]).

Let M be a real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, that is, a hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Levi Civita covariant derivative of (M, g). We denote by $\mathcal{C}$ and $\mathcal{Q}$ the maximal complex and quaternionic subbundle of the tangent bundle TM of M, respectively. Now let us put

$$JX = \phi X + \eta(X)N, \quad J_iX = \phi_iX + \eta_i(X)N \tag{2}$$

for any tangent vector field X of a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$, where ϕX denotes the tangential component of JX and N a unit normal vector field of M in $SU_{2,m}/S(U_2{\cdot}U_m)$.

From the Kähler structure J of $SU_{2,m}/S(U_2{\cdot}U_m)$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X,\xi) \tag{3}$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_i of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_iJ_{i+1} = J_{i+2} = -J_{i+1}J_i$ in section 1, induces an almost contact metric 3-structure $(\phi_i, \xi_i, \eta_i, g)$ on M as follows:

$$\begin{aligned}&\phi_i^2X = -X + \eta_i(X)\xi_i, \quad \eta_i(\xi_i) = 1, \quad \phi_i\xi_i = 0, \\ &\phi_{i+1}\xi_i = -\xi_{i+2}, \quad \phi_i\xi_{i+1} = \xi_{i+2}, \\ &\phi_i\phi_{i+1}X = \phi_{i+2}X + \eta_{i+1}(X)\xi_i, \\ &\phi_{i+1}\phi_iX = -\phi_{i+2}X + \eta_i(X)\xi_{i+1}\end{aligned} \tag{4}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_iJ = JJ_i$, $i = 1, 2, 3$ in section 1 and (2), the relation between these two contact metric structures (ϕ, ξ, η, g) and $(\phi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$, can be given by

$$\begin{aligned}&\phi\phi_iX = \phi_i\phi X + \eta_i(X)\xi - \eta(X)\xi_i, \\ &\eta_i(\phi X) = \eta(\phi_iX), \quad \phi\xi_i = \phi_i\xi.\end{aligned} \tag{5}$$

On the other hand, from the parallelism of Kähler structure J, that is, $\widetilde{\nabla}J = 0$ and the quaternionic Kähler structure $\mathfrak{J}$ (see (1)), together with Gauss and Weingarten formulas it follows that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{6}$$

$$\nabla_X\xi_i = q_{i+2}(X)\xi_{i+1} - q_{i+1}(X)\xi_{i+2} + \phi_iAX, \tag{7}$$

$$(\nabla_X\phi_i)Y = -q_{i+1}(X)\phi_{i+2}Y + q_{i+2}(X)\phi_{i+1}Y + \eta_i(Y)AX - g(AX,Y)\xi_i. \tag{8}$$

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Combining these formulas, we find the following:

$$\begin{aligned}\nabla_X(\phi_i\xi) &= \nabla_X(\phi\xi_i) = (\nabla_X\phi)\xi_i + \phi(\nabla_X\xi_i)\\ &= q_{i+2}(X)\phi_{i+1}\xi - q_{i+1}(X)\phi_{i+2}\xi + \phi_i\phi AX - g(AX,\xi)\xi_i + \eta(\xi_i)AX.\end{aligned} \tag{9}$$

Finally, using the explicit expression for the Riemannian curvature tensor $\bar{R}$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ in [2] the Codazzi equation takes the form

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = -\frac{1}{2}\Big[&\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\\ &+ \sum_{i=1}^{3}\{\eta_i(X)\phi_i Y - \eta_i(Y)\phi_i X - 2g(\phi_i X,Y)\xi_i\} + \sum_{i=1}^{3}\{\eta_i(\phi X)\phi_i\phi Y - \eta_i(\phi Y)\phi_i\phi X\}\\ &+ \sum_{i=1}^{3}\{\eta(X)\eta_i(\phi Y) - \eta(Y)\eta_i(\phi X)\}\xi_i\Big],\end{aligned} \tag{10}$$

for any vector fields X and Y on M.

3 Proof of Theorem 1

In this section, we want to give a complete proof of our Theorem 1.

Let M be a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ satisfying

$$A\phi_i X = \phi_i AX, \tag{11}$$

where $i = 1,2,3$ for any tangent vector field X on M. By putting $X = \xi_i$ into (11), and applying ϕ_i to (11), we have

$$A\xi_i = \eta_i(A\xi_i)\xi_i = \kappa_i\xi_i. \tag{12}$$

Also by substituting $X = \xi_{i+1}$ into (11) and using (4), we have $\kappa_1 = \kappa_2 = \kappa_3$. Thus from now on we will denote $\kappa = \kappa_1 = \kappa_2 = \kappa_3$.

Remark 3.1. *By* (12)*, the commuting condition $A\phi_i = \phi_i A$, $i = 1,2,3$ naturally gives $A\mathcal{Q} \subset \mathcal{Q}$.*

Lemma 3.2. *Let M be a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$. If M has commuting shape operator, that is, $A\phi_i = \phi_i A$, $i = 1,2,3$, then the Reeb vector field ξ belongs to either the 3-dimensional distribution $\mathcal{Q}^\perp = \{\xi_1,\xi_2,\xi_3\}$ or the orthogonal complement, that is, the quaternionic maximal subbundle $\mathcal{Q}$ such that $T_xM = \mathcal{Q}\oplus\mathcal{Q}^\perp$, $x\in M$.*

Proof. By taking the inner product of the equation of Codazzi (10) with ξ_1, we obtain

$$\begin{aligned}g\big((\nabla_X A)Y,\xi_1\big) - g\big((\nabla_Y A)\xi_1,X\big) &= -\eta(X)\eta_1(\phi Y) + \eta(Y)\eta_1(\phi X) + g(\phi X,Y)\eta(\xi_1) - \eta_2(X)\eta_3(Y)\\ &\quad + \eta_3(X)\eta_2(Y) + g(\phi_1 X,Y) - \eta_2(\phi X)\eta_3(\phi Y) + \eta_2(\phi Y)\eta_3(\phi X).\end{aligned} \tag{13}$$

On the other hand, by differentiation of $A\xi_1 = \kappa\xi_1$ and using (8), we have

$$(\nabla_X A)\xi_1 = (X\kappa)\xi_1 + \kappa\phi_1 AX - A\phi_1 AX. \tag{14}$$

Interchange X and Y in (14) and combining them, we get

$$\begin{aligned}g\big((\nabla_X A)Y,\xi_1\big) - g\big((\nabla_Y A)\xi_1,X\big) =&(X\kappa)\eta_1(Y) - (Y\kappa)\eta_1(X)\\ &+ \kappa g\big((\phi_1 A + A\phi_1)X,Y\big) - 2g(A\phi_1 AX,Y)\end{aligned} \tag{15}$$

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Combining (13) and (15), we have

$$\begin{aligned} &-\eta(X)\eta_1(\phi Y)+\eta(Y)\eta_1(\phi X)+g(\phi X,Y)\eta(\xi_1)-\eta_2(X)\eta_3(Y)\\ &\quad+\eta_3(X)\eta_2(Y)+g(\phi_1 X,Y)-\eta_2(\phi X)\eta_3(\phi Y)+\eta_2(\phi Y)\eta_3(\phi X)\\ &=(X\kappa)\eta_1(Y)-(Y\kappa)\eta_1(X)+\kappa g\big((\phi_1 A+A\phi_1)X,Y\big)-2g(A\phi_1 AX,Y). \end{aligned} \tag{16}$$

Let us show the fact that $\xi \in \mathcal{Q}$ or $\xi \in \mathcal{Q}^{\perp}$ from the assumption in our lemma. In order to do this, let us put $\xi = \eta(X)X + \eta(Z)Z$ for some unit $X \in \mathcal{Q}$ and $Z \in \mathcal{Q}^{\perp}$. Then we may put $Z = \xi_3 \in \mathcal{Q}^{\perp}$ without loss of generality and then

$$\xi = \eta(X)X + \eta(\xi_3)\xi_3 \tag{**}$$

gives $\eta(\xi_1) = 0 = \eta(\xi_2)$. Thus $\xi \perp \xi_1, \xi_2$, i.e., $\xi_1, \xi_2 \in \mathcal{C}$, where $\mathcal{C}$ denotes an orthogonal complement of ξ in T_xM.

On the other hand, by the assumption of commuting property, that is, $A\phi_i = \phi_i A$, we know that $A\mathcal{Q} = \mathcal{Q}$. By virtue of this fact, for any $X \in \mathcal{Q}$ such that $AX = \lambda X$ from (16), we have

$$(\kappa-2\lambda)A\phi_1 X+(\lambda\kappa)\phi_1 X+(X\kappa)\xi_1-\eta(X)\phi\xi_1-\eta(\phi_1 X)\xi-\eta_2(\phi X)\phi\xi_3+\eta_3(\phi X)\phi\xi_2=0. \tag{17}$$

From this, taking the inner product with ξ_2 and using $X \in \mathcal{Q}$ and (**), then by the commuting condition $A\phi_1 = \phi_1 A$, we have

$$\eta(X)\eta_3(\xi) = 0. \tag{18}$$

Thus, we get a complete proof of our Lemma 3.2. □

Now let us show another lemma as follows

Lemma 3.3. *Let M be a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$ satisfying $A\phi_i = \phi_i A$, $i = 1, 2, 3$. Then the Reeb vector field ξ becomes a principal vector field.*

Proof. First, we suppose $\xi = \xi_i \in \mathcal{Q}^{\perp}$. This means $A\xi = A\xi_i = \kappa\xi_i = \kappa\xi$ and ξ is a principal vector field.

Next, in the case of $\xi \in \mathcal{Q}$, by differentiating $g(\xi, \xi_i) = 0$, we obtain

$$g(\nabla_Y \xi, \xi_i) + g(\xi, \nabla_Y \xi_i) = 0. \tag{19}$$

From this, using (6) and (7), we get $2g(\phi AY, \xi_i) = 0$ for any tangent vector field Y on M, which is equivalent to $0 = A\phi\xi_i = A\phi_i\xi = \phi_i A\xi$. Applying ϕ_i and using $\eta_i(A\xi) = 0$, we have

$$A\xi = 0. \tag{20}$$

So in both cases, the Reeb vector field ξ becomes principal, that is, $A\mathcal{C} \subset \mathcal{C}$. □

By virtue of Lemmas 3.2 and 3.3, we conclude that M has a principal Reeb vector field ξ and $g(A\mathcal{Q}, \mathcal{Q}^{\perp}) = 0$. Then by a theorem due to Benrdt and Suh [2], M is congruent to an open part of hypersurfaces either of Type (A), (B), (C), or (D) in Theorem A mentioned in the introduction. So by using Propositions in [2], we want to give a complete proof of our Theorem 1.

In the case of $\xi \in \mathcal{Q}^{\perp}$ (i.e., $JN \in \mathfrak{J}N$), type (A) (resp., type (C_1)) stands for a tube around totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ (resp., a horosphere whose center at infinity with $JX \in \mathfrak{J}X$ is singular). In [2] Berndt and Suh gave some information related to the shape operator A of type (A) and type (C_1) as follows

Proposition A. *Let M be a connected real hypersurface in complex hyperbolic two-plane Grassamnnian $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \in \mathfrak{J}N$, then M is either of type (A) or type (C_1) and*

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(A) *M has exactly four distinct constant principal curvatures*

$$\alpha = 2\coth(2r),\ \beta = \coth(r),\ \lambda_1 = \tanh(r),\ \lambda_2 = 0,$$

(r denote the radius of M centered at $SU_{2,m-1}/S(U_2U_{m-1})$) and the corresponding eigenspaces are

$$T_\alpha = TM \ominus \mathcal{C},\ T_\beta = \mathcal{C} \ominus \mathcal{Q},\ T_{\lambda_1} = E_{-1},\ T_{\lambda_2} = E_{+1}.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are complex (with respect to J) and totally complex (with respect to $\mathfrak{J}$).

(C_1) *M has exactly three distinct constant principal curvatures*

$$\alpha = 2,\ \beta = 1,\ \lambda = 0$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus \mathcal{C},\ T_\beta = (\mathcal{C} \ominus \mathcal{Q}) \oplus E_{-1},\ T_\lambda = E_{+1}.$$

Here, E_{+1} and E_{-1} are the eigenbundles of $\phi\phi_1|_{\mathcal{Q}}$ with respect to the eigenvalues $+1$ and -1, respectively.

By using Proposition A, let us check whether the shape operator A on type (A) (resp., type (C_1)) satisfies the condition $A\phi_i = \phi_i A$, $i = 1, 2, 3$.

For $i = 1$ and $X = \xi_2$ into the given condition, we have $-\alpha\xi_3 = -\beta\xi_3$. For type (A) (resp., type (C_1)), this means that $r = 0$ (resp., $-2 = -\alpha = -\beta = -1$) which makes a contradiction.

Remark 3.4. *The shape operator A of real hypersurfaces type (A) (resp., type (C_1)) in $SU_{2,m}/S(U_2{\cdot}U_m)$ does not satisfy the condition $A\phi_i = \phi_i A$, $i = 1, 2, 3$.*

Let us suppose that $\xi \in \mathcal{Q}$ (i.e., $JN \perp \mathfrak{J}N$). Related to this condition, Suh [8] proved:

Theorem B. *Let M be a Hopf hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, with the Reeb vector field belonging to the maximal quaternionic subbundle $\mathcal{Q}$. Then one of the following statements holds*

(B) *M is an open part of a tube around a totally geodesic $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$,*

(C_2) *M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of Type $JN \perp \mathfrak{J}N$, or*

(D) *The normal bundle νM of M consists of singular tangent vectors of Type $JX \perp \mathfrak{J}X$.*

By virtue of this result, we assert that a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$ satisfying the hypotheses in our Theorem 1 is locally congruent to an open part of one of the model spaces mentioned in Theorem B. Hereafter, let us check whether the shape operator A of a model space of type (B), type (C_2) or type (D) satisfies our conditions. In order to do this, let us introduce the following proposition given by Berndt and Suh [2].

Proposition B. *Let M be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \perp \mathfrak{J}N$, then one of the following statements holds:*

(B) *M has five (four for $r = \sqrt{2}\tanh^{-1}(1/\sqrt{3})$ in which case $\alpha = \lambda_2$) distinct constant principal curvatures*

$$\alpha = \sqrt{2}\tanh(\sqrt{2}r),\ \beta = \sqrt{2}\coth(\sqrt{2}r),\ \gamma = 0,$$
$$\lambda_1 = \frac{1}{\sqrt{2}}\tanh(\frac{1}{\sqrt{2}}r),\ \lambda_2 = \frac{1}{\sqrt{2}}\coth(\frac{1}{\sqrt{2}}r),$$

(r denote the radius of M centered at $\mathbb{H}H^n$) and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C},\ T_\beta = TM \ominus \mathcal{Q},\ T_\gamma = J(TM \ominus \mathcal{Q}) = JT_\beta.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are invariant under $\mathfrak{J}$ and are mapped onto each other by J. In particular, the quaternionic dimension of $SU_{2,m}/S(U_2U_m)$ must be even.

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(C_2) *M has exactly three distinct constant principal curvatures*

$$\alpha = \beta = \sqrt{2},\ \gamma = 0,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}),\ T_\gamma = J(TM \ominus \mathcal{Q}),\ T_\lambda = \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

(D) *M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \beta = \sqrt{2},\ \gamma = 0,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}),\ T_\gamma = J(TM \ominus \mathcal{Q}),\ T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$. Thus, the corresponding multiplicities are

$$m(\alpha) = 4, \quad m(\gamma) = 3, \quad m(\lambda), \quad m(\mu).$$

By using Proposition B, let us check whether the shape operator A on type (B) (resp., type (C_2) or type (D)) satisfies the condition $A\phi_i = \phi_i A$, $i = 1, 2, 3$.

For $i = 1$ and $X = \phi_1 \xi$ into the given condition, we have $\alpha = 0$. For type (B) (resp., type (C_2) or type (D)), this means that $r = 0$ (resp., $\sqrt{2} = \alpha = 0$ in type (C_2) or type (D)) which makes a contradiction.

Remark 3.5. *The shape operator A of real hypersurfaces type (B) (resp., type (C_2) or type (D)) in $SU_{2,m}/S(U_2{\cdot}U_m)$ does not satisfy the condition $A\phi_i = \phi_i A$, $i = 1, 2, 3$.*

Summing up all documents mentioned above, we give a complete proof of our Theorem 1 in the introduction.

Finally, let us mention a brief proof of our Theorem 2 in the introduction. We put

$$\xi = \eta(X)X + \eta(Z)Z \tag{21}$$

for any $X \in \mathcal{Q}$ and $Z \in \mathcal{Q}^\perp$. Then without loss of generality we are able to choose a vector Z in such a way that

$$Z = \xi_3 \in \mathcal{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}.$$

From the expression of the Reeb vector field $\xi = \eta(X)X + \eta(\xi_3)\xi_3$, it follows that $\eta(\xi_1) = 0 = \eta(\xi_2)$. This implies $\xi_1, \xi_2 \in \mathcal{C}$. From the condition that $A\phi_\nu = \phi_\nu A$, $\nu = 2, 3$ on $\mathcal{C}$, we have

$$A\xi_3 = A\phi_1\xi_2 = \phi_1 A\xi_2 = \kappa_2\phi_1\xi_2 = \kappa_2\xi_3.$$

This means that all structure vector fields ξ_i, $i = 1, 2, 3$ are principal vectors with the same principal curvatures, that is, $\kappa_1 = \kappa_2 = \kappa_3$. This implies $A\mathcal{Q} \subset \mathcal{Q}$. Then by a theorem due to Benrdt and Suh [2], M is congruent to one of an open part of hypersurfaces of Type (A), (B), (C), or (D). By using the same method as in the proof of Theorem 1, we can give a contradiction in each case mentioned above.

Remark 3.6. *In the case of compact two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, the shape operator A of real hypersurfaces of type (B) satisfies the condition $A\phi_i = \phi_i A$, $i = 1, 2, 3$. But when we consider a real hypersurfaces in non-compact two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$, the situation is different from the compact case. In non-compact case, we give a non-existence theorem for hypersurfaces satisfying the commuting condition.*

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