A generalized and unified approach to the approximation of fuzzy numbers and its arithmetic and characteristics

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Abstract

In this paper we propose a general method for the approximation of an arbitrary fuzzy number. This method, which is constructive, recovers and properly extends some well-known approximations such as those obtained in terms of polygonal fuzzy numbers or simple fuzzy numbers. We prove the convergence of the general method and study the properties of the approximation operator, such as its compatibility with arithmetic operations of fuzzy numbers and with some of their important characteristics. In addition to this, we illustrate the method with some particularly interesting cases by providing algorithms, of great simplicity for practical use and apply them to some numerical examples. Furthermore, the approximations we construct are particularly simple from the point of view of fuzzy arithmetic and preserve some of their most important characteristics.

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1 Introduction

The study of fuzzy sets and in particular of the subset of fuzzy numbers has aroused enormous interest in scientific literature in recent decades due to the power that these numbers have to model uncertainty situations in many different fields of research.

The difficulty of extending real models to models with uncertainty using fuzzy numbers, as well as the complexity of fuzzy number calculations, has led to the development of a large number of techniques to approximate arbitrary fuzzy numbers by means of simpler ones.

Fuzzy number approximation has been approached over the years from several perspectives, depending on the intended use of the approximation.

The introductions of papers [11] and [28] provide a good review of the state of the art. Several papers focus on the approximation of fuzzy numbers by means of simpler ones such as intervals or triangular, trapezoidal, hexagonal... fuzzy numbers, so that both their description and the operations between them are reduced to a finite number of parameters (see [3, 4, 7, 10, 19, 20, 35]). These simplifications, operational from the point of view of fuzzy number computation, present the problem of the loss of information involved in the proposed approximation. To solve this problem, different modifications of the aforementioned methods have been proposed, preserving some important characteristics of fuzzy numbers. We can mention, without being overly exhaustive, the following: [4, 5, 9, 11, 12, 20, 21, 22, 23, 33, 34, 36].

Some other authors propose approximations by means of polynomial interpolation techniques or splines, attempting to ensure that the interpolant maintains different properties of shape when combined with the operations of the fuzzy numbers (see [8, 24, 25, 32]).

In most of these papers, the initial idea is to fix a finite number of α-levels and obtain the best approximation, for a suitable metric, in the set of fuzzy numbers of a certain type associated with these α-levels, sometimes preserving some of the characteristics, such as the core, the expected interval, etc. In this paper, however, we remove the initial restriction of fixing a priori the starting α-levels. In addition, we construct a sequence of approximating projections which, when applied to a fuzzy number, converges to it for a suitable family of metrics, which includes those classically used. The sequence of approximations thus generated for a fuzzy number is compatible with fuzzy arithmetic as well as with the convergence of the main characteristics associated with that metric. In this paper, we deal with several aspects of fuzzy numbers, all of them related to the approximation of fuzzy numbers by means of computationally easy-to-handle fuzzy numbers, described by simple algorithms and encompassing the two previous perspectives.

These ideas, which constitute the contributions of this paper, are developed in more detail below.

First, we introduce a family of metrics $d_X^{(\cdot)}$ in the set $\mathbb{F}_X$ of fuzzy numbers u such that its lower and upper branches $\underline{u}$ and $\overline{u}$ are in a space X of real functions defined on $[0, 1]$. Such a family encompasses directly, or except equivalences, to the most usual ones, such as the fuzzy Hausdorff distance ([26]) or the Euclidean distance (see, in essence, [18]). In particular, all the approximation results we obtain for this family of distances apply to all the usual ones.

Next, we focus on obtaining the fundamental result, Theorem 4.3, in which we approximate the fuzzy numbers in a metric space $(\mathbb{F}_X, d_X^{(\cdot)})$ by means of others in that space that are simpler, using as a fundamental tool Schauder bases. These bases have been successfully used in another uncertainty context (see [2]). More precisely, given a fuzzy number $u \in \mathbb{F}$ we construct a sequence of simple fuzzy numbers that always converges to u. Our approach avoids the possible ill-conditioned problems of some of the best approximation methods that start from a closed convex subset. Moreover, it allows us to approximate any fuzzy number, and establish convergence statements for all types of fuzzy numbers, unlike the results obtained in the above works.

For example, when $X = L_p[0,1]$, then $\mathbb{F}_X$ coincides with $\mathbb{F}$ and we are able to approximate any fuzzy number by means of another simple fuzzy number, in the sense that it is obtained as a limit of a sequence of simple fuzzy numbers. Moreover, we give an explicit algorithm for the approximation of an arbitrary fuzzy number by means of an arbitrary simple fuzzy number, which is straightforward and for which no additional computation is required, and in which the passage from an n^{th} approximation to the $(n+1)^{th}$ approximation is done by simply adding an additional term consisting of a step fuzzy number, unlike in other works, in which the computation has to be redone.

Another application of Theorem 4.3 is given for the set $\mathbb{F}_{C[0,1]}$ endowed with the fuzzy Hausdorff distance, in which a fuzzy number u with continuous branches, is obtained as the limit of a sequence of very simple fuzzy numbers, the so–called polygonal ones (see [3]). The corresponding algorithm has the same characteristics as those for arbitrary fuzzy numbers.

We should also note, with respect to the above algorithms for $\mathbb{F}$ in general and for $\mathbb{F}_{C[0,1]}$, that they are compatible with the arithmetic of fuzzy numbers –in particular, with the operations of addition, product by scalars and generalised Hukuhara difference– in the sense that the approximation of the operation coincides with the operation of the approximation. Moreover, we are able to control the distance of a particular operation to its approximation as a function of the elements of the operation and its approximation.

Finally, we generalise the convergence results of the approximations obtained in regard to the usual parameters of a fuzzy number, such as value, ambiguity, expected interval and expected value.

The paper is structured as follows. In Section 2 we compile the fundamental concepts and results related to the fuzzy numbers that we use. Section 3 focuses on introducing a family of metrics in the set of fuzzy numbers including, among others, the classical Haussdorf distance or the Euclidean distance. Section 4 is devoted to describing the proposed approximation method, including some numerical examples. Moreover, we study the properties of the method in Section 5 and end with some conclusions in Section 6.

2 Basic fuzzy concepts

We start by recalling the classical concept of a fuzzy number. We should first mention that, in relation to what follows, given a subset A of a topological space, we write $\mathrm{cl}\,(A)$ for its closure. In addition, we consider in $\mathbb{R}$ only its usual topology. A *fuzzy number* (see, *e.g.*, [6]) is a mapping $u : \mathbb{R} \longrightarrow [0,1]$ that is

i) *normal*, *i.e.*, there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,

ii) upper semi-continuous,

iii) *fuzzy-convex*, that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$, $x, y \in \mathbb{R}$, $\lambda \in [0,1]$,

iv) and compactly supported, in the sense that $\mathrm{cl}\,\{x \in \mathbb{R} : \ u(x) > 0\}$ is compact.

We denote the set of all fuzzy numbers by $\mathbb{F}$.

Note that fuzzy-convexity is nothing more than quasi-concavity, a basic notion in other contexts such as convex analysis.

We recall an important type of fuzzy number that illustrates the above definition. A fuzzy number $u \in \mathbb{F}$ is said to be a *simple fuzzy number* ([35]) provided that there exist $m, n \geq 2$, $r_1, \ldots, r_{m-1}, s_1, \ldots, s_{n-1} \in (0,1)$ and $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{R}$ with

$$r_1 < \cdots < r_{m-1}, \quad s_1 < \cdots < s_{n-1} \quad \text{and} \quad a_1 < a_m \leq b_n < \cdots < b_1,$$

and in such a way that

$$u(x) = \begin{cases} r_1, & \text{if } x \in [a_1, a_2) \\ r_2, & \text{if } x \in [a_2, a_3) \\ \vdots & \vdots \\ r_{m-1}, & \text{if } x \in [a_{m-1}, a_m) \\ 1, & \text{if } x \in [a_m, b_n] \\ s_{n-1}, & \text{if } x \in (b_n, b_{n-1}] \\ \vdots & \vdots \\ s_2, & \text{if } x \in (b_3, b_2] \\ s_1, & \text{if } x \in (b_2, b_1] \\ 0, & \text{if } x \notin [a_1, b_1] \end{cases}.$$

A related fuzzy number concept is now given: For a fuzzy number u and a real $0 \leq \alpha \leq 1$, the *α-level set* of u (see, *e.g.*, [6]) is

$$[u]^\alpha := \{x \in \mathbb{R} : \ u(x) \geq \alpha\}$$

whenever $0 < \alpha \leq 1$, while

$$[u]^0 := \mathrm{cl}\,\{x \in \mathbb{R} : \; u(x) > 0\}.$$

The 1-level set $[u]^1 = \{x \in \mathbb{R} : u(x) \geq 1\}$ is called the *core* of u.

The versatility of the level sets is given by this result which, in essence, guarantees that a fuzzy number is uniquely determined by its level sets:

Theorem 2.1 ([29]) *Let $u \in \mathbb{F}$ and for each $0 \leq \alpha \leq 1$, let $[u]^\alpha$ be its α-level set. Then*

i) *for any $\alpha \in [0,1]$, $[u]^\alpha$ is a closed interval of $\mathbb{R}$, $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$,*

ii) *$[u]^{\alpha_2} \subset [u]^{\alpha_1}$ provided that $0 \leq \alpha_1 \leq \alpha_2 \leq 1$,*

iii) *for each sequence $\{\alpha_n\}_{n\geq 1}$ in $[0,1]$ that converges from below to $\alpha \in (0,1]$ we have that*

$$\bigcap_{n=1}^{\infty} [u]^{\alpha_n} = [u]^\alpha,$$

iv) *and for any sequence $\{\alpha_n\}_{n\geq 1}$ in $[0,1]$ that converges from above to 0 there holds*

$$\mathrm{cl}\left(\bigcup_{n=1}^{\infty} [u]^{\alpha_n}\right) = [u]^0.$$

And conversely, given a family $\{[u]^\alpha : \; \alpha \in [0,1]\}$ of subsets of $\mathbb{R}$ fulfilling conditions i) *to* iv)*, there exists a unique $u \in \mathbb{F}$ such that for any $\alpha \in [0,1]$, $[u]^\alpha$ is its α-level set.*

A typical example of the use of the above result is the following notion: For an $m \geq 2$ and a partition $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_m = 1$ of the interval $[0,1]$, a fuzzy number $u \in \mathbb{F}$ is a *polygonal fuzzy number* associated with such a partition ([3]) provided that

$$i = 1,\ldots,m, \;\; \alpha_i < \alpha \leq \alpha_{i+1} \;\Rightarrow\; [u]^\alpha = \left(1 - \frac{\alpha - \alpha_i}{\alpha_{i+1} - \alpha_i}\right)[u]^{\alpha_i} + \frac{\alpha - \alpha_i}{\alpha_{i+1} - \alpha_i}[u]^{\alpha_{i+1}}.$$

This concept of *polygonal fuzzy numbers* includes to that of *trapezoidal fuzzy numbers* and thus that of *triangular fuzzy numbers* ([6]).

The relationship between fuzzy numbers and the intervals established in Theorem 2.1 allows us to offer another representation of a fuzzy number as a pair of functions with some suitable properties.

Theorem 2.2 ([17]) *Let u be a fuzzy number with level sets $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$, $0 \leq \alpha \leq 1$. Then, the functions $\underline{u}, \overline{u} : [0,1] \to \mathbb{R}$, defined at each $0 \leq \alpha \leq 1$ as the endpoints of the α-level set $[u]^\alpha$,*

$$\underline{u}(\alpha) := \underline{u}_\alpha, \quad \textit{and} \quad \overline{u}(\alpha) := \overline{u}_\alpha,$$

satisfy the following properties::

i) *$\underline{u}$ is bounded, non-decreasing, left-continuous in $(0,1]$ and right-continuous at 0,*

ii) *$\overline{u}$ is a bounded, non-increasing, left-continuous in $(0,1]$ and right-continuous at 0,*

iii) *and $\underline{u}(1) \leq \overline{u}(1)$.*

And conversely, given two functions $\underline{u}, \overline{u} : [0,1] \to \mathbb{R}$ that satisfy the above conditions i) *to* iii)*, there is a unique fuzzy number $u \in \mathbb{F}$ with $\underline{u}$ and $\overline{u}$ as its endpoints of its α-level sets $[u]^\alpha$. Moreover, u is explicitly determined at each $x \in [0,1]$ by the expression*

$$u(x) = \begin{cases} 0, & \text{if } x \notin [u]^0 \\ \sup\{\alpha \in [0,1] : x \in [u]^\alpha\}, & \text{if } x \in [u]^0 \end{cases}.$$

The representation of a fuzzy number u in terms of the functions $\underline{u}$ and $\overline{u}$ is called the *LU representation* and we refer to $\underline{u}$ and $\overline{u}$ as the *lower* and *upper branches* of u, respectively.

For instance, an equivalent definition of a polygonal fuzzy number in terms of lower and upper branches can be found in [11].

Now we deal with a different issue: we collect the most important indices related to a fuzzy number, *i.e.*, those real numbers that capture some information contained in a fuzzy number in order to simplify the task of representing and handling it (see [13, 14]). Let $u \in \mathbb{F}$:

i) If $c : [0,1] \longrightarrow [0,1]$ is a *reducing function*, that is, a non-decreasing function with $c(0) = 0$ and $c(1) = 1$, then the *ambiguity* of u related to c is given by

$$\mathrm{Amb}_c(u) := \int_0^1 c(\alpha)(\underline{u}(\alpha) - \overline{u}(\alpha))\, d\alpha,$$

while the *value* of u with respect to c is the real number

$$\mathrm{Val}_c(u) := \int_0^1 c(\alpha)(\underline{u}(\alpha) + \overline{u}(\alpha))\, d\alpha.$$

When, for any $\alpha \in [0,1]$, $c(\alpha) = 1$, we simply write Amb and Val instead of Amb_c and Val_c, respectively.

ii) The *expected interval* of the fuzzy number u is the compact interval

$$\mathrm{EI}(u) := \left[\int_0^1 \underline{u}(\alpha)\, d\alpha, \int_0^1 \overline{u}(\alpha)\, d\alpha \right],$$

and the *expected value* of u is the middle of the previous interval, *i.e.*,

$$\mathrm{EV}(u) := \frac{1}{2} \int_0^1 (\underline{u}(\alpha) + \overline{u}(\alpha))\, d\alpha.$$

Thus, the ambiguity of $u \in \mathbb{F}$ can be seen as a measure of its vagueness, its value as a characteristic value of u, its expected interval as an interval containing other significant value of u, its integral, and finally, the expected value of u as a representative value of such an interval.

We conclude this section by presenting some elementary aspects of the arithmetic of fuzzy numbers, which essentially reduces them to that of real compact intervals (see, for instance, [6]). So we start with the latter: interval addition, scalar-interval multiplication and (generalized Hukuhara) interval difference. Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be two real compact intervals and $\lambda \in \mathbb{R}$. Let us recall that the *addition* of A and B, denoted by $A + B$, is the interval

$$A + B := [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$$

and the *scalar multiplication* of λ and A is defined as

$$\lambda A := \{\lambda a : \ a \in A\} = [\min\{\lambda \underline{a}, \lambda \overline{a}\}, \max\{\lambda \underline{a}, \lambda \overline{a}\}].$$

With regard to the interval subtraction, there are several definitions in the literature. One of the most popular is the *generalized Hukuhara difference* (gH-difference, for short), see [31]. The gH-difference of A and B works like this:

$$A \ominus_{gH} B = C \Longleftrightarrow \begin{cases} & (a)\ A = B + C, \\ or & (b)\ B = A + (-1)C. \end{cases}$$

Clearly, $A \ominus_{gH} A = [0, 0]$ and furthermore, the gH-difference of two intervals always exists and

$$A \ominus_{gH} B = [\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}].$$

We recall the definition of the addition, the scalar-fuzzy number multiplication and the gH-difference of fuzzy numbers. As previously mentioned, it is a matter of translating interval arithmetic by means of the interval representation of fuzzy numbers given by the level sets. Specifically, if $u, v \in \mathbb{F}$ and $\lambda \in \mathbb{R}$, then the addition of u and v, $u + v$, the scalar-fuzzy number multiplication of λ and u, λu, and, when it exists, the gH-difference of u and v, $u \ominus_{gH} v$, are defined as those fuzzy numbers whose α-level sets, for each $\alpha \in [0, 1]$, are, respectively,

$$[u + v]^\alpha := [u]^\alpha + [v]^\alpha = [\underline{u}_\alpha + \underline{v}_\alpha \ , \ \overline{u}_\alpha + \overline{v}_\alpha],$$

$$[\lambda u]^\alpha := \lambda[u]^\alpha = [\min\{\lambda\underline{u}_\alpha, \lambda\overline{u}_\alpha\}, \max\{\lambda\underline{u}_\alpha, \lambda\overline{u}_\alpha\}]$$

and

$$[u \ominus_{gH} v]^\alpha := [u]^\alpha \ominus_{gH} [v]^\alpha = [\min\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}, \max\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}] .$$

Let us recall that the gH-difference of two fuzzy numbers u, v, is the fuzzy number w, if it exists, such that

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} & (i)\ u = v + w, \\ \text{or} & (ii)\ v = u + (-1)w \end{cases} .$$

3 Metrics in the space of fuzzy numbers

Our purpose in this section is to provide a unified treatment of the usual metrics in the space of fuzzy numbers introducing suitable normed spaces. Moreover, we also provide a large family of metrics for $\mathbb{F}$ and even for some relevant subsets of it.

In the literature, several metrics are considered in the set $\mathbb{F}$ of fuzzy numbers. The best known and most commonly used metric on that set $\mathbb{F}$ is the Hausdorff distance (see, *e.g.* [26, 15]), which is derived from the classical Hausdorff–Pompeiu distance between compact and convex subsets of $\mathbb{R}^n$, in particular, for compact intervals $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$:

$$d_H(A, B) := \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}.$$

The *fuzzy Hausdorff* distance $D_\infty : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{R}$ is defined for each $u, v \in \mathbb{F}$ as

$$D_\infty(u, v) := \sup_{\alpha\in[0,1]} \max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{u}_\alpha|\}.$$

Thus, $D_\infty(u, v)$ is a uniform version of d_H when applied to the level sets of u and v:

$$D_\infty(u, v) = \sup_{\alpha\in[0,1]} \{d_H([u]^\alpha, [v]^\alpha)\}.$$

Other interesting metrics have been introduced in $\mathbb{F}$ for different purposes. We highlight, on the one hand, the Euclidean distance d_2 defined at each $u, v \in \mathbb{F}$ by

$$d_2(u, v) := \sqrt{\int_0^1 (\underline{u}(\alpha) - \underline{v}(\alpha))^2 d\alpha + \int_0^1 (\overline{u}(\alpha) - \overline{v}(\alpha))^2 d\alpha},$$

which has been used in [35] to show that the space of simple fuzzy numbers is dense in the space of fuzzy numbers with regard to that metric. Here we also consider the wider family for each $1 \le p < \infty$

$$d_p(u, v) := \left(\int_0^1 |\underline{u}(\alpha) - \underline{v}(\alpha)|^p d\alpha + \int_0^1 |\overline{u}(\alpha) - \overline{v}(\alpha)|^p d\alpha\right)^{1/p},$$

whenever $u, v \in \mathbb{F}$.

On the other hand, we also highlight the distance $\rho_p : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{R}$, $1 \leq p < \infty$, introduced in [18], where the author introduces a fuzzy number ranking method based on it:

$$\rho_p(u,v) := \max\left\{ \left(\int_0^1 |\underline{u}(\alpha) - \underline{v}(\alpha)|^p \, d\alpha \right)^{1/p}, \left(\int_0^1 |\overline{u}(\alpha) - \overline{v}(\alpha)|^p \, d\alpha \right)^{1/p} \right\}, \qquad (u, v \in \mathbb{F}).$$

Now we proceed to provide the above-mentioned unified treatment by introducing an appropriate normed space, because the constructive approximation method we will design is based on the use of suitable Schauder bases in certain Banach spaces. Thus, let $X \subset \mathbb{R}^{[0,1]}$ and let

$$\mathbb{F}_X := \{ u \in \mathbb{F} : \ \underline{u}, \overline{u} \in X \}.$$

For example, $\mathbb{F}_{C[0,1]}$ corresponds to the set of those fuzzy numbers $u \in \mathbb{F}$ with $\underline{u}, \overline{u} \in C[0,1]$. There is an important consideration to be taken into account regarding this set: the fact that a fuzzy number $u \in \mathbb{F}$ is continuous has nothing to do with the continuity of $\underline{u}$ and $\overline{u}$, that $u \in C(\mathbb{R})$ is independent of $u \in \mathbb{F}_{C[0,1]}$. Indeed, if $u \in \mathbb{F}_{C[0,1]}$ and $\underline{u}$ (or $\overline{u}$) is constant on a proper subinterval of $[0,1]$, then $u \notin C(\mathbb{R})$, and a similar argument works in the opposite direction.

It should also be noted that for any $u \in \mathbb{F}$ we have that $\underline{u}, \overline{u} \in L_\infty[0,1]$: $\underline{u}, \overline{u}$ are measurable because they are monotone, and furthermore, making use of monotonicity again, we have that

$$\min\{\underline{u}(0), \overline{u}(1)\} \leq \underline{u}, \overline{u} \leq \max\{\overline{u}(0), \underline{u}(1)\}.$$

Therefore,

$$\mathbb{F}_{L_\infty[0,1]} = \mathbb{F},$$

and since for any $1 \leq p \leq \infty$, $L_\infty[0,1] \subset L_p[0,1]$, then

$$\mathbb{F}_{L_p[0,1]} = \mathbb{F}.$$

We now consider specific X sets, real normed spaces of real-valued functions defined on $[0,1]$. In addition, let $\|\cdot\|^{(\cdot)}$ be a norm in $\mathbb{R}^2$ satisfying the monotonicity property

$$0 \leq x_1 \leq x_1', \ 0 \leq x_2 \leq x_2' \ \Rightarrow \ \|(x_1, x_2)\|^{(\cdot)} \leq \|(x_1', x_2')\|^{(\cdot)}. \tag{3.1}$$

Proposition 3.1 *Assume that X is a real normed space of real-valued functions defined on $[0,1]$, endowed with its norm $\|\cdot\|$, and that $\|\cdot\|^{(\cdot)}$ is a norm in $\mathbb{R}^2$ fulfilling the monotonicity condition* (3.1)*. Then the mapping $d_X^{(\cdot)} : \mathbb{F}_X \times \mathbb{F}_X \longrightarrow \mathbb{R}$ given by*

$$d_X^{(\cdot)}(u,v) := \| \left(\|\underline{u} - \underline{v}\|, \|\overline{u} - \overline{v}\| \right) \|^{(\cdot)}, \qquad (u, v \in \mathbb{F}_X), \tag{3.2}$$

defines a distance in the set $\mathbb{F}_X$.

Proof. The non-negativeness, non-degeneracy and symmetry of $d_X^{(\cdot)}$ are clearly satisfied, so it suffices to prove the triangle inequality, thus let us fix $u, v, w \in \mathbb{F}_X$. Then, in view of the triangle inequality for the norms $\|\cdot\|$ and $\|\cdot\|^{(\cdot)}$, the condition (3.1) and the definition (3.2), we arrive at

$$\begin{array}{rcl} d_X^{(\cdot)}(u,w) & = & \|\,(\|\underline{u}-\underline{w}\|, \|\overline{u}-\overline{w}\|)\,\|^{(\cdot)} \\ & \leq & \|\,(\|\underline{u}-\underline{v}\|+\|\underline{v}-\underline{w}\|, \|\overline{u}-\overline{v}\|+\|\overline{v}-\overline{w}\|)\,\|^{(\cdot)} \\ & = & \|\,(\|\underline{u}-\underline{v}\|, \|\overline{u}-\overline{v}\|)+(\|\underline{v}-\underline{w}\|, \|\overline{v}-\overline{w}\|)\,\|^{(\cdot)} \\ & \leq & \|\,(\|\underline{u}-\underline{v}\|, \|\overline{u}-\overline{v}\|)\,\|^{(\cdot)}+\|\,(\|\underline{v}-\underline{w}\|, \|\overline{v}-\overline{w}\|)\,\|^{(\cdot)} \\ & = & d_X^{(\cdot)}(u,v)+d_X^{(\cdot)}(v,w). \end{array}$$

□

The norms $\|\cdot\|^{(\cdot)}$ to be used in this paper will satisfy the condition (3.1). In particular, the following ones will be useful for our purposes: for $(x_1, x_2) \in \mathbb{R}^2$,

$$\|(x_1,x_2)\|^{(\infty)} := \max\{|x_1|, |x_2|\},$$

and if $1 \leq p < \infty$,

$$\|(x_1,x_2)\|^{(p)} := (x_1^p + x_2^p)^{\frac{1}{p}}.$$

Remark 3.2 The condition (3.1) is essential for $\left(\mathbb{F}_X, d_X^{(\cdot)}\right)$ to be a metric space and it is worth noting that not all norms of $\mathbb{R}^2$ verify it. For example, if "co" denotes "convex hull", let $C := \mathrm{co}\{\pm(2,2), \pm(0,1)\}$ and $\|\cdot\|^{(\cdot)}$ be its associated Minkowski functional, *i.e.*,

$$\|(x,y)\|^{(\cdot)} = \inf\{t > 0 : (x,y) \in tC\}, \qquad ((x,y) \in \mathbb{R}^2).$$

Then,

$$\|(2,2)\|^{(\cdot)} = 1 \quad \text{but} \quad \|(0,3/2)\|^{(\cdot)} = 3/2.$$

Regarding the notation in Proposition 3.1, wherein some confusion can be derived from the use of different norms in X, we can specify a concrete one $\|\cdot\|$ for the distance by writing $d_{(X,\|\cdot\|)}^{(\cdot)}$ instead of $d_X^{(\cdot)}$.

Hereinafter, the real normed spaces X of real-valued functions defined on $[0,1]$ will typically be the Banach space $C[0,1]$, endowed with its usual sup-norm $\|\cdot\|_\infty$, or the Banach space $L_p[0,1]$, $(1 \leq p \leq \infty)$, with its usual norm $\|\cdot\|_p$, or even the normed space $C[0,1]$ with the norm $\|\cdot\|_p$. In this last case, we denote the corresponding distance associated with a suitable norm $\|\cdot\|^{(\cdot)}$ in $\mathbb{R}^2$ by $d_{(C[0,1],\|\cdot\|_p)}^{(\cdot)}$.

The norms commonly used in the literature, and mentioned earlier in this section, belong to the family of distances $d_X^{(\cdot)}$ in $\mathbb{F}$ (recall that $\mathbb{F} = \mathbb{F}_{L_p[0,1]}$, for $1 \le p \le \infty$) defined in (3.2):

$$D_\infty = d^{(\infty)}_{L_\infty[0,1]}$$

and for $1 \le p < \infty$,

$$\rho_p = d^{(\infty)}_{L_p[0,1]} \quad \text{and} \quad d_p = d^{(p)}_{L_p[0,1]}.$$

Moreover, their inherit distances in $F_{C[0,1]}$ are, respectively,

$$D_\infty = d^{(\infty)}_{C[0,1]}$$

and if $1 \le p < \infty$,

$$\rho_p = d^{(\infty)}_{(C[0,1],\|\cdot\|_p)} \quad \text{and} \quad d_p = d^{(p)}_{(C[0,1],\|\cdot\|_p)}.$$

Remark 3.3 We note that for a given $1 \le p < \infty$, the distances $d^{(p)}_{L_p[0,1]}$ and $d^{(\infty)}_{L_p[0,1]}$ in $\mathbb{F}$ are equivalent, since any two norms in $\mathbb{R}^2$ are. On the other hand, the distance D_p, defined at each $u, v \in \mathbb{F}$ as

$$D_p(u,v) := \left(\int_0^1 d_H([u]^\alpha, [v]^\alpha)^p \, d\alpha\right)^{\frac{1}{p}} = \left(\int_0^1 \max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{u}_\alpha|\}^p \, d\alpha\right)^{\frac{1}{p}}$$

(see [6]), is not of the type $d_X^{(\cdot)}$. However, there holds

$$d^{(\infty)}_{L_p[0,1]}(u,v) \le D_p(u,v) \le 2^{\frac{1}{p}} d^{(\infty)}_{L_p[0,1]}(u,v).$$

Therefore, for a fixed $1 \le p < \infty$, the topological properties that we establish below in terms of $d^{(\infty)}_{L_p[0,1]}$ or $d^{(p)}_{L_p[0,1]}$, specifically, the convergence of certain sequences, are equally applicable to D_p.

Remark 3.4 Even if X is a Banach space, the metric space $\left(\mathbb{F}_X, d_X^{(\cdot)}\right)$ is not complete in general. For example, $(\mathbb{F}, d^{(\infty)}_{L_\infty[0,1]})$ is (see [15]), unlike $(\mathbb{F}, d^{(p)}_{L_p[0,1]})$ which is not (see [6]). Note that we are making use of equality $\mathbb{F}_{L_p[0,1]} = \mathbb{F}$.

4 Approximation of a fuzzy number

In this section we compile our main results of the approximation of fuzzy numbers. To do so, we recall some basic analytical concepts and results.

A sequence $\{s_n\}_{n\geq 1}$ in a real Banach space X is a *Schauder basis* provided that for any element x there exists a unique sequence of real numbers $\{\lambda_n\}_{n\geq 1}$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n s_n.$$

For each $n \in \mathbb{N}$, we denote by $P_n : X \longrightarrow X$ the n^{th}*-basis projection*, defined at such an $x \in X$ as

$$P_n\left(\sum_{k=1}^{\infty} \lambda_k s_k\right) := \sum_{k=1}^{n} \lambda_k s_k,$$

and by $s_n^* : X \longrightarrow \mathbb{R}$ the n^{th}*-coordenate functional*, given at such an $x \in X$ by

$$s_n^*\left(\sum_{n=1}^{\infty} \lambda_n s_n\right) := \lambda_n.$$

For any $n \in \mathbb{N}$, both P_n and s_n^* are continuous (see, for instance, [1]), and so, as a consequence, for all $x \in X$,

$$\lim_{n\to\infty} \|P_n(x) - x\| = 0.$$

Moreover, there holds

$$1 \leq \inf_{n\in\mathbb{N}} \|P_n\| \leq \sup_{n\in\mathbb{N}} \|P_n\| < \infty$$

(once again, [1] is a suitable reference for this fact). The real number

$$\sup_{n\in\mathbb{N}} \|P_n\| \in [1, +\infty)$$

is known as the *basic constant* of the Schauder basis $\{s_n\}_{n\geq\mathbb{N}}$ and when it is 1, the basis is said to be *monotone*. For example, in a separable (any space with a base must be clearly separable) Hilbert space, every orthogonal basis is a Schauder basis, which is monotone when, in addition, the basis is orthonormal. Although not all Banach spaces have a Schauder basis (see [16]), every separable classical Banach space, as well as any separable Sobolev space, have a Schauder basis ([27, 30]). Because of their importance in what follows, we highlight two of these bases: the so-called *Haar* and *Faber–Schauder* systems, which are the classical Schauder bases in the spaces $L_p[0,1]$, $(1 \leq p < \infty)$, and $C[0,1]$, respectively.

Example 4.1 Let $t_0 = 0$, $t_1 = 1$ and for $n > 1$, write $n = 2^i + k$ with $0 \leq i$ and $1 \leq k \leq 2^i$, and define $t_n = \frac{2k-1}{2^{i+1}}$. The *Haar functions* on $[0,1]$ are defined as

$$h_1(t) := 1, \qquad (0 \leq t \leq 1),$$

and for $n \geq 2$, with i, k and n as we have just indicated,

$$h_n(t) := \begin{cases} 1, & \text{if } \dfrac{2k-2}{2^{i+1}} \leq t < \dfrac{2k-1}{2^{i+1}} \\ \\ -1, & \text{if } \dfrac{2k-1}{2^{i+1}} \leq t \leq \dfrac{2k}{2^{i+1}} \\ \\ 0, & \text{otherwise} \end{cases}.$$

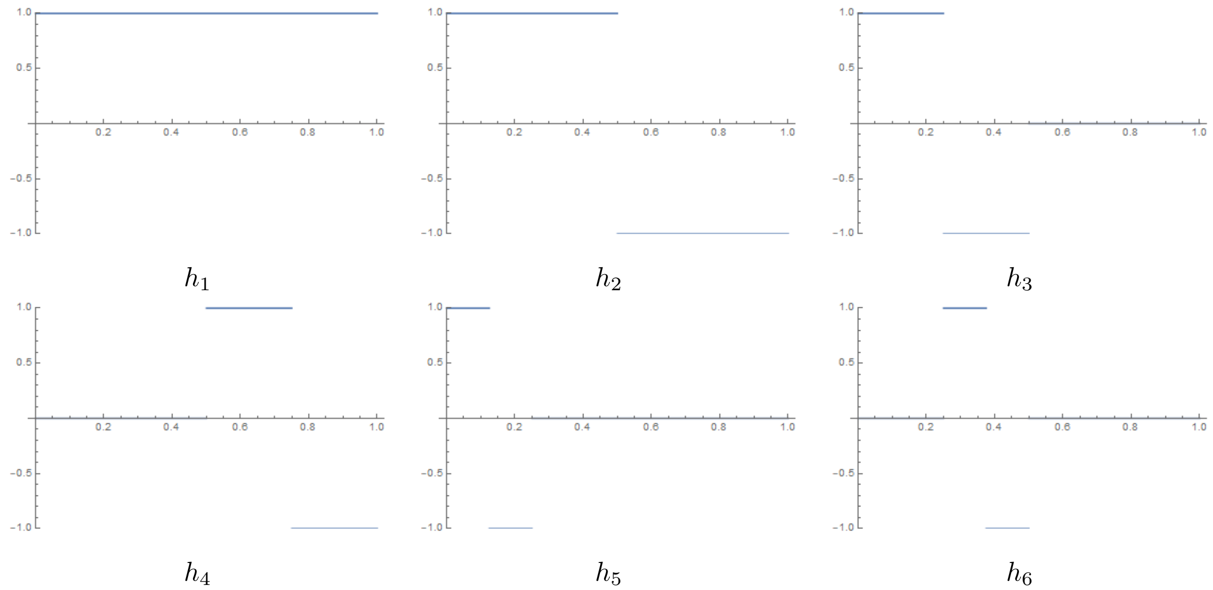


Figure 1: First functions of the Haar system.

We show some of these functions in Figure 1.

The *Haar system* $\{h_n\}_{n\geq 1}$ is a monotone Schauder basis in $L_p[0,1]$, for $1 \leq p < \infty$, and it is easy to check that if $n \geq 1$, then the n^{th}-coordenate functional is given, at any $y \in L_p[0,1]$, as

$$h_n^*(y) = \frac{\displaystyle\int_0^1 y(t) h_n(t)\, dt}{\displaystyle\int_0^1 h_n(t) h_n(t)\, dt}.$$

In order to describe its sequence of projections, let us first recall that the *characteristic function* $\mathbf{1}_A : X \longrightarrow \mathbb{R}$ of a non-empty subset A of a set X is given by

$$\mathbf{1}_A(x) := \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in X \backslash A \end{cases}.$$

Then, for any $y \in L_p[0,1]$ and $n \geq 1$, the projections adopt the alternative form

$$P_n(y) = \sum_{j=1}^{n} h_j^*(y) h_j = \sum_{j=1}^{n} \left(|I_j|^{-1} \int_{I_j} y(t)\, dt \right) \mathbf{1}_{I_j}, \tag{4.3}$$

with $I_j = [t'_{j-1}, t'_j]$, where $t'_0, t'_1, \ldots, t'_n$ is the increasing rearrangement of the set $\{t_0, \ldots, t_n\}$ (see [30]).

Example 4.2 Let $\{t_n\}_{n\geq 1}$ be a dense sequence in $[0,1]$ with $t_1 = 0$ and $t_2 = 1$. Let

$$f_1(t) := 1, \qquad (0 \leq t \leq 1),$$

and for $n \geq 2$, let f_n be the piecewise linear continuous function on $[0,1]$ with nodes at $\{t_m : 1 \leq m \leq n\}$, uniquely determined by the relations

$$f_n(t_n) = 1 \quad \text{and} \quad f_n(t_m) = 0, \quad \text{for } m < n.$$

The sequence $\{f_n\}_{n\geq 1}$ is known as the *Faber–Schauder system* in $C[0,1]$, and it is easy to check that it is a monotone Schauder basis and that for each $y \in C[0,1]$, the coordenate functionals are

$$f_1^*(y) = y(t_1)$$

and for all $n \geq 2$,

$$f_n^*(y) = y(t_n) - \sum_{m=1}^{n-1} f_m^*(y) f_m(t_n).$$

It is worth mentioning that this Schauder basis is interpolatory, in the sense that

$$y \in C[0,1],\ n \in \mathbb{N},\ m \leq n \ \Rightarrow\ P_n(y)(t_m) = y(t_m).$$

For example, for a dyadic distribution of points $\{t_n\}_{n\geq 1}$ in [0,1], with $t_1 = 0$ and $t_2 = 1$ the first functions of the Faber–Schauder are collected in Figure 2.

Our objective is to use the Schauder bases as a tool to approximate fuzzy numbers. The following result establishes conditions in the Schauder bases that allow us to approximate a fuzzy number.

Theorem 4.3 *Let $\{e_n\}_{n\geq 1}$ be a Schauder basis in a real Banach space $(X, \|\cdot\|)$ of real-valued functions defined on $[0,1]$, whose sequence of associated projections we denote by $\{P_n\}_{n\geq 1}$, in such a way that*

i) *given $n \in \mathbb{N}$, the function e_n is bounded, left-continuous on $(0,1]$ and right-continuous at 0;*

ii) *if $g \in X$ is non-increasing and $g(1) \geq 0$, then*

$$0 \leq \inf_{n\geq 1} P_n(g)(1);$$

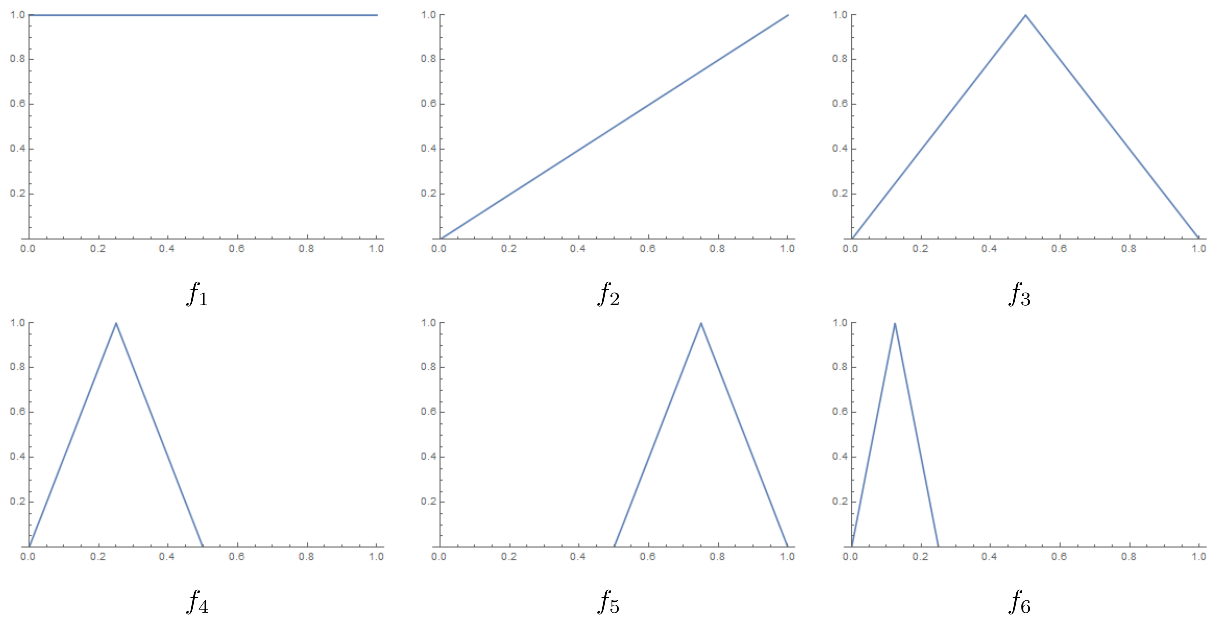


Figure 2: First functions of the Faber-Schauder system on the set of dyadic nodes.

iii) *and when $g \in X$ is a non-decreasing function and $n \in \mathbb{N}$, $P_n(g)$ is also non-decreasing.*

Let us suppose further that $\|\cdot\|^{(\cdot)}$ is a norm in $\mathbb{R}^2$ fulfilling the monotonicity condition (3.1) *and that $\mathbb{F}_X$ is endowed with the distance $d_X^{(\cdot)}$ defined by* (3.2)*. Then:*

a) *For each $u \in \mathbb{F}_X$ and $n \in \mathbb{N}$, $P_n(\underline{u})$ and $P_n(\overline{u})$ define the lower and upper branches of a fuzzy number $\mathbf{P_n}(u) \in \mathbb{F}_X$, that is, for any $\alpha \in [0,1]$,*

$$\underline{\mathbf{P_n}(u)}(\alpha) = P_n(\underline{u})(\alpha) \quad \text{and} \quad \overline{\mathbf{P_n}(u)}(\alpha) = P_n(\overline{u})(\alpha).$$

b) *If $u \in \mathbb{F}_X$, then the sequence $\{\mathbf{P_n}(u)\}_{n\geq 1}$ approximates u in the sense of the metric $d_X^{(\cdot)}$, i.e.,*

$$\lim_{n\to\infty} d_X^{(\cdot)}(u, \mathbf{P_n}(u)) = 0.$$

c) *For any $n \in \mathbb{N}$, the mapping $\boldsymbol{P_n} : \mathbb{F}_X \longrightarrow \mathbb{F}_X$ is a Lipschitz continuous projection. More specifically,*

$$\mathbf{P_n} \circ \mathbf{P_n} = \mathbf{P_n}$$

and if $u, v \in \mathbb{F}_X$, then

$$d_X^{(\cdot)}(\mathbf{P_n}(u), \mathbf{P_n}(v)) \leq M d_X^{(\cdot)}(u, v),$$

where M is the basic constant of the basis $\{e_n\}_{n\geq 1}$. In particular, if $\{e_n\}_{n\geq 1}$ is monotone, then $\mathbf{P_n}$ is non-expansive.

Proof. In order to establish the validity of a), we check the hypotheses of Theorem 2.2. Thus, let $u \in \mathbb{F}_X$ and $n \in \mathbb{N}$. As $P_n(\underline{u}), P_n(\overline{u}) \in \operatorname{span}\{e_1, \ldots, e_n\}$ and, in view of hypothesis i), the functions $e_1, \ldots, e_n$ are bounded, left-continuous on $(0,1]$ and right-continuous at 0, so are $P_n(\underline{u})$ and $P_n(\overline{u})$. Furthermore,

$$P_n(\underline{u})(1) \leq P_n(\overline{u})(1),$$

since $\overline{u}-\underline{u}$ is non-increasing and $(\overline{u}-\underline{u})(1) \geq 0$, so, according to our assumption ii), $P_n(\overline{u}-\underline{u})(1) \geq 0$ and the linearity of P_n yields the above inequality. And, in view of iii) and the linearity of P_n, $P_n(\underline{u})$ is non-decreasing and $P_n(\overline{u})$ is non-increasing.

Moreover, we fix $u \in \mathbb{F}_X$ and we must prove that

$$\lim_{n \to \infty} d_X^{(\cdot)}(u, \mathbf{P_n}(u)) = 0,$$

so, let $\varepsilon > 0$. The fact that $\{e_n\}_{n \geq 1}$ is a Schauder basis in X provides us with an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\max\{\|\underline{u} - P_n(\underline{u})\|, \|\overline{u} - P_n(\overline{u})\|\} < \frac{\varepsilon}{\|(1,1)\|^{(\cdot)}},$$

and then

$$\begin{aligned} d_X^{(\cdot)}(u, \mathbf{P_n}(u)) &= \|(\|\underline{u} - P_n(\underline{u})\|, \|\overline{u} - P_n(\overline{u})\|)\|^{(\cdot)} \\ &< \left\| \left(\frac{\varepsilon}{\|(1,1)\|^{(\cdot)}}, \frac{\varepsilon}{\|(1,1)\|^{(\cdot)}} \right) \right\|^{(\cdot)} \\ &= \frac{\varepsilon}{\|(1,1)\|^{(\cdot)}} \|(1,1)\|^{(\cdot)} \\ &= \varepsilon, \end{aligned}$$

which all together gives us the proof of b).

And finally, given $n \in \mathbb{N}$, it is clear that $\mathbf{P_n}$ is a projection in $\mathbb{F}_X$, according to i) and the fact that P_n is a projection in X. In addition, let M be the basic constant of $\{e_n\}_{n \geq 1}$. Then

$$\|P_n(\underline{u}) - P_n(\underline{v})\| \leq M\|\underline{u} - \underline{v}\|$$

and

$$\|P_n(\overline{u}) - P_n(\overline{v})\| \leq M\|\overline{u} - \overline{v}\|,$$

hence

$$\begin{aligned} d_X^{(\cdot)}(\mathbf{P_n}(u), \mathbf{P_n}(v)) &= \left\| \left(\|\underline{\mathbf{P_n}(u)} - \underline{\mathbf{P_n}(u)}\|, \|\overline{\mathbf{P_n}(u)} - \overline{\mathbf{P_n}(u)}\| \right) \right\|^{(\cdot)} \\ &\leq \|(M\|\underline{u} - \underline{v}\|, M\|\overline{u} - \overline{v}\|)\|^{(\cdot)} \\ &\leq \leq M\|(\|\underline{u} - \underline{v}\|, \|\overline{u} - \overline{v}\|)\|^{(\cdot)} \\ &= M d_X^{(\cdot)}(u, v), \end{aligned}$$

which states the M-Lipschitz continuity of $\mathbf{P_n}$ mentioned in c). $\square$

In Section 5 we will analyse the properties of the proposed approximations for a given fuzzy number $u \in \mathbb{F}_X$, the sequence of projections $\{\mathbf{P_n}\}_{\mathbf{n}\geq 1}$, their arithmetic and some of their advantages.

The next two subsections deal with the respective algorithms to approximate, on the one hand, an arbitrary fuzzy number using a suitable modification of the Haar system, and on the other hand, any fuzzy number in $\mathbb{F}_{C[0,1]}$ from the Faber–Schauder system.

4.1 Approximation of an arbitrary fuzzy number

We proceed to approximate an arbitrary fuzzy number from Theorem 4.3. As we mentioned in Section 3, given any fuzzy number $u \in \mathbb{F}$, its functions $\underline{u}, \overline{u}$ belong to $L_p[0,1]$, with $p \in [1,\infty]$. Therefore, in the separable case $(1 \leq p < \infty)$ we can consider the approximations $\mathbf{P_n}(u)$ generated from the Haar system, although we must modify that Schauder basis in order that the functions chosen verify the condition i) of Theorem 4.3. As for the Haar system, we fix $t_0 = 0$, $t_1 = 1$ and for $n > 1$, as $n = 2^i + k$ for some $0 \leq i$ and some $1 \leq k \leq 2^i$, we take $t_n = \frac{2k-1}{2^{i+1}}$. The *modified Haar system* consists of those functions on $[0,1]$ that are defined as

$$\hat{h}_1(t) := 1, \qquad (0 \leq t \leq 1), \tag{4.4}$$

and if $n \geq 2$, then

$$\hat{h}_{2^i+1}(t) := \begin{cases} 1, & \text{if } 0 \leq t \leq \dfrac{1}{2^{i+1}} \\ -1, & \text{if } \dfrac{1}{2^{i+1}} < t \leq \dfrac{2}{2^{i+1}} \\ 0, & \text{otherwise} \end{cases}, \tag{4.5}$$

while for $2 \leq k \leq 2^i$,

$$\hat{h}_{2^i+k}(t) := \begin{cases} 1, & \text{if } \dfrac{2k-2}{2^{i+1}} < t \leq \dfrac{2k-1}{2^{i+1}} \\ -1, & \text{if } \dfrac{2k-1}{2^{i+1}} < t \leq \dfrac{2k}{2^{i+1}} \\ 0, & \text{otherwise} \end{cases}. \tag{4.6}$$

It is clear that the slight modification we have made to the original Haar system determines a Schauder basis in $L_p[0,1]$, since each new function $\hat{h}_n$ is equal almost everywhere to the corresponding h_n of the Haar system, but verifying the conditions i), ii) and iii) of Theorem 4.3:

i) It is obvious that each basis function $\hat{h}_n$ is bounded, left-continuous in $(0,1]$ and right-continuous at 0.

ii) Suppose that $g \in L_p[0,1]$ is non-increasing and that $g(1) \geq 0$. Then, given $n \in \mathbb{N}$, according to the description of the projection P_n in (4.3) (it is defined in the same way almost everywhere as for the Haar system), and maintaining its notation, we have that

$$P_n(g)(1) = |I_n|^{-1} \int_{I_n} g(t)\, dt,$$

therefore $P_n(g)(1) \geq 0$.

iii) Let $g \in L_p[0,1]$ be a non-decreasing function, $n \in \mathbb{N}$ and $t, \tilde{t} \in [0,1]$ with $t \leq \tilde{t}$. Making use again of the expression of P_n in (4.3), let us initially consider that t and $\tilde{t}$ are in the same subinterval I_j. In such a case, it is clear that $P_n(g)(t) = P_n(g)(\tilde{t})$. If, on the other hand, t and $\tilde{t}$ belong to different subintervals, let us say $t \in I_i$ and $\tilde{t} \in I_k$, then $i < k$ and

$$\begin{aligned} P_n(g)(t) &= |I_i|^{-1} \int_{I_i} g(\xi)\, d\xi \\ &\leq P_n(g)(\tilde{t}) \\ &= |I_k|^{-1} \int_{I_k} g(\xi)\, d\xi, \end{aligned}$$

since the projections are the average values of the function in the respective intervals.

Let us observe that the modified Haar system $\{\hat{h}_n\}_{n\geq 1}$ is a monotone Schauder basis, since the Haar system is, and so, the generated projections $\mathbf{P_n}$ are non-expansive, thanks to Theorem 4.3.

We set the modified Haar system $\{\hat{h}_n\}_{n\geq 1}$ defined in (4.4), (4.5) and (4.6) as the Schauder basis. Therefore, in view of Theorem 4.3, for an arbitrary $u \in \mathbb{F}$ (let us recall that $\mathbb{F} = \mathbb{F}_{L_p[0,1]}$), the sequence of approximations $\{\mathbf{P_n}(u)\}_{n\geq 1}$ converges to u, which extends the previously stated result in [35], where the convergence of the sequence of approximations is proven only when the fuzzy number is in $\mathbb{F}_{C[0,1]}$.

We describe below an easy algorithm to approximate an arbitrary fuzzy number $u \in \mathbb{F}$ by means of a simple fuzzy number. The distance d fixed in the algorithm can be any of those described previously such as $d^{(\cdot)}_{L_p[0,1]}$ with $p \in [1, \infty)$.

Approximation algorithm of an arbitrary fuzzy number

Input: Functions $\underline{u}$ and $\overline{u}$, dyadic nodes $\{t_n\}_{n\geq 0}$, modified Haar system $\{\hat{h}_n\}_{n\geq 1}$, distance d, tolerance $\varepsilon > 0$.

Set $a_1 \leftarrow \int_0^1 \underline{u}(\alpha)\, d\alpha$, $b_1 \leftarrow \int_0^1 \overline{u}(\alpha)\, d\alpha$

Set $a_2 \leftarrow \int_0^1 \underline{u}(\alpha)\hat{h}_2(\alpha)\, d\alpha$, $b_2 \leftarrow \int_0^1 \overline{u}(\alpha)\hat{h}_2(\alpha)\, d\alpha$

Set $P(\underline{u}) \leftarrow a_1\hat{h}_1 + a_2\hat{h}_2$, $P(\overline{u}) \leftarrow b_1\hat{h}_1 + b_2\hat{h}_2$

Set $\mu_1 \leftarrow a_1 + a_2$, $\mu_2 \leftarrow a_1 - a_2$, $\eta_1 \leftarrow b_1 + b_2$, $\eta_2 \leftarrow b_1 - b_2$

for $i = 1, 2, \ldots$

  for $k = 1, 2, \ldots, 2^i$

    Set $n \leftarrow 2^i + k$

    Set

$$a_n \leftarrow \frac{\int_0^1 \underline{u}(\alpha)\hat{h}_n(\alpha)\ d\alpha}{\int_0^1 \hat{h}_n(\alpha)\hat{h}_n(\alpha)\ d\alpha} \qquad b_n \leftarrow \frac{\int_0^1 \overline{u}(\alpha)\hat{h}_n(\alpha)\ d\alpha}{\int_0^1 \hat{h}_n(\alpha)\hat{h}_n(\alpha)\ d\alpha}$$

    for $j = n, n-1, \ldots, 2k+1$

      Set $\mu_j = \mu_{j-1}$, $\eta_j = \eta_{j-1}$

    end (for)

    Set $\mu_{2k} \leftarrow \mu_{2k-1} - a_n$, $\eta_{2k} \leftarrow \eta_{2k-1} - a_n$

    Set $\mu_{2k-1} \leftarrow \mu_{2k-1} + a_n$, $\eta_{2k-1} \leftarrow \eta_{2k-1} + a_n$

    Set $P(\underline{u}) \leftarrow P(\underline{u}) + a_n\hat{h}_n$, $P(\overline{u}) \leftarrow P(\overline{u}) + b_n\hat{h}_n$

    Calculate $d(\mathbf{P_n}(u), u)$

    if $d(\mathbf{P_n}, u) < \varepsilon$

      Set $i_0 \leftarrow i$

      Set $k_0 \leftarrow k$

      Set $N \leftarrow n$

      stop

  end (for)

end (for)

Output: i_0, k_0, N, $\{a_j\}_{j=1}^N$, $\{b_j\}_{j=1}^N$, $\{\mu_j\}_{j=1}^N$, $\{\eta_j\}_{j=1}^N$, $P(\underline{u}), P(\overline{u})$.

Once we have finished the algorithm, we obtain the coefficients $\{a_j\}_{j=1}^N, \{b_j\}_{j=1}^N$, of the projections $P_N(\underline{u})$ and $P_N(\overline{u})$, respectively, in the modified Haar basis, as well as the coefficients, $\{\mu_j\}_{j=1}^N, \{\eta_j\}_{j=1}^N$, which allow us to rewrite the projections and recover the simple fuzzy number $\mathbf{P_N}(u)$ explicitly. For this, we note by $\{t'_j\}_{j=0}^{N+1}$ the rearrangement of $\{t_j\}_{j=0}^{N+1}$ in increasing order, and so we have that

$$P_N(\underline{u})(\alpha) = \sum_{j=1}^{N} \mu_j \mathbf{1}_{I_j}(\alpha), \quad P_N(\overline{u})(\alpha) = \sum_{j=1}^{N} \eta_j \mathbf{1}_{I_j}(\alpha), \quad (\alpha \in [0,1]),$$

where $I_1 = [t'_0, t'_1]$ and, for $j = 2, 3, \ldots N$, $I_j = (t'_{j-1}, t'_j]$. The simple fuzzy number $\mathbf{P_N}(u)(x)$ is then

$$\mathbf{P_N}(u)(x) = \begin{cases} 0, & \text{if } x < \mu_1 \\ \dfrac{j}{2^{i_0+1}}, & \text{if } \mu_j \le x < \mu_{j+1}, \qquad j = 1, 2, \ldots, 2k_0 - 1 \\ \dfrac{i - k_0}{2^{i_0}}, & \text{if } \mu_i \le x < \mu_{i+1}, \qquad i = 2k_0, 2k_0 + 1, \ldots, N - 1 \\ 1, & \text{if } \mu_N \le x \le \eta_N \\ \frac{N-k_0-i}{2^{i_0}}, & \text{if } \eta_{N-i+1} < x \le \eta_{N-i}, \quad i = 1, 2, \ldots, N - 2k_0 - 1 \\ \dfrac{N - j}{2^{i_0+1}}, & \text{if } \eta_{N-i+1} < x \le \eta_{N-i}, \quad j = N - 2k_0, \ldots, N - 1 \\ 0, & \text{if } x > \eta_1 \end{cases}.$$

Let us observe that the simple fuzzy number $\mathbf{P_N}(u)$ given in Theorem 4.3 is obtained for approximating any fuzzy number $u \in \mathbb{F}$, unlike that which can be found in other papers, where some additional conditions are assumed, such as the strict monotonicity of $\underline{u}$ and $\overline{u}$ ([35]).

Example 4.4 Let us consider the fuzzy number

$$u(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \le x < 1 \\ \frac{1}{4}, & 1 \le x < 2 \\ \frac{3x}{4} - \frac{5}{4}, & 2 \le x \le 3 \\ 1 - \frac{1}{4}(x-3)^2, & 3 < x \le 5 \\ 0, & x > 5 \end{cases}.$$

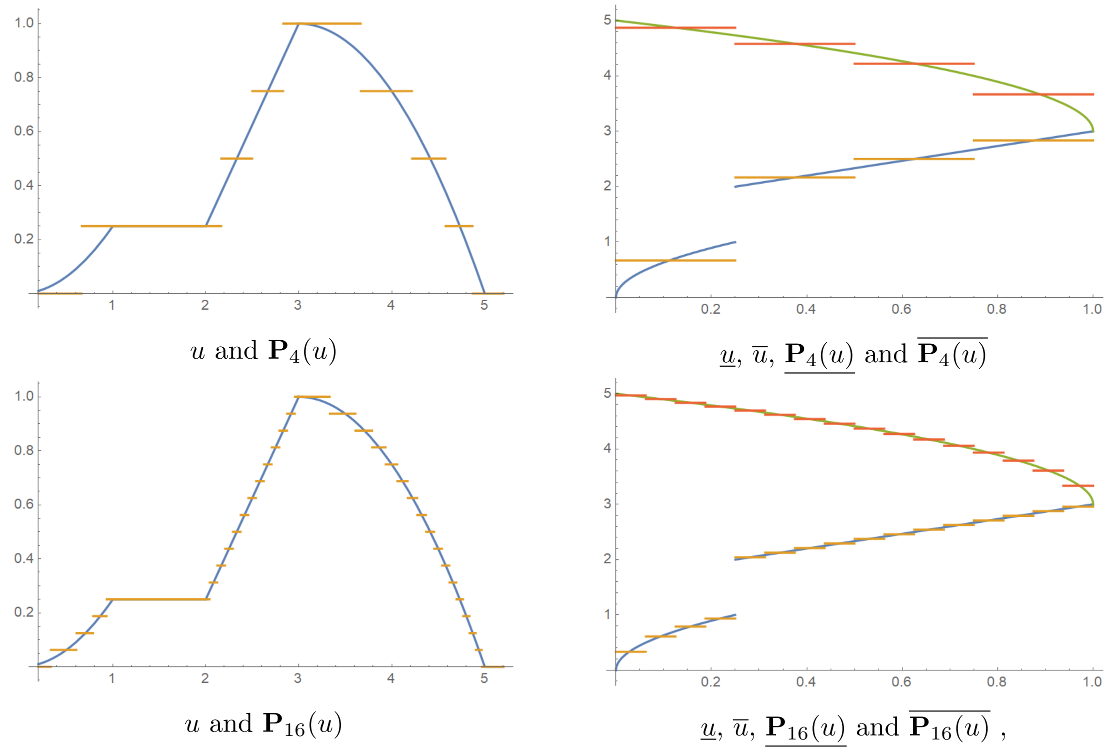
We approximate u for the cases $N = 4$ and $N = 16$ using the Haar system, we obtain

$$d^{(2)}_{L_2[0,1]}(u, \mathbf{P_4}(u)) = 0.204578$$

and

$$d^{(2)}_{L_2[0,1]}(u, \mathbf{P_{16}}(u)) = 0.0592643.$$

In Figure 3, we show the graphs of $\underline{u}$, $\overline{u}$, $\underline{\mathbf{P}_N(u)}$ and $\overline{\mathbf{P}_N(u)}$ for $N = 4$ and $N = 16$, as well as the fuzzy number u and the approximations above.



u and $\mathbf{P}_4(u)$ — $\underline{u}$, $\overline{u}$, $\underline{\mathbf{P}_4(u)}$ and $\overline{\mathbf{P}_4(u)}$

u and $\mathbf{P}_{16}(u)$ — $\underline{u}$, $\overline{u}$, $\underline{\mathbf{P}_{16}(u)}$ and $\overline{\mathbf{P}_{16}(u)}$,

Figure 3

4.2 Approximation of a fuzzy number in $\mathbb{F}_{C[0,1]}$

We now focus on the approximation of a fuzzy number that satisfies an additional condition, its lower and upper branches being continuous, that is, a fuzzy number $u \in \mathbb{F}_{C[0,1]}$. Then, we can consider the approximation $\mathbf{P_n}(u)$ obtained by means of the Faber–Schauder system $\{f_n\}_{n\geq 1}$ associated with the nodes $\{t_n\}_{n\geq 1}$, since such a Schauder basis clearly satisfies the hypotheses of Theorem 4.3: taking into account that $P_n(\underline{u})$ and $P_n(\overline{u})$ are the continuous piecewise linear functions that interpolate $\underline{u}$ and $\overline{u}$, respectively, at nodes $\{t_i\}_{i=1}^n$, so that, $\mathbf{P_n}(u)$ is a polygonal fuzzy number. In particular, as this Schauder basis is monotone, it follows from Theorem 4.3 the non-expansiveness of each projection $\mathbf{P_n}$.

All this is compiled in the following easy algorithm, where the goodness of approximation is measured with the fuzzy Hausdorff distance $d^{(\infty)}_{C[0,1]}$, which will be denoted, for the sake of simplicity, by d.

Approximation algorithm of a fuzzy number in $\mathbb{F}_{C[0,1]}$

Input: Functions $\underline{u}$ and $\overline{u}$, $\{t_n\}_{n\geq 1}$ dense sequence in $[0,1]$ with $t_1 = 0$, and $t_2 = 1$, Faber–Schauder system associated with $\{t_n\}_{n\geq 1}$, distance d, tolerance $\varepsilon > 0$.

Set $a_1 \leftarrow \underline{u}(t_1)$, $b_1 \leftarrow \overline{u}(t_1)$,

Set $P(\underline{u}) \leftarrow a_1 f_1$, $P(\overline{u}) \leftarrow b_1 f_1$

for $i = 2, 3 \ldots$

Set $a_i \leftarrow \underline{u}(t_i) - \sum_{j=1}^{i-1} a_j f_j(t_i)$, $b_i \leftarrow \overline{u}(t_i) - \sum_{j=1}^{i-1} b_j f_j(t_i)$

Set $P(\underline{u}) \leftarrow P(\underline{u})(x) + a_i f_i$, $P(\overline{u}) \leftarrow P(\overline{u}) + b_i f_i$

Calculate $d(\mathbf{P_i}(u), u)$

if $d(\mathbf{P_i}(u), u) < \varepsilon$

Set $N \leftarrow i$

stop

end (for)

Output: $N, \{a_j\}_{j=1}^N$, $\{b_j\}_{j=1}^N$, $P(\underline{u})$, $P(\overline{u})$.

Remark 4.5 When a fuzzy number $u \in \mathbb{F}_{C[0,1]}$ additionally satisfies that $\underline{u}, \overline{u}$ are Lipschitz continuous, with Lipschitz constant $\underline{L}$ and $\overline{L}$, respectively, then clearly

$$\|\underline{u} - P_n(\underline{u})\| \leq 2\underline{L} \max_{i=2,\ldots,n}(t_i - t_{i-1}) \quad \text{and} \quad \|\overline{u} - P_n(\overline{u})\| \leq 2\overline{L} \max_{i=2,\ldots,n}(t_i - t_{i-1}),$$

so, not only $d^{(\infty)}_{C[0,1]}(u, \mathbf{P_n}(u)) \to 0$ as $n \to \infty$, as stated in Theorem 4.3, but also

$$d^{(\infty)}_{C[0,1]}(u, \mathbf{P_n}(u)) \leq 2 \max\{\underline{L}, \overline{L}\} \max_{i=2,\ldots,n}(t_i - t_{i-1}).$$

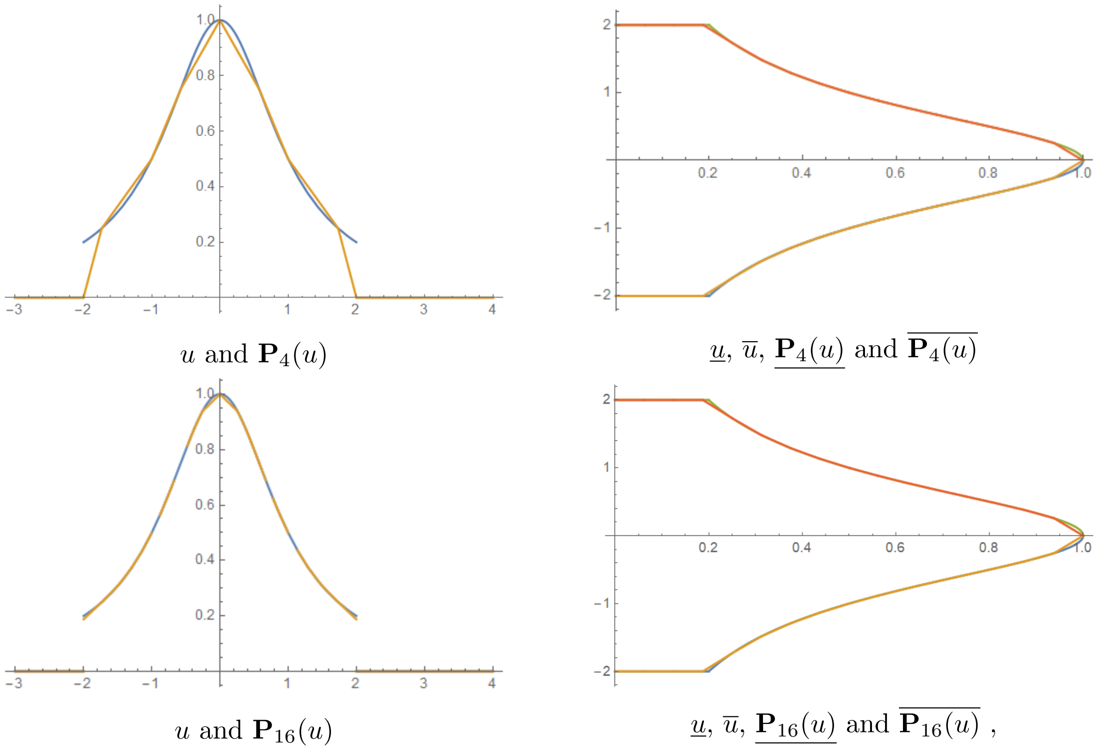
Obviously, we could consider other metrics, for instance, $d^{(2)}_{(C[0,1],\|\cdot\|_2)}$ thus recovering [11, Proposition 1].

Example 4.6 Consider a fuzzy number $v \in \mathbb{F}_{C[0,1]}$ defined as

$$v(x) = \begin{cases} \frac{1}{2+x^2}, & -2 \leq x \leq 2 \\ 0, & x \leq -2 \text{ or } 2 \leq x \end{cases}.$$

We use the Faber-Schauder system over the dyadic partition to approximate v considering $N = 4$ and 16 and we obtain

$$d^{(\infty)}_{C[0,1]}(v, \mathbf{P_4}(v)) = 0.214359$$



u and $\mathbf{P}_4(u)$ $\quad$ $\underline{u}$, $\overline{u}$, $\underline{\mathbf{P}_4(u)}$ and $\overline{\mathbf{P}_4(u)}$

u and $\mathbf{P}_{16}(u)$ $\quad$ $\underline{u}$, $\overline{u}$, $\underline{\mathbf{P}_{16}(u)}$ and $\overline{\mathbf{P}_{16}(u)}$,

Figure 4

and

$$d^{(\infty)}_{C[0,1]}(v, \mathbf{P_{16}}(v)) = 0.0614435.$$

The graphs associated with these approximations are shown in Figure 4.

5 Arithmetic and properties of the approximations

In this section, we focus our attention on the study of the compatibility between the approximations obtained in Theorem 4.3 for a fuzzy number and the usual operations of fuzzy arithmetic. We also analyse how such approximations allow us to obtain easy approximations for the ambiguity, value, expected interval and expected value of any fuzzy number. We begin with the first of these issues. The obtained result generalises [35, Theorem 6] and [3, Proposition 21].

Proposition 5.1 *With the notation and, under the assumptions of Theorem 4.3, let* $u, v \in \mathbb{F}_X$*,* $\lambda \in \mathbb{R}$ *and* $n \in \mathbb{N}$*. Then, we have that*

i)

$$\mathbf{P_n}(u+v) = \boldsymbol{P}_\mathbf{n}(u) + \mathbf{P_n}(v)$$

and

$$d_X^{(\cdot)}(u+v,\mathbf{P_n}(u+v)) \leq d_X^{(\cdot)}(u,\mathbf{P_n}(u)) + d_X^{(\cdot)}(v,\mathbf{P_n}(v)).$$

In particular,

$$\mathbf{P_n}(u+\lambda) = \mathbf{P_n}(u) + \lambda$$

and

$$d_X^{(\cdot)}(u+\lambda,\mathbf{P_n}(u+\lambda)) \leq d_X^{(\cdot)}(u,\mathbf{P_n}(u)).$$

ii)

$$\mathbf{P_n}(\lambda u) = \lambda \mathbf{P_n}(u)$$

and

$$d_X^{(\cdot)}(\lambda u,\mathbf{P_n}(\lambda u)) = |\lambda| d_X^{(\cdot)}(u,\mathbf{P_n}(u)).$$

iii) *Suppose that* $u \ominus_{gH} v$ *exists. Then* $\mathbf{P_n}(u) \ominus_{gH} \mathbf{P_n}(v)$ *exists,*

$$\mathbf{P_n}(u) \ominus_{gH} \mathbf{P_n}(v) = \mathbf{P_n}(u \ominus_{gH} v)$$

and

$$d_X^{(\cdot)}(u \ominus_{gH} v,\mathbf{P_n}(u \ominus_{gH} v)) \leq d_X^{(\cdot)}(u,\mathbf{P_n}(u)) + d_X^{(\cdot)}(v,\mathbf{P_n}(v)).$$

Proof. Let $u, v \in \mathbb{F}_X$, $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$.

First of all, we deal with the addition. The additivity of $\mathbf{P_n}$, $\mathbf{P_n}(u+v) = \mathbf{P_n}(u) + \mathbf{P_n}(v)$, follows from that of the projection P_n, theorem 4.3 a) and the definition of the interval sum, which equivalently yield

$$\alpha \in [0,1] \ \Rightarrow \ [\mathbf{P_n}(u+v)]^\alpha = [\mathbf{P_n}(u)]^\alpha + [\mathbf{P_n}(v)]^\alpha.$$

As a consequence, and taking into account the monotonicity condition (3.1), we arrive at

$$\begin{aligned} d_X^{(\cdot)}(u+v,\mathbf{P_n}(u+v)) &= \left\| \left(\|\underline{u} + \underline{v} - \underline{\mathbf{P_n}(u)} - \underline{\mathbf{P_n}(v)}\|, \|\overline{u} + \overline{v} - \overline{\mathbf{P_n}(u)} - \overline{\mathbf{P_n}(v)}\| \right) \right\|^{(\cdot)} \\ &\leq \left\| \left(\|\underline{u} - \underline{\mathbf{P_n}(u)}\| + \|\underline{v} - \underline{\mathbf{P_n}(v)}\|, \|\overline{u} - \overline{\mathbf{P_n}(u)}\| + \|\overline{v} - \overline{\mathbf{P_n}(v)}\| \right) \right\|^{(\cdot)} \\ &\leq d_X^{(\cdot)}(u,\mathbf{P_n}(u)) + d_X^{(\cdot)}(v,\mathbf{P_n}(v)). \end{aligned}$$

The other statement is obvious, since $\mathbf{P_n}(\lambda) = \lambda$.

Regarding the scalar-fuzzy number multiplication, $\mathbf{P_n}(\lambda u) = \lambda \mathbf{P_n}(u)$, that is,

$$\alpha \in [0,1] \ \Rightarrow \ [\mathbf{P_n}(\lambda u)]^\alpha = \lambda \mathbf{P_n}(u)]^\alpha,$$

it is fulfilled, because, due to the homogeneity of P_n,

$$\begin{aligned}
[\mathbf{P_n}(\lambda u)]^\alpha &= [\underline{\mathbf{P_n}(\lambda u)}(\alpha), \overline{\mathbf{P_n}(\lambda u)}(\alpha)] \\
&= [\min\{\lambda\underline{\mathbf{P_n}(u)}(\alpha), \lambda\overline{\mathbf{P_n}(u)}(\alpha)\}, \max\{\lambda\underline{\mathbf{P_n}(u)}(\alpha), \lambda\overline{\mathbf{P_n}(u)}(\alpha)\}] \\
&= \lambda[\min\{\underline{\mathbf{P_n}(u)}(\alpha), \overline{\mathbf{P_n}(u)}(\alpha)\}, \max\{\underline{\mathbf{P_n}(u)}(\alpha), \overline{\mathbf{P_n}(u)}(\alpha)\}] \\
&= \lambda[\mathbf{P_n}(u)]^\alpha.
\end{aligned}$$

And as a result,

$$\begin{aligned}
d_X^{(\cdot)}(\lambda u, \mathbf{P_n}(\lambda u)) &= \left\| \Big(\|\lambda \underline{u} - \lambda \underline{\mathbf{P_n}(u)}\|, \|\lambda \overline{u} - \lambda \overline{\mathbf{P_n}(u)}\| \Big) \right\|^{(\cdot)} \\
&= |\lambda| \left\| \Big(\|\underline{u} - \underline{\mathbf{P_n}(u)}\|, \|\overline{u} - \overline{\mathbf{P_n}(u)}\| \Big) \right\|^{(\cdot)} \\
&= |\lambda| d_X^{(\cdot)}(u, \mathbf{P_n}(u)).
\end{aligned}$$

And finally, for the gH-difference of the fuzzy numbers u and v, we assume that it exists and write $w := u \ominus_{gH} v$. Then, if $u = v + w$, by i), $\mathbf{P_n}(u) = \mathbf{P_n}(v) + \mathbf{P_n}(w)$, while when $v = u + (-1)w$, i) and ii) imply that

$$\mathbf{P_n}(v) = \mathbf{P_n}(u) + (-1)\mathbf{P_n}(w).$$

Therefore, $\mathbf{P_n}(u) \ominus_{gH} \mathbf{P_n}(v)$ exists and

$$\mathbf{P_n}(u) \ominus_{gH} \mathbf{P_n}(v) = \mathbf{P_n}(u \ominus_{gH} v).$$

To conclude, we prove the validity of the preciously mentioned control of the distance between $u \ominus_{gH} v$ and $\mathbf{P_n}(u \ominus_{gH} v)$, and, as we have just done, we make a distinction according to the form taken by the gH-difference w. Thus, on the one hand, if $u = v + w$, then, for any $\alpha \in [0, 1]$, there holds

$$[u]^\alpha = [v]^\alpha + [w]^\alpha,$$

and so,

$$[\mathbf{P_n}(u)]^\alpha = [\mathbf{P_n}(v)]^\alpha + [\mathbf{P_n}(w)]^\alpha,$$

which implies

$$\underline{w}(\alpha) = \underline{u}(\alpha) - \underline{v}(\alpha) \quad \text{and} \quad \overline{w}(\alpha) = \overline{u}(\alpha) - \overline{v}(\alpha),$$

and

$$\underline{\mathbf{P_n}(w)}(\alpha) = \underline{\mathbf{P_n}(u)}(\alpha) - \underline{\mathbf{P_n}(v)}(\alpha) \quad \text{and} \quad \overline{\mathbf{P_n}(w)}(\alpha) = \overline{\mathbf{P_n}(u)}(\alpha) - \overline{\mathbf{P_n}(v)}(\alpha),$$

respectively. Hence,

$$\begin{aligned}
d_X^{(\cdot)}(u \ominus_{gH} v, \mathbf{P_n}(u \ominus_{gH} v)) &= \left\| \Big(\|\underline{u \ominus_{gH} v} - \underline{\mathbf{P_n}(u \ominus_{gH} v)}\|, \|\overline{u \ominus_{gH} v} - \overline{\mathbf{P_n}(u \ominus_{gH} v)}\| \Big) \right\|^{(\cdot)} \\
&= \left\| \Big(\|\underline{u} - \underline{v} - \underline{\mathbf{P_n}(u)} + \underline{\mathbf{P_n}(v)}\|, \|\overline{u} - \overline{v} - \overline{\mathbf{P_n}(u)} + \overline{\mathbf{P_n}(v)}\| \Big) \right\|^{(\cdot)} \\
&\leq \left\| \Big(\|\underline{u} - \underline{\mathbf{P_n}(u)}\| + \|\underline{v} - \underline{\mathbf{P_n}(v)}\|, \|\overline{u} - \overline{\mathbf{P_n}(u)}\| + \|\overline{v} - \overline{\mathbf{P_n}(v)}\| \Big) \right\|^{(\cdot)} \\
&= \left\| \Big(\|\underline{u} - \underline{\mathbf{P_n}(u)}\|, \|\overline{u} - \overline{\mathbf{P_n}(u)}\| \Big) + \Big(\|\underline{v} - \underline{\mathbf{P_n}(v)}\|, \|\overline{v} - \overline{\mathbf{P_n}(v)}\| \Big) \right\|^{(\cdot)} \\
&\leq d_X^{(\cdot)}(u, \mathbf{P_n}(u)) + d_X^{(\cdot)}(v, \mathbf{P_n}(v)).
\end{aligned}$$

On the other hand, suppose that $v = u + (-1)w$ and let $\widehat{w} := (-1)w$. Then $v = u + \widehat{w}$, hence $v \ominus_{gH} u = \widehat{w}$ and so $v \ominus_{gH} u = (-1)(u \ominus_{gH} v)$. Thus, it follows from ii) and the above reasoning that

$$\begin{aligned} d_X^{(\cdot)}(u \ominus_{gH} v, \mathbf{P_n}(u \ominus_{gH} v)) &= d_X^{(\cdot)}((-1)(u \ominus_{gH} v), \mathbf{P_n}((-1)(u \ominus_{gH} v)) \\ &= d_X^{(\cdot)}(v \ominus_{gH} u, \mathbf{P_n}(v \ominus_{gH} u)) \\ &\leq d_X^{(\cdot)}(v, \mathbf{P_n}(v)) + d_X^{(\cdot)}(u, \mathbf{P_n}(u)). \end{aligned}$$

□

The operations of fuzzy addition, scalar-fuzzy number multiplication and gH-difference of fuzzy numbers are, in general, impossible to compute explicitly. Proposition 5.1 is particularly interesting in this respect, since, when an arbitrary fuzzy number $u \in \mathbb{F}$ is approximated by means of $\mathbf{P_n}(u)$, obtained from the Haar system, the arithmetic of these approximations is extremely easy, since it reduces to that of simple fuzzy numbers. For the same reason, it makes the approximations of the fuzzy numbers in $\mathbb{F}_{C[0,1]}$, from the Faber–Schauder system, particularly suitable for approximating the arithmetic in that metric space.

We have previously commented that, in the study of fuzzy numbers, it is useful to work with different parameters associated with them, such as the core, value, ambiguity, expected interval and expected value. We now prove that the approximations introduced in this paper work well with these parameters and, in particular, Proposition 5.4 extends the related result presented in [11].

Taking into account that the core of u and $\mathbf{P_n}(u)$ are explicitly described as $[u]^1 = [\underline{u}(1), \overline{u}(1)]$ and $[\mathbf{P_n}(u)]^1 = [P_n(\underline{u})(1), P_n(\overline{u})(1)]$, in the case presented in section 4.1, if $u \in \mathbb{F}$, then,

$$[u]^1 = \lim_{n\to\infty} [\mathbf{P_n}(u)]^1,$$

where the limit is understood in terms of the Haussdorff distance. To illustratethis statement, it is sufficient to note that, with the notation of Example 4.1, for $n \in \mathbb{N}$

$$P_n(\underline{u})(1) = |I_n|^{-1} \int_{I_n} \underline{u}(t)\, dt, \quad \text{and} \quad P_n(\overline{u})(1) = |I_n|^{-1} \int_{I_n} \overline{u}(t)\, dt,$$

with $I_n = [t'_{n-1}, 1]$. Since the sequences $\{P_n(\underline{u})(1)\}_{n\in\mathbb{N}}$ and $\{P_n(\overline{u})(1)\}_{n\in\mathbb{N}}$ are bounded and, if $n < m$, then $t'_{n-1} \leq t'_{m-1}$, and from the monotony and left-continuity at 1 of $\underline{u}$ and $\overline{u}$, we deduce the above claim.

On the other hand, in the case described in section 4.2, the approximations $\mathbf{P_n}(u)$ preserve the core, *i.e.*, if $u \in \mathbb{F}_{C[0,1]}$ and $n \in \mathbb{N}$, then

$$[u]^1 = [P_n(\underline{u})(1), P_n(\overline{u})(1)] = [\mathbf{P_n}(u)]^1,$$

because of the interpolation property of the Faber–Schauder basis.

In the following results, the limit of the expected interval is understood in terms of the Hausdorff distance, that is, convergence of the extremes of the intervals to those of the limit interval.

Proposition 5.2 *Let $u \in \mathbb{F}$ and suppose that $\{\mathbf{P_n}(u)\}_{n\geq 1}$ is the sequence of approximations associated with a norm in $\mathbb{R}^2$ satisfying* (3.1) *and a Schauder basis in* $(L_p[0,1], \|\cdot\|_p)$, $p \in [1, \infty)$, *under the assumptions of Theorem 4.3. Then,*

$$\lim_{n\to\infty} \mathrm{EI}(\mathbf{P_n}(u)) = \mathrm{EI}(u) \quad \textit{and} \quad \lim_{n\to\infty} \mathrm{EV}(\mathbf{P_n}(u)) = \mathrm{EV}(u).$$

If, in addition, $c : [0,1] \longrightarrow [0,1]$ is a reducing function, then

$$\lim_{n\to\infty} \mathrm{Val}_c(\mathbf{P_n}(u)) = \mathrm{Val}_c(u) \quad \textit{and} \quad \lim_{n\to\infty} \mathrm{Amb}_c(\mathbf{P_n}(u)) = \mathrm{Amb}_c(u).$$

Proof. Given $1 \leq p < \infty$, we can observe, on the one hand, that for any $u \in \mathbb{F}$,

$$\lim_{n\to\infty} \|\underline{u} - P_n(\underline{u})\|_p = 0 = \lim_{n\to\infty} \|\overline{u} - P_n(\overline{u})\|_p,$$

and, on the other hand, that if $c \in L_\infty[0,1]$, then the functional $\Phi : (L_p[0,1], \|\cdot\|_p) \longrightarrow \mathbb{R}$ defined at each $u \in \mathbb{F}$ by

$$\Phi(u) := \int_0^1 cu,$$

is well-defined and continuous (Hölder inequality). $\square$

Remark 5.3 When the considered Schauder basis in $L_p[0,1]$ is the modified Haar system, a direct application of its expression of the projections given in (4.3) – is the same as for the Haar system– it yields for each $n \in \mathbb{N}$

$$\mathrm{EI}(\mathbf{P_n}(u)) = \mathrm{EI}(u), \quad \mathrm{EV}(\mathbf{P_n}(u)) = \mathrm{EV}(u), \quad \mathrm{Val}(\mathbf{P_n}(u)) = \mathrm{Val}(u)$$

and

$$\mathrm{Amb}(\mathbf{P_n}(u)) = \mathrm{Amb}(u).$$

The continuous counterpart of Proposition 5.2 reads as follows:

Proposition 5.4 *Assume that, for a given norm in* $\mathbb{R}^2$*, fulfilling* (3.1)*, a Schauder basis in* $(C[0,1], \|\cdot\|_\infty)$ *satisfying the assumptions in Theorem 4.3, and a fuzzy number* $u \in \mathbb{F}$*,* $\{\mathbf{P_n}(u)\}_{n\geq 1}$ *is the corresponding sequence of approximations of* u*. Then,*

$$\lim_{n\to\infty} \mathrm{EI}(\mathbf{P_n}(u)) = \mathrm{EI}(u) \quad \textit{and} \quad \lim_{n\to\infty} \mathrm{EV}(\mathbf{P_n}(u)) = \mathrm{EV}(u).$$

Furthermore,

$$\lim_{n\to\infty} \mathrm{Val}_c(\mathbf{P_n}(u)) = \mathrm{Val}_c(u) \quad \textit{and} \quad \lim_{n\to\infty} \mathrm{Amb}_c(\mathbf{P_n}(u)) = \mathrm{Amb}_c(u),$$

whenever $c : [0,1] \longrightarrow [0,1]$ *is a reducing function.*

Proof. It is sufficient to apply arguments similar to those used in the previous proof in the continuous case. □

Remark 5.5 The validity of the results in the two previous propositions is also guaranteed, in a more general way, when considering the set $\mathbb{F}_X$, where X is a Banach space of real-valued functions on $[0,1]$ such that the functional $\Phi : X \longrightarrow \mathbb{R}$ given at each $u \in \mathbb{F}_X$ by

$$\Phi(u) := \int_0^1 cu,$$

is well-defined and continuous.

It is worth mentioning that the calculations of the approximations for each of these numbers and the expected interval are immediate when c is not complicated and either the Haar or the Faber-Schauder system is considered.

6 Conclusions

In this paper we propose a general method that, given an arbitrary fuzzy number u, allows us to obtain another fuzzy number $\mathbf{P_n}(u)$ which is simpler and as close as we want to the number u, in the sense of a wide family of distances including those defined in the literature.

This general method provides us with a wide range of concrete methods that verify, among other things, the good properties requested in [12]. In particular, the approximations proposed in sections 4.1 and 4.2, for example, verify that $\mathbf{P_n}(u)$ is an easy–to–handle fuzzy number. In addition, by its very construction, the sequence $\{\mathbf{P_n}(u)\}_{n\geq 1}$ converges to u in the appropriate metrics for each problem, including those most commonly used in the literature on the subject.

Moreover, when dealing with the space $\mathbb{F}$ in section 4.1, we assume no additional hypotheses on u in order to guarantee the convergence of $\{\mathbf{P_n}(u)\}_{n\geq 1}$, unlike taht which is done in [35].

As for the third property, as proved in propositions 5.2 and 5.4, the convergence of important characteristics of fuzzy numbers is also given, not only for $\mathbb{F}_{C[0,1]}$, as it is stated in [11].

We also note that the proposed approximations work well with arithmetic operations, which is certainly a very good property to add to the above list. In particular, we prove, in proposition 5.1, that $\mathbf{P_n}(u)+\mathbf{P_n}(v)$, $\lambda\mathbf{P_n}(u)$ and $\mathbf{P_n}(u)\ominus_{gH}\mathbf{P_n}(v)$ provide good approximations of $u+v$, λu and $u\ominus_{gH} v$ respectively. In addition to this, we should also add that the arithmetic operations between the projections are reduced to simple operations between the coefficients (real numbers) of the projections.

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References

[1] F. Albiac, N.J. Kalton, *Topics in Banach space theory*, Second edition, Graduate Texts in Mathematics 233, Springer, Cham, 2016.

[2] M. Arana-Jiménez, M.I. Berenguer, D. Gámez, A.I. Garralda-Guillem, M. Ruiz Galán, *A perturbed collage theorem and its application to inverse interval integral problems*, Communications in Nonlinear Science and Numerical Simulation 90 (2020), 105365, 10 pages.

[3] A.D. Báez-Sánchez, A.C. Moretti, M.A. Rojas-Medar, *On polygonal fuzzy sets and numbers*, Fuzzy Sets and Systems 209 (2012), 54–65.

[4] A. Ban, *Approximation of fuzzy numbers by trapezoidal fuzzy numbers preserving the expected interval*, Fuzzy Sets and Systems 159 (2008), 1327–1344.

[5] A. Ban, A. Brândaş, L. Coroianu, C. Negruţiu, O. Nica, *Approximations of fuzzy numbers by trapezoidal fuzzy numbers preserving the ambiguity and value*, Computers & Mathematics with Applications 61 (2011), 1379–1401.

[6] B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic*, Studies in Fuzziness and Soft Computing 295, Springer, 2013.

[7] S. Chanas, *On the interval approximation of a fuzzy number*, Fuzzy Sets and Systems 122 (2001), 353–356.

[8] J.E. Chen, K.N. Otto, *Constructing membership functions using interpolation and measurement theory*, Fuzzy Sets and Systems 73 (1995), 313–327.

[9] L. Coroianu, *Best Lipschitz constant of the trapezoidal approximation operator preserving the expected interval*, Fuzzy Sets and Systems 165 (2011), 81–97.

[10] L. Coroianu, M. Gagolewski, P. Grzegorzewski, *Nearest piecewise linear approximation of fuzzy numbers*, Fuzzy Sets and Systems 233 (2013), 26–51.

[11] L. Coroianu, M. Gagolewski, P. Grzegorzewski, *Piecewise linear approximation of fuzzy numbers: Algorithms, arithmetic operations and stability of characteristics*, Soft Computing 23 (2019), 9491–9505.

[12] L. Coroianu, M. Gagolewski, P. Grzegorzewski, M. Adabitabar Firozja, T. Houlari, *Piecewise linear approximation of fuzzy numbers preserving the support and core*, In: A. Laurent et al. (eds.), *Information Processing and Management of Uncertainty in Knowledge-Based Systems*, Part II, vol. 443, Springer, 2014, 244–254.

[13] M. Delgado, M.A. Vila, W. Voxman, *A fuzziness measure for fuzzy numbers: Applications*, Fuzzy Sets and Systems 94 (1998), 205–216.

[14] D. Dubois, H. Prade, *The mean value of a fuzzy number*, Fuzzy Sets and Systems 24 (1987), 279–300.

[15] P. Diamond, P. Kloeden, *Metric topology of fuzzy numbers and fuzzy analysis*, In: D. Dubois, H. Prade, et al. (eds.), *Handbook Fuzzy Sets* vol. 7, Kluwer Academic Publishers, Dordrecht, 2000, 583–641.

[16] P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Mathematica 130 (1973), 309–317.

[17] R. Goestschel, W. Voxman, *Elementary fuzzy calculus*, Fuzzy Sets and Systems 18 (1986), 31–43.

[18] P. Grzegorzewski, *Metrics and orders in space of fuzzy numbers*, Fuzzy Sets and Systems 97 (1998), 83–94.

[19] P. Grzegorzewski, *Nearest interval approximation of a fuzzy number*, Fuzzy Sets and Systems 130 (2002), 321–330.

[20] P. Grzegorzewski, *Trapezoidal approximations of fuzzy numbers preserving the expected interval. Algorithms and properties*, Fuzzy Sets and Systems 159 (2008), 1354–1364.

[21] P. Grzegorzewski, K. Pasternak-Winiarska, *Bi-symmetrically weighted trapezoidal approximations of fuzzy numbers*, ISDA 2009 - 9th International Conference on Intelligent Systems Design and Applications, (2009), 318.

[22] P. Grzegorzewski, L. Stefanini, *Nonlinear-shaped approximation of fuzzy numbers*, 2009 International Fuzzy Systems Association World Congress and 2009 European Society for Fuzzy Logic and Technology Conference, IFSA-EUSFLAT 2009, Proceedings, (2009), 1535.

[23] P. Grzegorzewski, *Algorithms for trapezoidal approximations of fuzzy numbers preserving the expected interval*, Studies in Fuzziness and Soft Computing 249 (2010), 85–98.

[24] M.L. Guerra, L. Stefanini, *Approximate fuzzy arithmetic operations using monotonic interpolations*, Fuzzy Sets and Systems 150 (2005), 5–33.

[25] M.L. Guerra, L. Stefanini, *A comparison index for interval ordering based on generalized Hukuhara difference*, Soft Computing 16 (2012), 1931–1943.

[26] O. Kaleva, *On the convergence of fuzzy sets*, Fuzzy Sets and Systems 17 (1985), 53–65.

[27] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. I. Sequence spaces*, Springer-Verlag, Berlin-New York, 1977.

[28] V.L.G. Nayagam, J. Murugan, *Hexagonal fuzzy approximation of fuzzy numbers and its applications in MCDM*, Complex & Intelligent Systems 7 (2021), 1459–1487.

[29] C. Negoita, D. Ralescu, *Application of Fuzzy Sets to System Analysis*, Wiley, New York, 1975.

[30] A. Pełczyński, M. Wojciechowski, *Sobolev spaces*, In: Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, 1361–1423.

[31] L. Stefanini, *A generalization of Hukuhara difference and division for interval and fuzzy arithmetic*, Fuzzy Sets and Systems 161 (2010), 1564–1584.

[32] L. Stefanini, L. Sorini, M.L. Guerra, *Parametric representation of fuzzy numbers and application to fuzzy calculus*, Fuzzy Sets and Systems 157 (2006), 2423–2455.

[33] C. Veeramani, C. Duraisamy, M. Sumathi, *Nearest symmetric trapezoidal fuzzy number approximation preserving expected interval*, International Journal of Uncertainty, Fuzziness and Knowlege-Based Systems 21 (2013), 777–794.

[34] Z. Wang, H. Xie, L. Niu, *Trapezoidal approximation-preserving the fuzziness of a fuzzy number*, Proceedings 2012 9th International Conference on Fuzzy Systems and Knowledge Discovery, FSKD 2012, 191.

[35] G. Wang, J. Li, *Approximations of fuzzy numbers by step type fuzzy numbers*, Fuzzy Sets and Systems 310 (2017), 47–59.

[36] C.T. Yeh, *Trapezoidal and triangular approximations preserving the expected interval*, Fuzzy Sets and Systems 159 (2008), 1345–1353.