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Research Article

Orthogonally Additive and Orthogonality Preserving Holomorphic Mappings between C*-Algebras

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We study holomorphic maps between C*-algebras A and B, when $f : B_A(0,\varrho) \to B$ is a holomorphic mapping whose Taylor series at zero is uniformly converging in some open unit ball $U = B_A(0,\delta)$. If we assume that f is orthogonality preserving and orthogonally additive on $A_{sa} \cap U$ and $f(U)$ contains an invertible element in B, then there exist a sequence (h_n) in B^{**} and Jordan *-homomorphisms $\Theta, \widetilde{\Theta} : M(A) \to B^{**}$ such that $f(x) = \sum_{n=1}^{\infty} h_n \widetilde{\Theta}(a^n) = \sum_{n=1}^{\infty} \Theta(a^n) h_n$ uniformly in $a \in U$. When B is abelian, the hypothesis of B being unital and $f(U) \cap \mathrm{inv}(B) \neq \emptyset$ can be relaxed to get the same statement.

1. Introduction

The description of orthogonally additive n-homogeneous polynomial on $C(K)$-spaces and on general C*-algebras, developed by Benyamini et al. [1], Pérez-García and Villanueva [2], and Palazuelos et al. [3], respectively (see also [4, 5], [6, Section 3] and [7]), made functional analysts study and explore orthogonally additive holomorphic functions on $C(K)$-spaces (see [8, 9]) and subsequently on general C*-algebras (cf. [10]).

We recall that a mapping f from a C*-algebra A into a Banach space B is said to be *orthogonally additive* on a subset $U \subseteq A$ if for every a, b in U with $a \perp b$, and $a + b \in U$ we have $f(a+b) = f(a) + f(b)$, where elements a, b in A are said to be *orthogonal* (denoted by $a \perp b$) whenever $ab^* = b^*a = 0$. We will say that f is *additive on elements having zero product* if for every a, b in A with $ab = 0$, we have $f(a+b) = f(a) + f(b)$. Having this terminology in mind, the description of all n-homogeneous polynomials on a general C*-algebra, A, which are orthogonally additive on the self-adjoint part, A_{sa}, of A reads as follows (see Section 2 for concrete definitions not explained here).

Theorem 1 (see [3]). *Let A be a C*-algebra and B a Banach space, $n \in \mathbb{N}$, and let $P : A \to B$ be an n-homogeneous polynomial. The following statements are equivalent.*

(a) *There exists a bounded linear operator $T : A \to B$ satisfying*

$$P(a) = T(a^n), \quad (1)$$

for every $a \in A$ and $\|P\| \le \|T\| \le 2\|P\|$.

(b) *P is additive on elements having zero products.*

(c) *P is orthogonally additive on A_{sa}.*

The task of replacing n-homogeneous polynomials by polynomials or by holomorphic functions involves a higher difficulty. For example, as noticed by Carando et al. [8, Example 2.2], when K denotes the closed unit disc in $\mathbb{C}$, there is no entire function $\Phi : \mathbb{C} \to \mathbb{C}$ such that the mapping $h : C(K) \to C(K)$, $h(f) = \Phi \circ f$ factorizes all degree-2 orthogonally additive scalar polynomials over $C(K)$. Furthermore, similar arguments show that defining $P : C([0,1]) \to \mathbb{C}$, $P(f) = f(0) + f(1)^2$, we cannot find a triplet

$(\Phi, \alpha_1, \alpha_2)$, where $\Phi : C[0,1] \to \mathbb{C}$ is a *-homomorphism and $\alpha_1, \alpha_2 \in \mathbb{C}$, satisfying that $P(f) = \alpha_1 \Phi(f) + \alpha_2 \Phi(f^2)$ for every $f \in C([0,1])$.

To avoid the difficulties commented above, Carando et al. introduce a factorization through an $L_1(\mu)$ space. More concretely, for each compact Hausdorff space K, a holomorphic mapping of bounded type $f : C(K) \to \mathbb{C}$ is orthogonally additive if and only if there exist a Borel regular measure μ on K, a sequence $(g_k)_k \subseteq L_1(\mu)$, and a holomorphic function of bounded type $h : C(K) \to L_1(\mu)$ such that $h(a) = \sum_{k=0}^{\infty} g_k a^k$ and

$$f(a) = \int_K h(a)\, d\mu, \tag{2}$$

for every $a \in C(K)$ (cf. [8, Theorem 3.3]).

When $C(K)$ is replaced with a general C*-algebra A, a holomorphic function of bounded type $f : A \to \mathbb{C}$ is orthogonally additive on A_{sa} if and only if there exist a positive functional φ in A^*, a sequence (ψ_n) in $L_1(A^{**}, \varphi)$, and a power series holomorphic function h in $\mathscr{H}_b(A, A^*)$ such that

$$h(a) = \sum_{k=1}^{\infty} \psi_k \cdot a^k, \qquad f(a) = \langle 1_{A^{**}}, h(a) \rangle = \int h(a)\, d\varphi, \tag{3}$$

for every a in A, where $1_{A^{**}}$ denotes the unit element in A^{**} and $L_1(A^{**}, \varphi)$ is a noncommutative L_1-space (cf. [10]).

A very recent contribution due to Bu et al. [11] shows that, for holomorphic mappings between $C(K)$ spaces, we can avoid the factorization through an $L_1(\mu)$-space by imposing additional hypothesis. Before stating the detailed result, we will set down some definitions.

Let A and B be C*-algebras. When $f : U \subseteq A \to B$ is a map and the condition

$$a \perp b \Longrightarrow f(a) \perp f(b)$$
$$(\text{resp., } ab = 0 \Longrightarrow f(a) f(b) = 0) \tag{4}$$

holds for every $a, b \in U$, we will say that f *preserves orthogonality* or it is *orthogonality preserving* (resp., f *preserves zero products*) on U. In the case $A = U$ we will simply say that f is *orthogonality preserving* (resp., f *preserves zero products*). Orthogonality preserving bounded linear maps between C*-algebras were completely described in [12, Theorem 17] (see [6] for completeness).

The following Banach-Stone type theorem for zero product preserving or orthogonality preserving holomorphic functions between $C_0(L)$ spaces is established by Bu et al. in [11, Theorem 3.4].

Theorem 2 (see [11]). *Let L_1 and L_2 be locally compact Hausdorff spaces and let $f : B_{C_0(L_1)}(0, r) \to C_0(L_2)$ be a bounded orthogonally additive holomorphic function. If f is zero product preserving or orthogonality preserving, then there exist a sequence $(\mathcal{O}_n)$ of open subsets of L_2, a sequence (h_n) of bounded functions from $L_2 \cup \{\infty\}$ into $\mathbb{C}$, and a mapping $\varphi : L_2 \to L_1$ such that for each natural n the function h_n is continuous and nonvanishing on $\mathcal{O}_n$ and*

$$f(a)(t) = \sum_{n=1}^{\infty} h_n(t) (a(\varphi(t)))^n, \quad (t \in L_2), \tag{5}$$

uniformly in $a \in B_{C_0(L_1)}(0, r)$.

The study developed by Bu et al. is restricted to commutative C*-algebras or to orthogonality preserving and orthogonally additive, n-homogeneous polynomials between general C*-algebras. The aim of this paper is to extend their study to holomorphic maps between general C*-algebras. In Section 4, we determine the form of every orthogonality preserving and orthogonally additive holomorphic function from a general C*-algebra into a commutative C*-algebra (see Theorem 16).

In the wider setting of holomorphic mappings between general C*-algebras, we prove the following: let A and B be C*-algebras with B unital and let $f : B_A(0, \varrho) \to B$ be a holomorphic mapping whose Taylor series at zero is uniformly converging in some open unit ball $U = B_A(0, \delta)$. Suppose f is orthogonality preserving and orthogonally additive on $A_{sa} \cap U$ and $f(U)$ contains an invertible element. Then there exist a sequence (h_n) in B^{**} and Jordan *-homomorphisms $\Theta, \widetilde{\Theta} : M(A) \to B^{**}$ such that

$$f(x) = \sum_{n=1}^{\infty} h_n \widetilde{\Theta}(a^n) = \sum_{n=1}^{\infty} \Theta(a^n) h_n, \tag{6}$$

uniformly in $a \in U$ (see Theorem 18).

The main tool to establish our main results is a newfangled investigation on orthogonality preserving pairs of operators between C*-algebras developed in Section 3. Among the novelties presented in Section 3, we find an innovating alternative characterization of orthogonality preserving operators between C*-algebras which complements the original one established in [12] (see Proposition 14). Orthogonality preserving pairs of operators are also valid to determine orthogonality preserving operators and orthomorphisms or local operators on C*-algebras in the sense employed by Zaanen [13] and Johnson [14], respectively.

2. Orthogonally Additive, Orthogonality Preserving, and Holomorphic Mappings on C*-Algebras

Let X and Y be Banach spaces. Given a natural n, a (continuous) n-homogeneous polynomial P from X to Y is a mapping $P : X \to Y$ for which there is a (continuous) n-linear symmetric operator $A : X \times \cdots \times X \to Y$ such that $P(x) = A(x, \ldots, x)$, for every $x \in X$. All polynomials considered in this paper are assumed to be continuous. By a 0-homogeneous polynomial we mean a constant function. The symbol $\mathscr{P}(^n X, Y)$ will denote the Banach space of all continuous n-homogeneous polynomials from X to Y, with norm given by $\|P\| = \sup_{\|x\| \leq 1} \|P(x)\|$.

Throughout the paper, the word operator will always stand for a bounded linear mapping.

We recall that, given a domain U in a complex Banach space X (i.e., an open, connected subset), a function f from U to another complex Banach space Y is said to be *holomorphic* if the Fréchet derivative of f at z_0 exists for every point z_0 in U. It is known that f is holomorphic in U if and only if for each $z_0 \in X$ there exists a sequence $(P_k(z_0))_k$ of polynomials from X into Y, where each $P_k(z_0)$ is k-homogeneous, and a neighborhood V_{z_0} of z_0 such that the series,

$$\sum_{k=0}^{\infty} P_k(z_0)(y - z_0), \tag{7}$$

converges uniformly to $f(y)$ for every $y \in V_{z_0}$. Homogeneous polynomials on a C*-algebra A constitute the most basic examples of holomorphic functions on A. A holomorphic function $f : X \to Y$ is said to be of *bounded type* if it is bounded on all bounded subsets of X; in this case its Taylor series at zero, $f = \sum_{k=0}^{\infty} P_k$, has infinite radius of uniform convergence, that is, $\limsup_{k\to\infty}\|P_k\|^{1/k} = 0$ (compare [15, Section 6.2], see also [16]).

Suppose $f : B_X(0,\delta) \to Y$ is a holomorphic function and let $f = \sum_{k=0}^{\infty} P_k$ be its Taylor series at zero which is assumed to be uniformly convergent in $U = B_X(0,\delta)$. Given $\varphi \in Y^*$, it follows from Cauchy's integral formula that, for each $a \in U$, we have

$$\varphi P_n(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{\varphi f(\lambda a)}{\lambda^{n+1}} d\lambda, \tag{8}$$

where γ is the circle forming the boundary of a disc in the complex plane $D_{\mathbb{C}}(0, r_1)$, taken counterclockwise, such that $a + D_{\mathbb{C}}(0, r_1)a \subseteq U$. We refer to [15] for the basic facts and definitions used in this paper.

In this section we will study orthogonally additive, orthogonality preserving, and holomorphic mappings between C*-algebras. We begin with an observation which can be directly derived from Cauchy's integral formula. The statement in the next lemma was originally stated by Carando et al. in [8, Lemma 1.1] (see also [10, Lemma 3]).

Lemma 3. *Let $f : B_A(0,\varrho) \to B$ be a holomorphic mapping, where A is a C*-algebra and B is a complex Banach space, and let $f = \sum_{k=0}^{\infty} P_k$ be its Taylor series at zero, which is uniformly converging in $U = B_A(0,\delta)$. Then the mapping f is orthogonally additive on U (resp., orthogonally additive on $A_{sa} \cap U$ or additive on elements having zero product in U) if and only if all the P_k's satisfy the same property. In such a case, $P_0 = 0$.*

We recall that a functional φ in the dual of a C*-algebra A is *symmetric* when $\varphi(a) \in \mathbb{R}$, for every $a \in A_{sa}$. Reciprocally, if $\varphi(b) \in \mathbb{R}$ for every symmetric functional $\varphi \in A^*$, the element b lies in A_{sa}. Having this in mind, our next lemma also is a direct consequence of Cauchy's integral formula and the power series expansion of f. A mapping $f : A \to B$ between C*-algebras is called *symmetric* whenever $f(A_{sa}) \subseteq B_{sa}$, or equivalently, $f(a) = f(a)^*$, whenever $a \in A_{sa}$.

Lemma 4. *Let $f : B_A(0,\varrho) \to B$ be a holomorphic mapping, where A and B are C*-algebras, and let $f = \sum_{k=0}^{\infty} P_k$ be its Taylor series at zero, which is uniformly converging in $U = B_A(0,\delta)$. Then the mapping f is symmetric on U (i.e., $f(A_{sa} \cap U) \subseteq B_{sa}$) if and only if P_k is symmetric (i.e., $P_k(A_{sa}) \subseteq B_{sa}$) for every $k \in \mathbb{N} \cup \{0\}$.*

Definition 5. Let $S, T : A \to B$ be a couple of mappings between two C*-algebras. One will say that the pair (S,T) is orthogonality preserving on a subset $U \subseteq A$ if $S(a) \perp T(b)$ whenever $a \perp b$ in U. When $ab = 0$ in U implies $S(a)T(b) = 0$ in B, we will say that (S,T) preserves zero products on U.

We observe that a mapping $T : A \to B$ is orthogonality preserving in the usual sense if and only if the pair (T,T) is orthogonality preserving. We also notice that (S,T) is orthogonality preserving (on A_{sa}) if and only if (T,S) is orthogonality preserving (on A_{sa}).

Our next result assures that the n-homogeneous polynomials appearing in the Taylor series of an orthogonality preserving holomorphic mapping between C*-algebras are pairwise orthogonality preserving.

Proposition 6. *Let $f : B_A(0,\varrho) \to B$ be a holomorphic mapping, where A and B are C*-algebras, and let $f = \sum_{k=0}^{\infty} P_k$ be its Taylor series at zero, which is uniformly converging in $U = B_A(0,\delta)$. The following statements hold.*

(a) *The mapping f is orthogonally preserving on U (resp., orthogonally preserving on $A_{sa} \cap U$) if and only if $P_0 = 0$ and the pair (P_n, P_m) is orthogonality preserving (resp., orthogonally preserving on A_{sa}) for every $n, m \in \mathbb{N}$.*

(b) *The mapping f preserves zero products on U if and only if $P_0 = 0$ and for every $n, m \in \mathbb{N}$, the pair (P_n, P_m) preserves zero products.*

Proof. (a) The "if" implication is clear. To prove the "only if" implication, let us fix $a, b \in U$ with $a \perp b$. Let us find two positive scalars r, C such that $a, b \in B(0,r)$ and $\|f(x)\| \leq C$ for every $x \in B(0,r) \subset \overline{B}(0,r) \subseteq U$. From the Cauchy estimates we have $\|P_m\| \leq C/r^m$, for every $m \in \mathbb{N} \cup \{0\}$. By hypothesis $f(ta) \perp f(tb)$, for every $r > t > 0$, hence

$$\begin{aligned} &P_0(ta) P_0(tb)^* + P_0(ta)\left(\sum_{k=1}^{\infty} P_k(tb)\right)^* \\ &\quad + \left(\sum_{k=1}^{\infty} P_k(ta)\right)\left(\sum_{k=0}^{\infty} P_k(tb)\right)^* = 0, \end{aligned} \tag{9}$$

and by homogeneity

$$\begin{aligned} P_0(a) P_0(b)^* = &-P_0(a)\left(\sum_{k=1}^{\infty} t^k P_k(b)\right)^* \\ &+ \left(\sum_{k=1}^{\infty} t^k P_k(a)\right)\left(\sum_{k=0}^{\infty} t^k P_k(b)\right)^*. \end{aligned} \tag{10}$$

Letting $t \to 0$, we have $P_0(a)P_0(b)^* = 0$. In particular, $P_0 = 0$.

We will prove by induction on n that the pair (P_j, P_k) is orthogonality preserving on U for every $1 \leq j, k \leq n$. Since $f(ta)f(tb)^* = 0$, we also deduce that

$$P_1(ta)P_1(tb)^* + P_1(ta)\left(\sum_{k=2}^{\infty} P_k(tb)\right)^* + \left(\sum_{k=2}^{\infty} P_k(ta)\right)\left(\sum_{k=1}^{\infty} P_k(tb)\right)^* = 0, \tag{11}$$

for every $(\min\{\|a\|, \|b\|\})/r > t > 0$, which implies that

$$t^2 P_1(a)P_1(b)^* = -tP_1(a)\left(\sum_{k=2}^{\infty} t^k P_k(b)\right)^* - \left(\sum_{k=2}^{\infty} t^k P_k(a)\right)\left(\sum_{k=1}^{\infty} t^k P_k(b)\right)^*, \tag{12}$$

for every $(\min\{\|a\|, \|b\|\})/r > t > 0$, and hence

$$\|P_1(a)P_1(b)^*\| \leq tC\|P_1(a)\| \sum_{k=2}^{\infty} \frac{\|b\|^k}{r^k} t^{k-2} + tC^2 \left(\sum_{k=2}^{\infty} \frac{\|a\|^k}{r^k} t^{k-2}\right)\left(\sum_{k=1}^{\infty} \frac{\|b\|^k}{r^k} t^{k-1}\right). \tag{13}$$

Taking limit in $t \to 0$, we get $P_1(a)P_1(b)^* = 0$. Let us assume that (P_j, P_k) is orthogonality preserving on U for every $1 \leq j, k \leq n$. Following the argument above we deduce that

$$\begin{aligned} P_1(a)P_{n+1}(b)^* + P_{n+1}(a)P_1(b)^* &= -tP_1(a)\left(\sum_{j=n+2}^{\infty} t^{j-n-2} P_j(b)\right)^* \\ &- t\sum_{k=2}^{n} t^{k-2} P_k(a)\left(\sum_{j=n+1}^{\infty} t^{j-n-1} P_j(b)\right)^* \\ &- tP_{n+1}(a)\left(\sum_{j=2}^{\infty} t^{j-2} P_j(b)\right)^* \\ &- t\left(\sum_{k=n+2}^{\infty} t^{k-n-2} P_k(a)\right)\left(\sum_{j=1}^{\infty} t^{j-1} P_j(b)\right)^*, \end{aligned} \tag{14}$$

for every $(\min\{\|a\|, \|b\|\})/r > |t| > 0$. Taking limit in $t \to 0$, we have

$$P_1(a)P_{n+1}(b)^* + P_{n+1}(a)P_1(b)^* = 0. \tag{15}$$

Replacing a with sa $(s > 0)$ we get

$$sP_1(a)P_{n+1}(b)^* + s^{n+1}P_{n+1}(a)P_1(b)^* = 0, \tag{16}$$

for every $s > 0$, which implies that

$$P_1(a)P_{n+1}(b)^* = 0. \tag{17}$$

In a similar manner we prove that $P_k(a)P_{n+1}(b)^* = 0$, for every $1 \leq k \leq n+1$. The equalities $P_k(b)^* P_j(a) = 0$ $(1 \leq j, k \leq n+1)$ follow similarly.

We have shown that for each $n, m \in \mathbb{N}$, $P_n(a) \perp P_m(b)$ whenever $a, b \in U$ with $a \perp b$. Finally, taking a, $b \in A$ with $a \perp b$, we can find a positive ρ such that $\rho a, \rho b \in U$ and $\rho a \perp \rho b$, which implies that $P_n(\rho a) \perp P_m(\rho b)$ for every $n, m \in \mathbb{N}$, witnessing that (P_n, P_m) is orthogonality preserving for every $n, m \in \mathbb{N}$.

The proof of (b) follows in a similar manner. □

We can obtain now a corollary which is a first step toward the description of orthogonality preserving, orthogonally additive, and holomorphic mappings between C*-algebras.

Corollary 7. *Let $f : B_A(0, \varrho) \to B$ be a holomorphic mapping, where A and B are C*-algebras and let $f = \sum_{k=0}^{\infty} P_k$ be its Taylor series at zero, which is uniformly converging in $U = B_A(0, \delta)$. Suppose f is orthogonality preserving and orthogonally additive on (resp., orthogonally additive and zero products preserving) $A_{sa} \cap U$. Then there exists a sequence (T_n) of operators from A into B satisfying that the pair (T_n, T_m) is orthogonality preserving on A_{sa} (resp., zero products preserving on A_{sa}) for every $n, m \in \mathbb{N}$ and*

$$f(x) = \sum_{n=1}^{\infty} T_n(x^n), \tag{18}$$

uniformly in $x \in U$. In particular every T_n is orthogonality preserving (resp., zero products preserving) on A_{sa}. Furthermore, f is symmetric if and only if every T_n is symmetric.

Proof. Combining Lemma 3 and Proposition 6, we deduce that $P_0 = 0$, P_n is orthogonally additive on A_{sa}, and (P_n, P_m) is orthogonality preserving on A_{sa} for every n, m in $\mathbb{N}$. By Theorem 1, for each natural n there exists an operator $T_n : A \to B$ such that $\|P_n\| \leq \|T_n\| \leq 2\|P_n\|$ and

$$P_n(a) = T_n(a^n), \tag{19}$$

for every $a \in A$.

Consider now two positive elements $a, b \in A$ with $a \perp b$ and fix $n, m \in \mathbb{N}$. In this case there exist positive elements c, d in A with $c^n = a$ and $d^m = b$ and $c \perp d$. Since the pair (P_n, P_m) is orthogonality preserving on A_{sa}, we have $T_n(a) = T_n(c^n) = P_n(c) \perp P_m(d) = T_m(d^m) = T_m(b)$. Now, noticing that given a, b in A_{sa} with $a \perp b$, we can write $a = a^+ - b^-$ and $b = b^+ - b^-$, where a^σ and b^τ are positive, $a^+ \perp a^-$, $b^+ \perp b^-$, and $a^\sigma \perp b^\tau$; for every $\sigma, \tau \in \{+, -\}$, we deduce that $T_n(a) \perp T_m(b)$. This shows that the pair (T_n, T_m) is orthogonality preserving on A_{sa}.

When f is orthogonally additive on A_{sa} and zero products preserving, then the pair (T_n, T_m) is zero products preserving on A_{sa} for every $n, m \in \mathbb{N}$. The final statement is clear from Lemma 4. □

It should be remarked here that if a mapping $f : B_A(0, \delta) \to B$ is given by an expression of the form in (18) which uniformly converges in $U = B_A(0, \delta)$, where (T_n) is a sequence of operators from A into B such that the pair (T_n, T_m) is orthogonality preserving on A_{sa} (resp., zero products preserving on A_{sa}) for every $n, m \in \mathbb{N}$, then f is orthogonally additive and orthogonality preserving on $A_{sa} \cap U$ (resp., orthogonally additive on $A_{sa} \cap U$ and zero products preserving).

3. Orthogonality Preserving Pairs of Operators

Let A and B be two C*-algebras. In this section we will study those pairs of operators $S, T : A \to B$ satisfying that S, T and the pair (S, T) preserve orthogonality on A_{sa}. Our description generalizes some of the results obtained by Wolff in [17] because a (symmetric) mapping $T : A \to B$ is orthogonality preserving on A_{sa} if and only if the pair (T, T) enjoys the same property. In particular, for every *-homomorphism $\Phi : A \to B$, the pair (Φ, Φ) preserves orthogonality. The same statement is true whenever Φ is a *-antihomomorphism, or a Jordan *-homomorphism, or a triple homomorphism for the triple product $\{a, b, c\} = (1/2)(ab^*c + cb^*a)$.

We observe that S, T being symmetric implies that (S, T) is orthogonality preserving on A_{sa} if and only if (S, T) is zero products preserving on A_{sa}. We shall present here a newfangled and simplified proof which is also valid for pairs of operators.

Let a be an element in a von Neumann algebra M. We recall that the *left* and *right support projections* of a (denoted by $l(a)$ and $d(a)$) are defined as follows: $l(a)$ (resp., $d(a)$) is the smallest projection $p \in M$ (resp., $q \in M$) with the property that $pa = a$ (resp., $aq = a$). It is known that when a is Hermitian $d(a) = l(a)$ is called the *support* or *range projection* of a and is denoted by $s(a)$. It is also known that, for each $a = a^*$, the sequence $(a^{1/3^n})$ converges in the strong *-topology of M to $s(a)$ (cf. [18, Sections 1.10 and 1.11]).

An element e in a C*-algebra A is said to be a *partial isometry* whenever $ee^*e = e$ (equivalently, ee^* or e^*e is a projection in A). For each partial isometry e, the projections ee^* and e^*e are called the left and right support projections associated with e, respectively. Every partial isometry e in A defines a Jordan product and an involution on $A_e(e) := ee^*Ae^*e$ given by $a \bullet_e b = (1/2)(ae^*b + be^*a)$ and $a^{\sharp_e} = ea^*e$ $(a, b \in A_2(e))$. It is known that $(A_2(e), \bullet_e, \sharp_e)$ is a unital JB*-algebra with respect to its natural norm and e is the unit element for the Jordan product $\bullet_e$.

Every element a in a C*-algebra A admits a *polar decomposition* in A^{**}; that is, a decomposes uniquely as follows: $a = u|a|$, where $|a| = (a^*a)^{1/2}$ and u is a partial isometry in A^{**} such that $u^*u = s(|a|)$ and $uu^* = s(|a^*|)$ (cf. [18, Theorem 1.12.1]). Observe that $uu^*a = au^*u = u$. The unique partial isometry u appearing in the polar decomposition of a is called the range partial isometry of a and is denoted by $r(a)$. Let us observe that taking $c = r(a)|a|^{1/3}$, we have $cc^*c = a$. It is also easy to check that for each $b \in A$ with $b = r(a)r(a)^*b$ (resp., $b = br(a)^*r(a)$) the condition $a^*b = 0$ (resp., $ba^* = 0$) implies $b = 0$. Furthermore, $a \perp b$ in A if and only if $r(a) \perp r(b)$ in A^{**}.

We begin with a basic argument in the study of orthogonality preserving operators between C*-algebras whose proof is inserted here for completeness reasons. Let us recall that for every C*-algebra A, the *multiplier algebra* of A, $M(A)$, is the set of all elements $x \in A^{**}$ such that for each $Ax, xA \subseteq A$. We notice that $M(A)$ is a C*-algebra and contains the unit element of A^{**}.

Lemma 8. *Let A and B be C*-algebras and let $S, T : A \to B$ be a pair of operators.*

(a) *The pair (S, T) preserves orthogonality (on A_{sa}) if and only if the pair $(S^{**}|_{M(A)}, T^{**}|_{M(A)})$ preserves orthogonality (on $M(A)_{sa}$).*

(b) *The pair (S, T) preserves zero products (on A_{sa}) if and only if the pair $(S^{**}|_{M(A)}, T^{**}|_{M(A)})$ preserves zero products (on $M(A)_{sa}$).*

Proof. (a) The "if" implication is clear. Let a, b be two elements in $M(A)$ with $a \perp b$. We can find two elements c and d in $M(A)$ satisfying $cc^*c = a$, $dd^*d = b$, and $c \perp d$. Since $cxc \perp dyd$, for every x, y in A, we have $T(cxc) \perp T(dyd)$ for every $x, y \in A$. By Goldstine's theorem we find two bounded nets (x_λ) and (y_μ) in A, converging in the weak* topology of A^{**} to c^* and d^*, respectively. Since $T(cx_\lambda c)T(dy_\mu d)^* = T(dy_\mu d)^*T(cx_\lambda c) = 0$, for every λ, μ, T^{**} is weak*-continuous, the product of A^{**} is separately weak*-continuous, and the involution of A^{**} is also weak*-continuous, we get $T^{**}(cc^*c)T^{**}(dd^*d) = T^{**}(a)T^{**}(b)^* = 0 = T^{**}(b)^*T^{**}(a)$ and hence $T^{**}(a) \perp T^{**}(b)$, as desired.

The proof of (b) follows by a similar argument. □

Proposition 9. *Let $S, T : A \to B$ be operators between C*-algebras such that (S, T) is orthogonality preserving on A_{sa}. Let us denote $h := S^{**}(1)$ and $k := T^{**}(1)$. Then the identities,*

$$S(a)T(a^*)^* = S(a^2)k^* = hT((a^2)^*)^*,$$
$$T(a^*)^*S(a) = k^*S(a^2) = T((a^2)^*)^*h, \tag{20}$$
$$S(a)k^* = hT(a^*)^*, \qquad k^*S(a) = T(a^*)^*h,$$

hold for every $a \in A$.

Proof. By Lemma 8, we may assume, without loss of generality, that A is unital. (a) for each $\varphi \in B^*$, the continuous bilinear form $V_\varphi : A \times A \to \mathbb{C}$, $V_\varphi(a, b) = \varphi(S(a)T(b^*)^*)$ is orthogonal; that is, $V_\varphi(a, b) = 0$, whenever $ab = 0$ in A_{sa}. By Goldstein's theorem [19, Theorem 1.10], there exist functionals $\omega_1, \omega_2 \in A^*$ satisfying that

$$V_\varphi(a, b) = \omega_1(ab) + \omega_2(ba), \tag{21}$$

for all $a, b \in A$. Taking $b = 1$ and $a = b$ we have

$$\varphi(S(a)k^*) = V_\varphi(a, 1) = V_\varphi(1, a) = \varphi(hT(a)^*),$$
$$\varphi(S(a)T(a)^*) = \varphi(S(a^2)k^*) = \varphi(hT(a^2)^*), \tag{22}$$

for every $a \in A_{sa}$, respectively. Since φ was arbitrarily chosen, we get, by linearity, $S(a)k^* = hT(a^*)^*$ and $S(a)T(a^*)^* = S(a^2)k^* = hT((a^2)^*)^*$, for every $a \in A$. The other identities follow in a similar way but replacing $V_\varphi(a, b) = \varphi(S(a)T(b^*)^*)$ with $V_\varphi(a, b) = \varphi(T(b^*)^*S(a))$. □

Lemma 10. *Let $J_1, J_2 : A \to B$ be Jordan *-homomorphism between C*-algebras. The following statements are equivalent.*

(a) *The pair (J_1, J_2) is orthogonality preserving on A_{sa}.*

(b) *The identity*

$$J_1(a) J_2(a) = J_1(a^2) J_2^{**}(1) = J_1^{**}(1) J_2(a^2), \quad (23)$$

holds for every $a \in A_{sa}$,

(c) *The identity,*

$$J_1^{**}(1) J_2(a) = J_1(a) J_2^{**}(1), \quad (24)$$

holds for every $a \in A_{sa}$.

Furthermore, when J_1^{**} is unital, $J_2(a) = J_1(a) J_2^{**}(1) = J_2^{**}(1) J_1(a)$, for every a in A.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) have been established in Proposition 9. To see (c) $\Rightarrow$ (a), we observe that $J_i(x) = J_i^{**}(1)J_i(x)J_i^{**}(1) = J_i(x)J_i^{**}(1) = J_i^{**}(1)J_i(x)$, for every $x \in A$. Therefore, given $a, b \in A_{sa}$ with $a \perp b$, we have $J_1(a)J_2(b) = J_1(a)J_1^{**}(1)J_2(b) = J_1(a)J_1(b)J_2^{**}(1) = 0$. □

In [17, Proposition 2.5], Wolff establishes a uniqueness result for *-homomorphisms between C*-algebras showing that for each pair (U, V) of unital *-homomorphisms from a unital C*-algebra A into a unital C*-algebra B, the condition (U, V) orthogonality preserving on A_{sa} implies $U = V$. This uniqueness result is a direct consequence of our previous lemma.

Orthogonality preserving pairs of operators can be also used to rediscover the notion of orthomorphism in the sense introduced by Zaanen in [13]. We recall that an operator T on a C*-algebra A is said to be an *orthomorphism* or a *band preserving* operator when the implication $a \perp b \Rightarrow T(a) \perp b$ holds for every $a, b \in A$. We notice that when A is regarded as an A-bimodule, an operator $T : A \to A$ is an orthomorphism if and only if it is a *local operator* in the sense used by Johnson in [14, Section 3]. Clearly, an operator $T : A \to A$ is an orthomorphism if and only if (T, Id_A) is orthogonality preserving. The following noncommutative extension of [13, Theorem 5] follows from Proposition 9.

Corollary 11. *Let T be an operator on a C*-algebra A. Then T is an orthomorphism if and only if $T(a) = T^{**}(1)a = aT^{**}(1)$, for every a in A; that is, T is a multiple of the identity on A by an element in its center.*

We recall that two elements a, and b in a JB*-algebra A are said to *operator commute* in A if the multiplication operators M_a and M_b commute, where M_a is defined by $M_a(x) := a \circ x$. That is, a and b operator commute if and only if $(a \circ x) \circ b = a \circ (x \circ b)$ for all x in A. A useful result in Jordan theory assures that self-adjoint elements a and b in A generate a JB*-subalgebra that can be realized as a JC*-subalgebra of some $B(H)$ (compare [20]) and, under this identification, a and b commute as elements in $L(H)$ whenever they operator commute in A, equivalently, $a^2 \circ b = 2(a \circ b) \circ a - a^2 \circ b$ (cf. Proposition 1 in [21]).

The next lemma contains a property which is probably known in C*-algebra, we include an sketch of the proof because we were unable to find an explicit reference.

Lemma 12. *Let e be a partial isometry in a C*-algebra A and let a, and b be two elements in $A_2(e) = ee^*Ae^*e$. Then a, b operator commute in the JB*-algebra $(A_2(e), \bullet_e, \sharp_e)$ if and only if ae^* and be^* operator commute in the JB*-algebra $(A_2(ee^*), \bullet_{ee^*}, \sharp_{ee^*})$, where $x \bullet_{ee^*} y = x \circ y = (1/2)(xy + yx)$, for every $x, y \in A_2(ee^*)$. Furthermore, when a and b are hermitian elements in $(A_2(e), \bullet_e, \sharp_e)$, a, and b operator commute if and only if ae^* and be^* commute in the usual sense (i.e., $ae^*be^* = be^*ae^*$).*

Proof. We observe that the mapping $R_{e^*} : (A_2(e), \bullet_e) \to (A_2(ee^*), \bullet_{ee^*}), x \mapsto xe^*$, is a Jordan *-isomorphism between the above JB*-algebras. So, the first equivalence is clear. The second one has been commented before. □

Our next corollary relies on the following description of orthogonality preserving operators between C*-algebras obtained in [12] (see also [6]).

Theorem 13 (see [12, Theorem 17], [6, Theorem 4.1 and Corollary 4.2]). *If T is an operator from a C*-algebra A into another C*-algebra B the following are equivalent.*

(a) *T is orthogonality preserving (on A_{sa}).*

(b) *There exists a unital Jordan *-homomorphism $J : M(A) \to B_2^{**}(r(h))$ such that $J(x)$ and $h = T^{**}(1)$ operator commute and*

$$T(x) = h \bullet_{r(h)} J(x), \quad \text{for every } x \in A, \quad (25)$$

where $M(A)$ is the multiplier algebra of A, $r(h)$ is the range partial isometry of h in B^{**}, $B_2^{**}(r(h)) = r(h)r(h)^*B^{**}r(h)^*r(h)$, and $\bullet_{r(h)}$ is the natural product making $B_2^{**}(r(h))$ a JB*-algebra.

Furthermore, when T is symmetric, h is hermitian and hence $r(h)$ decomposes as orthogonal sum of two projections in B^{**}.

Our next result gives a new perspective for the study of orthogonality preserving (pairs of) operators between C*-algebras.

Proposition 14. *Let A and B be C*-algebras. Let $S, T : A \to B$ be operators and let $h = S^{**}(1)$ and $k = T^{**}(1)$. Then the following statements hold.*

(a) *The operator S is orthogonality preserving if and only if there exist two Jordan *-homomorphisms $\Phi, \widetilde{\Phi} : M(A) \to B^{**}$ satisfying $\Phi(1) = r(h)r(h)^*$, $\widetilde{\Phi}(1) = r(h)^*r(h)$, and $S(a) = \Phi(a)h = h\widetilde{\Phi}(a)$, for every $a \in A$.*

(b) *S, T and (S, T) are orthogonality preserving on A_{sa} if and only if the following statements hold.*

(b1) *There exist Jordan *-homomorphisms $\Phi_1, \widetilde{\Phi}_1, \Phi_2, \widetilde{\Phi}_2 : M(A) \to B^{**}$ satisfying $\Phi_1(1) = r(h)r(h)^*$, $\widetilde{\Phi}_1(1) = r(h)^*r(h)$, $\Phi_2(1) = r(k)$*

$r(k)^*, \widetilde{\Phi}_2(1) = r(k)^* r(k), S(a) = \Phi_1(a)h = h\widetilde{\Phi}_1(a)$, *and* $T(a) = \Phi_2(a)k = k\widetilde{\Phi}_2(a)$, *for every* $a \in A$.

(b2) *The pairs* (Φ_1, Φ_2) *and* $(\widetilde{\Phi}_1, \widetilde{\Phi}_2)$ *are orthogonality preserving on* A_{sa}.

Proof. The "if" implications are clear in both statements. We will only prove the "only if" implication.

(a) By Theorem 13, there exists a unital Jordan *-homomorphism $J_1 : M(A) \to B_2^{**}(r(h))$ such that $J_1(x)$ and h operator commute in the JB*-algebra $(B_2^{**}(r(h)), \bullet_{r(h)})$ and

$$S(x) = h \bullet_{r(a)} J_1(a) \quad \text{for every } a \in A. \tag{26}$$

Fix $a \in A_{sa}$. Since h and $J_1(a)$ are hermitian elements in $(B_2^{**}(r(h)), \bullet_{r(h)})$ which operator commute, Lemma 12 assures that $hr(h)^*$ and $J_1(a)r(h)^*$ commute in the usual sense of B^{**}; that is,

$$hr(h)^* J_1(a) r(h)^* = J_1(a) r(h)^* hr(h)^*, \tag{27}$$

or equivalently,

$$hr(h)^* J_1(a) = J_1(a) r(h)^* h. \tag{28}$$

Consequently, we have

$$S(a) = h \bullet_{r(h)} J_1(a) = hr(h)^* J_1(a) = J_1(a) r(h)^* h, \tag{29}$$

for every $a \in A$. The desired statement follows by considering $\Phi_1(a) = J_1(a)r(h)^*$ and $\widetilde{\Phi}_1(a) = r(h)^* J_1(a)$.

(b) The statement in (b1) follows from (a). We will prove (b2). By hypothesis, given a, b in A_{sa} with $a \perp b$, we have

$$\begin{aligned} 0 = S(a) T(b)^* &= \left(h\widetilde{\Phi}_1(a)\right)\left(k\widetilde{\Phi}_2(b)\right)^* \\ &= h\widetilde{\Phi}_1(a)\widetilde{\Phi}_2(b)^* k^*. \end{aligned} \tag{30}$$

Having in mind that $\widetilde{\Phi}_1(A) \subseteq r(h)^* r(h) B^{**}$ and $\widetilde{\Phi}_2(A) \subseteq B^{**} r(k)^* r(k)$, we deduce that $\widetilde{\Phi}_1(a)\widetilde{\Phi}_2(b)^* = 0$ (compare the comments before Lemma 8), as we desired. In a similar fashion we prove $\widetilde{\Phi}_2(b)^*\widetilde{\Phi}_1(a) = 0$, $\Phi_2(b)^*\Phi_1(a) = 0 = \Phi_1(a)\Phi_2(b)^*$.

□

4. Holomorphic Mappings Valued in a Commutative C*-Algebra

The particular setting in which a holomorphic function is valued in a commutative C*-algebra B provides enough advantages to establish a full description of the orthogonally additive, orthogonality preserving, and holomorphic mappings which are valued in B.

Proposition 15. *Let* $S, T : A \to B$ *be operators between* C*-*algebras with* B *commutative. Suppose that* S, T *and* (S, T) *are orthogonality preserving, and let us denote* $h = S^{**}(1)$ *and* $k = T^{**}(1)$. *Then there exists a Jordan* *-*homomorphism* $\Phi : M(A) \to B^{**}$ *satisfying* $\Phi(1) = r(|h|+|k|)$, $S(a) = \Phi(a)h$, *and* $T(a) = \Phi(a)k$, *for every* $a \in A$.

Proof. Let $\Phi_1, \widetilde{\Phi}_1, \Phi_2, \widetilde{\Phi}_2 : M(A) \to B^{**}$ be the Jordan *-homomorphisms satisfying (b1) and (b2) in Proposition 14. By hypothesis, B is commutative, and hence $\Phi_i = \widetilde{\Phi}_i$ for every $i = 1, 2$ (compare the proof of Proposition 14). Since the pair (Φ_1, Φ_2) is orthogonality preserving on A_{sa}, Lemma 10 assures that

$$\Phi_1^{**}(1)\Phi_2(a) = \Phi_1(a)\Phi_2^{**}(1), \tag{31}$$

for every $a \in A_{sa}$. In order to simplify notation, let us denote $p = \Phi_1^{**}(1)$ and $q = \Phi_2^{**}(1)$.

We define an operator $\Phi : M(A) \to B^{**}$, given by

$$\Phi(a) = pq\Phi_1(a) + p(1-q)\Phi_1(a) + q(1-p)\Phi_2(a). \tag{32}$$

Since $p\Phi_2(a) = \Phi_1(a)q$, it can be easily checked that Φ is a Jordan *-homomorphism such that $S(a) = \Phi(a)h$ and $T(a) = \Phi(a)k$, for every $a \in A$. □

Theorem 16. *Let* $f : B_A(0, \varrho) \to B$ *be a holomorphic mapping, where* A *and* B *are* C*-*algebras with* B *commutative and let* $f = \sum_{k=0}^{\infty} P_k$ *be its Taylor series at zero, which is uniformly converging in* $U = B_A(0, \delta)$. *Suppose* f *is orthogonality preserving and orthogonally additive on* $A_{sa} \cap U$ *(equivalently, orthogonally additive on* $A_{sa} \cap U$ *and zero products preserving). Then there exist a sequence* (h_n) *in* B^{**} *and a Jordan* *-*homomorphism* $\Phi : M(A) \to B^{**}$ *such that*

$$f(x) = \sum_{n=1}^{\infty} h_n \Phi(a^n) = \sum_{n=1}^{\infty} h_n \Phi(a^n), \tag{33}$$

uniformly in $a \in U$.

Proof. By Corollary 7, there exists a sequence (T_n) of operators from A into B satisfying that the pair (T_n, T_m) is orthogonality preserving on A_{sa} (equivalently, zero products preserving on A_{sa}) for every $n, m \in \mathbb{N}$ and

$$f(x) = \sum_{n=1}^{\infty} T_n(x^n), \tag{34}$$

uniformly in $x \in U$. Denote $h_n = T_n^{**}(1)$.

We will prove now the existence of the Jordan *-homomorphism Φ. We prove, by induction, that for each natural n, there exists a Jordan *-homomorphism $\Psi_n : M(A) \to B^{**}$ such that $r(\Psi_n(1)) = r(|h_1| + \cdots + |h_n|)$ and $T_k(a) = h_k \Psi_n(a)$ for every $k \leq n$, $a \in A$. The statement for $n = 1$ follows from Corollary 7 and Proposition 14. Let us assume that our statement is true for n. Since for every k, m in $\mathbb{N}$, T_k, T_m, and the pair (T_k, T_m) are orthogonality preserving, we can easily check that $T_{n+1}, T_1 + \cdots + T_n$ and $(T_{n+1}, T_1 + \cdots + T_n) = (T_{n+1}, (h_1 + \cdots + h_n)\Psi_n)$ are orthogonality preserving.

By Proposition 15, there exists a Jordan *-homomorphism $\Psi_{n+1} : M(A) \to B^{**}$ satisfying $r(\Psi_{n+1}(1)) = r(|h_1| + \cdots + |h_n| + |h_{n+1}|)$, $T_{n+1}(a) = h_{n+1}\Psi_{n+1}(a^{n+1})$ and $(T_1 + \cdots + T_n)(a) = (h_1 + \cdots + h_n)\Psi_{n+1}(a)$ for every $a \in A$. Since, for each $1 \leq k \leq n$,

$$\begin{aligned} h_k\Psi_{n+1}(a) &= h_k r(|h_1| + \cdots + |h_n| + |h_{n+1}|)\Psi_{n+1}(a) \\ &= h_k r(|h_1| + \cdots + |h_n|)\Psi_{n+1}(a) \\ &= h_k r(|h_1| + \cdots + |h_n|)\Psi_n(a) = h_k\Psi_n(a) = T_k(a), \end{aligned} \tag{35}$$

for every $a \in A$, as desired.

Let us consider a free ultrafilter $\mathscr{U}$ on $\mathbb{N}$. By the Banach-Alaoglu theorem, any bounded set in B^{**} is relatively weak*-compact, and thus the assignment $a \mapsto \Phi(a) := w^* - \lim_{\mathscr{U}}\Psi_n(a)$ defines a Jordan *-homomorphism from $M(A)$ into B^{**}. If we fix a natural k, we know that $T_k(a) = h_k\Psi_n(a)$, for every $n \geq k$ and $a \in A$. Then it can be easily checked that $T_k(a) = h_k\Phi(a)$, for every $a \in A$, which concludes the proof. □

The Banach-Stone type theorem for orthogonally additive, orthogonality preserving, and holomorphic mappings between commutative C*-algebras, established in Theorem 2 (see [11, Theorem 3.4]), is a direct consequence of our previous result.

5. Banach-Stone Type Theorems for Holomorphic Mappings between General C*-Algebras

In this section we deal with holomorphic functions between general C*-algebras. In this more general setting we will require additional hypothesis to establish a result in the line of the above Theorem 16.

Given a unital C*-algebra A, the symbol inv(A) will denote the set of invertible elements in A. The next lemma is a technical tool which is needed later. The proof is left to the reader and follows easily from the fact that inv(A) is an open subset of A.

Lemma 17. *Let $f : B_A(0,\varrho) \to B$ be a holomorphic mapping, where A and B are C*-algebras with B unital and let $f = \sum_{k=0}^{\infty} P_k$ be its Taylor series at zero, which is uniformly converging in $U = B_A(0,\delta)$. Let us assume that there exists $a_0 \in U$ with $f(a_0) \in inv(B)$. Then there exists $m_0 \in \mathbb{N}$ such that $\sum_{k=0}^{m_0} P_k(a_0) \in inv(B)$.*

We can now state a description of those orthogonally additive, orthogonality preserving, and holomorphic mappings between C*-algebras whose image contains an invertible element.

Theorem 18. *Let $f : B_A(0,\varrho) \to B$ be a holomorphic mapping, where A and B are C*-algebras with B unital and let $f = \sum_{k=0}^{\infty} P_k$ be its Taylor series at zero, which is uniformly converging in $U = B_A(0,\delta)$. Suppose f is orthogonality preserving and orthogonally additive on $A_{sa} \cap U$ and $f(U) \cap inv(B) \neq \emptyset$. Then there exist a sequence (h_n) in B^{**} and Jordan *-homomorphisms $\Theta, \widetilde{\Theta} : M(A) \to B^{**}$ such that*

$$f(a) = \sum_{n=1}^{\infty} h_n \widetilde{\Theta}(a^n) = \sum_{n=1}^{\infty} \Theta(a^n) h_n, \tag{36}$$

uniformly in $a \in U$.

Proof. By Corollary 7 there exists a sequence (T_n) of operators from A into B satisfying that the pair (T_n, T_m) is orthogonality preserving on A_{sa} for every $n, m \in \mathbb{N}$ and

$$f(x) = \sum_{n=1}^{\infty} T_n(x^n), \tag{37}$$

uniformly in $x \in U$.

Now, Proposition 14 (a), applied to T_n $(n \in \mathbb{N})$, implies the existence of sequences (Φ_n) and $(\widetilde{\Phi}_n)$ of Jordan *-homomorphisms from $M(A)$ into B^{**} satisfying $\Phi_n(1) = r(h_n)r(h_n)^*$, $\widetilde{\Phi}_n(1) = r(h_n)^* r(h_n)$, where $h_n = T_n^{**}(1)$, and

$$T_n(a) = \Phi_n(a) h_n = h_n \widetilde{\Phi}_n(a), \tag{38}$$

for every $a \in A$, $n \in \mathbb{N}$. Moreover, from Proposition 14 (b), the pairs (Φ_n, Φ_m) and $(\widetilde{\Phi}_n, \widetilde{\Phi}_m)$ are orthogonality preserving on A_{sa}, for every $n, m \in \mathbb{N}$.

Since $f(U) \cap \text{inv}(B) \neq \emptyset$, it follows from Lemma 17 that there exists a natural m_0 and $a_0 \in A$ such that

$$\sum_{k=1}^{m_0} P_k(a_0) = \sum_{k=1}^{m_0} \Phi_k(a_0^k) h_k = \sum_{k=1}^{m_0} h_k \widetilde{\Phi}_k(a_0^k) \in \text{inv}(B). \tag{39}$$

We claim that $r(r(h_1)^* r(h_1) + \cdots + r(h_{m_0})^* r(h_{m_0})) = 1$ in B^{**}. Otherwise, we find a nonzero projection $q \in B^{**}$ satisfying

$$r\left(r(h_1)^* r(h_1) + \cdots + r(h_{m_0})^* r(h_{m_0})\right) q = 0. \tag{40}$$

Since $r(h_i)^* r(h_i) \leq r(r(h_1)^* r(h_1) + \cdots + r(h_{m_0})^* r(h_{m_0}))$, this would imply that

$$\left(\sum_{k=1}^{m_0} P_k(a_0)\right) q = \left(\sum_{k=1}^{m_0} \Phi_k(a_0^k) h_k\right) q = 0, \tag{41}$$

contradicting that $\sum_{k=1}^{m_0} P_k(a_0) = \sum_{k=1}^{m_0} \Phi_k(a_0^k) h_k$ is invertible in B.

Consider now the mapping $\Psi = \sum_{k=1}^{m_0} \widetilde{\Phi}_k$. It is clear that, for each natural n, Ψ, $\widetilde{\Phi}_n$, and the pair $(\Psi, \widetilde{\Phi}_n)$ are orthogonality preserving. Applying Proposition 14 (b), we deduce the existence of Jordan *-homomorphisms $\Theta, \widetilde{\Theta}, \Theta_n, \widetilde{\Theta}_n : M(A) \to B^{**}$ such that (Θ, Θ_n) and $(\widetilde{\Theta}, \widetilde{\Theta}_n)$ are orthogonality preserving, $\Theta(1) = r(k)r(k)^*$, $\widetilde{\Theta}(1) = r(k)^* r(k)$, $\Theta_n(1) = r(h_n)r(h_n)^*$, $\widetilde{\Theta}_n(1) = r(h_n)^* r(h_n)$,

$$\begin{gathered} \Psi(a) = \Theta(a) k = k\widetilde{\Theta}(a), \\ \widetilde{\Phi}_n(a) = \Theta_n(a) r(h_n)^* r(h_n) = r(h_n)^* r(h_n) \widetilde{\Theta}_n(a), \end{gathered} \tag{42}$$

for every $a \in A$, where $k = \Psi(1) = r(h_1)^* r(h_1) + \cdots + r(h_{m_0})^* r(h_{m_0})$. The condition $r(k) = 1$, proved in the previous paragraph, shows that $\Theta(1) = 1$. Thus, since $(\widetilde{\Theta}, \widetilde{\Theta}_n)$ is orthogonality preserving, the last statement in Lemma 10 proves that

$$\widetilde{\Theta}_n(a) = \widetilde{\Theta}_n(1)\widetilde{\Theta}(a) = \widetilde{\Theta}(a)\widetilde{\Theta}_n(1), \quad (43)$$

for every $a \in A$, $n \in \mathbb{N}$. The above identities guarantee that

$$\widetilde{\Phi}_n(a) = \Theta(a) r(h_n)^* r(h_n) = r(h_n)^* r(h_n) \widetilde{\Theta}(a), \quad (44)$$

for every $a \in A$, $n \in \mathbb{N}$.

A similar argument to the one given above, but replacing $\widetilde{\Phi}_k$ with Φ_k, shows the existence of a Jordan *-homomorphism $\Theta : M(A) \to B^{**}$ such that

$$\Phi_n(a) = \Theta(a) r(h_n) r(h_n)^* = r(h_n) r(h_n)^* \Theta(a), \quad (45)$$

for every $a \in A$, $n \in \mathbb{N}$, which concludes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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