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Research article

Hardy potential versus lower order terms in Dirichlet problems: regularizing effects†

David Arcoya[1,*], **Lucio Boccardo**[2] **and Luigi Orsina**[3]

[1] Departamento de Análisis Matemático, Universidad de Granada, Avda. de Fuente Nueva, s/n, 18071 Granada, Spain
[2] Sapienza Università di Roma – Istituto Lombardo, P.le A. Moro 2, Roma, Italy
[3] Sapienza Università di Roma, P.le A. Moro 2, Roma, Italy

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* **Correspondence:** Email: darcoya@ugr.es; Tel: +34958243153; Fax: +34958243272.

Abstract: In this paper, dedicated to Ireneo Peral, we study the regularizing effect of some lower order terms in Dirichlet problems despite the presence of Hardy potentials in the right hand side.

Keywords: Laplace equation; Hardy potentials; summability of solutions

En recuerdo de Ireneo: ‘Sed breves con las malas noticias’

1. Introduction and statement of the results

In the paper [8], Ireneo Peral and coauthors proved an existence and summability result on the solutions of the Dirichlet problem

$$\begin{cases} -\mathrm{div}(M(x)\,\nabla u) = B\,\dfrac{u}{|x|^2} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$, such that 0 belongs to Ω, $M : \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix such that

$$M(x)\,\xi\cdot\xi \geq \alpha\,|\xi|^2\,, \qquad |M(x)| \leq \beta\,, \tag{1.2}$$

for almost every x in Ω and for every ξ in $\mathbb{R}^N$, with $0 < \alpha \le \beta$, $B > 0$ and f belongs to some Lebesgue space $L^m(\Omega)$.

If $B = 0$, the summability results by G. Stampacchia (see [11]) state that the weak solutions u in $W_0^{1,2}(\Omega)$ of (1.1) are bounded if $m > \frac{N}{2}$; while they belong to $L^{m^{**}}(\Omega)$, with $m^{**} = \frac{Nm}{N-2m}$, when $2_* = \frac{2N}{N+2} \le m < \frac{N}{2}$.

If $B > 0$, the differential operator

$$A(v) = -\mathrm{div}(M(x)\,\nabla v) - B\,\frac{v}{|x|^2}\,,$$

may no longer be coercive, so that both existence and summability results for (1.1) may not be true. However, we recall the following result due to Hardy:

Proposition 1.1 (Hardy inequality)**.** *If v belongs to $W_0^{1,2}(\Omega)$, then*

$$\Big(\frac{N-2}{2}\Big)^2 \int_\Omega \frac{|v|^2}{|x|^2} \le \int_\Omega |\nabla v|^2\,. \tag{1.3}$$

Moreover $\mathcal{H}^2 = (\frac{N-2}{2})^2$ is optimal and is not achieved (for the proof, see [10] or [9]).

Thanks to Hardy inequality, if $0 < B < \alpha\,\mathcal{H}^2$, then the differential operator $A(v)$ is coercive, so that existence and summability results for (1.1) can be proved. However, with respect to the case $B = 0$, there is an important difference: the summability of the solution depends not only on the summability $L^m(\Omega)$ of the datum f, but also on the "size" of B. Indeed, in [8] it is proved that if $1 < m < \frac{N}{2}$, and if

$$0 < B < \alpha\,\frac{N\,(m-1)\,(N-2m)}{m^2}\,, \tag{1.4}$$

then there exists a (weak, or distributional, depending on whether $m \ge 2_*$ or $m < 2_*$) solution u of (1.1), with u belonging to $L^{m^{**}}(\Omega)$. Note that if m tends to $\frac{N}{2}$, or if m tends to 1, then B tends to zero, and that if $m = 2_*$, then the condition on B becomes $0 < B < \alpha\,\mathcal{H}^2$. In particular, observe that if f only belongs to $L^1(\Omega)$, and $B > 0$, neither existence nor summability results can be proved for Eq (1.1). Note also that, as it is proved in [8], if $0 < B < \alpha\,\mathcal{H}^2$, and f belongs to $L^m(\Omega)$, with $m > \frac{N}{2}$ (the classic threshold in order to have bounded solutions), then there exists a solution u in $W_0^{1,2}(\Omega)$ of Eq (1.1), but such solution *never belongs* to $L^\infty(\Omega)$.

In some recent papers (see [2], as well as [3]), the problem

$$\begin{cases} -\mathrm{div}(M(x)\,\nabla u) + a(x)\,u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

was studied when $a(x) \ge 0$ is a function in $L^1(\Omega)$ such that there exists $Q > 0$ such that

$$|f(x)| \le Q\,a(x)\,. \tag{1.6}$$

Under this assumption, the authors proved the existence of a weak solution u in $W_0^{1,2}(\Omega)$ of (1.5), with the property that

$$|u(x)| \le Q\,,$$

so that u belongs to $L^\infty(\Omega)$, even though the datum f may only be a function in $L^1(\Omega)$. This is clearly in sharp contrast with the existence results for the case $a(x) \equiv 0$, where the solution u does not in general belong to $W_0^{1,2}(\Omega)$, nor it is bounded.

The purpose of this paper is to prove existence and summability results for the boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\,\nabla u) + a(x)\,u = B\,\frac{u}{|x|^2} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

with $B > 0$, and $a(x)$ and $f(x)$ such that (1.6) holds. In other words, we will study whether assumption (1.6) (which yields existence of bounded solutions if $B = 0$) allows to improve the results of [8] as far as existence and summability of solutions is concerned. As we will see, if no further assumptions on $a(x)$ with respect to the function $B/|x|^2$ are made, then existence of solutions in $W_0^{1,2}(\Omega)$ for (1.7) follows for every $0 < B < \alpha\,\mathcal{H}^2$, with solutions that become more and more summable as B approaches zero.

In order to state our first result, let us define $2^* = \frac{2N}{N-2}$ and the function $F : [2^*, +\infty) \to \mathbb{R}$ by

$$F(t) = \alpha\,\frac{N}{t}\Big(N - 2 - \frac{N}{t}\Big). \tag{1.8}$$

We remark that $F(2^*) = \alpha\,\mathcal{H}^2$, that F is strictly decreasing (so that $t = 2^*$ is a maximum for F on $[2^*, +\infty)$), and that $F(t)$ tends to zero as t tends to infinity (see Figure 1).

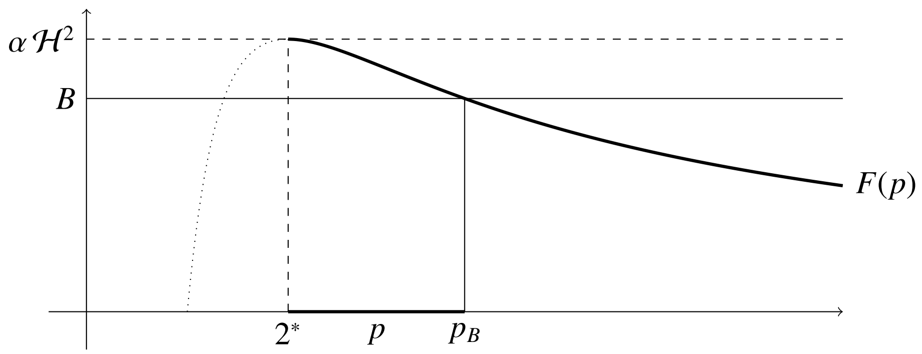

Figure 1. Summability of the solution u.

The following result will be proved in Section 2.

Theorem 1.2. *Let $a(x) \geq 0$ and $f(x)$ in $L^1(\Omega)$ be such that (1.6) holds. If $B > 0$ is such that*

$$0 < B < \alpha\,\mathcal{H}^2\,, \tag{1.9}$$

then there exists a unique weak solution u of Eq (1.7), that is a function u in $W_0^{1,2}(\Omega)$ such that $a(x)\,u$ belongs to $L^1(\Omega)$ and

$$\int_\Omega M(x)\,\nabla u \cdot \nabla\varphi + \int_\Omega a(x)\,u\,\varphi = B\int_\Omega \frac{u\,\varphi}{|x|^2} + \int_\Omega f(x)\,\varphi\,, \tag{1.10}$$

for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Furthermore, if $p_B > 2^*$ is the unique solution of the equation $F(p_B) = B$ on $(2^*, +\infty)$, then u belongs to $L^p(\Omega)$ for every $2^* \leq p < p_B$.

Remark 1.3. Observe that condition (1.6) allows the coefficient $a(x)$ to vanish in a subset of Ω. The results of Theorem 1.2 can be compared with those of [1] and [4]. In [1] (dedicated to Ireneo Peral for his 70th birthday), existence and L^p-regularity of solutions for the equation

$$-\mathrm{div}(M(x)\,\nabla u) + a\,|u|^{r-2}\,u = B\,\frac{u}{|x|^2} + f(x)\,,$$

is proved for any $B > 0$, under the assumption $a > 0$ (so that a cannot vanish), and $r > 2^*$: note that in this case the lower order term has a much stronger growth with respect to the one in Eq (1.7). These results were then generalized in [4, Theorem 2.1], where existence of solutions for the equation

$$-\mathrm{div}(M(x)\,\nabla u) + a(x)\,|u|^{r-2}\,u = B\,\frac{u}{|x|^2} + f(x)\,,$$

is proved for B even larger than $\alpha\,\mathcal{H}^2$ and, roughly speaking, the nonnegative coefficient $a(x)$ can vanish in a set of positive measure in the interior of Ω, under the assumptions $r > 2$ and $a(x)\,|f(x)|^{\frac{r}{r-1}}$ belongs to $L^1(\Omega)$. Note that also in this case we have that the lower order term grows more than linearly, but that the datum f need not be "bounded" with respect to $a(x)$.

Remark 1.4. Note that if B tends to $\alpha\,\mathcal{H}^2$, then p_B tends to 2^*, while if B tends to zero then p_B tends to infinity. Note also that, in contrast with what happens in the case $B = 0$, the value of the constant Q in (1.6) has no influence on the summability of the solution.

Remark 1.5. We remark the similarity between the summability result of Theorem 1.2, and the summability result of the paper [8] quoted before. In this latter paper, if $m > \frac{2N}{N+2}$ and $B > 0$ are such that (1.4) holds, then the weak solution u in $W_0^{1,2}(\Omega)$ of Eq (1.1) belongs to $L^{m^{**}}(\Omega)$. In our Theorem 1.2, the weak solution u in $W_0^{1,2}(\Omega)$ of Eq (1.7) belongs to $L^p(\Omega)$ for every p such that $B < F(p)$ (this inequality is equivalent to inequality $p < p_B$). If we choose $p = m^{**} = \frac{Nm}{N-2m}$, the condition $B < F(p)$ means

$$B < F(m^{**}) = \alpha\,\frac{N}{m^{**}}\Big(N - 2 - \frac{N}{m^{**}}\Big) = \alpha\,\frac{N\,(m-1)\,(N-2m)}{m^2}\,,$$

which is exactly (1.4). Thus, the same assumption on B which yields solutions in $L^{m^{**}}(\Omega)$ for equation (1.1), yields solutions in $L^{m^{**}}(\Omega)$ for Eq (1.7): note however that in the case of Eq (1.7) the datum f only belongs to $L^1(\Omega)$.

We now remark that the function $a(x)$ belongs to $L^1(\Omega)$, while the function $B/|x|^2$ belongs to $L^m(\Omega)$, for every $1 \le m < \frac{N}{2}$, so that it is more summable than $a(x)$. This means that it may happen that the function $a(x)$ dominates the function $B/|x|^2$. In this case, for every $B > 0$ we are going to prove that there exist weak solutions u of Eq (1.7), which belong to $L^\infty(\Omega)$. Our result is the following, and will be proved in Section 3.

Theorem 1.6. *Let $a(x) \ge 0$ and $f(x)$ in $L^1(\Omega)$ be such that (1.6) holds. If $B > 0$ and $\rho > 1$ are such that*

$$a(x) \ge \rho\,\frac{B}{|x|^2}\,, \tag{1.11}$$

then there exists a unique weak solution u in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of (1.7).

Suppose now that $a(x) = C/|x|^2$, with $C > 0$. If $C > B$, the result of Theorem 1.6 states that there exist bounded weak solutions u of (1.7); if $C = B$ then any weak solution u of (1.7) is also a solution of

$$-\mathrm{div}(M(x)\,\nabla u) = f(x)\,,$$

with $|f(x)| \le Q\,B/|x|^2$: it is well known from the results of Stampacchia that in this case u may not be in $L^\infty(\Omega)$. This shows that condition (1.11) (with $\rho > 1$) is somehow necessary in order to have bounded solutions.

If $C < B < \alpha\mathcal{H}^2$ one can only apply Theorem 1.2 to deduce the existence of weak solutions u of Eq (1.7), with u in $L^p(\Omega)$ for every $p < p_B$. In Section 4 we are going to prove that the result of Theorem 1.2 is in some sense sharp: for every $p > p_B$ there exists $C_p < B$, such that for $a(x) = C_p/|x|^2$ (and a suitable function $f(x)$ satisfying (1.6)) there exists a weak solution u of Eq (1.7) such that *u does not belong* to $L^p(\Omega)$.

In the final section of this paper, we will study the boundary value problem associated to a nonlinear quasilinear equation with a lower order term with quadratic growth with respect to the gradient, namely

$$\begin{cases} -\mathrm{div}(M(x)\,\nabla u) + g(u)\,|\nabla u|^2 = B\,\dfrac{u}{|x|^2} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

where $f(x)$ is a function in $L^1(\Omega)$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$ and $g(t)\,t$ is increasing on $(0, +\infty)$ (and decreasing on $(-\infty, 0)$). Also in this case, as in the case of Theorem 1.6, we will prove that the lower order term $g(u)\,|\nabla u|^2$ "dominates" the term $B\,u/|x|^2$, so that existence of solutions in $W^{1,2}_0(\Omega)$ will follow for every $B > 0$.

2. Proof of Theorem 1.2

In what follows, we will denote by C any constant depending on the data of the problem (such as N, Ω, α, β, ...) but never on the approximation parameter n.

Proof. Let n in $\mathbb{N}$ be fixed, and define

$$a_n(x) = \frac{a(x)}{1 + \frac{Q}{n}\,a(x)}\,, \qquad f_n(x) = \frac{f(x)}{1 + \frac{1}{n}\,|f(x)|}\,, \tag{2.1}$$

with $Q > 0$ given by (1.6). Note that, since the function $t \mapsto \frac{t}{1+\frac{Q}{n}\,t}$ is increasing for $t > 0$, from (1.6) it follows that

$$|f_n(x)| = \frac{|f(x)|}{1 + \frac{1}{n}\,|f(x)|} \le \frac{Q\,a(x)}{1 + \frac{1}{n}\,Q\,a(x)} = a_n(x)\,,$$

so that (1.6) is satisfied by $f_n(x)$ and $a_n(x)$ for every n in $\mathbb{N}$. A straightforward application of the Schauder theorem yields that for every n in $\mathbb{N}$ there exists a weak solution u_n in $W^{1,2}_0(\Omega)$ of the equation

$$\begin{cases} -\mathrm{div}(M(x)\,\nabla u_n) + a_n(x)\,u_n = \dfrac{B}{|x|^2 + \frac{1}{n}}\,\dfrac{u_n}{1 + \frac{1}{n}\,|u_n|} + f_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Furthermore, since the right hand side is bounded by $B\,n^2 + n$, and since $a_n(x) \geq 0$, the function u_n belongs to $L^\infty(\Omega)$ thanks to the results by G. Stampacchia (see [11]).

We are going to prove that if $B > 0$ satisfies assumption (1.9) then the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. In order to do that, we choose u_n as test function in the weak formulation for Eq (2.2) to deduce that

$$\int_\Omega M(x)\,\nabla u_n \cdot \nabla u_n + \int_\Omega a_n(x)\,u_n^2 = \int_\Omega \frac{B}{|x|^2 + \frac{1}{n}}\,\frac{u_n^2}{1 + \frac{1}{n}\,|u_n|} + \int_\Omega f_n(x)\,u_n\,.$$

Using (1.2) and (1.6), we obtain from the previous identity that

$$\alpha \int_\Omega |\nabla u_n|^2 + \int_\Omega a_n(x)\,u_n^2 \leq B \int_\Omega \frac{u_n^2}{|x|^2} + Q \int_\Omega a_n(x)\,|u_n|\,, \tag{2.3}$$

which implies, thanks to Hardy inequality (1.3), that

$$\alpha \int_\Omega |\nabla u_n|^2 + \int_\Omega a_n(x)\,|u_n|\,(|u_n| - Q) \leq \frac{B}{\mathcal{H}^2} \int_\Omega |\nabla u_n|^2\,. \tag{2.4}$$

We now observe that since $t\,(t - Q) \geq -Q^2$ for every $0 \leq t \leq Q$ and that $a(x) \geq 0$, we have

$$\begin{aligned}\int_\Omega a_n(x)\,|u_n|\,(|u_n| - Q) &= \int_{\{|u_n|\leq Q\}} a_n(x)\,|u_n|\,(|u_n| - Q) + \int_{\{|u_n|>Q\}} a_n(x)\,|u_n|\,(|u_n| - Q)\\ &\geq \int_{\{|u_n|\leq Q\}} a_n(x)\,|u_n|\,(|u_n| - Q) \geq -Q^2 \int_\Omega a_n(x)\\ &\geq -Q^2 \int_\Omega a(x)\,.\end{aligned}$$

Using this inequality in (2.4) we obtain that

$$\Big(\alpha - \frac{B}{\mathcal{H}^2}\Big) \int_\Omega |\nabla u_n|^2 \leq Q^2 \int_\Omega a(x)\,. \tag{2.5}$$

Thanks to assumption (1.9), and to the fact that $a(x)$ belongs to $L^1(\Omega)$, from (2.5) we obtain that the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, as desired. Therefore, there exists a function u in $W_0^{1,2}(\Omega)$ such that, up to subsequences, the sequence $\{u_n\}$ converges to u weakly in $W_0^{1,2}(\Omega)$, weakly in $L^{2^*}(\Omega)$, and almost everywhere in Ω.

Thanks again to the boundedness of $\{u_n\}$ in $W_0^{1,2}(\Omega)$, and to Hardy inequality (1.3), the sequence $\{\frac{u_n^2}{|x|^2}\}$ is bounded in $L^1(\Omega)$; therefore, from (2.3) (dropping the positive first term) and from Young inequality we have that there exists $C > 0$ such that

$$\int_\Omega a_n(x)\,u_n^2 \leq C + Q \int_\Omega a_n(x)\,|u_n| \leq C + \frac{1}{2} \int_\Omega a_n(x)\,u_n^2 + C \int_\Omega a_n(x) \leq C + \frac{1}{2} \int_\Omega a_n(x)\,u_n^2\,.$$

Therefore, the sequence $\{a_n(x)\,u_n^2\}$ is bounded in $L^1(\Omega)$. Let now E be a measurable subset of Ω. Then, for $k > 0$ we have

$$\int_E a_n(x)\,|u_n| = \int_{E\cap\{|u_n|\leq k\}} a_n(x)\,|u_n| + \int_{E\cap\{|u_n|>k\}} a_n(x)\,|u_n| \leq k \int_E a_n(x) + \frac{1}{k} \int_\Omega a_n(x)\,u_n^2 \leq k \int_E a(x) + \frac{C}{k}$$

where we have used the boundedness of $\{a_n(x)\, u_n^2\}$ in $L^1(\Omega)$ in the last passage. Let now $\varepsilon > 0$ be fixed, and choose $k > 0$ large enough to have that $\frac{C}{k} < \varepsilon$. Once $k > 0$ has been chosen, let meas(E) be small enough in order to have

$$k \int_E a(x) < \varepsilon\,.$$

Such a choice of E is possible since $a(x)$ belongs to $L^1(\Omega)$. We have thus proved that if meas(E) is small enough, then

$$\int_E a_n(x)\,|u_n| < 2\varepsilon\,, \qquad \forall n \in \mathbb{N}\,,$$

that is, that the sequence $\{a_n(x)\, u_n\}$ is uniformly equi-integrable. Since it is almost everywhere convergent to $a(x)\, u$, Vitali theorem implies that

the sequence $\{a_n(x)\, u_n\}$ strongly converges to $a(x)\, u$ in $L^1(\Omega)$.

This convergence, the convergences already proved on the sequence $\{u_n\}$, and the strong convergence of $\frac{B}{|x|^2+\frac{1}{n}}$ to $\frac{B}{|x|^2}$ in $L^s(\Omega)$, for every $s < \frac{N}{2}$, imply that one can pass to the limit in the identities

$$\int_\Omega M(x)\,\nabla u_n \cdot \nabla\varphi + \int_\Omega a_n(x)\, u_n\, \varphi = \int_\Omega \frac{B}{|x|^2 + \frac{1}{n}}\, \frac{u_n\, \varphi}{1 + \frac{1}{n}\,|u_n|} + \int_\Omega f_n(x)\,\varphi\,,$$

for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, to have (1.10) holds true.

Once existence of a weak solution has been proved, we turn now to uniqueness. Suppose that u and v are two weak solutions of (1.7), and define $w = u - v$. Since w belongs to $W_0^{1,2}(\Omega)$, from Hardy inequality (1.3) it follows that $\frac{w^2}{|x|^2}$ belongs to $L^1(\Omega)$. For $k > 0$, and t in $\mathbb{R}$, let us define

$$T_k(t) = \max(-k, \min(t, k))\,, \qquad G_k(t) = t - T_k(t) = (|t| - k)^+ \operatorname{sgn}(t)\,, \tag{2.6}$$

and consider

$$S_k(x) = B\, \frac{T_k(w(x))\, G_k(w(x))}{|x|^2}\,.$$

Since we have that S_k tends to zero almost everywhere in Ω, and since

$$0 \le S_k(x) = B\, \frac{T_k(w(x))\, G_k(w(x))}{|x|^2} \le B\, \frac{[w(x)]^2}{|x|^2} \in L^1(\Omega)\,,$$

by Lebesgue theorem we have that the sequence $\{S_k\}$ tends to zero strongly in $L^1(\Omega)$. Observe now that w is a weak solution of

$$-\operatorname{div}(M(x)\,\nabla w) + a(x)\, w = B\, \frac{w}{|x|^2}\,,$$

that is, we have

$$\int_\Omega M(x)\,\nabla w \cdot \nabla\varphi + \int_\Omega a(x)\, w\, \varphi = B \int_\Omega \frac{w\,\varphi}{|x|^2}\,,$$

for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Choosing $\varphi = T_k(w)$ we obtain, using (1.2), dropping a positive term, and recalling that $w = T_k(w) + G_k(w)$, that

$$\alpha \int_\Omega |\nabla T_k(w)|^2 \le B \int_\Omega \frac{w\, T_k(w)}{|x|^2} = B \int_\Omega \frac{[T_k(w)]^2}{|x|^2} + B \int_\Omega \frac{T_k(w)\, G_k(w)}{|x|^2}\,.$$

Recalling the definition of S_k, and using Hardy inequality (1.3), the previous inequality implies that

$$\alpha \int_\Omega |\nabla T_k(w)|^2 \le \frac{B}{\mathcal{H}^2} \int_\Omega |\nabla T_k(w)|^2 + \int_\Omega S_k(x)\,,$$

which yields that

$$\Big(\alpha - \frac{B}{\mathcal{H}^2}\Big) \int_\Omega |\nabla T_k(w)|^2 \le \int_\Omega S_k(x)\,.$$

Recalling that $0 < B < \alpha\mathcal{H}^2$, and letting k tend to infinity, we obtain from the above inequality, using that the sequence $\{S_k\}$ tends to zero in $L^1(\Omega)$, that

$$0 \le \Big(\alpha - \frac{B}{\mathcal{H}^2}\Big) \int_\Omega |\nabla w|^2 \le 0\,,$$

which then implies that $w = 0$, and so $u = v$.

We now turn to the second part of the result. Since we already know that there exists a solution u in $W_0^{1,2}(\Omega)$, in order to show that u belongs to $L^p(\Omega)$, for every $2^* \le p < p_B$, it is enough to prove that the sequence $\{u_n\}$ is bounded in $L^p(\Omega)$ for every $2^* \le p < p_B$. To this aim, let $\gamma \ge 1$, and choose $|u_n|^{2\gamma-2}\, u_n$ as test function in the weak formulation of Eq (2.2) (this can be done since u_n belongs to $L^\infty(\Omega)$ for every n in $\mathbb{N}$). We have

$$(2\gamma - 1) \int_\Omega M(x)\,\nabla u_n \cdot \nabla u_n\, |u_n|^{2\gamma-2} + \int_\Omega a_n(x)\, |u_n|^{2\gamma} = \int_\Omega \frac{B}{|x|^2 + \frac{1}{n}}\, \frac{|u_n|^{2\gamma}}{1 + \frac{1}{n}\,|u_n|} + \int_\Omega f_n(x)\, |u_n|^{2\gamma-2}\, u_n\,.$$

Using (1.2) and (1.6) we obtain from the previous identity that

$$\alpha\,(2\gamma - 1) \int_\Omega |\nabla u_n|^2\, |u_n|^{2\gamma-2} + \int_\Omega a_n(x)\, |u_n|^{2\gamma} \le B \int_\Omega \frac{|u_n|^{2\gamma}}{|x|^2} + Q \int_\Omega a_n(x)\, |u_n|^{2\gamma-1}\,. \tag{2.7}$$

We now remark that

$$|\nabla u_n|^2\, |u_n|^{2\gamma-2} = \frac{1}{\gamma^2}\, |\nabla |u_n|^\gamma|^2\,,$$

so that (2.7) can be rewritten as

$$\alpha\, \frac{2\gamma - 1}{\gamma^2} \int_\Omega |\nabla |u_n|^\gamma|^2 + \int_\Omega a_n(x)\, |u_n|^{2\gamma-1}\, (|u_n| - Q) \le B \int_\Omega \frac{(|u_n|^\gamma)^2}{|x|^2}\,. \tag{2.8}$$

Thus, using Hardy inequality (1.3), from (2.8) we deduce that

$$\Big(\alpha\, \frac{2\gamma - 1}{\gamma^2} - \frac{B}{\mathcal{H}^2}\Big) \int_\Omega |\nabla |u_n|^\gamma|^2 + \int_\Omega a_n(x)\, |u_n|^{2\gamma-1}\, (|u_n| - Q) \le 0\,. \tag{2.9}$$

Since $t^{2\gamma-1}\,(t - Q) \ge -Q^{2\gamma}$ for every $0 \le t \le Q$, we have

$$\int_\Omega a_n(x)\, |u_n|^{2\gamma-1}\, (|u_n| - Q) \ge \int_{\{|u_n| \le Q\}} a_n(x)\, |u_n|^{2\gamma-1}\, (|u_n| - Q) \ge -Q^{2\gamma} \int_\Omega a_n(x) \ge -Q^{2\gamma} \int_\Omega a(x)\,,$$

so that from (2.9) we obtain that

$$\Big(\alpha\, \frac{2\gamma - 1}{\gamma^2} - \frac{B}{\mathcal{H}^2}\Big) \int_\Omega |\nabla |u_n|^\gamma|^2 \le Q^{2\gamma} \int_\Omega a(x)\,. \tag{2.10}$$

If we now assume that $\gamma \geq 1$ is such that

$$\alpha \, \frac{2\gamma - 1}{\gamma^2} - \frac{B}{\mathcal{H}^2} > 0 \,, \tag{2.11}$$

from (2.10) it follows that the sequence $\{|u_n|^\gamma\}$ is bounded in $W_0^{1,2}(\Omega)$. Thanks to Sobolev embedding, this implies that the sequence $\{|u_n|^\gamma\}$ is bounded in $L^{2^*}(\Omega)$, so that the sequence $\{u_n\}$ is bounded in $L^{2^*\gamma}(\Omega)$.

Summing up, we have that if (2.11) holds, that is if

$$B < \alpha \, \frac{2\gamma - 1}{\gamma^2} \, \mathcal{H}^2 = \alpha \, \frac{2\gamma - 1}{(2\gamma)^2} \, (N-2)^2 \,, \tag{2.12}$$

then the sequence $\{u_n\}$ is bounded in $L^{2^*\gamma}(\Omega)$. Setting $p = 2^*\gamma$ we have, after some straightforward simplifications, that

$$\alpha \, \frac{2\gamma - 1}{(2\gamma)^2} \, (N-2)^2 = \alpha \, \frac{N}{p} \Big(N - 2 - \frac{N}{p} \Big) = F(p) \,.$$

Recalling that by definition $F(p_B) = B$, we have from (2.12) that the sequence $\{u_n\}$ is bounded in $L^p(\Omega)$ for every $p \geq 2^*$ such that $F(p_B) < F(p)$; since F is decreasing on $[2^*, +\infty)$, we have that the sequence $\{u_n\}$ is bounded in $L^p(\Omega)$ for every $2^* \leq p < p_B$, as desired. □

3. Proof of Theorem 1.6

Proof. In this case, by assumption (1.11), for any n in $\mathbb{N}$, we slightly modify the approximate problems (2.2) and we consider the solution u_n of

$$\begin{cases} -\mathrm{div}(M(x)\,\nabla u_n) + a_n(x)\,u_n = \dfrac{B}{|x|^2 + \frac{\rho\,Q\,B}{n}} \, \dfrac{u_n}{1 + \frac{1}{n}\,|u_n|} + f_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Observe that since the function $t \mapsto \frac{t}{1+\frac{Q}{n}\,t}$ is increasing, and since (1.11) holds, we have

$$a_n(x) = \frac{a(x)}{1 + \frac{Q}{n}\,a(x)} \geq \frac{\rho\,\frac{B}{|x|^2}}{1 + \frac{Q}{n}\,\frac{\rho\,B}{|x|^2}} = \frac{\rho\,B}{|x|^2 + \frac{\rho\,Q\,B}{n}} = \rho \, \frac{B}{|x|^2 + \frac{\rho\,Q\,B}{n}} \,,$$

so that

$$\frac{B}{|x|^2 + \frac{\rho\,Q\,B}{n}} \leq \frac{1}{\rho}\,a_n(x) \,, \qquad \forall n \in \mathbb{N} \,. \tag{3.2}$$

Let $k > 0$ and choose $G_k(u_n)$ as test function in the weak formulation of (3.1) (recall that the function $G_k(t)$ is defined by (2.6)). We have

$$\int_\Omega M(x)\,\nabla u_n \cdot \nabla G_k(u_n) + \int_\Omega a_n(x)\,u_n\,G_k(u_n) = \int_\Omega \frac{B}{|x|^2 + \frac{\rho\,Q\,B}{n}} \, \frac{u_n\,G_k(u_n)}{1 + \frac{1}{n}\,|u_n|} + \int_\Omega f_n(x)\,G_k(u_n) \,.$$

Using (1.2) and (1.6), as well as (3.2), from the above identity we obtain that

$$\alpha \int_\Omega |\nabla G_k(u_n)|^2 + \int_\Omega a_n(x)\,|u_n|\,|G_k(u_n)| \leq \int_\Omega \frac{B}{|x|^2 + \frac{\rho\,Q\,B}{n}} \, |u_n|\,|G_k(u_n)| + Q \int_\Omega a_n(x)\,|G_k(u_n)|$$

$$\leq \frac{1}{\rho} \int_\Omega a_n(x)\,|u_n|\,|G_k(u_n)| + Q \int_\Omega a_n(x)\,|G_k(u_n)|\,.$$

From the above inequality we obtain, dropping a positive term, that

$$\int_\Omega a_n(x)\left[\left(1 - \frac{1}{\rho}\right)|u_n| - Q\right]|G_k(u_n)| \leq 0\,.$$

Choosing $k > 0$ such that $(1 - \frac{1}{\rho})\,k > Q$, we therefore have that

$$0 \leq \int_\Omega a_n(x)\left[\left(1 - \frac{1}{\rho}\right)|u_n| - Q\right]|G_k(u_n)| \leq 0\,,$$

from which it follows that $G_k(u_n) = 0$; that is, $|u_n| \leq k$ almost everywhere in Ω, which implies that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$. Once this boundedness has been proved, choosing u_n as test function in the weak formulation of Eq (3.1), and using (1.2), one has (dropping a positive term) that

$$\alpha \int_\Omega |\nabla u_n|^2 \leq B \int_\Omega \frac{u_n^2}{|x|^2} + \int_\Omega |f_n(x)|\,|u_n| \leq C\,\|u_n\|^2_{L^\infty(\Omega)} \leq C\,,$$

so that the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. From these estimates, and reasoning as in the proof of Theorem 1.2, it follows that the weak limit u of the sequence $\{u_n\}$ in $W_0^{1,2}(\Omega)$ is a weak solution u of (4.1) that belongs to $L^\infty(\Omega)$. Uniqueness is then proved as in the proof of Theorem 1.2. □

Remark 3.1. Note that any weak solution of (1.7) is also a weak solution of

$$-\mathrm{div}(M(x)\,\nabla u) + b(x)\,u = f(x)\,,$$

where

$$b(x) = \left(a(x) - \frac{B}{|x|^2}\right).$$

Since under assumption (1.11) we have that

$$b(x) \geq \left(1 - \frac{1}{\rho}\right)a(x) = A(x)\,,$$

and since if $|f(x)| \leq Q\,a(x)$ one also has that $|f(x)| \leq \boldsymbol{Q}\,A(x)$, with

$$\boldsymbol{Q} = \frac{Q}{1 - \frac{1}{\rho}}\,,$$

the boundedness result of Theorem 1.6 can also be obtained using the boundedness result of [2]. It is by the convenience of the reader that we have given a self contained proof of the above theorem.

4. An example

As stated in the Introduction, we are going to prove that if $\alpha = 1$, then for every $0 < B < \mathcal{H}^2$, and for every $p > p_B$ there exist $a_p(x) \geq 0$, with $a_p(x) \leq B/|x|^2$, and $f_p(x)$, such that (1.6) holds, for which there exists a weak solution u in $W_0^{1,2}(\Omega)$ of

$$-\Delta u + a_p(x)\, u = B\, \frac{u}{|x|^2} + f_p(x)\,,$$

with u that does not belong to $L^p(\Omega)$. Therefore, Theorem 1.2 is sharp since the summability of the solution u can be at most $L^{p_B}(\Omega)$, and not better.

In order to prove the result, let $\Omega = B_1(0)$, let $0 < B < \mathcal{H}^2$, and let $p > p_B > 2^*$; since $F(p_B) = B$, and F is decreasing, we have that $B > F(p)$. Define

$$u(x) = \frac{1}{|x|^{\frac{N}{p}}} - 1\,,$$

and observe that u is the weak solution in $W_0^{1,2}(\Omega)$ of the equation

$$\begin{cases} -\Delta u = F(p)\, \dfrac{u}{|x|^2} + \dfrac{F(p)}{|x|^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Define

$$C_p = B - F(p)\,, \quad a_p(x) = \frac{C_p}{|x|^2} \qquad \text{and} \qquad Q_p = \frac{F(p)}{B - F(p)}\,, \quad f_p(x) = \frac{Q_p}{|x|^2}\,.$$

Thanks to these definitions, we have that $0 \leq a_p(x) \leq B/|x|^2$, that $|f_p(x)| \leq Q_p\, a_p(x)$, and that u, which does not belong to $L^p(\Omega)$, is a weak solution of

$$\begin{cases} -\Delta u + a_p(x)\, u = B\, \dfrac{u}{|x|^2} + f_p(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Since such weak solution is unique by Theorem 1.2, we have proved that the result of that theorem is sharp.

5. Lower order terms with quadratic growth with respect to the gradient

The result of Theorem 1.6 states that if the lower order term $a(x)$ dominates the Hardy potential $B/|x|^2$, then existence of bounded solutions follows for any $B > 0$. The same result is true if one considers gradient dependent lower order terms having quadratic growth. Our result is the following.

Theorem 5.1. *Let $B > 0$, and let $f(x)$ be a function in $L^1(\Omega)$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(0) = 0$ and that $g(t)\, t$ is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$. Then there exists a weak solution u in $W_0^{1,2}(\Omega)$ of the boundary value problem* (1.12), *that is: $g(u)\, |\nabla u|^2$ belongs to $L^1(\Omega)$ and*

$$\int_\Omega M(x)\, \nabla u \cdot \nabla \varphi + \int_\Omega g(u)\, |\nabla u|^2\, \varphi = B \int_\Omega \frac{u\, \varphi}{|x|^2} + \int_\Omega f(x)\, \varphi\,, \tag{5.1}$$

for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Proof. Let n in $\mathbb{N}$, let $f_n(x) = T_n(f(x))$, and let u_n in $W_0^{1,2}(\Omega)$ be a solution of

$$\begin{cases} -\mathrm{div}(M(x)\,\nabla u_n) + \dfrac{u_n}{n} + g(u_n)\,|\nabla u_n|^2 = \dfrac{B}{|x|^2 + \frac{1}{n}}\,\dfrac{u_n}{1 + \frac{1}{n}\,|u_n|} + f_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

The existence of u_n follows from the results of [7], where it is also proved that u_n belongs to $L^\infty(\Omega)$ for every n in $\mathbb{N}$ (note that the right hand side of the equation is bounded by $B\,n^2 + n$).

We now follow [5] (see also [6]) and choose $T_1(u_n)$ as test function in the weak formulation of (5.2). We obtain, dropping a positive term and using (1.2),

$$\alpha \int_\Omega |\nabla T_1(u_n)|^2 + \int_\Omega T_1(u_n)\, g(u_n)|\nabla u_n|^2 \le \int_\Omega \frac{B}{|x|^2 + \frac{1}{n}}\,\frac{u_n\,T_1(u_n)}{1 + \frac{1}{n}\,|u_n|} + \int_\Omega |f_n(x)|\,|T_1(u_n)|\,.$$

From this inequality it follows, using that $g(t)\,t$ is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$, that

$$\alpha \int_{\{|u_n|\le 1\}} |\nabla u_n|^2 + \max\{g(1), |g(-1)|\} \int_{\{|u_n|>1\}} |\nabla u_n|^2 \le B \int_\Omega \frac{|u_n|}{|x|^2} + \int_\Omega |f(x)|\,.$$

Defining $\mu = \min(\alpha, \max\{g(1), |g(-1)|\})$, from the above inequality it follows that

$$\mu \int_\Omega |\nabla u_n|^2 \le B \int_\Omega \frac{|u_n|}{|x|^2} + \int_\Omega |f(x)|\,. \tag{5.3}$$

We now observe that by Hölder, Sobolev and Young inequalities we have

$$\int_\Omega \frac{|u_n|}{|x|^2} \le \Big(\int_\Omega |u_n|^{2^*}\Big)^{\frac{1}{2^*}} \Big(\int_\Omega \frac{1}{|x|^{\frac{4N}{N+2}}}\Big)^{\frac{N+2}{2N}} \le C\Big(\int_\Omega |\nabla u_n|^2\Big)^{\frac{1}{2}} \le C \int_\Omega |\nabla u_n|^2 + C\,,$$

where in the second to last passage we have used that $N > 2$ so that $1/|x|^{\frac{4N}{N+2}}$ belongs to $L^1(\Omega)$. Using this inequality in (5.3) we have that

$$\mu \int_\Omega |\nabla u_n|^2 \le BC \int_\Omega |\nabla u_n|^2 + \int_\Omega |f(x)| + C\,,$$

from which it follows that

$$\text{the sequence } \{u_n\} \text{ is bounded in } W_0^{1,2}(\Omega).$$

Passing to a subsequence if necessary, we may assume the sequence $\{u_n\}$ converges to a function u weakly in $W_0^{1,2}(\Omega)$, strongly in $L^\rho(\Omega)$ for every $\rho < 2^*$, and almost everywhere in Ω. From these convergences it follows that

$$\frac{B}{|x|^2 + \frac{1}{n}}\,\frac{u_n}{1 + \frac{1}{n}\,|u_n|} \quad \text{strongly converges in } L^\sigma(\Omega), \text{ for every } \sigma < 2_*.$$

In particular, it converges in $L^1(\Omega)$, so that one can repeat the proof of [5] to have that

$$\text{the sequence } \{u_n\} \text{ strongly converges in } W_0^{1,2}(\Omega), \tag{5.4}$$

which in turn implies that

$$\text{the sequence } \{\nabla u_n\} \text{ almost everywhere converges to } \nabla u. \tag{5.5}$$

In order to pass to the limit in the weak formulation of (5.2), we need to deal with the lower order term with quadratic growth with respect to the gradient: we are going to prove that

$$\text{the sequence } \{g(u_n)\,|\nabla u_n|^2\} \text{ strongly converges in } L^1(\Omega) \text{ to } g(u)\,|\nabla u|^2. \tag{5.6}$$

Since we already know that $g(u_n)\,|\nabla u_n|^2$ almost everywhere converges to $g(u)\,|\nabla u|^2$ as a consequence of the almost everywhere convergence of u_n, of the continuity of $g(t)$, and of (5.5), to prove (5.6), by Vitali theorem, it suffices to show the equi-integrability of the sequence $\{g(u_n)\,|\nabla u_n|^2\}$. In order to do that, let $h > 0$, $k > 0$ and choose $\frac{1}{h}\,T_h[G_k(u_n)]$ as test function in the weak formulation of (5.2). Dropping the positive first term, and letting h tend to 0, we obtain (see [5])

$$\int_{\{|u_n|>k\}} |g(u_n)|\,|\nabla u_n|^2 \le B \int_{\{|u_n|>k\}} \frac{|u_n|}{|x|^2} + \int_{\{|u_n|>k\}} |f(x)|\,. \tag{5.7}$$

Since $|u_n|/|x|^2$ is compact in $L^1(\Omega)$, and since $\text{meas}(\{|u_n| > k\})$ tends to zero as k tends to infinity uniformly in n in $\mathbb{N}$, from (5.7) we have that

$$\lim_{k\to+\infty} \int_{\{|u_n|>k\}} |g(u_n)|\,|\nabla u_n|^2 = 0\,, \qquad \text{uniformly in } n \text{ in } \mathbb{N}. \tag{5.8}$$

Let now E be a measurable subset of Ω; for every $k > 0$ we have

$$\begin{aligned}\int_E |g(u_n)|\,|\nabla u_n|^2 &= \int_{E\cap\{|u_n|\le k\}} |g(u_n)|\,|\nabla u_n|^2 + \int_{E\cap\{|u_n|>k\}} |g(u_n)|\,|\nabla u_n|^2 \\ &\le \max_{[-k,k]} |g(t)| \int_E |\nabla u_n|^2 + \int_{\{|u_n|>k\}} |g(u_n)|\,|\nabla u_n|^2\,.\end{aligned}$$

We now fix $\varepsilon > 0$ and use (5.8) to choose $k > 0$ large enough in order to have

$$\int_{\{|u_n|>k\}} |g(u_n)|\,|\nabla u_n|^2 < \varepsilon\,.$$

Once $k > 0$ has been chosen, we use (5.4) in order to choose $\text{meas}(E)$ small enough to have

$$\max_{[-k,k]} |g(t)| \int_E |\nabla u_n|^2 < \varepsilon\,.$$

Therefore, for every $\varepsilon > 0$ we have that if $\text{meas}(E)$ is small enough, then

$$\int_E |g(u_n)|\,|\nabla u_n|^2 < 2\varepsilon\,, \qquad \forall n \in \mathbb{N}\,,$$

which proves the equi-integrability of the sequence $\{g(u_n)\,|\nabla u_n|^2\}$, which implies that (5.6) holds true.

Having proved all these convergences, we can pass to the limit in the identities

$$\int_\Omega M(x)\,\nabla u_n \cdot \nabla\varphi + \int_\Omega g(u_n)\,|\nabla u_n|^2\,\varphi = \int_\Omega \frac{B}{|x|^2 + \frac{1}{n}}\,\frac{u_n\,\varphi}{1 + \frac{1}{n}\,|u_n|} + \int_\Omega f_n(x)\,\varphi\,,$$

which hold for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, to have that u is such that (5.1) holds. □

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Conflict of interest

The authors declare no conflict of interest.

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