Applied Numerical Mathematics 191 (2023) 29–44

Contents lists available at ScienceDirect

Applied Numerical Mathematics

journal homepage: www.elsevier.com/locate/apnum

Non-uniform UE-spline quasi-interpolants and their application to the numerical solution of integral equations ☆

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ARTICLE INFO

Article history:
Received 24 July 2022
Received in revised form 20 March 2023
Accepted 6 May 2023
Available online 18 May 2023

Keywords:
UE-splines
Marsden's identity
Quasi-interpolation
Error estimate
Numerical integration
Fredholm integral equation
Hammerstein integral equation
Nyström method

ABSTRACT

A construction of Marsden's identity for UE-splines is developed and a complete proof is given. With the help of this identity, a new non-uniform quasi-interpolant that reproduces the spaces of polynomial, trigonometric and hyperbolic functions are defined. Efficient quadrature rules based on integrating these quasi-interpolation schemes are derived and analyzed. Then, a quadrature formula associated with non-uniform quasi-interpolation along with Nyström's method is used to numerically solve Hammerstein and Fredholm integral equations. Numerical results that illustrate the effectiveness of these rules are presented.

1. Introduction

Univariate spline quasi-interpolants (abbr. QIs) are a very important tool in data approximation, and numerical solution of ordinary and partial differential equations and integral equations [13–15]. The most popular techniques to construct quasi-interpolating splines are blossoming (polar forms and Marsden identities). In [11], a general version of Marsden's identity based on the blossoming technique is presented. Univariate polynomial spline QIs on uniform partitions of bounded intervals have been extensively studied (see, for instance, [10,12,18–21] and the references therein). In [17], some general Marsden identities for trigonometric splines are established and then formulae for the coefficients of QIs are established via blossoming. Another study of quasi-interpolation (also QI for short) has been done in [23], where the authors introduced some new results concerning the symmetric function of the difference of two finite sets. These results have been well used to construct explicitly discrete and differential QIs. In [6,7,9], the authors present new results then used to construct uniform algebraic trigonometric and hyperbolic B-spline QIs. Recently, in [16] it has been established the Marsden's identity related to uniform generalized (UE-) B-splines from which QI schemes reproducing the spaces of generalized functions are

☆ This work was supported by the Toubkal project: Toubkal/20/101 - Campus France: 43594NB.

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https://doi.org/10.1016/j.apnum.2023.05.006

derived. For non-uniform partitions also Marsden's identities are derived and used to define non-uniform QIs (see, e.g., [5,8]). Moreover, the performance of this numerical schemes is tested.

The purpose of the first part of this work is to develop a Marsden's identity related the UE-splines [24]. It is then used to construct explicitly new non-uniform QIs depending of a parameter ω. The main feature is that they combine polynomial, trigonometric and hyperbolic QIs in a single formulation. More precisely, if ω=0, we get non-uniform polynomial spline QIs. On the other hand, if ω is a real number, then non-uniform trigonometric spline QIs are obtained. Finally, if ω is a pure imaginary complex number, then non-uniform hyperbolic spline QIs result. The advantage of this QI is that it reproduces functions $\cos(nx)$, $\cosh(nx)$, $\sin(nx)$, $\sinh(nx)$, $\exp(nx)$, $\exp(-nx)$, with $n > 0$.

Many studies have shown the effectiveness of QI operators in several areas. They are used to solve integral equations numerically, in particular Hammerstein's and Fredholm's integral equations (see, e.g., [1,2,5,16]). In [1], an efficient iteration algorithm for Fredholm integral equations of the second kind based on spline QIs is presented. The same authors propose in [2] a discrete spline QI defined on a bounded interval for the numerical solution of linear Fredholm integral equations of the second kind with a smooth kernel by collocation and a modified Kulkarni's method together with its Sloan's iterated version. In [16], the authors deal with Fredholm integral equations using general quadratic and cubic QIs for solving. In [5], Hammerstein integral equations are solved numerically via Nyström method and quadrature rules based on non-uniform polynomial QI, showing that these QIs are efficient compared with the uniform case since this latter produces increasing errors near the boundary of the interval.

In the second part of this paper, we are going to use the non-uniform UE-spline QIs developed in the first part to construct an efficient quadrature rule, to be used afterward for solving Fredholm and Hammerstein integral equations via Nyström method.

They are defined over non-uniform partitions, which gives the possibility and reproduce a large set of functions. These easy and low cost QIs give good results for the approximation of the solution of integral equations as shown by the numerical results. The parameter ω can be adjusted by the user according to the type of problem considered, so that a single programme can handle different cases.

The rest of this work is organized as follows. In Section 2, we briefly recall some results on the construction of UE-splines and we give their properties. The main goal in Section 3 is to establish the Marsden's identity related to the UE-spline basis, which will be extremely important for the construction of the QI schemes in Section 4. We also study the error estimates associated to the QIs and some of their derivatives. In Section 5, the quadrature rules associated with the above QIs are derived. In section 6, the numerical solution of Hammerstein and Fredholm equations is dealt by using the approximation methods obtained, and numerical results are also given. The paper ends with some conclusions.

2. UE-spline bases

Let us briefly recall the definition and the construction of UE-splines (see [24] for more details).

Definition 1. Let Ξ be a given knot sequence $\{\zeta_j\}_{j\in\mathbb{Z}}$ with $\zeta_j \leqslant \zeta_{j+1}$ and $\boldsymbol{\omega} := \{\omega_j\}_{j\in\mathbb{Z}}$ be a sequence of non-zero real or pure imaginary frequency parameters. The basis functions $N_{j,2}$ defined as

$$N_{j,2}(\zeta) := \begin{cases} \frac{\sin\omega_j(\zeta-\zeta_j)}{\sin\omega_j(\zeta_{j+1}-\zeta_j)}, & \zeta_j \leqslant \zeta < \zeta_{j+1}, \\ \frac{\sin\omega_i(\zeta_{i+2}-\zeta)}{\sin\omega_i(\zeta_{i+2}-\zeta_{i+1})}, & \zeta_{j+1} \leqslant \zeta < \zeta_{j+2}, \\ 0, & \text{otherwise}, \end{cases}$$

are said to be UE-B-splines of order 2. For $k \geq 3$, functions defined recursively by

$$N_{j,k}(\zeta) = \int_{-\infty}^{\zeta} \left(\delta_{j,k-1}N_{j,k-1}(s) - \delta_{j+1,k-1}N_{j+1,k-1}(s)\right) \mathrm{d}s,$$

with $\delta_{j,k-1} := \left(\int_{-\infty}^{+\infty} N_{j,k-1}(\zeta)\mathrm{d}\zeta\right)^{-1}$, $j \in \mathbb{Z}$, are called UE-B-splines of order k.

Remark 2. If $\omega_j = 0$, we set $0/0 = 0$, and $\delta_{j,k}N_{j,k}(\zeta) = 0$ if $N_{j,k}(\zeta) = 0$. Otherwise, we compute it by the l'Hôpital rule. Additionally, we set $\delta_{j,k}N_{j,k}(\zeta)$ to satisfy

$$\int_{-\infty}^{\zeta} \delta_{j,k}N_{j,k}(\zeta)d\zeta = \begin{cases} 1, & \zeta \geq \zeta_{j+k}, \\ 0, & \zeta < \zeta_j, \end{cases}$$

if $N_{j,k}(\zeta) = 0$.

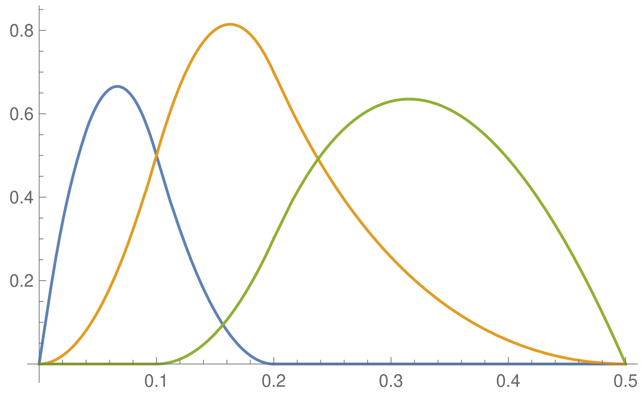

Fig. 1. UE-B-splines of order 3 with $\zeta = \{0, 0, 0.1, 0.2, 0.5\}$ and non-uniform parameters $\boldsymbol{\omega} = \{1, 7\mathrm{i}, 3, 2\mathrm{i}\}$.

UE-spline bases inherit all those nice properties of common B-splines and NURBS.

Proposition 3. *The following properties hold.*

- *Partition of unity:* $\sum_{j\in\mathbb{Z}} N_{j,k}(\zeta) = 1$, $k \geqslant 3$.
- *Positivity:* $N_{j,k}(\zeta) > 0$ *for* $\zeta \in (\zeta_j, \zeta_{j+k})$ *and* $\zeta_{j+k} > \zeta_j$.
- *Local support:* $N_{j,k}(\zeta) = 0$ *for* $\zeta \notin [\zeta_j, \zeta_{j+k}]$.
- *Linear independence: Functions* $N_{j,k}$, $j \in \mathbb{Z}$, *are linearly independent on the real line if the multiplicity of each knot of* Ξ *is less than* $k+1$.
- *Regularity:* $N_{j,k}$ *is* $C^{k-r_\ell-1}$ *continuous at knot* ζ_ℓ *with* r_ℓ *the number of times* ζ_ℓ *appears in the knot sequence.*
- *Derivative:* $N'_{j,k}(\zeta) = \delta_{j,k-1} N_{j,k-1}(\zeta) - \delta_{j+1,k-1} N_{j+1,k-1}(\zeta)$.

For a given set $\{P_j\}_{1\le j\le n} \subset \mathbb{R}^2$ of control points, $n \ge k-1$, the UE-spline curve of order k corresponding to the knot vector Ξ is defined on $[\zeta_{k-1}, \zeta_{n+1}]$ as

$$P(\zeta) = \sum_{j=1}^{n} N_{j,k}(\zeta) P_j.$$

Among other, UE-spline curves have the following properties: differentiation, shape local control, convex hull property, and geometric invariance. In addition, UE-spline bases and curves have also some other superior properties suitable for modelling and computation, especially polynomial-like computation of derivatives and integral, strong representation ability of analytical curves and surfaces, and subdivision property.

The explicit expression of UE-B-splines or order 3 is easily calculated, obtaining that

$$N_{j,3}(\zeta) = \begin{cases} \frac{(A_j^1(\zeta,\omega_j)-1)B_j^1(\omega_j)}{C_j^{0.5}(\omega_j)+C_{j+1}^{0.5}(\omega_j)}, & \zeta_j \leqslant \zeta < \zeta_{j+1}, \\ \frac{-A_{j+2}^1(\zeta,\omega_j)B_{j+1}^1(\omega_j)+C_j^{0.5}(\omega_j)-\frac{1}{C_{j+1}^1(\omega_j)}}{C_j^{0.5}(\omega_j)+C_{j+1}^{0.5}(\omega_j)} - \frac{(A_{j+1}^1(\zeta,\omega_{j+1})-1)B_{j+1}^1(\omega_{j+1})}{C_{j+1}^{0.5}(\omega_{j+1})+C_{j+2}^{0.5}(\omega_{j+1})}, & \zeta_{j+1} \leqslant \zeta < \zeta_{j+2}, \\ \sin^2\left(\frac{1}{2}\omega_{j+1}(\zeta-\zeta_{j+3})\right)\csc\left(\frac{1}{2}\omega_{j+1}(\zeta_{j+3}-\zeta_{j+1})\right)A_{j+1}^{0.5}(\zeta_{j+2},\omega_{j+1})B_{j+2}^{0.5}(\omega_{j+1}), & \zeta_{j+2} \leqslant \zeta < \zeta_{j+3}, \\ 0, & \text{otherwise}, \end{cases}$$

where $A_j^a(\zeta, z) := \cos(az(\zeta - \zeta_j))$, $B_j^a(z) := \csc(az(\zeta_{j+1} - \zeta_j))$, $C_j^a(z) := \tan\left(az(\zeta_j - \zeta_{j+1})\right)$, a being a positive real number. Fig. 1 shows the plots of UE-B-splines of order 3 on the interval $[\zeta_j, \zeta_{j+3}]$ with $\boldsymbol{\omega} = \{1, 7\mathbf{i}, 3, 2\mathbf{i}\}$.

When $\omega_j = \omega$ for all $j \in \mathbb{Z}$, those UE-B-splines become

$$N_{j,3}(\zeta) = \begin{cases} \frac{(A_j^1(\zeta)-1)B_j^1}{C_j^{0.5}+C_{j+1}^{0.5}}, & \zeta_j \leqslant \zeta < \zeta_{j+1}, \\ \frac{-A_{j+2}^1(\zeta)B_{j+1}^1+C_j^{0.5}-\frac{1}{C_{j+1}^1}}{C_j^{0.5}+C_{j+1}^{0.5}} - \frac{(A_{j+1}^1(\zeta)-1)B_{j+1}^1}{C_{j+1}^{0.5}+C_{j+2}^{0.5}}, & \zeta_{j+1} \leqslant \zeta < \zeta_{j+2}, \\ \sin^2\left(\frac{1}{2}\omega(\zeta-\zeta_{j+3})\right)\csc\left(\frac{1}{2}\omega(\zeta_{j+3}-\zeta_{j+1})\right)A_{j+1}^{0.5}(\zeta_{j+2})B_{j+2}^{0.5}, & \zeta_{j+2} \leqslant \zeta < \zeta_{j+3}, \\ 0, & \text{otherwise}, \end{cases}$$

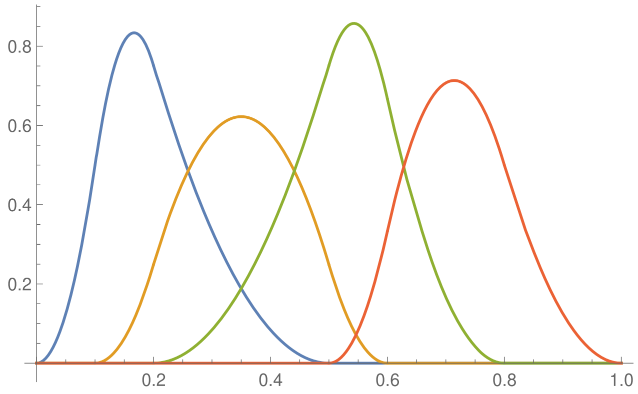

Fig. 2. UE-B-splines of order 3 with $\zeta = \{0, 0.1, 0.2, 0.5, 0.6, 0.8, 1\}$ and uniform parameters $\omega = 1$.

where $A_j^a(\zeta) := \cos(a\omega(\zeta - \zeta_j))$, $B_j^a := \csc(a\omega(\zeta_{j+1} - \zeta_j))$ and $C_j^a := \tan\left(a\omega(\zeta_j - \zeta_{j+1})\right)$.

Fig. 2 shows trigonometric splines with $\omega = 1$ of UE-spline basis of order 3, supported on $[\zeta_j, \zeta_{j+3}]$. Any reference to ω is omitted when $\omega = 1$.

In the rest of this paper, we will focus on the case UE-B-splines of order $k = 3$ and parameters equal to ω to define the UE-quasi-interpolant.

3. Marsden's identity

In this section, we give the main results of the paper. We will look for a suitable UE-spline representation of the elements of the space, i.e. a Marsden's Identity.

Theorem 4. *For all ζ, the following identities hold:*

1. $1 = \sum_{j\in\mathbb{Z}} N_{j,3}(\zeta)$;
2. $\cos(\omega\zeta) = \sum_{j\in\mathbb{Z}} \gamma_j^\omega N_{j,3}(\zeta)$;
3. $\sin(\omega\zeta) = \sum_{j\in\mathbb{Z}} \delta_j^\omega N_{j,3}(\zeta)$,

where

$$\gamma_j^\omega := \frac{\cos\left(\frac{1}{2}\omega(\zeta_{j+1} + \zeta_{j+2})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{j+1} - \zeta_{j+2})\right)} \quad \textit{and} \quad \delta_j^\omega := \frac{\sin\left(\frac{1}{2}\omega(\zeta_{j+1} + \zeta_{j+2})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{j+1} - \zeta_{j+2})\right)}.$$

Proof.

1. See Proposition 3.
2. For $\zeta \in \mathbb{R}$ there exists $p \in \mathbb{Z}$ such that $\zeta \in [\zeta_p, \zeta_{p+1}]$. Therefore, only $N_{p,3}(\zeta)$, $N_{p-1,3}(\zeta)$ and $N_{p-2,3}(\zeta)$ are non zero. Then,

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \gamma_j^\omega N_{j,3}(\zeta) &= \gamma_{p-2}^\omega N_{p-2,3}(\zeta) + \gamma_{p-1}^\omega N_{p-1,3}(\zeta) + \gamma_p^\omega N_{p,3}(\zeta) \\
&= \frac{\cos\left(\frac{1}{2}\omega(\zeta_{p-1} + \zeta_p)\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{p-1} - \zeta_p)\right)} \left(\sin^2\left(\frac{1}{2}\omega(\zeta - \zeta_{p+1})\right) \csc\left(\frac{1}{2}\omega(\zeta_{p+1} - \zeta_{p-1})\right) A_{p-1}^{0.5}(\zeta_p) B_p^{0.5}\right) \\
&+ \frac{\cos\left(\frac{1}{2}\omega(\zeta_p + \zeta_{p+1})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_p - \zeta_{p+1})\right)} \left(\frac{-A_{p+1}^1(\zeta)B_p^1 + C_{p-1}^{0.5} - \frac{1}{C_p^1}}{C_{p-1}^{0.5} + C_p^{0.5}} - \frac{(A_p^1(\zeta) - 1)B_p^1}{C_p^{0.5} + C_{p+1}^{0.5}}\right) \\
&+ \frac{\cos\left(\frac{1}{2}\omega(\zeta_{p+1} + \zeta_{p+2})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{p+1} - \zeta_{p+2})\right)} \left(\frac{(A_p^1(\zeta) - 1)B_p^1}{C_p^{0.5} + C_{p+1}^{0.5}}\right).
\end{aligned}$$

Let

$$E_p := \frac{\cos\left(\frac{1}{2}\omega(\zeta_{p-1}+\zeta_p)\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{p-1}-\zeta_p)\right)}\left(\sin^2\left(\frac{1}{2}\omega(\zeta-\zeta_{p+1})\right)\csc\left(\frac{1}{2}\omega(\zeta_{p+1}-\zeta_{p-1})\right)A^{0.5}_{p-1}(\zeta_p)B^{0.5}_p\right),$$
$$F_p := \frac{\cos\left(\frac{1}{2}\omega(\zeta_p+\zeta_{p+1})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_p-\zeta_{p+1})\right)}\frac{-A^1_{p+1}(\zeta)B^1_p + C^{0.5}_{p-1} - \frac{1}{C^1_p}}{C^{0.5}_{p-1}+C^{0.5}_p} - \frac{(A^1_p(\zeta)-1)B^1_p}{C^{0.5}_p+C^{0.5}_{p+1}},$$
$$G_p := \frac{\cos\left(\frac{1}{2}\omega(\zeta_{p+1}+\zeta_{p+2})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{p+1}-\zeta_{p+2})\right)}\left(\frac{(A^1_p(\zeta)-1)B^1_p}{C^{0.5}_p+C^{0.5}_{p+1}}\right).$$

By converting E_p, F_p and G_p into exponential forms, we get

$$E_p = \frac{\exp(-\mathrm{i}\zeta\omega)\left(\exp(\mathrm{i}\zeta\omega)-\exp(\mathrm{i}\zeta_p\omega)\right)^2\left(-1+\exp(\mathrm{i}\omega(\zeta_{p+1}+\zeta_{p+2}))\right)}{2\left(\exp(\mathrm{i}\zeta_p\omega)-\exp(\mathrm{i}\zeta_{p+1}\omega)\right)\left(\exp(\mathrm{i}\zeta_p\omega)-\exp(\mathrm{i}\zeta_{p+2}\omega)\right)},$$
$$\begin{aligned}F_p = {} & \frac{\left(-2\exp(\mathrm{i}\omega(\zeta_{p-1}+\zeta+\zeta_p))+\exp(\mathrm{i}\omega(\zeta_{p-1}+2\zeta))+\exp(\mathrm{i}\omega(\zeta_{p-1}+2\zeta_{p+1}))\right)}{2\left(\exp(\mathrm{i}\zeta_{p-1}\omega)-\exp(\mathrm{i}\zeta_{p+1}\omega)\right)\left(\exp(2\mathrm{i}\zeta_{p+1}\omega)-\exp(2\mathrm{i}\zeta_p\omega)\right)} \\ & \times \exp(-\mathrm{i}\zeta\omega)\left(1+\exp(\mathrm{i}\omega(\zeta_p+\zeta_{p+1}))\right) \\ & + \frac{\exp(-\mathrm{i}\zeta\omega)\left(1+\exp(\mathrm{i}\omega(\zeta_p+\zeta_{p+1}))\right)\left(\exp(\mathrm{i}\omega(2\zeta+\zeta_p))-2\exp(\mathrm{i}\omega(\zeta+2\zeta_{p+1}))+\exp(\mathrm{i}\omega(\zeta_p+2\zeta_{p+1}))\right)}{2\left(\exp(\mathrm{i}\zeta_{p-1}\omega)-\exp(\mathrm{i}\zeta_{p+1}\omega)\right)\left(\exp(2\mathrm{i}\zeta_{p+1}\omega)-\exp(2\mathrm{i}\zeta_p\omega)\right)} \\ & - \frac{\left(-1+\exp(\omega(\zeta-\zeta_p))\right)^2\exp(-\mathrm{i}\omega(\zeta-\zeta_{p+1}))\left(1+\exp(\omega(\zeta_p+\zeta_{p+1}))\right)\left(1+\exp(\omega(\zeta_{p+1}-\zeta_{p+2}))\right)}{2\left(\exp(\zeta_p\omega)+\exp(\zeta_{p+1}\omega)\right)\left(-1+\exp(\omega(\zeta_{p+1}-\zeta_p))\right)\left(\exp(\omega(\zeta_{p+1}-\zeta_p))-\exp(\omega(\zeta_{p+1}-\zeta_{p+2}))\right)},\end{aligned}$$
$$G_p = \frac{\exp(-\mathrm{i}\zeta\omega)\left(1+\exp(\mathrm{i}\omega(\zeta_{p-1}+\zeta_p))\right)\left(\exp(\mathrm{i}\zeta\omega)-\exp(\mathrm{i}\zeta_{p+1}\omega)\right)^2}{2\left(\exp(\mathrm{i}\zeta_{p+1}\omega)-\exp(\mathrm{i}\zeta_{p-1}\omega)\right)\left(\exp(\mathrm{i}\zeta_{p+1}\omega)-\exp(\mathrm{i}\zeta_p\omega)\right)}.$$

After calculation and simplification we find

$$E_p + F_p + G_p = \frac{\exp(-\mathrm{i}\omega\zeta)+\exp(\mathrm{i}\omega\zeta)}{2} = \cos(\omega\zeta),$$

and the proof is complete.

3. The proof is similar to the one presented in 2. □

Corollary 5. *For any non-negative integer m, we have*

1. $\exp(\pm\mathrm{i}\omega\zeta) = \sum_{j\in\mathbb{Z}}\vartheta^\omega_j N_{j,3}(\zeta)$, *where* $\vartheta^\omega_j := \frac{\exp\left(\pm\mathrm{i}\frac{1}{2}\omega(\zeta_{j+1}+\zeta_{j+2})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{j+1}-\zeta_{j+2})\right)}$.
2. $$\cos^m(\omega\zeta) = \begin{cases} \frac{1}{2^{m-1}}\left(\sum_{j\in\mathbb{Z}}\left(\frac{1}{2}\binom{m}{m/2}+\sum_{p=0}^{(m-2)/2}\binom{m}{p}\gamma_j^{(m-2p)\omega}\right)N_{j,3}(\zeta)\right), & m \text{ even},\\ \frac{1}{2^{m-1}}\left(\sum_{j\in\mathbb{Z}}\left(\sum_{p=0}^{(m-1)/2}\binom{m}{p}\gamma_j^{(m-2p)\omega}\right)N_{j,3}(\zeta)\right), & m \text{ odd}.\end{cases}$$
3. $$\sin^m(\omega\zeta) = \begin{cases} \frac{(-1)^{\frac{m}{2}}}{2^{m-1}}\left(\sum_{j\in\mathbb{Z}}\left(\frac{1}{2}\binom{m}{m/2}(-1)^{\frac{m}{2}}+\sum_{p=0}^{(m-2)/2}\binom{m}{p}(-1)^p\gamma_j^{(m-2p)\omega}\right)N_{j,3}(\zeta)\right), & m \text{ even},\\ \frac{(-1)^{\frac{m-1}{2}}}{2^{m-1}}\left(\sum_{j\in\mathbb{Z}}\left(\sum_{p=0}^{(m-1)/2}\binom{m}{p}(-1)^p\delta_j^{(m-2p)\omega}\right)N_{j,3}(\zeta)\right), & m \text{ odd}.\end{cases}$$

Proof. 1. By using Theorem 4 we have

$$\begin{aligned}\exp(\pm\mathrm{i}\omega\zeta) &= \cos(\omega\zeta)\pm\mathrm{i}\sin(\omega\zeta)\\ &= \sum_{j\in\mathbb{Z}}\left(\gamma^\omega_j\pm\mathrm{i}\delta^\omega_j\right)N_{j,3}(\zeta)\end{aligned}$$

$$= \sum_{i\in\mathbb{Z}} \left(\frac{\cos\left(\frac{1}{2}\omega(\zeta_{j+1}+\zeta_{j+2})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{j+1}-\zeta_{j+2})\right)} \pm \mathrm{i} \frac{\sin\left(\frac{1}{2}\omega(\zeta_{j+1}+\zeta_{j+2})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{j+1}-\zeta_{j+2})\right)} \right) N_{j,3}(\zeta)$$

$$= \sum_{j\in\mathbb{Z}} \frac{\exp\left(\pm \mathrm{i}\frac{1}{2}\omega(\zeta_{j+1}+\zeta_{j+2})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{j+1}-\zeta_{j+2})\right)} N_{j,3}(\zeta).$$

2. For a positive number m, it is satisfied that

$$\cos^m(\omega\zeta) = \left(\frac{\exp(\mathrm{i}\omega\zeta)+\exp(-\mathrm{i}\omega\zeta)}{2}\right)^m$$

$$= \frac{\sum_{p=0}^{m} \binom{m}{p} \exp(\mathrm{i}p\omega\zeta)\exp(-\mathrm{i}(m-p)\omega\zeta)}{2^m}$$

$$= \frac{\sum_{p=0}^{m} \binom{m}{p} \exp(-\mathrm{i}(m-2p)\omega\zeta)}{2^m}$$

$$= \frac{\exp(\mathrm{i}m\omega\zeta)+\cdots+\exp(-\mathrm{i}m\omega\zeta)}{2^m}.$$

Then, for m even, by using Euler's formula, we obtain

$$\cos^m(\omega\zeta) = \frac{1}{2^{m-1}} \left(\frac{1}{2}\binom{m}{m/2} + \sum_{p=0}^{(m-2)/2} \binom{m}{p} \cos((m-2p)\omega\zeta) \right).$$

From Theorem 4, we get

$$\cos^m(\omega\zeta) = \frac{1}{2^{m-1}} \left(\frac{1}{2}\binom{m}{m/2} \sum_{j\in\mathbb{Z}} N_{j,3}(\zeta) + \sum_{p=0}^{(m-2)/2} \binom{m}{p} \sum_{j\in\mathbb{Z}} \gamma_i^{(m-2p)\omega} N_{j,3}(\zeta) \right),$$

and the claim holds. By using a similar technique, we get the result for m odd.

3. The proof of this result is similar to the previous one. □

4. Non-uniform UE-spline quasi-interpolants

In this section we describe how the Marsden's Identity introduced in Section 3 is useful for the construction of UE QIs that reproduce the space Γ_3^ω spanned by 1, $\cos(\omega\zeta)$ and $\sin(\omega\zeta)$. We also give the error estimates associated with these QI operators.

4.1. Constructing non-uniform UE-spline QIs

Let $\Xi := \{\zeta_j, j=0,\dots,n\}$ be a set of knots that produces in general a non-uniform partition of the interval $I := [a,b]$. The unified extended space of order 3 associated with this partition is defined by

$$\Omega_3(I) := \left\{ s \in \mathcal{C}^1(I) : s_{|[\zeta_j,\zeta_{j+1}]} \in \Gamma_3^\omega, j=0,\dots,n-1 \right\}.$$

By adding to Ξ multiple nodes at the extremities, namely, $\zeta_{-2}=\zeta_{-1}=\zeta_0$ and $\zeta_{n+2}=\zeta_{n+1}=\zeta_n$, a basis $\{N_{j,3}, j=-2,\dots,n-1\}$ is defined for $\Omega_3(I)$ with B-splines N_j, $j=0,\dots,n-3$, given in Section 2. For the remaining ones, see [16].

Definition 6. Let $\lambda_{-2}^\omega,\dots,\lambda_{n-1}^\omega$ be a set of linear functionals and f be a function defined on I. Then, the operator Q^ω defined as

$$Q^\omega f(\zeta) = \sum_{j=-2}^{n-1} \lambda_j^\omega(f) N_{j,3}(\zeta) \tag{1}$$

is called UE quasi-interpolation operator of order 3.

We are interested in linear functionals λ_j such that $\lambda_j(f)$ is a linear combination of values of f at knots lying in the support of N_j, or in a neighbourhood of it. For $j = 0, \dots, n-3$, they can be written in the form

$$\lambda_j^\omega(f) = \alpha_{j,1}^\omega f(\zeta_j) + \alpha_{j,2}^\omega f(\zeta_{j+1}) + \alpha_{j,3}^\omega f(\zeta_{j+2}), \tag{2}$$

where the coefficients in $\alpha_j^\omega := \left(\alpha_{j,\ell}^\omega\right)_{1\le\ell\le3}$ are determined so that Q^ω be exact on Γ_3^ω, i.e. $Q^\omega p = p$, for all $p \in \Gamma_3^\omega$. Using the fact that each element of Γ_3^ω can be expressed in terms of $N_{j,3}(\zeta)$ (see Theorem 4), the exactness of Q^ω is equivalent to the following conditions:

$$\sum_{\ell=1}^{3}\alpha_{j,\ell}^\omega = 1,\ \sum_{\ell=1}^{3}\alpha_{j,\ell}^\omega \sin\left(\omega\zeta_{j+\ell-1}\right) = \frac{\sin\left(\frac{1}{2}\omega(\zeta_{j+1}+\zeta_{j+2})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{j+1}-\zeta_{j+2})\right)},\ \sum_{\ell=1}^{3}\alpha_{j,\ell}^\omega \cos\left(\omega\zeta_{j+\ell-1}\right) = \frac{\cos\left(\frac{1}{2}\omega(\zeta_{j+1}+\zeta_{j+2})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_{j+1}-\zeta_{j+2})\right)}.$$

This system has the unique solution $\left\{\alpha_{j,\ell}^\omega\right\}_{1\le\ell\le3}$ given by

$$\begin{aligned}
\alpha_{j,1}^\omega &= \frac{1-\cos(\omega(\zeta_{j+1}-\zeta_{j+2}))}{\cos(\omega(\zeta_j-\zeta_{j+1}))+\cos(\omega(\zeta_j-\zeta_{j+2}))-\cos(\omega(\zeta_{j+2}-\zeta_{j+1}))-1},\\
\alpha_{j,2}^\omega &= \frac{\cos\left(\frac{1}{2}\omega(\zeta_j+\zeta_{j+1}-2\zeta_{j+2})\right)-\cos\left(\frac{1}{2}\omega(\zeta_j-\zeta_{j+1})\right)}{\cos\left(\frac{1}{2}\omega(\zeta_j+\zeta_{j+1}-2\zeta_{j+2})\right)-\cos\left(\frac{1}{2}\omega(\zeta_j-3\zeta_{j+1}+2\zeta_{j+2})\right)},\\
\alpha_{j,3}^\omega &= \frac{1}{2}\sin\left(\frac{1}{2}\omega(\zeta_j-\zeta_{j+1})\right)\csc\left(\frac{1}{2}\omega(\zeta_j-\zeta_{j+2})\right)\sec\left(\frac{1}{2}\omega(\zeta_{j+1}-\zeta_{j+2})\right).
\end{aligned} \tag{3}$$

For the boundary B-splines, see [16]. Therefore, expression (1) becomes

$$Q^\omega f(\zeta) = \sum_{j=-2}^{n-1}\left(\alpha_{j,1}^\omega f(\zeta_j) + \alpha_{j,2}^\omega f(\zeta_{j+1}) + \alpha_{j,3}^\omega f(\zeta_{j+2})\right)N_{j,3}(\zeta), \tag{4}$$

with values of $\alpha_{j,\ell}^\omega$, $j = 0, \dots, n-3$, given in (3) (for the remaining ones, see [16]).

When the knots are uniformly distributed, say $\zeta_{j+1} - \zeta_j = h$, then it generalizes the QI of order 3 introduced in [16]. For example for $\zeta_j = jh$ and $\zeta_j = \frac{1}{2}(ih + (i+1)h)$, the QI in (4) becomes the QI operators $\mathcal{Q}_3^{1,\omega}$ and $\mathcal{Q}_3^{3,\omega}$ defined in [16] and given by

$$\begin{aligned}
\mathcal{Q}_3^{1,\omega} f(\zeta) &= \sum_{j=-2}^{n-1}\left(\frac{f\left(\zeta_{j+2}\right)-f\left(\zeta_j\right)}{2(1+\cos(h\omega))} + f\left(\zeta_{j+1}\right)\right)N_{j,3}(\zeta),\\
\mathcal{Q}_3^{3,\omega} f(\zeta) &= \sum_{j=-2}^{n-1}\left(\left(1-2\phi_h^\omega\right)f\left(\zeta_{j+1}\right)+\phi_h^\omega\left(f\left(\zeta_j\right)+f\left(\zeta_{j+2}\right)\right)\right)N_{j,3}(\zeta),
\end{aligned} \tag{5}$$

with $\phi_h^\omega := -\frac{1}{8}\sec\left(\frac{h\omega}{2}\right)\sec\left(\frac{\omega h}{4}\right)$.

4.2. Error estimates

Let $\mathcal{L}_k^\omega$ be the linear differential operator defined by $\mathcal{L}_k^\omega := \mathbf{D}^{k-2}(\mathbf{D}^2+\omega^2)$, whose null space is Γ_k^ω, and

$$L_p^k(I) := \left\{f : \mathbf{D}^{k-1}f \text{ is absolutely continuous on } I \text{ and } \mathbf{D}^{k-1}f \in L_p(I)\right\}$$

where $L_p[a,b] = \{f : f \text{ is measurable on } [a,b] \text{ and } \|f\|_p < \infty\}$, with

$$\|f\|_{L_{[a,b]}} := \|f\|_\rho = \left[\int_a^b |f(x)|^p dx\right]^{1/p}, \quad 1 \leqslant p < \infty.$$

Now, for $k \ge 3$ consider the Green's function $\mathcal{G}_k^\omega$ corresponding to $\mathcal{L}_k^\omega$ and given by

$$\mathcal{G}_k^\omega(x,y) := \begin{cases} (-1)^{\frac{k+2}{2}}\frac{1}{\omega^{k-1}}\sin(\omega(x-y)_+) + (-1)^{\frac{k}{2}}\frac{1}{\omega^{k-1}}\sum_{\ell=0}^{\frac{k-4}{2}}(-1)^\ell\frac{(\omega(x-y)_+)^{2\ell+1}}{(2\ell+1)!}, & \text{for } k \text{ even},\\ (-1)^{\frac{k-1}{2}}\frac{1}{\omega^{k-1}}\cos(\omega(x-y)_+) + (-1)^{\frac{k+1}{2}}\frac{1}{\omega^{k-1}}\sum_{\ell=0}^{\frac{k-3}{2}}(-1)^\ell\frac{(\omega(x-y)_+)^{2\ell+1}}{(2\ell)!}, & \text{for } k \text{ odd}.\end{cases} \tag{6}$$

Proposition 7. *It holds*

1. $\mathcal{G}_k^{\omega}(x,x)=0$.
2. $\left(\mathbf{D}^{\ell}\right)_x \mathcal{G}_k^{\omega}(x,y)=\mathcal{G}_{k-\ell}^{\omega}(x,y)$, $\ell=0,\ldots,k-2$.
3. $D^{\ell}\mathcal{G}_k^{\omega}(x,y)|_{x=y}=\delta_{\ell,k-1}$, $\ell=0,\ldots,k-1$.
4. $\mathcal{L}_k^{\omega}\mathcal{G}_k^{\omega}(x,y)=0$, $x\neq y$.

Proof. See the proof of Proposition 1 in [16]. □

Theorem 8. *Let $f\in L_1^3(I)$, then*

$$f(x)=s_f(x)+\int_a^b \mathcal{G}_3^{\omega}(x,y)\mathcal{L}_3^{\omega}f(y)dy,$$

where s_f is the unique element in Γ_3^{ω} such that

$$\mathbf{D}^{\ell-1}f(a)=\mathbf{D}^{\ell-1}s_f(a),\ \ell=1,2,3.$$

Proof. See the proof of Theorem 4 in [16]. □

Lemma 9. *There exists a positive constant C such that*

$$\left|\lambda_j^{\omega}(f)\right|\le C3^{\frac{1}{q}}\underline{h}^{\frac{-1}{p}}\|f\|_{L_p[\zeta_j,\zeta_{j+3}]},\ j=-2,\ldots,n-1,$$

for all $1\le p,q\le\infty$, with $\underline{h}:=\min_j h_j$ and $h_j:=\zeta_{j+1}-\zeta_j$.

Proof. From equation (2), coefficients $\lambda_j^{\omega}(f)$ are defined by $\sum_{\ell=1}^{3}\alpha_{j,\ell}^{\omega},f\left(\zeta_{j,\ell-1}\right)$. By using the Holder's inequality, we get

$$\left|\lambda_j^{\omega}(f)\right|\le\left(\sum_{\ell=1}^{3}\left|\alpha_{j,\ell}^{\omega}\right|^q\right)^{\frac{1}{q}}\left(\sum_{\ell=1}^{3}\left|f\left(\zeta_{j,\ell-1}\right)\right|^p\right)^{\frac{1}{p}}\le\left\|\alpha_j^{\omega}\right\|_{\infty}3^{\frac{1}{q}}\left(\sum_{\ell=1}^{3}\left|f\left(\zeta_{j,\ell-1}\right)\right|^p\right)^{\frac{1}{p}},$$

where $\left\|\alpha_j^{\omega}\right\|_{\infty}:=\max_{1\le\ell\le3}\left|\alpha_{j,\ell}^{\omega}\right|$. On the other hand, we have

$$\left(\sum_{\ell=1}^{3}\left|f\left(\zeta_{j,\ell-1}\right)\right|^p\right)^{\frac{1}{p}}\le\underline{h}^{\frac{-1}{p}}\|f\|_{L_p[\zeta_j,\zeta_{j+3}]}.$$

Hence, inequality (4.15) follows. □

Theorem 10 *(Markov inequality [22]). There exists a constant $\mathcal{M}$, depending only on N_L, such that for any interval $J\subseteq I$ of length $h<(b-a)/2$,*

$$\|D^{\ell}u\|_{L_p(J)}\le\mathcal{M}h^{-j+\frac{1}{p}-\frac{1}{q}}\|u\|_{L_q(I)}\tag{7}$$

for all $u\in N_L$ and $1\le p,\ q\le\infty$.

Lemma 11. *There exists a constant depending only on Γ_3^{ω} such that*

$$|D^{\ell}N_{j,k}|\le C_1\underline{h}^{-\ell},\quad \ell=1,\ 2.\tag{8}$$

Proof. By applying Theorem 10 to $N_{j,3}$ for each interval $[\zeta_p,\zeta_{p+1}]$, it holds

$$\|D^{\ell}N_{j,3}\|_{L_\infty[\zeta_p,\zeta_{p+1}]}\le\mathcal{M}\underline{h}^{-\ell}\|N_{j,3}\|_{L_\infty[\zeta_p,\zeta_{p+1}]}.$$

Since each $N_{j,3}$ is a positive and compactly supported function, we deduce that its uniform norm is bounded and, consequently the inequality (8) follows. □

Theorem 12. *Let $1 \le p \le q \le \infty$, and suppose $f \in L_p^3(I)$. Then, for all $\ell = 0,\ 1,\ 2$, we have*

$$\|D^\ell(f - \mathcal{Q}^\omega f)(\zeta)\|_{L_q(I)} \le \frac{C_2 \bar{h}^{3+\frac{1}{q}} \underline{h}^{-\ell-\frac{1}{p}} 3^{\frac{1}{q}}}{2} \left\|\mathcal{L}_3^\omega f\right\|_{L_p(I)}, \tag{9}$$

where $\overline{h} := \max_j h_j$.

Proof. Assume that $\zeta \in I_\mu = [\zeta_\mu, \zeta_{\mu+1}[$. Let s_f be the unique function in Γ_3^ω such that $D^\ell s_f(\zeta) = D^\ell f(\zeta)$, $\ell = 0, 1, 2$. As $\Gamma_3^\omega \subset \Omega_3(I)$, we have $\mathcal{Q}^\omega s_f = s_f$. By Lemma 11, we obtain

$$\begin{aligned}
\left|D^\ell \left(f - \mathcal{Q}^\omega f\right)(\zeta)\right| &= \left|D^\ell \mathcal{Q}^\omega \left(s_f - f\right)(\zeta)\right|, \\
&\le \sum_{j=-2}^{n-1} \left|\lambda_j^\omega (s_f - f)\right| \left|D^\ell N_{j,3}(\zeta)\right|, \\
&\le C 3^{\frac{1}{q}} \sum_{j=\mu-2}^{\mu} \|s_f - f\|_{L_p[\zeta_j, \zeta_{j+3}]} \left|D^\ell N_{j,3}(\zeta)\right|, \\
&\le C C_1 3^{\frac{1}{q}} \underline{h}^{-\ell-\frac{1}{p}} \sum_{j=\mu-2}^{\mu} \|s_f - f\|_{L_p[\zeta_j, \zeta_{j+3}]}.
\end{aligned}$$

It remains to estimate $s_f - f$. Applying the Hölder inequality to the Taylor expansion described in Theorem 8, we get

$$|s_f - f| \le \|\mathcal{G}_3^\omega(x, \cdot)\|_{L_{p'}[\zeta,x]} \|\mathcal{L}_3^\omega f\|_{L_p([\zeta,x])},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Now we will estimate the norm of $\mathcal{G}_3^\omega$ by using (6) and Minkowski inequality.

Indeed, we have

$$\|\mathcal{G}_3^\omega(x, .)\|_{L_{p'}[\zeta,x]} \le \frac{\mathcal{M}_1 \bar{h}^{3-\frac{1}{p}} 3^{\frac{-1}{p'}}}{2!}.$$

Therefore,

$$\|s_f - f\|_{L_p[\zeta_j, \zeta_{j+3}]} \le \frac{\mathcal{M}_1 \bar{h}^3 3^{-\frac{1}{p'}}}{2} \|\mathcal{L}_k^\omega f\|_{L_p([\zeta_j, \zeta_{j+3}])}.$$

Thus, we deduce that

$$\begin{aligned}
\|D^\ell(f - \mathcal{Q}_3^\omega f)(\zeta)\|_{L_q[\zeta_\mu, \zeta_{\mu+1}]} &\le \frac{C C_1 \mathcal{M}_1 \bar{h}^{3+\frac{1}{q}} \underline{h}^{-\ell-\frac{1}{p}} 3^{\frac{1}{q}} 3^{\frac{1}{p}-1}}{2} \sum_{j=\mu-2}^{\mu} \|\mathcal{L}_3^\omega f\|_{L_p([\zeta_j, \zeta_{j+3}])} \\
&\le \frac{C_2 \bar{h}^{3+\frac{1}{q}} \underline{h}^{-\ell-\frac{1}{p}} 3^{\frac{1}{q}}}{2} \sum_{j=\mu-2}^{\mu} \|\mathcal{L}_3^\omega f\|_{L_p([\zeta_j, \zeta_{j+3}])},
\end{aligned}$$

where $C_2 = C C_1 \mathcal{M}_1$. Finally, by summing over $\mu = 0, \ldots, n-1$, and applying Jensen inequality (see Remark 6.2, [22]), the claim follows. □

5. Quadrature formula

In this section we calculate and estimate the convergence order of quadrature formula associated with the non-uniform UE-spline quasi-interpolant described in this paper.

For any continuous function f, the quadrature formula associated with the spline QI on a non-uniform partition is obtained by integrating the QI in (4), i.e.

$$\mathcal{I}_{\mathcal{Q}^\omega}(f) := \int_a^b \mathcal{Q}^\omega f(s) ds = \sum_{j=-2}^{n-1} \lambda_j^\omega \varphi_j^\omega, \tag{10}$$

where $\varphi_j^\omega := \int_a^b N_{j,3}(s) ds$, which can be written as

$$\varphi_j^{\omega} := \mathcal{A}_j^{\omega} + \mathcal{B}_j^{\omega} + \mathcal{C}_j^{\omega},\ -2 \le j \le n-1, \tag{11}$$

with

$$\mathcal{A}_j^{\omega} := \frac{\omega\left(\zeta_j - \zeta_{j+1}\right)\csc\left(\omega\left(\zeta_{j+1}-\zeta_j\right)\right)+1}{\omega\left(\tan\left(\frac{1}{2}\omega\left(\zeta_j-\zeta_{j+1}\right)\right)+\tan\left(\frac{1}{2}\omega\left(\zeta_{j+1}-\zeta_{j+2}\right)\right)\right)},$$

$$\mathcal{B}_j^{\omega} = \frac{\frac{1-\omega(\zeta_{j+1}-\zeta_{j+2})\left(\tan\left(\frac{1}{2}\omega(\zeta_{j+1}-\zeta_j)\right)+\cot(\omega(\zeta_{j+1}-\zeta_{j+2}))\right)}{\tan\left(\frac{1}{2}\omega(\zeta_{j+1}-\zeta_j)\right)-\tan\left(\frac{1}{2}\omega(\zeta_{j+1}-\zeta_{j+2})\right)}+\frac{\omega(\zeta_{j+1}-\zeta_{j+2})\csc(\omega(\zeta_{j+1}-\zeta_{j+2}))-1}{\tan\left(\frac{1}{2}\omega(\zeta_{j+1}-\zeta_{j+2})\right)+\tan\left(\frac{1}{2}\omega(\zeta_{j+2}-\zeta_{j+3})\right)}}{\omega},$$

$$\mathcal{C}_j^{\omega} = \frac{1}{2}\left(\frac{\sin\left(\omega\left(\zeta_{j+2}-\zeta_{j+3}\right)\right)}{\omega} - \zeta_{j+2} + \zeta_{j+3}\right)\cos\left(\frac{1}{2}\omega\left(\zeta_{j+2}-\zeta_{j+1}\right)\right)$$
$$\times \csc\left(\frac{1}{2}\omega\left(\zeta_{j+3}-\zeta_{j+1}\right)\right)\csc\left(\frac{1}{2}\omega\left(\zeta_{j+3}-\zeta_{j+2}\right)\right),$$

for $j = 0, \ldots, n-3$. The values of φ_{-2}^{ω}, φ_{-1}^{ω}, φ_{n-2}^{ω} and φ_{n-1}^{ω} can be calculated as in [16, subs. 5.1] and specific expressions for $\mathcal{A}_j^{\omega}$, $\mathcal{B}_j^{\omega}$ and $\mathcal{C}_j^{\omega}$.

In order to estimate the convergence order of the quadrature formula associated with the non-uniform UE-spline quasi-interpolant described in this paper, the QI given in (4) is rewritten in quasi-Lagrange form to get

$$\mathcal{Q}^{\omega} f(\zeta) := \sum_{j=0}^{n} \bar{N}_{j,3}(\zeta) f\left(\zeta_j\right), \tag{12}$$

where $\bar{N}_{j,3}(\zeta) := \alpha_{j-2,3}^{\omega} N_{j-2,3}(\zeta) + \alpha_{j-1,2}^{\omega} N_{j-1,3}(\zeta) + \alpha_{j,1}^{\omega} N_{j,3}(\zeta)$, with coefficients $\left\{\alpha_{j,\ell}^{\omega}\right\}_{1\le\ell\le3}$ given in (3).

From (2) it follows that for any non-uniform partition Ξ of I, the infinity norm of $\mathcal{Q}^{\omega}$ is bounded by $\max\limits_{-2\le j\le n-1} \left\|\alpha_j^{\omega}\right\|_1$, where $\|v\|_1$ stands for the ℓ_1-norm of the vector $v \in \mathbb{R}^3$.

By integrating (12), the quadrature formula associated with the proposed QI $\mathcal{Q}^{\omega} f$ given in (10) can be written as

$$\mathcal{I}_{\mathcal{Q}^{\omega}}(f) = \sum_{j=-2}^{n-1} \bar{\varphi}_j^{\omega} f(\zeta_j), \tag{13}$$

where $\bar{\varphi}_j^{\omega} := \int\limits_a^b \bar{N}_{j,3} = \int\limits_{\zeta_{j-2}}^{\zeta_{j+3}} \bar{N}_{j,3}$.

To estimate the convergence order of the quadrature formula associated with the proposed QI, we recall the Lebesgue's Lemma (see e.g. [8] Proposition 4.1).

Proposition 13. *If $\mathcal{U}$ is a projection of $\mathcal{X}$ onto $\mathcal{Y}$, and if $\mathcal{E}(f)$ is the error of approximation of f by $\mathcal{Y}$, then*

$$\|f - \mathcal{U}f\| \le (1 + \|\mathcal{U}\|)\mathcal{E}(f).$$

Finally, Theorem 10 and Proposition 13 lead to the following result.

Theorem 14. *There exists a constant C_3 such that for all $f \in L_1^3(I)$ and for all partitions of I it holds*

$$\left\|f - \mathcal{Q}^{\omega} f\right\|_{\infty,I} \le C_3 \bar{h}^3 \left\|\mathcal{L}_3^{\omega} f\right\|_{\infty}.$$

Proof. Let t be in a neighbourhood of x, and $f := g + \int_t^x \mathcal{G}_3^{\omega}(x,y)\mathcal{L}_3^{\omega} f(y)dy$. Then,

$$\|f - g\| \le \left\|\mathcal{L}_3^{\omega} f\right\|_{\infty,I} \left|\int\limits_t^x \mathcal{G}_3^{\omega}(x,y)dy\right|,$$

where g is the unique element in Γ_3^{ω} such that

$$\mathbf{D}^{\ell-1} f(\zeta) = \mathbf{D}^{\ell-1} g(\zeta),\ \ell = 1, 2, 3.$$

Using a Taylor expansion of order three, we get

$$\left|\int_t^x \mathcal{G}_3^{\omega}(x,y)dy\right| \leq \frac{\mathcal{M}_2\bar{h}^3}{2!},$$

and the proof is complete. □

From Theorem 14, we immediately deduce that

Theorem 15. *There exists a constant C_4 such that for all $f \in L_1^3(I)$ and for all partitions Ξ of I it holds*

$$|\mathcal{E}_{\mathcal{Q}^{\omega}}(f,I)| \leq C_4\bar{h}^4 \left\|\mathbf{L}_3^{\omega} f\right\|_{\infty},$$

where $\mathcal{E}_{\mathcal{Q}^{\omega}}$ is the error associated with the quadrature formula based on $\mathcal{Q}^{\omega}$.

Analyzing the stability of the previous quadrature formula is a very complex task, but it is addressed next in the uniform case. From (12), each function $\bar{N}_{j,3}$ involved in the quasi-Lagrange representation of $\mathcal{Q}^{\omega} f$ can be expressed as

$$\bar{N}_{j,3}(\zeta) := \begin{cases} \alpha_{j-2,3}^{\omega} N_{j-2,3}(\zeta), & \zeta_{j-2} \leq \zeta < \zeta_{j-1}, \\ \alpha_{j-2,3}^{\omega} N_{j-2,3}(\zeta) + \alpha_{j-1,2}^{\omega} N_{j-1,3}(\zeta), & \zeta_{j-1} \leq \zeta < \zeta_{j}, \\ \alpha_{j-2,3}^{\omega} N_{j-2,3}(\zeta) + \alpha_{j-1,2}^{\omega} N_{j-1,3}(\zeta) + \alpha_{j,1}^{\omega} N_{j,3}(\zeta), & \zeta_{j} \leq \zeta < \zeta_{j+1}, \\ \alpha_{j-1,2}^{\omega} N_{j-1,3}(\zeta) + \alpha_{j,1}^{\omega} N_{j,3}(\zeta), & \zeta_{j+1} \leq \zeta < \zeta_{j+2}, \\ \alpha_{j,1}^{\omega} N_{j,3}(\zeta), & \zeta_{j+2} \leq \zeta < \zeta_{j+3}, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, the value $\bar{\varphi}_j^{\omega}$ in equation (13) is determined by integrating $\bar{N}_{j,3}$ over each interval $\left[\zeta_p, \zeta_{p+1}\right]$, $p = j-2, \ldots, j+2$. For knots uniformly spaced with step length h, by using equation (5) we have

$$\int_{\zeta_{j-2}}^{\zeta_{j-1}} \bar{N}_{j,3}(s)ds = \frac{h}{\cos(h\omega)-1} + \frac{\cot\left(\frac{h\omega}{2}\right)}{\omega} + h,$$

$$\int_{\zeta_{j-1}}^{\zeta_j} \bar{N}_{j,3}(s)\,ds = \frac{1}{2}\left(h\csc^2(h\omega) - \frac{\cot(h\omega)}{\omega}\right),$$

$$\int_{\zeta_j}^{\zeta_{j+1}} \bar{N}_{j,3}(s)\,ds = \frac{\csc(h\omega)(h\omega\csc(h\omega)-1)}{4\omega},$$

$$\int_{\zeta_{j+1}}^{\zeta_{j+2}} \bar{N}_{j,3}(s)\,ds = \frac{\csc(h\omega)(-\cos(h\omega)+h\omega(2\cot(h\omega)+\csc(h\omega))-2)}{2\omega},$$

$$\int_{\zeta_{j+2}}^{\zeta_{j+3}} \bar{N}_{j,3}(s)\,ds = -\frac{\csc(h\omega)(h\omega\csc(h\omega)-1)}{4\omega},$$

with $\csc(\zeta) := \frac{1}{\sin(\zeta)}$. After some calculations, it follows that $\bar{\varphi}_j^{\omega} = \int_{\zeta_{j-2}}^{\zeta_{j+3}} \bar{N}_{j,3}(s)ds = h$. The remaining weights are also positive (see [16]). Therefore, as the quasi-interpolation operator $\mathcal{Q}^{\omega}$ is exact on Γ_3^{ω}, it holds $\sum_{j=-2}^{n-1}\left|\bar{\varphi}_j^{\omega}\right| = \sum_{j=-2}^{n-1}\bar{\varphi}_j^{\omega} = 1$ and the quadrature formula is stable.

6. Numerical solution of integral equations

This section deals with the performance of the quadrature rules presented in Section 5 in solving Fredholm and Hammerstein integral equations in combination with Nyström method.

6.1. Application to Hammerstein integral equations

In this subsection we consider Hammerstein integral equations of the form

$$u(x) - \int_I k(x,\zeta)g(\zeta,u(\zeta))d\zeta = f(x), \quad x \in I, \tag{14}$$

where the kernel $k(x,\zeta)$ and the function $f(x)$ are given, and $g(\zeta,u(\zeta))$ is a non-linear function of then unknown solution $u(\zeta)$ of the integral equation. The existence and the uniqueness of solution to this type of integral equations have been investigated in the literature by many authors (see, e.g., [25]). For $g \in \mathcal{C}(I)$, the integral operator can be approximated as

$$\int_I k(x,\zeta)g(\zeta,u(\zeta))d\zeta \approx \sum_{k=0}^{n} \bar{\varphi}_j^{\omega} k(x,\zeta_k)\, g\left(\zeta_j, u_n(\zeta_k)\right).$$

Thus, we approximate (14) by

$$u_n(x) - \sum_{k=0}^{n} \bar{\varphi}_j^{\omega} k(x,\zeta_k)\, g\left(\zeta_j, u_n(\zeta_k)\right) = f(x), \quad x \in I \tag{15}$$

We solve this equation by collocation to yield to the system of non-linear equations

$$u_n(\zeta_i) - \sum_{k=0}^{n} \bar{\varphi}_j^{\omega} k\left(\zeta_j,\zeta_k\right) g\left(\zeta_k, u_n(\zeta_k)\right) = f\left(\zeta_j\right), \quad j = 0,\dots,n+1,$$

with unknowns $u_n(\zeta_k)$, $k = 0,\dots,n$, and then obtain the approximate solution as

$$u_n(x) = f(x) + \sum_{k=0}^{n} \bar{\varphi}_k^{\omega} k(x,\zeta_k)\, g\left(\zeta_k, u_n(\zeta_k)\right). \tag{16}$$

Theorem 16. *Let $u \in \mathcal{C}^3(I)$, $g \in \mathcal{C}^3\left(I \times \mathbb{R}\right)$ and $k \in \mathcal{C}^3\left(I^2\right)$ such that g is p-Lipschitz continuous with respect to the second variable. Let M be the matrix with entries $M_{j,k} := \left|\bar{\varphi}_j^{\omega} k\left(\zeta_j,\zeta_k\right)\right|$, and suppose that $\|M\|_\infty < 1$. Then for n sufficiently large, the error between the solution u of the integral equation (14) and the solution u_n of the approximate integral equation (15) satisfies*

$$\|u - u_n\|_{\infty,I} \le C_5 \bar{h}^4,$$

where C_5 is a finite constant independent of f and $\bar{h}$.

Proof. Let

$$e_n(x) := u(x) - u_n(x) = \int_a^b k(x,\zeta)g(\zeta,u(\zeta))d\zeta - \sum_{k=0}^{n} \bar{\varphi}_k^{\omega} k(x,\zeta_k)\, g\left(\zeta_k, u_n(\zeta_k)\right).$$

Then,

$$e_n(x) = \int_a^b k(x,\zeta)g(\zeta,u(\zeta))d\zeta - \sum_{k=0}^{n} \bar{\varphi}_k^{\omega} k(x,\zeta_k)\, g\left(\zeta_k, u(\zeta_k)\right) + \sum_{k=0}^{n} \bar{\varphi}_k^{\omega} k(x,\zeta_k)\left[g\left(\zeta_k, u(\zeta_k)\right) - g\left(\zeta_k, u_n(\zeta_k)\right)\right].$$

According to Theorem 15, and by using that $g(\zeta,u)$ is p-Lipschitz continuous with respect to u, we get

$$|e_n(x)| \le C_4 \bar{h}^4 \left\|\mathcal{L}_3^{\omega} k(x,\cdot)g(\cdot,u(\cdot))\right\|_\infty + p\sum_{k=0}^{n} \left|\bar{\varphi}_k^{\omega} k(x,\zeta_k)\right| |e_n(\zeta_k)|.$$

Setting $e_{n,j} := e_n\left(\zeta_j\right)$ and $\mathcal{R} := \left\{\mathcal{R}_j\right\}$ with $\mathcal{R}_j := \left\|\mathcal{L}_3^{\omega} k\left(\zeta_j,\cdot\right) g(\cdot,u(\cdot))\right\|_\infty$, then

$$\left|e_{n,j}\right| \le C_4 \bar{h}^4 \mathcal{R}_j + M_{j,k}\left|e_{n,k}\right|,$$

which is equivalent to

$$(\mathcal{I} - M)\,|e_n| \le C_4 \bar{h}^4 \mathcal{R}.$$

Table 1
Three test datasets for the Hammerstein integral equations.

j	$k_j(x,\zeta)$	$g_j(\zeta,u)$	$u_j(x)$	$f_j(x)$
1	$\cos(\pi x)\sin(\pi\zeta)$	u^2	$\sin(\pi x)$	$\sin(\pi x)-\frac{4}{3\pi}\cos(\pi x)$
2	$-x$	e^u	x	e^x
3	$-e^{x-2\zeta}$	u^3	e^x	e^{x+1}

Table 2
Results for examples 1, 2 and 3 shown in Table 1.

n	$\omega=\pi$		$\omega=1$		$\omega=\mathrm{i}$	
	$\|u_1-u_{n,1}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_1-u_{n,1}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_1-u_{n,1}\|_{\infty,I}$	$\mathcal{NCO}$
8	0	–	5.26(-6)	–	6.37(-6)	–
16	0	–	3.77(-7)	3.80	4.47(-7)	3.83
32	0	–	2.54(-8)	3.89	2.93(-8)	3.93
64	0	–	1.65(-9)	3.94	1.89(-9)	3.95
128	0	–	1.04(-10)	3.98	1.18(-10)	4.0

n	$\omega=3$		$\omega=1$		$\omega=\mathrm{i}$	
	$\|u_2-u_{n,2}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_2-u_{n,2}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_2-u_{n,2}\|_{\infty,I}$	$\mathcal{NCO}$
8	1.34(−7)	–	2.35(-6)	–	3.05(-6)	–
16	1.22(−8)	4.02	1.66(-7)	3.82	2.08(-7)	3.87
32	7.16(−10)	4.03	1.12(-8)	3.88	1.35(-8)	3.94
64	4.33(−11)	4.02	7.34(-10)	3.93	8.55(-10)	3.98
128	2.92(−12)	4.03	4.65(-11)	3.98	5.38(-11)	3.99

n	$\omega=\mathrm{i}$		$\omega=1$	
	$\|u_3-u_{n,3}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_3-u_{n,3}\|_{\infty,I}$	$\mathcal{NCO}$
8	0	–	1.06(−6)	–
16	0	–	7.24(−8)	3.87
32	0	–	4.81(−9)	3.91
64	0	–	3.09(−10)	3.96
128	0	–	1.94(−11)	3.99

From [4, Thm. 2.3.1], we deduce that $|e_n| \le C_4\bar{h}^4(\mathcal{I}-M)^{-1}\mathcal{R}$. Therefore

$$\|e_n\|_\infty \le C\frac{\|\mathcal{R}\|_\infty}{1-\|M\|_\infty}\bar{h}^4,$$

which concludes the proof. □

Numerical tests

To illustrate the results established in the above result, we consider three integral equations whose kernels and independent terms are given in Table 1, as well as the corresponding solutions. Since the use of uniform partitions produces increasing errors near the boundary of the interval, we choose extrema of Chebyshev polynomials of the first kind in order to have more knots near a and b.

Let us consider the partition Ξ of I given by the extrema of Chebyshev polynomials of the first kind of degree n on the interval I, namely

$$\zeta_k = a+\frac{b-a}{2}\left(\cos\left(\frac{(n-k)\pi}{n}\right)+1\right),\ k=0,\dots,n. \tag{17}$$

The solutions to the nonlinear systems provided by the proposed method have been approximated by using Mathematica.

For different values of n, we present in Table 2 the maximum errors $\|u_k-u_{n,k}\|_{\infty,I}$ of the approximate solution obtained by using our method with chosen values of ω. The numerical convergence orders $\mathcal{NCO}$ are easily computed by the formula

$$\mathcal{NCO} := \log_2\frac{\|u_k-u_{n,k}\|_{\infty,I}}{\|u_k-u_{2n,k}\|_{\infty,I}}. \tag{18}$$

In order to give a comparison with other methods, for the last example in Table 3 the numerical results provided by the proposed method are shown and compared with those obtained in [3] from the iterated versions of the superconvergent degenerate kernel and Nyström methods, and having the same order of convergence. We also included the $\mathcal{NCO}$ associated with each method, also calculated by using (18).

Table 3
Comparison of results of last example in Table 1 obtained by the method presented here, the Superconvergent degenerate kernel (SDK for short) method and Nyström method [3].

N	New method ($\omega = i$)	$\mathcal{NCO}$	SDK method [3]	$\mathcal{NCO}$	Nyström method [3]	$\mathcal{NCO}$
8	0	–	1.01(-6)	–	4.76(-7)	–
16	0	–	6.30(-8)	4.00	2.97(-8)	4.00
32	0	–	3.83(-9)	4.04	1.83(-9)	4.02
64	0	–	2.40(-10)	4.00	1.14(-10)	4.01
128	0	–	–	–	–	–

Table 4
Seven test datasets for the Fredholm integral equations.

j	I_j	$k_j(x,\zeta)$	$u_j(x)$	$f_j(x)$
1	$[0,1]$	$\cos(\pi x\zeta)$	e^{-x}	$e^{-x} - \frac{e+\pi x\sin(\pi x)-\cos(\pi x)}{(\pi^2x^2+1)e}$
2	$[0,1]$	$e^{x\zeta}$	e^{x}	$e^{x} - \frac{e^{x+1}-1}{x+1}$
3	$[0,1]$	$\ln(1+x+\zeta)$	$1-x+x^2-x^3$	$x\left(\frac{1}{4}x^3+\frac{4}{3}x^2+3x+4\right)\ln\left(\frac{x+1}{x+2}\right)$ $+\frac{25}{12}\ln(x+1)-\frac{8}{3}\ln(x+2)$
			$-\frac{3}{4}x^3+\frac{47}{24}x^2+\frac{1}{12}x+\frac{385}{144}$	
4	$[0,1]$	$e^{x\zeta}$	$e^{-x}\cos x$	$e^{-x}\cos x - \frac{((x-1)\cos 1+\sin 1 e^{x}-e(x-1)}{e(x^2-2x+2)}$
5	$[0,\pi]$	$\cos(\zeta+x)$	$\cos(50x)$	$\cos(50x) - \frac{2}{2499}\sin x$
6	$[0,\pi]$	$e^{x\zeta}$	$x^2\cos(50x)$	$-\frac{1}{(2500+x^2)^3}\left(-2x\left(-7500+x^2\right)\right.$ $+e^{\pi x}\left(2x\left(-7500+x^2\right)+\pi^2x\left(2500+x^2\right)^2\right)$ $\left.+2\pi e^{\pi x}\left(6250000-x^4\right)\right)$ $+x^2\cos(50x)$
7	$[0,\pi]$	$\zeta+x$	$e^{-x}\cos(50x)$	$-\frac{e^{-\pi}\left(2499-2501\pi-2501x+e^{\pi}(-2499+2501x)\right)}{6255001}$ $+e^{-x}\cos(50x)$

6.2. The case of linear Fredholm integral equations of the second kind

Consider the linear Fredholm integral equation of the second kind

$$\lambda u(x) - \int_I k(x,\zeta)u(\zeta)d\zeta = f(x),\ x \in I,$$

where $k(x,\zeta)$ is a enough regular kernel. It is a particular case of Hammerstein equation, so that (16) leads to

$$u_n(x) = \frac{1}{\lambda}\left(f(x) + \sum_{k=0}^{n} \bar{\varphi}_k^{\omega} k\left(x,\zeta_k\right) u_u\left(\zeta_k\right)\right),$$

where the values $u_u\left(\zeta_k\right)$ are computed by solving a system of linear equations.

Numerical tests

The performance of the proposed method is also tested for linear Fredholm integral equations. Table 4 shows the kernels and independent terms of seven examples, as well as their intervals and solutions.

Let us consider the partition Ξ of I given by (17). The solutions to the systems provided by the proposed method have been approximated by using Mathematica. For different values of n, for each text equation we present in Table 5 the maximum errors $\left\|u_k - u_{n,k}\right\|_{\infty,I}$ of the approximate solution obtained by using the proposed method with the chosen values of ω. The numerical convergence orders are computed by (18).

7. Conclusions

In this paper, we have constructed new non-uniform quasi-interpolation schemes that reproduce polynomials, trigonometric and hyperbolic functions. The main tool for this construction is the Marsden's identity, established and proved in Section 3. As an application of the introduced quasi-interpolants, we have built a quadrature rule that has been applied for solving Fredholm linear integral equations of the second kind and non-linear Hammerstein integral equations via Nyström method. According to these numerical tests, this rule method has excellent performance and provides a high degree of accuracy.

Table 5
Estimated infinity norms for the numerical approximations to the solutions of the tested integral equations.

n	$\omega=-\mathrm{i}$		$\omega=1$		$\omega=\mathrm{i}$	
	$\|u_1-u_{n,1}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_1-u_{n,1}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_1-u_{n,1}\|_{\infty,I}$	$\mathcal{NCO}$
8	0	–	6.54(-6)	–	2.41(-5)	–
16	0	–	4.50(-7)	3.86	1.71(-6)	3.81
32	0	–	3.01(-8)	3.90	1.17(-7)	3.86
64	0	–	1.93(-9)	3.96	7.72(-9)	3.92
128	0	–	1.22(-10)	3.98	4.92(-10)	3.97

n	$\omega=\mathrm{i}$		$\omega=1$	
	$\|u_2-u_{n,2}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_2-u_{n,2}\|_{\infty,I}$	$\mathcal{NCO}$
8	0	–	2.07(-6)	–
16	0	–	1.41(-7)	3.87
32	0	–	9.51(-9)	3.89
64	0	–	6.19(-10)	3.94
128	0	–	3.92(-11)	3.98

n	$\omega=3\mathrm{i}$		$\omega=1$		$\omega=\mathrm{i}$	
	$\|u_3-u_{n,3}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_3-u_{n,3}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_3-u_{n,3}\|_{\infty,I}$	$\mathcal{NCO}$
8	2.44(−6)	–	1.05(-5)	–	7.04(-6)	–
16	1.42(−7)	4.08	7.18(-7)	3.87	4.74(-7)	3.89
32	8.89(−9)	4.01	4.77(-8)	3.91	3.10(-8)	3.93
64	5.55(−10)	4.02	3.04(-9)	3.97	1.96(-9)	3.98
128	3.42(−11)	4.03	1.91(-10)	3.99	1.23(-10)	3.99

n	$\omega=4$		$\omega=1$		$\omega=\mathrm{i}$	
	$\|u_4-u_{n,4}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_4-u_{n,4}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_4-u_{n,4}\|_{\infty,I}$	$\mathcal{NCO}$
8	2.44(−6)	–	7.09(-6)	–	5.21(-5)	–
16	1.32(−7)	4.02	5.05(-7)	3.81	3.68(-6)	3.82
32	8.79(−9)	4.03	3.47(-8)	3.86	2.53(-7)	3.86
64	5.45(−10)	4.02	2.34(-9)	3.89	3.40(-9)	3.93
128	3.22(−11)	4.03	1.49(-10)	3.97	3.77(-10)	3.98

n	$\omega=50$		$\omega=1$		$\omega=\mathrm{i}$	
	$\|u_5-u_{n,5}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_5-u_{n,5}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_5-u_{n,5}\|_{\infty,I}$	$\mathcal{NCO}$
8	0	–	6.63(-5)		4.05(-5)	
16	0	–	3.75(-5)	3.75	3.01(-6)	3.75
32	0	–	2.82(-6)	3.80	2.13(-7)	3.82
64	0	–	5.49(-8)	3.88	1.42(-8)	3.90
128	0	–	3.57(-10)	3.94	9.12(-10)	3.96

n	$\omega=2$		$\omega=1$		$\omega=\mathrm{i}$	
	$\|u_6-u_{n,6}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_6-u_{n,6}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_6-u_{n,6}\|_{\infty,I}$	$\mathcal{NCO}$
8	3.01(−7)	–	2.23(-6)	–	5.61(-6)	–
16	1.82(−8)	3.97	1.48(-7)	3.91	3.86(-7)	3.86
32	1.10(−9)	4.00	9.64(-9)	3.94	2.60(-8)	3.89
64	7.38(−11)	4.00	6.15(-10)	3.97	1.67(-9)	3.96
128	4.27(−12)	4.00	3.84(-11)	4.0	1.05(-10)	3.98

n	$\omega=5\mathrm{i}$		$\omega=1$		$\omega=\mathrm{i}$	
	$\|u_7-u_{n,7}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_7-u_{n,7}\|_{\infty,I}$	$\mathcal{NCO}$	$\|u_7-u_{n,7}\|_{\infty,I}$	$\mathcal{NCO}$
8	3.01(−7)	–	1.29(-6)	–	2.16 (-6)	–
16	1.35(−8)	3.97	9.13(-7)	3.82	1.55(-7)	3.80
32	1.10(−9)	4.00	6.28(-8)	3.86	1.06(-8)	3.87
64	7.28(−11)	4.00	4.17(-9)	3.91	6.95(-10)	3.93
128	4.35(−12)	4.00	2.66(-10)	3.97	4.37(-11)	3.99

Acknowledgement

The authors thank the referees for their very helpful remarks and suggestions. Funding for open access charge: Universidad de Granada / CBUA.

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