Isoperimetric inequalities in convex bodies

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Granada, 24 de abril de 2014

Director de la Tesis

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Dedicated to the memory of my father:

Emmanouil Vernadakis (1946-2008)

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Contents

CHAPTER 1

Introduction and preliminaries

1.1. Introduction

In this work we consider the *isoperimetric problem* of minimizing perimeter under a given volume constraint inside a *convex set* C. The perimeter considered here will be the one relative to the interior of C.

A way to deal with this problem is to consider the *isoperimetric profile* I_C of C, i.e., the function assigning to each $0 < v < |C|$ the infimum of the relative perimeter of the sets inside C of volume v. The isoperimetric profile can be interpreted as an optimal isoperimetric inequality in C. A minimum for this problem will be called an *isoperimetric region*.

The isoperimetric profile of convex bodies with smooth boundary has been intensively considered. Many results are known, such as the concavity of the isoperimetric profile, Sternberg and Zumbrun [70], the concavity of the $(\frac{n+1}{n})$ power of the isoperimetric profile, Kuwert [43], the connectedness of the reduced boundary of the isoperimetric regions [70], the behavior of the isoperimetric profile for small volumes, Bérard and Meyer [10], or the behavior of isoperimetric regions for small volumes, Fall [25]. See also [8], [9] and [54]. The results in all these papers make a strong use of the regularity of the boundary. In particular, in [70] and [43], the $C^{2,\alpha}$ regularity of the boundary implies a strong regularity of the isoperimetric regions up to the boundary, except in a singular set of large Hausdorff codimension, that allows the authors to apply the classical first and second variation formulas for volume and perimeter. The convexity of the boundary then implies the concavity of the profile and the connectedness of the regular part of the free boundary.

Up to our knowledge, the only known results for non-smooth boundary are the ones by Bokowski and Sperner [12] on isoperimetric inequalities for the Minkowski content in Euclidean convex bodies, the isoperimetric inequality for convex cones by Lions and Pacella [46] using the Brunn-Minkowski inequality, with the characterization of isoperimetric regions by Figalli and Indrei [27], the extension of Levy-Gromov inequality, [35, App. C], to arbitrary convex sets given by Morgan [52], and the extension of the concavity of the $(\frac{n+1}{n})$ power of the isoperimetric profile to arbitrary convex bodies by E. Milman [49, § 6]. In his work on the isoperimetric profile for small volumes in the *boundary* of a polytope, Morgan mentions that his techniques can be adapted to handle the case of small volumes in a solid polytope, [51, Remark 3.11], without uniqueness, see Remark after Theorem 3.8 in [51]. We recall that isoperimetric inequalities outside a convex set with smooth boundary have been obtained in [19], [17], [18]. Previous estimates on least perimeter in convex bodies have been obtained by Dyer and Frieze [22], Kannan, Lovász and Simonovits [41] and Bobkov [11]. In the initial stages of this research the authors were greatly influenced by the paper of Bokowski and Sperner [12], see also [15]. This work is divided into two different parts: in the first one the authors characterize the isoperimetric regions in a ball (for the Minkowski content) using spherical symmetrization, see also [2] and [61]. In the second part, given a convex body C so that there is a closed ball $\bar{B}(x,r) \subset C$, they build a map between $\bar{B}(x,r)$ and C, which transform the volume and the perimeter in a controlled way, allowing them to transfer the isoperimetric inequality of the ball to C. This map is not bilipschitz, but can be modified to satisfy this property.

In Chapter 2 we deal only with compact convex bodies. The contents of this Chapter corresponds to [62]. First we extend some of the results already known for Euclidean convex bodies with smooth boundary to arbitrary convex bodies, and prove new results for the isoperimetric profile. We begin by considering the Hausdorff and Lipschitz convergences in the space of convex bodies. We prove in Theorem 2.4 that a sequence C_i of convex bodies that converges to a convex body C in Hausdorff distance also converges in Lipschitz distance. This is done by considering a "natural" sequence of bilipschitz maps $f_i : C \to C_i$, defined by (2.6), and proving that $\operatorname{Lip}(f_i)$, $\operatorname{Lip}(f_i^{-1}) \to 1$. These maps are modifications of the one used by Bokowski and Sperner in [12] and have the following key property, see Corollary 2.9: if $\bar{B}(0,2r) \subset C \cap C'$, $C \cup C' \subset \bar{B}(0,R)$ and $f : C \to C'$ is the considered map then $\operatorname{Lip}(f)$, $\operatorname{Lip}(f^{-1})$ are bounded above by a constant depending only on R/r. This implies, see Theorem 2.20, a uniform non-optimal isoperimetric inequality for all convex bodies with bounded quotient circumradius/inradius. We also prove in Theorem 2.8 that Lipschitz convergence implies convergence in the weak Hausdorff topology (modulo isometries).

Using Theorem 2.4 we prove in Theorem 2.10 the pointwise convergence of the normalized isoperimetric profiles. This implies, Corollary 2.11, through approximation by smooth convex bodies, the concavity of the isoperimetric profile I_C and of the function $I_C^{(n+1)/n}$ for an arbitrary convex body. As observed by Bayle [8, Thm. 2.3.10], the concavity of $I_C^{(n+1)/n}$ implies the strict concavity of I_C. This is an important property that implies the connectedness of an isoperimetric region and of its complement, Theorem 2.15. By standard properties of concave functions, we also obtain in Corollary 2.13 the uniform convergence of the normalized isoperimetric profiles J_C, and of their powers $J_C^{(n+1)/n}$ in compact subsets of the interval $(0,1)$. Using the bilipschitz maps constructed in the first section, we show in Theorem 2.21 that a uniform relative isoperimetric inequality, and hence a Poincaré inequality, holds in metric balls of small radius in C.

Using this relative isoperimetric inequality we prove in Theorem 2.26 a key result on the density of an isoperimetric region and its complement, similar to the ones obtained by Leonardi and Rigot [44], which are in fact based on ideas by David and Semmes [20] for quasi-minimizers of the perimeter. Theorem 2.26 is closer to a "clearing out" result as in

Massari and Tamanini [48, Thm. 1] (see also [45]) than to a concentration type argument as in Morgan's [53, § 13.7]. One of the consequences of Theorem 2.26 is a uniform lower density result, Corollary 2.29. The estimates obtained in Theorem 2.26 are stable enough to allow passing to the limit under Hausdorff convergence. Hence we can improve the L^1 convergence of isoperimetric regions and show in Theorem 2.32 that this convergence is in Hausdorff distance (see [73, § 1.3] and [3, Thm. 2.4.5]). We can prove the convergence of the free boundaries in Hausdorff distance in Theorem 2.34 as well. As a consequence, we are able to show in Theorem 2.33 that, given a convex body C, for every $0 < v < |C|$, there always exists an isoperimetric region with connected free boundary.

Finally, in the last section of Chapter 2 we consider the isoperimetric profile for small volumes. In the smooth boundary case, Fall [25] showed that for sufficiently small volume, the isoperimetric regions are small perturbations of geodesic spheres centered at a global maximum of the mean curvature, and derived an asymptotic expansion for the isoperimetric profile. We show in Theorem 2.40 that the isoperimetric profile of a convex set for small volumes is asymptotic to the one of its smallest tangent cone, i.e., the one with the smallest solid angle, and that rescaling isoperimetric regions to have volume 1 makes them subconverge in Hausdorff distance to an isoperimetric region in this convex cone, which is a geodesic ball centered at some apex by the recent result of Figalli and Indrei [27]. Although in the interior of the convex set we can apply Allard's regularity result for rectifiable varifolds, obtaining high order convergence of the boundaries of isoperimetric sets, we do not dispose of any regularity result at the boundary to ensure convergence up to the boundary (unless both the set and its limit tangent cone have smooth boundary [38]). As a consequence of Theorem 2.40, we show in Theorem 2.42 that the only isoperimetric regions of sufficiently small volume inside a convex polytope are geodesic balls centered at the vertices whose tangent cones have the smallest solid angle. The same result holds when the convex set is locally a cone at the points of the boundary with the smallest solid angle. A similar result for the *boundary* of the polytope was proven by Morgan [51].

In Chapter 3 we deal with convex cylinders and cylindrically bounded convex bodies. The contents of this Chapter correspond to [65]. A cylindrically bounded convex set is always included and asymptotic, in a sense to be precised later, to a *convex right cylinder*, a set of the form $K \times \mathbb{R}$, where $K \subset \mathbb{R}^n$ is a (compact) convex body. Here we have identified $\mathbb{R}^n$ with the hyperplane $x_{n+1} = 0$ of $\mathbb{R}^{n+1}$. In this work we first consider the more general convex cylinders of the form $C = K \times \mathbb{R}^q$, where $K \subset \mathbb{R}^m$ is an arbitrary convex body with interior points, and $\mathbb{R}^m \times \mathbb{R}^q = \mathbb{R}^{n+1}$, and prove a number of results for their isoperimetric profiles. No assumption on the regularity of ∂C will be made. Existence of isoperimetric regions is obtained in Proposition 3.2 following the scheme of proof by Galli and Ritoré [28], which essentially needs a uniform local relative isoperimetric inequality [62], a doubling property on $K \times \mathbb{R}^q$ given in Lemma 1.9, an upper bound for the isoperimetric profile of C given in (2.36), and a well-known deformation controlling the perimeter in terms of the volume. A proof of existence of isoperimetric regions in Riemannian manifolds with compact quotient under their isometry groups was previously given by Morgan [53]. Regularity results in the interior follow from Gonzalez, Massari and Tamanini [32] and Morgan [50], but no boundary regularity result is known for general convex bodies. We also prove in Proposition 3.5 that the isoperimetric profile I of a convex cylinder, as well as its power $I^{(n+1)/n}$, are concave functions of the volume, a strong result that implies the connectedness of isoperimetric regions. Further assuming $C^{2,\alpha}$ regularity of the boundary of C, we prove in Theorem 3.6 that, for an isoperimetric region $E \subset C$, either the closure of $\partial E \cap \text{int}(C)$ is connected, or $E \subset K \times \mathbb{R}$ is a slab. This follows from the connectedness of isoperimetric regions and from the results by Stredulinsky and Ziemer [71]. Next we consider small and large volumes. For small volumes, following Ritoré and Vernadakis [62], we show in Theorem 2.40 that the isoperimetric profile of a convex cylinder for small volumes is asymptotic to the one of its narrowest tangent cone. As a consequence, we completely characterize the isoperimetric regions of small volumes in a convex prism, i.e, a cylinder $P \times \mathbb{R}^q$ based on a convex polytope $P \subset \mathbb{R}^m$. Indeed, we show in Theorem 2.42 that the only isoperimetric regions of sufficiently small volume inside a convex prism are geodesic balls centered at the vertices with tangent cone of the smallest possible solid angle. For large volumes, we shall assume that C is a *right* convex cylinder, i.e., $p = 1$. Adapting an argument by Duzaar and Stephen [21] to the case when ∂K is not smooth, we prove in Theorem 3.9 that for large volumes the only isoperimetric regions in $K \times \mathbb{R}$ are the slabs $K \times I$, where $I \subset \mathbb{R}$ is a compact interval.

In the second part of Chapter 3 we apply the previous results for right convex cylinders to obtain properties of the isoperimetric profile of cylindrically bounded convex bodies. In Theorem 3.11 we show that the isoperimetric profile of a cylindrically bounded convex body C approaches, when the volume grows, that of its asymptotic half-cylinder. We also show the continuity of the isoperimetric profile in Proposition 3.14. Further assuming $C^{2,\alpha}$ regularity of both the cylindrically bounded convex body C and of its asymptotic cylinder, we prove the concavity of $I_C^{(n+1)/n}$ and existence of isoperimetric regions of large volume in Proposition 3.15. The final result of the second chapter, Theorem 3.22, implies that translations of isoperimetric regions of unbounded volume converge in Hausdorff distance to a half-slab in the asymptotic half-cylinder. The same convergence result holds for their free boundaries, that converge in Hausdorff distance to a flat $K \times \{t\}$, $t \in \mathbb{R}^+$. Theorem 3.22 is obtained from a clearing-out result for isoperimetric regions of large volume proven in Theorem 3.18 and its main consequence, lower density estimates for isoperimetric regions of large volume given in Proposition 3.19. Such lower density bounds provide an alternative proof of Theorem 3.9, given in Corollary 3.21.

In Chapter 4 we deal with unbounded convex body with non-degenerate asymptotic cone and their subcategory of conically bounded convex sets. The contents of this Chapter correspond to [64]. We have organized this Chapter into four sections. The next one contains basic preliminaries, while Sections 3.1 and 3.2 cover the already mentioned results for cylinders and cylindrically bounded sets, respectively.

Given an unbounded convex body C, a classical notion in the theory of convex sets is that the *asymptotic cone* of C, or tangent cone at infinity, defined by $C_\infty = \bigcap_{\lambda>0} \lambda C$. We shall say that C_∞ is *non-degenerate* when $\dim C_\infty = \dim C = n+1$. Assuming C has a non-degenerate asymptotic cone, we can extract useful information on the isoperimetric profile I_C of C but, unfortunately, we need a stronger control on the large scale geometry of C to get a more precise information on the geometry of large isoperimetric regions in C. Thus we are led to consider *conically bounded convex sets*. We shall say that a convex set C is conically bounded if there exists a non-degenerate cone C^∞ containing C, the *exterior asymtotic cone* of C, so that the Hausdorff distance of $C_t = C \cap \{x_{n+1} = t\}$ and $(C^\infty)_t$ goes to zero when t goes to infinity. When C is conically bounded, C^∞ coincides with C_∞ up to translation. There are examples of convex sets C with non-degenerate asymptotic cone that are not conically bounded. In convex cones, this isoperimetric problem has been considered by Lions and Pacella [46], Ritoré and Rosales [60] and Figalli and Indrei [27]. Outside convex bodies, possibly unbounded, isoperimetric inequalities have been established by Choe and Ritoré [19], and Choe, Ghomi and Ritoré [17], [18].

We have organized Chapter 4 into several sections. In Section 4.1, we consider convex bodies C with non-degenerate asymptotic cone C_∞ and we prove in Theorem 4.6 that the isoperimetric profile I_C of C is always bounded from below by the isoperimetric profile of I_{C_∞}, and that I_C and I_{C_∞} are asymptotic. The inequality $I_C \geqslant I_{C_\infty}$ is interesting since it implies that the isoperimetric inequality of the convex cone C_∞ also holds in C. We also show the continuity of the isoperimetric profile of C in Lemma 4.7.

In Section 4.2, we consider conically bounded convex bodies with smooth boundary. The boundary of its exterior asymptotic cone out of the vertex is not regular in general as it follows from the discussion at the beginning of Section 4.2. Assuming the regularity of this convex cone, we prove existence of isoperimetric regions for all volumes in Proposition 4.12, and the concavity of the isoperimetric profile I_C and of its power $I_C^{(n+1)/n}$ in Proposition 4.13. It is well-known [43] that the concavity of $I_C^{(n+1)/n}$ implies the connectedness of isoperimetric regions in C. In a similar way to [62] we prove a "clearing-out" result in Proposition 4.17, and a lower density bound in Corollary 4.18, that allow us to show in Theorem 4.19 a key convergence result: if we have a sequence isoperimetric regions in C whose volumes go to infinity, then scaling them down to have constant volume, we have convergence of the scaled isoperimetric regions in *Hausdorff distance* to a ball in the exterior asymptotic cone. Moreover, the boundaries of the scaled isoperimetric regions also converge in Hausdorff distance to the spherical cap that bounds this ball. This convergence can be improved to higher order convergence using Allard type estimates for varifolds using the estimate in Lemma 4.20.

In Section 4.3, we consider conically bounded sets of revolution. These sets are foliated, out of a compact set, by a family of spherical caps whose mean curvatures go to 0 by Lemma 4.21. Using the results in the previous Section and an argument based on the Implicit Function Theorem, we show in Theorem 4.25 that large isoperimetric regions are spherical caps meeting the boundary of the unbounded convex body in an orthogonal way.

In Chapter 5 we consider isoperimetric regions of large volume in the product of a compact Riemannian manifold with a Euclidean space and we refer to the introduction there. The contents of this Chapter correspond to the manuscript [63]. We consider the *isoperimetric problem* of minimizing perimeter under a given volume constraint inside $M \times \mathbb{R}^k$, where $\mathbb{R}^k$ is k-dimensional Euclidean space and M is an m-dimensional compact Riemmanian manifold without boundary. The dimension of the product manifold $N = M \times \mathbb{R}^k$ will be $n = m + k$. Our main result is the following

THEOREM 1.1. *Let M be a compact Riemannian manifold. There exists a constant $v_0 > 0$ such that any isoperimetric region in $M \times \mathbb{R}^k$ of volume $v \geqslant v_0$ is isometric to a tubular neighborhood of $M \times \{0\}$.*

This result, in case $k = 1$, was first proven by Duzaar and Steffen [21, Prop. 2.11]. As observed by Frank Morgan, an alternative proof for $k = 1$ can be given using the monotonicity formula and properties of the isoperimetric profile of $M \times \mathbb{R}$. Gonzalo [33] considered the general problem in his Ph.D. Thesis. In $\mathbb{S}^1 \times \mathbb{R}^k$, the result follows from the classification of isoperimetric regions by Pedrosa and Ritoré [57]. Large isoperimetric regions in asymptotically flat manifolds have been recently characterized by Eichmair and Metzger [23]. Gonzalo also gave a proof of Theorem 1.1 in his recent paper [34].

In our proof we use symmetrization and prove in Corollary 5.6 that an anisotropic scaling of symmetrized isoperimetric regions of large volume L^1-converge to a tubular neighborhood of $M \times \{0\}$. This convergence can be improved in Lemma 5.8 to Hausdorff convergence of the boundaries from density estimates on tubes, obtained in Lemma 5.7. Results of White [74] and Grosse-Brauckmann [36] on stable submanifolds then imply that the scaled boundaries are cylinders, Theorem 5.10. For small dimensions, it is also possible to use a result by Morgan and Ros [55] to get the same conclusion only using L^1-convergence. Once it is shown that the symmetrized set is a tube, it is not difficult to show that the original isoperimetric region is also a tube.

The arguments in Chapter 5 are still valid when the Riemannian manifold M has smooth non-empty boundary. In particular, Theorem 1.1 holds when M is replaced by a convex body $C \subset \mathbb{R}^m$ with smooth boundary. A way of extending this result for general C would be to obtain a geometric estimate on the constant v_0.

1.2. Preliminaries

Throughout this work we shall denote by $C \subset \mathbb{R}^{n+1}$ a compact convex set with non-empty interior. We shall call such a set a *convex body*. If compact is replaced by closed and unbounded we shall say that C in an *unbounded convex body*. Note that this terminology does not agree with some classical texts such as Schneider [68]. As a rule, basic properties of convex sets which are stated without proof in this paper can be easily found in Schneider's monograph.

The Euclidean distance in $\mathbb{R}^{n+1}$ will be denoted by d, and the r-dimensional Hausdorff measure of a set E by $H^r(E)$. The volume of a set E is its $(n+1)$-dimensional Hausdorff measure and we shall denote it by $|E|$. We shall denote the closure of E by $\operatorname{cl}(E)$ or $\overline{E}$ and the topological boundary by ∂E. The open ball of center x and radius $r > 0$ will be denoted by $B(x,r)$, and the corresponding closed ball by $\bar{B}(x,r)$.

Given $x \in C$ and $r > 0$, we define the *intrinsic ball* $B_C(x,r) = B(x,r) \cap C$, and the corresponding closed ball $\bar{B}_C(x,r) = C \cap \bar{B}(x,r)$. For $E \subset C$, the *relative boundary* of E in the interior of C is $\partial_C E = \partial E \cap \operatorname{int} C$.

In the space of convex bodies one may consider two different notions of convergence. Given a convex body C, and $r > 0$, we define $C_r = \{p \in \mathbb{R}^{n+1} : d(p,C) \leqslant r\}$. The set C_r is the tubular neighborhood of radius r of C and is a closed convex set. Given two convex sets C, C', we define its *Hausdorff distance* $\delta(C,C')$ by

$$\delta(C,C') = \inf\{r > 0 : C \subset (C')_r, C' \subset C_r\}. \tag{1.1}$$

The space of convex bodies with the Hausdorff distance is a metric space. Bounded sets in this space are relatively compact by Blaschke's Selection Theorem, [68, Thm. 1.8.4]. We shall say that a sequence $\{C_i\}_{i\in\mathbb{N}}$ of convex bodies converges to a convex body C in Hausdorff distance if $\lim_{i\to\infty} \delta(C_i,C) = 0$.

Given two convex bodies C, $C' \subset \mathbb{R}^{n+1}$, we define its *weak Hausdorff distance* $\delta_S(C,C')$ by

$$\delta_S(C,C') = \inf\{\delta(C,h(C')) : h \in \operatorname{Isom}(\mathbb{R}^{n+1})\}. \tag{1.2}$$

The weak Hausdorff distance is non-negative, symmetric, and satisfies the triangle inequality. Moreover, $\delta_S(C,C') = 0$ if and only if there exists $h \in \operatorname{Isom}(\mathbb{R}^{n+1})$ such that $C = h(C')$.

A map $f : (X,d) \to (X',d')$ between metric spaces is *lipschitz* if there exists a constant $L > 0$ so that

$$d'(f(x),f(y)) \leqslant L\, d(x,y), \tag{1.3}$$

for all x, $y \in X$. Sometimes we will refer to such a map as an L-lipschitz map. The smallest constant satisfying (1.3), sometimes called the dilatation of f, will be denoted by $\operatorname{Lip}(f)$. A lipschitz function on (X,d) is a lipschitz map $f : X \to \mathbb{R}$, where we consider on $\mathbb{R}$ the Euclidean distance. A map $f : X \to Y$ is bilipschitz if both f and f^{-1} are lipschitz maps.

Given two convex bodies C, C', we define its *Lipschitz distance* d_L by

$$d_L(C,C') = \inf_{f\in \operatorname{Lip}(C,C')} \{\log(\max\{\operatorname{Lip}(f),\operatorname{Lip}(f^{-1})\})\}, \tag{1.4}$$

where $\operatorname{Lip}(C,C')$ is the set of bilipschitz maps from C to C'. We shall say that a sequence $\{C_i\}_{i\in\mathbb{N}}$ of convex bodies converges in Lipschitz distance to a convex body C if $\lim_{i\to\infty} d_L(C_i,C) = 0$. The Lipschitz distance is non-negative, symmetric and satisfies the triangle inequality. Moreover, $d_L(C,C') = 0$ if and only if C and C' are isometric. If a sequence $\{C_i\}_{i\in\mathbb{N}}$ converges to C is the lipschitz sense, then there is a sequence of bilipschitz maps $f_i : C_i \to C$ such that

$$\lim_{i\to\infty} \log(\max\{\mathrm{Lip}(f_i), \mathrm{Lip}(f_i^{-1})\}) = 0.$$

This implies $\lim_{i\to\infty} \max\{\mathrm{Lip}(f_i), \mathrm{Lip}(f_i^{-1})\} = 1$. As $1 \leqslant \mathrm{Lip}(f_i)\,\mathrm{Lip}(f_i^{-1})$, we obtain that both $\mathrm{Lip}(f_i)$, $\mathrm{Lip}(f_i^{-1}) \to 1$. Conversely, if there is a sequence of bilipschitz maps $f_i : C_i \to C$ such that $\lim_{i\to\infty} \mathrm{Lip}(f_i) = \lim_{i\to\infty} \mathrm{Lip}(f_i^{-1}) = 1$ then $\lim_{i\to\infty} d_L(C_i, C) = 0$.

If M, N are subsets of Euclidean spaces and $f : M \to N$ is a lipschitz map, then $g : \lambda M \to \lambda N$ defined by $g(x) = \lambda f(\frac{x}{\lambda}), x \in \lambda M, \lambda > 0$, is a lipschitz map so that $\mathrm{Lip}(g) = \mathrm{Lip}(f)$. This yields the very useful consequence

$$d_L(\lambda M, \lambda N) = d_L(M, N), \qquad \lambda > 0. \tag{1.5}$$

For future reference, we list the following properties of lipschitz maps and functions

Lemma 1.2.

(i) *Let f be a lipschitz function on (X, d) so that $|f| \geqslant M > 0$. Then $1/f$ is a lipschitz function and* $\mathrm{Lip}(1/f) \leqslant \mathrm{Lip}(f)/M^2$.
(ii) *Let f_1, f_2 be lipschitz functions on (X, d). Then $f_1 + f_2$ is a lipschitz function and* $\mathrm{Lip}(f_1 + f_2) \leqslant \mathrm{Lip}(f_1) + \mathrm{Lip}(f_2)$.
(iii) *Let f_1, f_2 be lipschitz functions on (X, d) so that $|f_i| \leqslant M_i$, $i = 1, 2$. Then $f_1 f_2$ is a lipschitz function and* $\mathrm{Lip}(f_1 f_2) \leqslant M_1 \mathrm{Lip}(f_2) + M_2 \mathrm{Lip}(f_1)$.
(iv) *If $\lambda : (X, d) \to \mathbb{R}$ is lipschitz with $|\lambda| \leqslant L'$, and $f : (X, d) \to \mathbb{R}^n$ is lipschitz with $|f| < M'$, then* $\mathrm{Lip}(\lambda f) \leqslant M' \mathrm{Lip}(\lambda) + L' \mathrm{Lip}(f)$.
(v) *If f_i are lipschitz maps that converge pointwise to a lipschitz map f, then* $\mathrm{Lip}(f) \leqslant \liminf_{i\to\infty} \mathrm{Lip}(f_i)$.

1.2.1. Sets of finite perimeter and isoperimetric regions. Given $E \subset C$, we define the *relative perimeter* of E in $\mathrm{int}(C)$, by

$$P_C(E) = \sup\Big\{ \int_E \mathrm{div}\,\xi \, dH^{n+1}, \xi \in \Gamma_0(C), \, |\xi| \leqslant 1 \Big\},$$

where $\Gamma_0(C)$ is the set of smooth vector fields with compact support in $\mathrm{int}(C)$. We shall say that E has *finite perimeter* in C if $P_C(E) < \infty$. A set E of finite perimeter in $\mathrm{int}(C)$ satisfies $P(E) \leqslant P_C(E) + H^n(\partial C)$ and so is a Cacciopoli set in $\mathbb{R}^{n+1}$. Observe that we are only taking into account the $\mathscr{H}^n$-measure of ∂E inside the interior of C. We define the *isoperimetric profile* of C by

The *volume* of E is defined as the $(n+1)$-dimensional Hausdorff measure of E and will be denoted by $|E|$. The r-dimensional Hausdorff measure will be denoted by H^r.

For $t \geqslant 0$, let $E(t)$ denote the set of points of density t of E in C

$$E(t) = \{x \in C : \lim_{r\to 0} \frac{|E \cap B_C(x, r)|}{|B_C(x, r)|} = t\}.$$

Since $|E \cap \partial C| = 0$, we have that $|E(t)| = |E(t) \cap \mathrm{int}(C)|$. By Lebesgue- Besicovitch Theorem we have $|E(1)| = |E|$ and similarly $|E(0)| = |C \setminus E|$.

Given a finite perimeter set, we define the sets

$$\begin{aligned} E_1 &= \{x \in C : \exists\, r > 0 \text{ such that } |B_C(x,r) \setminus E| = 0\}, \\ E_0 &= \{x \in C : \exists\, r > 0 \text{ such that } |B_C(x,r) \cap E| = 0\}, \\ \partial_{*C} E &= \{x \in C : |B_C(x,r) \setminus E| > 0 \text{ and } |B_C(x,r) \cap E| > 0, \text{ for all } r > 0\}, \end{aligned}$$

the measure theoretical interior, exterior and relative boundary of E in C, respectively. By [24, § 5.8] (see also [31]), there holds

$$P_C(E) = H^n(\partial_{*C} E). \tag{1.6}$$

The behavior of the Hausdorff measure [14, § 1.7.2] with respect to lipschitz maps is well known.

If $C, C' \subset \mathbb{R}^{n+1}$ are convex bodies (possible unbounded) and $f : C \to C'$ is a Lipschitz map, then, for every $s > 0$ and $E \subset C$, by the definition of Hausdorff measure, we get $H^s(f(E)) \leqslant \mathrm{Lip}(f)^s H^s(E)$. Furthermore, $f(\partial_{*C} E) = \partial_{*f(C)}(f(E))$. Thus

LEMMA 1.3. *Let $C, C' \subset \mathbb{R}^{n+1}$ and $f : C \to C'$ a bilipschitz map then we have*

$$\begin{aligned} \mathrm{Lip}(f^{-1})^{-n}\, P_C(E) \leqslant P_{f(C)}(f(E)) \leqslant \mathrm{Lip}(f)^n\, P_C(E), \\ \mathrm{Lip}(f^{-1})^{-(n+1)}\, |E| \leqslant |f(E)| \leqslant \mathrm{Lip}(f)^{n+1}\, |E|. \end{aligned} \tag{1.7}$$

REMARK 1.4. Let M_i, $i = 1, 2, 3$ be metric spaces and $f_i : M_i \to M_{i+1}$, $i = 1, 2$ be lipschitz maps, then $\mathrm{Lip}(f_2 \circ f_1) \leqslant \mathrm{Lip}(f_1)\mathrm{Lip}(f_2)$. Consequently if $g : M_1 \to M_2$ is a bilipschitz map, then $1 \leqslant \mathrm{Lip}(g)\mathrm{Lip}(g^{-1})$.

REMARK 1.5. If $f : C_1 \to C_2$ is a bilipschitz map between subsets of $\mathbb{R}^{n+1}$, then $g : \lambda C_1 \to \lambda C_2$, defined by $g(x) = \lambda f(\frac{x}{\lambda})$, is also bilipschitz and satisfies $\mathrm{Lip}(f) = \mathrm{Lip}(g)$, $\mathrm{Lip}(f^{-1}) = \mathrm{Lip}(g^{-1})$.

Given $C \subset \mathbb{R}^{n+1}$, the *isoperimetric profile* of C is the function I_C defined by

$$I_C(v) = \inf\Big\{P_C(E) : E \subset C, |E| = v\Big\}. \tag{1.8}$$

We shall say that $E \subset C$ is an *isoperimetric region* if $P_C(E) = I_C(|E|)$. The *renormalized isoperimetric profile* of C is

$$Y_C = I_C^{(n+1)/n}. \tag{1.9}$$

We shall denote by $J_C : [0,1] \to \mathbb{R}^+$ the *normalized isoperimetric profile* function

$$J_C(\lambda) = I_C(\lambda\, |C|). \tag{1.10}$$

We shall also denote by $y_C : [0,1] \to \mathbb{R}^+$ the function

$$y_C = J_C^{(n+1)/n}. \tag{1.11}$$

Standard results of Geometric Measure Theory imply that isoperimetric regions exist in a convex body. The following basic properties are well known.

LEMMA 1.6. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body. Consider a sequence $\{E_i\}_{i\in\mathbb{N}} \subset C$ of subsets with finite perimeter in the interior of C.*

(i) *If E_i converges to a set $E \subset C$ with finite perimeter in $\operatorname{int}(C)$ in the $L^1(\operatorname{int}(C))$ sense, then $P_C(E) \leqslant \liminf_{i\to\infty} P_C(E_i)$*
(ii) *If $P_C(E_i)$ is uniformly bounded from above, then there exists a set $E \subset C$ of finite perimeter in $\operatorname{int}(C)$ such that a subsequence of $\{E_i\}_{i\in\mathbb{N}}$ converges to E in the $L^1(\operatorname{int}(C))$ sense.*
(iii) *Isoperimetric regions exist in C for every volume.*
(iv) *I_C is continuous.*

PROOF. Properties (i), (ii) and (iii) follow from the lower semicontinuity of perimeter [31, Thm. 1.9] and compactness [31, Thm. 1.19]. The continuity of the isoperimetric profile was proven in [29, Lemma 6.2]. □

For a convex body C, the continuity of the isoperimetry profile of C will be a trivial consequence of the concavity of I_C proven in Corollary 2.11.

The known results on the regularity of isoperimetric regions are summarized in the following Lemma.

LEMMA 1.7 ([32], [37], [70, Thm. 2.1]). *Let $C \subset \mathbb{R}^{n+1}$ a convex body and $E \subset C$ an isoperimetric region. Then $\partial E \cap \operatorname{int}(C) = S_0 \cup S$, where $S_0 \cap S = \emptyset$ and*

(i) *S is an embedded C^∞ hypersurface of constant mean curvature.*
(ii) *S_0 is closed and $H^s(S_0) = 0$ for any $s > n-7$.*

Moreover, if the boundary of C is of class $C^{2,\alpha}$ then $\operatorname{cl}(\partial E \cap \operatorname{int}(C)) = S \cup S_0$, where

(iii) *S is an embedded $C^{2,\alpha}$ hypersurface of constant mean curvature*
(iv) *S_0 is closed and $H^s(S_0) = 0$ for any $s > n-7$*
(v) *At points of $S \cap \partial C$, S meets ∂C orthogonally.*

PROPOSITION 1.8 ([60, Thm. 2.1]). *Let C be an unbounded convex body and $v > 0$. Then there exists a finite perimeter set $E \subset C$ (possibly empty), with $|E| = v_1 \leqslant v$, $P_C(E) = I_C(v_1)$, and a diverging sequence $\{E_i\}_{i\in\mathbb{N}}$ of finite perimeter sets such that $|E_i| \to v_2$ and $v_1 + v_2 = v$. Moreover*

$$I_C(v) = P_C(E) + \lim_{i\to\infty} P_C(E_i) \tag{1.12}$$

LEMMA 1.9. *Let $C \subset \mathbb{R}^{n+1}$ be an unbounded convex body. Then C is a doubling metric space with a constant depending only on n.*

PROOF. Let $x \in C$, $r > 0$ and K denote the convex cone with vertex x which subtended by $\partial B_C(x,r)$ then

$$
\begin{aligned}
|B_C(x,2r)| &= |B_C(x,2r) \setminus B_C(x,r)| + |B_C(x,r)| \\
&\leqslant |B_K(x,2r) \setminus B_K(x,r)| + |B_C(x,r)| \\
&\leqslant |B_K(x,2r)| + |B_C(x,r)| \\
&= 2^{n+1}|B_K(x,r)| + |B_C(x,r)| \\
&= (2^{n+1}+1)|B_C(x,r)|.
\end{aligned} \tag{1.13}
$$

□

We shall say that a cone is *regular* if its boundary is C^2 out of the vertices.

PROPOSITION 1.10. *Let C be a regular convex cone and $\{E_i\}_{i\in\mathbb{N}} \subset C$ a diverging sequence of finite perimeter sets with $\lim_{i\to\infty}|E_i| = v$. Then $\liminf_{i\to\infty} P_C(E_i) \geqslant I_H(v)$.*

PROOF. The proof is modeled on [60, Thm. 3.4], where the sets of the diverging sequence were assumed to have the same volume. If one looks at the proof, will see that this is not an issue. □

CHAPTER 2

Convex bodies

2.1. Hausdorff and Lipschitz convergence in the space of convex bodies

As a first step in our study of the isoperimetric profile of a convex body, we need to prove that Hausdorff convergence of convex bodies implies Lipschtz convergence. We shall also prove the converse replacing the Hausdorff distance by the weak Hausdorff distance as defined in (1.2). We need first some preliminary results for convex sets.

Given a convex body $C \subset \mathbb{R}^n$ containing 0 in its interior, its *radial function* $\rho(C,\cdot):\mathbb{S}^n \to \mathbb{R}$ is defined by

$$\rho(C,u) = \max\{\lambda \geqslant 0 : \lambda u \in C\}.$$

From this definition it follows that $\rho(C,u)u \in \partial C$ for all $u \in \mathbb{S}^n$.

Lemma 2.1. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body so that $B(0,r) \subset C \subset B(0,R)$. Then the radial function $\rho(C,\cdot):\mathbb{S}^n \to \mathbb{R}$ is R^2/r-lipschitz.*

Proof. Let C^* be the polar body of C, [68, § 1.6]. Theorem 1.6.1 in [68] implies that $(C^*)^* = C$ and that $B(0,1/R) \subset C^* \subset B(0,1/r)$. Let $h(C^*,\cdot)$ be the support function of C^*. Using $(C^*)^* = C$, Remark 1.7.7 in [68] implies

$$\rho(C,u) = \frac{1}{h(C^*,u)}.$$

By Lemma 1.8.10 in [68] the function $h(C^*,\cdot)$ is $1/r$-lipschitz. Since $h(C^*,\cdot) \geqslant 1/R$, we conclude from Lemma 1.2 that $\rho(C,\cdot)$ is an R^2/r-lipschitz function. $\square$

Lemma 2.2. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of convex bodies converging in Hausdorff distance to a convex body C. We further assume that there exist r, $R>0$ such that $B(0,r) \subset \operatorname{int}(C_i) \subset B(0,R)$ for all $i \in \mathbb{N}$, and $B(0,r) \subset \operatorname{int}(C) \subset B(0,R)$. Then*

$$\lim_{i\to\infty} \sup_{u\in\mathbb{S}^n} |\rho(C_i,u) - \rho(C,u)| = 0.$$

Proof. We reason by contradiction. Assume there exists $\varepsilon > 0$ and $u_i \in \mathbb{S}^n$ so that a subsequence satisfies

$$|\rho(C_i,u_i) - \rho(C,u_i)| \geqslant \varepsilon.$$

Passing again to a subsequence we may assume that $u_i \to u \in \mathbb{S}^n$. We define

$$x_i = \rho(C_i, u_i)u_i \in \partial C_i, \qquad y_i = \rho(C, u_i)u_i \in \partial C.$$

Since $\rho(C_i, \cdot)$ and $\rho(C, \cdot)$ are uniformly bounded, we may extract again convergent subsequences $x_i \to x$ and $y_i \to y$. Since ∂C is closed, we have $y \in \partial C$. Since $C_i \to C$ in Hausdorff distance, we have $x \in \partial C$ (it is straightforward to check that $x \notin \mathbb{R}^{n+1} \setminus C$, and that $x \notin \operatorname{int}(C)$ by Lemma 1.8.14 in [68]). Since $|x_i - y_i| \geqslant \varepsilon$ we get $|x - y| \geqslant \varepsilon$, but both x, y belong to the ray emanating from 0 with direction u. This is a contradiction since $0 \in \operatorname{int}(C)$, [68, Lemma 1.1.8]. □

LEMMA 2.3. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of convex functions defined on a convex open set C and converging uniformly on C to a convex function f.*

(i) *Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence such that $x = \lim_{i\to\infty} x_i$. If $\nabla f_i(x_i)$, $\nabla f(x)$ exist for all $i \in \mathbb{N}$, then $\nabla f_i(x_i) \to \nabla f(x)$.*
(ii) $\operatorname{Lip}(f_i - f) \to 0$.
(iii) *If g is a convex function defined in a convex body C, then*

$$\operatorname{Lip}(g) = \sup_{z\in D} |\nabla g(z)|,$$

where D is the subset of C (dense and of full measure) where ∇g exists.

PROOF. The proof of (i) is taken from [66, Thm. 25.7]. We give it for completeness. Assume that $\nabla f_i(x_i)$ does not converge to $\nabla f(x)$. Then there exists $y \in \mathbb{R}^n$ and $\varepsilon > 0$ such that either

$$\begin{aligned} &\langle \nabla f_i(x_i), y\rangle - \langle \nabla f(x), y\rangle \geqslant \varepsilon, \text{ or} \\ &\langle \nabla f_i(x_i), y\rangle - \langle \nabla f(x), y\rangle \leqslant -\varepsilon, \end{aligned} \tag{2.1}$$

holds for a subsequence.

Let us assume that the second inequality in (2.1) holds for a subsequence. For simplicity, we assume it holds for the whole sequence. Thus we have $\langle \nabla f_i(x_i), y\rangle \leqslant \langle \nabla f(x), y\rangle - \varepsilon$ for any index i. Multiplying this inequality by $t < 0$ we obtain $\langle \nabla f_i(x_i), ty\rangle \geqslant \big(\langle \nabla f(x), y\rangle - \varepsilon\big)\, t$. From this inequality and the convexity of f_i we get

$$f_i(x_i + ty) - f_i(x_i) \geqslant \langle \nabla f_i(x_i), ty\rangle \geqslant \big(\langle f(x), y\rangle - \varepsilon\big)\, t.$$

Letting $i \to \infty$, taking into account that $f_i \to f$ uniformly, we find

$$\frac{f(x+ty) - f(x)}{t} \leqslant \langle \nabla f(x), y\rangle - \varepsilon$$

Taking limits when $t \uparrow 0$ we get $\langle \nabla f(x), y\rangle \leqslant \langle \nabla f(x), y\rangle - \varepsilon$, and we reach a contradiction. The case of the first inequality in (2.1) is treated in the same way. This proves (i).

To prove (ii) we also reason by contradiction. So we assume there exists $\varepsilon > 0$ so that $\operatorname{Lip}(f_i - f) > \varepsilon$ holds for a subsequence. For simplicity, we assume that every index i satisfies this inequality. We can find sequences $\{x_i\}_{i\in\mathbb{N}}$, $\{y_i\}_{i\in\mathbb{N}}$ such that $x_i \neq y_i$ and

$$|(f_i - f)(x_i) - (f_i - f)(y_i)| > \varepsilon\,|x_i - y_i| \quad \text{for all } i \in \mathbb{N}. \tag{2.2}$$

Passing again to a subsequence if necessary, we assume that there are points x, y such that $x = \lim_{i\to\infty} x_i$, $y = \lim_{i\to\infty} y_i$.

We observe that it can be assumed that both ∇f_i and ∇f are defined H^1-almost everywhere in the segment $[x_i, y_i]$: otherwise we consider a right circular cylinder $D \times [x_i, y_i]$ of axis $[x_i, y_i]$ so that, in every segment parallel to $[x_i, y_i]$ of height $|x_i - y_i|$, inequality (2.2) is satisfied by its extreme points. Since the set where the gradients ∇f_i, ∇f exist has full H^{n+1}-measure in $D \times [x_i, y_i]$, [66, Thm. 25.4], Fubini's Theorem implies that H^n-almost everywhere in D, the gradients are H^1-almost everywhere defined. We replace $[x_i, y_i]$ by one of such segments if necessary.

For $\lambda \in [0,1]$, and $i \in \mathbb{N}$, we define convex functions u_i, v_i by

$$u_i(\lambda) := \frac{f_i(x_i + \lambda(y_i - x_i)) - f_i(x_i)}{|y_i - x_i|}, \qquad v_i(\lambda) := \frac{f(x_i + \lambda(y_i - x_i)) - f(x_i)}{|y_i - x_i|}. \tag{2.3}$$

Hence (2.2) is equivalent to

$$\lim_{i\to\infty} (u_i(1) - v_i(1)) \geqslant \varepsilon \tag{2.4}$$

We easily find

$$(u_i(\lambda) - v_i(\lambda))' = f_i'(x_i + \lambda(y_i - x_i); \frac{x_i - y_i}{|x_i - y_i|}) - f'(x_i + \lambda(y_i - x_i); \frac{x_i - y_i}{|x_i - y_i|}), \tag{2.5}$$

where the derivative $f'(p;u)$ of the convex function f at the point p in the direction of u is defined as in [66, p. 213]. At the points where both $\nabla f_i, \nabla f$ exist we get

$$(u_i(\lambda) - v_i(\lambda))' = \langle (\nabla f_i - \nabla f)(x_i + \lambda(y_i - x_i)), \frac{x_i - y_i}{|x_i - y_i|} \rangle,$$

and

$$|(u_i(\lambda) - v_i(\lambda))'| \leqslant |\nabla f_i(x_i + \lambda(y_i - x_i)) - \nabla f(x_i + \lambda(y_i - x_i))|.$$

By (i) and [66, Thm. 25.5] we have $\lim_{i\to\infty}(u_i(\lambda) - v_i(\lambda))' = 0$. By [66, Thm. 10.6], $\mathrm{Lip}(f_i)$ is uniformly bounded. So $(u_i - v_i)'$ is bounded by a constant by (iii). Then by the Dominated Convergence Theorem, [66, Corollary 24.2.1], and the fact that $u_i(0) = v_i(0) = 0$, we get

$$\lim_{i\to\infty} (u_i(1) - v_i(1)) = \lim_{i\to\infty} \int_0^1 (u_i(\lambda) - v_i(\lambda))' d\lambda = 0,$$

which, together with (2.4), gives a contradiction. Hence $\lim_{i\to\infty} \mathrm{Lip}(f_i - f) = 0$.

To prove (iii), let $z \in D$. There is $w \in \mathbb{S}^n$ such that $|\nabla g(z)| = \langle \nabla g(z), w \rangle$. Hence

$$|\nabla g(z)| = \left| \lim_{\lambda\to 0} \frac{g(z + \lambda w) - g(z)}{\lambda} \right| \leqslant \sup_{x\neq y} \frac{|g(x) - g(y)|}{|x - y|} = \mathrm{Lip}(g).$$

To prove the reverse inequality, take $x, y \in C$ and assume for the moment that ∇g exists H^1-almost everywhere in the segment $[x, y]$. Then by [66, Corollary 24.2.1] we have

$$|g(x) - g(y)| = \left| \int_0^1 \langle \nabla g(x + \lambda(y - x)), y - x \rangle d\lambda \right| \leqslant \sup_{z \in D} |\nabla g(z)||x - y|$$

If ∇g does not exist H^1-almost everywhere in the segment $[x, y]$, we can make an approximation argument, as in the proof of (ii), with segments parallel to $[x, y]$, where ∇g exists H^1- almost everywhere, to conclude the proof. □

Now we prove that Hausdorff convergence of a sequence of convex bodies implies Lipschitz convergence.

THEOREM 2.4. *Let $\{C_i\}_{i \in \mathbb{N}}$ be a sequence of convex bodies in $\mathbb{R}^{n+1}$ that converges in Hausdorff distance to a convex body C. Then $\{C_i\}_{i \in \mathbb{N}}$ converges to C in Lipschitz distance.*

PROOF. Translating the whole sequence and its limit we assume that $0 \in \operatorname{int}(C)$. Let $r > 0$ so that $\bar{B}(0, 2r) \subset \operatorname{int}(C)$. By [68, Lemma 1.8.14] and the convergence of C_i to C in Hausdorff distance, there exists $i_0 \in \mathbb{N}$ such that $\bar{B}(0, r) \subset \operatorname{int}(C_i)$ for $i \geqslant i_0$. Let us denote by ρ_i and ρ the radial functions $\rho(C_i, \cdot)$ and $\rho(C, \cdot)$, respectively. Since the sequence $\{C_i\}_{i \in \mathbb{N}}$ converges to C in Hausdorff distance, there exists $R > 0$ so that $\bigcup_{i \in \mathbb{N}} C_i \cup C \subset B(0, R)$.

For $i \geqslant i_0$, we define a map $f_i : C \to C_i$ by

$$f_i(x) = \begin{cases} x, & |x| \leqslant r, \\ r\,\dfrac{x}{|x|} + (|x| - r)\,\dfrac{\rho_i\left(\frac{x}{|x|}\right) - r}{\rho\left(\frac{x}{|x|}\right) - r}\,\dfrac{x}{|x|}, & |x| \geqslant r. \end{cases} \tag{2.6}$$

Using Lemmata 1.2 and 2.1 we obtain that f_i is a lipschitz function. The inverse mapping can be defined exchanging the roles of ρ_i and ρ to conclude that f_i is a bilipschitz map. The function f_i can be rewritten as

$$f_i(x) = x + \left(1 - \frac{\rho_i\left(\frac{x}{|x|}\right) - r}{\rho\left(\frac{x}{|x|}\right) - r}\right)(r - |x|)\,\frac{x}{|x|}, \qquad |x| \geqslant r. \tag{2.7}$$

To show that the sequence $\{C_i\}_{i \in \mathbb{N}}$ converges in Lipschitz distance to C, it is enough to prove that both $\operatorname{Lip}(f_i)$, $\operatorname{Lip}(f_i^{-1})$ converge to 1. We shall show that

$$\lim_{i \to \infty} \operatorname{Lip}\left(1 - \frac{\rho_i\left(\frac{x}{|x|}\right) - r}{\rho\left(\frac{x}{|x|}\right) - r}\right) = 0, \tag{2.8}$$

and the corresponding inequality interchanging ρ_i and ρ. From (2.8) and the expression of f_i given by (2.7) we would get $\limsup_{i \to \infty} \operatorname{Lip}(f_i) \leqslant 1$. Since $\operatorname{Lip}(f_i) \geqslant \operatorname{Lip}(f_i|_{\bar{B}(0,r)}) = 1$ we obtain $1 \leqslant \liminf_{i \to \infty} \operatorname{Lip}(f_i)$. Crossing both inequalities we would have $\lim_{i \to \infty} \operatorname{Lip}(f_i) = 1$. The same argument would work for f_i^{-1}.

Let us now prove (2.8). In what follows we shall assume that ρ, ρ_i have $\mathbb{S}^n$ as their domain of definition. As $\rho - r$ is bounded from below, again by Lemma 1.2, it is enough to prove $\lim_{i\to\infty} \mathrm{Lip}(\rho_i - \rho) = 0$. Let us denote by h_i^*, h^* the support functions of the polar sets C_i^*, C^* of C_i, C, respectively. By [68, Remark 1.7.7], $h_i^* = 1/\rho_i$. Since ρ_i is uniformly bounded from below, again by Lemma 1.2, it is enough to check that that $\mathrm{Lip}(h_i^* - h^*) \to 0$. By Lemma 2.2, the convex functions h_i^* converge pointwise to h^*. Lemma 2.3 then implies that $\mathrm{Lip}(h_i^* - h^*) = 0$. $\square$

REMARK 2.5. Observe that the map given by (2.6) is defined in all of $\mathbb{R}^{n+1}$ and takes C onto C_i and $\mathbb{R}^{n+1} \setminus C$ onto $\mathbb{R}^{n+1} \setminus C_i$.

REMARK 2.6. If $f : C_1 \to C_2$ is a bilipschitz map between convex bodies of $\mathbb{R}^{n+1}$, then $g : \lambda C_1 \to \lambda C_2$, defined by $g(x) = \lambda f(\frac{x}{\lambda})$, is also bilipschitz and satisfies $\mathrm{Lip}(f) = \mathrm{Lip}(g)$, $\mathrm{Lip}(f^{-1}) = \mathrm{Lip}(g^{-1})$.

REMARK 2.7. Let C, $C' \subset \mathbb{R}^{n+1}$ two convex bodies so that $\delta(C, C') > 0$, $d_L(C, C') > 0$ (it is enough to consider two non-isometric convex bodies). For $i \in \mathbb{N}$, we have

$$d_L(iC, iC') = d_L(i^{-1}C, i^{-1}C') = d_L(C, C').$$

On the other hand

$$\delta(iC, iC') = i\,\delta(C, C') \to +\infty; \qquad \delta(i^{-1}C, i^{-1}C') = i^{-1}\delta(C, C') \to 0.$$

Hence Lipschitz and Hausdorff distances will not be equivalent in a subset of the space of convex bodies unless we impose uniform bounds on the circumradius and the inradius.

Now we prove that the convergence of a sequence of convex bodies in Lipschitz distance, together with an upper bound on the circumradii of the elements of the sequence, implies the convergence of a subsequence in Hausdorff distance to a convex body isometric to the Lipschitz limit. We recall that Lipschitz convergence implies Gromov-Hausdorff convergence, see [35, Prop. 3.7], [14, Ex. 7.4.3].

THEOREM 2.8. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of convex bodies converging to a convex body C in Lipschitz distance. Then $\{C_i\}_{i\in\mathbb{N}}$ converges to C in weak Hausdorff distance.*

PROOF. Let $f_i : C \to C_i$ be a sequence of bilipschitz maps with $\mathrm{Lip}(f_i)$, $\mathrm{Lip}(f_i^{-1}) \to 1$. Then $\mathrm{diam}(C_i)$ are uniformly bounded, so that translating the sets C_i we may assume they are uniformly bounded. Applying the Arzelà-Ascoli Theorem, a subsequence of f_i uniformly converges to a lipschitz map $f : C \to \mathbb{R}^{n+1}$. We shall assume the whole sequence converges. The sequence $C_i = f_i(C)$ converges to the compact set $f(C)$ in the sense of Kuratowski [4, Def. 4.4.13] and so converges to $f(C)$ in Hausdorff distance by [4, Prop. 4.4.14]. To check that C_i converges to $f(C)$ in the sense of Kuratowski we take $x = \lim_{k\to\infty} f_{i_k}(x_{i_k})$, with $x_{i_k} \in C$, and we extract a convergent subsequence of x_{i_k} to some $x_0 \in C$ to get $x = f(x_0) \in f(C)$; on the other hand, every $x \in f(C)$ is the limit of the sequence of points $f_i(x) \in C_i$.

Since $f_i \to f$ and $\mathrm{Lip}(f_i) \to 1$, Lemma 1.2 implies $\mathrm{Lip}(f) \leqslant 1$ and $|f(x)-f(y)| \leqslant |x-y|$ for any $x, y \in C$. On the other hand, taking limits when $i \to \infty$ in the inequalities

$$|x-y| = |f_i^{-1}(f_i(x)) - f_i^{-1}(f_i(y))| \leqslant \mathrm{Lip}(f_i^{-1})\,|f_i(x)-f_i(y)|$$

we get $|x-y| \leqslant |f(x)-f(y)|$ and so f is an isometry. This arguments shows that any subsequence of $\{C_i\}_{i\in\mathbb{N}}$ has a convergent subsequence in weak Hausdorff distance to C, which is enough to conclude that $\lim_{i\to\infty} \delta_S(C_i, C) = 0$. □

In the next result we shall obtain a geometric upper bound for the lipschitz constant of the map built in the proof of Theorem 2.4. Observe that the the same bound holds for the inverse mapping, which satisfies the same geometrical condition.

Corollary 2.9. *Let C, $C' \subset \mathbb{R}^{n+1}$ be convex bodies so that $\bar{B}(0,2r) \subset C \cap C'$, $C \cup C' \subset \bar{B}(0,R) \subset \mathbb{R}^{n+1}$. Let $f : C \to C'$ be the bilipschitz map defined by*

$$f(x) = \begin{cases} x, & |x| \leqslant r, \\ r\,\dfrac{x}{|x|} + (|x|-r)\,\dfrac{\rho'(\frac{x}{|x|}) - r}{\rho(\frac{x}{|x|}) - r}\,\dfrac{x}{|x|}, & |x| \geqslant r. \end{cases} \tag{2.9}$$

Then we have

$$1 \leqslant \mathrm{Lip}(f), \mathrm{Lip}(f^{-1}) \leqslant 1 + \frac{R}{r}\Big(\frac{R}{r} - 1\Big)\Big(\frac{R^2}{r^2} + 1\Big). \tag{2.10}$$

Proof. By Lemma 1.2 we get $\mathrm{Lip}(f) \geqslant \mathrm{Lip}(f|_{\{|x|\leqslant r\}}) = 1$ and the same argument is valid for f^{-1} as well. So in what is follows we assume that $|x| \geqslant r$. Observe that $x \in \mathbb{R}^{n+1} \setminus B(0,r) \mapsto r\frac{x}{|x|}$ is the metric projection onto the convex set $\{|x| \leqslant r\}$ and so has Lipschitz constant 1, thus

$$\mathrm{Lip}\,\Big(\frac{x}{|x|}\Big) \leqslant 1/r. \tag{2.11}$$

We denote by ρ, ρ' the radial functions of C, C' respectively. Let us estimate first the Lipschitz constant of the map

$$x \in \mathbb{R}^{n+1} \setminus B(0,r) \mapsto \frac{\rho'(\frac{x}{|x|}) - r}{\rho(\frac{x}{|x|}) - r}.$$

By Lemma 1.2 (i), (iii),(vii), and (2.11) we get

$$\mathrm{Lip}\,\Big(\frac{\rho'(\frac{x}{|x|}) - r}{\rho(\frac{x}{|x|}) - r}\Big) \leqslant \frac{1}{r}\frac{R^2}{r}\frac{1}{r} + (R-r)\frac{R^2}{r}\frac{1}{r}\frac{1}{r} = \frac{R^2}{r^3} + (R-r)\frac{R^2}{r^4}. \tag{2.12}$$

As the above function is bounded from above by $\frac{R-r}{r}$, and $x \mapsto \frac{x}{|x|}$ is bounded from above by 1, having Lipschitz constant no larger than $1/r$ by (2.11), Lemma 1.2 (iv) then implies

$$\mathrm{Lip}\,\Big(\frac{\rho'(\frac{x}{|x|}) - r}{\rho(\frac{x}{|x|}) - r}\Big)\frac{x}{|x|} \leqslant \frac{R^2}{r^3} + (R-r)\frac{R^2}{r^4} + \frac{R-r}{r}\frac{1}{r}. \tag{2.13}$$

Thus, as the above function is bounded from above by $\frac{R-r}{r}$, and $x \mapsto |x| - r$ is bounded from above by $R-r$, having Lipschitz constant no larger than 1, then from Lemma 1.2 (iv) we get

$$\begin{aligned} \operatorname{Lip}(f) &\leqslant 1+(R-r)\Big(\frac{R^2}{r^3}+(R-r)\frac{R^2}{r^4}+\frac{R-r}{r^2}\Big)+\frac{R-r}{r} \\ &\leqslant 1+\Big(\frac{R-r}{r}\Big)\Big(\frac{R^2}{r^2}+\Big(\frac{R-r}{r}\Big)\frac{R^2}{r^2}+\frac{R-r}{r}+1\Big) \\ &\leqslant 1+\Big(\frac{R}{r}-1\Big)\Big(\frac{R^3}{r^3}+\frac{R}{r}\Big). \end{aligned} \tag{2.14}$$

□

2.2. The isoperimetric profile in the space of convex bodies

Using the results of the previous Section, we shall prove in this one that, when a sequence of convex bodies converges in Hausdorff distance to a convex body, then the normalized isoperimetric profiles defined by (1.10) and (1.11) converge uniformly to the normalized isoperimetric profiles of the limit convex body. This has some consequences: the isoperimetric profile I_C of a convex body C, and its power $I_C^{(n+1)/n}$, even with non-smooth boundary, are concave. This would imply that isoperimetric regions and their complements are connected, and also the connectedness of the free boundaries when the boundary is of class $C^{2,\alpha}$.

THEOREM 2.10. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of convex bodies in $\mathbb{R}^{n+1}$ that converges to a convex body $C \subset \mathbb{R}^{n+1}$ in Hausdorff distance. Then J_{C_i} converges to J_C pointwise in $[0,1]$. Consequently, also y_{C_i} converges pointwise to y_C.*

PROOF. For $\lambda\in\{0,1\}$ we have $J_{C_i}(\lambda)=J_C(\lambda)=0$. Let us fix some $\lambda\in(0,1)$. Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of isoperimetric regions in C_i with $|E_i| = \lambda\,|C_i|$, see Lemma 1.6. By the regularity lemma 1.7, $P_C(E_i)=H^n(\partial E_i\cap \operatorname{int}(C_i))$. By the continuity of the volume with respect to the Hausdorff distance, we have $\lim_{i\to\infty}|E_i| = \lambda\,|C|$.

Theorem 2.4 implies the existence of a sequence of bilipschitz maps $f_i : C_i \to C$ so that $\lim_{i\to\infty}\operatorname{Lip}(f_i) = \lim_{i\to\infty}\operatorname{Lip}(f_i)^{-1} = 1$. Lemma 1.3 yields

$$\begin{gathered} \frac{1}{\operatorname{Lip}(f_i^{-1})^{n+1}}\,|E_i| \leqslant |f_i(E_i)| \leqslant \operatorname{Lip}(f_i)^{n+1}\,|E_i|, \\ \frac{1}{\operatorname{Lip}(f_i^{-1})^{n}}\,P_{C_i}(E_i) \leqslant P_C(f_i(E_i)) \leqslant \operatorname{Lip}(f_i)^{n}\,P_{C_i}(E_i). \end{gathered}$$

So $\{f_i(E_i)\}_{i\in\mathbb{N}}$ is a sequence of finite perimeter sets in C with $\lim_{i\to\infty}|f_i(E_i)| = \lambda\,|C|$, and $\liminf_{i\to\infty} P_{C_i}(E_i) = \liminf_{i\to\infty} P_C(f_i(E_i))$. From Lemma 1.6 we have

$$\begin{aligned} J_C(\lambda) &\leqslant \lim_{i\to\infty} I_C(|f_i(E_i)|) \leqslant \liminf_{i\to\infty} P_C(f_i(E_i)) \\ &= \liminf_{i\to\infty} P_{C_i}(E_i) = \liminf_{i\to\infty} J_{C_i}(\lambda). \end{aligned}$$

Let us prove now that $J_C(\lambda) \geqslant \limsup_{i\to\infty} J_{C_i}(\lambda)$. We shall reason by contradiction assuming that $J_C(\lambda) < \limsup J_{C_i}(\lambda)$. Passing to a subsequence we can suppose that $\{J_{C_i}(\lambda)\}_{i\in\mathbb{N}}$ converges. So let us assume $J_C(\lambda) < \lim_{i\to\infty} J_{C_i}(\lambda)$. Let $E \subset C$ be an isoperimetric region with $|E| = \lambda\,|C|$. Consider a point p in the regular part of $\partial E \cap \operatorname{int}(C)$. We take a vector field in $\mathbb{R}^{n+1}$ with compact support in a small neighborhood of p that does not intersect the singular set of ∂E. We choose the vector field so that the deformation $\{E_t\}_{t\in\mathbb{R}}$ induced by the associated flow strictly increases the volume in the interval $(-\varepsilon,\varepsilon)$, i.e., $t \mapsto |\,E_t|$ is strictly increasing in $(-\varepsilon,\varepsilon)$. Taking a smaller ε if necessary, the first variation formulas of volume and perimeter imply the existence of a constant $M > 0$ so that

$$|H^n(\partial E_t \cap \operatorname{int}(C)) - H^n(\partial E \cap \operatorname{int}(C)| \leqslant M\,||E_t| - |\,E|| \tag{2.15}$$

holds for all $t \in (-\varepsilon,\varepsilon)$. Reducing ε again if necessary we may assume

$$H^n(\partial E \cap \operatorname{int}(C)) + M\,||E_t| - |\,E|| < \lim_{i\to\infty} J_{C_i}(\lambda). \tag{2.16}$$

(recall we are supposing $H^n(\partial E \cap \operatorname{int}(C)) = J_C(\lambda) < \lim_{i\to\infty} J_{C_i}(\lambda)$).

For every $i \in \mathbb{N}$, consider the sets $\{f_i^{-1}(E_t)\}_{t\in(-\varepsilon,\varepsilon)}$. Since

$$\frac{1}{\operatorname{Lip}(f_i)^{n+1}}\,|E_t| \leqslant |f_i^{-1}(E_t)| \leqslant \operatorname{Lip}(f_i^{-1})^{n+1}\,|E_i|,$$

$|E_{-\varepsilon/2}| < \lambda\,|C|$, $|E_{\varepsilon/2}| > \lambda\,|C|$ by the monotonicity of the function $t \mapsto |\,E_t|$ in $(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})$, the Lipschitz constants $\operatorname{Lip}(f_i), \operatorname{Lip}(f_i^{-1})$ converge to 1 when $i \to \infty$, and $\lim_{i\to\infty} |C_i|/|C| = 1$, there exists $i_0 \in \mathbb{N}$ such that

$$|f_i^{-1}(E_{\varepsilon/2})| > \lambda\,|C_i|, \qquad |f_i^{-1}(E_{-\varepsilon/2})| < \lambda\,|C_i|,$$

for all $i \geqslant i_0$. Since $t \mapsto |\,f_i^{-1}(E_t)|$ is continuous, for every $i \geqslant i_0$, there exists $t(i) \in (-\frac{\varepsilon}{2},\frac{\varepsilon}{2})$ so that $|f_i^{-1}(E_{t(i)})| = \lambda\,|C_i|$, and we have

$$\begin{aligned}
P_{C_i}(f_i^{-1}(E_{t(i)})) &\leqslant \operatorname{Lip}(f_i^{-1})\,P_C(E_{t(i)}) \\
&\leqslant \operatorname{Lip}(f_i^{-1})\,\big(P_C(E) + M\,||E_t| - |\,E||\big) \\
&< J_{C_i}(\lambda),
\end{aligned}$$

for i large enough, using (2.16) and $\operatorname{Lip}(f_i^{-1}) \to 1$. This contradiction shows

$$J_C(\lambda) \geqslant \limsup_{i\to\infty} J_{C_i}(\lambda),$$

and hence $J_C(\lambda) = \lim_{i\to\infty} J_{C_i}(\lambda)$. □

Theorem 2.10 allows us to extend properties of the isoperimetric profile for convex bodies with smooth boundary to arbitrary convex bodies. The following result was first proven by E. Milman

COROLLARY 2.11 ([49, Corollary 6.11]). *Let $C \subset \mathbb{R}^{n+1}$ be a convex body. Then y_C is a concave function. As a consequence, the functions Y_C, I_C and J_C are concave.*

PROOF. Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of convex bodies with smooth boundaries that converges to C in Hausdorff distance. The functions y_{C_i} are concave by the results of Kuwert [43], see also [9, Remark 3.3]. By Theorem 2.10, $y_{C_i} \to y_C$ pointwise in $[0,1]$ and so y_C is concave. Since Y_C is the composition of y_C with an affine function, we conclude that Y_C is also concave. As the composition of a concave function with an increasing concave function is concave, it follows that $I_C = Y_C^{n/(n+1)}$, $J_C = y_C^{n/(n+1)}$ are concave as well. □

REMARK 2.12. The concavity of the isoperimetric profile of an Euclidean convex body with $C^{2,\alpha}$ boundary was proven by Sternberg and Zumbrum [70], see also [9]. Kuwert later extended this result by showing the concavity of $I_C^{(n+1)/n}$ for convex sets with C^2 boundary.

COROLLARY 2.13. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of convex bodies in $\mathbb{R}^{n+1}$ that converges to a convex body $C \subset \mathbb{R}^{n+1}$ in the Hausdorff topology. Then J_{C_i} (resp. y_{C_i}) converges to J_C (resp. y_C) uniformly on compact subsets of $(0,1)$.*

PROOF. By Theorem 2.10 we have that $J_{C_i} \to J_C$ pointwise. By [66, Thm. 10.8], this convergence is uniform on compact sets of $(0,1)$. □

COROLLARY 2.14. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of convex bodies in $\mathbb{R}^{n+1}$ that converges to a convex body C in the Hausdorff topology. Let $v_i \in [0,|C_i|]$, $v \in [0,|C|]$ so that $v_i \to v$. Then $I_{C_i}(v_i) \to I_C(v)$.*

PROOF. First we consider the case $v = 0$. For i sufficiently large, consider Euclidean geodesic balls $B_i \subset \operatorname{int}(C_i)$ of volume v_i. Letting $v_i \to 0$ and taking into account that $I_C(0) = 0$, we are done. The case $v = |C|$ is handled taking the complements $C \setminus B_i$ of the balls.

Now assume that $0 < v < |C|$. Let $w_i = v_i/|C_i|$ and $w = v/|C|$. Then by the continuity of the volume with respect to the Hausdorff distance [68, Thm. 1.8.16] we get $w_i \to w$. Take $\varepsilon > 0$ such that $[w-\varepsilon, w+\varepsilon] \subset (0,1)$. For large i we have

$$\begin{aligned} |J_{C_i}(w_i) - J_C(w)| &\leqslant |J_{C_i}(w_i) - J_C(w_i)| + |J_C(w_i) - J_C(w)| \\ &\leqslant \sup_{x\in[w-\varepsilon,w+\varepsilon]} |J_{C_i}(x) - J_C(x)| + |J_C(w_i) - J_C(w)|. \end{aligned}$$

By Corollary 2.13, J_{C_i} converges to J_C uniformly on $[w-\varepsilon, w+\varepsilon]$ and, as J_C is continuous [29], we get $J_{C_i}(w_i) \to J_C(w)$. From the definition of J, w_i, and w the proof follows. □

THEOREM 2.15. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body, and $E \subset C$ an isoperimetric region. Then E and $C \setminus E$ are connected.*

PROOF. We shall prove that the function I_C satisfies

$$I_C(v_1 + v_2) < I_C(v_1) + I_C(v_2), \tag{2.17}$$

whenever v_1, $v_2 > 0$. To prove (2.17) we shall use the concavity of Y_C showed in Corollary 2.11 and the fact that $Y_C(0) = 0$ to obtain

$$\frac{Y_C(v_1+v_2)}{v_1+v_2} \leqslant \min\left\{\frac{Y_C(v_1)}{v_1}, \frac{Y_C(v_2)}{v_2}\right\},$$

what implies

$$Y_C(v_1+v_2) \leqslant Y_C(v_1) + Y_C(v_2),$$

as in [8, Lemma B.1.4]. Raising to the power $n/(n+1)$ we get

$$I_C(v_1+v_2) \leqslant (I_C(v_1)^{(n+1)/n} + I_C(v_2)^{(n+1)/n})^{n/(n+1)} < I_C(v_1) + I_C(v_1),$$

where the last inequality follows from $(a+b)^q < a^q + b^q$, for a, $b > 0$, $q \in (0,1)$, cf. [39, (2.12.2)]. This proves (2.17).

If $E \subset C$ were a disconnected isoperimetric region, then $E = E_1 \cup E_2$, with $|E| = |E_1| + |E_2|$, and $P_C(E) = P_C(E_1) + P_C(E_2)$, and we should have

$$I_C(v) = P_C(E) = P_C(E_1) + P_C(E_2) \geqslant I_C(v_1) + I_C(v_2),$$

which is a contradiction to (2.17). If $E \subset C$ is an isoperimetric region, then $C \setminus E$ is an isoperimetric region and so connected as well. $\square$

In case the boundary of C is of class $C^{2,\alpha}$, Sternberg and Zumbrun [70] obtained a expression for the second derivative of the perimeter with respect to the volume in formula (2.31) inside Theorem 2.5 of [70]. Using this formula they obtained in their Theorem 2.6 that a local minimizer E of perimeter (in a L^1 sense) has the property that the closure of $\partial E \cap \operatorname{int}(C)$ is either connected or it consists of a union of parallel planar (totally geodesic) components meeting ∂C orthogonally with that part of C lying between any two such totally geodesic components consisting of a cylinder. If E is an isoperimetric region so that the closure of $\partial E \cap \operatorname{int}(C)$ consists on more than one totally geodesic component, then Theorem 2.6 in [70] implies that either E or its complement in C is disconnected, a contradiction to Theorem 2.15. So we have proven

THEOREM 2.16. *Let C be a convex body with $C^{2,\alpha}$ boundary, and $E \subset C$ an isoperimetric region. Then the closure of $\partial E \cap \operatorname{int}(C)$ is connected.*

From the concavity of I_C the following properties of the isoperimetric profile of I_C follow. Similar properties can be found in [7], [40], [59], [67] and [54].

PROPOSITION 2.17. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body. Then*

(i) *I_C can be extended continuously to $[0, |C|]$ so that $I_C(0) = I_C(|C|) = 0$.*
(ii) *$I_C : [0, |C|] \to \mathbb{R}^+$ is a positive concave function, symmetric with respect to $|C|/2$, increasing up to $|C|/2$ and decreasing from $|C|/2$. Left and right derivatives $(I_C)'_-(v)$,*

$(I_C)'_+(v)$, exist for every $v \in (0,|C|)$. Moreover, I_C is differentiable H^1-almost everywhere and we have

$$I_C(v) = \int_0^v (I_C)'_-(w)\,dw = \int_0^v (I_C)'_+(w)\,dw = \int_0^v I'_C(w)\,dw,$$

for every $v \in [0,|C|]$.

(iii) *If $E \subset C$ is an isoperimetric region of volume $v \in (0,|C|)$, and H is the (constant) mean curvature of the regular part of $\partial E \cap \operatorname{int}(C)$, then*

$$(I_C)'_+(v) \leqslant H \leqslant (I_C)'_-(v).$$

In particular, if I_C is differentiable at v, then the mean curvature of every isoperimetric region of volume v equals $I'_C(v)$.

Proof. By Theorem 2.10 we have that I_C is a symmetric, positive, concave function, increasing up to the midpoint and then decreasing. By [66, Thm. 24.1], side derivatives exist for all volumes. By [66, Thm. 25.3] differentiability almost everywhere, and absolute continuity [66, Cor. 24.2.1] hold, from where the proof of (i) follows.

To prove (ii), take an isoperimetric region $E \subset C$ of volume v and constant mean curvature H. By the regularity lemma 1.7 we can find an open subset U contained in the regular part of ∂E. Take a nontrivial C^1 function $u \geqslant 0$ with compact support in U that produces an inward normal variation $\{\phi_t\}$ for t small. By the first variation of volume and perimeter we get

$$\frac{d}{dt}\bigg|_{t=0} |\phi_t(E)| = -\int_{\partial E} u, \qquad \frac{d}{dt}\bigg|_{t=0} P_C(\phi_t(E)) = -\int_{\partial E} Hu.$$

So we get $|\phi_t(E)| < |E|$ for $t > 0$ and $|\phi_t(E)| > |E|$ for $t < 0$. As $P_C(\phi_t(E)) \leqslant I_C(|\phi_t(E)|$, we have

$$(I_C)'_-(v) = \lim_{\lambda \uparrow 0} \frac{I_C(v+\lambda) - I_C(v)}{\lambda} \geqslant \frac{dP_C(\phi_t(E))}{d|\phi_t(E)|} = H.$$

Similarly replacing u by $-u$ we get $\lambda > 0$ we find.

$$(I_C)'_+(v) = \lim_{\lambda \downarrow 0} \frac{I_C(v+\lambda) - I_C(v)}{\lambda} \leqslant \frac{dP_C(\phi_t(E))}{d|\phi_t(E)|} = H$$

□

Finally, we shall prove in Theorem 2.20 that convex bodies with uniform quotient circumradius/inradius satisfy a uniform relative isoperimetric inequality invariant by scaling. A similar result was proven by Bokowski and Sperner [12, Satz 3] using a map different from (2.6). A consequence of Theorem 2.20 is the existence of a uniform Poincaré inequality for balls of small radii inside convex bodies that will be proven in Theorem 2.21 and used in the next Section. First we prove the following Lemma.

LEMMA 2.18. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body and $0 < v_0 < |C|$. We have*

$$I_C(v) \geqslant \frac{I_C(v_0)}{v_0^{n/(n+1)}}\, v^{n/(n+1)}, \tag{2.18}$$

for all $0 \leqslant v \leqslant v_0$. As a consequence, we get

$$I_C(v) \geqslant \frac{I_C(|C|/2)}{(|C|/2)^{n/(n+1)}} \min\{v, |C|-v\}^{n/(n+1)}, \tag{2.19}$$

for all $0 \leqslant v \leqslant |C|$.

PROOF. Since $Y_C = I_C^{(n+1)/n}$ is concave and $Y_C(0) = 0$ we get

$$\frac{Y_C(v)}{v} \geqslant \frac{Y_C(v_0)}{v_0},$$

for $0 < v \leqslant v_0$. Raising to the power $n/(n+1)$ we obtain (2.18). If $0 \leqslant v \leqslant |C|/2$ then (2.19) is simply (2.18). If $|C|/2 \leqslant v \leqslant |C|$, then $0 \leqslant |C| - v \leqslant |C|/2$, we apply (2.18) to $|C| - v$ with $v_0 = |C|/2$ and we take into account that $I_C(v) = I_C(|C| - v)$ to prove (2.19). $\square$

REMARK 2.19. If a set E is isoperimetric in C of volume $|C|/2$, then λE is isoperimetric in λC with volume $|\lambda C|/2$ and perimeter $P_{\lambda C}(\lambda E) = \lambda^n P_C(E)$. So the constant in (2.19) satisfies

$$M_C = \frac{I_C(|C|/2)}{(|C|/2)^{n/(n+1)}} = \frac{I_{\lambda C}(|\lambda C|/2)}{(|\lambda C|/2)^{n/(n+1)}},$$

for any $\lambda > 0$. Hence all dilated convex sets λC, with $\lambda > 0$, satisfy the same isoperimetric inequality

$$I_{\lambda C}(v) \geqslant M_C \min\{v, |\lambda C| - v\}^{n/(n+1)},$$

for $0 < v < |\lambda C|$.

THEOREM 2.20. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body, $x, y \in C$, $0 < r < R$, such that $\bar{B}(y,r) \subset C \subset \bar{B}(x,R)$. Then there exists a constant $M > 0$, only depending on R/r and n, such that*

$$I_C(v) \geqslant M \min\{v, |C| - v\}^{n/(n+1)}, \tag{2.20}$$

for all $0 \leqslant v \leqslant |C|$.

PROOF. Since $\bar{B}(y,r) \subset C \subset \bar{B}(x,R)$ we can construct a bilipschitz map $f : C \to \bar{B}(x,R)$ as in (2.9). Take $0 < v < |C|$. By Lemma 1.6, there exists an isoperimetric set $E \subset C$ of volume v. By Lemma 1.3 we have

$$\begin{aligned} I_C(v) = P_C(E) &\geqslant (\operatorname{Lip} f)^{-n} P_{B(x,R)}(f(E)), \\ |\bar{B}(x,R) \setminus f(E)| &\geqslant (\operatorname{Lip} f^{-1})^{-(n+1)}(|C \setminus E|), \\ |f(E)| &\geqslant (\operatorname{Lip}(f^{-1})^{-(n+1)}|E|. \end{aligned}$$

We know [31, Cor. 1.29] that for $f(E) \subset \bar{B}(x,R)$ we have the isoperimetric inequality

$$P_{\bar{B}(x,R)}(f(E)) \geqslant M(n) \min\{|f(E)|, |\bar{B}(x,R)| - |f(E)|\}^{n/(n+1)},$$

where $M(n)$ is a constant that only depends on the dimension n. So we get

$$I_C(v) \geqslant M(n)\left((\operatorname{Lip} f)(\operatorname{Lip} f^{-1})\right)^{-n} \min\{v, |C|-v\}^{n/(n+1)}.$$

As $\bar{B}(x,R) \subset \bar{B}(y,2R)$, Corollary 2.9 provides upper bounds of $\operatorname{Lip}(f)$, $\operatorname{Lip}(f^{-1})$ only depending on R/r. This completes the proof of the Proposition. □

THEOREM 2.21. *Let $C \subset \mathbb{R}^{n+1}$ a convex body. Given $r_0 > 0$, there exist positive constants M, ℓ_1, only depending on r_0 and C, and a universal positive constant ℓ_2 so that*

$$I_{\bar{B}_C(x,r)}(v) \geqslant M \min\{v, |\bar{B}_C(x,r)|-v\}^{n/(n+1)}, \tag{2.21}$$

for all $x \in C$, $0 < r \leqslant r_0$, and $0 < v < |\bar{B}_C(x,r)|$. Moreover

$$\ell_1 r^{n+1} \leqslant |\bar{B}_C(x,r)| \leqslant \ell_2 r^{n+1}, \tag{2.22}$$

for any $x \in C$, $0 < r \leqslant r_0$.

PROOF. To prove (2.21) we only need an upper estimate of the quotient of r over the inradius of $\bar{B}(x,r)$ by Theorem 2.20. By the compactness of C we deduce that

$$\inf_{x\in C} \operatorname{inr}(\bar{B}_C(x,r_0)) > 0 \tag{2.23}$$

Hence, for every $x \in C$, we always can find a point $y(x) \in \bar{B}_C(x,r_0)$ and a positive constant $\delta > 0$ independent of x such that,

$$\bar{B}(y(x),\delta) \subset \bar{B}_C(x,r_0) \subset \bar{B}(x,r_0). \tag{2.24}$$

Now take $0 < r \leqslant r_0$. Let $0 < \lambda \leqslant 1$ so that $r = \lambda r_0$, and denote by $h_{x,\lambda}$ the homothety of center x and radius λ. Then we have $h_{x,\lambda}(\bar{B}(y(x),\delta)) \subset h_{x,\lambda}(\bar{B}_C(x,r_0))$ and so

$$\bar{B}(h_{x,\lambda}(y(x)),\lambda\delta) \subset \bar{B}_{h_{x,\lambda}(C)}(x,\lambda r_0) \subset \bar{B}_C(x,\lambda r_0),$$

since $h_{x,\lambda}(C) \subset C$ as $x \in C$, $0 < \lambda \leqslant 1$, and C is convex. Again by Theorem 2.20, a relative isoperimetric inequality is satisfied in $\bar{B}_C(x,r)$ with a constant M that only depends on r_0/δ.

We now prove (2.22). Since $|\bar{B}_C(x,r)| \leqslant |\bar{B}(x,r)|$, it is enough to take $\ell_2 = \omega_{n+1} = |\bar{B}(0,1)|$. For the remaining inequality, using the same notation as above, we have

$$\begin{aligned}
|\bar{B}(x,r)\cap C| &= |\bar{B}(x,\lambda r_0)\cap C| \geqslant |h_{x,\lambda}(\bar{B}(x,r_0)\cap C)| \\
&= \lambda^{n+1}|\bar{B}(x,r_0)\cap C| \geqslant \lambda^{n+1}|\bar{B}(y(x),\delta)| \\
&= \omega_{n+1}(\delta/r_0)^{n+1}\, r^{n+1},
\end{aligned}$$

and we take $\ell_1 = \omega_{n+1}(\delta/r_0)^{n+1}$. □

2.3. Convergence of isoperimetric regions

Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of convex bodies converging in Hausdorff distance to a convex body C, and $\{E_i\}_{i\in\mathbb{N}}$ a sequence of isoperimetric regions in C_i of volumes v_i weakly converging to some isoperimetric region $E\subset$ of volume $v=\lim_{i\to\infty} v_i$. The main result in this Section is that E_i converges to E in Hausdorff distance, and also their relative boundaries. As a byproduct, we shall also prove that there exists always in C an isoperimetric region with connected boundary. It is still an open question to show that *every* isoperimetric region on a convex body has connected boundary.

We prove first a finite number of Lemmata

LEMMA 2.22. *Let C be a convex body, and $\lambda>0$. Then*

$$I_{\lambda C}(\lambda^{n+1}v)=\lambda^n I_C(v), \tag{2.25}$$

for all $0\leqslant v\leqslant \min\{|C|,|\lambda C|\}$.

PROOF. For v in the above conditions we get

$$\begin{aligned} I_{\lambda C}(\lambda^{n+1}v) &= \inf\{P_{\lambda C}(\lambda E):\lambda E\subset\lambda C,\ |\lambda E|=\lambda^{n+1}v\} \\ &= \inf\Big\{\lambda^n P_C(E): E\subset C,\ |E|=v\Big\} \\ &= \lambda^n I_C(v). \end{aligned}$$

□

REMARK 2.23. Lemma 2.22 implies

$$Y_{\lambda C}(\lambda^{n+1}v)=\lambda^{n+1}Y_C(v) \tag{2.26}$$

for any $\lambda>0$ and $0\leqslant v\leqslant\min\{|C|,|\lambda C|\}$.

LEMMA 2.24. *Let C be a convex body, $\lambda\geqslant 1$. Then*

$$I_{\lambda C}(v)\geqslant I_C(v) \tag{2.27}$$

for all $0\leqslant v\leqslant |C|$.

PROOF. Let $Y_{\lambda C}=I_{\lambda C}^{(n+1)/n}$. We know from Corollary 2.11 that Y_C is a concave function with $Y_{\lambda C}(0)=0$. Since $\lambda\geqslant 1$, for $v>0$ we have

$$\frac{Y_{\lambda C}(v)}{v}\geqslant\frac{Y_{\lambda C}(\lambda^{n+1}v)}{\lambda^{n+1}v},$$

what implies, using (2.26),

$$\lambda^{n+1}Y_{\lambda C}(v)\geqslant Y_{\lambda C}(\lambda^{n+1}v)=\lambda^{n+1}Y_C(v).$$

This proves (2.27). □

In a similar way to [44, p. 18], given a convex body C and $E \subset C$, we define a function $h : C \times (0,+\infty) \to (0,\frac{1}{2})$ by

$$h(E,C,x,R) = \frac{\min\{|E \cap B_C(x,R)|, |B_C(x,R) \setminus E|\}}{|B_C(x,R)|}, \tag{2.28}$$

for $x \in C$ and $R > 0$. When E and C are fixed, we shall simply denote

$$h(x,R) = h(E,C,x,R). \tag{2.29}$$

Lemma 2.25. *For any $v > 0$, consider the function $f_v : [0,v] \to \mathbb{R}$ defined by*

$$f_v(s) = s^{-n/(n+1)} \left(\left(\frac{v-s}{v} \right)^{n/(n+1)} - 1 \right).$$

Then there is a constant $0 < c_2 < 1$ that does not depends on v so that $f_v(s) \geqslant -(1/2)\, v^{-n/(n+1)}$ for all $0 \leqslant s \leqslant c_2\, v$.

Proof. By continuity, $f_v(0) = 0$. Observe that $f_v(v) = -v^{-n/(n+1)}$ and that, for $s \in [0,1]$, we have $f_v(sv) = f_1(s)\, v^{-n/(n+1)}$. The derivative of f_1 in the interval $(0,1)$ is given by

$$f_1'(s) = \frac{n}{n+1} \, \frac{(s-1) + (1-s)^{n/(n+1)}}{s-1} \, s^{-1-n/(n+1)},$$

which is strictly negative and so f_1 is strictly decreasing. Hence there exists $0 < c_2 < 1$ such that $f_1(s) \geqslant -1/2$ for all $s \in [0,c_2]$. This implies $f_v(s) = f_1(s/v)\, v^{-n/(n+1)} \geqslant -(1/2)\, v^{-n/(n+1)}$ for all $s \in [0, c_2 v]$. $\square$

Now we prove a key density result for isoperimetric regions.

Theorem 2.26. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body, and $E \subset C$ an isoperimetric region of volume $0 < v < |C|$. Choose ε so that*

$$0 < \varepsilon < \min \left\{ \frac{v}{\ell_2}, \frac{|C|-v}{\ell_2}, c_2 v, c_2(|C|-v), \frac{I_C(v)^{n+1}}{\ell_2 8^{n+1} v^n}, \frac{I_C(v)^{n+1}}{\ell_2 8^{n+1} (|C|-v)^n} \right\}, \tag{2.30}$$

where c_2 is the constant in Lemma 2.25.

Then, for any $x \in C$ and $R \leqslant 1$ so that $h(x,R) \leqslant \varepsilon$, we get

$$h(x,R/2) = 0. \tag{2.31}$$

Moreover, in case $h(x,R) = |E \cap B_C(x,R)| |B_C(x,R)|^{-1}$, we get $|E \cap B_C(x,R/2)| = 0$ and, in case $h(x,R) = |B_C(x,R) \setminus E| |B_C(x,R)|^{-1}$, we have $|B_C(x,R/2) \setminus E| = 0$.

Proof. From Lemma 2.18 we get

$$I_C(w) \geqslant c_1 w^{n/(n+1)}, \qquad \text{where} \qquad c_1 = v^{-n/(n+1)} I_C(v), \tag{2.32}$$

for all $0 \leqslant w \leqslant v$.

Assume first that

$$h(x,R)=\frac{|E\cap B_C(x,R)|}{|B_C(x,R)|}.$$

Define $m(t)=|E\cap B_C(x,t)|, 0<t\leqslant R$. Thus $m(t)$ is a non-decreasing function. For $t\leqslant R\leqslant 1$ we get

$$m(t)\leqslant m(R)=|E\cap B_C(x,R)|=h(x,R)\,|B_C(x,R)|\leqslant h(x,R)\,\ell_2 R^{n+1}\leqslant \varepsilon\ell_2<v, \tag{2.33}$$

by (2.30). So we obtain $(v-m(t))>0$.

By the coarea formula, when $m'(t)$ exists, we get

$$m'(t)=\frac{d}{dt}\int_0^t H^n(E\cap\partial B_C(x,s))ds=H^n(E\cap\partial B_C(x,t)), \tag{2.34}$$

where we have denoted $\partial B_C(x,t)=\partial B(x,t)\cap\operatorname{int}(C)$. Define

$$\lambda(t)=\frac{v^{1/(n+1)}}{(v-m(t))^{1/(n+1)}},\qquad E(t)=\lambda(t)(E\setminus B_C(x,t)). \tag{2.35}$$

Then $E(t)\subset\lambda(t)C$ and $|E(t)|=|E|=v$. By Lemma 2.24, we get $I_{\lambda(t)C}\geqslant I_C$ since $\lambda(t)\geqslant 1$. Combining this with [75, Cor. 5.5.3], equation (2.34), and elementary properties of the perimeter functional, we get

$$\begin{aligned}
I_C(v)&\leqslant I_{\lambda(t)C}(v)\leqslant P_{\lambda(t)C}(E(t))=\lambda^n(t)\,P_C(E\setminus B_C(x,t))\\
&\leqslant \lambda^n(t)\big(P_C(E)-P(E,B_C(x,t))+H^n(E\cap\partial B_C(x,t))\big)\\
&\leqslant \lambda^n(t)\big(P_C(E)-P_C(E\cap B_C(x,t))+2H^n(E\cap\partial B_C(x,t))\big)\\
&\leqslant \lambda^n(t)\big(I_C(v)-c_1 m(t)^{n/(n+1)}+2m'(t)\big),
\end{aligned} \tag{2.36}$$

where c_1 is the constant in (2.32). Multiplying both sides by $I_C(v)^{-1}\lambda(t)^{-n}$ we find

$$\lambda(t)^{-n}-1+\frac{c_1}{I_C(v)}m(t)^{n/(n+1)}\leqslant\frac{2}{I_C(v)}m'(t). \tag{2.37}$$

Set

$$a=\frac{2}{I_C(v)},\qquad b=\frac{c_1}{I_C(v)}=\frac{1}{v^{n/(n+1)}}. \tag{2.38}$$

From the definition (2.35) of $\lambda(t)$ we get

$$f(m(t))\leqslant am'(t)\qquad H^1\text{-a.e}, \tag{2.39}$$

where

$$\frac{f(s)}{s^{n/(n+1)}}=b+\frac{\left(\frac{v-s}{v}\right)^{n/(n+1)}-1}{s^{n/(n+1)}}. \tag{2.40}$$

By Lemma 2.25, there exists a universal constant $0<c_2<1$, not depending on v, so that

$$\frac{f(s)}{s^{n/n+1}}\geqslant b/2\qquad\text{whenever}\qquad 0<s\leqslant c_2 v. \tag{2.41}$$

Since $\varepsilon \leqslant c_2 v$ by (2.30), equation (2.41) holds in the interval $[0,\varepsilon]$. If there were $t \in [R/2,R]$ such that $m(t)=0$ then, by monotonicity of $m(t)$, we would conclude $m(R/2)=0$ as well. So we assume $m(t)>0$ in $[R/2,R]$. Then by (2.39) and (2.41), we get

$$b/2a \leqslant \frac{m'(t)}{m(t)^{n/n+1}}, \qquad H^1\text{-a.e.}$$

Integrating between $R/2$ and R we get by (2.33)

$$bR/4a \leqslant (m(R)^{1/(n+1)} - m(R/2)^{1/(n+1)}) \leqslant m(R)^{1/(n+1)} \leqslant (\varepsilon\ell_2)^{1/(n+1)}R.$$

This is a contradiction, since $\varepsilon\ell_2 < (b/4a)^{n+1} = I_C(v)^{n+1}/(8^{n+1}v^n)$ by (2.30). So the proof in case $h(x,R) = |E \cap B_C(x,R)|\,(|B_C(x,R))|^{-1}$ is completed.

For the remaining case, when $h(x,R) = |B_C(x,R)|^{-1}|B_C(x,R) \setminus E|$, we replace E by $C \setminus E$, which is also an isoperimetric region, and we are reduced to the previous case. □

Remark 2.27. Case $h(x,R) = |B_C(x,R)|^{-1}|B_C(x,R) \setminus E|$ is treated in [44] in a completely different way using the monotonicity of the isoperimetric profile in Carnot groups.

We define the sets

$$\begin{aligned} E_1 &= \{x \in C : \exists\, r > 0 \text{ such that } |B_C(x,r) \setminus E| = 0\}, \\ E_0 &= \{x \in C : \exists\, r > 0 \text{ such that } |B_C(x,r) \cap E| = 0\}, \\ S &= \{x \in C : h(x,r) > \varepsilon \text{ for all } r \leqslant 1\}. \end{aligned}$$

In the same way as in Theorem 4.3 of [44] we get

Proposition 2.28. *Let ε be as in Theorem 2.26. Then we have*

(i) *E_0, E_1 and S form a partition of C.*
(ii) *E_0 and E_1 are open in C.*
(iii) *$E_0 = E(0)$ and $E_1 = E(1)$.*
(iv) *$S = \partial E_0 = \partial E_1$, where the boundary is taken relative to C.*

As a consequence we get the following two corollaries

Corollary 2.29 (Lower density bound). *Let $C \subset \mathbb{R}^{n+1}$ be a convex body, and $E \subset C$ an isoperimetric region of volume v. Then there exists a constant $M > 0$, only depending on ε, on Poincaré constant for $r \leqslant 1$, and on an Ahlfors constant ℓ_1, such that*

$$P(E, B_C(x,r)) \geqslant M r^n, \tag{2.42}$$

for all $x \in \partial E_1$ and $r \leqslant 1$.

PROOF. If $x \in \partial E_1$, the choice of ε and the relative isoperimetric inequality (2.21) give

$$\begin{aligned} P(E, B_C(x,r)) &\geqslant M \min\{|E \cap B_C(x,r)|, |B_C(x,r) \setminus E|\}^{n/(n+1)} \\ &= M\,(|B_C(x,r)|\,h(x,r))^{n/(n+1)} \geqslant M(|B_C(x,r)|\,\varepsilon)^{n/(n+1)} \\ &\geqslant M\,(\ell_1\varepsilon)^{n/(n+1)}\,r^n. \end{aligned}$$

This implies the desired inequality. □

REMARK 2.30. If C_i is a sequence of convex bodies converging to a convex body C in Hausdorff distance, and $E_i \subset C_i$ is a sequence of isoperimetric regions converging weakly to an isoperimetric region $E \subset C$ of volume $0 < v < |C|$, then a constant $M > 0$ in (2.42) can be chosen independently of $i \in \mathbb{N}$. In fact, by (2.30), the constant ε only depends on $|E_i|$, $|C_i| - |E_i|$, and $I_{C_i}(|E_i|)$, which are uniformly bounded since $|C_i| \to |C|$ and $|E_i| \to |E|$. By the convergence in Hausdorff distance of C_i to C, both a lower Ahlfors constant ℓ_1 and a Poincaré constant can be chosen uniformly for all $i \in \mathbb{N}$.

REMARK 2.31. The classical monotonicity formula for rectifiable varifolds [69] can be applied in the interior of C to get the lower bound (2.42) for small r. Assuming C^2 regularity of the boundary of C (convexity is no longer needed), a monotonicity formula for varifolds with free boundary under boundedness condition on the mean curvature have been obtained by Grüter and Jost [38]. This monotonicity formula implies the lower density bound (2.42).

Now we prove that isoperimetric regions also converge in Hausdorff distance to their weak limits, which are also isoperimetric regions. It is necessary to choose a representative of the isoperimetric regions in the class of finite perimeter so that Hausdorff convergence makes sense: we simply consider the closure of the set E_1 of points of density one.

THEOREM 2.32. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of convex bodies that converges in Hausdorff distance to a convex body C. Let $E_i \subset C_i$ be a sequence of isoperimetric regions of volumes $v_i \to v \in (0, |C|)$. Let $f_i : C_i \to C$ be a sequence of bilipschitz maps with $\mathrm{Lip}(f_i), \mathrm{Lip}(f_i^{-1}) \to 1$.*

Then there is an isoperimetric set $E \subset C$ such that a subsequence of $f_i(E_i)$ converges to E in Hausdorff distance. Moreover, E_i converges to E in Hausdorff distance.

PROOF. The sequence $\{f_i(E_i)\}_{i\in\mathbb{N}}$ has uniformly bounded perimeter and so a subsequence, denoted in the same way, converges in $L^1(C)$ to a finite perimeter set E, which has volume v. The set E is isoperimetric in C since the sets E_i are isoperimetric in C_i and $I_{C_i}(v_i) \to I_C(v)$ by Corollary 2.14.

By Remark 2.30, we can choose $\varepsilon > 0$ so that Theorem 2.26 holds with this ε for all $i \in \mathbb{N}$. Choosing a smaller ε if necessary we get that, for any $x \in C$ and $0 < r \leqslant 1$, whenever $h(f_i(E_i), C, x, r) \leqslant \varepsilon$, we get $h(f_i(E_i), C, x, r/2) = 0$.

We now prove that $f_i(E_i) \to E$ in Hausdorff distance. As $\chi_{f_i(E_i)} \to \chi_E$ in $L^1(C)$, we can choose a sequence $r_i \to 0$ so that

$$|f_i(E_i) \triangle E| < r_i^{n+2}. \tag{2.43}$$

Now fix some $0 < r < 1$ and assume that, for some subsequence, there exist $x_i \in f_i(E_i) \setminus E_r$, where $E_r = \{x \in C : d(x, E) \leqslant r\}$. Choose i large enough so that $r_i < \min\{\frac{\ell_1}{2}, r\}$. Then, by (2.43),

$$|f_i(E_i) \cap B_C(x_i, r_i)| \leqslant |f_i(E_i) \setminus E| \leqslant |f_i(E_i) \triangle E| < r_i^{n+2} < \frac{\ell_1 r_i^{n+1}}{2} \leqslant \frac{|B_C(x_i, r_i)|}{2}. \tag{2.44}$$

So, for i large enough, we get

$$h(f_i(E_i), C, x_i, r_i) = \frac{|f_i(E_i) \cap B_C(x_i, r_i)|}{|B_C(x_i, r_i)|} < \ell_1^{-1} r_i \leqslant \varepsilon.$$

By Theorem 2.26, we conclude that $|f_i(E_i) \cap B_C(x, r_i/2)| = 0$. The normalization condition imposed on the isoperimetric regions implies a contradiction that shows that $f_i(E_i) \subset (E)_r$ for i large enough. In a similar way we get that $E \subset f_i(E_i)_r$, which proves that the Hausdorff distance between E and $f_i(E_i)$ is less than an arbitrary $r > 0$. So $f_i(E_i) \to E$ in Hausdorff distance.

Now we prove $\delta(E_i, E) \to 0$. By the triangle inequality we have

$$\delta(E_i, E) \leqslant \delta(f_i(E_i), E) + \delta(f_i(E_i), E_i).$$

It only remains to show that $\delta(f_i(E_i), E_i) \to 0$. For $x \in E_i$ we have

$$\operatorname{dist}(f_i(x), E_i) \leqslant |f_i(x) - x|.$$

Assume that $r > 0$ is as in definition (2.6) of f_i. Recall that $B(0, 2r) \subset C_i \cap C$ and that $C_i \cup C \subset B(0, R)$. Then by (2.7) we get $|f_i(x) - x| = 0$ if $|x| \leqslant r$ and

$$|f_i(x) - x| \leqslant \frac{(R - r)}{r} \left| \rho_i\Big(\frac{x}{|x|}\Big) - \rho\Big(\frac{x}{|x|}\Big) \right|$$

if $|x| \geqslant r$. Lemma 2.2 then implies the existence of a sequence of positive real numbers $\varepsilon_i \to 0$ such that $|f_i(x) - x| \leqslant \varepsilon_i$ for all $x \in E_i$. We conclude that

$$f_i(E_i) \subset (E_i)_{\varepsilon_i}.$$

Writing $E_i = f_i^{-1}(f_i(E_i))$ and reasoning as above with f_i^{-1} instead of f_i we obtain

$$E_i \subset (f_i(E_i))_{\varepsilon_i},$$

By the definition of the Hausdorff distance δ, we get $\delta(f_i(E_i), E_i) \to 0$. □

Recall that in Theorem 2.16 we showed that the boundaries of isoperimetric regions in convex sets with $C^{2,\alpha}$ boundary are connected. For arbitrary convex sets we have the following

THEOREM 2.33. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body. For every volume $0 < v < |C|$ there exists an isoperimetric region in C of volume v with connected boundary.*

We shall use the following result in the proof of Theorem 2.33.

THEOREM 2.34. *Let $\{C_i\}_{i\in\mathbb{N}}$ a sequence of convex bodies converging in Hausdorff distance to a convex body C, and let $E_i \subset C_i$ be a sequence of isoperimetric regions converging in Hausdorff distance to an isoperimetric region $E \subset C$.*

Then a subsequence of $\operatorname{cl}(\partial E_i \cap \operatorname{int}(C_i))$ *converges to* $\operatorname{cl}(\partial E \cap \operatorname{int}(C))$ *in Hausdorff distance as well.*

PROOF OF THEOREM 2.33. Let $C_i \subset \mathbb{R}^{n+1}$ be convex bodies with $C^{2,\alpha}$ boundary converging to C in Hausdorff distance. Let $E_i \subset C_i$ be isoperimetric regions of volumes approaching v. By Theorem 2.32, a subsequence of the sets E_i converges to E in Hausdorff distance, where $E \subset C$ is an isoperimetric region of volume v. By Theorem 2.34, a subsequence of the sets $\operatorname{cl}(\partial E_i \cap \operatorname{int}(C_i))$ converges to $\operatorname{cl}(\partial E \cap \operatorname{int}(C))$ in Hausdorff distance. Theorem 2.16 implies that the sets $\operatorname{cl}(\partial E_i \cap \operatorname{int}(C_i))$ are connected. By Proposition A.1.7 in [42], $\operatorname{cl}(\partial E \cap \operatorname{int}(C))$ is connected as well. □

PROOF OF THEOREM 2.34. We shall prove that that the sequence $\{\operatorname{cl}(\partial E_i \cap \operatorname{int}(C_i))\}_{i\in\mathbb{N}}$ converges to $\operatorname{cl}(\partial E \cap \operatorname{int}(C))$ in Kuratowski sense [4, 4.4.13]

1. If $x = \lim_{j\to\infty} x_{i_j}$ for some subsequence $x_{i_j} \in \operatorname{cl}(\partial E_{i_j} \cap \operatorname{int}(C_i))$, then $x \in \operatorname{cl}(\partial E \cap \operatorname{int}(C))$, and
2. If $x \in \operatorname{cl}(\partial E \cap \operatorname{int}(C))$, then there exists a sequence $x_i \in \operatorname{cl}(\partial E \cap \operatorname{int}(C))$ converging to x.

Assume 1 does not hold. To simplify the notation we shall assume that $x = \lim_{i\to\infty} x_i$, with $x_i \in \operatorname{cl}(\partial E_i \cap \operatorname{int}(C_i))$. If $x \notin \operatorname{cl}(\partial E \cap \operatorname{int}(C))$ we had $x \in \operatorname{int}(E) \cup \operatorname{int}(C \setminus E)$. If $x \in \operatorname{int}(E)$, then there exists $r > 0$ such that $|B(x,r) \cap (C \setminus E)| = 0$. Since $x_i \to x$, and E_i, C_i converge to E, C in Hausdorff sense, respectively, we conclude by [4, Proposition 4.4.14] that $\bar{B}(x_i,r) \cap (C_i \setminus E_i) \to \bar{B}(x,r) \cap (C \setminus E)$ in the Hausdorff sense as well. Thus by [16, Lemma III.1.1] we get

$$\limsup_{i\to\infty} |B(x_i,r) \cap (C_i \setminus E_i)| \leqslant |B(x,r) \cap (C \setminus E)| = 0.$$

Now if $\varepsilon > 0$ is as in Theorem 2.26, we get $|B(x_i,r) \cap (C_i \setminus E_i)| \leqslant \varepsilon$ for all large $i \in \mathbb{N}$ which implies $|B(x_i,r/2) \cap (C_i \setminus E_i)| = 0$. This contradicts the fact that $x_i \in \operatorname{cl}(\partial E_i \cap \operatorname{int}(C_i))$. Assuming $x \in C \setminus E$ and arguing similarly we would find $|B(x_i,r/2) \cap \operatorname{int}(E_i)| = 0$. Thus $x \in \operatorname{cl}(\partial E \cap \operatorname{int}(C))$.

Assume now that 2 does not hold. Then there exists $x \in \operatorname{cl}(\partial E \cap \operatorname{int}(C))$ so that no sequence in $\operatorname{cl}(\partial E_i \cap \operatorname{int}(C_i))$ converges to x. We may assume that, passing to a subsequence if necessary, that there exists $\eta > 0$ so that $B_C(x,\eta)$ does not contain any point in $\operatorname{cl}(\partial E_i \cap \operatorname{int}(C_i))$. The radius η can be chosen less than ε. Reasoning as in Case 1, we conclude that either $B_C(x,\eta/2) \cap E_i = \emptyset$ or $B_C(x,\eta/2) \cap (C \setminus E_i) = \emptyset$. □

2.4. The asymptotic isoperimetric profile of a convex body

In this section we shall prove that isoperimetric regions of small volume inside a convex body concentrate near boundary points whose tangent cone has the smallest possible solid angle. This will be proven by rescaling the isoperimetric regions and then studying their convergence, as in Morgan and Johnson [54]. We shall recall first some results on convex cones.

Let $K \subset \mathbb{R}^{n+1}$ be a closed convex cone with vertex p . Let $\alpha(K) = H^n(\partial B(p,1) \cap \operatorname{int}(K))$ be the *solid angle* of K. It is known that the geodesic balls centered at the vertex are isoperimetric regions in K, [46], [60], and that they are the only ones [27] for general convex cones, without any regularity assumption on the boundary. The invariance of K by dilations centered at some vertex yields

$$I_K(v) = I_K(1)\, v^{n/(n+1)} = \alpha(K)^{1/(n+1)}\,(n+1)^{n/(n+1)} v^{n/(n+1)}. \tag{2.45}$$

Consequently the isoperimetric profile of a convex cone is completely determined by its solid angle.

We define the tangent cone C_p of a (possibly unbounded) convex body C at a given boundary point $p \in \partial C$ as the closure of the set

$$\bigcup_{\lambda>0} h_{p,\lambda}(C),$$

where $h_{p,\lambda}$ denotes the dilation of center p and factor λ. The solid angle $\alpha(C_p)$ of C_p will be denoted by $\alpha(p)$. Tangent cones to convex bodies have been widely considered in convex geometry under the name of supporting cones [68, § 2.2] or projection cones [13]. In the following result, we prove the lower semicontinuity of the solid angle of tangent cones in convex modies.

LEMMA 2.35. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body, $\{p_i\}_{i\in\mathbb{N}} \subset \partial C$ so that $p = \lim_{i\to\infty} p_i$. Then*

$$\alpha(p) \leqslant \liminf_{i\to\infty} \alpha(p_i). \tag{2.46}$$

In particular, this implies the existence of points in ∂C whose tangent cones are minima of the solid angle function.

PROOF. We may assume that $\alpha(p_i)$ converges to $\liminf_{i\to\infty} \alpha(p_i)$ passing to a subsequence if necessary. Since the sequence $C_{p_i} \cap \overline{B}(p_i,1)$ is bounded for the Hausdorff distance, we can extract a subsequence (denoted in the same way) converging to a convex body $C_\infty \subset \overline{B}(p,1)$. It is easy to check that C_∞ is the intersection of a closed convex cone K_∞ of vertex p with $\overline{B}(p,1)$, and that $C_p \subset K_\infty$. By the continuity of the volume with respect to the Hausdorff distance we have

$$\alpha(p) = |C_p \cap \overline{B}(p,1)| \leqslant |C_\infty| = \lim_{i\to\infty} |C_{p_i} \cap \overline{B}(p_i,1)| = \lim_{i\to\infty} \alpha(p_i),$$

yielding (2.46). To prove the existence of tangent cones with the smallest solid angle, we simply take a sequence $\{p_i\}_{i\in\mathbb{N}}$ of points at the boundary of C so that $\alpha(p_i)$ converges to $\inf\{\alpha(p) : p \in \partial C\}$, we extract a convergent subsequence, and we apply the lower semicontinuity of the solid angle function.

□

The isoperimetric profiles of tangent cones which are minima of the solid angle function coincide. The common profile will be denoted by $I_{C_{\min}}$.

PROPOSITION 2.36 ([62, Proposition 6.2]). *Let $C \subset \mathbb{R}^{n+1}$ be a convex body (possibly unbounded), and $p \in \partial C$. Then every intrinsic ball in C centered at p has no more perimeter than an intrinsic ball of the same volume in C_p. Consequently*

$$I_C(v) \leqslant I_{C_p}(v), \tag{2.47}$$

for all $0 < v < |C|$. Furthermore, if C is bounded then

$$I_C(v) \leqslant I_{C_{\min}}(v), \tag{2.48}$$

for all $0 \leqslant v \leqslant |C|$.

REMARK 2.37. A closed half-space $H \subset \mathbb{R}^{n+1}$ is a convex cone with the largest possible solid angle. Hence, for any convex body $C \subset \mathbb{R}^{n+1}$, we have

$$I_C(v) \leqslant I_H(v), \tag{2.49}$$

for all $0 < v < |C|$.

REMARK 2.38. Proposition 2.36 implies that $E \cap \partial C \neq \emptyset$ when $E \subset C$ is isoperimetric. Since in case $E \cap \partial C$ is empty, then E is an Euclidean ball. Moreover, as the isoperimetric profile of Euclidean space is strictly larger than that of the half-space, a set whose perimeter is close to the value of the isoperimetric profile of C must touch the boundary of C.

PROOF OF PROPOSITION 2.36. Let $0 < v < |C|$ and $p \in \partial C$. Let $r > 0$ such that $|B_C(p,r)| = v$. The closure of the set $\partial B(p,r) \cap \operatorname{int}(C)$ is a geodesic sphere of the closed cone K_p of vertex p subtended by the closure of $\partial B(p,r) \cap \operatorname{int}(C)$. If $S = \partial B(p,r) \cap \operatorname{int}(C)$ then $S = \partial B(p,r) \cap \operatorname{int}(K_p)$ as well. By the convexity of C, $B(p,r) \cap \operatorname{int}(K_p) \subset B(p,r) \cap \operatorname{int}(C)$ and so $v_0 = H^{n+1}(B(p,r) \cap \operatorname{int}(K_p)) \leqslant v$. Since $K_p \subset C_p$, (2.45) implies $H^n(S) \leqslant I_{C_p}(v_0)$. So we have

$$I_C(v) \leqslant P_C(B_C(p,r)) = H^n(S) \leqslant I_{C_p}(v_0) \leqslant I_{C_p}(v),$$

as I_{C_p} is an increasing function. This proves (2.47). Now if C is bounded we choose $p \in \partial C$ such that $I_{C_p} = I_{C_{\min}}$ to prove (2.48). □

We now prove the following result which strongly depends on the paper by Figalli and Indrei [27].

LEMMA 2.39. *Let $K \subset \mathbb{R}^{n+1}$ be a closed convex cone. Consider a sequence of sets E_i of finite perimeter in* $\operatorname{int}(K)$ *such that $v_i = |E_i| \to v$. Then*

$$\liminf_{i\to\infty} P_K(E_i) \geqslant I_K(v). \tag{2.50}$$

If equality holds, then there is a family of vectors x_i such that $x_i + K \subset K$, and $x_i + E_i$ converges to a geodesic ball centered at 0 *of volume v.*

PROOF. We assume $K = \mathbb{R}^k \times \tilde{K}$, where $k \in \mathbb{N} \cup \{0\}$ and $\tilde{K}$ is a closed convex cone which contains no lines so that 0 is an apex of $\tilde{K}$. Inequality (2.50) follows from $P_K(E_i) \geqslant I_K(v_i)$ and the continuity of I_K. Let $B(w)$ be the geodesic ball in K centered at 0 of volume $w > 0$. If equality holds in (2.50) then

$$\mu(E_i) = \left(\frac{P_K(E_i)}{I_K(v_i)} - 1\right) \to 0.$$

Define s_i by the equality $|B(v_i)| = |s_i B(v)|$. Obviously $s_i \to 1$. By Theorem 1.2 in [27] there is a sequence of points $x_i \in \mathbb{R}^k \times \{0\}$ such that

$$\left(\frac{|E_i \triangle (s_i B(v) + x_i)|}{|E_i|}\right) \leqslant C(n, B(v)) \left(\sqrt{\mu(E_i)} + \frac{1}{i}\right).$$

Since $\mu(E_i) \to 0$, and $|E_i| \to v > 0$, taking limsup we get $|E_i \triangle (s_i B(v) + x_i)| \to 0$ and so $|(E_i - x_i) \triangle B(v)| \to 0$, which proves the result. □

THEOREM 2.40. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body. Then*

$$\lim_{v\to 0} \frac{I_C(v)}{I_{C_{\min}}(v)} = 1. \tag{2.51}$$

Moreover, a rescaling of a sequence of isoperimetric regions of volumes approaching 0 *has a convergent subsequence in Hausdorff distance to a geodesic ball centered at some vertex in a tangent cone with the smallest solid angle. The same convergence result holds for their free boundaries.*

PROOF. To prove (2.51) we first observe that the invariance of the tangent cone by dilations implies that (2.48) is valid for every λC with $\lambda > 0$, i. e., $I_{\lambda C} \leqslant I_{C_{\min}}$. So we get

$$\limsup_{i\to\infty} I_{\lambda_i C}(v) \leqslant I_{C_{\min}}(v), \tag{2.52}$$

for any sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of positive numbers such that $\lambda_i \to \infty$ and any $v > 0$.

Consider now a sequence $\{E_i\}_{i\in\mathbb{N}} \subset C$ of isoperimetric regions of volumes $v_i \to 0$ and $p_i \in E_i \cap \partial C$. Translating the convex set and passing to a subsequence we may assume that $p_i \to 0 \in \partial C$. Let $\lambda_i = v_i^{-1/(n+1)}$. Then $\lambda_i \to \infty$ and $\lambda_i E_i$ are isoperimetric regions in $\lambda_i C$ of volume 1. By Theorem 2.15, the sets $\lambda_i E_i$ are connected. We claim that

$$\sup_{i\in\mathbb{N}} \operatorname{diam}(\lambda_i E_i) < \infty. \tag{2.53}$$

If claim holds, since $p_i \to 0$, there is a sequence $\tau_i \to 0$ such that $E_i \subset C \cap \overline{B}(0, \tau_i)$. Let $q \in \operatorname{int}(C \cap \overline{B}(0,1))$ and $B_q \subset \operatorname{int}(C \cap \overline{B}(0,1))$ a Euclidean geodesic ball. Now consider a solid cone K_q with vertex q such that $0 \in \operatorname{int}(K_q)$ and $K_q \cap C_0 \cap \partial B(0,1) = \emptyset$. Let $s > 0$ so that $\bar{B}(0,s) \subset K_q$. Taking $r_i = s^{-1}\tau_i$, $i \in \mathbb{N}$, we have

$$r_i^{-1}E_i \subset \bar{B}(0, r_i^{-1}\tau_i) = \bar{B}(0,s) \subset K_q.$$

As the sequence $r_i^{-1}C \cap \bar{B}(0,1)$ converges in Hausdorff distance to $C_0 \cap \bar{B}(0,1)$ we construct, using Theorem 2.4, a family of bilipschitz maps $h_i : r_i^{-1}C \cap \bar{B}(0,1) \to C_0 \cap \bar{B}(0,1)$ using the ball B_q. So h_i is the identity in B_q and it is extended linearly along the segments leaving from q. By construction, the maps h_i have the additional property

$$P_{C_0}(h_i(r_i^{-1}E_i)) = P_{C_0 \cap \bar{B}(0,1)}(h_i(r_i^{-1}E_i)). \tag{2.54}$$

So the sequence of bilipschitz maps $g_i : \lambda_i C \cap \bar{B}(0, \lambda_i r_i) \to C_0 \cap \bar{B}(0, \lambda_i r_i)$, obtained as in Remark 1.5 with the property $\operatorname{Lip}(h_i) = \operatorname{Lip}(g_i)$ and $\operatorname{Lip}(h_i) = \operatorname{Lip}(g_i^{-1})$ satisfies

$$P_{C_0}(g_i(\lambda_i E_i)) = P_{C_0 \cap \bar{B}(0,\lambda_i r_i)}(g_i(\lambda_i E_i)).$$

This property and Lemma 1.3 imply

$$\begin{aligned} \lim_{i\to\infty} |g_i(\lambda_i E_i)| &= \lim_{i\to\infty} |\lambda_i E_i|, \\ \lim_{i\to\infty} P_{C_o}(g_i(\lambda_i E_i)) &= \lim_{i\to\infty} P_{\lambda_i C}(\lambda_i E_i). \end{aligned} \tag{2.55}$$

From these equalities, the continuity of I_{C_0}, and the fact that $\lambda_i E_i \subset \lambda_i C$ are isoperimetric regions of volume 1, we get

$$I_{C_0}(1) \leqslant \liminf_{i\to\infty} I_{\lambda_i C}(1).$$

combining this with (2.52) and the minimal property of $C_{\min}$ we deduce

$$\limsup_{i\to\infty} I_{\lambda_i C}(1) \leqslant I_{C_{\min}}(1) \leqslant I_{C_0}(1) \leqslant \liminf_{i\to\infty} I_{\lambda_i C}(1).$$

Thus

$$I_{C_0}(1) = I_{C_{\min}}(1) = \lim_{i\to\infty} I_{\lambda_i C}(1). \tag{2.56}$$

By (2.45), we deduce that C_0 has minimum solid angle. Finally, from (2.56), (2.22), and the fact that $\lambda C_0 = C_0$ we deduce

$$1 = \lim_{i\to\infty} \frac{I_{\lambda_i C}(1)}{I_{C_0}(1)} = \lim_{i\to\infty} \frac{\lambda_i^n I_C(1/\lambda_i^{n+1})}{\lambda_i^n I_{C_0}(1/\lambda_i^{n+1})} = \lim_{i\to\infty} \frac{I_C(v_i)}{I_{C_0}(v_i)}.$$

So it remains to prove (2.53) to conclude the proof. For this it is enough to prove

$$P_{\lambda_i C}(F_i, B_{\lambda_i C}(x,r)) \geqslant M r^n, \tag{2.57}$$

for any $0 < r \leqslant 1$, $x \in C$, and any isoperimetric region $F_i \subset \lambda_i C$ of volume 1. The constant $M > 0$ is independent of i.

To prove (2.57), observe first that the constant M in the relative isoperimetric inequality (2.21) is invariant by dilations and, if the factor of dilation is chosen larger than 1 then the estimate $r \leqslant r_0$ is uniform. The same argument can be applied to a lower Ahlfors constant ℓ_1. The constant $\ell_2 = \omega_{n+1} = |\bar{B}(0,1)|$ is universal and does not depend on the convex set.

Now we modify the proof of Theorem 2.26 to show that there exists some $\varepsilon > 0$, independent of i, so that if $h(\lambda_i E_i, \lambda_i C, x, r) \leqslant \varepsilon$ then $h(\lambda_i E_i, \lambda_i C, x, r/2) = 0$, for $0 < r \leqslant 1$.

First we treat the case

$$h(F_i, \lambda_i C, x, R) = \frac{|F_i \cap B_{\lambda_i C}(x, R)|}{|B_{\lambda_i C}(x, R)|}.$$

By Theorem 2.26, since $I_C(1) \leqslant I_{\lambda_i C}(1)$ for all $i \in \mathbb{N}$, it is enough to take

$$0 < \varepsilon \leqslant \min \left\{ \frac{1}{\ell_2}, c_2, \frac{I_C(1)^{n+1}}{\ell_2 8^{n+1}} \right\}.$$

Now when

$$h(F_i, \lambda_i C, x, R) = \frac{|B_{\lambda_i C}(x, R) \setminus F_i|}{|B_{\lambda C}(x, R)|},$$

we proceed as in the proof of Case 1 of Lemma 4.2 in [44]. For λ_i large enough we have $1 + \ell_2 = |\lambda_i E_i| + \ell_2 < |\lambda_i C|/2$. As $I_{\lambda_i C}$ is increasing in the interval $(0, |\lambda_i C|/2]$ the proof of Case 1 in Lemma 4.2 of [44] provides an $\varepsilon > 0$ independent of i.

As in Remark 2.30 we conclude the existence of $M > 0$ independent of i so that (2.57) holds.

Now, if $\operatorname{diam}(\lambda_i E_i)$ is not uniformly bounded, (2.57) implies that $P_{\lambda_i C}(\lambda_i E_i)$ is unbounded. But this contradicts the fact that $P_{\lambda_i C}(\lambda_i E_i) = I_{\lambda_i C}(1) \leqslant I_{C_{\min}}(1)$ for all i.

Finally we prove that $\lambda_i E_i$ converges to E in Hausdorff distance, where $E \subset C_0$ is a geodesic ball of volume 1 centered at 0. By (2.55), $\{g_i(\lambda_i E_i)\}_{i \in \mathbb{N}}$ is a minimizing sequence in C_0 of volume 1. By Lemma 2.39, translating the whole sequence $\{g_i(\lambda_i E_i)\}_{i \in \mathbb{N}}$ if necessary we may assume it is uniformly bounded and so a subsequence of $g_i(\lambda_i E_i) \to E$ in $L^1(C_0)$. Theorem 2.32 implies the Hausdorff convergence of the isoperimetric regions. Theorem 2.34 implies the convergence of the free boundaries. □

From Theorem 2.40 we easily get

COROLLARY 2.41. *Let $C, K \subset \mathbb{R}^{n+1}$ be convex bodies, with $I_{C_{\min}} > I_{K_{\min}}$. Then for small volumes we have $I_C > I_K$.*

For polytopes we are able to show which are the isoperimetric regions for small volumes. The same result holds for any convex set so that there is $r > 0$ such that, at every point $p \in \partial C$ with tangent cone of minimum solid angle we have $B(p, r) \cap C_p = B(p, r) \cap C$.

THEOREM 2.42. *Let $P \subset \mathbb{R}^{n+1}$ be a convex polytope. For small volumes the isoperimetric regions in P are geodesic balls centered at vertices with the smallest solid angle.*

PROOF. Let $\{E_i\}_{i \in \mathbb{N}}$ be a sequence of isoperimetric regions in P with $|E_i| \to 0$. By Theorem 2.40, a subsequence of E_i is close to some vertex x in P. Since $\operatorname{diam}(E_i) \to 0$ we can suppose that, for small enough volumes, the sets E_i are also subsets of the tangent cone P_x

and they are isoperimetric regions in P_x. By [27] the only isoperimetric regions in this cone are the geodesic balls centered at x. These geodesic balls are also subsets of P. $\square$

REMARK 2.43. In [25] Fall considered the partitioning problem of a domain with smooth boundary in a smooth Riemannian manifold. He showed that, for small enough volume, the isoperimetric regions are concentrated near the maxima of the mean curvature function and that they are asymptotic to half-spheres. The techniques used in this paper are similar to the ones used by Nardulli [56] in his study of isoperimetric regions of small volume in compact Riemannian manifolds. See also [54, Thm. 2.2].

PROPOSITION 2.44. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body and $\{E_i\}_{i\in\mathbb{N}}$ a sequence of isoperimetric regions with $|E_i| \to 0$. Assume that $0 \in \partial C$ and that C_0 is a tangent cone with the smallest solid angle. Let $\lambda_i > 0$ be so that $|\lambda_i E_i| = 1$, and let $E \subset C_0$ be the geodesic ball in C_0 centered at 0 of volume 1. Then, for every $x \in \partial E \cap \operatorname{int}(C_0)$ so that $B(x,r) \subset \operatorname{int}(C_0)$, the boundary $\partial \lambda_i E_i \cap B(x,r)$ is a smooth graph with constant mean curvature for i large enough.*

PROOF. We use Allard's Regularity Theorem for rectifiable varifolds, see [1], [69].

Assume $\{E_i\}_{i\in\mathbb{N}}$ is a sequence of isoperimetric regions of volumes $v_i \to 0$, and that $0 \in \partial C$ is an accumulation point of points in E_i. We rescale so that $|\lambda_i E_i| = 1$, project to C_0 (by means of the mapping g_i), and rescale again to get a minimizing sequence F_i in C_0 of volume 1. The sequence $\{F_i\}_{i\in\mathbb{N}}$ converges in $L^1(C_0)$ by Lemma 2.39.

If $v_i = |E_i| \to 0$ then $\lambda_i = v_i^{-1/(n+1)}$. Let H_i be the constant mean curvature of the reduced boundary of E_i. Then the mean curvature of the reduced boundary of $\lambda_i E_i$ is $\frac{1}{\lambda_i} H_i = v_i^{1/(n+1)} H_i$. Let us check that these values are uniformly bounded.

From (2.18) we get

$$I_C(v) \geqslant m v^{n/(n+1)}, \tag{2.58}$$

for all $0 < v < \frac{|C|}{2}$ with $m = I_C(|C|/2)/(|C|/2)^{n/(n+1)}$. We also have

$$I_C^{(n+1)/n}(v) \leqslant M v \tag{2.59}$$

for all $0 < v < |C|$. Here M can be chosen as a power of the isoperimetric constant of $C_{\min}$ or $\mathbb{H}^{n+1}$ since $I_C \leqslant I_{C_{\min}} \leqslant I_H$ by Proposition 2.36 and Remark 2.37. Since $Y_C = I_C^{(n+1)/n}$ is concave, given $h > 0$ small enough, using (2.59) we have

$$\frac{Y_C(v) - Y_C(v-h)}{h} \leqslant \frac{Y_C(v)}{v} \leqslant M.$$

Taking limits when $h \to 0$ we get

$$(Y_C)'_-(v) \leqslant M,$$

for all $0 < v < |C|$. By the chain rule

$$\left(\frac{n+1}{n}\right) I_C^{1/n}(v) (I_C)'_-(v) = (Y_C)'_-(v) \leqslant M.$$

Since the mean curvature H of any isoperimetric region of volume v satisfies $H \leqslant (I_C)'_-(v)$, using (2.58) we have

$$\left(\frac{n+1}{n}\right) m^{1/n} v^{1/(n+1)} H \leqslant \left(\frac{n+1}{n}\right) I_C^{1/n}(v)(I_C)'_-(v) = (Y_C)'_-(v) \leqslant M$$

So the quantity $v^{1/(n+1)}H$ is uniformly bounded for any $0 < v < |C|$. This implies that the constant mean curvature of the reduced boundary of the regions $\lambda_i E_i$ is uniformly bounded. □

CHAPTER 3

Cilindrically bounded convex bodies

3.1. Isoperimetric regions in cylinders

In this Section we consider the isoperimetric problem when the ambient space is a convex cylinder $K \times \mathbb{R}^q$, where $K \subset \mathbb{R}^m$ is a convex body. We shall assume that $m+q=n+1$. Existence of isoperimetric regions in $K \times \mathbb{R}^q$ can be obtained following the strategy of Galli and Ritoré for contact sub-Riemannian manifolds [28] with compact quotient under their contact isometry group. One of the basic ingredients in this strategy is the relative isoperimetric inequality in Proposition 3.1. A second one is the property that any unbounded convex body C is a doubling metric space

PROPOSITION 3.1. *Let $C = K \times \mathbb{R}^q$, where K is an m-dimensional convex body. Given $r_0 > 0$, there exist positive constants M, ℓ_1, only depending on r_0 and C, and a universal positive constant ℓ_2 so that*

$$P_{\overline{B}_C(x,r)}(v) \geqslant M \min\{v, |\overline{B}_C(x,r)| - v\}^{n/(n+1)}, \tag{3.1}$$

for all $x \in C$, $0 < r \leqslant r_0$, and $0 < v < |\overline{B}_C(x,r)|$, and

$$\ell_1 r^{n+1} \leqslant |\overline{B}_C(x,r)| \leqslant \ell_2 r^{n+1}, \tag{3.2}$$

for any $x \in C$, $0 < r \leqslant r_0$.

PROOF. Since the quotient of C by its isometry group is compact, the proof is reduced to that of Theorem 2.21. □

Using Lemma 1.9 and Proposition 2.36 we can show

PROPOSITION 3.2. *Consider the convex cylinder $C = K \times \mathbb{R}^q$, where $K \subset \mathbb{R}^m$ is a convex body. Then isoperimetric regions exist in $K \times \mathbb{R}^q$ for all volumes and they are bounded.*

PROOF. To follow the strategy of Galli and Ritoré [28] (see Morgan [53] for a slightly different proof for smooth Riemannian manifolds), we only need a relative isoperimetric inequality (3.1) for balls $\overline{B}_C(x,r)$ of small radius with a uniform constant; the doubling property (1.13); inequality (2.47) giving an upper bound of the isoperimetric profile; and a deformation of isoperimetric sets E by finite perimeter sets E_t satisfying

$$|H^n(\partial E_t \cap \operatorname{int}(C)) - H^n(\partial E \cap \operatorname{int}(C))| \leqslant M \, ||E_t| - |E||,$$

for small $|t|$ and some constant $M > 0$ not depending in t, which can be obtained by deforming the regular part of the boundary of E using the flow associated to a vector field with compact support.

Using all these ingredients, the proof of Theorem 6.1 in [28] applies to prove existence of isoperimetric regions in $K \times \mathbb{R}^q$. $\square$

Let us prove now the concavity of the isoperimetric profile of the cylinder and of its power $\frac{n+1}{n}$. We start by proving its continuity.

PROPOSITION 3.3. *Let $C = K \times \mathbb{R}^q$, where K is an m-dimensional convex body. Then I_C is non-decreasing and continuous.*

PROOF. Given $t > 0$, the smooth map $\varphi_t : C \to C$ defined by $\varphi_t(x, y) = (x, ty)$, $x \in C$, $y \in \mathbb{R}^q$, satisfies $|\varphi_t(E)| = t^q |E|$. When $t \leqslant 1$, we also have $P_C(\varphi_t(E)) \leqslant t^{q-1} P_C(E)$. This implies that the isoperimetric profile is a non-decreasing function. Hence it can only have jump discontinuities.

If E is an isoperimetric region of volume v, using a smooth vector field supported in the regular part of the boundary of E, one can find a continuous function f, defined in a neighborhood of v, so that $I \leqslant f$. This implies that I cannot have jump discontinuities at v. $\square$

LEMMA 3.4. *Let $\{K_i\}_{i\in\mathbb{N}}$ be a sequence of m-dimensional convex bodies converging to a convex body K in Hausdorff distance. Then $\{K_i \times \mathbb{R}^q\}_{i\in\mathbb{N}}$ converges to $K \times \mathbb{R}^q$ in lipschitz distance.*

PROOF. By Theorem 2.4, there exists a sequence of bilipschitz maps $f_i : K_i \to K$ such that $\operatorname{Lip}(f_i), \operatorname{Lip}(f_i^{-1}) \to 1$ as $i \to \infty$. For every $i \in \mathbb{N}$, define $F_i : K_i \times \mathbb{R}^q \to K \times \mathbb{R}^q$ by

$$F_i(x, y) = (f_i(x), y), \qquad (x, y) \in K_i \times \mathbb{R}^q. \tag{3.3}$$

Take now $(x_1, y_1), (x_2, y_2) \in K_i \times \mathbb{R}^q$. We have

$$\begin{aligned} |F_i(x_1, y_1) - F_i(x_2, y_2)|^2 &= |f_i(x_1) - f_i(x_2)|^2 + |y_1 - y_2|^2 \\ &\leqslant \max\{\operatorname{Lip}(f_i)^2, 1\}\left(|x_1 - x_2|^2 + |y_1 - y_2|^2\right) \\ &= \max\{\operatorname{Lip}(f_i)^2, 1\}\left|(x_1, y_1) - (x_2, y_2)\right|^2, \end{aligned} \tag{3.4}$$

where $|\cdot|$ is the Euclidean norm in the suitable Euclidean space. Hence we get

$$\limsup_{i\to\infty} \operatorname{Lip}(F_i) \leqslant 1$$

since $\lim_{i\to\infty} \operatorname{Lip}(f_i) = 1$. In a similar way we find $\limsup_{i\to\infty} \operatorname{Lip}(F_i^{-1}) \leqslant 1$. By Remark 1.4, we get $\operatorname{Lip}(F_i^{-1}) \operatorname{Lip}(F_i) \geqslant 1$ and the proof follows. $\square$

PROPOSITION 3.5. *Let $K \subset \mathbb{R}^m$ be a convex body and $C = K \times \mathbb{R}^q$. Then $I_C^{(n+1)/n}$ is a concave function. This implies that I_C is concave and every isoperimetric set in C is connected.*

PROOF. When the boundary of a convex cylinder C is smooth, its isoperimetric profile I_C and its power $I_C^{(n+1)/n}$ are known to be concave using a suitable deformation of an isoperimetric region and the first and second variations of perimeter and volume, as in Kuwert [43].

By approximation [68], there exists a sequence $\{K_i\}_{i\in\mathbb{N}}$ of convex bodies in $\mathbb{R}^m$ with C^∞ boundary such that $K_i \to K$ in Hausdorff distance. Set $C_i = K_i \times \mathbb{R}^q$. By Lemma 3.4, $C_i \to C$ in lipschitz distance. Fix now some $v > 0$. By Proposition 3.2, there is a sequence of isoperimetric sets $E_i \subset C_i$ of volume v. Thus arguing as in Theorem (2.10), using the continuity of the isoperimetric profile I_C, we get

$$I_C(v) \leqslant \liminf_{i\to\infty} I_{C_i}(v).$$

Again by Proposition 3.2 there exists an isoperimetric set $E \subset C$ of volume v. Arguing again as in (2.10), we obtain

$$I_C(v) \geqslant \limsup_{i\to\infty} I_{C_i}(v).$$

Combining both inequalities we get

$$I_C(v) = \lim_{i\to\infty} I_{C_i}(v).$$

So $I_C^{(n+1)/n}$, I_C are concave functions as they are pointwise limits of concave functions.

Connectedness of isoperimetric regions is a consequence of the concavity of $I_C^{(n+1)/n}$ as in Theorem 2.15. □

Assume now that the cylinder $C = K \times \mathbb{R}^q$ has $C^{2,\alpha}$ boundary. By Theorem 2.6 in Stredulinsky and Ziemer [71], a local minimizer of perimeter under a volume constraint has the property that either $\mathrm{cl}(\partial E \cap \mathrm{int}(C))$, the closure of $\partial E \cap \mathrm{int}(C)$, is either connected or it consists of a union of parallel (totally geodesic) components meeting ∂C orthogonally with the part of C lying between any two of such components consisting of a *right* cylinder. By the connectedness of isoperimetric regions proven in Proposition 3.5, E must be a slab in $K \times \mathbb{R}$. So we have proven the following

THEOREM 3.6. *Let $C = K \times \mathbb{R}^q$ be a convex cylinder with $C^{2,\alpha}$ boundary, and $E \subset C$ an isoperimetric region. Then either the closure of $\partial E \cap \mathrm{int}(C)$ is connected or E is an slab in $K \times \mathbb{R}$.*

Since the quotient of the cylinder $C = K \times \mathbb{R}^q$ by its isometry group is compact, then adapting Lemma 2.35 we get the existence of points in ∂C whose tangent cones are minima of the solid angle function. By (2.45), the isoperimetric profiles of tangent cones which are minima of the solid angle function coincide. The common profile will be denoted by $I_{C_{\min}}$.

Let us consider now the isoperimetric profile for small volumes. The following is inspired by Theorem 2.40, although we have simplified the proof.

THEOREM 3.7. *Let $C = K \times \mathbb{R}^q$, where $K \subset \mathbb{R}^m$ is a convex body. Then, after translation, isoperimetric regions of small volume are close to points with the narrowest tangent cone. Furthermore,*

$$\lim_{v\to 0} \frac{I_C(v)}{I_{C_{\min}}(v)} = 1. \tag{3.5}$$

PROOF. To prove (3.5), consider a sequence $\{E_i\}_{i\in\mathbb{N}} \subset C$ of isoperimetric regions of volumes $v_i \to 0$. By Proposition 3.5, the sets E_i are connected. The key of the proof is to show

$$\operatorname{diam}(E_i) \to 0. \tag{3.6}$$

To accomplish this we consider $\lambda_i \to \infty$ so that the isoperimetric regions $\lambda_i E \subset \lambda_i C$ have volume 1. Then we argue exactly as in Theorem 3.7. We first produce an elimination Lemma as in Theorem 2.26, with $\varepsilon > 0$ independent of λ_i, that yields a perimeter lower density bound Corollary 2.29, independent of λ_i. Hence the sequence $\{\operatorname{diam}(\lambda_i E_i)\}_{i\in\mathbb{N}}$ must be bounded, since otherwise applying the perimeter lower density bound we would get $P_{\lambda_i C}(\lambda_i E_i) \to \infty$, contradicting Proposition 2.36. Since $\{\operatorname{diam}(\lambda_i E_i)\}_{i\in\mathbb{N}}$ is bounded, (3.6) follows.

Translating each set of the sequence $\{E_i\}_{i\in\mathbb{N}}$, and eventually C, we may assume that E_i converges to $0 \in \partial K \times \mathbb{R}^k$ in Hausdorff distance. Taking $r_i = (\operatorname{diam}(E_i))^{1/2}$ we have $\operatorname{diam}(r_i^{-1}E_i) \to 0$ and so

$$r_i^{-1}E_i \to 0 \quad \text{in Hausdorff distance.} \tag{3.7}$$

Let $q \in \operatorname{int}(K \cap \overline{D}(0,1))$ and let D_q be an m-dimensional closed ball centered at q and contained in $\operatorname{int}(K \cap \overline{D}(0,1))$. As the sequence $r_i^{-1}K \cap \overline{D}(0,1)$ converges to $K_0 \cap \overline{D}(0,1)$ in Hausdorff distance, we construct, using Theorem 2.4, a family of bilipschitz maps $f_i : r_i^{-1}K \cap \overline{D}(0,1) \to K_0 \cap \overline{B}(0,1)$ with $\operatorname{Lip}(f_i)$, $\operatorname{Lip}(f_i^{-1}) \to 1$, where f_i is the identity on D_q and is extended linearly along the segments leaving from q. We define, as in Lemma 3.4, the maps $F_i : (r_i^{-1}K \cap \overline{D}(0,1)) \times \mathbb{R}^k \to (K_0 \cap \overline{D}(0,1)) \times \mathbb{R}^k$ by $F_i(x,y) = (f_i(x), y)$. These maps satisfy $\operatorname{Lip}(F_i)$, $\operatorname{Lip}(F_i^{-1}) \to 1$. Since (3.7) holds, the maps F_i have the additional property

$$P_{C_0}(F_i(r_i^{-1}E_i)) = P_{C_0\cap\overline{B}(0,1)}(F_i(r_i^{-1}E_i)), \qquad \text{for large } i \in \mathbb{N}. \tag{3.8}$$

Thus by Lemma 1.3 and (2.45) we get

$$\begin{aligned}
\frac{P_C(E_i)}{|E_i|^{n/(n+1)}} &= \frac{P_{r_i^{-1}C}(r_i^{-1}E_i)}{|r_i^{-1}E_i|^{n/(n+1)}} \\
&\geqslant \frac{P_{C_0}(F_i(r_i^{-1}E_i))}{|F_i(r_i^{-1}E_i)|^{n/(n+1)}} (\operatorname{Lip}(F_i)\operatorname{Lip}(F_i^{-1}))^{-n} \\
&\geqslant \alpha(C_0)^{1/(n+1)}(n+1)^{n/(n+1)}(\operatorname{Lip}(F_i)\operatorname{Lip}(F_i^{-1}))^{-n}
\end{aligned} \tag{3.9}$$

Since E_i are isoperimetric regions of volumes v_i, passing to the limit we get

$$\liminf_{i\to\infty} \frac{I_C(v_i)}{v_i^{n/(n+1)}} \geqslant \alpha(C_0)^{1/(n+1)}(n+1)^{n/(n+1)}.$$

From (2.45) we obtain,

$$\liminf_{i\to\infty} \frac{I_C(v_i)}{I_{C_0}(v_i)} \geqslant 1.$$

Combining this with (2.48) and the minimal property of $I_{C_{\min}}$ we deduce

$$\limsup_{i\to\infty} \frac{I_C(v_i)}{I_{C_0}(v_i)} \leqslant \limsup_{i\to\infty} \frac{I_C(v_i)}{I_{C_{\min}}(v_i)} \leqslant 1 \leqslant \liminf_{i\to\infty} \frac{I_C(v_i)}{I_{C_0}(v_i)}.$$

Thus

$$\lim_{i\to\infty} \frac{I_C(v_i)}{I_{C_{\min}}(v_i)} = 1. \tag{3.10}$$

By (2.45), we conclude that C_0 has minimum solid angle. □

A convex prism Π is a set of the form $P \times \mathbb{R}^q$ where $P \subset \mathbb{R}^m$ is a polytope. For convex prisms we are able to characterize the isoperimetric regions for small volumes.

Theorem 3.8. *Let $\Pi \subset \mathbb{R}^{n+1}$ be a convex prism. For small volumes the isoperimetric regions in Π are geodesic balls centered at vertices with the smallest solid angle.*

Proof. Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of isoperimetric regions in Π with $|E_i| \to 0$. By Theorem 3.7, after translation, a subsequence of E_i is close to some vertex x in Π. Since $\operatorname{diam}(E_i) \to 0$ we can assume that the sets E_i are also subsets of the tangent cone Π_x and they are isoperimetric regions in Π_x. By [27] the only isoperimetric regions in this cone are, after translation, the geodesic balls centered at x. These geodesic balls are also subsets of Π. □

To end this section, let us characterize the isoperimetric regions for large volume in the right cylinder $K \times \mathbb{R}$. We closely follow the proof by Duzaar and Stephen [21], which is slightly simplified by the use of Steiner symmetrization. The case of the cylinder $K \times \mathbb{R}^q$, with $q > 1$, is more involved and will be treated in a different paper.

We shall say that a set $E \subset K \times \mathbb{R}$ is *normalized* if, for every $x \in K$, the intersection $E \cap (\{x\} \times \mathbb{R})$ is a segment with midpoint $(x, 0)$.

Theorem 3.9. *Let $C = K \times \mathbb{R}$, where $K \subset \mathbb{R}^n$ is a convex body. Then there is a constant $v_0 > 0$ so that the slabs $K \times I$, where $I \subset \mathbb{R}$ is a compact interval, are the only isoperimetric regions of volume larger than or equal to v_0. In particular, $I_C(v) = 2H^n(K)$ for all $v \geqslant v_0$.*

Proof. The proof is modeled on [21, Prop 2.11]. By comparison with slabs we have $I_C(v) \leqslant 2H^n(K)$ for all $v > v_0$.

Let us assume first that $E \subset K \times \mathbb{R}$ is a normalized set of finite volume and $H^n(\partial_C E) \leqslant 2H^n(K)$, and let E^* be its orthogonal projection over $K_0 = K \times \{0\}$. We claim that, it $H^n(K_0 \setminus E^*) > 0$, then there is a constant $c > 0$ so that

$$H^n(\partial_C E) \geqslant c|E|. \tag{3.11}$$

For $t \in \mathbb{R}$, we define $E_t = E \cap (K \times \{t\})$. As E is normalized, we can choose $\tau > 0$ so that $H^n(E_t) \leqslant H^n(K)/2$ for $t \geqslant \tau$ and $H^n(E_t) > H^n(K)/2$ for $0 < t < \tau$.

For $t \geqslant \tau$ we apply the coarea formula and Lemma 2.18 to get

$$\begin{aligned} H^n(\partial_C E) &\geqslant H^n(\partial_C E \cap (K \times [t, \infty)) \\ &\geqslant \int_\tau^{+\infty} H^{n-1}(\partial_C E_s)\, ds \geqslant c_1 \int_\tau^{+\infty} H^n(E_s)\, ds \geqslant c_1 |E \cap (K \times [\tau, +\infty))|, \end{aligned} \tag{3.12}$$

where c_1 is a constant only depending on $H^n(K)/2$.

Let $S_t = K \times \{t\}$. For $0 < t < \tau$ we have

$$H^n(S_t \setminus E_t) \geqslant H^n(\partial_C E \cap (K \times (0, t))), \tag{3.13}$$

since otherwise

$$\begin{aligned} H^n(K) &= H^n(S_t \setminus E_t) + H^n(E_t) \\ &< H^n(\partial_C E \cap (K \times (0, t))) + H^n(\partial_C E \cap (K \times [t, +\infty))) \\ &\leqslant H^n(\partial_C E)/2, \end{aligned}$$

and we should get a contradiction to our assumption $H^n(\partial_C E) \leqslant 2H^n(K)$, what proves (3.13). So we obtain from (3.13) and Lemma 2.18

$$\begin{aligned} H^n(S_t \setminus E_t) &\geqslant H^n(\partial_C E \cap (K \times (0, t))) \\ &\geqslant \int_0^t H^{n-1}(\partial_C E \cap S_t) dt \\ &\geqslant c_2 \int_0^\tau H^n(S_t \setminus E_t)^{(n-1)/n} dt, \end{aligned} \tag{3.14}$$

where c_2 is a constant only depending on $H^n(K)/2$. Letting $y(t) = H^n(S_t \setminus E_t)$, inequality (3.14) can be rewritten as the integral inequality

$$y(t) \geqslant c_2 \int_0^t y(s)^{(n-1)/n} ds.$$

Since $H^n(K_0 \setminus E^*) > 0$ by assumption and E is normalized, we have $y(t) > 0$ for all $t > 0$, and so

$$2H^n(K) \geqslant H^n(S_\tau \setminus E_\tau) = y(\tau) \geqslant \frac{c_2^n}{n^n} \tau^n,$$

what implies

$$\tau \leqslant \frac{n}{c_2 (2H^n(K))^{1/n}}. \tag{3.15}$$

We finally estimate

$$|E \cap (K \times [0,\tau])| = \int_0^\tau H^n(E_t)\,dt \leqslant 2H^n(E_0)\,\tau \leqslant \frac{n}{c_2\,(2H^n(K))^{1/n}} H^n(\partial_C E). \tag{3.16}$$

Combining (3.12) and (3.16), we get (3.11). This proves the claim.

Let now $E \subset K \times \mathbb{R}$ be an isoperimetric region of large enough volume v. Following Talenti [72] or Maggi [47], we may consider its Steiner symmetrized $\operatorname{sym} E$. The set $\operatorname{sym} E$ is normalized and we have $|E| = |\operatorname{sym} E|$ and $P_C(\operatorname{sym} E) \leqslant P_C(E)$. Of course, since E is an isoperimetric region we have $P_C(\operatorname{sym} E) = P_C(E)$. If $H^n(K_0 \setminus E^*) > 0$, then (3.11) implies

$$P_C(E) = P_C(\operatorname{sym} E) = H^n(\partial_C(\operatorname{sym} E)) \geqslant c\,|\operatorname{sym} E| = c\,|E|,$$

providing a contradiction since $I_C \leqslant 2\,H^n(K)$.

We conclude that $H^n(K_0 \setminus E^*) = 0$ and that E is the intersection of the subgraph of a function $u : K \to \mathbb{R}$ and the epigraph of a function $v : K \to \mathbb{R}$. The perimeter of E is then given by

$$P_C(E) = \int_K \sqrt{1 + |\nabla u|^2}\, dH^n + \int_K \sqrt{1 + |\nabla v|^2}\, dH^n \geqslant 2H^n(K),$$

with equality if and only if $\nabla u = \nabla v = 0$. Hence u, v are constant functions and E is a slab. □

As a consequence we have

Corollary 3.10. *Let $K \subset \mathbb{R}^n$ be a convex body and $C = K \times [0,\infty)$. Then there is a constant $v_0 > 0$ such that any isoperimetric region in M with volume $v \geqslant v_0$ is the slab $K \times [0,b]$, where $b = v/H^n(K)$. In particular, $I_C(v) = H^n(K)$ for $v \geqslant v_0$.*

Proof. Just reflect with respect to the plane $x_{n+1} = 0$ and apply Theorem 3.9. Alternatively, the proof of Theorem 3.9 can also be adapted to handle this case. □

3.2. Cilindrically bounded convex bodies

We shall say that an unbounded convex body C is *cylindrically bounded* if there is a hyperplane Π such that the orthogonal projection $\pi : \mathbb{R}^{n+1} \to \Pi$ applies C onto a bounded convex set. After a rigid motion of $\mathbb{R}^{n+1}$ taking Π onto the hyperplane $\{x_{n+1} = 0\}$, we may assume there is a smallest compact convex set $K \subset \mathbb{R}^n \equiv \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ such that $C \subset K \times \mathbb{R}$. The set K is the closure of the orthogonal projection $\pi(C)$ over the hyperplane $x_{n+1} = 0$. We shall denote $K \times \mathbb{R}$ by C_∞ and we shall call it the *asymptotic cylinder* of C. Given a cylindrically bounded convex body $C \subset \mathbb{R}^n \times \mathbb{R}$ so that K is the closure of the orthogonal projection of C over $\mathbb{R}^n \times \{0\}$, we shall say that $C_\infty = K \times \mathbb{R}$ is the *asymptotic cylinder* of C. Assuming C is unbounded in the positive vertical direction, the asymptotic cylinder can be obtained as a Hausdorff limit of downward translations of C. Another property of C_∞ is the following: given $t \in \mathbb{R}$, define

$$C_t = C \cap (\mathbb{R}^n \times \{t\}). \tag{3.17}$$

Then the orthogonal projection of C_t to $\mathbb{R}^n \times \{0\}$ converges in Hausdorff distance to the basis K of the asymptotic cylinder when $t \uparrow +\infty$ by [68, Thm. 1.8.16]. In particular, this implies

$$\lim_{t\to+\infty} H^n(C_t) = H^n(K).$$

Let us prove now that the isoperimetric profile of I_C is asymptotic to the one of the half-cylinder

THEOREM 3.11. *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body with asymptotic cylinder $C_\infty = K \times \mathbb{R}$. Then*

$$\lim_{v\to\infty} I_C(v) = H^n(K). \tag{3.18}$$

PROOF. We assume that C is unbounded in the positive x_{n+1}-direction and consider the sets $\Omega(v) = C \cap (\mathbb{R}^n \times (-\infty, t(v)])$, where $t(v)$ is chosen so that $|\Omega(v)| = v$. Then

$$I_C(v) \leqslant P_C(\Omega(v)) \leqslant H^n(K),$$

and taking limits we get

$$\limsup_{v\to\infty} I_C(v) \leqslant H^n(K).$$

Let us prove now that

$$H^n(K) \leqslant \liminf_{v\to\infty} I_C(v). \tag{3.19}$$

Fix $\varepsilon > 0$. We consider a sequence of volumes $v_i \to \infty$ and a sequence $E_i \subset C$ of finite perimeter sets of volume v_i with smooth boundary, so that

$$P_C(E_i) \leqslant I_C(v_i) + \varepsilon. \tag{3.20}$$

We shall consider two cases. Recall that $(E_i)_t = E_i \cap (\mathbb{R}^n \times \{t\})$.

Case 1. $\liminf_{i\to\infty} \big(\sup_{t>0} H^n((E_i)_t)\big) = H^n(K)$.

This is an easy case. Since the projection over the horizontal hyperplane does not increase perimeter we get

$$I_C(v_i) + \varepsilon \geqslant P_C(E_i) \geqslant \sup_{t>0} H^n((E_i)_t).$$

Taking inferior limit, we get (3.19) since $\varepsilon > 0$ is arbitrary.

Case 2. $\liminf_{i\to\infty} \big(\sup_{t>0} H^n((E_i)_t)\big) < H^n(K)$.

In this case, passing to a subsequence, there exists $v_0 < H^n(K)$ such that $H^n((E_i)_t) \leqslant v_0$ for all t. By [68, Thm. 1.8.16] we have $H^n(C_t) \to H^n(K)$. Hence there exists $t_0 > 0$ such that $v_0 < H^n(C_t)$ for $t \geqslant t_0$. By Lemma 2.18, for $c_t = I_{C_t}(v_0)/v_0$, we get

$$I_{C_t}(v) \geqslant c_t v, \text{ for all } v \leqslant v_0,\ t \geqslant t_0.$$

Furthermore, as $I_{C_t}(v_0) \to I_K(v_0) > 0$ and $I_K(v_0) > 0$, we obtain the existence of $c > 0$ such that $c_t > c$ for t large enough. Taking t_0 larger if necessary we may assume $c_t > c$ holds when $t \geqslant t_0$. Thus for large $i \in \mathbb{N}$ we obtain

$$\begin{aligned}
|E_i| = \int_0^\infty H^n((E_i)_t)\,dt &\leqslant b + \int_{t_0}^\infty H^n((E_i)_t)\,dt \\
&\leqslant b + \int_{t_0}^\infty c_t^{-1} H^{n-1}((\partial E_i)_t)\,dt \\
&\leqslant b + c^{-1}\int_0^\infty H^{n-1}((\partial E_i)_t)\,dt \leqslant b + c^{-1} P_C(E_i),
\end{aligned}$$

where $b = t_0 H^n(K)$. So $P_C(E_i) \to \infty$ when $|E_i| \to \infty$. From (3.20) and $I_C \leqslant H^n(K)$ we get a contradiction. This proves that Case 2 cannot hold and so (3.19) is proven. $\square$

Let us show now that the isoperimetric profile of C is continuous and, when the boundary of C is smooth enough, that the isoperimetric profile I_C and its normalization $I_C^{(n+1)/n}$ are both concave non-decreasing functions. We shall need first some preliminary results.

PROPOSITION 3.12. *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex set, and $C_\infty = K \times \mathbb{R}$ its asymptotic cylinder. Consider a diverging sequence of finite perimeter sets $\{E_i\}_{i\in\mathbb{N}} \subset C$ such that $v = \lim_{i\to\infty} |E_i|$. Then*

$$\liminf_{i\to\infty} P_C(E_i) \geqslant I_{C_\infty}(v).$$

PROOF. Without lost of generality we assume $E_i \subset C \cap \{x_{n+1} \geqslant i\}$. Let $r > 0$ and $t_0 > 0$ so that the half-cylinder $B(0,r) \times [t_0, +\infty)$ is contained in $C \cap \{x_{n+1} \geqslant t_0\}$. Consider the horizontal sections $C_t = C \cap \{x_{n+1} = t\}$, $(C_\infty)_t = C_\infty \cap \{x_{n+1} = t\}$. We define a map $F : C \cap \{x_{n+1} \geqslant t_0\} \to C_\infty \cap \{x_{n+1} \geqslant t_0\}$ by

$$F(x,t) = (f_t(x), t),$$

where $f_t : C_t \to (C_\infty)_t$ is defined as in (2.6). For $i \in \mathbb{N}$, let $F_i = F|_{C\cap\{x_{n+1}\geqslant i\}}$. We will check that $\max\{\mathrm{Lip}(F_i), \mathrm{Lip}(F_i^{-1})\} \to 1$ when $i \to \infty$.

Take now $(x,t), (y,s) \in C \cap \{x_{n+1} \geqslant i\}$, and assume $t \geqslant s$, $i \geqslant t_0$. Then we have

$$
\begin{aligned}
|F(x,t) - F(y,s)| &= \left(|f_t(x) - f_s(y)|^2 + |t-s|^2\right)^{1/2} \\
&= \left(|f_t(x) - f_t(y) + f_t(y) - f_s(y)|^2 + |t-s|^2\right)^{1/2} \\
&= \left(|f_t(x) - f_t(y)|^2 + |f_t(y) - f_s(y)|^2 \right. \\
&\qquad \left. + 2\,|f_t(x) - f_t(y)||f_t(y) - f_s(y)| + |t-s|^2\right)^{1/2}
\end{aligned}
\tag{3.21}
$$

We have $|(f_t(x) - f_t(y))| \leqslant \mathrm{Lip}(f_t)|x-y|$. Theorem 2.4, we can write $\mathrm{Lip}(f_t) < (1+\varepsilon_i)$ for $t \geqslant i$, where $\varepsilon_i \to 0$ when $i \to \infty$. Hence

$$
|(f_t(x) - f_t(y))| \leqslant (1+\varepsilon_i)\,|x-y|, \qquad \text{for } t \geqslant i. \tag{3.22}
$$

We estimate now $|f_t(y) - f_s(y)|$. In case $|y| \leqslant r$, we trivially have $|f_t(y) - f_s(y)| = 0$. So we assume $|y| \geqslant r$. For $u \in \mathbb{S}^{n-1}$, consider the functions $\rho_t(u) = \rho(C_t, u)$, $\rho(u) = \rho(K,u)$. Observe that, for every $u \in \mathbb{S}^n$ orthogonal to $\partial/\partial x_{n+1}$, the 2-dimensional half-plane defined by u and $\partial/\partial x_{n+1}$ intersected with C is a 2-dimensional convex set, and the function $t \mapsto \rho_t(u)$ is concave with a horizontal asymptotic line at height $\rho(u)$. So we have, taking $u = y/|y|$,

$$
\frac{|f_t(y) - f_s(y)|}{|t-s|} = \frac{(|y|-r)}{|t-s|}\left|\frac{\rho_t(u)-r}{\rho(u)-r} - \frac{\rho_s(u)-r}{\rho(u)-r}\right| \leqslant \frac{\left|\rho_t(u) - \rho_s(u)\right|}{|t-s|},
$$

since $|y| - r \geqslant \rho(u) - r$. Using the concavity of $t \mapsto \rho_t(u)$ we get

$$
\frac{\left|\rho_t(u) - \rho_s(u)\right|}{|t-s|} \leqslant \left|\rho_i(u) - \rho_{i-1}(u)\right|, \qquad \text{for } t,s \geqslant i.
$$

Letting $\ell_i = \sup_{u \in \mathbb{S}^{n-1}} |\rho_i(u) - \rho_{i-1}(u)|$, we get

$$
|f_t(y) - f_s(y)| \leqslant \ell_i\,|t-s|. \tag{3.23}
$$

As C_∞ is the asymptotic cylinder of C we conclude that $\ell_i \to 0$ when $i \to \infty$.

From (3.21), (3.22), (3.23), and trivial estimates, we obtain

$$
|F_i(x,t) - F_i(y,s)| \leqslant \left((1+\varepsilon_i)^2 + \ell_i^2 + (1+\varepsilon_i)\,\ell_i\right)^{1/2} |x-y| \tag{3.24}
$$

Now $\varepsilon_i \to 0$ and $\ell_i \to 0$ as $i \to \infty$. Thus inequality (3.24) yields

$$
\limsup_{i\to\infty} \mathrm{Lip}(F_i) \leqslant 1.
$$

Similarly we find $\limsup_{i\to\infty} \mathrm{Lip}(F_i^{-1}) \leqslant 1$ and since $\mathrm{Lip}(F_i^{-1})\,\mathrm{Lip}(F_i) \geqslant 1$ by Remark 1.4, we finally get $\max\{\mathrm{Lip}(F_i), \mathrm{Lip}(F_i^{-1})\} \to 1$ when $i \to \infty$.

Thus we have

$$
\begin{gathered}
v = \lim_{i\to\infty} |E_i| = \lim_{i\to\infty} |F_i(E_i)|, \\
\liminf_{i\to\infty} P_C(E_i) = \liminf_{i\to\infty} P_{C_\infty}(F_i(E_i)).
\end{gathered}
\tag{3.25}
$$

Now from (3.25) and the continuity of I_{C_∞} we get

$$\liminf_{i\to\infty} P_C(E_i) = \liminf_{i\to\infty} P_{C_\infty}(F_i(E_i)) \geqslant I_{C_\infty}(v).$$

□

LEMMA 3.13. *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex set and $C_\infty = K \times \mathbb{R}$ its asymptotic cylinder. Let $E_\infty \subset C_\infty$ a bounded set of finite perimeter. Then there exists a sequence $\{E_i\}_{i\in\mathbb{N}} \subset C$ of finite perimeter sets such that $|E_i| = |E_\infty|$ and $\lim_{i\to\infty} P_C(E_i) = P_{C_\infty}(E_\infty)$.*

PROOF. Let $e_{n+1} = (0,\dots,0,1) \in \mathbb{R}^{n+1}$. We consider the truncated downward translations of C defined by

$$C_i = (-i\, e_{n+1} + C) \cap \{t \geqslant 0\},\ i \in \mathbb{N}.$$

These convex bodies have the same asymptotic cylinder and

$$\bigcup_{i\in\mathbb{N}} C_i = C_\infty \cap [0,\infty). \tag{3.26}$$

Translating E_∞ along the vertical direction if necessary we assume $E_\infty \subset \{t > 0\}$. Consider the sets $G_i = E_\infty \cap C_i$. For large indices G_i is not empty by (3.26). By the monotonicity of the Hausdorff measure we have $|G_i| \uparrow |E_\infty|$, and $H^n(\partial G_i \cap \operatorname{int}(C_i)) \uparrow H^n(\partial E_\infty \cap \operatorname{int}(C_\infty))$. As E_∞ is bounded, for large i we can find Euclidean geodesic balls $B_i \subset \operatorname{int}(C_i)$, disjoint from G_i, such that $|B_i| = |E_\infty| - |G_i|$. Obviously the volume and and the perimeter of these balls go to zero when i goes to infinity. Then $E_i = G_i \cup B_i$ are the desired sets. □

PROPOSITION 3.14. *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body with asymptotic cylinder $C_\infty = K \times \mathbb{R}$. Then I_C is continuous.*

PROOF. The continuity of the isoperimetric profile I_C at $v = 0$ is proven by comparison with geodesic balls intersected with C.

Fix $v > 0$ and let $\{v_i\}_{i\in\mathbb{N}}$ be a sequence of positive numbers converging to v. Let us prove first the lower semicontinuity of I_C. By the definition of isoperimetric profile, given $\varepsilon > 0$, there is a finite perimeter set E_i of volume v_i so that $I_C(v_i) \leqslant P_C(E_i) \leqslant I_C(v_i) + \frac{1}{i}$, for every $i \in \mathbb{N}$. Reasoning as in [60, Thm. 2.1], we can decompose $E_i = E_i^c \cup E_i^d$ into convergent and diverging pieces, and there is a finite perimeter set $E \subset C$, eventually empty, so that

$$\begin{aligned} |E_i| &= |E_i^c| + |E_i^d|, \\ P_C(E_i) &= P_C(E_i^c) + P_C(E_i^d), \\ |E_i^c| &\to |E|, \\ P_C(E) &\leqslant \liminf_{i\to\infty} P_C(E_i^c). \end{aligned} \tag{3.27}$$

Let $w_1 = |E|$. By Proposition 3.2, there exists an isoperimetric region $E_\infty \subset C_\infty$ of volume $|E_\infty| = w_2 = v - w_1$. By Proposition 3.12 we have $P_{C_\infty}(E_\infty) \leqslant \liminf_{i\to\infty} P_C(E_i^d)$. Hence

$$\begin{aligned} I_C(v) \leqslant I_C(w_1) + I_{C_\infty}(w_2) \leqslant P_C(E) + P_{C_\infty}(E_\infty) \\ &\leqslant \liminf_{i\to\infty} P_C(E_i^c) + \liminf_{i\to\infty} P_C(E_i^d) \\ &\leqslant \liminf_{i\to\infty} P_C(E_i) \\ &= \liminf_{i\to\infty} I_C(v_i). \end{aligned}$$

To prove the upper semicontinuity of I_C we will use a standard variational argument. Fix $\varepsilon > 0$. We can find a bounded set $E \subset C$ of volume v with $I_C(v) \leqslant P_C(E) \leqslant I_C(v) + \varepsilon$ and a smooth open portion $U \subset \partial_C E$ contained in the relative boundary. We construct a variation compactly supported in U of E by sets E_s so that $|E_s| = v + s$ for $s \in (-\delta, \delta)$. Then there is $M > 0$ so that

$$|H^n(\partial_C E_s) - H^n(\partial_C E)| \leqslant M\, ||E_s| - |\,E||.$$

Hence

$$\begin{aligned} I_C(v+s) &\leqslant H^n(\partial_C E_s) \leqslant H^n(\partial_C E) \\ &\leqslant I_C(v) + \varepsilon + M\,\big(|E_s| - |\,E|\big) \\ &= I_C(v) + \varepsilon + Ms. \end{aligned}$$

Taking a sequence $v_i \to v$ we get $\limsup_{i\to\infty} I_C(v_i) \leqslant I_C((v) + \varepsilon$. As ε is arbitrary we obtain the upper semicontinuity of I_C. □

Proposition 3.15. *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body with asymptotic cylinder $C_\infty = K \times \mathbb{R}$. Assume that both C and C_∞ have smooth boundary. Then isoperimetric regions exist on C for large volumes and have connected boundary. Moreover $I_C^{(n+1)/n}$ and so I_C are concave non-decreasing functions.*

Proof. Fix $v > 0$. By [60, Thm. 2.1] there exists an isoperimetric region $E \subset C$ (eventually empty) of volume $|E| = v_1 \leqslant v$, and a diverging sequence $\{E_i\}_{i\in\mathbb{N}}$ of finite perimeter sets of volume $v_2 = v - v_1$, such that

$$I_C(v) = P_C(E) + \lim_{i\to\infty} P_C(E_i) \tag{3.28}$$

By Proposition 3.2, there is an isoperimetric region $E_\infty \subset C_\infty$ of volume v_2. We claim

$$\lim_{i\to\infty} P_C(E_i) = P_{C_\infty}(E_\infty). \tag{3.29}$$

If (3.29) does not hold, then Proposition 3.12 implies $\liminf_{i\to\infty} P_C(E_i) > I_{C_\infty}(v_2)$, and Lemma 3.13 provides a sequence of finite perimeter sets in C, of volume v_2, approaching E_∞. This way we can build a minimizing sequence of sets of volume v whose perimeters converge to some quantity strictly smaller than $I_C(v)$, a contradiction that proves (3.29). From

(3.28) and (3.29) we get

$$I_C(v) = P_C(E) + P_{C_\infty}(E_\infty). \tag{3.30}$$

Reasoning as in the proof of Theorem 2.8 in [58], the configuration $E \cup E_\infty$ in the disjoint union of the sets C, C_∞ must be stationary and stable, since otherwise we could slightly perturb $E \cup E_\infty$, keeping constant the total volume, to get a set $E' \cup E'_\infty$ such that

$$P_C(E') + P_{C_\infty}(E'_\infty) < P_C(E) + P_{C_\infty}(E_\infty),$$

contradicting (3.30).

Now as C, C_∞ are convex and have smooth boundary, we can use a stability argument similar to that in [9, Proposition 3.9] to conclude that one of the sets E or E_∞ must be empty and the remaining one must have connected boundary. A third possibility, that $\partial_C E \cup \partial_{C_\infty} E_\infty$ consists of a finite number of hyperplanes intersecting orthogonally both C and C_∞, can be discarded since in this case E_∞ would be a slab with $P_{C_\infty}(E_\infty) = 2H^n(K) > I_C$.

If v is large enough so that isoperimetric regions in C_∞ are slabs, then the above argument shows existence of isoperimetric regions of volume v in C.

As I_C is always realized by an isoperimetric set in C or C_∞, the arguments in [9, Theorem 3.2] imply that the second lower derivative of $I_C^{(n+1)/n}$ is non-negative. As $I_C^{(n+1)/n}$ is continuous by Proposition 3.14, Lemma 3.2 in [54] implies that $I_C^{(n+1)/n}$ is concave and hence non-decreasing. Then I_C is also concave as a composition of $I_C^{(n+1)/n}$ with the concave non-increasing function $x \mapsto x^{n/(n+1)}$.

The connectedness of the isoperimetric regions in C follows easily as an application of the concavity of $I_C^{(n+1)/n}$, as in Theorem 2.15. □

The concavity of $I_C^{(n+1)/n}$ also implies the following Lemma. The proof in Lemma 2.18 for convex bodies also holds in our setting.

Lemma 3.16. *Let C be be a cylindrically bounded convex body with asymptotic cylinder C_∞. Assume that both C and C_∞ have smooth boundary. Let $\lambda \geqslant 1$. Then*

$$I_{\lambda C}(v) \geqslant I_C(v) \tag{3.31}$$

for all $0 \leqslant v \leqslant |C|$.

Our aim now is to get a density estimate for isoperimetric regions of large volume in Theorem 3.18. This estimate would imply the convergence of the free boundaries of large isoperimetric regions to hyperplanes in Hausdorff distance given in Theorem 3.22.

Proposition 3.17. *Let C be cylindrically bounded convex body with asymptotic cylinder C_∞. Given $r_0 > 0$, there exist positive constants M, ℓ_1, only depending on r_0 and C, C_∞, and a universal positive constant ℓ_2 so that*

$$P_{\bar{B}_C(x,r)}(v) \geqslant M \min\{v, |\bar{B}_C(x,r)| - v\}^{n/(n+1)}, \tag{3.32}$$

for all $x \in C$, $0 < r \leqslant r_0$, and $0 < v < |\bar{B}(x,r)|$. Moreover

$$\ell_1 r^{n+1} \leqslant |\bar{B}_C(x,r)| \leqslant \ell_2 r^{n+1}, \tag{3.33}$$

for any $x \in C$, $0 < r \leqslant r_0$.

Proof. Reasoning as in Theorem 2.21, it is enough to show

$$\Lambda_0 = \inf_{x \in C} \operatorname{inr}(\bar{B}_C(x,r_0)) > 0.$$

To see this consider a sequence $\{x_i\}_{i\in\mathbb{N}}$ so that $\operatorname{inr}(\bar{B}_C(x_i,r_0))$ converges to Λ_0. If $\{x_i\}_{i\in\mathbb{N}}$ contains a bounded subsequence then we can extract a convergent subsequence to some point $x_0 \in C$ so that $\Lambda_0 = \operatorname{inr}(\bar{B}(x_0,r_0)) > 0$. If $\{x_i\}_{i\in\mathbb{N}}$ is unbounded, we translate vertically the balls $\bar{B}_C(x_i,r_0)$ so that the new centers x_i' lie in the hyperplane $x_{n+1} = 0$. Passing to a subsequence we may assume that x_i' converges to some point $x_0 \in C_\infty$. By the proof of Proposition 3.12, we have Hausdorff convergence of the translated balls to $\bar{B}_{C_\infty}(x_0,r_0)$ and so $\Lambda_0 = \operatorname{inr}(\bar{B}_{C_\infty}(x_0,r_0)) > 0$. □

Theorem 3.18. *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body with asymptotic cylinder $C_\infty = K \times \mathbb{R}$. Assume that C, C_∞ have smooth boundary. Let $E \subset C$ an isoperimetric region of volume $v > 1$. Choose ε so that*

$$0 < \varepsilon < \left\{ {\ell_2}^{-1}, c_2, \frac{{\ell_2}^n}{8^{n+1}}, {\ell_2}^{-1}\left(\frac{I_C(1)}{4}\right)^{n+1} \right\}, \tag{3.34}$$

where c_2 is the constant in Lemma 2.25., and ℓ_1, ℓ_2 the constants in Proposition 3.17.

Then, for any $x \in C$ and $R \leqslant 1$ so that $h(x,R) \leqslant \varepsilon$, we get

$$h(x,R/2) = 0. \tag{3.35}$$

Moreover, in case $h(x,R) = |E \cap B_C(x,R)||B_C(x,R)|^{-1}$, we get $|E \cap B_C(x,R/2)| = 0$ and, in case $h(x,R) = |B_C(x,R) \setminus E||B_C(x,R)|^{-1}$, we have $|B_C(x,R/2) \setminus E| = 0$.

Proof. From the concavity of $I_C^{(n+1)/n}$ and the fact that $I_C(0) = 0$ we get, as in Lemma (2.18), the following inequality

$$I_C(w) \geqslant c_1 w^{n/(n+1)}, \qquad c_1 = I_C(1), \tag{3.36}$$

for all $0 \leqslant w \leqslant 1$.

Assume first that

$$h(x,R) = \frac{|E \cap B_C(x,R)|}{|B_C(x,R)|}.$$

Define $m(t) = |E \cap B_C(x,t)|, 0 < t \leqslant R$. Thus $m(t)$ is a non-decreasing function. For $t \leqslant R \leqslant 1$ we get

$$m(t) \leqslant m(R) = |E \cap B_C(x,R)| = h(x,R)\,|B_C(x,R)| \leqslant h(x,R)\,\ell_2 R^{n+1} \leqslant \varepsilon \ell_2 < 1, \tag{3.37}$$

by (3.34). Since $v > 1$, we get $v - m(t) > 0$.

By the coarea formula, when $m'(t)$ exists, we obtain

$$m'(t) = \frac{d}{dt}\int_0^t H^n(E \cap \partial_C B(x,s))ds = H^n(E \cap \partial_C B(x,t)). \tag{3.38}$$

Define

$$\lambda(t) = \frac{v^{1/(n+1)}}{(v-m(t))^{1/(n+1)}}, \qquad E(t) = \lambda(t)(E \setminus B_C(x,t)). \tag{3.39}$$

Then $E(t) \subset \lambda(t)C$ and $|E(t)| = |E| = v$. By Lemma 3.16, we get $I_{\lambda(t)C} \geqslant I_C$ since $\lambda(t) \geqslant 1$. Combining this with [75, Cor. 5.5.3], equation (3.38), and elementary properties of the perimeter functional, we have

$$\begin{aligned} I_C(v) &\leqslant I_{\lambda(t)C}(v) \leqslant P_{\lambda(t)C}(E(t)) = \lambda^n(t)P_C(E \setminus B_C(x,t)) \\ &\leqslant \lambda^n(t)\big(P_C(E) - P(E, B_C(x,t)) + H^n(E \cap \partial B_C(x,t))\big) \\ &\leqslant \lambda^n(t)\big(P_C(E) - P_C(E \cap B_C(x,t)) + 2H^n(E \cap \partial B_C(x,t))\big) \\ &\leqslant \lambda^n(t)\big(I_C(v) - c_1 m(t)^{n/(n+1)} + 2m'(t)\big), \end{aligned} \tag{3.40}$$

where c_1 is the constant in (3.36). Multiplying both sides by $I_C(v)^{-1}\lambda(t)^{-n}$ we find

$$\lambda(t)^{-n} - 1 + \frac{c_1}{I_C(v)}m(t)^{n/(n+1)} \leqslant \frac{2}{I_C(v)}m'(t). \tag{3.41}$$

As we have $I_C \leqslant H^n(K)$, and I_C is concave by Proposition 3.15, there exists a constant $\alpha > 0$ such that $I_C \geqslant \alpha$ for sufficient large volumes. Set

$$a = \frac{2}{\alpha} \geqslant \frac{2}{I_C(v)}, \quad \text{and} \quad b = \frac{c_1}{H^n(K)} \leqslant \frac{c_1}{I_C(v)}. \tag{3.42}$$

From the definition (3.39) of $\lambda(t)$ we get

$$f(m(t)) \leqslant am'(t) \qquad H^1\text{-a.e}, \tag{3.43}$$

where

$$\frac{f(s)}{s^{n/(n+1)}} = b + \frac{\left(\frac{v-s}{v}\right)^{n/(n+1)} - 1}{s^{n/(n+1)}}. \tag{3.44}$$

By Lemma 2.25, there exists a universal constant $0 < c_2 < 1$, not depending on v, so that

$$\frac{f(s)}{s^{n/n+1}} \geqslant b/2 \qquad \text{whenever} \qquad 0 < s \leqslant c_2. \tag{3.45}$$

Since $\varepsilon \leqslant c_2$ by (3.34), equation (3.45) holds in the interval $[0, \varepsilon]$. If there were $t \in [R/2, R]$ such that $m(t) = 0$ then, by monotonicity of $m(t)$, we would conclude $m(R/2) = 0$ as well. So we assume $m(t) > 0$ in $[R/2, R]$. Then by (3.43) and (3.45), we get

$$b/2a \leqslant \frac{m'(t)}{m(t)^{n/n+1}}, \qquad H^1\text{-a.e.}$$

Integrating between $R/2$ and R we get by (3.37)

$$bR/4a \leqslant (m(R)^{1/(n+1)} - m(R/2)^{1/(n+1)}) \leqslant m(R)^{1/(n+1)} \leqslant (\varepsilon \ell_2)^{1/(n+1)}R.$$

This is a contradiction, since $\varepsilon\ell_2 < (b/4a)^{n+1} = I_C(v)^{n+1}/(8^{n+1}v^n) \leqslant \ell_2^{n+1}/8^{n+1}$ by (3.34) and Proposition 2.36. So the proof in case $h(x,R) = |E \cap B_C(x,R)|\,(|B_C(x,R))|^{-1}$ is completed. For the remaining case, when $h(x,R) = |B_C(x,R)|^{-1}|B_C(x,R) \setminus E|$, we use Lemma 2.18 and the fact that I_C is non-decreasing proven in Proposition 3.15. Then we argue as in Case 1 in Lemma 4.2 of [44] to get

$$c_1/4 \leqslant (\varepsilon\ell_2)^{1/(n+1)}.$$

This is a contradiction, since $\varepsilon\ell_2 < (c_1/4)^{n+1}$ by assumption (3.34) □

Proposition 3.19. *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body and C_∞ its asymptotic cylinder. Assume that both C and C_∞ have smooth boundary. Then there exists a constant $c > 0$ such that, for each isoperimetric region E of volume $v > 1$,*

$$P(E, B_C(x,r)) \geqslant c r^n, \tag{3.46}$$

for $r \leqslant 1$ and $x \in \partial_C E$.

Proof. Let $E \subset C$ be an isoperimetric region of volume larger than 1. Choose $\varepsilon > 0$ satisfying (3.34). Since $x \in \partial_C E$ we have $\lim_{r\to 0} h(x,r) \neq 0$ and, by Theorem 3.18, $h(x,r) \geqslant \varepsilon$ for $0 < r \leqslant 1$. So we get

$$\begin{aligned} P(E, B_C(x,r)) &\geqslant M \min\{|E \cap B_C(x,r)|, |B_C(x,r) \setminus E|\}^{n/(n+1)} \\ &= M\,(|B_C(x,r)|\,h(x,r))^{n/(n+1)} \geqslant M(|B_C(x,r)|\,\varepsilon)^{n/(n+1)} \\ &\geqslant M\,(\ell_1\varepsilon)^{n/(n+1)}\,r^n. \end{aligned}$$

Inequality (3.46) follows by taking $c = M(\ell_1\varepsilon)^{n/(n+1)}$, which is independent of v. □

Remark 3.20. Theorem 3.18 and Proposition 3.19 also hold if C is a convex cylinder.

As a Corollary we obtain a new proof of Theorem 3.9

Corollary 3.21. *Let $C = K \times \mathbb{R}$, where $K \subset \mathbb{R}^n$ is a convex body. Then there is a constant $v_0 > 0$ so that $I_C(v) = 2H^n(K)$ for all $v \geqslant v_0$. Moreover, the slabs $K \times [t_1, t_2]$ are the only isoperimetric regions of volume larger than or equal to v_0.*

Proof. Let E be an isoperimetric region with volume

$$|E| > 2mr_0H^n(K), \tag{3.47}$$

where r_0, $c > 0$, are the constants in Proposition 3.19 (see also Remark 3.20), and $m > 0$ is chosen so that

$$mcr_0^n > 2H^n(K). \tag{3.48}$$

By results of Talenti on Steiner symmetrization for finite perimeter sets [72], we can assume that the boundary of E is the union of two graphs, symmetric with respect to a horizontal hyperplane, over a subset $K^* \subset K$. If $K^* = K$ then $P_C(E) \geqslant 2H^n(K)$, since the orthogonal projection over $K \times \{0\}$ is perimeter non-increasing. This implies $P_C(E) = 2H^n(K)$ and it follows, as in the proof of Theorem 3.9, that E is a slab.

So assume that K^* is a proper subset of K. Since $|E| > 2mr_0H^n(K)$, E cannot be contained in the slab $K \times [-r_0m, r_0m]$. Then as $\partial_C E$ is a union of two graphs over K^* we can find $x_j \in \partial_C E$, $1 \leqslant j \leqslant m$, so that the balls centered at these points are disjoint. Then by the lower density bound (3.46) we get

$$P_C(E) \geqslant \sum_{j=1}^{m} P(E, B_C(x_j, r_0)) \geqslant mcr_0^n > 2H^n(K), \tag{3.49}$$

a contradiction since $I_C \leqslant 2H^n(K)$. □

Recall that, in Corollary 3.10, we showed that, given a half-cylinder $K \times [0, \infty)$, there exists $v_0 > 0$ so that every isoperimetric region in $K \times [0, \infty)$ of volume larger than or equal to v_0 is a slab $K \times [0, b]$, where $b = v/H^n(K)$. We can use this result to obtain

THEOREM 3.22. *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body, $C_\infty = K \times \mathbb{R}$ its asymptotic cylinder and $C_\infty^+ = K \times [0, \infty)$. Assume that both C and C_∞ have smooth boundary. Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of isoperimetric regions with $\lim_{i\to\infty} |E_i| = \infty$. Then truncated downward translations of E_i converge in Hausdorff distance to a half-slab $K \times [0, b]$ in C_∞^+. The same convergence result holds for their free boundaries.*

PROOF. By Corollary 3.10, we can choose $v_0 > 0$ such that each isoperimetric region with volume $v \geqslant v_0$ in C_+^∞ is a half-slab $K \times [0, b(v)]$ of perimeter $H^n(K)$, where $b(v) = v/H^n(K)$.

Since $|E_i| \to \infty$, we can find vertical vectors y_i, with $|y_i| \to \infty$, so that $\Omega_i = (-y_i + E_i) \cap \{x_{n+1} \geqslant 0\}$ has volume v_0 for large enough $i \in \mathbb{N}$. We observe also that, by Proposition 3.19 and the fact that $I_C \leqslant H^n(K)$, the sets ∂E_i have uniformly bounded diameter.

Consider the convex bodies

$$C_i = (-y_i + C) \cap \{x_{n+1} \geqslant 0\}, \tag{3.50}$$

for $i \in \mathbb{N}$. The sets C_i have the same asymptotic cylinder C_∞ and we have

$$\bigcup_{i\in\mathbb{N}} C_i = C_\infty^+. \tag{3.51}$$

By construction we have

$$P_{C_i}(\Omega_i) \leqslant P_C(E_i) \leqslant H^n(K). \tag{3.52}$$

Since ∂E_i are uniformly bounded and $|\Omega_i| = v_0$, there exists a Euclidean geodesic ball B such that $\Omega_i \subset B$ for all $i \in \mathbb{N}$. By (3.51) the sequence of convex bodies $\{C_i \cap B\}_{i\in\mathbb{N}}$ converges to $C_\infty^+ \cap B$ in Hausdorff distance and, by Theorem 2.4, in lipschitz distance. Hence, by the proof of Theorem 2.4 and Lemma 1.6, we conclude there exists a finite perimeter set $\Omega \subset C_\infty^+$, such that

$$\Omega_i \xrightarrow{L^1} \Omega \quad \text{and} \quad P_{C_\infty^+}(\Omega) \leqslant \liminf_{i\to\infty} P_{C_i}(\Omega_i). \tag{3.53}$$

So we obtain from (3.52) and (3.53),

$$H^n(K) = I_{C_\infty^+}(v_0) \leqslant P_{C_\infty^+}(\Omega) \leqslant \liminf_{i\to\infty} P_{C_i}(\Omega_i) \leqslant \liminf_{i\to\infty} P_C(E_i) \leqslant H^n(K), \tag{3.54}$$

what implies that Ω is an isoperimetric region of volume v_0 in C_∞^+ and so it is a slab.

Furthermore, the arguments of Theorem 2.32 and Theorem 2.34 can be applied here to improve the L^1 convergence to Hausdorff convergence, both for the sets Ω_i and for their free boundaries. □

REMARK 3.23. The proof of Theorem 3.22 implies $\lim_{v\to\infty} I_C(v) = H^n(K)$. So we have a different proof of Theorem 3.11.

CHAPTER 4

Conically bounded convex bodies

4.1. Unbounded convex bodies with non-degenerate asymptotic cone

We define the *asymptotic cone* C_∞ of an unbounded convex body C by

$$C_\infty = \bigcap_{\lambda>0} \lambda C, \tag{4.1}$$

where $\lambda C = \{\lambda x : x \in C\}$ is the image of C under the homothety of center 0 and ratio λ. If $p \in \mathbb{R}^{n+1}$ and $h_{p,\lambda}$ is the homothety of center p and ratio λ then $\bigcap_{\lambda>0} h_{p,\lambda}(C) = p + C_\infty$ is a translation of C_∞. Hence the shape of the asymptotic cone is independent of the chosen origin. When C is bounded the set C_∞ defined by (4.1) is a point. It is known that λC converges, in the pointed Hausdorff topology, to the asymptotic cone C_∞ [14] and hence it satisfies $\dim C_\infty \leqslant \dim C$. We shall say that the asymptotic cone is *non-degenerate* if $\dim C_\infty = \dim C$. The solid paraboloid $\{z \geqslant x^2 + y^2\}$ and the cilindrically bounded convex set $\{z \geqslant (1-x^2-y^2)^{-1} : x^2+y^2 < 1\}$ are examples of unbounded convex bodies with the same degenerate asymptotic cone $\{(0,0,z) : z \geqslant 0\}$.

The main result in this Section is Theorem 4.6, where we prove that the isoperimetric profile I_C of an unbounded convex body C with non-degenerate asymptotic cone C_∞ is bounded from below by I_{C_∞} and that I_C and I_{C_∞} are asymptotic functions. We also prove the continuity of the isoperimetric profile I_C.

Assume now that $C \subset \mathbb{R}^{n+1}$ is an unbounded convex body and $0 \in C$. We denote

$$B_r = \bar{B}_C(0,r)$$

and

$$I_{C_r}(v) = \inf\{P_C(E) : E \subset B_r, |E| = v\}.$$

LEMMA 4.1. *Let C be an unbounded convex body. Then*

$$I_C = \inf_{r>0} I_{C_r}. \tag{4.2}$$

REMARK 4.2. Lemma 4.1 implies that, for every volume, there exists a minimizing sequence consisting of bounded sets.

PROOF. From the definition of I_{C_r} it follows that, for $0 < r < s$, we have $I_{C_s} \geqslant I_{C_r} \geqslant I_C$ in the common domain of definition. Hence $I_C \leqslant \inf_{r>0} I_{C_r}$.

In order to prove the opposite inequality we will be follow an argument in [60]. Fix $v>0$, and let $\{E_i\}_{i\in\mathbb{N}}$ be a minimizing sequence for volume v. This means $|E_i|=v$ and $\lim_{i\to\infty}P_C(E_i)=I_C(v)$.

For every $i\in\mathbb{N}$ we have $\lim_{r\to\infty}|E_i\setminus B_r|=0$. Thus for every $i\in\mathbb{N}$ there exists $R_i>0$ such that

$$|E_i\setminus B_{R_i}|<\frac{1}{i}.$$

We now define a sequence of real numbers $\{r_i\}_{i\in\mathbb{N}}$ by induction taking $r_1=R_1$ and $r_{i+1}=\max\{r_i,R_{i+1}+1\}+i$. Then $\{r_i\}_{i\in\mathbb{N}}$ satisfies

$$r_{i+1}-r_i\geqslant i\qquad\text{and}\quad |E_i\setminus B_{r_i}|<\frac{1}{i}.$$

By the coarea formula

$$\int_{r_i}^{r_{i+1}}H^n(E_i\cap\partial B_t)\,dt\leqslant\int_{\mathbb{R}}H^n(E_i\cap\partial B_t)\,dt=|E_i|=v.$$

Thus there exists $\rho(i)\in[r_i,r_{i+1}]$ so that $(r_{i+1}-r_i)H^n(E_i\cap\partial B_{\rho(i)})\leqslant v$, and so

$$H^n(E_i\cap\partial B_{\rho(i)})\leqslant\frac{v}{i}.$$

Now by Corollary 5.5.3 in [75] we have

$$P_C(E_i\cap B_{\rho(i)})\leqslant P(E_i,B_{\rho(i)})+H^n(E_i\cap\partial B_{\rho(i)}).$$

Let B_i^* be a sequence of Euclidean balls of volume $|B_i^*|=|E_i\setminus B_{\rho(i)}|$. Since $|B_i^*|\to 0$ when $i\to\infty$, the balls can be taken at positive distance of $E_i\cap B_{\rho(i)}$, but inside B_{2r_i} for i large enough. Hence

$$\begin{aligned}I_{C_{2r_i}}(v)&\leqslant P_C(E_i\cap B_{\rho(i)})+P(B_i^*)\\&\leqslant P_C(E_i,B_{\rho(i)})+H^n(E_i\cap\partial B_{\rho(i)})+P(B_i^*)\\&\leqslant P_C(E_i)+\frac{v}{i}+P(B_i^*).\end{aligned}$$

Taking limits when $i\to\infty$ we obtain $\inf_{r>0}I_{C_r}(v)\leqslant I_C(v)$. $\square$

The following is inspired by Theorem 2.21

Lemma 4.3. *Let $C\subset\mathbb{R}^{n+1}$ a convex body with non-degenerate asymptotic cone C_∞. Given $r_0>0$, there exist positive constants M, ℓ_1, only depending on r_0 and C_∞, and a universal positive constant ℓ_2 so that*

$$I_{\bar B_C(x,r)}(v)\geqslant M\min\{v,|\bar B_C(x,r)|-v\}^{n/(n+1)},\tag{4.3}$$

for all $x\in C$, $0<r\leqslant r_0$, and $0<v<|\bar B(x,r)|$. Moreover

$$\ell_1 r^{n+1}\leqslant|\bar B_C(x,r)|\leqslant\ell_2 r^{n+1},\tag{4.4}$$

for any $x\in C$, $0<r\leqslant r_0$.

PROOF. Fix $r_0 > 0$. Following Theorem 2.20, to show the validity of (4.3), we only need to obtain a lower bound δ for the inradius of $\bar{B}_C(x, r_0)$ independent of $x \in C$. Then a relative isoperimetric inequality is satisfied in $\bar{B}_C(x, r)$, for $0 < r < r_0$, with a constant M that only depends on r_0/δ.

Let C_∞ be the asymptotic cone of C with vertex at the origin, defined by

$$C_\infty = \bigcup_{\lambda > 0} \lambda C. \tag{4.5}$$

For every $x \in C$, we have $x + C_\infty = \bigcap_{\lambda>0} h_{x,\lambda}(C) = \bigcap_{1\geqslant\lambda>0} h_{x,\lambda}(C) \subset C$. Fix $r_0 > 0$ and $x \in C$. As $x + C_\infty \subset C$, we get $\bar{B}_{x+C_\infty}(x, r) \subset \bar{B}_C(x, r)$. Since C_∞ is non-degenerate, then we can pick $\delta > 0$ and $y \in C_\infty$ so that $B(y, \delta) \subset \bar{B}_{C_\infty}(0, r_0)$. Hence $B(x + y, \delta) \subset \bar{B}_{x+C_\infty}(x, r_0)$. This provides the desired uniform lower bound for the inradius of $\bar{B}(x, r_0)$.

We now prove (4.4). Since $|\bar{B}_C(x, r)| \leqslant |\bar{B}(x, r)|$, it is enough to take $\ell_2 = \omega_{n+1} = |\bar{B}(0, 1)|$. For the remaining inequality, using the same notation as above, we have

$$\begin{aligned} |\bar{B}(x, r) \cap C| &= |\bar{B}(x, \lambda r_0) \cap C| \geqslant |h_{x,\lambda}(\bar{B}(x, r_0) \cap C)| \\ &= \lambda^{n+1} |\bar{B}(x, r_0) \cap C| \geqslant \lambda^{n+1} |\bar{B}(y(x), \delta)| \\ &= \omega_{n+1} (\delta/r_0)^{n+1}\, r^{n+1}, \end{aligned}$$

and we take $\ell_1 = \omega_{n+1}(\delta/r_0)^{n+1}$. □

Arguing similarly as in the proof of Theorem 2.10 we obtain

LEMMA 4.4. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of (possibly unbounded) convex bodies converging to a convex body C in pointed Hausdorff distance. Let $E \subset C$ a bounded set of finite perimeter and volume $v > 0$, such that the set of regular points of $\partial_C E$ is open in $\partial_C E$ and has bounded mean curvature. If $v_i \to v$. Then there exists a sequence $\{E_i\}_{i\in\mathbb{N}}$ of bounded sets $E_i \subset C_i$ of finite perimeter in C_i with $|E_i| = v_i$ and $\lim_{i\to\infty} P_{C_i}(E_i) = P_C(E)$.*

PROOF. Let $B \subset \mathbb{R}^{n+1}$ be a closed Euclidean ball containing E in its interior. By hypothesis, the sequence $\{C_i \cap B\}_{i\in\mathbb{N}}$ converges in Hausdorff distance to $C \cap B$. As in Theorem 2.4, we consider a sequence $f_i : C_i \cap B \to C \cap B$ of bilipschitz maps with $\operatorname{Lip}(f_i)$, $\operatorname{Lip}(f_i^{-1}) \to 1$. Now we argue as in Theorem 2.10, defining the sets $E_i \subset C_i$ as the preimages by f_i of smooth perturbations of E supported in the regular part of $\partial_C E$, and such that $|E_i| = v_i$, and $\lim_{i\to\infty} P_{C_i}(E_i) = P_C(E)$. □

PROPOSITION 4.5. *Let $C \subset \mathbb{R}^{n+1}$ be a convex body with non-degenerate asymptotic cone. Then each isoperimetric region in C is bounded.*

PROOF. The proof follows using the doubling property, Lemma 1.9, and (4.3) as in Proposition 3.2 □

THEOREM 4.6. *Let C be a convex body with non-degenerate asymptotic cone C_∞. Then*

$$\frac{I_C}{I_{C_\infty}} \geqslant 1. \tag{4.6}$$

Moreover

$$\lim_{v\to\infty} \frac{I_C(v)}{I_{C_\infty}(v)} = 1. \tag{4.7}$$

PROOF. Fix $v > 0$ and let $E \subset C$ be any bounded set of finite perimeter and volume v. Let $q \in \operatorname{int}(C_\infty \cap \bar{B}(0,1))$ and $B_q \subset \operatorname{int}(C_\infty \cap \bar{B}(0,1))$ be a Euclidean geodesic ball. Now consider a solid cone K_q with vertex q such that $0 \in \operatorname{int}(K_q)$ and $K_q \cap C \cap \partial B(0,1) = \emptyset$. Let $r_i \uparrow \infty$. By definition of the asymptotic cone, $r_i^{-1}C \cap \bar{B}(0,1)$ converges to $C_\infty \cap \bar{B}(0,1)$ in Hausdorff distance. Thus we may construct, as in Theorem 2.4, a family of bilipschitz maps $f_i : r_i^{-1}C \cap \bar{B}(0,1) \to C_\infty \cap \bar{B}(0,1)$ which fix the points in the ball B_q, and such that

$$\operatorname{Lip}(f_i), \operatorname{Lip}(f_i^{-1}) \to 1. \tag{4.8}$$

So f_i is the identity in B_q and it is extended linearly along the segments leaving from q. For large enough $i \in \mathbb{N}$ we have, $E \subset C \cap B(0, r_i)$ and $r_i^{-1}E \subset K_q$, since $\operatorname{diam}(E) < \infty$. For this large i, by construction, the maps f_i have the additional property

$$P_{C_\infty}(f_i(r_i^{-1}E)) = P_{C_\infty \cap \bar{B}(0,1)}(f_i(r_i^{-1}E)). \tag{4.9}$$

For i large enough, $P_C(E) = P_C(E \cap B(0,r))$. Thus by Lemma 1.3, (2.45) and the above, we get

$$\begin{aligned}\frac{P_C(E)}{|E|^{n/(n+1)}} = \frac{P_{r_i^{-1}C}(r_i^{-1}E)}{|r_i^{-1}E|^{n/(n+1)}} &\geqslant \frac{P_{C_0}(f_i(r_i^{-1}E))}{|f_i(r_i^{-1}E)|^{n/(n+1)}} (\operatorname{Lip}(f_i)\operatorname{Lip}(f_i^{-1}))^{-n} \\ &\geqslant I_{C_\infty}(1)(\operatorname{Lip}(f_i)\operatorname{Lip}(f_i^{-1}))^{-n}.\end{aligned} \tag{4.10}$$

Passing to the limit we get,

$$\frac{P_C(E)}{|E|^{n/(n+1)}} \geqslant I_{C_\infty}(1). \tag{4.11}$$

Thus, by (2.45), for every $v \geqslant 0$, we obtain,

$$I_C(v) \geqslant I_{C_\infty}(v), \tag{4.12}$$

which implies (4.6).

Let us prove now (4.7). Let $\lambda_i \downarrow 0$, $i \in \mathbb{N}$. Since C_∞ is the asymptotic cone of each $\lambda_i C$ then the last inequality holds for every $\lambda_i C, i \in \mathbb{N}$. Passing to the limit we conclude

$$I_{C_\infty}(1) \leqslant \liminf_{i\to\infty} I_{\lambda_i C}(1).$$

Now consider a ball B_{C_∞} centered at a vertex of C_∞ of volume 1, which is an isoperimetric region by [46]. By Lemma 4.4, there exist a sequence $E_i \subset \lambda_i C$ of finite perimeter sets with $|E_i| = 1$ and such that $\lim_{i\to\infty} P_{\lambda_i C}(E_i) = P_C(B)$. So we get

$$I_{C_\infty}(1) \geqslant \limsup_{i\to\infty} I_{\lambda_i C}(1),$$

and we conclude

$$I_{C_\infty}(1) = \lim_{i\to\infty} I_{\lambda_i C}(1). \tag{4.13}$$

From (4.13), Lemma 2.22 and the fact that C_∞ is a cone we deduce

$$1 = \lim_{\lambda\to 0} \frac{I_{\lambda C}(1)}{I_{C_\infty}(1)} = \lim_{\lambda\to 0} \frac{\lambda^n I_C(1/\lambda^{n+1})}{\lambda^n I_{C_\infty}(1/\lambda^{n+1})} = \lim_{v\to\infty} \frac{I_C(v)}{I_{C_\infty}(v)},$$

as desired. □

We now prove the continuity of the isoperimetric profile of C. The proof of the following is adapted from [29, Lemma 6.2]

Lemma 4.7. *Let C be a convex body with non-degenerate asymptotic cone. Then I_C is continuous.*

Proof. Given $r > 0$ and $x \in C$, we get $B(x,r) \cap (x + C_\infty) \subset B(x,r) \cap C$. Thus

$$|B_C(x,r)| \geqslant |B_{x+C_\infty}(x,r)| = |B_{x+C_\infty}(x,1)|\, r^{n+1} = \ell_1 r^{n+1},$$

for all $x \in C$ and $r > 0$, where $\ell_1 = |B_{C_\infty}(0,1)|$.

Let $E \subset C$ a finite perimeter set and $r > 0$. We apply Fubini's Theorem to the function $C \times E \to \mathbb{R}$ defined by

$$(x,y) \mapsto \chi_{B_C(x,r)}(y)$$

to obtain

$$\int_C |B_C(x,r) \cap E|\, dx = \int_E |B_C(y,r)|\, dy \geqslant \ell_1 r^{n+1} |E|.$$

This implies the existence of some $x \in C$ (depending on E and $r > 0$) such that

$$|B_C(x,r) \cap E| \geqslant \ell_1 r^{n+1} \frac{|E|}{|C|}. \tag{4.14}$$

Fix now two volumes $0 < v_1 < v_2$. Define $r > 0$ by

$$\ell_1 r^{n+1} \frac{v_2}{|C|} = v_2 - v_1.$$

Fix $\varepsilon > 0$. From the definition of the isoperimetric profile, there exists a finite perimeter set $E \subset C$ of volume v_2 such that $P_C(E) \leqslant I_C(v_2) + \varepsilon$. From the above discussion, there exists $x \in C$ so that (4.14) holds. This implies

$$|E \setminus B_C(x,r)| \leqslant |E| - |B_C(x,r) \cap E| \leqslant v_2 - \ell_1 r^{n+1} \frac{v_0}{|C|} = v_1.$$

As the function $t \mapsto |E \setminus B_C(x,t)|$ is continuous and monotone, there exists $0 < s \leqslant r$ so that $|E \setminus B_C(x,s)| = v_1$. Hence we get

$$\begin{aligned} I_C(v_1) &\leqslant P_C(E \setminus B_C(x,s)) \leqslant P_C(E) + P_C(B_C(x,s)) \\ &\leqslant I_C(v_2) + \varepsilon + ms^n \leqslant I_C(v_2) + \varepsilon + mr^n \\ &\leqslant I_C(v_2) + \varepsilon + c\, v_1^{-n/(n+1)}(v_2 - v_1)^{n/(n+1)}, \end{aligned}$$

where $m > 0$ is the perimeter of a Euclidean geodesic sphere of radius 1 and $C > 0$ is explicitly computed from the definition of r. As ε was arbitrary, we get

$$I_C(v_1) \leqslant I_C(v_2) + c\, v_1^{-n/(n+1)}(v_2 - v_1)^{n/(n+1)}. \tag{4.15}$$

We now prove a second inequality. By Lemma 4.1, given $\varepsilon > 0$, there exists $R > 0$ and a finite perimeter set $E \subset \bar{B}_C(0,R)$ of volume v_0 such that $P_C(E) \leqslant I_C(v_1) + \varepsilon$. Now consider a Euclidean geodesic ball B of volume $v_2 - v_1$ in $\operatorname{int}(C) \setminus \bar{B}(0,R))$. We have

$$I_C(v_2) \leqslant P_C(E \cup B) = P_C(E) + P_C(B) \leqslant I_C(v_1) + \varepsilon + c\,(v_2 - v_1)^{n/(n+1)},$$

where $c' > 0$ is the Euclidean isoperimetric constante. Since $\varepsilon > 0$ is arbitrary, we get

$$I_C(v_2) \leqslant I_C(v_1) + c'\,(v_2 - v_1)^{n/(n+1)}. \tag{4.16}$$

Now the continuity of I_C follows from (4.15) and (4.16). □

4.2. Conically bounded convex bodies

Let $C \subset \mathbb{R}^{n+1}$ be an unbounded convex body that can be written as the epigraph of a non-negative convex function over the hyperplane $x_{n+1} = 0$. We shall say that C is a *conically bounded convex body* if, for every $t \geqslant 0$, the set $C_t = C \cap \{x_{n+1} = t\}$ is a convex body in the hyperplane $\{x_{n+1} = t\}$, and there exists a non-degenerate convex cone C^∞ including C such that

$$\lim_{t \to \infty} \max_{|u|=1} |\rho(C_t, u) - \rho((C_\infty)_t, u)| = 0. \tag{4.17}$$

We shall call C^∞ the *exterior asymptotic cone* of C. Because of our assumption of compactness of the slices C_t, the exterior asymptotic cone has a unique vertex. We have the following

LEMMA 4.8. *Let $C \subset \mathbb{R}^{n+1}$ be a conically bounded convex body. Then C_∞ and C^∞ coincide up to translation.*

PROOF. Assume C is the epigraph of the convex function $f : \mathbb{R}^n \to \mathbb{R}^+$, and let C^∞ be defined as the epigraph of the convex function $f^\infty : \mathbb{R}^n \to \mathbb{R}^+$. Since C^∞ is a cone, assuming the origin is a vertex, we have $\lambda f^\infty(x) = f^\infty(\lambda x)$ for any $\lambda > 0$ and $x \in \mathbb{R}^n$.

Let us compute now the asymptotic cone C_∞. From (4.5), the point $(x,y) \in \mathbb{R}^n \times \mathbb{R}$ belongs to C_∞ if and only if $(\mu x, \mu y) \in C$ for all $\mu > 0$. This is equivalent to $y \geqslant \mu^{-1} f(\mu x)$ for all $\mu > 0$. The family $\{f_\mu\}_{\mu>0}$, where f_μ is defined by $f_\mu(x) = \mu^{-1} f(\mu x)$, is composed of convex functions. The convexity of f and the fact that $f(0) = 0$ imply that $f_\mu(x) \leqslant f_\beta(x)$ when $\mu \leqslant \beta$. Hence the asymptotic cone of C is the epigraph of the convex function $f_\infty = \sup_{\mu>0} f_\mu = \lim_{\mu\to\infty} f_\mu$. Observe that $\lambda f_\infty(x) = f_\infty(\lambda x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$. Since $C \subset C^\infty$ we have $f \geqslant f^\infty$ and so

$$f_\infty(x) \geqslant f_\mu(x) = \mu^{-1} f(\mu x) \geqslant \mu^{-1} f^\infty(\mu x) = f^\infty(x).$$

Let us check now that $f_\infty = f^\infty$. Fix some $x \in \mathbb{R}^n \setminus \{0\}$ and let $u = x/|x|$. Then $(x, f(x)) \in \partial C_{f(x)}$ and $\rho(C_{f(x)}, u) = |x|$. If $\mu = f(x)/f^\infty(x)$ then $f^\infty(\mu x) = \mu f^\infty(x) = f(x)$. Hence $(\mu x, f^\infty(\mu x))$ belongs to $\partial(C^\infty)_{f(x)}$, and $\rho((C^\infty)_{f(x)}, u)$ is given by $\mu|x| = (f(x)/f_\infty(x))|x|$. Hence we have

$$|\rho(C_{f(x)}, u) - \rho((C^\infty)_{f(x)}, u)| = \left(\frac{f(x)}{f_\infty(x)} - 1\right)|x|.$$

Replacing x by λx we get

$$|\rho(C_{f(\lambda x)}, u) - \rho((C^\infty)_{f(\lambda x)}, u)| = \left(\frac{f(\lambda x)}{f_\infty(\lambda x)} - 1\right)\lambda|x|.$$

Letting $\lambda \to \infty$, we know that $f(\lambda x)$ converges to ∞ since $f(\lambda x) \geqslant \lambda f^\infty(x)$. By (4.17) we obtain

$$1 = \lim_{\lambda\to+\infty} \frac{f(\lambda x)}{f^\infty(\lambda x)} = \lim_{\lambda\to+\infty} \frac{\lambda^{-1} f(\lambda x)}{\lambda^{-1} f^\infty(\lambda x)} = \frac{f_\infty(x)}{f^\infty(x)}.$$

□

REMARK 4.9. It is not difficult to produce examples of unbounded convex bodies with non-degenerate asymptotic cones which are not conically bounded. Simply consider the epigraph in $\mathbb{R}^2$ of the convex function $f(x) = e^x - 1$. Its asymptotic cone is the quadrant $x \leqslant 0, y \geqslant 0$. On the other hand, there are no asymptotic lines to the graph of $f(x)$ when $x \to +\infty$.

Starting from this example we can produce higher dimensional ones: consider the reflection of $\{(x, f(x)) : x \geqslant 0\}$ with respect to the normal line $x + y = 0$ to the graph of $f(x)$ at $(0,0)$. This convex function can be used to produce higher dimensional unbounded convex bodies of revolution with non-degenerate asymptotic cone which are not conically bounded.

In this Section we shall obtain a number of results for conically bounded convex bodies with smooth boundary. Observe that this assumption does not guarantee that the asymptotic cone has smooth boundary out of the vertexes: simply consider the function in $\mathbb{R}^2$ defined by $f(x,y) = (1+x^2)^{1/2} + (1+y^2)^{1/2}$. The asymptotic cone of its epigraph can be computed as in the proof of Lemma 4.8 as $\{(x,y,z) \in \mathbb{R}^3 : z \geqslant f_\infty(x,y)\}$, where f_∞ is the limit, when $\mu \to \infty$, of the functions $f_\mu(p) = \mu^{-1} f(\mu p)$. In our case $f_\infty(x,y) = |x| + |y|$.

We shall say that a conically bounded convex body is *regular* if it has smooth boundary and its asymptotic cone has smooth boundary out of the vertexes.

The following elementary result on convex functions will be needed

LEMMA 4.10. *Let $a>0$, and $f:[0,+\infty)\to[0,+\infty)$ a convex function satisfying*
$$\lim_{x\to\infty} f(x)-(ax+b)=0.$$
Then, for every $x_0\geqslant 0$ and any $u_0\geqslant f(x_0)$, the halfline $\{(x,u_0+a(x-x_0)) : x\geqslant x_0\}$ is contained in the epigraph of f.

PROOF. Let us prove first that the function $x\mapsto (x-x_0)^{-1}(f(x)-u_0)$ is non-decreasing. Let $x_0<x<z$ so that $x=x_0+\lambda(z-x_0)$, with $\lambda=(x-x_0)/(z-x_0)$. By the concavity of f we get $f(x)=f(\lambda z+(1-\lambda)x_0)\leqslant \lambda f(z)+(1-\lambda)f(x_0)\leqslant \lambda f(z)+(1-\lambda)u_0$. Hence $f(x)-u_0\leqslant\lambda(f(z)-u_0)$, what implies
$$\frac{f(x)-u_0}{x-x_0}\leqslant\frac{f(z)-u_0}{x-x_0},$$
as we claimed.

For any $x>x_0$, the segment joining the points (x_0,u_0) and $(x,f(x))$ is contained in the epigraph of f by the concavity of f. Moreover, we have
$$\frac{f(x)-u_0}{x-x_0}\leqslant\frac{f(x)-f(x_0)}{x-x_0}=\frac{f(x)-ax-b}{x-x_0}-\frac{f(x_0)-ax-b}{x-x_0},$$
and taking limits we get
$$\lim_{x\to\infty}\frac{f(x)-u_0}{x-x_0}\leqslant a,$$
by the monotonicity of $x\mapsto (x-x_0)^{-1}(f(x)-u_0)$ and the asymptotic property of the line $ax+b$. So we conclude $f(x)-u_0\leqslant a(x-x_0)$ for all $x>x_0$, as claimed. □

PROPOSITION 4.11. *Let C be a regular conically bounded convex body, and $\{E_i\}_{i\in\mathbb{N}}$ a diverging sequence of finite perimeter sets with $\lim_{i\to\infty}|E_i|=v$. Then,*
$$\liminf_{i\to\infty} P_C(E_i)\geqslant I_H(v).$$

PROOF. Assume that 0 is the vertex of $C_\infty=C^\infty$. As usual, let $C_s=C\cap\{x_{n+1}=s\}$. The orthogonal projection of $\mathbb{R}^{n+1}$ over $\{x_{n+1}=0\}$ will be denoted by π. The balls considered in what follows will be n-dimensional.

For $t_0>0$ take a positive radius $r_0>0$ so that $B(0,r_0)\times\{t_0\}\subset \operatorname{int} C_{t_0}$. Is is an easy consequence of Lemma 4.10 that the cone of base $B(0,r_0)\times\{t_0\}$ with vertex 0, intersected with $t\geqslant t_0$, is contained in the interior of C. The section of this cone at height t is $B(0,tr_0/t_0)\times\{t\}$, and so $B(0,tr_0/t_0)\subset\operatorname{int}\pi(C_t)$.

We define $F: C\cap\{t\geqslant t_0\}\to C_\infty\cap\{t\geqslant t_0\}$ by
$$F(x,t)=(\tilde f_t(x),t),$$
where $\tilde f_t:\pi(C_t)\to\pi((C_\infty)_t)$ is the map defined by equation (2.6) which leaves fixed the points in the inner ball $B(0,tr_0/t_0)\subset\operatorname{int}\pi(C_t)$. For $i\geqslant t_0$, let $F_i=F|_{C\cap\{x_{n+1}\geqslant i\}}$.

Let us denote by h_λ the dilation in $\mathbb{R}^n$ of ratio $\lambda > 0$. Taking $\lambda = t_0/t$ we have

$$B(0, r_0) = h_\lambda(B(0, \tfrac{t}{t_0} r_0)) \subset \operatorname{int} h_\lambda(\pi(C_t)) \subset \operatorname{int} h_\lambda(\pi((C_\infty)_t)) = \operatorname{int} \pi((C_\infty)_{t_0}).$$

When $t \to \infty$, $h_\lambda(\pi(C_t)) \to \pi((C_\infty)_{t_0})$ in Hausdorff distance since C_∞ is the asymptotic cone of C. Let $f_t : h_\lambda(\pi(C_t)) \to \pi((C_\infty)_{t_0})$ be the family of maps given by (2.6) leaving fixed the ball $B(0, r_0)$ so that $\operatorname{Lip}(f_t)$, $\operatorname{Lip}(f_t^{-1}) \to 1$. It is immediate to show that $\tilde{f}_t = h_{\lambda^{-1}} \circ f_t \circ h_\lambda$ and that $\operatorname{Lip}(\tilde{f}_t) = \operatorname{Lip}(f_t)$, $\operatorname{Lip}(\tilde{f}_t^{-1}) = \operatorname{Lip}(f_t^{-1})$. We conclude that $\operatorname{Lip}(\tilde{f}_t)$, $\operatorname{Lip}(\tilde{f}_t^{-1}) \to 1$.

Let $t \geqslant s \geqslant i \geqslant t_0$. We estimate

$$\begin{aligned} |F(x,t) - F(y,s)| &= \left(|\tilde{f}_t(x) - \tilde{f}_s(y)|^2 + |t-s|^2\right)^{1/2} \\ &= \left(|\tilde{f}_t(x) - \tilde{f}_t(y) + \tilde{f}_t(y) - \tilde{f}_s(y)|^2 + |t-s|^2\right)^{1/2} \\ &= \left(|\tilde{f}_t(x) - \tilde{f}_t(y)|^2 + |\tilde{f}_t(y) - \tilde{f}_s(y)|^2 \right. \\ &\qquad \left. + 2\,|\tilde{f}_t(x) - \tilde{f}_t(y)||\tilde{f}_t(y) - \tilde{f}_s(y)| + |t-s|^2\right)^{1/2}. \end{aligned} \tag{4.18}$$

We have $|(\tilde{f}_t(x) - \tilde{f}_t(y))| \leqslant \operatorname{Lip}(\tilde{f}_t)|x-y|$. By Theorem 2.4, we can write $\operatorname{Lip}(\tilde{f}_t) < (1 + \varepsilon_i)$ for $t \geqslant i$, where $\varepsilon_i \to 0$ when $i \to \infty$. Hence

$$|\tilde{f}_t(x) - \tilde{f}_t(y)| \leqslant (1+\varepsilon_i)\,|x-y|, \qquad \text{for } t \geqslant i. \tag{4.19}$$

We estimate now $|\tilde{f}_t(y) - \tilde{f}_s(y)|$.

In case $|y| \leqslant sr_0/t_0 \leqslant tr_0/t_0$, we trivially have $|\tilde{f}_t(y) - \tilde{f}_s(y)| = 0$. Let us consider the case $|y| \geqslant tr_0/t_0 \geqslant sr_0/t_0$. Set $u = y/|y|$ and for every $t > 0$ denote $\rho_t(u) = \rho(C_t, u)$, $\tilde{\rho}_t(u) = \rho((C_\infty)_t, u)$ hence by(2.7) we have

$$\begin{aligned} |\tilde{f}_t(y) - \tilde{f}_s(y)| &= \left| \frac{(tr_0/t_0 - |y|)}{\tilde{\rho}_t(u) - tr_0/t_0} (\tilde{\rho}_t(u) - \rho_t(u)) - \frac{(sr_0/t_0 - |y|)}{\tilde{\rho}_s(u) - sr_0/t_0} (\tilde{\rho}_s(u) - \rho_s(u)) \right| \\ &\leqslant \frac{|sr_0/t_0 - |y||}{|\tilde{\rho}_s(u) - sr_0/t_0|} \left| (\tilde{\rho}_t(u) - \rho_t(u)) - (\tilde{\rho}_s(u) - \rho_s(u)) \right| \\ &\quad + \left|(\tilde{\rho}_t(u) - \rho_t(u))\right| \left| \frac{tr_0/t_0 - |y|}{\tilde{\rho}_t(u) - tr_0/t_0} - \frac{sr_0/t_0 - |y|}{\tilde{\rho}_s(u) - sr_0/t_0} \right| \\ &\leqslant \left| (\tilde{\rho}_t(u) - \rho_t(u)) - (\tilde{\rho}_s(u) - \rho_s(u)) \right| \\ &\qquad + M \left| \frac{tr_0/t_0 - |y|}{\tilde{\rho}_t(u) - tr_0/t_0} - \frac{sr_0/t_0 - |y|}{\tilde{\rho}_s(u) - sr_0/t_0} \right|, \end{aligned} \tag{4.20}$$

where we have used

$$\left. \frac{|sr_0/t_0 - |y||}{|\tilde{\rho}_s(u) - sr_0/t_0|} \right| \leqslant 1,$$

since $|y| \leqslant \tilde{\rho}_s(u)$ (because $y \in \pi(C_s) \subset \pi((C_\infty)_s)$), and $\left|(\tilde{\rho}_t(u) - \rho_t(u))\right| \leqslant M$ for $t > 1$, since $\sup_{u \in \mathbb{S}^{n-1}} |\tilde{\rho}_t(u) - \rho_t(u)| \to 0$ and so that M does not depend on i, u. For $u \in \mathbb{S}^{n-1}$, consider the functions $\rho_t(u) = \rho(C_t, u)$, $\tilde{\rho}_t(u) = \rho((C_\infty)_t, u)$. Observe that, for every $u \in \mathbb{S}^n$ orthogonal to $\partial/\partial x_{n+1}$, the 2-dimensional half-plane defined by u and $\partial/\partial x_{n+1}$ intersected with

C is a 2-dimensional convex set, and the function $t \mapsto \rho_t(u)$ is concave with asymptotic line the function $t \mapsto \tilde{\rho}_t(u)$. Thus the function $t \mapsto \rho_t(u) - \tilde{\rho}_t(u)$ is concave, because $t \mapsto \rho_t(u)$ is concave and $t \mapsto \tilde{\rho}_t(u)$ is affine, and so

$$\frac{\left|(\tilde{\rho}_t(u)-\rho_t(u))-(\tilde{\rho}_s(u)-\rho_s(u))\right|}{|t-s|} \leqslant \left|(\tilde{\rho}_i(u)-\rho_i(u))-(\tilde{\rho}_{i-1}(u)-\rho_{i-1}(u))\right|. \tag{4.21}$$

Thus by (4.17), the lipschitz constant of $t \mapsto (\tilde{\rho}_t(u)-\rho_t(u))|_{\{t\geqslant i\}}$ is independent of u and tends to 0 as $i \to +\infty$. So, only remains to estimate the second term in the right part of (4.20). To accomplish that, set

$$\rho(u) = \rho((C_\infty)_{t_0}, u) = \rho(h_{t_0/t}(\pi((C_\infty)_t), u) \quad \text{for every } u \in \mathbb{S}^{n-1}.$$

By the homogeneity of the radial function we get

$$\rho(u) = \frac{t_0}{t}\rho(\pi((C_\infty)_t), u) = \frac{t_0}{t}\tilde{\rho}_t(u) \quad \text{for every } t \geqslant t_0.$$

Consequently if R is the inradius of $(C_\infty)_{t_0}$, and u_0 such that $\rho(u_0) = \min_{u\in\mathbb{S}^{n-1}} \rho(u)$, then

$$\begin{aligned}
\left|\frac{tr_0/t_0-|y|}{\tilde{\rho}_t(u)-tr_0/t_0} - \frac{sr_0/t_0-|y|}{\tilde{\rho}_s(u)-sr_0/t_0}\right| &\leqslant \left|\frac{tr_0/t_0-|y|}{t/t_0\,\tilde{\rho}(u)-tr_0/t_0} - \frac{sr_0/t_0-|y|}{s/t_0\,\tilde{\rho}(u)-sr_0/t_0}\right| \\
&\leqslant \frac{|y|t_0}{\rho(u)-r_0}\left|\frac{1}{t}-\frac{1}{s}\right| \\
&\leqslant \frac{Rt_0}{\rho(u_0)-r_0}\left|\frac{1}{t}-\frac{1}{s}\right| \\
&\leqslant \frac{Rt_0}{\rho(u_0)-r_0}\,\frac{1}{i^2}|t-s|
\end{aligned} \tag{4.22}$$

Thus, the lipschitz constant of

$$t \mapsto \frac{tr_0/t_0-|y|}{\tilde{\rho}_t(u)-tr_0/t_0}\Big|_{\{t\geqslant i\}}$$

is independent of u and tends to 0 as $i \to +\infty$.

By the above discussion and (4.20), there exists ℓ_i for every $i \in \mathbb{N}$ such that $\ell_i \to 0$, and

$$|f_t(y) - f_s(y)| \leqslant \ell_i\,|t-s|. \tag{4.23}$$

From (4.18), (4.19), (4.23), and trivial estimates, we obtain

$$|F_i(x,t) - F_i(y,s)| \leqslant \left((1+\varepsilon_i)^2 + \ell_i^2 + (1+\varepsilon_i)\ell_i\right)^{1/2}|x-y| \tag{4.24}$$

Now $\varepsilon_i \to 0$ and $\ell_i \to 0$ as $i \to \infty$. Thus inequality (4.24) finally give us

$$\limsup_{i\to\infty} \mathrm{Lip}(F_i) \leqslant 1.$$

Similarly we find $\limsup_{i\to\infty} \mathrm{Lip}(F_i^{-1}) \leqslant 1$. From the general inequality $\mathrm{Lip}(F_i^{-1})\,\mathrm{Lip}(F_i) \geqslant 1$ we finally get that $\max\{\mathrm{Lip}(F_i), \mathrm{Lip}(F_i^{-1})\} \to 1$ when $i \to \infty$ (indeed we have just proved that $d_L(C \cap \{x_{n+1} \geqslant i\}, C^\infty \cap \{x_{n+1} \geqslant i\}) \to 0$).

Now in case that $|y| \geqslant tr_0/t_0$ but $|y| \leqslant sr_0/t_0$, we can find $t^* > 0$ such that $|y| = t^* r_0/t_0$, then as $\tilde{f}_t(y) = \tilde{f}_{t^*}(y) = y$, but in the same time $\tilde{f}_{t^*}(y)$ can have the expression of (2.6) then after a triangle inequality argument this case is reduced to the previous one. □

PROPOSITION 4.12. *Let C be a regular conically bounded convex body. Then isoperimetric regions exist in C for all volumes.*

PROOF. Fix $v > 0$. By Proposition 1.8, there exists $E \subset C$ (possibly empty) such that $|E| = v_1$, $P_C(E) = I_C(v_1)$, and a diverging sequence $\{E_i\}_{i \in \mathbb{N}}$ of finite perimeter sets such that $|E_i| \to v_2 = v - v_1$; moreover

$$I_C(v) = P_C(E) + \lim_{i \to \infty} P_C(E_i) \tag{4.25}$$

Assume now that $v_2 > 0$. From Proposition 4.11 we get $\lim P_C(E_i) \geqslant I_H(v_2)$. Now by Proposition 4.5, the set E is bounded and by Proposition 2.36 we can find an intrinsic ball $B \subset C$ with volume v_2 such that $E \cap B = \emptyset$ and $P_C(B) \leqslant I_H(v_2)$. Then (4.25) gives

$$I_C(v) = P_C(E) + \lim_{i \to \infty} P_C(E_i) \geqslant P_C(E) + I_H(v_2) \geqslant P_C(E) + P_C(B). \tag{4.26}$$

Thus $E \cup B$ is an isoperimetric region with volume v. □

PROPOSITION 4.13. *Let $C \subset \mathbb{R}^{n+1}$ be a conically convex set. Then I_C, Y_C are positive concave functions, and so they non-decreasing. Consequently, every isoperimetric region in C is connected.*

PROOF. By 4.12 isoperimetric regions exist for all volumes thus we can argue as in [9, Thm. 3.2] to conclude that the upper second derivative of Y_C is non-positive, where combining with the fact that Y_C is continuous 4.7, we deduce that Y_C is concave. And so is I_C as a composition of non-negative concave functions.

The connectedness of the isoperimetric regions is an implication of the concavity of Y_C, Theorem 2.15. □

COROLLARY 4.14. *Let $C \subset \mathbb{R}^{n+1}$ be a regular conically bounded convex body. Given any $v > 0$, any minimizing sequence for volume v converges to an isoperimetric region.*

PROOF. We reason by contradiction as in the proof of Proposition 4.12. Then we find an isoperimetric region in C consisting of two components E and B, a contradiction to Proposition 4.13. □

As a consequence we get, in the same way as in section 4, the two following lemmata,

LEMMA 4.15. *Let $C \subset \mathbb{R}^{n+1}$ be a regular conically bounded convex body, and $0 < v_0 < |C|$. Then*

$$I_C(v) \geqslant \frac{I_C(v_0)}{v_0^{n/(n+1)}} \, v^{n/(n+1)}, \tag{4.27}$$

for all $0 < v \leqslant v_0$.

LEMMA 4.16. *Let C be be a regular conically bounded convex body, $\lambda \geqslant 1$. Then*

$$I_{\lambda C}(v) \geqslant I_C(v) \tag{4.28}$$

for all $0 < v < |C|$.

Now we can prove the following density estimate.

PROPOSITION 4.17. *Let $C \subset \mathbb{R}^{n+1}$ be a regular conically bounded convex body, and $E \subset C$ an isoperimetric region of volume $0 < v < |C|$. Choose ε so that*

$$0 < \varepsilon < \min\left\{ {\ell_2}^{-1} v, c_2 v, \frac{\ell_2^n}{8^{n+1}}, {\ell_2}^{-1}\left(\frac{c_1}{4}\right)^{n+1} \right\}, \tag{4.29}$$

where $c_1 = v^{-n/(n+1)} I_C(v)$ and c_2 is the constant in Lemma 2.25.

Then, for any $x \in C$ and $R \leqslant 1$ so that $h(x,R) \leqslant \varepsilon$, we get

$$h(x,R/2) = 0. \tag{4.30}$$

Moreover, in case $h(x,R) = |E \cap B_C(x,R)||B_C(x,R)|^{-1}$, we get $|E \cap B_C(x,R/2)| = 0$ and, in case $h(x,R) = |B_C(x,R) \setminus E||B_C(x,R)|^{-1}$, we have $|B_C(x,R/2) \setminus E| = 0$.

PROOF. In case $h(x,R) = |E \cap B_C(x,R)||B_C(x,R)|^{-1}$ we argue as in [62, Prop. 4.9] to get

$$bR/4a \leqslant (m(R)^{1/(n+1)} - m(R/2)^{1/(n+1)}) \leqslant m(R)^{1/(n+1)} \leqslant (\varepsilon \ell_2)^{1/(n+1)} R.$$

This is a contradiction, since $\varepsilon \ell_2 < (b/4a)^{n+1} = I_C(v)^{n+1}/(8^{n+1} v^n) \leqslant \ell_2^{n+1}/8^{n+1}$ by (4.29) and Proposition 2.36. So the proof in case $h(x,R) = |E \cap B_C(x,R)|\,(|B_C(x,R))|^{-1}$ is completed.

For the remaining case, when $h(x,R) = |B_C(x,R)|^{-1}|B_C(x,R) \setminus E|$, using Lemma 2.18 and the fact that I_C is non-decreasing by Proposition 4.13, we argue as in Case 1 in Lemma 4.2 of [44] we get

$$c_1/4 \leqslant (\varepsilon \ell_2)^{1/(n+1)}$$

This is a contradiction, since $\varepsilon \ell_2 < (c_1/4)^{n+1}$ by (4.29). □

One of the consequences of Proposition 4.17 is the following lower density bound, which is usually obtained from the monotonicity formula.

COROLLARY 4.18 (Lower density bound). *Let $C \subset \mathbb{R}^{n+1}$ be a regular conically bounded convex body, and $E \subset C$ an isoperimetric region of volume v. Then there exists a constant $M > 0$, only depending on the constant ε in (4.29), on a Poincaré's constant for $r \leqslant 1$ as in (4.3), and on an Ahlfors constant ℓ_1 as in (4.4), such that*

$$P(E, B_C(x,r)) \geqslant M r^n, \tag{4.31}$$

for all $x \in \partial_C E_1$ and $r \leqslant 1$.

PROOF. Let $E \subset C$ be an isoperimetric region of volume $v > 1$, that exists by Proposition 4.12. The constant ε in (4.29) can be chosen independently of $v > 1$ since the quantity $\inf_{v \geqslant 1} v^{-n/(n+1)} I_C(v)$ is uniformly bounded from below by a positive constant because of (4.7). Then we have

$$\begin{aligned} P(E, B_C(x,r)) &\geqslant M \min\{|E \cap B_C(x,r)|, |B_C(x,r) \setminus E|\}^{n/(n+1)} \\ &= M\,(|B_C(x,r)|\, h(x,r))^{n/(n+1)} \geqslant M(|B_C(x,r)|\,\varepsilon)^{n/(n+1)} \\ &\geqslant M\,(\ell_1 \varepsilon)^{n/(n+1)}\, r^n, \end{aligned}$$

as claimed. □

So we have our convergence result

THEOREM 4.19. *Let $C \subset \mathbb{R}^{n+1}$ be a regular conically bounded convex body. Then a rescaling of a sequence of isoperimetric regions of volumes approaching infinity converges in Hausdorff distance to a geodesic ball centered at the vertex in the asymptotic cone. The same convergence result holds for their free boundaries.*

PROOF. Assume $0 \in \partial C$. Let $\{E_i\}_{i \in \mathbb{N}} \subset C$ be a sequence of isoperimetric regions of volumes $|E_i| \to \infty$, and let $\lambda_i \to 0$ so that $|\lambda_i E_i| = 1$. The sets $\Omega_i = \lambda_i E_i$ are isoperimetric regions in $\lambda_i C$, and they are connected by Proposition 4.13. We claim

$$\operatorname{diam}(\Omega_i) \leqslant c, \qquad \text{for all } i \text{ and some } c > 0. \tag{4.32}$$

If claim holds, let $q \in \operatorname{int}(\overline{B}_{C_\infty}(0,1))$ and $B_q \subset \operatorname{int}(\overline{B}_\infty(0,1))$ be a Euclidean geodesic ball. Consider a solid cone K_q with vertex q such that $0 \in \operatorname{int}(K_q)$ and $K_q \cap C \cap \partial B(0,1) = \emptyset$. By (4.32) we get $\operatorname{diam}(\lambda_i \Omega_i) \to 0$, and hence $\lambda_i \Omega_i \to 0$ in Hausdorff distance, what implies

$$\lambda_i \Omega_i \subset K_q,$$

for large enough $i \in \mathbb{N}$.

As the sequence $\lambda_i^2 C \cap \overline{B}(0,1)$ converges in Hausdorff distance to $C_\infty \cap \overline{B}(0,1)$, we construct using Theorem 2.4 a family of bilipschitz maps

$$f_i : \lambda_i^2 C \cap \overline{B}(0,1) \to C_\infty \cap \overline{B}(0,1)$$

so that f_i is the identity in B_q and it is extended linearly along the segments leaving from q. The maps f_i satisfy $\operatorname{Lip}(f_i)$, $\operatorname{Lip}(f_i^{-1}) \to 1$, and have the additional property

$$P_{C_\infty}(f_i(\lambda_i \Omega_i)) = P_{\overline{B}_{C_\infty}(0,1)}(f_i(\lambda_i \Omega_i)).$$

Then $g_i = \lambda_i f_i \lambda_i^{-1}$, defined from $\lambda_i C \cap \overline{B}(0, \lambda_i^{-1})$ to $C_\infty \cap \overline{B}(0, \lambda_i^{-1})$ satisfy the same properties $\operatorname{Lip}(g_i)$, $\operatorname{Lip}(g_i^{-1}) \to 1$ and $P_{C_\infty}(g_i(\Omega_i)) = P_{\overline{B}_{C_\infty}(0,\lambda_i^{-1})}(g_i(\Omega_i))$. From Lemma 1.3 we get

$$\begin{aligned} \lim_{i\to\infty} \operatorname{diam}(\Omega_i) &= \lim_{i\to\infty} \operatorname{diam}(g_i(\Omega_i)), \\ 1 = \lim_{i\to\infty} |\Omega_i| &= \lim_{i\to\infty} |g_i(\Omega_i)|, \\ \liminf_{i\to\infty} P_{\lambda_i C}(\Omega_i) &= \liminf_{i\to\infty} P_{C_\infty}(g_i(\Omega_i)). \end{aligned} \tag{4.33}$$

Consequently, by (4.32), the sets $g_i(\Omega_i)$ have uniformly bounded diameter. If the sequence of sets $\{g_i(\Omega_i)\}_{i\in\mathbb{N}}$ has a divergent subsequence, then (4.13), (4.33), and Proposition 1.10 imply

$$I_{C_\infty}(1) = \lim_{i\to\infty} I_{\lambda_i C}(1) = \liminf_{i\to\infty} P_{C_\infty}(g_i(\Omega_i)) \geqslant I_H(1), \tag{4.34}$$

and from (4.5) we would get that C_∞ is a half-space, a contradiction. Hence the sequence $\{g_i(\Omega_i)\}_{i\in\mathbb{N}}$ stays bounded, and we can apply the convergence results for convex bodies to obtain L^1-convergence of the sets Ω_i and improve, using the density estimates in Proposition 4.17, the L^1-convergence to Hausdorf convergence of the sets Ω_i and their boundaries, Theorem 2.32 and Theorem 2.34

So it only remains to prove (4.32) to conclude the proof. Since $(\lambda_i C)_\infty = C_\infty$ we can choose, using Lemma 4.3, a uniform Poincaré's constant for $r \leqslant 1$, and a uniform Ahlfors constant ℓ_1 for all $\lambda_i C$. Further, since $I_{\lambda_i C} \geqslant I_{C_\infty}$, the constant ε in (4.29) can be chosen uniformly for all $\lambda_i C$ as well. Consequently a lower density bound, as in Corollary 4.18, holds for all Ω_i with a uniform constant. Since the sets Ω_i are connected by Proposition 4.13, we conclude that $\operatorname{diam}(\Omega_i)$ are uniformly bounded, since otherwise (4.31) would imply that $P_{\lambda_i C}(\lambda_i E_i)$ goes to infinity. This way we obtain a contradiction, since by (2.49), we get $P_{\lambda_i C}(\lambda_i E_i) = I_{\lambda_i C}(1) \leqslant I_H(1)$ for all i. □

Since we are assuming smoothness of the boundaries of both the conically bounded set C and of its asymptotic cone C_∞ (out of the vertex), we can use density estimates for varifolds to improve the convergence. In particular, the mean curvatures of the boundaries of the isoperimetric regions satisfy a uniform estimate

LEMMA 4.20. *Let $C \subset \mathbb{R}^{n+1}$ be a regular conically bounded convex body, and $\{E_i\}_{i\in\mathbb{N}}$ a sequence of isoperimetric regions of volumes $v_i \to \infty$. Let H_i be the constant mean curvature of the regular part of the boundary of E_i. Then $H_i v_i^{1/(n+1)}$ is bounded.*

PROOF. It is known that the mean curvature H of the boundary of an isoperimetric region of volume v satisfies $H \leqslant (I_C')_-(v)$, where $(I_C')_-$ is the left derivative of the concave function I_C. Observe that there are constants $m, M > 0$ such that

$$m v^{n/(n+1)} \leqslant I_C(v) \leqslant M v^{n/(n+1)}, \qquad \text{for large } v.$$

The left inequality follows from inequality (4.6), $I_C \geqslant I_{C_\infty}$, and it is indeed true for any $v > 0$. The second one follows from (4.7), $\lim_{v\to\infty}(I_{C_\infty}^{-1} I_C)(v) = 1$.

For large v we have

$$v^{1/(n+1)} H \leqslant \left(\frac{1}{m}\right)^{1/n} I_C(v)^{1/n} (I_C')_-(v) = \left(\frac{1}{m}\right)^{1/n} \left(\frac{n}{n+1}\right) (Y_C)'_-(v),$$

where $Y_C = I_C^{(n+1)/n}$. Hence the estimate

$$(Y_C)'_-(v) = \lim_{h\to 0^+} \frac{Y_C(v-h) - Y_C(v)}{h} \leqslant \frac{Y_C(v)}{v} \leqslant M^{(n+1)/n}.$$

proves the result. □

4.3. Large isoperimetric regions in conically bounded convex bodies of revolution

In this Section we consider regular conically bounded sets of revolution in $\mathbb{R}^{n+1}$, generated by a smooth convex function $f : [0,+\infty) \to \mathbb{R}^+$ with $f(0) = f'(0) = 0$. We may think of f as the restriction to $[0,+\infty)$ of a smooth convex function $f : \mathbb{R} \to \mathbb{R}^+$ satisfying $f(x) = f(-x)$. For any $n \in \mathbb{N}$, the function f defines a convex body of revolution $C_f \subset \mathbb{R}^{n+1}$ as the set of points $(x,y) \in \mathbb{R}^n \times \mathbb{R}$ satisfying the inequality $y \geqslant f(|x|)$. As we shall see, the conical boundedness condition is equivalent to the existence of a constant $a > 0$ so that

$$\lim_{x\to\infty}(f(x) - ax) = 0.$$

This implies that the line $y = ax$ is an asymptote of the function f. For such a function, we have

$$\lim_{x\to\infty}\frac{f(x)}{x} = a.$$

and L'Hôpital's Rule implies

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\frac{f(x)}{x} = a,$$

and

$$\lim_{x\to\infty} x f''(x) = \lim_{x\to\infty}\frac{f'(x)}{\log(x)} = 0.$$

We have the following

LEMMA 4.21. *Given a smooth convex function $f : [0,+\infty) \to \mathbb{R}^+$ such that $f'(0) = 0$ and $\lim_{x\to+\infty}(f(x) - ax) = 0$ for some constant $a > 0$, we have*

- (i) *The set $C_f = \{x,y) \in \mathbb{R}^n \times \mathbb{R} : y \geqslant f(|x|)\}$ is conically bounded with asymptotic cone at infinity $(C_f)_\infty = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y \geqslant a|x|\}$.*
- (ii) *There exists a compact set $K \subset C_f$ so that $C_f \setminus K$ is foliated by spherical caps meeting ∂C_f in an orthogonal way.*
- (iii) *The mean curvature of the spherical caps is a non-increasing function (in the unbounded direction) and converges to* 0.

PROOF. Let us call $C^\infty = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y \geqslant a|x|\}$. Observe that Lemma 4.10 implies that $f(x) \geqslant ax$ for all $x \geqslant 0$ and so $C_f \subset C^\infty$. To show that the set C_f is conically bounded we compute $\rho((C_f)_{f(x)}, u) = x$, and $\rho((C^\infty)_{f(x)}, u) = f(x)/a$ for all $u \in \mathbb{S}^{n-1}$. Hence condition (4.17) is satisfied. We know that the asymptotic cone $(C_f)_\infty$ is the epigraph of the convex function $f_\infty(x) = \lim_{\mu\to\infty}\mu^{-1}f(\mu x) = ax$. This implies (i).

Let us prove (ii). For any $x > 0$, we consider the center $(0, c(x))$ and the radius $r(x)$ of the circle meeting the graph of f orthogonally at the point $(x, f(x))$. We have

$$c(x) = f(x) - x f'(x), \qquad r(x) = x(1 + f'(x)^2)^{1/2}.$$

It is easy to check that $c'(x) = -x f''(x) \leqslant 0$. If we define $g(x) = c(x) + r(x)$ and fix $x_0 > 0$, the circles around the one with center $(0, c(x_0))$ and radius $r(x_0)$ form a local foliation if $g'(x_0) > 0$. Since

$$g'(x) = x f''(x) \left(-1 + \frac{f'(x)}{(1 + f'(x)^2)^{1/2}} \right) + (1 + f'(x)^2)^{1/2},$$

taking limits we obtain

$$\lim_{x \to \infty} g'(x) = (1 + a^2)^{1/2} > 0.$$

So we conclude that there exists $x_m > 0$ so that the circles corresponding to points $x > x_m$ form a foliation meeting the boundary of the convex set in an orthogonal way. The corresponding bodies of revolution exhibit the same property. In these cases, there is a foliation outside a compact set whose leaves are spherical caps meeting orthogonally the boundary of the convex set.

To prove (iii), simply take into account that the mean curvature of the spheres is $r(x)^{-1} = x^{-1}(1 + f'(x)^2)^{-1/2}$ and $\lim_{x \to \infty} r(x)^{-1} = 0$. □

REMARK 4.22. Let C be a convex body of revolution generated by a convex function f satisfying $f'(0) = 0$. If we assume $\lim_{x \to \infty} x^{-1} f(x) = 0$ then $f \equiv 0$. This follows since the function f' is non-decreasing and satisfies $\lim_{x \to \infty} f'(x) = 0$. Hence a convex body of revolution cannot be asymptotic to a half-space unless it is a half-space.

Let (M, g_0) be a smooth Riemannian manifold with smooth boundary. Assume that Σ is an embedded hypersurface with constant mean curvature H_Σ and that $\partial \Sigma$ is contained in ∂M and meets ∂M in an orthogonal way. We shall assume that Σ is two-sided and so there is a unit normal N_Σ to Σ. The unit conormal to $\partial \Sigma$ will be denoted by ν_Σ.

Let X be a C^∞ complete vector field in M so that $X|_\Sigma = N$ and $X|_{\partial M}$ is tangent to ∂M. The flow $\{\varphi_t\}_{t \in \mathbb{R}}$ of X preserves the boundary of M and allows us to define "graphs" over Σ. If $u \in C^{2,\alpha}(\Sigma)$ has small enough $C^{2,\alpha}$ norm, then the graph of u, denoted by $\Sigma(u)$, is defined as the set $\{\varphi_{u(p)}(p) : p \in \Sigma\}$. For small $C^{2,\alpha}$ norm, $\Sigma(u)$ is an embedded hypersurface. Given a Riemannian metric g on M, we shall denote the unit normal to $\Sigma(u)$ in (M, g) by $N^g_{\Sigma(u)}$ and shall drop g when $g = g_0$. The unit conormal will be denoted by $\nu^g_{\Sigma(u)}$. Given g, the inner unit normal to the boundary of M will be denoted by $N^g_{\partial M}$. The laplacian on Σ, the Ricci curvature tensor, the second fundamental form of ∂M with respect to an inner normal, and the squared norm of the second fundamental form, with respect to a Riemannian metric g, will be denoted by Δ^g_Σ, Ric^g, II^g, $|\sigma^g|^2$, respectively. We shall drop the superscript g when $g = g_0$.

We shall use the following well-known result, compare with [5, Prop. 10]

PROPOSITION 4.23. *Let (M, g_0) be a Riemannian manifold with smooth boundary and $\Sigma \subset M$ an embedded hypersurface with constant mean curvature H_Σ such that $\partial \Sigma \subset \partial M$ meets ∂M*

in an orthogonal way. Assume that the free boundary problem

$$\begin{aligned} &\Delta_\Sigma u + (\mathrm{Ric}(N,N) + |\sigma|^2)u = 0, \qquad &&\text{on } \Sigma \\ &\frac{\partial u}{\partial \nu_\Sigma} + \mathrm{II}(N,N)u = 0, \qquad &&\text{on } \partial\Sigma \end{aligned} \tag{4.35}$$

has just the trivial solution. Then there is a neighborhood U of g_0 in $\mathrm{Riem}(M)$ and a neighborhood I of H_Σ so that for $(g,H) \in U \times I$, there is just one graph of class $C^{2,\alpha}$ with constant mean curvature H meeting ∂M in an orthogonal way in the Riemannian manifold (M,g).

Proof. The proof is an application of the Implicit Function Theorem for Banach spaces. Consider the map $\Phi : (\mathrm{Riem}(M) \times \mathbb{R}) \times C^{2,\alpha}(\Sigma) \longrightarrow C^{0,\alpha}(\Sigma) \times C^{1,\alpha}(\partial\Sigma)$ defined by

$$\Phi(g,H,u) = (H^g_{\Sigma(u)} - H_\Sigma, g(\nu^g_{\Sigma(u)}, N^g_{\partial M})).$$

The partial derivative $D_2\Phi$ with respect to the factor $C^{2,\alpha}(\Sigma)$ is given by

$$-D_2\Phi(g_0,H_0,0)(v) = \big(\Delta_\Sigma v + (\mathrm{Ric}(N,N) + |\sigma|^2)\,v, \frac{\partial v}{\partial \nu_\Sigma} + \mathrm{II}(N_\Sigma,N_\Sigma)\,v\big).$$

This map is injective by assumption and surjective by the Fredholm alternative. It is continuous and an isomorphism by Schauder estimates [30, Theorem 6.30 (6.77)]. Hence we can apply the Implicit Function Theorem for Banach spaces to conclude the proof. □

We shall also need the following

Lemma 4.24 ([6, Corollary 3.4]). *Let $\mathbb{S}^n(R) \subset \mathbb{R}^{n+1}$ and $B(r) \subset \mathbb{S}(R)$ be a geodesic ball (spherical cap) of radius $0 < r < \pi R/2$. Then the first nonzero Neumann eigenvalue $\mu(r)$ in $B(r)$ satisfies $\mu(r) > nR^{-2}$.*

Now we are in position to prove the main result in this Section

Theorem 4.25. *Let C be a conically bounded convex body of revolution. Then there exists $v_0 > 0$ such that any isoperimetric region $E \subset C$ of volume $|E| \geqslant v_0$ is a spherical cap meeting the boundary of C in an orthogonal way.*

Proof. By Remark 4.22, the asymptotic cone of C is not a half-space. Hence C is generated by a convex function f such that $\lim_{x\to\infty} x^{-1} f(x) = a > 0$. The asymptotic cone of C is C_∞ is the convex body of revolution generated by the function $f_\infty(x) = ax$.

Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of isoperimetric regions in C with $|E_i| \to \infty$. By Theorem 4.19, for $\lambda_i = v_i^{-1/(n+1)}$, the boundaries of $\lambda_i E_i$ converge in Hausdorff distance to a spherical cap $\Sigma \subset \mathbb{S}(R)$, of radius $0 < r < \pi R/2$, inside the asymptotic cone of C. Moreover, we can find a sequence of diffeomorphisms φ_i of class C^∞ applying a small tubular neighborhood of Σ into a subset of $\lambda_i C$ containing the boundary of $\lambda_i E_i$. The diffeomorphisms can be chosen to respect the orthogonal directions to the boundaries. The mean curvature of the boundary of $\lambda_i E_i$ is given by $H_i v_i^{1/(n+1)}$, which is uniformly bounded by Lemma 4.20, and so it is the mean curvature of $\varphi_i^{-1}(\lambda_i E_i)$ computed with respect to the metric $\varphi_i^* g_0$. The reduced boundary of $\varphi_i(\lambda_i E_i)$ is a stationary varifold with boundary because of the condition imposed to φ_i to respect the orthogonal directions of the boundaries. Since the perimeters of $\varphi_i(\lambda_i E_i)$ converge to the perimeter of Σ, we can use [38, Theorem 4.13] to get $C^{1,\delta}$-convergence of the boundaries, see Section 4.4. By elliptic regularity, the mean curvatures of the boundaries of $\varphi_i^{-1}(\lambda_i E_i)$, computed with respect to the metric $\varphi_i^* g_0$, also converge to the mean curvature of Σ, and the boundary of $\varphi_i(\lambda_i E_i)$ is the graph of a C^∞ function over Σ in the sense defined above.

The hypersurface $\Sigma \subset C_\infty$ is the boundary of an isoperimetric region in C_∞. On Σ we have $\operatorname{Ric}(N,N)+|\sigma|^2 = nR^{-2}$ and $\mathrm{II}(N,N)=0$. So the free boundary problem (4.35) is given by

$$\begin{aligned} \Delta u + nR^{-2}u &= 0, \qquad \text{on } \Sigma, \\ \frac{\partial u}{\partial \nu} &= 0, \qquad \text{on } \partial\Sigma. \end{aligned}$$

By Lemma 4.24 the first nonzero Neumann eigenvalue of the Laplacian on Σ is strictly larger than nR^{-2}, and so the only solution is $u=0$. Proposition 4.23 then implies that, for large enough $i \in \mathbb{N}$ so that $\varphi_i^* g_0$ is close to g_0 and the mean curvature of the boundary of $\lambda_i E_i$ is close to the one of Σ, there is only one such graph.

Consider now a sequence of spherical caps in C with the same mean curvature as the one of ∂E_i. Scaling down we have C^∞ convergence to Σ. By the uniqueness part of Proposition 4.23, we obtain that E_i is a spherical cap for i large enough. □

4.4. The result by Grüter and Jost

The version of Theorem 4.13 of the paper by Grüter and Jost we are going to use reads as follows

THEOREM 4.26. *For any n, $p \in \mathbb{N}$ such that $p > n$, $\eta > 0$, there exists $\gamma(n,p) > 0$, $\varepsilon(n,p,\eta) > 0$ with the following property.*

If $\rho \leqslant 1$, $B \subset \mathbb{R}^{n+1}$ is a hypersurface of class C^2 with $0 \in B$, $\bar{B} \cap B_1(0) = B$, and the radius of curvature κ of B satisfies

$$\kappa\rho \leqslant \varepsilon^2,$$

and if $V = v(M,\theta)$ is a rectifiable n-varifold with

$$\begin{gathered} spt\,\mu \subset \overline{B_1'(0)}, \qquad (\mu = \mu_V), \\ 0 \in spt\,\mu, \\ \theta \geqslant 1 \qquad \mu\text{-a.e.} \\ \frac{1}{\omega_n \rho^n}\,\mu(B_\rho(0)) \leqslant \frac{1}{2}\,(1+\varepsilon) \\ \int \operatorname{div}_M X\,d\mu = -\int X \cdot H\,d\mu \end{gathered}$$

for all $X \in C_c^1(B_1(0), \mathbb{R}^{n+1})$ with $X(b) \in \tau(b)$ for $b \in B$ and

$$\left(\int_{B_\rho(0)} |H|^p\,d\mu\right)^{1/p} \rho^{1-n/p} \leqslant \varepsilon,$$

then there is a $C^{1,\delta}$-function $u : B^n_{\gamma\rho}(0) \to \mathbb{R}$ and an isometry ℓ of $\mathbb{R}^{n+1}$ with

$$\begin{gathered} u(0) = 0 \\ \nu_{\ell B}(x) \subset T_x \operatorname{graph} u \qquad \text{for } x \in \ell B \cap \operatorname{graph} u \\ \operatorname{spt}\mu_{\ell_\# V} \cap B_{\gamma\rho}(0) = \operatorname{graph} u \cap B_{\gamma\rho}(0) \cap \overline{\ell B_1'(0)} \end{gathered}$$

and

$$\begin{gathered} \rho^{-1} \sup_{D^n_{\gamma\rho}(0)} |u| + \sup_{D^n_{\gamma\rho}(0)} |Du| + \rho^\delta \sup_{x,y \in D^n_{\gamma\rho}(0), x \neq y} |x-y|^{-\delta} |Du(x) - Du(y)| \leqslant c\eta, \\ \delta = \min\{\tfrac{1}{2}, 1 - n/p\} \text{ and } D^n_r(0) = p\,(\operatorname{graph} u \cap B_r(0) \cap \overline{\ell B_1'(0)}). \end{gathered}$$

We are going to apply this result to a sequence $\varphi_i(\lambda_i E_i)$, where φ_i is a sequence of diffeomorphisms converging to the identity in the C^k topology, where $k \in \mathbb{N}$ is arbitrarily large (even ∞), E_i is a sequence of isoperimetric regions in C, and $\lambda_i = |E_i|^{-1/(n+1)}$. We know that $\varphi_i(\lambda_i E_i)$ and their boundaries converge to a ball $E \subset C_\infty$ and also their boundaries converge in Hausdorff distance.

The set V is the reduced boundary of $\varphi_i(\lambda_i E_i)$, which is a varifold with uniformly bounded mean curvature, because $\lambda_i E_i$ has uniformly bounded mean curvature and φ_i converges to the identity. The support of μ is contained in the boundary of $\varphi_i(\lambda_i E_i)$, contained in the interior of C_∞. So the hypothesis $\operatorname{spt}\mu \subset \overline{B_1'(0)}$, is trivially satisfied ($B_1'(0)$ is one of the connected components of $B_1(0) \setminus B$). That $0 \in \operatorname{spt}\mu$ and $\theta = 1$ hold in our case. The hypotheses

$$\frac{1}{\omega_n \rho^n}\,\mu(B_\rho(0)) \leqslant \frac{1}{2}\,(1+\varepsilon)$$

is satisfied because of the L^1-convergence of $\varphi_i(\lambda_i E_i)$ to E, the lower semicontinuity of perimeter, and the regularity of ∂E. Condition

$$\int \operatorname{div}_M X \, d\mu = -\int X \cdot H \, d\mu$$

holds if we apply the boundary of $\lambda_i C$ to the boundary of C preserving the orthogonallity to the boundary. If H is uniformly bounded then, for any $p > n$, we would have

$$\left(\int_{B_\rho(0)} |H|^p \, d\mu \right)^{1/p} \rho^{1-n/p} \leqslant \rho \sup |H| \left(\frac{\mu(B_\rho(0))}{\rho^n} \right)^{1/p} \leqslant \rho \sup |H| \left(\frac{1+\varepsilon}{2} \right)^{1/p},$$

which is smaller than ε for ρ small enough. Hence we get the conclusion that the boundary of $\varphi_i(\lambda_i E_i)$ is a $C^{1,\delta}$-graph over the boundary of ∂E for i large enough.

CHAPTER 5

Large isoperimetric regions in the product of a compact manifold with Euclidean space

Here $N = M \times \mathbb{R}^k$, where M is a compact Riemannian manifold. Given a set $E \subset N$, their perimeter and volume will be denoted by $|E|$ and $P(E)$, respectively. We refer the reader to Maggi's book [47] for background on finite perimeter sets. The r-dimensional Hausdorff measure of a set E will be denoted by $H^r(E)$.

On $M \times \mathbb{R}^k$ we shall consider the anisotropic dilation of ratio $t > 0$ defined by

$$\varphi_t(p,x) = (p,tx), \qquad (p,x) \in M \times \mathbb{R}^k.$$

Since the Jacobian of the map φ_t is t^k we have

$$|\varphi_t(E)| = t^k|E|, \qquad E \subset M \times \mathbb{R}^k. \tag{5.1}$$

Let $\Sigma \subset M \times \mathbb{R}^k$ be an $(n-1)$-rectifiable set. At a regular point $p \in \Sigma$, the unit normal ξ can be decomposed as $\xi = av + bw$, with $a^2 + b^2 = 1$, v tangent to M and w tangent to $\mathbb{R}^k$. Then the Jacobian of $\varphi_t|\Sigma$ is equal to $t^{k-1}(t^2a^2 + b^2)^{1/2}$. For $t \geqslant 1$ we get

$$t^k H^{n-1}(\Sigma) \geqslant H^{n-1}(\varphi_t(\Sigma)) \geqslant t^{k-1}H^{n-1}(\Sigma), \tag{5.2}$$

and the reversed inequalities when $t \leqslant 1$. A similar property holds for the perimeter. Equality holds in the right hand side of (5.2) if and only if $a = 0$, what implies that ξ is tangent to $\mathbb{R}^k$.

An open ball of radius $r > 0$ and center $x \in \mathbb{R}^k$ will be denoted $D(x,r)$. If it is centered at the origin, then $D(r) = D(0,r)$. We shall also denote by $T(x,r)$ the set $M \times D(x,r)$, and by $T(r)$ the set $M \times D(r)$. Observe that $\varphi_t(T(x,r)) = T(tx,tr)$ and that $T(x,r)$ is the tubular neighborhood of radius $r > 0$ of $M \times \{x\}$. If $E \subset N$ and $r > 0$, we shall denote by E_r the set $E \cap \big(N \setminus T(r)\big)$.

Given any set $E \subset N$ of finite perimeter, we can replace it by a *normalized* set $\operatorname{sym} E$ by requiring $\operatorname{sym} E \cap (\{p\} \times \mathbb{R}^k) = \{p\} \times D(r(p))$, where $H^k(D(r(p))$ is equal to the H^k-measure of $\operatorname{sym} E \cap (\{p\} \times \mathbb{R}^k)$. For such a set we get

THEOREM 5.1. *In the above conditions, we have*

(1) $|\operatorname{sym} E| = |E|$,
(2) $P(\operatorname{sym} E) \leqslant P(E)$.

The proof of Theorem 5.1 is similar to the one of symmetrization in $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$ with respect to one of the factors, see Burago and Zalgaller [15] (or Maggi [47] for the case $m = 1$). The main ingredients are a corresponding inequality for the Minkowski content and approximation of finite perimeter sets by sets with smooth boundary.

Given $E \subset N$, we shall denote by E^* its orthogonal projection over M.

$$|T(r)| = \omega_k r^k H^m(M),$$
$$P(T(r)) = k\omega_k r^{k-1} H^m(M),$$

so that

$$P(T(r)) = k\left(\omega_k H^m(M)\right)^{1/k} |T(r)|^{(k-1)/k}. \tag{5.3}$$

Observe also that, in case E is normalized and $0 < r < s$, we have $(E_s)^* \subset (E_r)^*$.

The isoperimetric profile of $M \times \mathbb{R}^k$ is the function

$$I(v) = \inf\{P(E); |E| = v\}.$$

An isoperimetric region $E \subset M \times \mathbb{R}^k$ is one that satisfies $I(|E|) = P(E)$. Existence of isoperimetric regions in $M \times \mathbb{R}^k$ is guaranteed by a result of Frank Morgan [53, pp. 129], since the quotient of $M \times \mathbb{R}^k$ by its isometry group is compact. From his arguments, it also follows that isoperimetric regions are bounded in M. See also [28]. Observe that, from (5.3), we get

$$I(v) \leqslant k\left(\omega_k H^m(M)\right)^{1/k} v^{(k-1)/k}, \tag{5.4}$$

for any $v > 0$. The regularity of isoperimetric regions in Riemannian manifolds is well-known, see Morgan [50] and Gonzalez-Massari-Tamanini [32]. The boundary is regular except for a singular set of vanishing H^{n-7} measure.

Proposition 5.2. *The isoperimetric profile I of N is non-decreasing and continuous.*

Proof. Let $v_1 < v_2$, and $E \subset N$ an isoperimetric region of volume v_2. Let $0 < t < 1$ so that $|\varphi_t(E)| = v_1$. By (5.2) we have

$$I(v_1) \leqslant P(\varphi_t(E)) \leqslant P(E) = I(v_2).$$

This shows that I is non-decreasing.

Since I is a monotone function, it can only have jump discontinuities. If E is an isoperimetric region of volume v, using a smooth vector field supported in the regular part of the boundary of E, one can find a continuous function f, defined in a neighborhood of v, so that $I \leqslant f$. This implies that I cannot have jump discontinuities at v. $\square$

We shall also use the following well-known isoperimetric inequalities in M and $M \times \mathbb{R}^k$

Lemma 5.3 ([21]). *Given $0 < v_0 < H^m(M)$, there exist a constant $a(v_0) > 0$ such that*

$$H^{m-1}(\partial E) \geqslant a(v_0) H^m(E),$$

for any set $E \subset M$ satisfying $0 < H^m(E) < v_0$.

LEMMA 5.4. *Given $v_0 > 0$, there exists a constant $c(v_0) > 0$ so that*

$$I(v) \geqslant c(v_0)\, v^{(n-1)/n}, \tag{5.5}$$

for any $v \in (0, v_0)$.

Lemma 5.4 follows from the facts that $I(v)$ is strictly positive for $v > 0$ and is asymptotic to the Euclidean isoperimetric profile when v approaches 0.

5.1. Large isoperimetric regions in N

If $E \subset N$ is any finite perimeter set and $T(E)$ is the tube with the same volume as E, we define

$$E^- = E \cap T(E), \quad E^+ = E \setminus T(E)$$

Let $t > 0$, and $\Omega = \varphi_t(E)$. Since $\varphi_t(E^+) = \Omega^+$, (5.1) implies

$$\frac{|E^+|}{|E|} = \frac{|\Omega^+|}{|\Omega|}. \tag{5.6}$$

A similar equality holds replacing E^+ by E^-.

PROPOSITION 5.5. *Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of normalized sets with volumes $|E_i| \to \infty$. Let $v_0 > 0$ and $0 < t_i < 1$ so that $|\varphi_{t_i}(E_i)| = v_0$ for all $i \in \mathbb{N}$. Let T be the tube of volume v_0 around M_0.*

If $\varphi_{t_i}(E_i)$ does not converge to T in the L^1-topology, then there is a constant $c > 0$, only depending on $\{E_i\}_{i\in\mathbb{N}}$, so that, passing to a subsequence we get,

$$H^{n-1}(\partial E_i) \geqslant c|E_i|. \tag{5.7}$$

PROOF. Assume $T = M \times D(r)$, and set $\Omega_i = \varphi_{t_i}(E_i)$. As $|\Omega_i| = |T|$, we have the equality $2|\Omega_i^+| = |\Omega_i \triangle T|$. Since $|\Omega_i \triangle T|$ does not converge to 0, the sequence $|\Omega_i^+|$ does not converge to 0 either. Hence there exists a constant $c_1 > 0$ so that $\limsup_{i\to\infty}(|\Omega_i^+|/|\Omega_i|) > c_1$. From (5.6) we obtain

$$\limsup_{i\to\infty} \frac{|E_i^+|}{|E_i|} > c_1. \tag{5.8}$$

Now we claim that

$$\liminf_{i\to\infty} H^m((\Omega_i \cap \partial T)^*) < H^m(M). \tag{5.9}$$

To prove (5.9) we argue by contradiction. Assume that $\liminf_{i\to\infty} H^m((\Omega_i \cap \partial T)^*) = H^m(M)$. As Ω_i is normalized, we have $(\Omega_i \cap \partial T)^* \subset (\Omega_i \cap T)^*$ and so $(T \setminus \Omega_i) \subset (M \setminus (\Omega_i \cap \partial T)^*) \times D(r)$.

This implies $\limsup_{i\to\infty} |T \setminus \Omega_i| = 0$. Since $|\Omega_i| = |T|$, we get $\lim_{i\to\infty} |\Omega_i \triangle T| = 2\lim_{i\to\infty} |T \setminus \Omega_i| = 0$, a contradiction that proves the claim. Hence there exists $w \in (0, H^m(M))$ so that

$$\liminf_{i\to\infty} H^m((\Omega_i \cap \partial T)^*) < w. \tag{5.10}$$

Let $r_i > 0$ be the radius of the tube with the same volume as E_i. As $(E_i^+)^* = (\Omega_i^+)^*$ and E_i is normalized, we have

$$\liminf_{i\to\infty} H^m((E_i \cap \partial T(s))^*) < w, \qquad s \geqslant r_i. \tag{5.11}$$

The above arguments imply, replacing the original sequence by a subsequence, that

$$|E_i^+| > c_1 |E_i|, \qquad H^m((E_i \cap \partial T(s))^*) < w, \qquad i \in \mathbb{N}, s \geqslant r_i. \tag{5.12}$$

Let $a = a(w)$ be the constant in Lemma 5.3. For the elements of the subsequence satisfying (5.12) we have

$$\begin{aligned} H^{n-1}(\partial E_i) &\geqslant H^{n-1}(\partial E_i \cap (N \setminus T(r_i))) \\ &\geqslant \int_{r_i}^{\infty} H^{n-2}(\partial E_i \cap \partial T(s))\, ds \\ &\geqslant \int_{r_i}^{\infty} H^{n-2}(\partial (E_i \cap \partial T(s)))\, ds \\ &= \int_{r_i}^{\infty} H^{m-1}(\partial (E_i \cap \partial T(s))^*) H^{k-1}(\partial D(s))\, ds \\ &\geqslant \int_{r_i}^{\infty} a H^{m}((E_i \cap \partial T(s))^*) H^{k-1}(\partial D(s))\, ds \\ &= a \int_{r_i}^{\infty} H^{n-1}(E_i \cap \partial T(s))\, ds = a\,|E_i^+| > a\, c_1 |E_i|, \end{aligned}$$

what proves the result. In the previous inequalities we have used the coarea formula for the distance function to $M \times \{0\}$; that $\partial(E_i \cap \partial T(s)) \subset \partial E_i \cap \partial T(s)$, where the first ∂ denotes the boundary operator in $\partial T(s)$; the fact that for an $O(k)$-invariant set F we have $F \cap \partial T(s) = (F \cap \partial T(s))^* \times \partial D(s)$, and so $H^{r+k-1}(F \cap \partial T(s)) = H^r((F \cap \partial T(s))^*) H^{k-1}(\partial D(s))$; that $(\partial(E_i \cap \partial T(s)))^* = \partial(E_i \cap \partial T(s))^*$; and the isoperimetric inequality on M given in Lemma 5.3. □

Corollary 5.6. *Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of normalized isoperimetric sets with volumes $\lim_{i\to\infty} |E_i| = \infty$. Let $v_0 > 0$ and $0 < t_i < 1$ such that $\Omega_i = \varphi_{t_i}(E_i)$ has volume v_0 for all $i \in \mathbb{N}$. Then $\Omega_i \to T$ in the L^1-topology, where T is the tube of volume v_0.*

Proof. Regularity results for isoperimetric regions imply that $P(E_i) = H^{n-1}(\partial E_i)$. If Ω_i does not converge to T in the L^1-topology then, using (5.7) in Lemma 5.5 and (5.4), we get,

$$c\,|E_i| \leqslant P(E_i) \leqslant k\left(\omega_k H^m(M)\right)^{1/k} |E_i|^{(k-1)/k},$$

for a subsequence, thus yielding a contradiction by letting $i \to \infty$ since $|E_i| \to \infty$. $\square$

Using density estimates, we shall show now that the L^1 convergence of the scaled isoperimetric regions can be improved to Hausdorff convergence.

In a similar way to Theorem 2.26, we define a function $h : \mathbb{R}^k \times (0, +\infty) \to \mathbb{R}^+$ by

$$h(x,R) = \frac{\min\{|E \cap T(x,R)|, |T(x,R) \setminus E|\}}{R^n},$$

for $x \in \mathbb{R}^k$ and $R > 0$. We remark that the quantity $h(x,R)$ is not homogeneous in the sense of being invariant by scaling since $h(x,R) \leqslant \frac{1}{2}(k\omega_k H^m(M))R^{k-n}$, which goes to infinity when R goes to 0. When the set E should be explicitly mentioned, we shall write

$$h(E,x,R) = h(x,R).$$

Lemma 5.7. *Let $E \subset N$ be an isoperimetric region of volume $v > v_0$. Let $\tau > 1$ such that $\Omega = \varphi_\tau^{-1}(E)$ has volume v_0. Choose ε so that*

$$0 < \varepsilon < \left\{ v_0, \left(\frac{c(v_0)\, v_0^{1/k}}{2H^m(M)} \right)^n, \left(\frac{c(v_0)}{8n} \right)^n \right\}, \tag{5.13}$$

where $c(v_0)$ the one in (5.5).

Then, for any $x \in \mathbb{R}^k$ and $R \leqslant 1$ so that $h(\Omega, x, R) \leqslant \varepsilon$, we get

$$h(\Omega, x, R/2) = 0.$$

Moreover, in case $h(\Omega, x, R) = |\Omega \cap T(x,R))|R^{-n}$, we get $|\Omega \cap T(x,R/2)| = 0$ and, in case $h(\Omega, x, R) = |T(x,R) \setminus \Omega| R^{-n}$, we have $|T(x,R/2) \setminus \Omega| = 0$.

Proof. Using Lemma 5.4 we get a positive constant $c(v_0)$ so that (5.5) is satisfied, i.e., $I(w) \geqslant c(v_0)\, w^{(n-1)/n}$, for all $0 \leqslant w \leqslant v_0$.

Assume first that

$$h(x,R) = h(\Omega, x, r) = \frac{|\Omega \cap T(x,R)|}{R^n}.$$

Define

$$m(r) = |\Omega \cap T(x,r)|, \quad 0 < r \leqslant R.$$

The function $m(r)$ is non-decreasing and, for $r \leqslant R \leqslant 1$, we get

$$m(r) \leqslant m(R) \leqslant |\Omega \cap T(x,R)| \leqslant \varepsilon R^n \leqslant \varepsilon < v_0, \tag{5.14}$$

by (5.13). Hence $v_0 - m(r) > 0$ for $0 < r \leqslant R$.

By the coarea formula, when $m'(r)$ exists, we get

$$m'(r) = \frac{d}{dr} \int_0^r H^{n-1}(\Omega \cap \partial T(x,s))\, ds = H^{n-1}(\Omega \cap \partial T(x,r)).$$

Now define

$$\lambda(r) = \frac{v_0^{1/k}}{(v_0 - m(t))^{1/k}} = \frac{v^{1/k}}{(v - |T(\tau x, \tau r)|)^{1/k}} \geqslant 1,$$

and

$$\Omega(r) = \varphi_{\lambda(r)}(\Omega \setminus T(x,r)),$$

so that $|\Omega(r)| = |\Omega|$. Then

$$E(r) = \varphi_\tau(\Omega(r)) = \varphi_{\lambda(r)}(E \setminus T(\tau x, \tau r)),$$

and $|E(r)| = |E|$. Then, using (5.2) for $\lambda(r) \geqslant 1$ and standard properties of finite perimeter sets, we have

$$\begin{aligned} I(v) &\leqslant P(E(r)) \leqslant \lambda(r)^k \left(P(E \setminus T(\tau x, \tau r))\right) \\ &\leqslant \frac{v_0}{v_0 - m(r)} \left(P(E) - P(E \cap T(\tau x, \tau r)) + 2H^{n-1}(E \cap \partial T(\tau x, \tau r))\right). \end{aligned} \tag{5.15}$$

Since $\tau \geqslant 1$ and $E \cap \partial T(\tau x, \tau r)$ is part of a cylinder, using (5.2) again we get

$$P(E \cap T(\tau x, \tau r) \geqslant \tau^{k-1} P(\Omega \cap T(x,r)) \geqslant \tau^{k-1} c(v_0)\, m(r)^{(n-1)/n},$$
$$H^{n-1}(E \cap \partial T(\tau x, \tau r)) = \tau^{k-1} H^{n-1}(\Omega \cap \partial T(x,r)) = \tau^{k-1} m'(r),$$

Replacing them in (5.15), taking into account that $P(E) = I(v)$ and $\tau^k v_0 = v$, we have

$$\begin{aligned} 2m'(r) &\geqslant m(r)^{(n-1)/n} \left(c(v_0) - \frac{m(r)^{1/n}}{\tau^{k-1} v_0} I(v) \right) \\ &\geqslant m(r)^{(n-1)/n} \left(c(v_0) - \frac{m(r)^{1/n}}{v_0^{1/k}} \frac{I(v)}{v^{(k-1)/k}} \right) \\ &\geqslant m(r)^{(n-1)/n} \left(c(v_0) - \frac{\varepsilon^{1/n}}{v_0^{1/k}} (k\omega_k H^m(M)) \right) \\ &\geqslant \frac{c(v_0)}{2} m(r)^{(n-1)/n}, \end{aligned} \tag{5.16}$$

where we have used $m(r) \leqslant \varepsilon$, (5.4), and (4.29)

If there were $r \in [R/2, R]$ such that $m(r) = 0$ then, by the monotonicity of the function $m(r)$, we would conclude $m(R/2) = 0$ as well. So we assume $m(r) > 0$ in $[R/2, R]$. Then by (5.16), we get

$$\frac{c(v_0)}{4} \leqslant \frac{m'(t)}{m(t)^{(n-1)/n}}, \qquad H^1\text{-a.e.}$$

By (5.14) we get $m(R) \leqslant \varepsilon R^n$. Integrating between $R/2$ and R

$$c(v_0) R/8 \leqslant n\,(m(R)^{1/n} - m(R/2)^{1/n}) \leqslant n\, m(R)^{1/n} \leqslant n\, \varepsilon^{1/n} R.$$

This is a contradiction, since $\varepsilon < (c(v_0)/8n)^n$ by (4.29). So the proof in case $h(x,R) = |\Omega \cap T(x,R)| R^{-n}$ is completed.

Now we deal with the case $h(x,R)=|T(x,R)\setminus\Omega|R^{-n}$. Define

$$m(r)=|T(x,r)\setminus\Omega|.$$

Then $m(r)$ is a non-decreasing function and

$$m'(r)=H^{n-1}(\Omega^c\cap\partial T(x,r))=\frac{1}{\tau^{k-1}}H^{n-1}(E^c\cap\partial T(\tau x,\tau r)),\tag{5.17}$$

since $E^c\cap\partial T(\tau x,\tau r)$ is part of a tube. We also have $m(r)\leqslant m(R)\leqslant\varepsilon R^n\leqslant\varepsilon<v_0$ by (4.29). Observe that

$$P(E\cup T(\tau x,\tau r)\leqslant P(E)-P(T(\tau x,\tau r)\setminus E)+2H^{n-1}(E^c\cap\partial E(\tau x,\tau r)).\tag{5.18}$$

Since $\varphi_\tau(T(x,r)\setminus\Omega)=T(\tau x,\tau r)\setminus E$ and $\tau\geqslant 1$, we get

$$\begin{aligned}P(T(\tau x,\tau r)\setminus E)&=P(\varphi_\tau(T(x,r)\setminus\Omega))\\&\geqslant\tau^{k-1}P(T(x,r)\setminus\Omega)\geqslant\tau^{k-1}\,c(v_0)\,m(r)^{(n-1)/n}.\end{aligned}\tag{5.19}$$

Now, using that I is a non-decreasing function we easily obtain $P(E)=I(v)\leqslant I(|E\cup T(\tau x,\tau r)|)\leqslant P(E\cup T(\tau x,\tau r))$. We estimate $P(E\cup T(\tau x,\tau r))$ from (5.18). Using (5.19) and (5.17), we get

$$I(v)=P(E)\leqslant P(E\cup T(\tau x,\tau r))\leqslant I(v)-\tau^{k-1}c(v_0)\,m(r)^{(k-1)/k}+2\tau^{k-1}m'(r)\tag{5.20}$$

and so

$$\frac{c(v_0)}{2}\leqslant\frac{m'(r)}{m(r)^{(n-1)/n}},\qquad H^1\text{-a.e.}$$

By (5.14) we get $m(R)\leqslant\varepsilon R^n$. Integrating between $R/2$ and R

$$c(v_0)R/4\leqslant n\,(m(R)^{1/n}-m(R/2)^{1/n})\leqslant n\,m(R)^{1/n}\leqslant n\,\varepsilon^{1/n}R,$$

we get a contradiction since by (5.13) we have $\varepsilon<(c(v_0)/(8n))^n<(c(v_0)/(4n))^n$. This concludes the proof. □

Let $F\subset N$, then $F_r=\{x\in N: d(x,F)\leqslant r\}$. We improve now the L^1-convergence of normalized isoperimetric regions obtained in Corollary 5.6 to Hausdorff convergence of their boundaries

LEMMA 5.8. *Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of isoperimetric sets in N with $\lim_{i\to\infty}|E_i|=\infty$. Let $v_0>0$ and $\{t_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty}t_i=0$ and $|\Omega_i|=v_0$ for all $i\in\mathbb{N}$, where $\Omega_i=\varphi_{t_i}(E_i)$. Then for every $r>0$, $\partial\Omega_i\subset(\partial T)_r$, for large enough $i\in\mathbb{N}$, where T is the tube of volume v_0.*

PROOF. Since $|\Omega_i|=v_0$, using (5.13) we can choose a uniform $\varepsilon>0$ so that Lemma 5.7 holds with this ε for all Ω_i, $i\in\mathbb{N}$. This means that, for any $x\in N$ and $0<r\leqslant 1$, whenever $h(\Omega_i,x,r)\leqslant\varepsilon$ we get $h(\Omega_i,x,r/2)=0$.

As $\Omega_i\to T$ in $L^1(N)$ by Corollary 5.6, we can choose a sequence $r_i\to 0$ so that

$$|\Omega_i\triangle T|<r_i^{n+1}.\tag{5.21}$$

Now fix some $0 < r < 1$. We reason by contradiction assuming that, for some subsequence, there exist

$$x_i \in \partial\Omega_i \setminus (\partial T)_r. \tag{5.22}$$

We distinguish two cases.

First case: $x_i \in N \setminus T$, for a subsequence. Choosing i large enough, (5.22) implies $T(x_i, r_i) \cap T = \emptyset$ and (5.21) yields

$$|\Omega_i \cap T(x_i, r_i)| \leqslant |\Omega_i \setminus T| \leqslant |\Omega_i \triangle T| < r_i^{n+1}.$$

So, for i large enough, we get

$$h(\Omega_i, x_i, r_i) = \frac{|\Omega_i \cap T(x_i, r_i)|}{r_i^n} < r_i \leqslant \varepsilon.$$

By Lemma 5.7, we conclude that $|\Omega_i \cap T(x_i, r_i/2)| = 0$, a contradiction.

Second case: $x_i \in T$. Choosing i large enough, (5.22) implies $T(x_i, r_i) \subset T$ and so

$$|T(x_i, r_i) \setminus \Omega_i| \leqslant |T \setminus \Omega_i|, \quad \text{for every } r_i < r.$$

Then, by (5.21), we get

$$|T(x_i, r_i) \setminus \Omega_i| \leqslant |T \setminus \Omega_i| \leqslant |\Omega_i \triangle T| < r_i^{n+1}.$$

So, for i large enough, we get

$$h(\Omega_i, x_i, r_i) = \frac{|T(x_i, r_i) \setminus \Omega_i|}{r_i^n} < r_i \leqslant \varepsilon.$$

By Lemma 5.7, we conclude that $|T(x_i, r_i/2) \setminus \Omega_i| = 0$, and we get again contradiction that proves the Lemma. $\square$

5.2. Strict $O(k)$-stability of tubes with large radius

In his Section we consider the orthogonal group $O(k)$ acting on the product $M \times \mathbb{R}^k$ through the second factor.

Let $\Sigma \subset M \times \mathbb{R}^k$ be a compact hypersurface with constant mean curvature. It is well-known that Σ is a critical point of the area functional under volume-preserving deformations, and that Σ is a second order minima of the area under volume-preserving variations if and only if

$$\int_\Sigma \left(|\nabla u|^2 - q\,u^2\right) d\Sigma \geqslant 0, \tag{5.23}$$

for any smooth function $u : \Sigma \to \mathbb{R}$ with mean zero on Σ. In the above formula ∇ is the gradient on Σ and q is the function

$$q = \mathrm{Ric}(N, N) + |\sigma|^2,$$

where $|\sigma|^2$ is the sum of the squared principal curvatures in Σ, N is a unit vector field normal to Σ, and Ric is Ricci curvature on N.

A hypersurface satisfying (5.23) is usually called *stable* and condition (5.23) is referred to as *stability condition*. In case Σ is $O(k)$-invariant we can consider an equivariant stability condition: we shall say that Σ is *strictly $O(k)$-stable* if there exists a positive constant $\lambda > 0$ such that

$$\int_\Sigma (|\nabla u|^2 - q u^2)\, d\Sigma \geqslant \lambda \int_\Sigma u^2\, d\Sigma$$

for any function $u : \Sigma \to \mathbb{R}$ with mean zero which is $O(k)$-invariant.

We consider now the tube $T(r) = M \times D(r)$. The boundary of $T(r)$ is the cylinder $\Sigma(r) = M \times \partial D(r)$, which is $O(k)$-invariant, and has k principal curvatures equal to $1/r$. Hence its mean curvature is equal to k/r and the squared norm of the second fundamental form satisfies $|\sigma|^2 = k/r^2$. The inner unit normal to $\Sigma(r)$ is the normal to $\partial D(r)$ in $\mathbb{R}^k$ (it is tangent to the factor $\mathbb{R}^k$). This implies that $\mathrm{Ric}(N, N) = 0$.

We have the following result

LEMMA 5.9. *The cylinder $\Sigma(r)$ is strictly $O(k)$-stable if and only if*

$$r^2 > \frac{k}{\lambda_1(M)},$$

where $\lambda_1(M)$ is the first positive eigenvalue of the Laplacian in M.

PROOF. Let $\Sigma = \Sigma(r) = M \times D(r)$. Observe that an $O(k)$-invariant function on Σ is just a function $u : M \to \mathbb{R}$, that has mean zero on Σ if and only if $\int_M u\, dM = 0$. Hence

$$\begin{aligned}\int_\Sigma (|\nabla u|^2 - q u^2)\, d\Sigma &= k\omega_k r^{k-1} \int_M (|\nabla_M u|^2 - \frac{k}{r^2} u^2)\, dM \\ &\geqslant k\omega_k r^{k-1} \left(\lambda_1(M) - \frac{k}{r^2}\right) \int_M u^2\, dM \\ &= \left(\lambda_1(M) - \frac{k}{r^2}\right) \int_\Sigma u^2\, d\Sigma.\end{aligned}$$

This proves the Lemma. □

Using the results of White [74] and Grosse-Brauckmann [36], we deduce the following result

THEOREM 5.10. *Let T be a normalized tube so that $\Sigma = \partial T$ is a strictly $O(k)$-stable cylinder. Then there exists $r > 0$ so that any $O(k)$-invariant finite perimeter set E with $|E| = |T|$ and $\partial E \subset T_r$ has larger perimeter than T unless $E = T$.*

PROOF. Since Σ is strictly $O(k)$-stable, Grosse-Brauckmann [36, Lemma 5] implies that, for some $C > 0$, Σ has strictly positive second variation for the functional

$$F_C = \text{area} + H\,\text{vol} + \frac{C}{2}(\text{vol} - \text{vol}(T))^2,$$

in the sense that the second variation of F_C in the normal direction of a function u satisfies

$$\delta_u^2 F_C = \int_\Sigma (|\nabla u|^2 - q\,u^2)\,d\Sigma + C\left(\int_\Sigma u\,d\Sigma\right)^2 \geqslant \lambda \int_\Sigma u^2\,d\Sigma,$$

for any smooth $O(k)$-invariant function u (see the discussion in the proof of Theorem 2 in Morgan and Ros [55]). White's proof of Theorem 3 in [74] observes that a sequence of minimizers of F_C in tubular neighborhoods of radius $1/n$ of Σ are *almost minimizing* and hence $C^{1,\alpha}$ submanifolds that converge Hölder differentiably to Σ, contradicting the positivity of the second variation of Σ. Theorem 5.1 implies that the symmetrization of these minimizers are again minimizers. Thus we get a family of $O(k)$-minimizers of F_C converging Hölder differentiably to Σ, thus contradicting the strict $O(k)$-stability of Σ. $\square$

5.3. Proof of Theorem 1.1

First we claim that there exists $v_0 > 0$ such that, for any isoperimetric region E of volume $|E| \geqslant v_0$, the set $\text{sym}\,E_i$ is a tube.

To prove this, consider a sequence of isoperimetric regions $\{E_i\}_{i\in\mathbb{N}}$ with $\lim_{i\to\infty}|E_i| = \infty$. We know that $\{\text{sym}\,E_i\}_{i\in\mathbb{N}}$ are also isoperimetric regions. Let $T = M \times D$ be a strictly $O(k)$-stable tube, that exists by Lemma 5.9. For large i, we scale down the sets $\text{sym}\,E_i$ so that $\Omega_i = \varphi_{t_i}^{-1}(\text{sym}\,E_i)$ has the same volume as T. As $\text{sym}\,E_i$ is isoperimetric and $t_i > 1$, we get from (5.4) and (5.2) that $P(\Omega_i) \leqslant P(T)$. By Corollary 5.6, the sets $\{\partial\Omega_i\}_{i\in\mathbb{N}}$ converge to ∂T in Hausdorff distance. By Theorem 5.10, $\Omega_i = T$ and so $\text{sym}\,E_i$ is a tube. This proves the claim. In particular, $H^m(E \cap (\{p\} \times \mathbb{R}^k)) = H^m(D)$ for any $p \in M$.

Hence the isoperimetric profile satisfies $I(v) = C\,v^{(k-1)/k}$ for some constant $C > 0$ and $v \geqslant v_0$. We conclude

$$I(t^k v) = t^{k-1} I(v), \qquad t^k v \geqslant v_0. \tag{5.24}$$

Let E be an isoperimetric region with volume $|E| > v_0$, and $t < 1$ so that $t^k|E| = v_0$. Then

$$I(t^k|E|) \leqslant P(\varphi_t(E)) \leqslant t^{k-1}P(E) = t^{k-1}I(|E|)$$

by the inequality corresponding to (5.2) when $t \leqslant 1$. By (5.24), equality hold and the unit normal ξ to $\text{reg}(\partial E)$, the regular part of ∂E, is tangent to the $\mathbb{R}^k$ factor. This implies that the m-Jacobian of the restriction f of the projection $\pi_1 : M \times \mathbb{R}^k \to M$ to the regular part of ∂E is equal to 1. By Federer's coarea formula for rectifiable sets [26, 3.2.22] we get

$$H^{n-1}(\partial E) = \int_M H^{k-1}(f^{-1}(p))\,dH^m.$$

Assume that $\operatorname{sym} E$ is the tube $T(E) = M \times D$. The Euclidean isoperimetric inequality implies $H^{k-1}(f^{-1}(p)) \geqslant H^{k-1}(\{p\} \times \partial D)$ and so $H^{n-1}(\partial E) \geqslant H^{n-1}(\partial T(E))$, again by the coarea formula. As $P(E) = P(\operatorname{sym} E) = P(T(E))$, we get $H^{k-1}(f^{-1}(p)) = H^{k-1}(\partial D)$ for H^m-a.e. $p \in M$ and so $\pi_1^{-1}(p)$ is equal to a disc $\{p\} \times D_p$ for H^m- a.e. $p \in M$.

The fact that ξ is tangent to $\mathbb{R}^k$ in $\operatorname{reg}(\partial E)$ implies that $\operatorname{reg}(\partial E)$ is locally a cylinder of the form $U \times S$, where $U \subset M$ is an open set and $S \subset \mathbb{R}^k$ is a smooth hypersurface. Hence the discs D_p are centered at the same point, i.e., E is the translation of a normalized tube, what proves the theorem.

REMARK 5.11. The equivariant version of Theorem 2 in Morgan and Ros [55], together with Corollary 5.6, can be used to prove Theorem 1.1 for small dimension.

CHAPTER 6

Summary

In this thesis we study isoperimetric inequalities in convex bodies. We have divided the Thesis into five chapters. The first chapter includes the introduction and preliminaries.

In Chapter 2 we deal only with compact convex bodies and we consider the problem of minimizing the relative perimeter under a volume constraint in the interior of a convex body, i.e., a compact convex set in Euclidean space with interior points. We shall not impose any regularity assumption on the boundary of the convex body. Amongst other results, we shall prove the equivalence between Hausdorff and Lipschitz convergence, the continuity of the isoperimetric profile with respect to the Hausdorff distance, and the convergence in Hausdorff distance of sequences of isoperimetric regions and their free boundaries. We shall also describe the behavior of the isoperimetric profile for small volume, and the behavior of isoperimetric regions for small volume.

In Chapter 3 we consider the isoperimetric profile of convex cylinders $K \times \mathbb{R}^q$, where K is an m-dimensional convex body, and of cylindrically bounded convex sets, i.e, those with a relatively compact orthogonal projection over some hyperplane of $\mathbb{R}^{n+1}$, asymptotic to a right convex cylinder of the form $K \times \mathbb{R}$, with $K \subset R^n$. Results concerning the concavity of the isoperimetric profile, existence of isoperimetric regions, and geometric descriptions of isoperimetric regions for small and large volumes are obtained.

In Chapter 4 we consider the problem of minimizing the relative perimeter under a volume constraint in the interior of a conically bounded convex set, i.e., an unbounded convex body admitting an *exterior* asymptotic cone. Results concerning existence of isoperimetric regions, the behavior of the isoperimetric profile for large volumes, and a characterization of isoperimetric regions of large volume in conically bounded convex sets of revolution is obtained.

Finally In Chapter 5, given a compact Riemannian manifold M, we show that large isoperimetric regions in $M \times \mathbb{R}^k$ are tubular neighborhoods of $M \times \{x\}$ with $x \in \mathbb{R}^k$.

CHAPTER 7

Resumen

En esta tesis estudiamos desigualdades isoperimétricas en cuerpos convexos. Hemos dividido la tesis en cinco capítulos. El primer capitulo incluye la introducción y los preliminares.

En el capitulo 2 consideramos cuerpos convexos compactos y consideramos el problema de minimizar el perímetro relativo en el interior de un cuerpo convexo en el espacio Euclídeo i.e., un conjunto convexo con puntos interiores. No impondremos ninguna hipótesis sobre la regularidad de la frontera del cuerpo convexo. Unos de resultados son, la equivalencia entre la Hausdorff y Lipschitz la continuidad del perfil isoperimetric con respeto a la distancia de Hausdorff, y la convergencia en la distancia de Hausdorff de sucesiones de regiones isoperimétricas y de sus fronteras libres. También describiremos el comportamiento del perfil isoperimétrico para volúmenes pequeños, y el comportamiento de las regiones isoperimétricas para volúmenes pequeños.

En el capítulo 3 consideramos el perfil isoperimétrico de cilindros convexos $K\times\mathbb{R}^q$, donde K es un cuerpo convexo $m-$dimensional, y de cuerpos convexos cilíndricamente acotados, i.e., con una proyección relativamente compacta sobre algún hiperplano afín de $\mathbb{R}^{n+1}$, asintótico a un cilindro convexo de tipo $K\times\mathbb{R}$ con $K\subset\mathbb{R}^n$. Probaremos resultados sobre la concavidad de perfil isoperimétrico y la describiremos geométricamente las regiones isoperimétricas para volúmenes pequeños y grandes.

En el capítulo 4 consideramos el problema de minimizar el perímetro relativo bajo de restricción de volumen en el interior de un cuerpo convexo cónicamente acotado, i.e., un cuerpo convexo no acotado que admite un cono asintótico *exterior*. Se demostrarán resultados sobre la existencia de regiones isoperimétricas, el comportamiento del perfil isoperimétricas para volúmenes grandes, y se caracterizarán las regiones isoperimétricas para volúmenes grandes en cuerpos convexos cónicamente acotados de revolución.

Finalmente en el capítulo 5, demostramos que en una variedad Riemanniana compacta sin borde M, las únicas regiones isoperimétricas de volúmenes grandes en $M\times\mathbb{R}^k$ son entornos tubulares de $M\times\{x\}$ con $x\in\mathbb{R}^k$.

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