CP PHASES IN MODELS WITH SOME FERMION MASSES VANISHING AND/OR DEGENERATE*

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Dedicated to Wojciech Królikowski in honour of his 70th birthday

We count the number of CP breaking phases in models with $SU(2)_L \times U(1)_Y$ and $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ electroweak gauge groups and extended matter contents with some fermion masses vanishing and/or degenerate. Quarks and leptons, including Majorana neutrinos, are treated in a similar way. CP violation is characterized in the mass-eigenstate and in the weak-eigenstate bases. Necessary and sufficient conditions for CP conservation, invariant under weak basis redefinitions are also studied in these models. CP violating factors entering in physical observables and only invariant under phase redefinitions are discussed.

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1. Introduction

CP violation is related to the presence of complex phases in the mixing matrices describing the gauge couplings in the mass-eigenstate basis. (We do not consider other (Higgs) sources of CP violation.) However, not all phases in the mixing matrices are CP violating. Some of them can be eliminated redefining the fermion phases. In the standard model with three non-degenerate quark families the six phases defining the 3×3 unitary mixing matrix reduce to one after an appropriate fermion field phase redefinition. This was first realized by Kobayashi and Maskawa [1] and it is the simplest way to account for the observed CP violation [2]. In general if there are degenerate fermion masses, the number of CP violating phases is further reduced. In the standard model with three massless neutrinos the six phases defining the 3×3 unitary mixing matrix in the lepton sector can be eliminated. As a matter of fact not only all the phases but the three real mixing angles are non-physical. There is no mixing between lepton families and the three lepton numbers are conserved. These two cases are extreme, non-degenerate quark masses and degenerate and vanishing neutrino masses. Here we will examine the intermediate case. We allow for an arbitrary number of standard families and Majorana neutrinos with some fermion masses vanishing and/or degenerate. We consider models with the standard gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ and with its left-right extension $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_Y$, in turn.

Although the number of CP violating phases is more easily counted in the mass-eigenstate basis, CP violation can be also discussed in the weak-eigenstate basis [3, 4]. In this basis the mass matrices are in general non-diagonal and necessarily complex if CP is not conserved. The use of an invariant formulation for CP conservation is more convenient in this case. Necessary (and sufficient) conditions for CP conservation can be found which are independent of the choice of basis [5–8]. If one of these conditions is not fulfilled, CP is not conserved. However, some of them could be trivially satisfied if some fermion masses are vanishing or degenerate. We study this possibility in simple cases.

In Sec. 2 we count the number of CP breaking phases in $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge models with some vanishing and/or degenerate quark and lepton masses, including Majorana neutrinos. $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_Y$ gauge models are considered in Sec. 3. The necessary and sufficient conditions for CP conservation are discussed in Sec. 4 in simple models with some vanishing and/or degenerate fermion masses. In Sec. 5 we comment on the relevance of the invariants under fermion phase redefinitions. Sec. 6 is devoted to conclusions.

2. The standard model with n_L fermion families and n_R neutral fermion singlets

The n_L left-handed fields transform as $SU(2)_L$ doublets, whereas the n_R right-handed fields are $SU(2)_L$ singlets. Hence, only the left-handed fermions interact with the charged gauge boson W.

2.1. The quark sector

Let M_u and M_d be the $n_L \times n_L$ mass matrices for up and down quarks, respectively, in the weak-eigenstate basis ($n_L = n_R$). In general they are complex and can be diagonalized by unitary transformations

$$(M_u)_{\text{diag}} = U_L^{u\dagger} M_u U_R^u \,, \quad (M_d)_{\text{diag}} = U_L^{d\dagger} M_d U_R^d \,, \tag{1}$$

where $U_{L,R}^{u,d}$ are $n_L \times n_L$ unitary matrices. Thus, the Cabibbo–Kobayashi–Maskawa (CKM) mixing matrix [1] reads

$$U_{\text{CKM}} = U_L^{u\dagger} U_L^d \,. \tag{2}$$

The matrices $U_{L,R}^{u,d}$ in Eq. (1) are not uniquely determined. We can still perform unitary transformations $V_{L,R}^{u,d}$, which leave unchanged the diagonal mass matrices

$$(M_u)_{\text{diag}} = V_L^{u\dagger} \, (M_u)_{\text{diag}} \, V_R^u \,, \quad (M_d)_{\text{diag}} = V_L^{d\dagger} \, (M_d)_{\text{diag}} \, V_R^d \,, \tag{3}$$

but redefine the CKM matrix

$$U_{\text{CKM}} \to V_L^{u\dagger} U_{\text{CKM}} V_L^d = (U_L^u V_L^u)^\dagger \left(U_L^d V_L^d\right) . \tag{4}$$

How the matrices $V_{L,R}^u$ ($V_{L,R}^d$) look like depends on the properties of $(M_u)_{\text{diag}}$ ($(M_d)_{\text{diag}}$). Let us assume that $(M_u)_{\text{diag}}$ ($(M_d)_{\text{diag}}$) has l_u^0 (l_d^0) vanishing masses, l_u (l_d) > 1 degenerate masses $m_u (m_d)$, and $n_L - l_u^0 - l_u$ ($n_L - l_d^0 - l_d$) non-degenerate masses

$$(M_u)_{\text{diag}} = \begin{pmatrix} 0 & & & & & & & & \\ & \ddots & & & & & & 0 & \\ & & 0 & & & & & & \\ & & & m_u & & & & & \\ & & & & \ddots & & & & \\ & & & & & m_u & & & \\ & & & & & & m_1 & & \\ & 0 & & & & & & m_2 & \\ & & & & & & & & \ddots \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} l_u^0 \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} l_u \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} n_L - l_u^0 - l_u \end{matrix} \tag{5}$$

(analogously for $(M_d)_{\rm diag}$). Then

$$V_L^u = \begin{pmatrix} W^0_{uL} & & & & 0 \\ & W_{uL} & & & \\ & & \exp(i\delta_1) & & \\ & & & \ddots & \\ 0 & & & & \exp(i\delta_{n_L-l^0_u-l_u}) \end{pmatrix},$$

$$V_R^u = \begin{pmatrix} W^0_{uR} & & & & 0 \\ & W_{uL} & & & \\ & & \exp(i\delta_1) & & \\ & & & \ddots & \\ 0 & & & & \exp(i\delta_{n_L-l^0_u-l_u}) \end{pmatrix}, \qquad (6)$$

where $W^0_{uL,R}$ (W_{uL}) are $l^0_u \times l^0_u$ ($l_u \times l_u$) unitary matrices, and analogously for $V^d_{L,R}$.

Now we can count the number of CP violating phases. The CKM matrix is a $n_L \times n_L$ unitary matrix and is parametrized by $n_L\,(n_L-1)/2$ mixing angles and $n_L\,(n_L+1)/2$ complex phases. Not all of these parameters are physical but we can get rid of the unphysical ones as shown in Eq. (4) with an appropriate choice of $V^{u,d}_L$ in Eq. (6). Although known results [3] can be easily recovered as we do below, the general case is involved. It looks necessary to treat it with a computer [9].

- 2.1.1. The counting for $l^0_{u,d} = 0, 1$, $l_{u,d} = 0$ is the same as in the non-vanishing, non-degenerate standard case because $V^{u,d}_L$ have the same structure. The number of CP violating phases is equal to the number of phases in a $n_L \times n_L$ unitary matrix, $n_L\,(n_L+1)/2$, minus the number of phases in $V^{u,d}_L$, $2n_L$, plus 1 to avoid double counting of the common phase redefinition

$$\frac{n_L\,(n_L+1)}{2} - 2n_L + 1 = \frac{(n_L-1)\,(n_L-2)}{2}\,. \qquad (7)$$

For $n_L = 3$ one recovers the standard model result with one CP violating phase.

- 2.1.2. For l^0_u or $l_u = n_L - 1$ there is no CP violation. Using the decomposition of a $n_L \times n_L$ unitary matrix

$$U_{\rm CKM} = \begin{pmatrix} W^0_{uL} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} c_1 & -s_1 s_2 & -s_1 c_2 s_3 & \cdots & \cdots & -s_1 c_2 \ldots c_{n_L-1} \\ & c_2 & -s_2 s_3 & \cdots & \cdots & -s_2 c_3 .. c_{n_L-1} \\ & & c_3 & \cdots & \cdots & -s_3 c_4 . c_{n_L-1} \\ & & & \cdots & \cdots & \cdots \\ 0 & & & & c_{n_L-1} & -s_{n_L-1} \\ s_1 & c_1 s_2 & c_1 c_2 s_3 & \cdots & \cdots & c_1 c_2 \ldots c_{n_L-1} \end{pmatrix}$$

$$\times \begin{pmatrix} \exp(-i\delta_1) & & \\ & \ddots & \\ & & \exp(-i\delta_{n_L}) \end{pmatrix}, \tag{8}$$

where W^0_{uL} is a $(n_L - 1) \times (n_L - 1)$ unitary matrix, and Eqs (4), (6) U_{CKM} can be made always real, almost triangular and depending only on $n_L - 1$ mixing angles. For l^0_d or $l_d = n_L - 1$ we use the inverse unitary decomposition to prove that there is no CP violation either.

2.2. The lepton sector

n_L is the number of standard fermion families and n_R the number of right-handed neutral fermion singlets. Then the charged lepton mass matrix M_l is $n_L \times n_L$ and complex, and the neutrino mass matrix M_ν is $(n_L + n_R) \times (n_L + n_R)$, complex and symmetric. They can be diagonalized by unitary transformations

$$(M_l)_{\mathbf{diag}} = U^{l\dagger}_L M_l U^l_R\,, \quad (M_\nu)_{\mathbf{diag}} = U^T M_\nu U\,, \tag{9}$$

with $U^l_{L,R}$ and U $n_L \times n_L$ and $(n_L + n_R) \times (n_L + n_R)$ unitary matrices, respectively. Defining

$$U = \overbrace{\begin{pmatrix} U^*_L \\ U_R \end{pmatrix}}^{n_L+n_R} \begin{matrix} \}n_L \\ \}n_R \end{matrix}\,, \tag{10}$$

the mixing matrices in the charged and neutral currents can be written

$$K = U^\dagger_L U^l_L \quad \text{and} \quad \Omega = KK^\dagger\,, \tag{11}$$

respectively. The diagonalization conditions in Eq. (9) do not determine $U^l_{L,R}$ and U uniquely. One can still perform unitary transformations which leave unchanged the (diagonal) mass matrices

$$(M_l)_{\mathbf{diag}} = V^\dagger_L \, (M_l)_{\mathbf{diag}} \, V_R = \left(U^l_L V_L\right)^\dagger M_l \left(U^l_R V_R\right)\,,$$
$$(M_\nu)_{\mathbf{diag}} = V^T \, (M_\nu)_{\mathbf{diag}} \, V = (UV)^T \, M_\nu \, (UV)\,. \tag{12}$$

The form of $V_{L,R}$ and V depends on the fermion spectrum (degeneracy). For l_l^0 vanishing, l_l degenerate and $n_L - l_l^0 - l_l$ non-degenerate charged lepton masses and l_ν^0 vanishing, l_ν degenerate and $n_L + n_R - l_\nu^0 - l_\nu$ non-degenerate neutrino masses

$$V_L = \begin{pmatrix} W_L^0 & & & & 0 \\ & W_L & & & \\ & & \exp(i\alpha_1) & & \\ & & & \ddots & \\ 0 & & & & \exp(i\alpha_{n_L - l_l^0 - l_l}) \end{pmatrix},$$
$$V_R = \begin{pmatrix} W_R^0 & & & & 0 \\ & W_L & & & \\ & & \exp(i\alpha_1) & & \\ & & & \ddots & \\ 0 & & & & \exp(i\alpha_{n_L - l_l^0 - l_l}) \end{pmatrix},$$
$$V = \begin{pmatrix} W^0 & & & & & 0 \\ & W & & & & \\ & & \pm 1 & & & \\ & & & \pm 1 & & \\ & & & & \ddots & \\ 0 & & & & & \pm 1 \end{pmatrix}, \tag{13}$$

with $W_{L,R}^0, W_L, W^0$ and W $l_l^0 \times l_l^0, l_l \times l_l, l_\nu^0 \times l_\nu^0$ unitary and $l_\nu \times l_\nu$ real orthogonal matrices, respectively. The mixing matrices, however, do change under these transformations

$$K \to V^T K V_L\,, \quad \Omega \to V^T \Omega V^*\,. \tag{14}$$

The counting of CP violating phases in the lepton sector reduces to count the phases in K because the phases in Ω are not independent (see Eq. (11)). K is defined by the first n_L columns of a $(n_L + n_R) \times (n_L + n_R)$ unitary matrix. Thus it is parametrized by $n_L\,(n_L + 2n_R - 1)/2$ mixing angles and $n_L\,(n_L + 2n_R + 1)/2$ phases. But not all of them are physical. Eq. (14) allows to subtract the unphysical ones. The general counting is involved but it can be worked out with a computer [9].

- 2.2.1. If there are no vanishing or degenerate lepton masses, the number of CP breaking phases is equal to the number of phases parametrizing K minus the n_L phases fixing V_L [4, 8],

$$\frac{n_L\,(n_L + 2n_R + 1)}{2} - n_L = \frac{n_L\,(n_L + 2n_R - 1)}{2}\,. \tag{15}$$

If there is one massless neutrino, W^0 is one-dimensional and there is one phase less.

- 2.2.2. If, for instance, $n_L = n_R$ and there are n_L massless neutrinos, the number of CP violating phases in Eq. (15) is reduced by $n_L (n_L + 1)/2$, which is the number of phases of the unitary matrix W^0 in V. The subtraction of non-physical phases is more delicate when the n_L neutrinos have a common mass. The real and imaginary parts of K rotate independently because $W \subset V$ is a real orthogonal matrix in this case. Then with an appropriate choice of W $n_L (n_L - 1)/2$ entries in the $(n_L \times n_L)$ K upper half can be made real, reducing the number of phases by the same amount. The unitarity constraints and the charged lepton phase redefinitions can be chose to fix part of the phases in the K lower half and are already subtracted in Eq. (15). If the n_L massless or degenerate leptons are the charged ones, the number of CP violating phases in Eq. (15) is reduced by $n_L (n_L - 1)/2$, which is the number of phases in W_L^0 or W_L in V_L, $n_L (n_L + 1)/2$, minus n_L, the number of diagonal phases already subtracted in Eq. (15).
- 2.2.3. If $n_L = n_R = 1$ and both neutrinos have a common mass, CP is conserved. K, which is 2×1, depends on 2 complex numbers. With an appropriate choice of the 2×2 real orthogonal matrix $W = V$ the moduli of K_{11} and K_{21} can be made equal. Then redefining the charged lepton phase we can always assume $K = \begin{pmatrix} a \\ a^* \end{pmatrix}$. The phase of a, however, does not stand for CP violation. The lagrangian is invariant under complex conjugation and the interchange of both neutrinos. Similarly for $n_L = 2, n_R = 0$, with an appropriate choice of W the moduli of the 2×2 unitary matrix K can be made equal. Then redefining the charged lepton phases we can always assume $K = \begin{pmatrix} a & a \\ -a^* & a^* \end{pmatrix}$. The phase of a does not stand for CP violation either. The Lagrangian is invariant under complex conjugation, the interchange of both neutrinos and the change of sign of the charged lepton in the first column. If there are the two charged leptons which are degenerate (or massless), CP is also conserved because in this case V_L in Eq. (14) is an arbitrary 2×2 unitary matrix.

3. Left-right models with n fermion families

In this case $n_L = n_R = n$ and there are left-handed as well as right-handed charged currents.

3.1. The quark sector

Without any additional symmetry M_u and M_d are arbitrary complex matrices and the expressions in Sec. 2 are still valid. However, in addition to the CKM matrix for left-handed currents (see Eq. (2))

$$U^L_{\rm CKM} = U^{u\dagger}_L U^d_L\,, \tag{16}$$

there is a CKM mixing matrix for right-handed currents

$$U^R_{\rm CKM} = U^{u\dagger}_R U^d_R\,. \tag{17}$$

Both $n \times n$ unitary matrices can be redefined without modifying the (diagonal) mass matrices (see Eqs (3)–(6))

$$U^L_{\rm CKM} \to V^{u\dagger}_L U^L_{\rm CKM} V^d_L\,, \quad U^R_{\rm CKM} \to V^{u\dagger}_R U^R_{\rm CKM} V^d_R\,. \tag{18}$$

The counting of CP breaking phases in $U^L_{\rm CKM}$ is the same as the standard model counting in Sec. 2. On the other hand, the number of CP breaking phases in $U^R_{\rm CKM}$ can be reduced by an appropriate choice of $W^0_{u,d\ R}$ (see Eq. (6)), for the other entries in $V^{u,d}_R$ are fixed by the corresponding entries in $V^{u,d}_L$. However, one may choose to eliminate the non-physical phases in $U^{L,R}_{\rm CKM}$ in a different way. What matters it is the combined number of CP violating phases.

- 3.1.1. CP can be violated in a left-right model even for one family, $n = 1$, if both quarks are massive. This is so because the phase redefinition in Eq. (18) is the same for $U^L_{\rm CKM}$ and $U^R_{\rm CKM}$, as there are the same $V^u_L(V^d_L)$ and $V^u_R(V^d_R)$. Thus the number of CP violating phases is $2 - 1 = 1$. Similarly for three generations, $n = 3$, CP can be broken if there is at least one massive quark of each type.

3.2. The lepton sector

In this sector there are also right-handed currents and there are left- and right-handed mixing matrices (see Eq. (11))

$$K_L = U^\dagger_L U^l_L\,, \quad K_R = U^\dagger_R U^l_R\,,$$

and

$$\Omega_L = K_L K_L^\dagger\,, \quad \Omega_R = K_R K_R^\dagger\,. \tag{19}$$

The K matrices are $2n \times n$ and satisfy the orthogonality condition

$$K_L^T K_R = 0\,; \tag{20}$$

and the Ω matrices, which are $2n \times 2n$, are completely fixed by the K matrices. As in Eq. (14) the mass matrices remain unchanged, whereas

$$K_L \to V^T K_L V_L\,, \quad K_R \to V^\dagger K_R V_R\,. \tag{21}$$

The unphysical phases in K_L can be eliminated as in Sec. 2. The phases in K_R can be also eliminated using Eq. (21) but only if there is any freedom left after fixing V and V_R, which is related to V_L (Eq. (13)), to reduce the number of K_L phases. What matters is the combined number of CP violating phases in K_L and K_R.

3.2.1. For $n = 1$ there are in general 2 CP breaking phases, of the four phases in K_L and K_R one is fixed by Eq. (20) and another one is eliminated by an appropriate choice of V_L ($V_R = V_L$ if the charged lepton has a non-zero mass.) If the charged lepton is massless, V_R is independent of V_L and we can get rid of a third phase. Finally, if one neutrino is also massless, the fourth phase can be cancelled. If the two neutrinos have a common mass, we can always write

$$K_L = \begin{pmatrix} a \\ -a^* \end{pmatrix}, \quad K_R = \begin{pmatrix} a^* \\ a \end{pmatrix} \mathrm{e}^{i\beta}\,.$$

The β phase can be eliminated if the charged lepton is massless $(V_L \neq V_R)$. This form of $K_{L,R}$ is the same as in case 2.2.3 and CP is also conserved because the lagrangian is invariant under the same operations.

3.2.2. For $n > 1$ CP can be violated even in the case of n degenerate charged leptons and n massless plus n degenerate heavy neutrinos.

4. CP symmetry breaking in the weak basis

In the two previous Sections we have discussed CP symmetry breaking in the mass-eigenstate basis. CP violation can be also studied in the weak basis where gauge interactions are diagonal. CP conservation is then related to the specific form of quark and lepton mass matrices, M_u, M_d, M_l, M_ν [5–8]. If these are real, CP is conserved. However, they can be complex and CP be still conserved. This is so because we can perform unitary transformations on the fields which leave unchanged the gauge couplings but redefine the mass matrices. If there are only left-handed currents, these transformations on quarks and leptons read

$$\begin{aligned} u_L &\to X_L u_L, \\ d_L &\to X_L d_L, \\ u_R &\to X_R^u u_R, \\ d_R &\to X_R^d d_R, \\ \nu_L &\to Y_L \nu_L, \\ l_L &\to Y_L l_L, \\ l_R &\to Y_R^l l_R, \\ \nu_R &\to Y_R^\nu \nu_R, \end{aligned} \tag{22}$$

where X_L, X_R^u, X_R^d, Y_L and Y_R^l (Y_R^ν) are arbitrary $n_L \times n_L$ ($n_R \times n_R$) unitary matrices. If there are also right-handed currents,

$$X_R^u = X_R^d, \quad Y_R^l = Y_R^\nu . \tag{23}$$

Under these transformations the mass matrices M_u, M_d, M_l and

$$M_\nu = \begin{pmatrix} \overbrace{M_L}^{n_L} & \overbrace{M_D}^{n_R} \\ M_D^T & M_R \end{pmatrix} \begin{matrix} \}n_L \\ \}n_R \end{matrix} , \tag{24}$$

change

$$\begin{aligned} M_u &\to X_L M_u X_R^{u\dagger}, \\ M_d &\to X_L M_d X_R^{d\dagger}, \\ M_l &\to Y_L M_l Y_R^{l\dagger}, \\ M_L &\to Y_L M_L Y_L^T, \\ M_D &\to Y_L M_D Y_R^{\nu\dagger}, \\ M_R &\to Y_R^{\nu *} M_R Y_R^{\nu\dagger} . \end{aligned} \tag{25}$$

Then CP is conserved if and only if there exist unitary matrices $X_L, X_R^{u,d}, Y_L, Y_R^{l,\nu}$ such that

$$\begin{aligned} X_L M_u X_R^{u\dagger} &= M_u^*, \\ X_L M_d X_R^{d\dagger} &= M_d^*, \\ Y_L M_l Y_R^{l\dagger} &= M_l^*, \\ Y_L M_L Y_L^T &= M_L^*, \\ Y_L M_D Y_R^{\nu\dagger} &= M_D^*, \\ Y_R^{\nu *} M_R Y_R^{\nu\dagger} &= M_R^* . \end{aligned} \tag{26}$$

These conditions also suggest how to find other necessary (and sufficient) CP invariant constraints which are more useful in practice [5–8]. In this Section we discuss these constraints when some fermion masses are vanishing and/or degenerate (see Section 2, 3).

- 4.1. In the standard model with $n_L = 3$ generations of quarks the necessary and sufficient condition for CP conservation is [5, 6]

$$\mathrm{Tr}\left[M_u M_u^\dagger, M_d M_d^\dagger\right]^3 = 3\,\mathrm{Det}\left[M_u M_u^\dagger, M_d M_d^\dagger\right] = \\ -6i\left(m_t^2 - m_c^2\right)\left(m_t^2 - m_u^2\right)\left(m_c^2 - m_u^2\right)\left(m_b^2 - m_s^2\right)\left(m_b^2 - m_d^2\right) \\ \times\left(m_s^2 - m_d^2\right)\mathrm{Im}\left(U_{ud}U_{cs}U_{us}^*U_{cd}^*\right) = 0\,, \tag{27}$$

where m_f are the quark masses and U is the CKM matrix. CP can be violated if there is at most one massless quark of each type (case 2.1.1). However, this invariant is identically zero if two up or down quark masses are degenerate (case 2.1.2).

- 4.2. For leptons practical, necessary and sufficient CP invariant constraints were obtained in Ref. [8] in simple cases . For $n_L = n_R = 1$ the constraint for CP conservation is

$$\mathrm{Im\,Tr}\left(M_D^\dagger M_L M_D^* M_R\right) = \\ m_1 m_2\left(m_2^2 - m_1^2\right)\mathrm{Im}\left(K_{11}^2 K_{21}^{*2}\right) = 0\,, \tag{28}$$

where m_i are the neutrino masses and K is the mixing matrix in Eq. (11). If there is one massless neutrino or both neutrinos are degenerate, CP is conserved (cases 2.2.1 and 2.2.3, respectively). For $n_L = 2, n_R = 0$ the CP invariant constraint is

$$\mathrm{Im\,Det}\left(M_L M_l^* M_l^T M_L^\dagger - M_L M_L^\dagger M_l M_l^\dagger\right) = \\ m_1 m_2\left(m_2^2 - m_1^2\right)\left(m_\mu^2 - m_e^2\right)^2\mathrm{Im}\left(K_{11}^2 K_{21}^{*2}\right) = 0\,, \tag{29}$$

where $m_{1,2(e,\mu)}$ are the neutrino (charged lepton) masses and K is the mixing matrix. CP is conserved if there is a massless neutrino (case 2.2.1) or the neutrinos (charged leptons) are degenerate (case 2.2.3).

- 4.3. In left-right models with $n = 1$ there are two necessary and sufficient invariant constraints for CP conservation in the lepton sector

$$\mathrm{Im\,Tr}\left(M_l M_D^\dagger\right) = m_e\,\mathrm{Im}\left(m_1 K_{L11}K_{R11}^* + m_2 K_{L21}K_{R21}^*\right) = 0\,,$$

$$\mathrm{Im\,Tr}\left(M_l M_R^\dagger M_l^T M_L^\dagger - M_L M_D^* M_R M_D^\dagger\right) = \\ m_e^2\,\mathrm{Im}\left((m_1 K_{L11}^2 + m_2 K_{L21}^2)(m_1 K_{R11}^{*2} + m_2 K_{R21}^{*2})\right) \\ -m_1 m_2\left(m_2^2 - m_1^2\right)\mathrm{Im}(K_{L11}^{*2}K_{L21}^2) = 0\,, \tag{30}$$

where $m_{1,2(e)}$ are the neutrino (charged lepton) masses and $K_{L,R}$ are the mixing matrices. CP is conserved if the charged lepton is massless and there is one massless neutrino or both neutrinos are degenerate (case 3.2.1). Otherwise, CP can be broken.

As in the former examples we expect that the counting of CP breaking phases in Sections 2, 3 will be useful for searching for a set of necessary and sufficient, and also practical, CP invariant constraints.

5. Phase redefinition invariants

Physical observables can not depend on fermion field redefinitions. In the mass-eigenstate basis the only fermion field redefinitions left are the unitary transformations in Eqs (6), (13), which leave unchanged the diagonal mass matrices and redefine the mixing matrices (see Eqs (4), (14), (18), (21)). Then the corresponding observables can only depend on quantities invariant under these transformations. The simplest of these quantities are

$$\sum_{i,j} |T_{ij}|^2 , \tag{31}$$

where if i or j stands for a degenerate fermion, the sum (as the sums below) also includes the other fermions of the same type with the same mass. Otherwise, i and j can be any set of fermions and T is any mixing matrix, U, K, Ω. However, these expressions do not depend on any phase, and even if CP is conserved, they are in general non-zero. (Sums

$$\sum_{j} T_{ij}T^*_{kj} \tag{32}$$

with $i \neq j$ are not invariant because even for non-degenerate Majorana neutrinos they can transform with a sign (see Eq. (13)).) In left-right models there are also mixed bilinear invariants

$$\sum_{i,j} T_{Lij}T^*_{Rij} , \tag{33}$$

where T_L is a left-handed mixing matrix, U_L, K_L, Ω_L, and T_R its right-handed partner, U_R, K_R, Ω_R. These invariants depend in general on the CP breaking phases. The number of independent CP violating invariants is, of course, finite in specific models. In the left-right model in Section 4 with $n = 1$ there are two such invariants (see Eq. (30))

$$K_{L11}K^*_{R11} \quad \text{and} \quad K_{L21}K^*_{R21} . \tag{34}$$

A non-zero imaginary part of these invariants stands for CP non-conservation. In models with only left-handed mixing matrices we have to look for invariants of higher dimensions to observe CP violation. Possible invariants of dimension four are

$$\sum_{i,j,k,m} T_{ij} T_{km} T^*_{im} T^*_{kj} \,. \tag{35}$$

In the minimal standard model CP violation is characterized by a non-zero imaginary part of one of these invariants, for example (see Eq. (27)) $\mathrm{Im}\left(U_{ud} U_{cs} U^*_{us} U^*_{cd}\right)$.

6. Conclusions

We have discussed the number of independent CP breaking phases in models with $SU(2)_L \times U(1)_Y$ and $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ electroweak gauge groups and extended matter contents, paying special attention to the case of vanishing and/or degenerate fermion masses. Quarks and leptons, including Majorana neutrinos, are treated in a similar way. We have also revised the necessary and sufficient constraints for CP conservation in some simple models. Some of these constraints are identically zero when some fermion masses vanish or are degenerate. The knowledge of the number of independent CP violating phases and the study of these particular cases are a useful guide for the search of CP invariant constraints which are not only necessary but sufficient for CP conservation. Observables involving well-defined mass eigenstates depend on factors which are only invariant under phase redefinitions. We study the phase redefinition invariants of lowest dimension.

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