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What makes a task a problem in early childhood education?

This article begins with a theoretical discussion of the characteristics that a task should feature to be regarded as a mathematics problem suitable for pre-primary students. Those considerations are followed by a report of a classroom experience in which three problems involving quotative or partitive division were posed to pre-primary school pupils to determine the presence of otherwise of the respective characteristics. The findings show that the characteristics of pre-primary education problems depend on two factors: mathematical activity that engages pupils and a structure that favours both their understanding of the problem and the application and verification of the solutions.

Keywords: Characteristics of Problems, Division, Early childhood, Mathematical Problems, Pre-primary, Problem Solving

1 Introduction

Pre-primary education has become a growing concern in today's societies due to the impact of quality early education on the development of civic attitudes (OECD, 2016). As part of that development, early childhood mathematics education is a subject of interest for the scientific community. Research on pre-primary school children's aptitudes (Clements & Sarama, 2007; Mulligan & Vergnaud, 2006; Schoenfeld & Stipek, 2011) has prompted a number of mathematics teachers' organisations and groups to take a position on early childhood mathematics education (NAEYC & NCTM, 2010).

The significance of early mathematics learning is not associated with quality instruction, however (Clements & Sarama, 2013). Problem solving, for instance, is not included as a process to be developed in early childhood education, for it is regarded as too complex for pre-primaries. Most research on the subject has been conducted with primary, secondary school or higher education students (Lesh & Zawojewski, 2007), with very few, albeit promising, studies on early childhood. The findings of some of these studies show that suitable selection and use of problems and the related solving processes encourage skill development and help teachers gain insight into their pupils' thought processes (Charlesworth & Leali 2012; De Castro & Hernández, 2014; Matalliotaki, 2012).

Given that choosing classroom mathematics problems is no easy task, particularly for such young children, this study focused on the characteristics suitable problems should feature. The definition of such characteristics is first addressed by posing the question: what elements should characterise a pre-primary task for it to constitute a problem? A review of the literature reveals the consensuses reached from different theoretical postulates. A classroom experience is described, in which the characteristics defined were empirically contrasted by posing three problems to 5-year-old pupils. The three multiplicative structure problems studied were drawn from earlier research that identified problem-solving aptitudes in children at that age (Davis & Pepper, 1992; Nelson & Kirkpatrick, 1975).

2 Theoretical framework

A theoretical discussion of the characteristics of effective tasks and problems follows.

2. 1. Classroom mathematics problems in pre-primary education

As in all other stages of schooling, in pre-primary education problem solving is a key means of developing children's mathematical knowledge (Bristz & Richard, 1992; Castro & Castro, 2016). Solving meaningful problems contributes to the development of higher thought processes and the discovery of a series of strategies that further pupils' ability to solve new problems. Children acquire a sense of mathematical ideas by actively participating in the solution of a variety of mathematical problems (Britz & Richard, 1992). Problems have been characterised from a number of perspectives: educational (Kilpatrick, 1980), philosophical (Agre, 1982) and psychological (Mayer & Wittrock, 2006). One widely accepted definition describes problems as situations that involve a subject in a series of cognitive and non-cognitive, non-predetermined processes (Castro & Castro, 2016; NCTM, 2000; Reys, Lindquist, Lambdin & Smith, 2009; Van de Walle, 2003). For what one solver may be a problem, then, for another may be no more than a routine exercise for which there is an immediate answer. In early childhood, problem solver's consideration has greater emphasis in light of factors such as each child's cognitive development or the greater or lesser ex-ante stimulus received.

Facing challenges and consequently solving problems comes naturally to such young children (Britz & Richard, 1992). The world is new for them and they are innately curious and flexible when confronting situations for the first time (NCTM, 2000). Teachers should respect and stimulate that innate problem-solving inclination based on intuitive and informal mathematical knowledge with a view to expanding and consolidating such willingness. Against that backdrop, the question that might be posed is: what characteristics should a task feature to constitute a problem in pre-primary education?

2.2 Characterising problems in pre-primary education

Despite the establishment of a general consensus, the question of what constitutes a problem is the object of constant evolution and revision. In particular, authors such as Nelson & Kirkpatrick (1975), Van de Walle (2003), Yee (2009) and Lesh, English, Riggs

& Sevis (2013) have proposed lists of characteristics that an effective pre-primary task or problem should feature.

Nelson and Kirkpatrick (1975) listed seven characteristics, the first three of which stress the role of the situation. Their list includes: (a) mathematics significance, (b) the involvement of real objects, (c) the engagement of children's interest, (d) The role of action, (e) different levels of solutions, (f) the variety of physical embodiments, and (g) the possibility that children know when a problem has a solution.

Van der Walle (2003), in turn, identified three characteristics of a problematic task.

- What is problematic must be the mathematics. The task must focus pupils' attention on the mathematical ideas implied. Their interest must be sought not only through problems, but with the mathematics used.
- Tasks must be accessible to students. The degree of difficulty must be such that it affords opportunities to build learning sequences but should not entail inaccessible challenges. That calls for good diagnostics, for given the breadth of classroom variability the literature can provide no more than guidelines.
- Tasks must require pupils to justify and explain their answers and procedures.
- Tasks must include clear expectations on how ideas and the solution will be shared. The use of different formats must be explained through the use of a variety of representations (drawings, words and symbols).

Yee (2009) identified four non-elementary cognitive processes that must be required of good problem-solving tasks.

- Reasoning must be complex and non-algorithmic.

- Analysis of what needs to be done should be fostered and the use of heuristic strategies incentivised.
- Mathematical concepts, processes or relationships must be explored.
- The context must be understood to arouse interest and motivate pupils to seek a solution.

Along these lines, Lesh, English, Riggs and Sevis (2013) proposed two questions to test for problem suitability.

> (a) Do the children try to make sense of the problem using their own 'real life' experiences – instead of simply trying to do what they believe that some authority (such as the teacher) considers to be correct (even if it doesn't make sense to them)? (b) When the children are aware of several different ways of thinking about a given problem, are they themselves able to assess the strengths and weaknesses of these alternatives – without asking their teacher or some other authority? (p. 38)

The approach proposed here, which aims to identify tasks with real life, focuses on four characteristics.

- The result is not just a 'short answer'.
- Solvers must know who needs the result and why.
- Reaching the answer is a multi-stage process.
- The answer involves integrating ideas and procedures from several areas.

The aforementioned characterisations and others to be found in the literature concur in a number of significant points. One is the emphasis on children and their context, the wealth of mathematical ideas involved or the language used to understand and express solving procedures. By way of synthesis, in this study a pre-primary classroom problem is defined as one that features the following characteristics.

(1) C1. Reasoning. The problem must explore and develop mathematical ideas through reasoning, the use of strategies, as well as a number of trial and error cycles, rather than through algorithms.

(2) C2. Contexts. The problem must refer to situations familiar to the child. Situations need not necessarily be real from an adult's standpoint: stories, films, cartoon series are also acceptable.

(3) C3. Challenge. The task must induce the child to seek the solution. That may be furthered with different representations (verbal, physical, graphic), requiring the child to handle, transform or modify materials.

(4) C4. Multiple solutions. The problem must afford different levels of solutions, which must not consist in mere short answers.

(5) C5. Expandability. The mathematical structure must be applicable to a number of situations to enable children to generalise.

(6) C6. Comprehensibility. The problem must be understandable for all children, who must be convinced that they can solve it and know when they have found the solution.

A classroom experience was conducted to validate the aforementioned characteristics. To that end, three multiplicative word problems used in earlier research (Davis & Pepper, 1992; Nelson & Kirkpatrick, 1975) were selected and posed to several groups of pupils for analysis on the grounds of the characteristics proposed.

3. Method

This qualitative-descriptive study involved exploratory research in mathematics instruction. The methodology deployed, the population studied, the research design and the problems used are described in the sections below.

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3.1.1 Subjects

The subjects, 15 girls and 11 boys enrolled in the same class of third year, second level early childhood (pre-primary) education at a school in Granada, Spain, were all 5 to 6 years old. Whilst the participants were used to handling classroom manipulatives, they had not been taught partitive or quotative division.

3.1.2 Problems

Multiplicative structure, and more specifically quotative and partitive division, problems that had been used in earlier studies (Davis & Pepper, 1992; Nelson & Kirkpatrick, 1975) were chosen, and research has yielded promising results in connection with the ability of children of these ages to solve such problems (Matalliotaki, 2012). More specifically, three problems consisting in two exercises each were used.

Problem 1. The pirate panda activity (Davis & Pepper, 1992). The children were gathered around a table on which three plastic figurines representing pirates were set. They were given 12 coins and told: ‘three pirates want to share their booty equally. How many coins does each pirate get? Help them share.’ (Pirate1)

In the second exercise, a fourth pirate was set on the table and the children were told that he was entitled to the same number of coins as his buddies. They were asked:. ‘How many coins does each pirate get? Help them share.’ (Pirate2)

Figure 1. Team solving problem Pirate2

Problem 2. Loading and unloading (Nelson & Kirkpatrick, 1975, p. 83). Here the children worked with a working board, four buildings with unfinished roofs, a lorry and 12 square counters symbolising roof tiles. They were told that the lorry was to deliver the tiles to finish the roofs and asked how many tiles it had to deliver to each building (Loading1).

In the second exercise, the manipulatives were eight buildings with unfinished roofs, a lorry and tiles. The explanation was that the lorry had to deliver the tiles to fix the roofs and that each building needed three tiles. The questions they had to answer were: 'do you think there are enough tiles for them all? How many buildings will get enough tiles to fix the roof? How many won't get any?' (Loading2).

Figure 2. Team solving problem Loading1

Problem 3. The ferry (Nelson & Kirkpatrick, 1975, p.73).

Here the working board depicted a river, with a boat and 12 cars as manipulatives. The children were told that the boat would sail several times from one shore to the other, carrying three cars each time and asked: 'how many times will the boat have to cross the river to get all the cars on the other shore? Help the boat move all the cars'. (Ferry1)

Using the same manipulatives, in the second exercise the situation described was as follows. 'The boat sails from one shore to the other four times, always carrying the same number of cars. How many cars will it carry each time? Help the boat move all the cars.' (Ferry2)

Materials were prepared as necessary to be used by the pupils to solve the problems.

Figure 3. Team solving the Ferry problem

The problems were chosen on the grounds of their conformity with the theoretical characteristics defined earlier (see Table 1).

Table 1. Conformity of the problems analysed to the theory on the characteristics of suitable problems

C1	Presence of mathematical ideas: quotative and partitive division
	Accommodation of several solving strategies: trial and error in each phase
C2	Contextualisation: pirate figurines and coins (problem 1), mock-up and lorries (problem 2) and working board, boat and cars (problem 3)
C3	Oral and physical representation: use of manipulatives
C4	Multiple solutions: physical distribution of objects, with several levels of solution, from distribution in no order and subsequent ordering to a predetermined strategy using division
C5	Expandability: second exercises with other amounts to be distributed (with non-exact division) and divisors; generalisation to child's real-life situations involving distribution with quotative and partitive division
C6	Comprehensibility: adaptation of problem wording to children's language and verification of the solution through physical handling of objects

3.1.3 Procedure and data collection

The experience was conducted in the middle of the academic year. Subjects were assigned by their teacher to teams of four pupils each, in keeping with normal classroom dynamics. The teacher took each team separately to another classroom for the experience and explained the problems orally, using the respective manipulatives: figurines, trucks, tiles. She stood by the pupils as they worked and praised their performance. Problem 1 was posed to all teams, whilst some teams did only the first or second exercise in problems 2 and 3, further to the teacher's observations about the time devoted to each.

The sessions were video recorded using one fixed and one moving camera, the fixed to obtain an overview of each team and the moving to record details of the children's actions.

3.1.4 Data analysis

The categories for the deductive content analysis conducted (McMillan & Schumacher, 2010) were the six characteristics of pre-elementary classroom problems defined above:

reasoning (C1), context (C2), challenge (C3), multiple solutions (C4), expandability (C5), and comprehensibility (C6).

The units of analysis were each team's replies to and reactions in each problem, based on both the videos and their transcriptions. The problem-solving strategies proposed by Carpenter, Fennema, Franke, Levi & Empson (1999) and Davis & Pepper (1992), synthesised in Tables 2 and 3, were included as subcategories under the first category, mathematical reasoning. Analyses were conducted by two of the authors independently and the partial results were subsequently harmonised by all the authors.

Table 2. Partitive division strategies

Modelling strategies	S1	Count all the objects and distribute them one by one until none is left. Count the number of objects allocated to one of the resulting groups.
	2	Count all the objects and distribute them two by two (or three by three...) until none is left. Count the objects resulting from the allocation.
	S3	Begin to distribute the objects without counting the total, tallying the number allocated while some are still left and then distributing the remainder.
	S4	Count the total number of objects, allocate seven (more than appropriate) to one group and then make the necessary adjustments until the objects are equally distributed.
	S5	Divide the set of objects into equal subsets, allocating one subset to each group.
Counting strategies	S6	Skip count (3, 6, 9 ... 4, 8, 12), raising a finger or allocating one object with each number. If the last number called concurs with the number of objects to be distributed, the solution is the number of raised fingers.
Addition and subtraction strategies	SO	Find the solution by adding or subtracting.

Table 3. Quotative division strategies

Modelling strategies	S7	Count the total number of objects needed, create groups or group the objects (five-by-five, for instance) and count the number of groups.

	S8	Group the objects five-by-five; after creating several groups, count the total number of objects allocated and use the rest to continue the grouping process.
Counting strategies	S9	Skip count: 3, 6, 9 ... and use fingers, for instance, or the objects, to represent each number. Count one-by-one where necessary (3, 6, (7, 8, 9), 9) and compute the number of groups as the number of fingers or objects.

4 Results

Team performance in connection with the problems is described below under the categories/characteristics established as requisites for pre-primary classroom problems.

C1. Reasoning

A summary of the strategies used by the pupils to solve the problems is given in Table 4.

Table 4. Strategies used by six teams of pupils to solve the problems

	Team 1	Team 2	Team 3	Team 4	Team 5	Team 6
Pirate1 Partitive	S1*	S1*, S2	S1*	S1*	S1	S1*
Pirate2 Partitive	S1*	S4, S1, calculated mentally	S4	S4	S1*	S4
Loading1 Partitive	S1		S1		S1	S1*
Loading2 Quotative	S1*	SO, S5, calculated mentally		S4, S5, calculated mentally		
Ferry1 Quotative	S5	S9, S7	S7, calculated mentally	S7, S9, calculated mentally	S7, calculated mentally	S5
Ferry2 Partitive		Calculated mentally	Calculated mentally	S7 and S5	S4 and S5	

S1* = strategy based on strategy S1

Blank cells = exercise not done

Participants used both modelling and counting strategies, and in some cases mental calculation. Moreover, the fact that the pupils worked in teams induced the appearance of a new strategy, S1*, based on strategy S1 (in which the objects were allocated one by one until none was left, with the number of objects in the resulting groups providing the solution). S1* differed from that approach in that, as pupils worked in teams rather than individually, each team member allocated one object to each group. When the pupils failed to follow a consistent order, the resulting allocation was unequal and had to be adjusted. Two types of adjustment were used. In one, the pupils arranged the objects allocated to each group in columns to visualise the uneven distribution, reorganising the objects in keeping with the height of the columns. In the other, as the objects allocated were not arranged linearly, the number allocated to each group had to be counted to make the adjustment.

Figure 4. Strategy S1* in problem Pirate1

Another significant finding was that problems of the same type (partitive or quotative) were solved using different strategies. For instance, all the teams used strategy S1 or S1* to solve the (partitive) Pirate1 problem, whereas other strategies were brought into play to solve likewise partitive Ferry2. Other variables, such as the materials used or the context, were believed to prompt the use of one or another strategy. One clear example lies in Loading1, where the subjects interacted with a lorry that carried tiles by road. This quotative problem was solved using partitive strategies, on which the tiles were placed on the lorry and distributed among the various buildings. In contrast, quotative problem Ferry1, involving the carriage of cars across a river by a ship, was solved with quotative strategies: the cars were arranged into groups because the ship's unwieldy size made it difficult to move.

In some of the exercises, some teams verbalised their answer, alluding to mental calculations they failed to explain. When solving the Loading2 problem, in addition to using the strategies described, team 2 drew from the similarity with the preceding problem. That indication of some degree of generalisation of the problem structure was illustrated in the following exchange.

Teacher (explaining Loading2): 'you need three tiles for each roof. There are eight buildings. How many won't get tiles? I don't know how many tiles there are. Count them.'

Pupil 1: 'thirteen.'

Teacher: 'thirteen? Are you sure? Count again.'

Pupil 1: 'there are twelve.'

Teacher: 'what's going to happen, then?'

Pupil 1: 'four buildings won't get tiles.'

Pupil 2: 'four won't get a roof.'

Teacher: 'how did you figure that out?'

Pupil 2: 'since we did the playmobile thing before and there were twelve coins...'

Pupil 1: 'it's the same.'

Pupil 2: 'and there were four people, and now there are eight, so four buildings won't get a roof.'

The experience consequently showed that the problems used encouraged reasoning and accommodated several solving strategies.

C2. Context

The children were familiar with the elements used, pirates, lorries and boats, thanks to their presence in the media, stories or games. As ferries might be regarded as less common, the exercises were posed around more familiar objects. In the first problem, the context was the distribution of a treasure among pirates; in the second, a lorry delivering building tiles; and in the third, a boat to move cars. Nothing in the pupils' reactions denoted unfamiliarity with the situations, although they may have been less accustomed to moving cars on a boat.

C3. Challenge

Oral representation was used in all cases when introducing the problems to the pupils, together with the physical materials or manipulatives. The toys chosen, as shown in Figures 1, 2 and 3, were attractive and familiar to the children. With the exception of the boat in the ferry problem, which had to be large enough to accommodate the cars, all the materials could be readily handled. Some of the subjects opted to move the cars alone, instead of loading them on the boat.

C4. Multiple solutions

Pupils' answers were more than just short replies: they included an explanation of the criterion or procedure followed. The four levels of solutions identified in the replies are discussed below.

In the first level pupils solved the problem mentally, giving an oral reply without handling the material. An example follows.

> Pupil: 'I'd give them three each.'
> Teacher: 'why? How did you come up with that answer?'
> Pupil: (thinking)
> Teacher: 'how did you?'
> Pupil: 'it just occurred to me.'
> Teacher: 'how did you share them out?'
> Pupil: 'since there were twelve coins, they each got three.'

In a second level, subjects replied first and then used the material to verify the answer. One of the girls replied to Ferry1 with the following explanation.

> Pupil: 'four times.'
> Teacher: 'how did you figure that out?'
> Pupil (picking up the cars). 'Take three and ship them. Take [another] three and ship them (grouping the cars three-by-three). Take [another] three and ship them and take three [more] and ship them.'

On a third level, the solution was the result of handling the materials. Pupils used the material to solve the problem and then gave an oral reply.

The fourth level consisted of replies based on the materials only. The solution was displayed using the manipulatives, with no oral reply. The pupils regarded the problem as solved after handling the respective materials.

The data also showed that pupils depended on the teacher's evaluation to corroborate that theirs was the right solution.

C5. Expandability

Partitive and quotative division were the underlying mathematical structures in the problems. Although the pupils had not previously worked with division, they had mastered the following skills.

- They could compare amounts, count using their fingers or objects, perform simple mental calculations, count two-by-two, three-by-three...
- The were able to add.

> Teacher: 'the girl with the pony tail, how many does she have?'
> Several pupils: 'two.'
> Teacher: 'and the captain, how many does he have?'
> Several pupils: 'six.'

Teacher: 'is that fair?'

Several pupils: 'we took two away from him, two and [then another] two.'

- They could subtract.

'And there were four people, and now there are eight, four buildings won't get a roof.'

- They knew multiplication terminology:

Teacher: 'three times four then?'

Pupil 1: '12.'

The pupils used the solution to one problem to solve others with different data, denoting an ability to generalise.

Teacher: 'how many tiles do we need then?'

Pupil 1: 'more.'

Teacher: 'how many more?'

Pupil 1 (thinking as he gazes upward): 'twelve.'

Teacher: 'for what?'

Pupil 1: 'so each building can have three.'

One of the teams even generalised the problem structure, identifying a relationship between the problems posed.

Teacher: 'you need three tiles for each roof. There are eight buildings. How many buildings won't get fixed? I don't know how many tiles there are. Count them.'

Teacher: 'some are missing.'

Pupil: 'four buildings won't get tiles.'

Teacher: 'four won't get a roof.' 'How did you figure that out?'

Pupil: 'since before we did the playmobile thing and the coins, and there were twelve'. 'And it's the same.'…

C6. Comprehensibility

The language used was familiar to pupils, who harboured no doubts about problem wording or the action to be taken to solve them. Moreover, they were able to reach solutions with the materials provided. They verified their solutions by physically handling the materials.

> Teacher: 'does everybody understand?'
> All: 'yes.'
> Teacher: 'what do you need to do?'
> Pupil: 'put tiles on the roof.'

The pupils showed no signs of 'drawing a blank' when confronted with the problems. Although they had difficulties in verbalising their solving procedures, they were able to outline them. In some cases different strategies used by pupils on the same team were even identified as being the same.

> Teacher: '[pupil's name] found a solution. Why?'
> Pupil 1: 'I counted. Three cars each time, I say one (counting on his fingers).'
> Teacher: 'another three cars, you said.'
> Pupil 1: 'three (signing with his fingers).'
> Teacher: 'that seems to be to be a very good way to go about it.'
> Teacher: 'how did you go about it?'
> Pupil 2: 'counting the cars. To see if there were twelve. If there are twelve, it takes four times.'
> Teacher: 'what did you do?'
> Pupil 2: 'I was going to count three-by-three.'

5 Discussion

Initially, the children's actions corroborated the expectations about the theoretical characteristics attributed to the problems. Two sets of characteristics were identified, depending on the child's mathematical activity and understanding of problem structure.

The first set was associated with diversity in students' mathematical activity, such as posing several solving strategies (C1), different representations (C3) or different levels of solutions (C4). Whilst such diversity might be attributed, a priori, to a given problem, proof that the problems proposed favoured such a wealth of approaches was furnished by the present experience. More than one strategy, representation and solution level were detected in them all. Even strategies not predicted in the literature were observed, as a result of working simultaneously with several pupils. This set of characteristics supports Matalliotaki's (2012) findings, for at these ages children find it difficult to solve problems without some manner of representation, which is one of the factors that fuels their progress in developing solving strategies.

The other set of characteristics was associated with understanding the problem, its structure and reflection on the solution proposed. The contexts used were familiar (C2) and the language understandable (C6) and both proved to be suitable for introducing the meaning of a new mathematical concept (here, partitive and quotative division) based on prior knowledge (C5). All the teams proposed solutions and were able to solve the same problems with different data. That confirmed earlier research findings on the benefits of a problem-solving approach to mathematics teaching in mathematics learning (Cai, 2010).

6 Conclusions

This paper synthesises the characteristics associated with problems suitable for pre-primary education, drawing from prominent earlier research on the subject (Nelson & Kirkpatrick, 1975; Lesh et al., 2013; Van de Walle, 2003; Yee, 2009). The characteristics proposed were applied to analyse three problems put forward by other authors (Davis & Pepper, 1992; Nelson & Kirkpatrick, 1975) based on the reactions of 26 pre-primary 5-

to 6-year-olds. The problems selected were observed to conform to the characteristics defined, confirming their suitability as pre-primary problems. Moreover, the list of characteristics selected proved to be operational for the ex-ante recognition of the features of problems for that stage of schooling, a matter of practical utility, given that problem selection is one of teachers' major tasks (Britz & Richard, 1992; Lester & Cai, 2016).

The present empirical results showed that problem characteristics depend on two main factors: mathematical activity, which motivates pupils, and problem structure, which favours understanding the approach to, engagement in and verification of the solutions.

The variety of solving strategies, representations and levels of solutions observed was consistent with other findings on the use of multiplicative structures (Davis & Pepper, 1992; De Castro & Hernández, 2014; Desforgues & Desforgues, 1980; Matalliotaki, 2012).

The three problems studied entailed mathematical ideas, in which the meaning of division was shown to be equitable distribution, establishment of quotas and reiterated subtraction. The classroom experience revealed the presence of characteristic C1, reasoning, although differences were observed among the problems, attributed to the use of materials and working with several pupils at the same time. The presence of reasoning reported in earlier studies (De Castro & Hernández, 2014) was confirmed here by the prevalence of modelling strategies in the first problem, in which all the objects were first counted and then distributed one by one until none was left. Conversely, teamwork and children's reactions to the objects to be distributed prompted strategies not identified in earlier research involving individual work (Carpenter et al., 1999; Davis & Pepper, 1992). These new approaches favoured readjustment strategies and trial and error cycles, in which the initial allocation in problem 1 of more than the correct number of coins to one pirate was rectified by the necessary rearrangements. A similar situation arose in problem 2, in which the same strategies were observed even in the quotative exercise. Problem 3 was characterised by a prevalence of quotative strategies, however.

The materials provided also had a significant effect on pupils' use of one strategy or another. In problem 1, linear arrangement of the coins facilitated comparison with no need for counting. In problem 2, the presence of a lorry favoured one-by-one sharing over prior grouping of the elements to be distributed, whereas in problem 3 the physical unwieldiness of the boat prompted deployment of the latter strategy.

The representations used were conditioned by the material provided and the format of the oral reply called for by the teacher. The problems featured characteristic C3 (Challenge), as they engaged pupils in seeking a solution, which resulted from handling the materials. When the material was less convenient for representing the situation, such as in the ferry problem, oral representation prevailed. Pupils' prior experience in solving two problems with similar structures may have improved their performance in the third (Matalliotaki, 2012). Such a variety of representations favoured the presence of multiple solutions, which were not short answers. Up to four levels of solutions were identified, depending on whether the answer was obtained from mental calculation only, the manipulatives or combinations of the two.

The classroom experience showed that both real and fictitious contexts were familiar to the pupils, although shipping cars on a ferry was a somewhat less ordinary situation for them. The language used and actions to be performed were understandable and in all cases the pupils were able to validate their answers by physically distributing the materials. In some cases, in addition to expressing their confidence in the solution found, they recognised several of the strategies used as equivalent. The experience revealed that these pupils generalised the mathematical structure involved, and some teams even identified similarities among the problems. Despite having not been taught to divide, their knowledge of counting and arithmetic operations enabled them to deploy strategies apt for solving partitive and quotative division problems. That would support pupils' ability to understand and solve such problems much earlier than assumed in school curricula (De Castro & Hernández, 2012), providing the problems are posed in a manner suited to their age. There may be two explanations for that finding: one may have to do with the practice acquired by the pupils after solving more than one problem, further to Matalliotaki's (2012) contention that pupils' performance improves when solving problems with similar structures. The other explanation lies in some pupils' natural mathematical talent. As the present evidence is insufficient to support either explanation, however, further research is necessary.

Two main contributions are deemed to stem from this study. First, it synthesises the characteristics that define problems apt for pre-primary education, furnishing an enhanced operational approach to analysing new proposals. Second, the empirical findings highlight the role of working with groups of pupils and of the materials used in favouring strategic diversity and several levels of solutions and representations. Such diversity may be favoured by the problems themselves. Nonetheless, the ideas put forward by Britz and Richard (1992) and Lester and Cai (2016) on the teacher's importance in the emergence of different strategies are supported by the present findings, which revealed the significance of variables such as presenting the children with materials, the size of the objects used and working in teams. Quotative problems were solved with partitive strategies because of the format of the materials, revealing their decisive role in the meanings of the mathematical concepts introduced.

This study is subject to limitations, in particular as regards the small number of pupils involved and the use of three very specific problems. Nonetheless, the experience described is deemed to be repeatable in other classrooms for comparison with the results reported here. Future lines of research would include analysing the validity of other problems in terms of the characteristics proposed and contrasting the empirical findings, particularly in connection with the impact of materials on solving strategies.

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Table 1. Conformity of the problems analysed to the theory on the characteristics of suitable problems

C1	Presence of mathematical ideas: quotative and partitive division
	Accommodation of several solving strategies: trial and error in each phase
C2	Contextualisation: pirate figurines and coins (problem 1), mock-up and lorries (problem 2) and working board, boat and cars (problem 3)
C3	Oral and physical representation: use of manipulatives
C4	Multiple solutions: physical distribution of objects, with several levels of solution, from distribution in no order and subsequent ordering to a predetermined strategy using division
C5	Expandability: second exercises with other amounts to be distributed (with non-exact division) and divisors; generalisation to child's real-life situations involving distribution with quotative and partitive division
C6	Comprehensibility: adaptation of problem wording to children's language and verification of the solution through physical handling of objects

Table 2. Partitive division strategies

Modelling strategies	S1	Count all the objects and distribute them one by one until none is left. Count the number of objects allocated to one of the resulting groups.
	2	Count all the objects and distribute them two by two (or three by three...) until none is left. Count the objects resulting from the allocation.
	S3	Begin to distribute the objects without counting the total, tallying the number allocated while some are still left and then distributing the remainder.
	S4	Count the total number of objects, allocate seven (more than appropriate) to one group and then make the necessary adjustments until the objects are equally distributed.
	S5	Divide the set of objects into equal subsets, allocating one subset to each group.
Counting strategies	S6	Skip count (3, 6, 9 ... 4, 8, 12), raising a finger or allocating one object with each number. If the last number called concurs with the number of objects to be distributed, the solution is the number of raised fingers.
Addition and subtraction strategies	SO	Find the solution by adding or subtracting.

Table 3. Quotative division strategies

Modelling strategies	S7	Count the total number of objects needed, create groups or group the objects (five-by-five, for instance) and count the number of groups.
	S8	Group the objects five-by-five; after creating several groups, count the total number of objects allocated and use the rest to continue the grouping process.
Counting strategies	S9	Skip count: 3, 6, 9 ... and use fingers, for instance, or the objects, to represent each number. Count one-by-one where necessary (3, 6, (7, 8, 9), 9) and compute the number of groups as the number of fingers or objects.

Table 4. Strategies used by six teams of pupils to solve the problems

	Team 1	Team 2	Team 3	Team 4	Team 5	Team 6
Pirate1 Partitive	S1*	S1*, S2	S1*	S1*	S1	S1*
Pirate2 Partitive	S1*	S4, S1, calculated mentally	S4	S4	S1*	S4
Loading1 Partitive	S1		S1		S1	S1*
Loading2 Quotative	S1*	SO, S5, calculated mentally		S4, S5, calculated mentally		
Ferry1 Quotative	S5	S9, S7	S7, calculated mentally	S7, S9, calculated mentally	S7, calculated mentally	S5
Ferry2 Partitive		Calculated mentally	Calculated mentally	S7 and S5	S4 and S5	

S1* = strategy based on strategy S1

Blank cells = exercise not done