THE MEAN FIELD PROBLEM WITH SIGN CHANGING POTENTIALS

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por

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A mis padres,
Josefa y Rafael.

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Si olvido a alguien, disculpen la torpeza.

Contents

Introduction

0.1 Motivation of the problem

This thesis is concerned with Liouville type equations on compact surfaces. More precisely, our analysis is focused on three fundamental issues in the analysis of partial differential equations: existence, multiplicity and compactness of solutions.

The study of the Liouville equation dates back to 19th century through the work of the same Liouville, [84], in which the entire solutions of the equation

$$-\Delta u = 2e^{u} \quad \text{in} \quad \mathbb{R}^2 \tag{0.1}$$

are classified.

This kind of equations gave rise to a lot of interest in the middle of the 70's due to its geometric meaning. Let (Σ, g) be a surface Σ equipped with a certain metric g and $\tilde{g}$ a conformal metric to g on Σ, namely $\tilde{g} = ge^{v}$. If K_g, $K_{\tilde{g}}$ are the Gaussian curvatures relative to these metrics, then the logarithm of the conformal factor satisfies the equation

$$-\Delta_g v + 2K_g = 2K(x)e^{v} \quad \text{in} \quad \Sigma, \tag{0.2}$$

where $K = K_{\tilde{g}}$. Here Δ_g denotes the Laplace-Beltrami operator in (Σ, g).

On the other hand, the classical *Uniformization Theorem* asserts that every simply connected Riemann surface is conformally equivalent to one of the following three Riemann surfaces: the open unit disk, the complex plane or the Riemann sphere. As a consequence, we can conclude that every compact orientable surface carries a conformal metric with constant Gaussian curvature. Hence we can assume from now on that K_g is constant.

At this point, one may ask the following question: *given a function K defined on Σ, is there any conformal deformation of the metric g such that K becomes the Gaussian curvature of the new metric?* This problem, known as the *prescribed Gaussian curvature problem*, is reduced to study the existence of solutions for the equation (0.2). This problem was proposed by Kazdan and Warner for general surfaces in [72] and by Nirenberg in the special case of the standard sphere.

Integrating (0.2) and taking into account the *Gauss-Bonnet Theorem*, we obtain that

$$\lambda := 2\int_\Sigma K_g \, dV_g = 2\int_\Sigma K e^v \, dV_g = 4\pi\chi(\Sigma), \tag{0.3}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. By this formula, we can observe how the topology of the surface Σ gives necessary conditions on the choice of the function K. Indeed, the sign of the function K in at least some point of Σ is prescribed by $\chi(\Sigma)$.

If $\lambda \neq 0$, the problem (0.2) can be reformulated as follows

$$-\Delta_g u = \lambda\left(\frac{Ke^u}{\int_\Sigma Ke^u dV_g} - \frac{1}{|\Sigma|}\right) \quad \text{in} \quad \Sigma. \tag{0.4}$$

Obviously, any solution of (0.2) solves (0.4). Reversely, observe that (0.4) is invariant under addition of constants. So, given a solution of (0.4), we can add a constant appropriately to obtain a solution of (0.2).

This problem is the so-called *mean field equation* of Liouville type. Due to the invariance by addition of constants, sometimes it is helpful to suppose that

$$\int_\Sigma Ke^u dV_g = 1. \tag{0.5}$$

In the seminal work [115], Troyanov proposed the construction of conformal metrics with prescribed Gaussian curvature on surfaces with conical singularities. This problem can be considered as the *singular* analogue of the question discussed above. A metric $\tilde{g}$ defined on Σ admits a conical singularity of order $\alpha > -1$ at $p \in \Sigma$, if

$$\tilde{g} \sim |x-p|^{2\alpha} g \ \text{ as } \ x \to p. \tag{0.6}$$

In such case, Σ admits a tangent cone with vertex at the point p of total angle $\vartheta = 2\pi(1+\alpha)$.

Let $G(x,y)$ be the Green function of the Laplace-Beltrami operator on Σ associated to g, i.e.

$$-\Delta_g G(x,y) = \delta_y - \frac{1}{|\Sigma|} \quad \text{in} \quad \Sigma, \qquad \int_\Sigma G(x,y) dV_g = 0, \tag{0.7}$$

where δ_y denotes a Dirac delta at the point $y \in \Sigma$. Moreover, given $p_1, \dots, p_m \in \Sigma$ and $\alpha_1, \dots, \alpha_m \in (-1, +\infty)$ we define

$$h_m(x) = 4\pi \sum_{j=1}^m \alpha_j G(x, p_j) = 2 \sum_{j=1}^m \alpha_j \log\left(\frac{1}{d(x,p_j)}\right) + 2\pi\alpha_j H(x,p_j), \tag{0.8}$$

where H is the regular part of G.

Following the approach of [115], if K is the Gaussian curvature of a metric $\tilde{g}$ which admits $p_1, \dots, p_m \in \Sigma$ conical singularities with their corresponding orders $\alpha_1, \dots, \alpha_m \in (-1, +\infty)$, then v is a solution of (0.2) in $\Sigma \setminus \{p_1, \dots, p_m\}$ such that

$$2\pi\chi(\Sigma) + 2\pi \sum_{j=1}^m \alpha_j = \int_\Sigma K e^v dV_g, \tag{0.9}$$

and

$$v \sim 2\alpha \log|x - p_j| \quad \text{as} \quad x \to p_j. \tag{0.10}$$

Moreover, by the change of variable $u = v + h_m$, we obtain the equation

$$-\Delta_g u + 2K_g = 2\tilde{K}(x) e^u - \frac{4\pi}{|\Sigma|} \sum_{j=1}^m \alpha_j \quad \text{in} \quad \Sigma \setminus \{p_1, \dots, p_m\}, \tag{0.11}$$

where

$$\tilde{K} = K e^{-h_m}. \tag{0.12}$$

Observe that

$$\tilde{K}(x) \simeq d(x,p_j)^{2\alpha_j} K(x) \qquad \text{close to } p_j.$$

Obviously, K and $\tilde{K}$ have the same sign in $\Sigma \setminus \{p_1, \dots, p_m\}$.

In addition, it holds

$$2\pi\chi(\Sigma) + 2\pi\sum_{j=1}^{m}\alpha_j = \int_\Sigma \tilde{K}e^u dV_g.$$

By standard regularity theory, we can show that a weak solution u for (0.11) *in the whole domain* Σ is smooth. Observe that $e^v = e^{-h_m}e^u$ which is consistent with (0.6).

Therefore, we have proved that the *singular* prescribed Gaussian curvature problem amounts to solve the PDE

$$-\Delta_g v + 2K_g = 2K(x)e^v - 4\pi\sum_{j=1}^{m}\alpha_j\delta_{p_j} \quad \text{in} \quad \Sigma. \tag{0.13}$$

Next, integrating (0.13) and taking into account the *Gauss-Bonnet formula* (0.9) is verified. Let us introduce the parameter λ as

$$\lambda = 4\pi\chi(\Sigma) + 4\pi\sum_{j=1}^{m}\alpha_j = 2\int_\Sigma Ke^v dV_g. \tag{0.14}$$

Frequently, $\frac{\lambda}{4\pi}$ is called the Euler characteristic of the *singular surface* $(\Sigma, \tilde{g})$.

Recall that we can assume that K_g is a constant, so we can rewrite (0.11) as

$$-\Delta_g u = \lambda\left(\frac{\tilde{K}e^u}{\int_\Sigma \tilde{K}e^u dV_g} - \frac{1}{|\Sigma|}\right) \quad \text{in} \quad \Sigma, \tag{0.15}$$

where $\tilde{K}$ is defined in (0.12) if $\lambda \neq 0$. This problem is called the *singular mean field equation* of Liouville type.

During recent years, the relevance of Liouville type equations has experienced a great increase occasioned by its connection with many current physical theories. Next, we describe briefly some of them.

- **Periodic vortices in Electroweak theory of Glashow-Salam-Weinberg:** mean field problems of Liouville type are present in the study of vortex type configurations in the Electroweak theory of Glashow-Salam-Weinberg in the selfdual regime, ([76]). This theory unifies the description of weak and electromagnetic interaction between elementary particles. In the analysis of quantum electroweak instabilities, Ambjorn and Oleson modeled Electroweak vortices by virtue of the selfdual Bogomol'nyi type equations. Moreover, these problems can be rewritten as the following elliptic system

$$\begin{cases} -\Delta_g v &= 4g^2 e^v + g^2 e^w - 4\pi \sum_{i=1}^m \alpha_i \delta_{p_i}, \\ -\Delta_g w &= -2g^2 e^v - \frac{g^2}{2\cos^2\theta} e^w + \frac{g^2\varphi_0^2}{2\cos^2\theta}, \end{cases} \tag{0.16}$$

where $\theta \in (0, \frac{\pi}{2})$ indicates the Weinberg mixing angle, g is the coupling constant and φ_0 is the symmetry breaking parameter.

The expressions e^v and e^w make reference to the magnitude for two gauge fields, whereas the points p_i are the vortex points and $\alpha_i \in \mathbb{N}$ correspond to their multiplicity. Regarding periodic vortices, namely solutions of the system imposing periodic boundary conditions, it is possible to reduce these equations to the following mean field system with singular data

$$\begin{cases} -\Delta_g v_1 = \mu \frac{e^{v_1}}{\int_{\mathbb{T}^2} e^{v_1}} + (4\pi N - \mu) \frac{e^{v_2}}{\int_{\mathbb{T}^2} e^{v_2}} - 4\pi \sum_{i=1}^m \alpha_i \delta_{p_i} & \text{in} \quad \mathbb{T}^2, \\ \Delta_g v_2 = \frac{\mu}{2} \left(\frac{e^{v_1}}{\int_{\mathbb{T}^2} e^{v_1}} - \frac{1}{|\mathbb{T}^2|} \right) + \frac{4\pi N}{2\cos^2\theta} \left(\frac{e^{v_2}}{\int_{\mathbb{T}^2} e^{v_2}} - \frac{1}{|\mathbb{T}^2|} \right) & \text{in} \quad \mathbb{T}^2, \end{cases} \tag{0.17}$$

where $\mathbb{T}^2$ is the flat 2-torus. With respect to the previous system, the following relation holds

$$v_1 = v - \overline{v}, \qquad v_2 = w - \overline{w}, \qquad \mu = \frac{4\pi N - g^2\varphi_0^2|\mathbb{T}^2|}{\sin^2\theta}.$$

Now, if one fixes the component v_2, the solvability of the first equation of (0.17) ammounts to solve (0.15). Hence the existence of Electroweak periodic vortices depends strongly on the existence of solutions for the singular mean field equation. We refer the reader to [9, 35, 112, 113, 120] for a complete description and several results in this context.

- **Periodic vortices in Chern-Simons-Higgs theory:** The second physical application of Liouville mean field equations is connected to Chern-Simons-Higgs theory. As discussed in [54], Chern–Simons theories are relevant in the study of several physical phenomena, such as high critical temperature superconductivity, the quantum Hall effect or conformal field theory. Specifically, the abelian Chern-Simons gauge theory proposes a selfdual model whose electrodynamics, under the control of the Chern-Simons parameter, yield vortices which carry electric charge as well as magnetic flux (see [54, 76, 112]). Indeed, the study of the existence of periodic selfdual Chern-Simons vortices, leads to solve the following Liouville type equation ([65])

$$-\Delta u = \frac{1}{\varepsilon^2} e^u(1-e^u) - 4\pi \sum_{i=1}^{m} \alpha_i \delta_{p_i} \quad \text{in} \quad \mathbb{R}^2, \tag{0.18}$$

under the assumption $\int_{\mathbb{R}^2} e^u(1-e^u)\, dx \leq C$. Here the parameter ε is the Chern-Simons term and $p_1, \ldots, p_m$ are the vortex points corresponding to the zeroes of the Higgs field ϕ, with multiplicty $\alpha_i \in \mathbb{N}$. The value e^u defines the Higgs field ϕ by the relation $u = \log|\phi|^2$. In this context, consider u a solution of (0.18) such that $e^{u_\varepsilon} \to 0$ as $|z| \to +\infty$. This type of solutions are called *non-topological*, and give rise to vortexes asymptotically equilavent to the symmetric vacuum state $\phi = 0$. We point out that, in [101], it is proved that given a sequence of *non-topological* solutions u_ε such that $u_\varepsilon - 2\log\varepsilon$ is bounded, then it converges to a solution u of the singular mean field equation

$$-\Delta u = \lambda \left(\frac{\tilde{K} e^u}{\int_{\mathbb{T}^2} \tilde{K} e^u \, dV_g} - \frac{1}{|\mathbb{T}^2|} \right) \quad \text{in} \quad \mathbb{T}^2,$$

as $\varepsilon \to 0$, where λ is a positive parameter, $\tilde{K}$ is defined in (0.12) and $\mathbb{T}^2$ is the flat 2-torus.

- **Stationary solutions of a reaction-diffusion system:** Liouville type equations arise also in some biological processes such as chemotaxis. This is defined as the phenomena whereby a collection of organisms (cells or bacteria) moves according to the presence of certain chemicals. Such processes are tipically modelled by reaction-diffusion PDE's type, as Keller-Segel system, [74], which is written as

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u\nabla v) & \text{in} \quad \Omega \times (0,T), \\ \tau v_t = \Delta v - \alpha v + u & \text{in} \quad \Omega \times (0,T), \end{cases} \tag{0.19}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and α, τ are positive constants. For the system are given initial data and non-flux boundary conditions (homogeneous Neumann condition)

$$\begin{cases} \frac{\partial u}{\partial n} = 0 = \frac{\partial v}{\partial n} & \text{on} \quad \partial\Omega, \\ u(0,\cdot) = u_0, v(0,\cdot) = v_0, & \text{in} \quad \Omega, \end{cases} \tag{0.20}$$

System (0.19) models the dynamic of a population migration in the domain driven by the gradient of the chemical substances. The value u corresponds to the concentration of population, whereas v does to the density of the chemical substance. If $N = 2$, one can show that the stationary solutions satisfy the relation

$$\log u - v = \log \sigma, \qquad \text{with } \sigma > 0,$$

see [99] for more details. In this situation, the time-independent system (0.19)-(0.20) yields the Liouville type equation

$$\begin{cases} -\Delta v + \alpha v = \sigma e^u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

We refer the reader to [57, 107] for a further discussion and some results. A particular case of the last problem will be discussed in the subsection 0.4.1.

- **Stationary turbulence for Euler flow with vortices:** mean field equations arise also in statistical mechanics in the study of the turbulent behavior of a Euler flow with vortices of the same orientation. According to the vortex theory proposed by Onsager (see [102]), a finite dimensional Hamiltonian system can describe the Euler flow in a bounded planar domain Ω when the vorticity field is $\sum_{j=1}^{s} \beta_j \delta_{q_j}$, where q_j are the vortices and β_j their respective vorticity intensity.

One can try to analyze the stationary turbulence where the number of vortexes s tends to infinity and $\beta_j \to 0$ in such way that the total vorticity is bounded. In this way, it is proved that the limit of the vortex model is the equation

$$\begin{cases} -\Delta u = \lambda \frac{e^u}{\int_\Omega e^u\,dx} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ and $\frac{u}{\lambda}$ is the stream function for an Euler flow confined in the domain with vorticity $\frac{e^u}{\int e^u}$. In this model, $-\lambda$ corresponds to negative values of the statistical temperature, a range which is expected to describe the high energy (turbulent) behavior of the flow. For a rigorous derivation of the model, see [18, 75].

Now, consider an Euler flow in Ω under the influence of m sinks of vorticity $-4\pi\frac{\alpha_i}{\lambda}$ located at points p_i, which are in the opposite direction with respect to the location of the rest of vortices. Now, the limit of the singular vortex model is the problem with singular data

$$\begin{cases} -\Delta u = \lambda \frac{e^u}{\int_\Omega e^u} - 4\pi \sum_{i=1}^m \alpha_i \delta_{p_i} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Now the total vorticity equals $\frac{e^u}{\int e^u} - 4\pi \sum_{i=1}^m \frac{\alpha_i}{\lambda}\delta_{p_i}$. In [117], the authors give a rigorous deduction of this singular model.

We highlight that under these perspectives the restrictions (0.3) and (0.14) are not present.

0.2 An overview of the regular case

In this section we collect some fundamental results concerning equation (0.2). Let us first present the classification result of entire solutions given by Liouville in [84].

Let $\pi : \mathbb{S}^2 \to \overline{\mathbb{C}}$ be the stereographic projection and h a meromorphic function on $\mathbb{C}$ such that $h'(z) \neq 0$ and the order of its poles is 1. Consider the conformal map

$$\mathbb{C} \xrightarrow{h} \overline{\mathbb{C}} \xrightarrow{\pi^{-1}} \mathbb{S}^2,$$

where $\mathbb{C}$ is equipped with the Euclidean metric $|dz|^2$ and $\mathbb{S}^2$ with the round metric g_0. The pull-back metric of g_0 along $(\pi^{-1} \circ h)$ is

$$(\pi^{-1} \circ h)^* g_0 = \frac{4|h'(z)|^2}{(1 + |h(z)|^2)^2} |dz|^2.$$

Hence the function

$$\varphi(z) = \log \frac{4|h'(z)|^2}{(1 + |h(z)|^2)^2},$$

solve (0.1), where e^{φ} is the conformal factor of $\pi^{-1} \circ h$. Reversely, all solutions of (0.1) are of this form, see [120] for different proofs of this fact.

In particular, if a finite curvature condition holds, namely

$$\int_{\mathbb{R}^2} e^u \, dx < C, \tag{0.21}$$

then h must be a Möbius transformation. Therefore, the solutions of (0.1) yield to the family

$$\varphi_\mu(z) = \log \frac{4\mu}{(1 + \mu|z - z_0|^2)^2}, \qquad \text{with } z \in \mathbb{C}. \tag{0.22}$$

Notice that function φ_μ, known as *bubble*, coincides with the conformal factor of the stereographic map, composed with a translation and a dilation. This classification was given by Chen and Li in [25] relying on the moving planes method; Chou and Wan gave a simpler proof in [40] by means of complex analysis arguments. In addition, a solution of (0.1) satisfying (0.21) verifies

$$2 \int_{\mathbb{R}^2} e^u \, dx = 2 \int_{\mathbb{S}^2} 1 \, dV_{g_0} = 8\pi. \tag{0.23}$$

The free parameter μ involved in (0.22) just reflects the scale invariance of (0.1) under the transformation

$$\varphi_\mu(x) \mapsto \varphi_\mu(\mu x) + 2 \log \mu \quad \text{for any } \mu > 0.$$

The scale invariance allows $\varphi_\mu(x)$ to concentrate around the point z_0, where $\varphi_\mu(x)$ attains its maximum. Actually, $e^{\varphi_\mu} \rightharpoonup 4\pi\delta_{z_0}$ weakly in the measures sense as $\mu \to +\infty$. As we will discuss in the sequel, this concentration phenomena may occur in general for sequences of solutions Liouville type.

Now, let us review some contributions concerning equation (0.2) on compact surfaces without boundary.

If $\chi(\Sigma) = 0$, the solvability of the problem is completely settled in [72]. Specifically, the problem admits a non-trivial solution if, and only if, K changes sign and $\int_\Sigma K\, dV_g < 0$. In case $\chi(\Sigma) < 0$, besides the fact that K must be negative somewhere, other necessary conditions for the existence of solutions are also given in [72]. However, the problem is not completely settled. For instance, given a strictly negative K, Berger shows that (0.2) admits a solution. If one considers a function $K(x) = k(x) + \rho$, where $k(x) \leq 0$ and ρ is a small parameter, the problem is solvable. Different results in this direction are given in [3, 13, 50].

The problem of prescribing the Gaussian curvature on $\mathbb{S}^2$, proposed initially by Nirenberg, is more delicate. Indeed, the available results are partial, [3, 22, 23, 30–32, 72], and a complete answer on the existence question is still unknown. Kazdan and Warner deduce that a solution of (0.2) verifies the following necessary condition

$$\int_{\mathbb{S}^2} \nabla\zeta \cdot \nabla K e^u dV_{g_0} = 0, \qquad \text{where} \qquad -\Delta_{g_0}\zeta = 2\zeta \quad \text{in } \mathbb{S}^2. \tag{0.24}$$

If the function K is even, one can reformulate the problem in $\mathbb{R}P^2$. Actually, (0.2) admits a solution if and only if K is positive somewhere, see [97].

The inclusion of symmetric functions on $\mathbb{S}^2$ provides more conditions for the solvability of the Nirenberg problem and cases of non-existence. Given an axially symmetric function K, if (0.2) admits a solution, then (0.24) implies that K' changes sign. Kowever, Chen and Li show that this condition is not sufficient, see [30]. In particular, if one assumes that K is axially symmetric on $\mathbb{S}^2$, not constant and monotone in the region where $K > 0$, then there is no solution for (0.2).

The study of the case $\chi(\Sigma) > 0$ enables us to reformulate the problem (0.2) into (0.4), which admits a variational structure. A reasonable approach, meaningful also from the physical point of view, is to study (0.4) for any positive parameter λ independent of the restriction (0.3).

Another related significant issue is the study of compactness for solutions of Liouville type equations. Roughly speaking, given $\{u_n\}$ a sequence of solutions, one desires to find conditions that allow one to pass to the limit. By standard regularity, it is enough to show L^∞ boundedness. This problem is typically studied by means of a blow–up analysis, which determines for what values of λ the sequence u_n is uniformly bounded or may blow–up.

The main tool in this analysis is the classical blow–up alternative established first by Brezis and Merle, [17], and completed by Li and Shafir, [78], for the solutions of the problem

$$-\Delta u_n = V_n(x)e^{u_n} \quad \text{in} \quad \Omega, \tag{0.25}$$

where $\Omega \subset \mathbb{R}^2$ is an open domain and $0 < V_n(x) \in C^0(\overline{\Omega})$, satisfying

$$\int_\Omega e^{u_n}\, dx < C. \tag{0.26}$$

As in [17], we say that a point $q \in \Omega$ is a **blow–up** point relative to u_n if there exists $\{q_n\} \subset \Omega$ with $q_n \to q$ such that $u_n(q_n) \to +\infty$. In this way, let $\{u_n\}$ be a sequence of solutions of (0.4) and $V_n \to V$ in $C^0(\overline{\Omega})$ sense, then the following alternative holds, up to subsequence:

1. either u_n is uniformly bounded in $L^\infty_{loc}(\Omega)$;

2. or $u_n \to -\infty$ uniformly on compact sets of Ω;

3. or there exists a finite set $S = \{q_1, \ldots, q_r\} \subset \Omega$ of blow–up points.

   In such case, $u_n \to -\infty$ in compact sets of $\Omega \setminus S$ and $V_n e^{u_n} \rightharpoonup \sum_{i=1}^r \beta(q_i)\delta_{q_i}$ in the weak sense of measures where $\beta(q_i) = 8\pi m_i$ with $m_i \in \mathbb{N}$ and $i = 1, \ldots, r$.

By the blow–up procedure around a maximum of u_n, one is led to an entire solution of the global problem (0.1). This is the reason why the local mass $\beta(q_i)$ corresponds to multiples to the entire mass given by (0.23).

As observed by Wolansky, the third alternative holds with $\beta(q_i) = 8\pi$, if the oscillation is bounded on $\partial\Omega$, namely

$$\sup_{\partial\Omega} u_n - \inf_{\partial\Omega} u_n \leq C, \tag{0.27}$$

and $||\nabla V_n||_{L^\infty(\Omega)} \leq C$. In other words, Wolansky shows that around a blow–up point there is only one bubble.

This is still the case for sequences of solutions of (0.4) satisfying (0.5), as shown by Yan Yan Li ([77]). He uses the moving plane method outside the blow–up points to obtain the boundary condition (0.27). In any case, as a consequence of the above blow–up alternative, sequences of solutions form a compact set if $\lambda \neq 8k\pi$.

Here it is worth to comment some contributions which deal with the existence of blowing–up solutions for (0.4). Namely, given $\lambda = 8k\pi$, one can construct a sequence u_n of solutions of (0.4) such that $\lambda_n \to \lambda$ and with k blow–up points. These construction use singular perturbative methods à la Lyapunov-Schmidt, see [34, 36, 49, 56]. Those results complement the aforementioned compactness theorems.

The previous compactness result enables us to define the global Leray-Schauder degree of (0.4) for $\lambda \in (8k\pi, 8(k+1)\pi)$ with $k \in \mathbb{N}$ and $K > 0$. Because of its homotopy invariance, the degree is independent of the function K and the metric g. In fact, it follows from [77] that the degree only depends on λ and the topology of Σ, so it is denoted by $d(k, \chi(\Sigma))$. An accurate blow–up analysis, [34], allows Chen and Lin in [35] to obtain the explicit formula

$$d(k, \chi(\Sigma)) = \binom{k - \chi(\Sigma)}{k} = \begin{cases} \frac{(k-\chi(\Sigma))\cdots(2-\chi(\Sigma))(1-\chi(\Sigma))}{k!} & \text{if } \ k > 0, \\ 1 & \text{if } \ k = 0. \end{cases} \tag{0.28}$$

As Malchiodi shows in [88, 89], the Leray-Schauder degree can be interpreted by means of a variational formulation of (0.4). Moreover, following the ideas of the influential work of Djadli and Malchiodi ([53]) a very general existence result is obtained for every compact surface and for every positive K in $C^1(\Sigma)$ with $\lambda \neq 8k\pi$, [52, 88]. The proof uses min-max arguments, see subsection 0.4.3 for more details. For instance if $\Sigma = \mathbb{S}^2$, observe that by (0.28), $d(k, \chi(\mathbb{S}^2)) = 0$ with $k \geq 2$, whereas the variational approach yields existence of solutions.

In the works [45, 46], De Marchis proves a result of generic multiplicity of solutions of (0.4). Let $\lambda \in (8k\pi, 8(k+1)\pi)$ with $k \in \mathbb{N}$; then, for a generic choice of the metric g and the positive function K,

$$\#\{\text{solutions of } (0.4)\} \geq \begin{cases} p_k & \text{if} \quad \chi(\Sigma) = 2, \\ \sum_{r=0}^{k} \binom{k - r - \chi(\Sigma) + 1}{k - r} p_r & \text{if} \quad \chi(\Sigma) = 0. \end{cases} \tag{0.29}$$

where $p_0 = 1, p_{2m+1} = p_{2m} = \sum_{j=0}^{m} p_j$ for any $m \in \mathbb{N}$.

Let us point out that the above estimate on the number of solutions improves the one that suggests the Leray-Schauder degree, (0.28). As we see, in this case, Morse theory gives more information about the structure of the critical points than degree theory.

0.3 An overview of the singular case

This section is devoted to introduce different results concerning equation (0.15). Let us first consider the entire problem

$$-\Delta u = |x|^{2\alpha} e^u \quad \text{in} \quad \mathbb{R}^2, \tag{0.30}$$

satisfying the integral condition

$$\int_{\mathbb{R}^2} |x|^{2\alpha} e^u dx < C, \tag{0.31}$$

where $\alpha > -1$. The solutions of (0.30) such that (0.31) holds are classified by Prajapat and Tarantello in [104] and are given by the expression

$$\varphi_\mu(z) = \log \frac{8(1+\alpha)\mu}{(1+\mu|z^{1+\alpha} - c|^2)^2}, \quad \text{with } z \in \mathbb{C}, \tag{0.32}$$

where μ is a positive parameter, $c \in \mathbb{C}$; if $\alpha \notin \mathbb{N} \cup \{0\}$ then c must be equal to 0. In addition, a solution of (0.30) satisfying (0.31) verifies

$$\int_{\mathbb{R}^2} |x|^{2\alpha} e^u \, dx = 8(1+\alpha)\pi. \tag{0.33}$$

The functions φ_μ are scale invariant under the transformation

$$\varphi_\mu(z) \mapsto \varphi_\mu(\mu z) + 2(1+\alpha)\log\mu \quad \text{for any } \mu > 0.$$

As $\mu \to +\infty$, the following concentration phenomena holds

i) if $c = 0$, then

$$|x|^{2\alpha} e^{\varphi_\mu} \rightharpoonup 8(1+\alpha)\pi\delta_0 \text{ in the weak sense of measures;}$$

i) if $c \neq 0$ and $\alpha \in \mathbb{N} \cup \{0\}$, then

$$|x|^{2\alpha} e^{\varphi_\mu} \rightharpoonup \sum_{i=1}^{\alpha+1} 8\pi\delta_{q_i} \text{ in the weak sense of measures;}$$

where $\{q_1, \ldots, q_\alpha\}$ are the set of $(\alpha+1)$-roots of c.

Similarly as we have proceeded for the regular case, we discuss now problem (0.13). The case $\lambda \leq 0$ has been treated in [115], which obtains existence results analogous to the regular case ones, [12, 72]. Under the assumption $K < 0$, McOwen gives an existence result for surfaces with positive Euler characterstic, see [94].

Again, when one considers the equation on the sphere, the problem becomes delicate. For $m = 2$ and positive constant curvature, Troyanov ([114]) shows that (0.13) admits a solution only if $\alpha_1 = \alpha_2$, the so-called *american football*. In particular, (0.13) does not admit solutions for $m = 1$ (taking $\alpha_2 = 0$). In other words, the *tear drop* conical singularity on $\mathbb{S}^2$ does not admit constant curvature. Besides, for $m = 2$, Chen and Li ([26]) give necessary conditions on K for the solvability of (0.13); whereas Eremenko in [55] studies the case of prescribing constant positive curvature with three conical singularities. We also refer the reader to [95], where the authors give a criterion for the existence of a metric of constant curvature on $\mathbb{S}^2$ with conical singularities.

As we are interested in any positive parameter λ, we can transform equation (0.13) into (0.15), which admits variational structure. In addition, by the invariance under addition of constants of (0.15), let us suppose that

$$\int_\Sigma \tilde{K} e^u \, dV_g = 1. \tag{0.34}$$

Regarding the compactness question, the presence of the singularities implies that the corresponding critical value set becomes more involved. In fact, the condition $\lambda \notin 8\pi\mathbb{N}$ is no longer sufficient to guarantee uniformly upper boundedness of solutions. In [8, 9, 111] it is shown that if $K > 0$, then the solution set of the problem (0.15) is compact for $\lambda \notin \Lambda_m$, where Λ_m is the critical value set defined as follows

$$\Lambda_m = \left\{ 8\pi r + \sum_{j=1}^m 8\pi(1+\alpha_j) n_j \,|\, r \in \mathbb{N} \cup \{0\}, n_j \in \{0,1\} \right\} \setminus \{0\}. \tag{0.35}$$

This result is proved by a blow-up analysis of the sequence of solutions of the problem

$$-\Delta u_n = |x|^{2\alpha_n} V_n(x) e^{u_n} \quad \text{in} \quad \Omega, \tag{0.36}$$

where $0 < V_n \in C^1(\overline{\Omega})$, $\alpha_n > -1$, and the following finite integral condition holds:

$$\int_\Omega |x|^{2\alpha_n} V_n(x) e^{u_n} \, dx \le C. \tag{0.37}$$

Let u_n be a sequence of solutions of (0.36) satisfying (0.37), $V_n \to V$ in $C^0(\overline{\Omega})$ sense and $\alpha_n \to \alpha > -1$. Then, the following alternative holds, up to subsequence (see [9])

1. either u_n is uniformly bounded in $L^\infty_{loc}(\Omega)$;

2. or $u_n \to -\infty$ uniformly on compact sets of Ω;

3. or there exists a set $S = \{q_1, \ldots, q_r\} \subset \Omega$ of blow–up points.

In such case, $u_n \to -\infty$ in compact sets of $\Omega \backslash S$ and $|x|^{2\alpha_n} V_n e^{u_n} \rightharpoonup \sum_{i=1}^{r} \beta(q_i)\delta_{q_i}$ in the weak sense of measures, where $\beta(q_i) = 8\pi m_i$ if $q_i \neq 0$ and $\beta(q_i) = 8(1+\alpha)\pi + 8\pi\tilde{m}_i$ if $q_i = 0$ with $m_i \in \mathbb{N}$ and $\tilde{m}_i \in \mathbb{N} \cup \{0\}$ for any $i = 1, \ldots, r$.

Applying the blow–up technique around each point q_i, either (0.1) or (0.30) appears in the limit. Notice that the values 8π and $8(1+\alpha)\pi$ are the local masses that arise from the corresponding quantization (0.23) and (0.33).

Bartolucci and Tarantello prove that the oscillation bounded mean condition, (0.27), holds for solutions of (0.15), so the third statement of the previous blow–up alternative holds with $\beta(q_i) = 8\pi$ for $q_i \neq 0$ and $\beta(q_i) = 8(1+\alpha)\pi$ for $q_i = 0$.

The construction of blowing–up solutions for (0.15) according to the third blow–up alternative has been treated in [36]. This construction allows Chen and Lin to calculate the Leray-Schauder degree of (0.15), see [36]. Consider the function g

$$g(x) = (1 + x + x^2 + x^3 + \cdots)^{-\chi(\Sigma)+m} \prod_{i=1}^{m} \left(1 - x^{1+\alpha_i}\right) = 1 + b_1 x^{n_1} + b_2 x^{n_2} + \cdots ,$$

where $b_i \in \mathbb{Z}$ and $n_i \in \mathbb{R}^+$ with $n_i < n_j$ if $i < j$. In this case, the homotopy invariance of the degree implies that $d(k, \chi(\Sigma), m)$ is constant for every $\lambda \in (8n_i\pi, 8n_{i+1}\pi)$ for any $i = 0, 1, \ldots,$ and independent of the function K as long K is a C^1 positive function. Indeed,

$$d(k, \chi(\Sigma), m) = \sum_{i=0}^{m} b_i,$$

where $b_0 = 1$.

Let us now discuss the variational approach to (0.15). This problem is the Euler-Lagrange equation of the energy functional

$$J_\lambda(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 dV_g + \frac{\lambda}{|\Sigma|}\int_\Sigma u\, dV_g - \lambda \log \int_\Sigma \tilde{K} e^u dV_g, \tag{0.38}$$

defined in the domain

$$X = \left\{ u \in H^1(\Sigma) : \int_\Sigma \tilde{K} e^u \, dV_g > 0 \right\}. \tag{0.39}$$

By a proper modification of the Moser-Trudinger inequality, Troyanov proves that J_λ is coercive if $\lambda < 8\pi \min_{i=1,\dots,m}\{1, 1+\alpha_i\}$, then a solution can be found as a minimizer, see [115]. If $\lambda = 8\pi \min_{i=1,\dots,m}\{1, 1+\alpha_i\}$, J_λ is bounded below, but is no longer coercive. This case is discussed in [92].

Instead, J_λ does not remain bounded from below for $\lambda > 8\pi \min_{i=1,\dots,m}\{1, 1+\alpha_i\}$. In order to find saddle-type critical points, min-max arguments seem to be the natural technique to handle the problem. Under the assumption $K > 0$ and $\alpha_i > 0$ for every $i = 1, \dots, m$, Bartolucci, De Marchis and Malchiodi give a general positive answer to the existence question for surfaces with positive genus and $\lambda \in (8\pi, +\infty) \setminus \Lambda_m$, see [5].

The study of the multiplicity of solutions of (0.15) is studied in [5] for positive genus surfaces and positive potentials. In this way, using a Morse-theoretical approach, for $\lambda \in (8k\pi, 8(k+1)\pi) \setminus \Lambda_m$ and for a generic choice of (g, K), then

$$\#\{\text{solutions of } (0.15)\} \geq \binom{N+g-1}{g-1}, \tag{0.40}$$

where g is the genus of the surface Σ and $\chi(\Sigma) = 2 - 2g$.

In the special case of the standard sphere, also for the case of positive potentials, Malchiodi and Ruiz, ([90]), obtain an improvement on the Moser-Trudinger inequality which allows them to prove that (0.15) is solvable under some extra assumptions involving the order of the singularities in the case $\lambda \in (8\pi, 16\pi) \setminus \Lambda_m$. Moreover, Bartolucci, Lin and Tarantello, [6], prove that (0.15) does not admit solution assuming that K is a positive constant, $m = 1$, $\alpha_1 > 0$ and $\lambda \in (8\pi, 8\pi(1+\alpha_1))$. Applying an argument based on vanishing moments, in the case $m \geq 2$, Bartolucci and Malchiodi show existence for (0.15) if $\lambda \in (0, 8\pi \min_i\{1, 1+\alpha_i\}) \setminus 4\pi\mathbb{N}$ and $m \geq 2$, see [7]. In addition, if $m = 2$ and $\alpha_1 < \alpha_2$, the authors show that there is no solution for $\lambda \in (8\pi(1+\alpha_1), 8\pi(1+\alpha_2))$ and positive constant potentials.

The case of negative orders α_i has been less studied from a variational point of view. Actually, the unique contributions are given by Carlotto and Malchiodi, [21], and Carlotto, [20], for positive K. They find a topologic sufficient condition for the solvability of (0.15), whereas the first author deduce an algebraic criterion which implies such sufficient condition.

0.4 Objectives

In this thesis we present several results for the case in which the function K is allowed to change sign. The absence of sign restriction opens a large number of related current questions and engaging problems to solve. As far as we know, this case has not much been considered in the literature. For that reason, the questions to be analyzed in the present work are some of the most fundamental ones in the analysis of PDEs: existence, multiplicity and compactness of solutions. Let us point out that from a geometric point of view, there is no reason for K to be strictly positive.

Actually, this thesis contains the first studies on the sign changing case for general singular surfaces with an arbitrary number of conical singularities, [47, 48]. In fact, this question was proposed in Remark 2.8 of [5], which points out that *the difficulties are inherited by the lack of concentration-compactness-quantization results.*

Our study on the sign changing case has begun to generate a real interest. In fact, this situation has been treated recently by D'Aprile, De Marchis and Ianni using perturbative methods, [44].

0.4.1 Prescribing Gaussian curvature in a subdomain of the sphere

Let Ω be a subdomain of $\mathbb{S}^2$, considered with the standard metric. We will study the existence of solutions for the problem

$$\begin{cases} -\Delta_{g_0} u + 2 = 2K(x)e^u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.41}$$

where K is a continuous function defined on $\overline{\Omega}$.

It is important to first observe that with this boundary condition (0.41) is not invariant under conformal transformations of the sphere, as is the Nirenberg problem.

This problem has already been studied in [22, 61, 79, 118]. By integrating equation (0.41) we obtain that

$$\lambda := 2|\Omega| = 2\int_\Omega K(x)e^u dV_{g_0}. \tag{0.42}$$

In particular, no solution exists if K is negative. From now on we will assume

(A1) $K(x) > 0$ for some $x \in \overline{\Omega}$.

Moreover, (0.42) implies that (0.41) can be rewritten in the form

$$\begin{cases} -\Delta_{g_0} u = \lambda \left(\frac{Ke^u}{\int_\Omega Ke^u dV_{g_0}} - \frac{1}{|\Omega|} \right) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.43}$$

Problem (0.43) is the Euler-Lagrange equation of the energy functional

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dV_{g_0} + 2 \int_\Omega u \, dV_{g_0} - \lambda \log \int_\Omega Ke^u \, dV_{g_0}, \tag{0.44}$$

defined in the domain

$$X = \left\{ u \in H^1(\Omega) : \int_\Omega Ke^u \, dV_{g_0} > 0 \right\}. \tag{0.45}$$

Observe that assumption (A1) implies that X is not empty. As problem (0.43), the functional I_λ is invariant under addition of constants.

In [22] it was shown that I_λ is always bounded from below and coercive if $\lambda < 4\pi$, i.e. $|\Omega| < 2\pi$). Therefore, a solution is obtained by minimization. The case $\lambda = 4\pi$ is critical, I_λ is still bounded from below but loses coercivity. Moreover, the problem may present loss of compactness due to bubbling of solutions. If Ω is a hemisphere, for instance, one cass pass to a problem in $\mathbb{S}^2$ by reflection and apply the known results for the Nirenberg problem, see [22].

In this thesis we consider the case $\lambda \in (4\pi, 8\pi)$, that is, $|\Omega| > 2\pi$. The case

(Q1) $K(x) < 0$ for any $x \in \partial\Omega$,

was already treated in [61]. Under (Q1) I_λ is still bounded from below and coercive, and a solution can be found by minimization.

Instead, if $K(x) > 0$ on some point $x \in \partial\Omega$, then I_λ is no longer bounded from below. In order to find critical points of saddle type, min-max arguments appear as the natural technique to handle the problem. A first result in this direction was given in [118], where the existence of a solution for (0.41) is shown under the assumption

(Q2) $\partial\Omega$ is disconnected and $K(x) > 0$ for any $x \in \partial\Omega$.

In this chapter we extend the existence results of [61] and [118] under a unique general condition, namely

(A2) $K(x) \neq 0$ for all $x \in \partial\Omega$.

Theorem 0.4.1. *Assume (A1) and (A2). If Ω is a smooth domain of $\mathbb{S}^2$ such that $|\Omega| \in (2\pi, 4\pi)$, then problem (0.41) admits a solution.*

Let us emphasize that our assumption (A2) contains both (Q1) and (Q2) as particular cases. Moreover, our proofs fix some gaps in the proof of [118], as will be explained in Chapter 2. This result has been presented in the publication [86].

0.4.2 Prescribing Gaussian curvature in singular surfaces

We now address the problem (0.15) in the case in which K is a sign changing function. We give new existence and generic multiplicity results by means of variational methods. In order to obtain these results, it is unavoidable to establish a compactness property, also in this thesis.

Let us introduce a first hypothesis on K:

(H1) K is a sign changing $C^{2,\alpha}$ function with $\nabla K(x) \neq 0$ for any $x \in \Sigma$ with $K(x) = 0$.

Let us define the sets

$$\Sigma^+ = \{x \in \Sigma : \ K(x) > 0\}, \quad \Sigma^- = \{x \in \Sigma : \ K(x) < 0\}, \quad \Gamma = \{x \in \Sigma : \ K(x) = 0\}.$$

Assumption (H1) implies that the set of nodal curves Γ is regular and that

$$N^+ = \#\{\text{connected components of } \Sigma^+\} < +\infty. \tag{0.46}$$

In what follows we will also assume that

(H2) $p_j \notin \Gamma$ for all $j \in \{1, \dots, m\}$.

So we can suppose, up to reordering, that there exists $\ell \in \{0, \ldots, m\}$ such that

$$p_j \in \Sigma^+ \text{ for } j \in \{1 \ldots, \ell\}, \quad p_j \in \Sigma^- \text{ for } j \in \{\ell+1, \ldots, m\}. \tag{0.47}$$

As commented in previous sections, given a positive function K, the set of solutions is compact if λ does not belong to the critical set Λ_m defined in (0.35). In the next theorem we obtain an analogous conclusion without the sign restriction on K, with a slight modification on the set of critical values

$$\Lambda_\ell = \left\{ 8\pi r + \sum_{j=1}^{\ell} 8\pi(1+\alpha_j)n_j : \ r \in \mathbb{N} \cup \{0\}, n_j \in \{0,1\} \right\} \setminus \{0\}. \tag{0.48}$$

Theorem 0.4.2. *Assume that $\alpha_1, \ldots, \alpha_m > -1$ and let K_n be a sequence of functions with $K_n \to K$ in $C^{2,\alpha}$ sense, where K verifies (H1), (H2). Let u_n be a sequence of solutions of the problem*

$$-\Delta_g u_n = \tilde{K}_n e^{u_n} - f_n \quad in \quad \Sigma, \tag{0.49}$$

with $f_n \to f$ in $C^{0,\alpha}$ sense and $\tilde{K}_n = K_n e^{-h_m}$ with h_m given by (0.8)*. Then, up to a subsequence, the following alternative holds:*

1. *either u_n is uniformly bounded in $L^\infty(\Sigma)$;*
2. *or u_n diverges to $-\infty$ uniformly;*
3. *or there exists a finite set $S = \{q_1, \ldots, q_r\} \subset \Sigma^+$ of blow–up points.*

   In such case, $u_n \to -\infty$ in compact sets of $\Sigma \setminus S$ and $\tilde{K}_n e^{u_n} \rightharpoonup \sum_{i=1}^r \beta(q_i)\delta_{q_i}$ in the weak sense of measures where $\beta(q_i) = 8\pi$ if $q_i \notin \{p_1, \ldots, p_\ell\}$ and $\beta(q_i) = 8(1+\alpha_j)\pi$ if $q_i = p_j$ for some $1 \le j \le \ell$.

   Therefore, $\lim_{n\to+\infty} \int_\Sigma \tilde{K}_n e^{u_n} \in \Lambda_\ell$, defined in (0.48)*.*

We point out that that the conical singularities located in Σ^- do not play any role in the compactness result, and for that reason we replace Λ_m by Λ_ℓ in this case.

Observe also that equation (0.15) can be written in the form (0.49) by adding a suitable constant to $u_n = u$, if $K_n = K$ and $f_n = \frac{\lambda}{|\Sigma|}$.

The proof of Theorem 0.4.2 is an adaptation of the results [31, 33] for positive solutions. Let us emphasize, though, that in our case u_n need not be bounded from below and this causes several difficulties in our proofs, see Section 3.1 for more details.

For what concerns existence and multiplicity of solutions, we shall restrict ourselves to the case of **positive orders** α_j. Our proofs make use of variational methods. Indeed, problem (0.15) is the Euler-Lagrange equation of the energy functional (0.38).

Recall that if $\lambda < 8\pi$, then J_λ is coercive and a minimizer exists, whereas J_λ is not bounded from below if $\lambda > 8\pi$. This range is our main concern, and we shall use min-max scheme to find solutions of (0.15) which are saddle-type critical points of J_λ.

In order to state our existence result we introduce an additional assumption on K

(H3) $N^+ > k$ or Σ^+ has a connected component which is not simply connected,

where N^+ is defined in (0.46).

Theorem 0.4.3. *Let $\alpha_1, \dots, \alpha_\ell > 0$, with ℓ defined in (0.47), and $\lambda \in (8k\pi, 8(k+1)\pi) \setminus \Lambda_\ell$. If (H1), (H2), (H3) are satisfied then (0.15) admits a solution.*

For $K > 0$, then $\Sigma^+ = \Sigma$ and $N^+ = 1$, (H3) is then satisfied if the surface Σ has positive genus; this case has been covered in [5]. In this form we reobtain the result of [5].

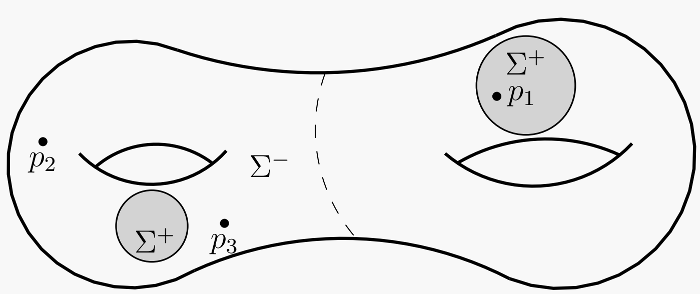

Figure 1

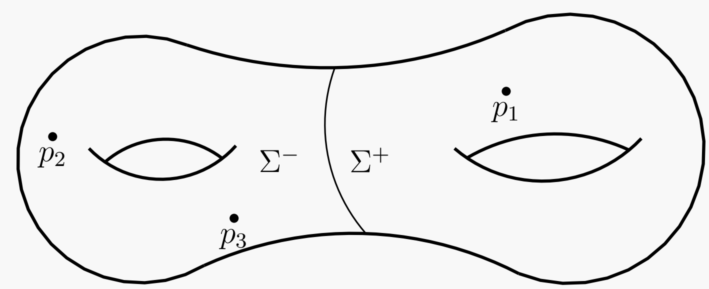

Figure 2

Figure 1 shows an example of potential K with $N^+ = 2$. In this situation, Theorem 0.4.3 only applies for the case $k = 1$. For the situation described by Figure 2, Σ^+ has a connected component which is non-contractible. Therefore, (H3) holds and Theorem 0.4.3 applies for any $k \in \mathbb{N}$.

If Σ^+ has trivial topology Theorem 0.4.3 is not applicable. We can give a result also in this case, following the ideas of [90], which considers positive potentials. For that, we define the set

$$\Theta_\lambda = \{p_j \in \Sigma^+ : \ \lambda < 8\pi(1 + \alpha_j)\}, \tag{0.50}$$

and we introduce the hypothesis

(H4) $\Theta_\lambda \neq \emptyset$.

Theorem 0.4.4. *Let $\alpha_1, \dots, \alpha_\ell \geq 0$, where ℓ is defined in (0.47), and $\lambda \in (8\pi, 16\pi) \setminus \Lambda_\ell$. If (H1), (H2), (H4) are satisfied then (0.15) admits a solution.*

In Figure 3 we show an example of applicability of Theorem 0.4.4 if $\lambda < 8\pi(1+\alpha_1)$.

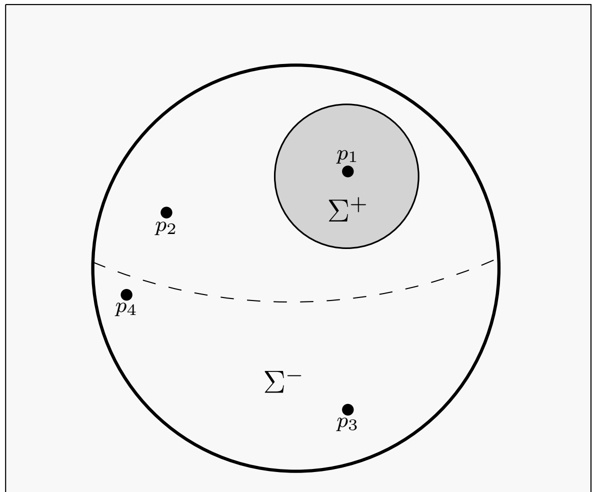

Figure 3

Remark 0.4.5. *There are many examples of applications of these results to the geometric problem commented in the Introduction. Just to show an example, consider the problem of choosing a conformal metric in* $\Sigma = \mathbb{T}^2$ *with Gaussian curvature* K *and one conical point* p *of order* α*. Assume that assumptions (H1), (H2) are satisfied. Then Theorem 0.4.3 implies that the problem is solvable if one of the following assumptions are satisfied*

1. $\alpha \in (k, k+1)$ *with* $k \in \mathbb{N}$ *and* Σ^+ *has more than* k *connected components.*

2. $\alpha \in (k, k+1)$ *with* $k \in \mathbb{N}$ *and* Σ^+ *has a component which is not simply connected.*

Let us now consider the same problem but with m *conical points, all of them of order* α*. Then Theorem 0.4.4 implies that the geometric problem is solvable if* $1 < m\,\alpha < 1 + \alpha$ *and at least one conical points is placed in* Σ^+*.*

Many other examples can be constructed.

In our next result for the special case $\Sigma = \mathbb{S}^2$, we present a class of functions K for which (0.15) does not admit solution. Actually, these functions satisfy (H1) and (H2) but neither (H3) nor (H4) are fulfilled. In order to make clear the statement of the theorem, we will not enter in details on the definition on K.

Theorem 0.4.6. *Let $p \in \mathbb{S}^2$ and $\alpha > 0$ with $m = 1$, $p_1 = p$, $\alpha_1 = \alpha$ and $\tilde{K} = e^{-h_1} K$, then there exists a family of functions K such that (H1) and (H2) hold but equation* (0.15) *does not admit a solution for $\lambda \in (8\pi, +\infty)$,*

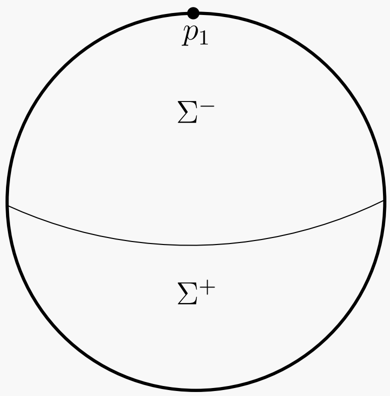

Figure 4

As it will be clear from the definition of K (see (3.92)) in the statement of the previous theorem $p \in \Sigma^-$ (then $J_\lambda = \emptyset$) and Σ^+ is contractible. In particular $N^+ = 1$. In this way, neither (H3) nor (H4) are verified. Figure 4 illustrates a sign changing function K defined in $\mathbb{S}^2$ such that $N^+ = 1$ and a conical singularity located at Σ^-.

This theorem can be considered as the singular extension of the non-existence theorem for the *regular* Nirenberg problem given by Chen and Li in [30], introduced in the subsection 0.2. In fact, the strategy is to choose a function K such that $\tilde{K}$ is sign changing, rotationally symmetric with respect to the point p, monotone in the region where it is positive and $\tilde{K}(-p) = \max_{\mathbb{S}^2} \tilde{K}$

As a consequence of Theorem 0.4.6, the function K can not be realized as the Gaussian curvature of any conformal metric with one singularity.

Finally, we present two multiplicity results for generic choices of the couple (K, g), which cover the situations studied by our existence theorems, Theorem 0.4.3 and Theorem 0.4.4, under nondegeneracy assumptions on the solutions. Roughly speaking, the number of solutions increases as the topology of Σ^+ becomes more involved.

The statements of such results are rather cumbersome and by that reason they are postponed to Section 3.3. In the proofs, some tools of algebraic topology are needed to compute the Betti numbers of the low sublevels of the energy functional.

The results described above have been published in two papers, [47], [48].

0.4.3 Methodology: strategy of the proofs

As commented previously, our proofs use min-max arguments to show existence of critical points of the associated energy functionals. We present an outline which provides a common strategy for the proofs and recalls some well-known facts which encompass the used arguments.

- ***Existence:*** Regarding existence Theorems 0.4.1, 0.4.3 and 0.4.4, we follow the main basis of Morse theory, which, intuitively, asserts that the topology of the sublevels of a functional does not change if there are no critical points. Let E be an open subset in a Hilbert space and $\mathcal{F} \in \mathcal{C}^1(E, \mathbb{R})$, we denote the sublevel

  $$\mathcal{F}^a = \{e \in E : \mathcal{F}(e) \leq a\},$$

  where $a \in \mathbb{R}$. Roughly speaking, a topological change of the sublevels implies the existence of a critical point. In this way, our first goal is to obtain a precise topological description of the sublevels of I_λ, and J_λ. We will see that the very low sublevels of the energy functionals have non-trivial topology, while the very high ones are trivial, what confirms the existence of a topological variation between high and low levels.

  In a certain sense, functions at low energy level tend to concentrate around a finite number of points, and we use this point configurations to study the topology of $\mathcal{F}^{-L}$, with large L enough. Therefore, the principal purpose is to find a compact non-contractible topological space $\mathcal{Z}$, space of weighted of point configurations, to describe the topology of $\mathcal{F}^{-L}$. We shall construct a continuous map Ψ which projects $\mathcal{F}^{-L}$ into $\mathcal{Z}$ and a reverse one Φ, such that the composition

  $$\mathcal{Z} \xrightarrow{\Phi} \mathcal{F}^{-L} \xrightarrow{\Psi} \mathcal{Z} \tag{0.51}$$

is homotopically equivalent to the identity map on $\mathcal{Z}$. In this situation, it is said that $\mathcal{F}^{-L}$ dominates $\mathcal{Z}$ (see [64], page 528). As a consequence, the topology of $\mathcal{F}^{-L}$ is richer than that of $\mathcal{Z}$. Indeed, the non-contractibility of $\mathcal{Z}$ implies that $\Phi(\mathcal{Z})$ is not contractible in $\mathcal{F}^{-L}$. Moreover, the map induced by Φ

$$H_q(\mathcal{Z}) \xrightarrow{\Phi*} H_q(\mathcal{F}^{-L}), \quad \text{for } q \in \mathbb{N},$$

is injective, where $H_q(\mathcal{Z})$ is the homology group with $\mathbb{Z}_2$ coefficients.

The topological variation of the sublevels implies the existence of a Palais-Smale sequence $\{u_n\}$, namely $I_\lambda(u_n) \to c_\lambda > -\infty$ and $I'_\lambda(u_n) \to 0$, where c_λ is the min-max value, see [4] for instance. However, this fact does not imply directly the existence of a critical point. In fact, the Palais-Smale property is not known to hold in this kind of problems. Fortunately, this difficulty can be overcome by using the so-called monotonicity trick of Struwe, [109], which guarantees the boundedness, and hence convergence, of the Palais-Smale sequence for almost all values of the parameter λ. To extend the existence of critical points for the rest of values of the parameter, a compactness property for the solutions is needed. This issue is commented below.

- ***Compactness:*** Taking into account what commented above, we are led with the following problem: given u_n a sequence of solutions of (0.43) or (0.15) for $\lambda = \lambda_n \to \lambda_0$, is it uniformly bounded?

  As we have introduced in the Sections 0.2 and 0.3, this question has been addressed in [17, 78] for the regular problem, and in [8, 9] for the equation with vortices, always for positive potentials $K(x)$. Here, the assumption on the positivity of K is not just a technical issue, as can be inferred from some recent examples of blowing-up solutions in [13, 50]. Those solutions concentrate around local maxima of K at 0 level, a situation which, a priori, could be reproduced in the proposed problems. However, the assumptions on where and how K can change sign, (A2) and (H1) respectively, allows us to rule out this phenomena.

  In the first problem we conclude compactness by energy estimates. This argument seems to be completely new in this kind of problems, but cannot be interpreted as an usual compactness result due to the extra assumption on bounded energy level. Moreover, this energy comparison argument is restricted to the specific form of (0.4) and does not work for more general problems.

In order to study the compactness question for (0.4), we adopt a different strategy. First we derive uniform integral estimates in subsets of Σ^+ or Σ^-, which allow one to obtain a priori estimates in the the region $\{x \in \Sigma : \ K(x) < -\delta\}$, for $\delta > 0$ small. Then the moving plane technique is used to compare the values of u on both sides of the nodal curve Γ. This, together with the aforementioned integral estimate, implies boundedness in a neighborhood of Γ. Conclusively, we rely on blow–up analysis in the region $\{x \in \Sigma : \ K(x) > \delta\}$ and the quantization results, introduced in the previous section, can be applied.

- ***Multiplicity:*** The estimates of the number of solutions are given in generic terms. In other words, the multiplicity results are valid under the assumption that all solutions are non-degenerate. A transversality argument, see [106] for instance, guarantees that this is the case for a generic choice of (g, K). More precisely, for (g, K) in an open and dense subset of $\mathcal{M}^2 \times C^{2,\alpha}(\Sigma)$, where $\mathcal{M}^2$ stand for the space of all $C^{2,\alpha}$ Riemannian metrics on Σ equipped with the $C^{2,\alpha}$ norm.

  Under these conditions, we can employ the weak Morse inequalities, which, together with the computations of the homology of a pair, enables us to prove that

$$\#\{\text{critical points of } I_\lambda \text{ in } \{a \leq I_\lambda \leq b\} \ \} \geq \sum_{q \geq 0} dim \left(H_q(I_\lambda^b, I_\lambda^a) \right).$$

  The above formula suggests to study rigorously the homology of the high and low sublevels. Hence, we can make use of the topological description of the sublevels given in the existence part. Indeed, by (0.51),

$$\sum_{q \geq 0} dim \left(H_q(I_\lambda^b, I_\lambda^a) \right) \geq \sum_{q \geq 0} dim(H_q(\mathcal{Z}).$$

  To deduce the Betti numbers of $\mathcal{Z}$, we need to make use of some tools from algebraic topology. Actually, one of the main problems is to study the homogy groups of $\mathcal{Z}$, which will be the set of barycenters on a disjoint union. We bypass the difficulty through a formula which relates the homology of the barycenters on a disjoint union to the homology of the barycenters on the disjoint spaces, see Proposition 1.2.2.

Chapter 1

Notation and preliminaries

In this chapter we fix the notation used in this thesis and collect some preliminary known results.

From now on (Σ, g) is a compact surface without boundary equipped with a Riemannian metric g, whereas $(\mathbb{S}^2, g_0)$ is the 2-sphere equipped with the standard metric. We denote by $d(x, y)$ the distance between two points $x, y \in \Sigma$ induced by the ambient metric. The symbol $B_p(r)$ stands for the open ball of radius $r > 0$ and center $p \in \Sigma$, $A_p(r, R)$ denotes the corresponding open annulus and

$$\Omega^r = \{x \in \Sigma : \, d(x, \Omega) < r\}.$$

Given $f \in L^1(\Sigma)$, we denote the mean value of f by $\fint_\Sigma f = \frac{1}{|\Sigma|} \int_\Sigma f$, where $|\Sigma|$ is the area of Σ.

Let us recall the definition of the energy functionals

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dV_g + 2 \int_\Omega u \, dV_g - \lambda \log \int_\Omega K e^u \, dV_g, \tag{1.1}$$

defined in the domain

$$X = \left\{ u \in H^1(\Omega) : \int_\Omega K e^u \, dV_g > 0 \right\}. \tag{1.2}$$

Here Ω is a subdomain of $\mathbb{S}^2$. Moreover, we have defined

$$J_\lambda(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 dV_g + \frac{\lambda}{|\Sigma|}\int_\Sigma u\, dV_g - \lambda \log \int_\Sigma \tilde{K} e^u dV_g, \tag{1.3}$$

defined in the domain

$$X = \left\{ u \in H^1(\Sigma) : \int_\Sigma \tilde{K} e^u \, dV_g > 0 \right\}, \tag{1.4}$$

where Σ is a compact surface without boundary. The Euler-Lagrange equations of those functionals are (0.43) and (0.4), respectively.

The symbol $\amalg$ will be employed to denote the disjoint union of sets; while $\triangle$ stands for the symmetric difference of sets. If two topological spaces X, Y are homeomorphic, we will write $X \cong Y$.

Throughout the thesis, the sign $\simeq$ will refer to homotopy equivalences, while $\cong$ refers to homeomorphisms between topological spaces or isomorphisms between groups.

Given a metric space M and $k \in \mathbb{N}$, we denote by $Bar_k(M)$ the set of formal barycenters of order k on M, namely the following family of unit measures supported in at most k points

$$Bar_k(M) = \left\{ \sum_{i=1}^k t_i \delta_{x_i} \,:\, t_i \in [0,1], \, \sum_{i=1}^k t_i = 1, \, x_i \in M \right\}. \tag{1.5}$$

We consider $Bar_k(M)$ as a topological space with the weak* topology of measures. Positive constants are denoted by C, and the value of C is allowed to vary from formula to formula. If we want to stress the dependence of the constants on some parameters, we include subscripts to C, as C_ε. Moreover, we will write $o_\alpha(1)$ to denote quantities that tend to 0 as $\alpha \to 0$ or $\alpha \to +\infty$; whereas we will use the symbol $O(1)$ for bounded quantities.

1.1 Analytic preliminaries

A powerful tool in our study is the Moser-Trudinger inequalities and its variations, which will allow us to deduce fundamental properties about the functionals I_λ and

J_λ.

Let us start by recalling the classical Moser-Trudinger inequality for compact surfaces, which was established in [96].

Theorem 1.1.1. *Let Σ be a compact surface without boundary, then there exists a universal constant $C > 0$ such that*

$$\int_\Sigma e^{4\pi u^2} \leq C,$$

for any $u \in H^1(\Sigma)$ with $\int_\Sigma |\nabla u|^2 \leq 1$ and $\int_\Sigma u = 0$.

Now, we introduce a *weaker* version of the previous Moser-Trudinger inequality:

Proposition 1.1.2. *Let Σ be a compact surface without boundary, then there exists $C > 0$ such that*

$$\log \int_\Sigma e^u dV_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 \, dV_g + C \,, \qquad \forall\, u \in H^1(\Sigma) \quad \textit{with} \int_\Sigma u \, dV_g = 0. \quad (1.6)$$

Proof. Clearly,

$$u \leq \frac{1}{4a} + a \cdot u^2 \qquad \forall a > 0.$$

Choosing $a = \frac{4\pi}{\int_\Sigma |\nabla u|^2}$, we have

$$e^u \leq \exp\left\{ \frac{\int_\Sigma |\nabla u|^2}{16\pi} \right\} \exp\left\{ \frac{4\pi u^2}{\int_\Sigma |\nabla u|^2} \right\}.$$

Integrating this inequality on Σ and taking logarithms, one obtains

$$\log \int_\Sigma e^u \, dV_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 \, dV_g + \log \int_\Sigma e^{\frac{4\pi u^2}{\int_\Sigma |\nabla u|^2}} \, dV_g.$$

It suffices to apply Theorem 1.1.2 to the function $\frac{u}{\sqrt{\int_\Sigma |\nabla u|^2}}$ to conclude the proof. □

The constant multiplying the Dirichlet energy is optimal. In other words, for α less than $\frac{1}{16\pi}$, using the standard bubbles (0.22) peaked at some point of Σ, one can check that (1.6) does not hold. (See Lemma 2.1.2 and Appendix)

As a direct consequence of the previous proposition, we have that

$$J_\lambda(u) \geq \frac{8\pi - \lambda}{16\pi} \int_\Sigma |\nabla u|^2 \, dV_g - C,$$

for all $u \in X$. In particular, J_λ is coercive for $\lambda \in (0, 8\pi)$, and a solution for (0.15) can be found as a minimizer.

For larger values of the parameter λ the previous inequality does not give any information. In particular, it can be seen that the functional is not bounded from below for $\lambda > 8\pi$. See Lemma 3.2.4 and Lemma 3.2.11.

Next we introduce a localized version of the Moser-Trudinger inequality related to Proposition 1.1.2.

Proposition 1.1.3. *Let Σ be a compact surface with or without boundary, $\Sigma_1 \subset \Sigma$ and $\delta > 0$ such that $(\Sigma_1)^\delta \cap \partial\Sigma = \emptyset$. Then, for any $\varepsilon > 0$ there exists a constant $C = C_{\varepsilon,\delta}$ such that for all $u \in H^1(\Sigma)$ with $\int_\Sigma u \, dV_g = 0$,*

$$16\pi \log \int_{\Sigma_1} e^u \, dV_g \leq \int_{(\Sigma_1)^\delta} |\nabla_g u|^2 \, dV_g + \varepsilon \int_\Sigma |\nabla_g u|^2 \, dV_g + C. \tag{1.7}$$

Proof. Our argument follows closely the proof used in Theorem 2.1 in [26] (see also [90]).

First, we consider a smooth cutoff function g with values into $[0, 1]$ satisfying

$$\begin{cases} g(x) = 1, & \forall x \in \Sigma_1, \\ g(x) = 0, & \forall x \in \Sigma \setminus (\Sigma_1)^{\delta/2}. \end{cases} \tag{1.8}$$

Clearly, $gu \in H^1_0(\Sigma)$. Applying inequality (1.1.1) to gu we obtain

$$16\pi \log \int_{\Sigma_1} e^u \, dV_g \leq 16\pi \log \int_\Sigma e^{gu} \, dV_g \leq \int_\Sigma |\nabla(gu)|^2 \, dV_g + C.$$

Using the Leibnitz rule to the gradient we have

$$\int_\Sigma |\nabla(gu)|^2 \, dV_g \leq \int_{(\Sigma_1)^\delta} |\nabla u|^2 \, dV_g + 2 \int_\Sigma gu \nabla g \nabla u \, dV_g + C_\delta \int_\Sigma u^2 \, dV_g. \tag{1.9}$$

By Cauchy's inequality,

$$\int_{\Sigma} gu\nabla g\nabla u\, dV_g \leq \varepsilon \int_{\Sigma} |\nabla u|^2 \, dV_g + C_{\varepsilon,\delta} \int_{\Sigma} u^2 \, dV_g. \tag{1.10}$$

Combining (1.9) and (1.10), we get

$$16\pi \log \int_{\Sigma_1} e^u \, dV_g \leq \int_{(\Sigma_1)^\delta} |\nabla u|^2 \, dV_g + \varepsilon \int_{\Sigma} |\nabla u|^2 \, dV_g + C_{\varepsilon,\delta} \int_{\Sigma} u^2 \, dV_g. \tag{1.11}$$

Let us now estimate the last term of (1.11). Take η such that $|\{x \in \Sigma : u(x) \geq a\}| = \eta$. Let $(u-a)^+ = max\{0, u-a\}$ and applying (1.11), we obtain

$$\begin{aligned} 16\pi \log \int_{\Sigma_1} e^u \, dV_g &\leq 16\pi \log \left\{ e^a \, dV_g \int_{\Sigma_1} e^{(u-a)^+} \, dV_g \right\} \\ \leq 16\pi a + \int_{(\Sigma_1)^\delta} dV_g \, |\nabla u|^2 \, dV_g &+ \varepsilon \int_{\Sigma} |\nabla u|^2 \, dV_g + C_{\varepsilon,\delta} \int_{\Sigma} \left((u-a)^+\right)^2 \, dV_g. \end{aligned} \tag{1.12}$$

By Hölder and Sobolev inequalities

$$\int_{\Sigma} \left((u-a)^+\right)^2 \, dV_g \leq \eta^{1/2} \left(\int_{\Sigma} \left((u-a)^+\right)^4 \right)^{1/2} \, dV_g \leq c\eta^{1/2} \int_{\Sigma} |\nabla u|^2 \, dV_g, \tag{1.13}$$

and by Poincaré inequality

$$a\eta \leq \int_{a \leq u} u \, dV_g \leq \int_{\Sigma} |u| \, dV_g \leq c \left(\int_{\Sigma} |\nabla u|^2 \right)^{1/2} \, dV_g. \tag{1.14}$$

Hence for every $\delta > 0$, from (1.14) by Cauchy's inequality,

$$a \leq \delta \int_{\Sigma} |\nabla u|^2 \, dV_g + \frac{c^2}{4\delta\eta^2}. \tag{1.15}$$

Finally, let η satisfying

$$\eta^{1/2} \geq \frac{C_{\varepsilon,\delta}}{\varepsilon}. \tag{1.16}$$

From (1.12), (1.13), (1.15) and (1.16), we conclude the proof.

□

As a consequence of the last result, we present a version of the Chen-Li inequality [26] (see also [3]). This version was first stated in [118]. Roughly speaking, it states that if e^u is *spread* in several regions of Σ, one can obtain a better constant in the weak Moser-Trudinger inequality. This is essential in the min-max scheme developed in Section 3.2.

Lemma 1.1.4. *Let $\varepsilon > 0$, $\delta > 0$, $\ell \in \mathbb{N}$, $\gamma \in (0, \frac{1}{\ell+1})$ and $\Sigma_1, \ldots, \Sigma_{\ell+1}$ be subsets of Σ with $(\Sigma_i)^\delta \cap (\Sigma_j)^\delta = \emptyset$, for $i \neq j$. If*

$$\frac{\int_{\Sigma_i} e^u \, dV_g}{\int_\Sigma e^u \, dV_g} \geq \gamma, \qquad \text{for } i = 1, \ldots, \ell + 1, \tag{1.17}$$

then, there exists a constant $C = C_{\tilde{\varepsilon},\delta,\gamma}$ such that for all $u \in H^1(\Sigma)$ with $\int_\Sigma u \, dV_g = 0$

$$(16(\ell + 1)\pi - \varepsilon) \log \int_\Sigma e^u \, dV_g \leq \int_\Sigma |\nabla u|^2 \, dV_g + C. \tag{1.18}$$

Proof. First, we take $\Sigma_1, \ldots, \Sigma_{\ell+1}$ verifying (1.17) and apply Proposition 1.1.3 for each one

$$16\pi \log \int_\Sigma e^u \, dV_g \leq 16\pi \log \left(\frac{1}{\gamma} \int_{\Sigma_i} e^u \, dV_g \right) \leq \int_{(\Sigma_i)^\delta} |\nabla u|^2 \, dV_g + \varepsilon \int_\Sigma |\nabla u|^2 \, dV_g + C_{\varepsilon,\delta,\gamma}$$

with $i = 1, \ldots, \ell + 1$.

Finally, we make the addition in i,

$$\begin{aligned} 16\pi\ell \log \int_\Sigma e^u \, dV_g &\leq \int_{\bigcup_{i=1}^{\ell} (\Sigma_i)^\delta} |\nabla u|^2 \, dV_g + (\ell + 1)\varepsilon \int_\Sigma |\nabla u|^2 \, dV_g + C_{\varepsilon,\delta,\gamma} \\ &\leq (1 + \ell\varepsilon) \int_\Omega |\nabla u|^2 \, dV_g + C_{\varepsilon,\delta,\gamma}, \end{aligned}$$

concluding the proof.

□

On the other hand, we can introduce analogous versions of the previous inequalities for subdomains of some surface Σ. The first inequality is the classic Moser-Trudinger inequality for domains, see [96, 97, 116]. The second one is the Proposition 2.3 of [22], which was conceived for plane domains and so for subdomains of any general compact surface.

Theorem 1.1.5. *Let Ω be a compact surface with smooth boundary, then there exists $C > 0$ such that*

$$\int_\Omega e^{4\pi u^2}\, dV_g \leq C, \quad \forall u \in H_0^1(\Omega), \tag{1.19}$$

and

$$\int_\Omega e^{2\pi u^2}\, dV_g \leq C, \quad \forall u \in H^1(\Omega) \quad \textit{with} \quad \int_\Omega u\, dV_g = 0. \tag{1.20}$$

Repeating the proof of Proposition 1.1.2, we can show the following *weaker* inequalities as a consequence of the previous ones.

Proposition 1.1.6. *Let Ω be a compact surface with smooth boundary, then there exists $C > 0$ such that*

$$\log \int_\Omega e^u\, dV_g \leq \frac{1}{16\pi} \int_\Omega |\nabla u|^2\, dV_g + C, \quad \forall u \in H_0^1(\Omega), \tag{1.21}$$

and

$$\log \int_\Omega e^u\, dV_g \leq \frac{1}{8\pi} \int_\Omega |\nabla u|^2\, dV_g + C, \quad \forall u \in H^1(\Omega) \quad \textit{with} \quad \int_\Omega u\, dV_g = 0. \tag{1.22}$$

The constant in (1.22) appears multiplied by two in relation to (1.21), since we can center a bubble on a point of $\partial\Omega$, so that its volume and Dirichlet energy are divided approximately by two. However, this is not allowed in (1.21) because of its boundary condition.

As an easy application of the previous proposition, we have

$$I_\lambda(u) \geq \frac{4\pi - \lambda}{8\pi} \|u\|_{H^1(\Omega)}^2 + C,$$

for all $u \in X$. In particular, I_λ is coercive for $\lambda \in (0, 4\pi)$, and a solution for (0.43) can be found as a minimizer.

The following localized version of Moser-Trudinger type inequalities will be of use.

Proposition 1.1.7. *Let $\varepsilon > 0$, $\delta > 0$ and $\Omega_1 \subset \Omega$ such that $(\Omega_1)^\delta \cap \partial\Omega = \emptyset$. Then, there exists a constant $C = C_{\varepsilon,\delta}$, such that for every $u \in H^1(\Omega)$ with $\int_\Sigma u\, dV_g = 0$,*

$$16\pi \log \int_{\Omega_1} e^u \, dV_g \leq \int_{(\Omega_1)^\delta} |\nabla u|^2 \, dV_g + \varepsilon \int_\Omega |\nabla u|^2 \, dV_g + C.$$

Proof. The result follows directly from the proof of Proposition 1.1.3, applying inequality (1.21). □

Proposition 1.1.8. *Let Ω be a compact surface with smooth boundary and $\Omega_1 \subset \Omega$. For any $\varepsilon > 0$, $\delta > 0$, there exists a constant $C = C_{\varepsilon,\delta}$, such that for every $u \in H^1(\Omega)$ with $\int_\Omega u\, dV_g = 0$,*

$$8\pi \log \int_{\Omega_1} e^u \, dV_g \leq \int_{(\Omega_1)^\delta} |\nabla u|^2 \, dV_g + \varepsilon \int_\Omega |\nabla u|^2 \, dV_g + C.$$

Proof. We use an analogous argument to the one used in Proposition 1.1.3. Let g as defined in (1.8) and applying (1.22) to gu, we obtain

$$8\pi \log \int_{\Omega_1} e^u \, dV_g \leq 8\pi \log \int_\Omega e^{gu} \, dV_g \leq \int_\Omega |\nabla (gu)|^2 \, dV_g + \overline{gu} + C.$$

Now, we estimate the average of gu as

$$\overline{gu} \leq C_\delta + C \int_\Omega u^2 \, dV_g.$$

Then, as we did in (1.11), we have

$$8\pi \log \int_{\Omega_1} e^u \, dV_g \leq \int_{(\Omega_1)^\delta} |\nabla u|^2 \, dV_g + \varepsilon \int_\Omega |\nabla u|^2 \, dV_g + C \int_\Omega u^2 \, dV_g + C.$$

It suffices to estimate $\int_\Omega u^2 \, dV_g$ exactly in the same way from the previous proof.

□

Observe the difference between the choice of Ω_1 in both propositions. Whereas in the first result Ω_1 stays away from the boundary of Ω, there is no restriction in that sense in the second one.

Finally, we introduce the Chen-Li inequality related to Proposition 1.1.7 and Proposition 1.1.8. In a certain sense, it states that if e^u is *spread* into two regions of Ω or if e^u has *mass inside* Ω, then the energy functional is bounded from below.

Lemma 1.1.9. *Let $\varepsilon > 0$, $\delta > 0$ and $0 < \gamma < 1/2$. Let Ω_1, Ω_2 and S be subsets of Ω, where Ω is a compact surface with smooth boundary, such that $(\Omega_1)^\delta \bigcap (\Omega_2)^\delta = \emptyset$ and $S^\delta \bigcap \partial\Omega = \emptyset$. If*

$$\frac{\int_{\Omega_1} e^u \, dV_g}{\int_\Omega e^u \, dV_g} \geq \gamma, \qquad \frac{\int_{\Omega_2} e^u \, dV_g}{\int_\Omega e^u \, dV_g} \geq \gamma, \tag{1.23}$$

or

$$\frac{\int_S e^u \, dV_g}{\int_\Omega e^u \, dV_g} \geq \gamma, \tag{1.24}$$

then, there exists a constant $C = C_{\varepsilon,\delta,\gamma}$, such that for all $u \in H^1(\Omega)$ satisfying $\int_\Omega u \, dV_g = 0$,

$$(16\pi - \varepsilon) \log \int_\Omega e^u \, dV_g \leq \int_\Omega |\nabla u|^2 \, dV_g + C.$$

Proof. For S satisfying (1.24), this is just Proposition 1.1.4 with $\ell = 1$.

Take Ω_1, Ω_2 verifying (1.23) and apply Proposition 1.1.8 to each one

$$8\pi \log \int_\Omega e^u \, dV_g \leq 8\pi \log \left(\frac{1}{\gamma} \int_{\Omega_i} e^u \, dV_g \right) \leq \int_{\Omega_i^\delta} |\nabla u|^2 \, dV_g + \varepsilon \int_\Omega |\nabla u|^2 \, dV_g + C_{\varepsilon,\delta,\gamma}$$

with $i = 1, 2$.

Finally, we add both expressions

$$\begin{aligned} 16\pi \log \int_\Omega e^u \, dV_g &\leq \int_{\Omega_1^\delta \bigcup \Omega_2^\delta} |\nabla u|^2 \, dV_g + 2\varepsilon \int_\Omega |\nabla u|^2 \, dV_g + C_{\varepsilon,\delta,\gamma} \\ &\leq (1 + 2\varepsilon) \int_\Omega |\nabla u|^2 \, dV_g + C_{\varepsilon,\delta,\gamma}), \end{aligned}$$

concluding the proof.

□

Lemma 1.1.10. *Let Ω be a smooth compact surface (with or without boundary) and $\{u_n\} \subset H^1(\Omega)$ a sequence such that*

$$u_n \rightharpoonup u_0 \in H^1(\Omega).$$

Then

$$e^{u_n} \to e^{u_0} \quad \text{in } L^p(\Omega) \text{ with } p \in [1, \infty)$$

Proof.

Let $a, b \in \mathbb{R}$, by the mean value theorem we have that

$$\left|e^a - e^b\right| \le e^{|a|+|b|} |a - b| . \tag{1.25}$$

Consider $\{u_n\} \subset H^1(\Omega)$, $u_0 \in H^1(\Omega)$ and $p \in [1, \infty)$, by (1.25) one obtains

$$|e^{u_n} - e^{u_0}|^p \le e^{p(|u_n|+|u_0|)} |u_n - u_0|^p . \tag{1.26}$$

Now, integrating the last inequality and applying Hölder then

$$\int_\Omega |e^{u_n} - e^{u_0}|^p \le \int_\Omega e^{p(|u_n|+|u_0|)} |u_n - u_0|^p \le \left(\int_\Omega e^{2p(|u_n|+|u_0|)}\right)^{1/2} \left(\int_\Omega |u_n - u_0|^{2p}\right)^{1/2} . \tag{1.27}$$

On the other hand, for some $\epsilon > 0$ we get the following inequality

$$e^u \le e^{\frac{u^2 \epsilon}{2}} e^{\frac{1}{2\epsilon}} = C_1 e^{\frac{u^2 \epsilon}{2}} \tag{1.28}$$

which allows to take C_0 such that

$$0 \le \frac{\epsilon}{2} \le \frac{2\pi}{\int_\Omega |\nabla(2p\,|u_n| + |u_0|)|^2} \tag{1.29}$$

Combining (1.28), (1.29) and (1.27) and applying the Moser-Trudinger inequality (1.20) (or Theorem 1.1.1)

$$\int_\Omega e^{2p(|u_n|+|u_0|)} \le C_1 \int_\Omega e^{\frac{\epsilon}{2}(2p(|u_n|+|u_0|))^2} \le C_1 \int_\Omega e^{\frac{2\pi(|u_n|+|u_0|))^2}{\int_\Omega |\nabla(|u_n|+|u_0|)|^2}} \le C_2,$$

By the *Rellich-Kondrachov Theorem*, $u_n \to u_0$ in $L^{2p}(\Omega)$. Therefore,

$$\int_\Omega |u_n - u_0|^{2p} \to 0,$$

which immediately implies

$$\int_\Omega |e^{u_n} - e^{u_0}|^p \to 0,$$

as we wanted. □

1.2 Topological and Morse-theoretical preliminaries

In this subsection we recall a classical theorem on Morse inequalities. Furthermore we give a short review of basic notions of algebraic topology needed to get the multiplicity estimates. Finally, we state a recent result concerning the topology of barycenter sets with disconnected base space.

Given a pair of spaces (X, A) we will denote by $H_q(X, A)$ the relative q-th homology group with coefficient in $\mathbb{Z}_2$ and by $\tilde{H}_q(X) = H_q(X, x_0)$ the reduced homology with coefficient in $\mathbb{Z}_2$, where $x_0 \in X$. We adopt the convention that $\tilde{H}_q(X) = 0$ for any $q < 0$.
Finally, if X, Y, are two topological spaces and $f : X \to Y$ is a continuous function, we will denote by $f_* : H_q(X) \to H_q(Y)$ the pushforward morphism induced by f.

Let us first recall a result in Infinite Morse theory, see e.g. Theorem 4.3 [19].

Theorem 1.2.1. *Suppose that H is a Hilbert manifold, $I \in C^2(H;\mathbb{R})$ satisfies the (PS)-condition at any level $c \in [a, b]$, where a, b are regular values for I. If all the critical points of I in $\{a \le I \le b\}$ are nondegenerate, then*

$$\#\{\textit{critical points of } I \textit{ in } \{a \le I \le b\} \textit{ with index } q\} \ge \dim(H_q(\{I \le b\}, \{I \le a\}))$$

for any $q \geq 0$.

In what follows we collect some well-known definitions and results in algebraic topology and we refer to [64] for further details.

Wedge sum. Given spaces C and D with chosen points $c_0 \in C$ and $d_0 \in D$, then the wedge sum $C \vee D$ is the quotient of the disjoint union $C \amalg D$ obtained by identifying c_0 and d_0 to a single point. If $\{c_0\}$ (resp. $\{d_0\}$) is a closed subspace of C (resp. D) and is a deformation retract of some neighborhood in C (resp. D), then

$$\tilde{H}_q(C \vee D) \cong \tilde{H}_q(C) \bigoplus \tilde{H}_q(D), \tag{1.30}$$

see [64, Corollary 2.25].

Unreduced suspension. The unreduced suspension (often, as in [64], denoted by SC) is defined to be

$$\Sigma C = (C \times [0,1])/\{(c_1, 0) \simeq (c_2, 0) \text{ and } (c_1, 1) \simeq (x_2, 1) \text{ for all } c_1, c_2 \in C\}. \tag{1.31}$$

For the reduced homology of the unreduced suspension the following formula holds, [64, page 132, ex. 20],

$$\tilde{H}_{q+1}(\Sigma C) \cong \tilde{H}_q(C). \tag{1.32}$$

Join. The join of two spaces C and D is the space of all segments joining points in C to points in D. It is denoted by $C * D$ and is the identification space

$$C * D = C \times [0,1] \times D/(c,0,d) \sim (c',0,d), (c,1,d) \sim (c,1,d') \qquad \forall\, c,\, c' \in C, \forall\, d,\, d' \in D.$$

Being $C * D \simeq \Sigma(C \vee D)$, [64, page 20, ex. 24], we have that

$$\tilde{H}_q(C * D) \cong \tilde{H}_q(\Sigma(C \vee D)). \tag{1.33}$$

At last, we present a recent result obtained in [1, Theorem 5.19] concerning the space of formal barycenters on a disjoint union of spaces.

Proposition 1.2.2. *For C, D two disjoint connected spaces and $k \geq 2$, $Bar_k(C \amalg D)$ has the homology of*

$$\begin{aligned} &Bar_k(C) \vee \Sigma Bar_{k-1}(C) \vee Bar_k(D) \vee \Sigma Bar_{k-1}(D) \vee \\ &\vee \bigvee_{\ell=1}^{k-1} (Bar_{k-\ell}(C) * Bar_\ell(D)) \vee \bigvee_{\ell=2}^{k-1} (\Sigma Bar_{k-\ell}(C)) * Bar_{\ell-1}(D). \end{aligned}$$

Chapter 2

The mean field problem on a subdomain of the sphere

This chapter is devoted to the proof of Theorem 0.4.1. This theorem claims the existence of solutions for the Neumann boundary problem

$$\begin{cases} -\Delta_{g_0} u + 2 = 2K(x)e^u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{S}^2$ and K satisfies

(A1) $K(x) > 0$ for some $x \in \overline{\Omega}$;

(A2) $K(x) \neq 0$ for all $x \in \partial\Omega$.

As explained in the introduction, (2.1) is equivalent to the mean field equation

$$\begin{cases} -\Delta_{g_0} u = \lambda \left(\frac{Ke^u}{\int_\Omega Ke^u dV_{g_0}} - \frac{1}{|\Omega|} \right) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $\lambda = 2|\Omega|$. Therefore, Theorem 0.4.1 can be obtained as a direct consequence of the following result:

Theorem 2.0.1. *Let $\lambda \in (4\pi, 8\pi)$ and assume (A1) and (A2). Then, problem (2.2) admits a solution.*

This result has been presented in the publication [86].

At first, let us denote the boundary points such that K is strictly positive by

$$(\partial\Omega)^+ = \{x \in \partial\Omega : K(x) > 0\}.$$

By assumption (A2), the set $(\partial\Omega)^+$ is a union of simple closed curves. Moreover, if $(\partial\Omega)^+ \neq \emptyset$, the set is not contractible.

Let us recall that the underlying energy functional of (2.2) is

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dV_{g_0} + 2\int_\Omega u \, dV_{g_0} - \lambda \log \int_\Omega K e^u \, dV_{g_0},$$

defined in the domain

$$X = \left\{ u \in H^1(\Omega) : \int_\Omega K e^u \, dV_{g_0} > 0 \right\}.$$

The proof of Theorem 2.0.1 uses the variational argument introduced in subsection 0.4.3. Roughly speaking, we will show that functions u with very low energy level concentrate around a point of $\mathcal{Z} = (\partial\Omega)^+$. This fact allows us to map continuously I_λ^{-L} into $(\partial\Omega)^+$ for large values L. Next, we can construct the reverse projection employing *bubbles* around any fixed point $p \in (\partial\Omega)^+$. The nontrivial topology of $(\partial\Omega)^+$ implies that the low energy levels of I_λ are also not contractible. In that situation, the min-max structure, jointly with the monotonicty trick of Struwe, yields the existence of a critical point of I_λ for almost every value of λ.

Finally, a compactness criterion via energy estimates concludes the proof. Typically, compactness of solutions is obtained via a quantization result, in the spirit of Brezis-Merle and Li-Shafrir [17, 78]. However, the fact that K may change sign is a serious obstacle for this quantization; this question will be considered in depth in Section 3.1.

In [118] it is claimed that if u_n is an unbounded sequence of solutions of (2.2) with $\lambda = \lambda_n$, then

$$\lambda_n \to 4k\pi, k \in \mathbb{N}.$$

However the derivation of this result in [118] is correct only for strictly positive

K. Indeed, even for K vanishing at a point, other limit values can be achieved, as shown in [9]. Observe, moreover, that in our setting no assumption is made on the set of zeroes of K, apart from being disjoint with $\partial\Omega$.

Here we bypass this problem by noting that our solutions have bounded energy. This energy control implies already a certain concentration behavior of the sequence of solutions, if unbounded. Since K is strictly positive on $(\partial\Omega)^+$, the blow–up quantization of [78] yields the desired contradiction.

2.1 Proof of Theorem 0.4.1

The following proposition will become crucial not only in the min-max argument, but also for the compactness result (see Proposition 2.1.9). Its proof is based on an energy comparison argument and an application of the Moser-Trudinger type inequalities, stated in Chapter 1.

Proposition 2.1.1. *Let λ be a fixed constant in $(4\pi, 8\pi)$, $\{u_n\}$ a sequence in X such that $I_\lambda(u_n) < C$. If*

$$\|u_n\|_{H^1(\Omega)} \to \infty, \tag{2.3}$$

then, up to a subsequence,

$$\frac{e^{u_n}}{\int_\Sigma e^{u_n}} \rightharpoonup \delta_p \quad \textit{with } p \in (\partial\Omega)^+.$$

Proof. Let $C > 0$ be such that $K(x) < C$ for all $x \in \Omega$. We define

$$E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dV_{g_0} + 2\int_\Omega u \, dV_{g_0} - \lambda \log \int_\Omega C e^u \, dV_{g_0}, \quad \text{in } H^1(\Omega).$$

Clearly,

$$I_\lambda(u) > E_\lambda(u),$$

for all $u_n \in X$, hence $E_\lambda(u_n)$ is also bounded from above. We introduce the measures

$$\sigma_n = \frac{e^{u_n}}{\int_\Omega e^{u_n} \, dV_{g_0}},$$

which satisfies that $\sigma_n \rightharpoonup \sigma$ in the sense of weak convergence of measures, up to subsequence.

Let $\tau > 0$ and the open subset $\Omega_\tau = \{x \in \Omega : d(x, \partial\Omega) > \tau\}$. Suppose $\sigma(\Omega_\tau) > 0$. By the weak convergence of measures, we have

$$0 < \sigma(\Omega_\tau) \leq \liminf_{n\to\infty} \sigma_n(\Omega_\tau).$$

From this inequality, since (1.24) holds, we can use Lemma 1.1.9 for any $\varepsilon > 0$ to obtain

$$E_\lambda(u_n) > \left(\frac{1}{2} - \frac{\lambda}{16\pi - \varepsilon}\right) \|u_n\|^2_{H^1(\Omega)} + C'. \tag{2.4}$$

By (2.3), (2.4) contradicts the boundedness of $E_\lambda(u_n)$. Therefore $\sigma(\Omega_\tau) = 0$ for every $\tau > 0$, i.e., σ is supported in $\partial\Omega$.

Now, assume by contradiction that there exist $p, q \in supp(\sigma)$ with $p, q \in \partial\Omega$ and $p \neq q$. Given $r > 0$ such that $B_p(2r) \cap B_q(2r) = \emptyset$, there exists $\varepsilon_0 > 0$ which verifies $\sigma(\hat{B}_p(r)) > 2\varepsilon_0$ and $\sigma(\hat{B}_q(r)) > 2\varepsilon_0$ where we define $\hat{B}_p(r) = B_p(r) \cap \overline{\Omega}$.

This last fact implies that $\sigma_n(\hat{B}_p(r)) > \varepsilon_0$ and $\sigma_n(\hat{B}_q(r)) > \varepsilon_0$ for some $n \in \mathbb{N}$. Now, we can apply Lemma 1.1.9 for any $\varepsilon > 0$ to obtain again (2.4) which violates the hypothesis on the boundedness of the energy level.

So,

$$\sigma = \delta_p,$$

as claimed. Moreover, since $u_n \in X$,

$$0 < \frac{\int_\Omega K e^{u_n}\, dV_{g_0}}{\int_\Omega e^{u_n}\, dV_{g_0}} \to K(p),$$

which implies that $p \in (\partial\Omega)^+$.

□

Let us observe that Proposition 2.1.1 yields easily the existence of a solution if K is negative in $\partial\Omega$. Indeed, in such case $(\partial\Omega)^+ = \emptyset$, and Proposition 1.1.9 implies that I_λ is coercive. It is well-known that it is also weak lower semicontinuous, and therefore it attains its infimum. In this way we reobtain the result of [61].

Therefore, we will assume in what follows that K is positive on some connected component of Ω, so that $(\partial\Omega)^+ \neq \emptyset$.

In such case, we are able to construct functions with arbitrary low energy level, as follows.

Lemma 2.1.2. *For any $\mu > 0$ and $p \in (\partial\Omega)^+$, let us define*

$$\varphi_{\mu,p} : \Omega \to \mathbb{R},\ \varphi_{\mu,p}(x) = 2\log\left(\frac{\mu}{1+\mu^2 d(x,p)^2}\right).$$

Then, for any $L > 0$, there exists $\mu(L)$ such that for any $\mu \geq \mu(L)$, $p \in (\partial\Omega)^+$, $I_\lambda(\varphi_{\mu,p}) < -L$.

Proof. See the Appendix.

□

The above lemma implies, in particular, that I_λ is unbounded from below. But it gives much more information: indeed, given any $L > 0$, we can choose μ so that the following continuous map is well-defined

$$\begin{aligned} \Phi_\mu : (\partial\Omega)^+ &\to I_\lambda^{-L} \\ p &\mapsto \varphi_{\mu,p}. \end{aligned}$$

Observe that those functions concentrate, as $\mu \to +\infty$, around $p \in (\partial\Omega)^+$. Now we plan to show that, indeed, any function u in a low sublevel of I_λ must behave in that fashion. This idea is made explicit by a reverse map, that is, a continuous map $\Psi : I_\lambda^{-L} \to (\partial\Omega)^+$, for L large. This map, together with Φ_μ, will give us useful information about the topology of low energy sub-levels of I_λ.

First, let us introduce the center of mass of the function e^u, defined as

$$P(u) = \frac{\int_\Omega x e^u \, dV_{g_0}}{\int_\Omega e^u \, dV_{g_0}} \in \mathbb{R}^3. \tag{2.5}$$

In our next result we show that low sub-levels have center of mass in an arbitrary small neighborhood of Ω^+ in $\mathbb{R}^3$, denoted by

$$N((\partial\Omega)^+, \delta) = \{x \in \mathbb{R}^3 : \ \mathrm{dist}(x, (\partial\Omega)^+) < \delta\},$$

where dist refers to the Euclidean distance in $\mathbb{R}^3$.

Proposition 2.1.3. *Given any $\delta > 0$, there exists $L(\delta) > 0$ such that for any $L > L(\delta)$, we have that $P\left(I_\lambda^{-L}\right) \subset N((\partial\Omega)^+, \delta)$.*

Proof. Take $u_n \in X$ with $I_\lambda(u_n) \to -\infty$. Obviously, it must be an unbounded sequence. By Proposition 2.1.1,

$$\frac{e^{u_n}\, dV_{g_0}}{\int_\Omega e^{u_n}} \rightharpoonup \delta_p,\ p \in (\partial\Omega)^+ \Rightarrow P(u_n) = \frac{\int_\Omega x e^{u_n}\, dV_{g_0}}{\int_\Omega e^{u_n}\, dV_{g_0}} \to p.$$

□

Because of the smoothness of Ω, there exists $\delta_0 > 0$ and a continuous retraction

$$\Pi : N((\partial\Omega)^+, \delta_0) \to (\partial\Omega)^+.$$

In view of Proposition 2.1.3, there exists $L_0 = L_{\delta_0}$ such that for any $L > L_0$, we can define the reverse map

$$\Psi = \Pi \circ P : I_\lambda^{-L} \to (\partial\Omega)^+. \tag{2.6}$$

Next proposition will be the key point for our min-max argument.

Proposition 2.1.4. *Fix any $L > L_0$ and take $\mu > \mu(L)$ where $\mu(L)$ is given in Lemma 2.1.2. Then the composition $\Psi \circ \Phi_\mu : (\partial\Omega)^+ \to (\partial\Omega)^+$ is homotopically equivalent to the identity map. Moreover, $\Phi_\mu((\partial\Omega)^+)$ is not contractible in I_λ^{-L}.*

Proof. Let us define the homotopy

$$\begin{aligned} H : [0,1] \times (\partial\Omega)^+ &\to (\partial\Omega)^+ \\ (t,p) &\mapsto H(t,p) = \Psi \circ \Phi_{\mu(t)}(p) = \Psi \circ \varphi_{\mu(t),p}, \end{aligned}$$

where $\mu(0) = \mu$ and $\mu(t)$ is an increasing continuous function with $\mu(t) \to \infty$ as $t \to 1$.

Let us show first that $H(t, \cdot) \to Id|_{(\partial\Omega)^+}$ as $t \to 1$. Take $p_n \to p \in (\partial\Omega)^+$, $\mu_n \to +\infty$; by the proof of Lemma 2.1.2,

$$\frac{e^{\varphi_{\mu_n,p_n}} \, dV_{g_0}}{\int_\Omega e^{\varphi_{\mu_n,p_n}} \, dV_{g_0}} \rightharpoonup \delta_p.$$

As a consequence,

$$P \circ \varphi_{\mu_n,p_n} \to p.$$

The second assertion of Proposition 2.1.4 follows easily from the former and the fact that $(\partial\Omega)^+$ is a non-contractible set.

□

Take any $v \in X$ fixed, and define:

$$\mathcal{C} = \left\{ \log\{t \exp\{\varphi_{\mu,p}\} + (1-t)\exp\{v\}\} : p \in (\partial\Omega)^+, t \in [0,1] \right\}.$$

It is easy to check that $\mathcal{C}$ is contained in X. As topology is concerned, $\mathcal{C}$ is a cone with base $\Phi_\mu((\partial\Omega)^+) \sim (\partial\Omega)^+$, so that $\partial\mathcal{C} = \Phi_\mu((\partial\Omega)^+)$. In other words, $\mathcal{C}$ is the union of a finite number of circular cones, each of them containing a connected component of $\Phi_\mu((\partial\Omega)^+)$ in its base, such that their vertices coincide at v.

We now define the min-max value of I_λ on suitable deformations of $\mathcal{C}$, namely

Definition 2.1.5.

$$\alpha_\lambda = \inf_{\eta \in \Gamma} \max_{u \in \mathcal{C}} I_\lambda(\eta(u)),$$

with

$$\Gamma = \left\{ \eta : \mathcal{C} \to X \ \textit{continuous}: \ \eta(u) = u \ \forall \ u \in \Phi_\mu((\partial\Omega)^+) \right\}.$$

Lemma 2.1.6. *$\alpha_\lambda \geq -L_0$, where L_0 is given in the definition of (2.6).*

Proof. Take $L > L_0$; for any deformation $\eta \in \Gamma$, $\partial\mathcal{C} = \Phi_\mu((\partial\Omega)^+)$ is contractible in $\eta(\mathcal{C})$. Moreover, Proposition 2.1.4 establishes that $\Phi_\mu((\partial\Omega)^+)$ is not contractible in I_λ^{-L}. Therefore, $\eta(\mathcal{C}) \nsubseteq I_\lambda^{-L}$, that is, there exists $\hat{u} \in \mathcal{C}$ with $I_\lambda(\eta(\hat{u})) \geq -L$. This concludes the proof.

□

Therefore, take $L > L_0$ and $\mu > \mu(L)$ where $\mu(L)$ is given in Lemma 2.1.2. Lemma 2.1.6 implies that $\alpha_\lambda > \max\{I_\lambda(u) : \; u \in \Phi_\mu((\partial\Omega)^+)\}$, which provides us with a min-max structure. Therefore, we can conclude the existence of a Palais-Smale sequence at level α_λ. However, as commented in subection 0.4.3, the boundedness of Palais-Smale sequences is still unknown for this kind of problems. The derivation of a solution follows an argument first used by Struwe, [109]. This argument has been used many times in this and other types of problems, see [51–53], so we will be sketchy. An essential ingredient is the following lemma

Lemma 2.1.7. *The function $\lambda \mapsto \frac{\alpha_\lambda}{\lambda}$ is monotonically decreasing.*

Proof. Just observe that, for $\lambda < \lambda'$,

$$\frac{I_\lambda(u)}{\lambda} - \frac{I_{\lambda'}(u)}{\lambda'} = \frac{1}{2}\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)\int_\Omega |\nabla u|^2 \, dV_{g_0} \geq 0.$$

Since α_λ is a min-max value for I_λ, the previous estimate implies the monotonicity of $\frac{\alpha_\lambda}{\lambda}$.

□

In this setting, we obtain the following:

Proposition 2.1.8. *There exists a set $E \subset (4\pi, 8\pi)$ such that*

1. *$(4\pi, 8\pi) \setminus E$ has zero Lebesgue measure, and*
2. *for any $\lambda \in E$ there exists a solution u_λ of (2.2) with $I_\lambda(u_\lambda) = \alpha_\lambda$.*

Proof. Define

$$E = \{\lambda \in (4\pi, 8\pi) : \quad \text{the map } \lambda \mapsto \alpha_\lambda \text{ is differentiable at } \lambda\}.$$

By Lemma 2.1.7, $(4\pi, 8\pi) \setminus E$ has zero measure. Fixed $\lambda \in E$, take $\varepsilon > 0$ sufficiently small. Observe that the above min-max scheme is valid for values of the parameter in the interval $(\lambda - \varepsilon, \lambda + \varepsilon)$. In this situation, it is well-known that there exists a sequence u_n satisfying:

1. u_n is bounded in $H^1(\Omega)$,

2. $I_\lambda(u_n) \to \alpha_\lambda$,

3. $I'_\lambda(u_n) \to 0$.

That is, for almost all values of λ we can assure the existence of a *bounded* (PS) sequence. This kind of argument was first devised in [109] (see also [51–53]).

Since u_n is bounded, up to a subsequence, $u_n \rightharpoonup u_\lambda$. Standard arguments show then that actually $u_n \to u_\lambda$ strongly and that u_λ is a critical point for I_λ, see Lemma 1.1.10.

□

So far, we have proved the existence of a solution for (2.2) for almost all values of $\lambda \in (4\pi, 8\pi)$. Now, our intention is to extend this existence result for any $\lambda \in (4\pi, 8\pi)$.

Proposition 2.1.9. *Let λ_n, $\lambda_0 \in (4\pi, 8\pi)$, $\lambda_n \to \lambda_0$, and u_n solutions of (2.2) for $\lambda = \lambda_n$. Assume also that $I_{\lambda_n}(u_n)$ is bounded from above. Then, up to a subsequence, $u_n \to u_0$, and u_0 is a solution of (2.2) for $\lambda = \lambda_0$.*

Proof. If u_n is bounded, up to a subsequence, $u_n \rightharpoonup u_0$. In Lemma 1.1.10 it is proved that the convergence is strong and that u_0 is the required solution.

Assume now that u_n is unbounded. By Proposition 2.1.1, there exists $p \in (\partial\Omega)^+$ with

$$\frac{e^{u_n}}{\int_\Omega e^{u_n}\, dV_{g_0}} \rightharpoonup \delta_p.$$

Clearly,

$$\frac{\int_\Omega K(x) e^{u_n}\, dV_{g_0}}{\int_\Omega e^{u_n}\, dV_{g_0}} \to K(p) > 0.$$

Take $\tau > 0$ so that $K(x) > 0$ in $B_p(\tau) \cap \Omega$. First, observe that

$$\frac{\int_{\Omega \setminus B_p(\tau)} K(x) e^{u_n}\, dV_{g_0}}{\int_\Omega K(x) e^{u_n}\, dV_{g_0}} \to 0. \tag{2.7}$$

Moreover, by the quantization result of Li-Shafrir in [78] (see also [107] for the Neumann boundary case), we obtain that

$$\lambda_n \frac{\int_{B_p(\tau)\cap\Omega} K(x) e^{u_n}\, dV_{g_0}}{\int_\Omega K(x) e^{u_n}\, dV_{g_0}} \to 4k\pi,\ \ k \in \mathbb{N}. \tag{2.8}$$

Equations (2.7), (2.8) imply that $\lambda_n \to 4k\pi$ with $k \in \mathbb{N}$, a contradiction.

□

We can now finish the proof of Theorem 2.0.1. Take any $\lambda_0 \in (4\pi, 8\pi)$ and $\lambda_n \in E$, $\lambda_n \to \lambda$. Let u_n denote the solutions of (2.2) for $\lambda = \lambda_n$ given by Proposition 2.1.8. Recall that $I_{\lambda_n}(u_n) = \alpha_{\lambda_n}$, which is bounded (for instance, by Lemma 2.1.7). Proposition 2.1.9 allows us to conclude.

2.2 Final remarks and open problems

Remark 2.2.1. *The arguments of the proofs works perfectly well if Ω is a subdomain of any compact surface Λ, and g_0 is any Riemannian metric on Λ. In this general case, though, equation (2.2) loses its geometrical interpretation.*

Observe that we can assume that Λ is isometrically embedded in $\mathbb{R}^k$; therefore, the barycenter map (2.5) would take values in $\mathbb{R}^k$, and $N((\partial\Omega)^+, \delta)$ would denote the corresponding neighborhood in $\mathbb{R}^k$. Those are the only modifications needed in order to adapt the above arguments to this general setting.

Remark 2.2.2. *A natural extension of the problem studied in this chapter is its singular version, i.e. to add a linear combination of Dirac delta measures located at points of Ω in (0.41). More precisely, the Neumann boundary problem*

$$\begin{cases} -\Delta_{g_0} u + 2 = 2K(x)e^u - 4\pi \sum_{i=1}^m \alpha_i \delta_{p_i} & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

where $p_1, \cdots, p_m \in \Omega$ and $\alpha_i > -1$ for any $i = 1, \cdots, m$. The previous equation allows us to seek a metric of Ω with $p_1, \dots, p_m \in \Omega$ conical points with $\alpha_i > -1$ order such that K the is Gauss curvature of Ω.

Integrating (2.9)

$$\lambda = 2|\Omega| + 4\pi \sum_{i=1}^m \alpha_i = 2\int_\Omega K(x)e^u \, dV_{g_0}.$$

If $\alpha_i > 0$ for any $i = 1, \dots, m$, $\lambda \in (4\pi, 8\pi)$ and (A1), (A2), our approach prove that this problem admits solution. The same ideas of the proof of Theorem 2.0.1 *work also in this case.*

Open Problem 2.2.3. *In this chapter we have established a compactness property assuming that solutions have bounded level energy. This fact is shown a posteriori by the min-max scheme. A possible problem is to deduce a complete compactness result without the energy level condition, in spirit of Theorem* (0.4.2)*. This result would enable us to establish the generic a multiplicity of* (2.2)*, employing the same topological description of the low sublevels.*

Following the proof of our compactness result, one can show that solution sequences of (2.2) *(or* (2.9)*) remain* L^∞*-bounded* $\Omega \setminus (\partial\Omega)^\delta$ *for every* $\delta > 0$ *if* $\lambda \in (4\pi, 8\pi)$ *(*$\lambda \in (4\pi, 8\pi) \setminus \Lambda_m$*). However, our argument does not apply at* $\partial\Omega$*, as in the upcoming chapter, see Section 3.3.*

Open Problem 2.2.4. *Another related problem is to try to weaken the hypothesis on the sign of* K *on the boundary, namely (A2). If one allows* K *to change sign on* $\partial\Omega$*, one can adapt the proof of Proposition 3.2.1 with minor modifications, in order to project continuously* I_λ^{-L} *into* $(\partial\Omega)^+$*. As* K *changes sign on* $\partial\Omega$*, then* $(\partial\Omega)^+$ *is no longer a union of connected sets homeomorphic to the unit circle. As we have seen, if we choose some* K *such that* $(\partial\Omega)^+$ *is not contractible, min-max structure guarantees the existence of a critical point for almost every value of the parameter* λ*. Again, a compactness result would be needed to conclude the existence of solution.*

Another question to be analyzed is the case in which $(\partial\Omega)^+$ *contractible, where the min-max argument does not work. Under symmetry assumptions on* K*, one could try to prove a non-existence result in the spirit of Theorem 0.4.6.*

Chapter 3

The singular mean field problem on compact surfaces

We dedicate this chapter to study the existence, compactness and multiplicity of solutions for the singular mean field problem

$$-\Delta_g u = \lambda \left(\frac{\tilde{K} e^u}{\int_\Sigma \tilde{K} e^u dV_g} - \frac{1}{|\Sigma|} \right) \quad \text{in} \quad \Sigma, \tag{3.1}$$

where Σ is an arbitrary compact surface without boundary. Recall that $\tilde{K} = K e^{-h_m}$ where h_m is given by the expression (0.8) and K is a sign changing potential. The positive integer m corresponds to the number of conical singularities located at different points $p_i \in \Sigma$ with order α_i; whereas $\ell \leq m$ is the number of singularities located at the region where K is strictly positive. Previously, the nodal sets have been denoted by

$$\Sigma^+ = \{x \in \Sigma : \, K(x) > 0\}, \quad \Sigma^- = \{x \in \Sigma : \, K(x) < 0\}, \quad \Gamma = \{x \in \Sigma : \, K(x) = 0\}.$$

Let us recall the hypotheses on K and the singular points p_i's

(H1) K is a sign changing $C^{2,\alpha}$ function with $\nabla K(x) \neq 0$ for any $x \in \Sigma$ with $K(x) = 0$;

(H2) $p_j \notin \Gamma$ for all $j \in \{1, \ldots, m\}$;

(H3) $N^+ > k$ or Σ^+ has a connected component which is not simply connected;

(H4) $\Theta_\lambda \neq \emptyset$;

where the set Θ_λ is defined as

$$\Theta_\lambda = \{p_j \in \Sigma^+ : \ \lambda < 8\pi(1 + \alpha_j)\}. \tag{3.2}$$

We will follow the strategy formulated in Section 0.4.3, which has been already used in the previous chapter. This approach is mainly based on the variational formulation of the equation (3.1). Actually, solutions of the singular mean field problem can be found as critical points of the energy functional

$$J_\lambda(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 dV_g + \frac{\lambda}{|\Sigma|}\int_\Sigma u\, dV_g - \lambda \log \int_\Sigma \tilde{K} e^u dV_g, \tag{3.3}$$

defined in the domain

$$X = \left\{ u \in H^1(\Sigma) : \ \int_\Sigma \tilde{K} e^u \, dV_g > 0 \right\}. \tag{3.4}$$

Without loss of generality, we can restrict its domain to functions with 0 mean. In other words, we can consider J_λ defined in $\bar{X}$,

$$\bar{X} = \left\{ u \in X : \ \int_\Sigma u \, dV_g = 0 \right\}. \tag{3.5}$$

This chapter is organized as follows. In Section 3.1, we show a compactness result for singular Liouville type equations. Section 3.2 is devoted to describe the topology of the energy sublevels of J_λ. From Section 3.1 and Section 3.2 we conclude the existence results.In Section 3.3 we state the multiplicity results which have been announced in the final part of the Introduction. These results cover the setting of the two existence theorems. Finally, we focus on constructing a class of functions such that the problem does not admit any solution. This argument employs the so-called moving sphere method on entire solutions, see Section 3.4.

These results have been presented in two publications [47], [48].

3.1 Compactness of solutions

For sign changing functions K the first related compactness result is [31]. That paper is concerned with the scalar curvature prescription problem, a higher dimension analogue of our problem which has also attracted much attention in the literature. Later, an evolution of this technique has been given in [33]. The general idea is to first derive uniform integral estimates, which allow one to obtain a priori estimates in the the region $\{x \in \Sigma : \; K(x) < -\delta\}$, for $\delta > 0$ small. Then the moving plane technique is used to compare the values of u on both sides of the nodal curve $\Gamma = \{x \in \Sigma : \; K(x) = 0\}$. This, together with the integral estimate, implies boundedness in a neighborhood of Γ. Finally, one reiles on [8, 9, 78] for the region $\{x \in \Sigma : \; K(x) > \delta\}$.

The approach of [31] has been partially adapted to problem (3.1) in [32]. However, this result uses the stereographic projection to pass to a global problem in the plane and is hence restricted to $\Sigma = \mathbb{S}^2$. Moreover, the derivation of the integral estimate [32, Lemma 2.2] is not completely clear. One of the goals of this section is to settle the question of compactness: we show compactness for (3.1) in any compact surface under assumption (H1).

Our approach follows the ideas of [33]. The main difficulty with respect to [33] comes from the fact that u_n is not positive nor uniformly bounded from below, a priori. This is a problem for the integral estimate in [32, 33], and also for the use of the moving plane method near the nodal curve. In our proofs we first estimate the negative part of u_n by using Kato inequality. This is the key for the proof of the integral estimate and is also essential to perform the comparison argument by the moving plane method.

Let us now recall Theorem 0.4.2. Assume that u_n is a sequence of solutions of

$$-\Delta_g u_n = \tilde{K}_n e^{u_n} - f_n \quad \text{in} \quad \Sigma, \tag{3.6}$$

where $f_n \to f$ in $C^{0,\alpha}$ sense and $\tilde{K}_n = K_n e^{-h_m}$. Recall that the function h_m has already defined as

$$h_m(x) = 4\pi \sum_{j=1}^{m} \alpha_j G(x, p_j),$$

where $G(x, p_j)$ are the Green functions, solutions to (0.7), and $\alpha_j > -1$ for any $j = 1, \dots, m$. In addition, it is supposed that $K_n \to K$ in $C^{2,\alpha}$ sense and the function K satisfies (H1), (H2). Then, up to a subsequence, one of the following alternatives holds

1. either u_n is uniformly bounded in $L^\infty(\Sigma)$;
2. or u_n diverges to $-\infty$ uniformly;
3. or there exists a finite set $S = \{q_1, \dots, q_r\} \subset \Sigma^+$ of blow–up points.

   In such case, $u_n \to -\infty$ in compact sets of $\Sigma \setminus S$ and $\tilde{K}_n e^{u_n} \rightharpoonup \sum_{i=1}^r \beta(q_i)\delta_{q_i}$ in the weak sense of measures where $\beta(q_i) = 8\pi$ if $q_i \notin \{p_1, \dots, p_\ell\}$ and $\beta(q_i) = 8(1+\alpha_j)\pi$ if $q_i = p_j$ for some $1 \le j \le \ell$

   Therefore, $\lim_{n\to+\infty} \int_\Sigma \tilde{K}_n e^{u_n} \in \Lambda_\ell$, where

$$\Lambda_\ell = \left\{ 8\pi r + \sum_{j=1}^{\ell} 8\pi(1+\alpha_j)n_j : \ r \in \mathbb{N} \cup \{0\}, n_j \in \{0,1\} \right\} \setminus \{0\}. \tag{3.7}$$

We begin by establishing the following propositions.

Proposition 3.1.1. *Given $\delta > 0$, there exists $C > 0$ such that $u_n(x) \le C$ for all $x \in \Sigma^- \setminus \Gamma^\delta$, $n \in \mathbb{N}$.*

Proposition 3.1.2. *There exist $\varepsilon, C > 0$, such that $u_n(x) \le C$ for all $n \in \mathbb{N}$ and $x \in \Gamma^\varepsilon$.*

The proof of Theorem 0.4.2 will be finally accomplished by studying the possible blow-up of the sequence u_n in $\Sigma^+ \setminus \Gamma^\varepsilon$.

One of the difficulties in our study is that we do not know a priori whether the term

$$\int_\Sigma |\tilde{K}_n| e^{u_n} dV_g \tag{3.8}$$

is bounded or not. By standard regularity results, this would give a priori $W^{1,p}$ estimates ($p \in (1,2)$) on u_n. Instead, if we integrate (3.6) we only obtain that

$\int_\Sigma \tilde{K}_n e^{u_n}$ is bounded. Observe that if $K_n > 0$, then (3.8) holds directly by integrating (3.6).

Our first lemma shows that such kind of estimate is indeed possible for $u_n^- = \min\{u_n, 0\}$. This fact will be useful to prove both Propositions 3.1.1 and 3.1.2.

Lemma 3.1.3. *Under the conditions of Theorem 0.4.2, define* $v_n = u_n^- - \fint_\Sigma u_n^-$. *Then there exists* $C > 0$ *such that*

a) $\|v_n\|_{W^{1,p}} \leq C$ *for any* $p \in (1, 2)$;

b) $v_n(x) \geq -C$ *for any* $x \in \Sigma$.

Proof. We apply the well-known Kato inequality to the operator Δ_g (see for instance [108])

$$-\Delta_g u_n^- \geq \left(\tilde{K}_n e^{u_n} - f_n\right) \chi_{\{u_n \leq 0\}} \geq -C g(x), \tag{3.9}$$

where

$$g(x) = 1 + \sum_{\substack{j \geq \ell+1 \\ \alpha_j < 0}} d(x, p_j)^{2\alpha_j}.$$

Observe that $g \in L^q(\Sigma)$ for $q \in [1, 1+\delta)$ if $\delta > 0$ is sufficiently small.

Since the Radon measures $\mu_n = -\Delta u_n^- \geq -C g(x)$ are given as a divergence (in the sense of distributions), then $\int_\Sigma d\,\mu_n = 0$. From that we conclude that $\int_\Sigma d\,|\mu_n|$ is bounded. By elliptic regularity estimates, v_n is bounded in $W^{1,p}(\Sigma)$ for any $p \in (1, 2)$.

For the second part we use the Green representation for v_n. Let $G(x, y)$ be the Green function for the operator Δ_g in Σ; observe that $G(x, y) = -\frac{1}{2\pi}\log(rd(x, y)) + \tilde{H}(x, y)$, where $\tilde{H} : \Sigma \times \Sigma \to \mathbb{R}$ is a bounded function. Here we have chosen $r \in (0, diam(\Sigma)^{-1})$. Then,

$$\begin{aligned} v_n(x) = \int_\Sigma G(x, y) d\,\mu_n(y) &= -\frac{1}{2\pi}\int_\Sigma \log(rd(x, y)) d\,\mu_n^+(y) \\ &-\frac{1}{2\pi}\int_\Sigma \log(rd(x, y)) d\,\mu_n^-(y) + \int_\Sigma \tilde{H}(x, y) d\,\mu_n(y). \end{aligned}$$

By the choice of $r > 0$,

$$-\frac{1}{2\pi}\int_\Sigma \log(rd(x,y))d\,\mu_n^+(y) \geq 0.$$

Moreover, by (3.9),

$$-\frac{1}{2\pi}\int_\Sigma \log(rd(x,y))d\,\mu_n^-(y) \geq -C\frac{1}{2\pi}\int_\Sigma \log(rd(x,y))g(y)\,dy \geq -C,$$

and finally

$$\left|\int_\Sigma \tilde{H}(x,y)d\,\mu_n(y)\right| \leq \|\tilde{H}\|_{L^\infty}\int_\Sigma d|\mu_n| \leq C.$$

□

As a first consequence of Lemma 3.1.3, we present an integral estimate in domains entirely contained in the positive or negative region. This result is an extension of the Chen-Li integral estimate for positive solutions, see [33]. In our case u_n may change sign, but we can perform the estimate thanks to Lemma 3.1.3.

Lemma 3.1.4. *Under the conditions of Theorem 0.4.2, for every open subdomain Σ_0 completely contained in Σ^+ or Σ^-, there exists $C > 0$ so that*

$$\left|\int_{\Sigma_0} \tilde{K}_n e^{u_n} dV_g\right| \leq C.$$

Proof. Take Σ_1 a smooth domain such that $\overline{\Sigma_0} \subset \Sigma_1 \subset \overline{\Sigma_1} \subset \Sigma^\pm$. Let φ be the first eigenfunction of the Laplace operator in Σ_1, that is,

$$\begin{cases} -\Delta\varphi = \lambda_1\varphi & \text{in } \Sigma_1, \\ \varphi > 0 & \text{in } \Sigma_1, \\ \varphi = 0 & \text{on } \partial\Sigma_1. \end{cases}$$

Next, we multiply (3.6) by φ^2, and integrate by parts over Σ_1 to obtain

$$\int_{\Sigma_1} \tilde{K}_n \varphi^2 e^{u_n} = -\int_{\Sigma_1} u_n \Delta(\varphi^2) + O(1). \tag{3.10}$$

Let us denote $f = \Delta(\varphi^2) = 2(|\nabla\varphi|^2 - \lambda_1\varphi^2)$. Observe that $\int_{\Sigma_1} f = 0$. Then

$$\int_{\Sigma_1} u_n^- f = \int_{\Sigma_1} \left(u_n^- - \fint_{\Sigma} u_n^- \right) f,$$

so that, by Lemma 3.1.3, a),

$$\left| \int_{\Sigma_1} u_n^- f \right| \leq \left\| u_n^- - \fint_{\Sigma} u_n^- \right\|_{L^1(\Sigma_1)} \|f\|_{L^\infty(\Sigma_1)} \leq C. \tag{3.11}$$

On the other hand, for any $\gamma > 0$,

$$\int_{\Sigma_1} u_n^+ |f| \leq C \int_{\Sigma_1} u_n^+ = C \int_{\Sigma_1} u_n^+ \frac{|\varphi^2 \tilde{K}_n|^\gamma}{|\varphi^2 \tilde{K}_n|^\gamma}.$$

By Young inequality we obtain

$$\int_{\Sigma_1} u_n^+ |f| \leq \varepsilon \int_{\Sigma_1} |u_n^+|^{\frac{1}{\gamma}} \varphi^2 |\tilde{K}_n| + C_\varepsilon \int_{\Sigma_1} \frac{1}{|\varphi^2 \tilde{K}_n|^{\frac{\gamma}{1-\gamma}}}. \tag{3.12}$$

We can take $\gamma > 0$ sufficiently small so that the second integral term in the right hand side is finite (recall that, by Hopf principle, $\varphi \sim d(x, \partial\Sigma_1)$ near the boundary). Then, by (3.10), (3.11) and (3.12)

$$\int_{\Sigma_1} |\tilde{K}_n| \varphi^2 e^{u_n} \leq C + \varepsilon \int_{\Sigma_1} |u_n^+|^{\frac{1}{\gamma}} \varphi^2 |\tilde{K}_n|.$$

We now use the inequality $(t^+)^{\frac{1}{\gamma}} \leq C + e^t$ to conclude that

$$\int_{\Sigma_1} |\tilde{K}_n| \varphi^2 e^{u_n} \leq C,$$

finishing the proof.

□

In order to prove Proposition 3.1.1, we will need the following result, which is based on a mean value inequality for subharmonic functions.

Lemma 3.1.5. *Let w be a function defined in $\Sigma_0 \subset \Sigma$, $x_0 \in \Sigma_0$, and assume that $-\Delta_g w(x) \leq -A$ for all $x \in \Sigma_0$, for some positive value $A > 0$. Take $R > 0$ such that*

$$R < \min \left\{ \frac{1}{5} d(x_0, \partial\Sigma_0),\ \frac{1}{2} diam(\Sigma_0) \right\}.$$

Then there exists $C > 0$ depending only on Σ_0 and A such that

$$\sup_{x \in B_{x_0}(R/4)} w(x) \leq C \left(1 + \fint_{B_x(R)} w \right).$$

Proof. Define v as the solution of the problem

$$\begin{cases} -\Delta_g v = A, & \text{in } \Sigma_0, \\ v = 0, & \text{on } \partial\Sigma_0. \end{cases}$$

Clearly v is smooth and $w+v$ is a subharmonic function. We now apply the mean value inequality for subharmonic functions (see [80, Theorem 2.1] for its version on manifolds) to conclude.

□

Proof of Proposition 3.1.1. Take $\Sigma_0 \subset \overline{\Sigma_0} \subset \Sigma^-$, $x \in \Sigma_0$ and fix $r > 0$ sufficiently small. We apply Lemma 3.1.5 to $w = u^+$ and we obtain

$$\sup_{B_x(r)} u_n^+(x) \leq C + C \int_{B_x(4r)} u_n^+ = C + C \int_{B_x(4r)} \frac{u_n^+}{p} \frac{-\tilde{K}_n^{1/p}(x)}{-\tilde{K}_n^{1/p}(x)}$$

$$\leq C + C \int_{B_x(4r)} e^{\frac{u_n}{p}} \leq C + C \left(\int_{B_x(4r)} -\tilde{K}_n(x) e^{u_n} \right)^{1/p} \left(\int_{B_x(4r)} \frac{1}{-\tilde{K}_n^{\frac{1}{p-1}}(x)} \right)^{\frac{p-1}{p}}.$$

It suffices to choose a large enough p and use Lemma 3.1.4 to conclude that $\sup\limits_{B_x(R)} u_n^+(x) < C$.

□

We now turn our attention to Proposition 3.1.2. The proof follows the argument of [33], with the main difference that our solutions u_n are not positive. This difficulty can be bypassed thanks to the following lemma, whose proof is based on Lemma 3.1.3.

Lemma 3.1.6. *Under the hypotheses of Theorem 0.4.2, and given $\delta > 0$, there exists $C > 0$ such that*

$$u_n(x_0) - u_n(x_1) \leq C \tag{3.13}$$

for every $n \in \mathbb{N}$, $x_0 \in \Sigma^- \setminus \Gamma^\delta$, $x_1 \in \Sigma$. *Moreover, for any* $r_0 > 0$, *there exists* $C > 0$ *such that*

$$|\nabla u_n(x)| \leq C \qquad \forall\, x \in \Sigma^- \setminus (\Gamma^\delta \cup \bigcup_{i=\ell+1}^{m} B_{p_i}(r_0)). \tag{3.14}$$

Proof. By Lemma 3.1.3, b), we have that

$$u_n(x_1) - \fint_\Sigma u_n^- \geq u_n^-(x_1) - \fint_\Sigma u_n^- \geq C. \tag{3.15}$$

Taking into account Lemma 3.1.5, we have that for small $r > 0$,

$$u_n(x_0) - \fint_\Sigma u_n^- \leq C \left(1 + \fint_{B_{x_0}(r)} \left(u_n(x) - \fint_\Sigma u_n^-\right)\right)$$

Moreover, by Proposition 3.1.1, $u_n(x) \leq u_n^-(x) + C$ for all $x \in B_{x_0}(r)$. Making use of Lemma 3.1.3, a), we conclude

$$u_n(x_0) - \fint_\Sigma u_n^- \leq C \left(1 + \fint_{B_{x_0}(r)} \left|u_n^-(x) - \fint_\Sigma u_n^-\right|\right) \leq C \tag{3.16}$$

This, together with (3.15), allows us to show (3.13).

We not turn our attention to the proof of (3.14). Given $r_0 > 0$, take any $p > 2$ and fix x such that $B_x(r) \subset \Sigma^- \setminus (\Gamma^\delta \cup \bigcup_{i=\ell+1}^m B_{p_i}(r_0))$. Recall the inequality (see [59, Theorem 9.11])

$$\left\|u_n - \fint_\Sigma u_n^-\right\|_{W^{2,p}(B_x(\frac{r}{2}))} \leq C \left(||\tilde{K}_n e^{u_n} - f_n||_{L^p(B_x(r))} + \left\|u_n^- - \fint_\Sigma u_n^-\right\|_{L^p(B_x(r))}\right).$$

Combining (3.16) and Lemma 3.1.3, b), $u_n - \fint_\Sigma u_n^- \in L^\infty(B_x(r))$, whereas Proposition 3.1.1 implies that $\tilde{K}_n e^{u_n} - f_n$ is uniformly bounded. Therefore $u_n - \fint_\Sigma u_n^- \in W^{2,p}(B_x(\frac{r}{2}))$. In particular (3.14) holds.

□

Proof of Proposition 3.1.2. The proof is of local nature, so that we can restrict ourselves (by using isothermal coordinates) to planar domains.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain and u_n be a solution sequence for the problem

$$-\Delta u_n = W_n e^{u_n} \qquad \text{in } \Omega, \tag{3.17}$$

with $W_n \to W$ in $C^{2,\alpha}(\overline{\Omega})$. Assumption (H1) is translated to W in the form

$$W \text{ is a } C^{2,\alpha}(\Omega) \text{ function, sign changing and } \nabla W(x) \neq 0 \text{ in } \Gamma.$$

Our proof is based on the method of moving planes, which allows us to compare the values of u_n close to Γ. For the sake of clarity, we drop the subindex n in the notation of the rest of this proof.

By the assumptions on W for small $\delta > 0$, there exists $\beta > 0$ s.t.

$$|\nabla W(x)| \geq \beta \text{ for any } x \text{ with } |W(x)| \leq \delta. \tag{3.18}$$

First of all, we can transform the region through a Kelvin transform, a translation and a rotation. We define the new system of coordinates as $x = (x_1, x_2)$ and $x_1 = \gamma(x_2)$ which corresponds to the curve Γ. Let Ω_ε be the region enclosed by the curve $\partial_l \Omega_\varepsilon = \{x \mid x_1 - \gamma(x_2) = \varepsilon\}$ and the half-plane $\{x_1 \geq -2\varepsilon\}$ along with its reflection image about the line $x_1 = -2\varepsilon$. It is possible to choose the transformation such that, for some $\varepsilon > 0$ small, the following hold (see figure):

(i) x_0 becomes the origin;

(ii) Ω_ε is located to the left of the line $x_1 = \varepsilon$ and it is tangent to it;

(iii) $\partial_l \Omega_\varepsilon$ is uniformly convex;

(iv) $\frac{\partial W}{\partial x_1} \leq -\frac{1}{2}\beta$, for every $x \in \Omega_\varepsilon$;

(v) $\overline{\Omega_\varepsilon} \cap \{p_{\ell+1}, \ldots, p_m\} = \emptyset$.

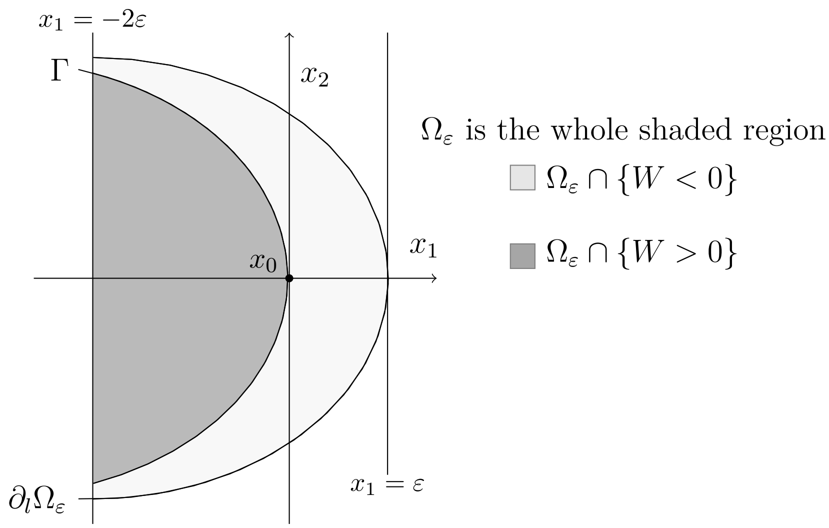

Figure 5

Define $m = \min_{x \in \partial_l \Omega_\varepsilon} u(x)$, $M = \max_{x \in \partial_l \Omega_\varepsilon} u(x)$ and $\tilde{u}$ as a C^2 extension of u from $\partial_l \Omega_\varepsilon$ to the whole $\partial \Omega_\varepsilon$ such that $m \leq \tilde{u} \leq M$ and, by (3.14), $|\nabla \tilde{u}| \leq C$. Let w be the harmonic function

$$\begin{cases} \Delta w = 0, & \text{in } \Omega_\varepsilon, \\ w = \tilde{u}, & \text{on } \partial\Omega_\varepsilon, \\ m \leq w \leq M & \text{in } \Omega_\varepsilon. \end{cases} \tag{3.19}$$

Due to (3.13), the oscillation of u on $\partial_l \Omega_\varepsilon$ is bounded, i.e.,

$$M - m = \max_{\partial_l \Omega_\varepsilon} u - \min_{\partial_l \Omega_\varepsilon} u \leq C. \tag{3.20}$$

Consequently, the oscillation of w is also bounded in Ω_ε. We also define a new auxiliary function v as

$$v(x) = u(x) - w(x) + C_0(\varepsilon + \gamma(x_2) - x_1), \tag{3.21}$$

for some $C_0 > 0$ to be determined. It is clear that the function v verifies

$$\Delta v + f(x, v(x)) - C_0 \gamma''(x_2) = 0, \qquad \text{in } \Omega_\varepsilon, \tag{3.22}$$

with

$$f(x, v(x)) = W(x) e^{v(x)+w(x)-C_0(\varepsilon+\gamma(x_2)-x_1)}.$$

We claim that for a suitable C_0

$$v(x) \geq 0 \text{ in } \Omega_\varepsilon \quad \text{ and } \quad v(x) = 0 \text{ on } \partial_l \Omega_\varepsilon. \tag{3.23}$$

The boundary condition is direct. In order to prove the first part, we distinguish two cases:

- **Case 1**: $\frac{\varepsilon}{2} < x_1 - \gamma(x_2) \leq \varepsilon$

  Taking into account (v), by (3.14) we have that

  $$\left| \frac{\partial u}{\partial x_1} \right| \leq C \text{ and } \left| \frac{\partial w}{\partial x_1} \right| \leq C.$$

  Consequently,

  $$\frac{\partial v}{\partial x_1} = \frac{\partial u}{\partial x_1} - \frac{\partial w}{\partial x_1} - C_0 \leq C - C_0. \tag{3.24}$$

  It suffices to choose C_0 sufficiently large to obtain that $\frac{\partial v}{\partial x_1}$ is negative. Since $v = 0$ on $\partial_l \Omega_\varepsilon$, it is clear that (3.23) holds.

- **Case 2**: $x_1 - \gamma(x_2) \leq \frac{\varepsilon}{2}$ and $x_1 \geq -2\varepsilon$

  By (3.13), we have that

  $$v(x) = u(x) - w(x) + C_0(\varepsilon + \gamma(x_2) - x_1) \geq \min_{\Omega_\varepsilon} u - \max_{\partial_l \Omega_\varepsilon} u + C_0 \frac{\varepsilon}{2} \geq -C + C_0 \frac{\varepsilon}{2}.$$

  So, choosing C_0 sufficiently large, (3.23) holds.

Now we are ready to apply the method of moving planes to v in the x_1 direction. Thus, we start from $x_1 = \varepsilon$ and move the line perpendicular to x_1-axis towards the left. Namely, let $T_\lambda = \{x \in \mathbb{R}^2 : \ x_1 \geq \lambda\}$ the half-plane, $M_\lambda = \{x \in \mathbb{R}^2 : \ x_1 = \lambda\}$ its boundary and $x_\lambda = (2\lambda - x_1, x_2)$ the reflection point of x with respect to the line M_λ.

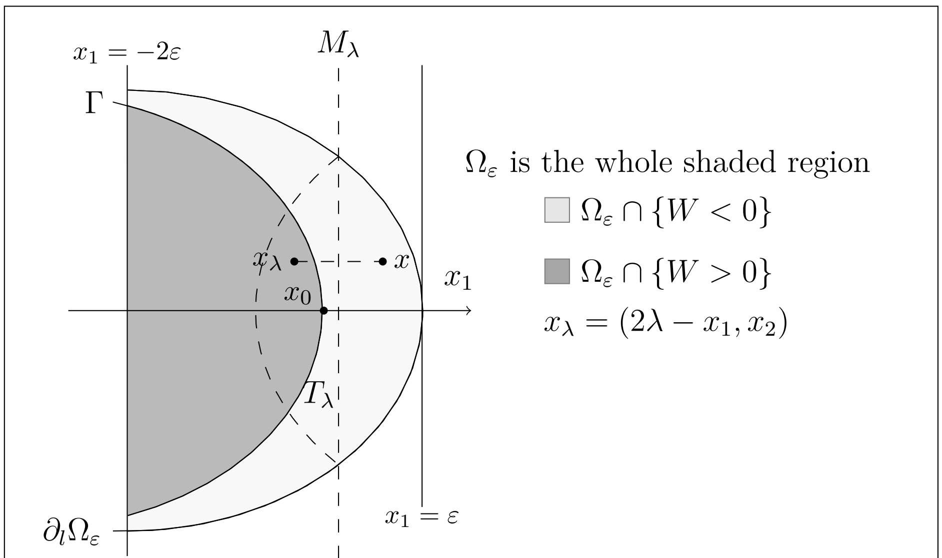

Figure 6

Our goal is to prove that

$$v(x_\lambda) \geq v(x), \tag{3.25}$$

for every $x \in T_\lambda \cap \Omega_\varepsilon$ for $\lambda \in \left[\frac{\varepsilon}{2} - \varepsilon_1, \varepsilon\right]$, with some $\varepsilon_1 \in (0, \varepsilon)$ to be determined. By (3.23) and (3.24), (3.25) holds for $\lambda \in \left(3\frac{\varepsilon}{4}, \varepsilon\right]$.

By a standard argument (see [58]), we can affirm that the moving planes argument can be carried on provided that

$$f(x, v) \leq f(x_\lambda, v) \quad \text{for every } x \in \Omega_\varepsilon \quad \text{with} \quad \varepsilon > \lambda > \frac{\varepsilon}{2} - \varepsilon_1. \tag{3.26}$$

It is easy to check that (3.26) holds if

$$\frac{\partial f(x, v(x))}{\partial x_1} = e^u \left(\frac{\partial W}{\partial x_1} + W \left(\frac{\partial w}{\partial x_1} + M'\tau \right) \right) \leq 0, \qquad \text{if } x \in \Omega_\varepsilon,\ x_1 > -\varepsilon_1. \quad (3.27)$$

In other words, if f is monotone decreasing along the direction $(1,0)$, near x_0.

If $W(x) \leq 0$, it is enough to choose $C_0 > -\frac{\partial w}{\partial x_1}$ to verify

$$\frac{\partial W}{\partial x_1} + W \left(\frac{\partial w}{\partial x_1} + C_0 \right) \leq W \left(\frac{\partial w}{\partial x_1} + C_0 \right) \leq 0.$$

In the case that $W(x) > 0$ by the assumptions on W, for every $\varepsilon_1 > 0$ there exists a neighborhood V_{ε_1} of Γ such that $W(x) \leq \varepsilon_1$.

Since $\frac{\partial w}{\partial x_1} + C_0$ is bounded from above, then

$$\frac{\partial W}{\partial x_1} + W \left(\frac{\partial w}{\partial x_1} + C_0 \right) \leq -\frac{\beta}{2} + \varepsilon_1 \left(\frac{\partial w}{\partial x_1} + C_0 \right) \leq -\frac{\beta}{2} + \varepsilon_1 C,$$

therefore we can take ε_1 small enough to obtain the desired conclusion. We choose $\varepsilon_1 < \varepsilon$.

In this way, the method of moving planes works up to $\lambda = \frac{\varepsilon}{2} - \varepsilon_1$. Therefore, (3.25) implies that $v(x)$ is monotone decreasing in the $(1,0)-$direction. In fact, we can repeat the previous argument rotating the x_1-axis by a small angle. Thus, there exists a fixed cone Δ_0 such that for any $x \in B_{x_0}(\varepsilon_1/4)$ we have

$$v(y) \geq v(x),\ \forall y \in \Delta_x,$$

where Δ_x denotes a translation of the cone Δ_0 with x at its vertex. By (3.21), we can transform the previous inequality into

$$u(y) + C(\varepsilon_1) \geq u(x) \quad \forall\, y \in \Delta_{x_0}. \quad (3.28)$$

The proof can be concluded combining (3.28) and the integral estimate given in Lemma 3.1.4. □

Proof of Theorem 0.4.2. Take $\varepsilon > \delta > 0$ and the open set $\Sigma_1 = \Sigma^+ \setminus \overline{\Gamma^\delta}$, where ε is given by Proposition 3.1.2. By Propositions 3.1.1 and 3.1.2, u_n is uniformly bounded from above in $\Sigma \setminus \Sigma_1$. Moreover, by Lemma 3.1.4, $\int_{\Sigma_1} \tilde{K}_n e^{u_n}$ is bounded. By the compactness criterion of [9], stated in Section 0.3 there are two possibilities:

Case 1: u_n is bounded from above in Σ. As a consequence, $\tilde{K}_n e^{u_n}$ is bounded in $L^\infty(\Sigma)$. Elliptic regularity estimates imply that $u_n - \fint_\Sigma u_n$ is bounded in $W^{1,p}(\Sigma)$ for all $p > 1$. If $\fint_\Sigma u_n$ is bounded, we obtain 1); if, on the contrary, $\fint_\Sigma u_n$ diverges negatively, we obtain 2).

Case 2: The sequence u_n is not bounded from above. Applying the results of [9] concerning the blow–up analysis for (3.6) in Σ_1, we can assume that there exists a finite blow-up set $S = \{q_1, \dots, q_r\} \subset \Sigma_1$. Moreover, by enlarging δ if necessary, we can assume that $u_n \to -\infty$ uniformly in $\partial\Sigma_1$, and

$$\tilde{K}_n e^{u_n} \rightharpoonup \sum_{i=1}^{r} \beta(q_i)\delta_{q_i} \quad \text{in the sense of weak convergence of measures in } \overline{\Sigma_1},$$

with $\beta(q_i) \geq 8\pi$.

Now, let us define v the solution of the problem

$$\begin{cases} -\Delta v = C_1 & \text{in } \Sigma \setminus \Sigma_1, \\ v = 0 & \text{on } \partial\Sigma_1. \end{cases}$$

where C_1 is an upper bound of the term $\tilde{K}_n e^{u_n}$ in $\Sigma \setminus \Sigma_1$. Standard regularity results imply that $v \in L^\infty(\Sigma \setminus \Sigma_1)$. By the maximum principle, for any $C > 0$ there exists $n_0 \in \mathbb{N}$ such that $u_n \leq v - C$ in $\Sigma \setminus \Sigma_1$ for $n \geq n_0$. This implies that $u_n \to -\infty$ uniformly in $\Sigma \setminus \Sigma_1$; in particular,

$$\tilde{K}_n e^{u_n} \rightharpoonup \sum_{i=1}^{r} \beta(q_i)\delta_{q_i} \quad \text{in the sense of weak convergence of measures in } \Sigma. \tag{3.29}$$

It is worth to point out that, at this point of the proof, we cannot apply yet the quantization part of the concentration-compactness Theorem of [9] if we do not check the mean oscillation condition on $\partial\Sigma_1$.

By (3.29), employing the Green's representation formula for u_n, we have that

$$u_n - \overline{u}_n \to \sum_{i=1}^{r} \beta(q_i) G(x, q_i) + h_m,$$

uniformly on compact sets of $\Sigma \setminus S$, where h_m is defined in (0.8). In this way, the sequence $u_n - \overline{u}_n$ admits uniformly bounded mean oscillation on any compact set of $\Sigma \setminus (S \cup \{p_1, \ldots, p_\ell\})$. Indeed, there exists a constant $C > 0$ such that

$$\max_{\partial \Sigma_1} u_n - \min_{\partial \Sigma_1} u_n < C.$$

By virtue of this condition, we can apply the quantization result of [9] to conclude that $\beta(q_i) = 8\pi$ if $q_i \notin \{p_1, \ldots, p_\ell\}$ and $\beta(q_i) = 8\pi(1 + \alpha_i)$ if $q_i = p_i \in \{p_1, \ldots, p_\ell\}$. Moreover, up to subsequence, we obtain that

$$\lim_{k \to +\infty} \int_{\Sigma_1} \tilde{K}_n e^{u_n} \in \Lambda_\ell.$$

□

3.2 Two Existence Results

In this section we will find solutions of (3.1) as critical points of the energy functional J_λ. This will be accomplished, as commented in Section 0.4.3, by a careful study of the topology of low energy levels of J_λ. Indeed we will find a certain topological space $\mathcal{Z}$, L sufficiently large and two continuous maps:

$$\mathcal{Z} \xrightarrow{\Phi} J^{-L} \xrightarrow{\Psi} \mathcal{Z}, \tag{3.30}$$

whose composition is homotopically equivalent to the identity map. In order to obtain existence of solutions for (3.1), it suffices that the set $\mathcal{Z}$ is not contractible. Instead, for multiplicity, a deeper knowledge of the topology of $\mathcal{Z}$ will be needed (see next section).

By Lemma 1.1.4 functions u at a very low energy level will concentrate around at most k points if $\lambda \in (8k\pi, 8(k+1)\pi)$. Being more specific, we will show that if $J_\lambda(u_n) \to -\infty$, then

$$\frac{K^+e^{u_n}}{\int_\Sigma K^+e^{u_n}} \rightharpoonup \sum_{i=1}^{k} t_i\delta_{p_i},$$

where $t_i \geq 0$, $\sum_i t_i = 1$. Moreover, the points p_i belong to $\overline{\Sigma^+}$. We denote by $Bar_k(\overline{\Sigma^+})$ the set of such configurations. We shall also use a retraction from $\overline{\Sigma^+}$ onto a subset Z, which allows us to avoid the singular points. Hence the set $Bar_k(Z) \simeq Bar_k(\overline{\Sigma^+})$ will play the role of $\mathcal{Z}$. The first map in (3.30) is built by means of certain test functions, whereas the second uses a convenient projection and topological retractions from $\overline{\Sigma^+}$ onto Z.

This procedure has been carried out in [5, 52, 53] for the case of positive potentials K. The main difference with respect to the positive case is in the fact that the points of concentration are restricted to the region Σ^+. This fact changes dramatically the topology of the barycenter set and also the existence result obtained. We conclude by showing that assumption (H3) implies that $Bar_k(\overline{\Sigma^+})$ is not contractible.

For the special case $k = 1$, we are able to give a more accurate description of J_λ^{-L} depending on the order of the conical points. That argument will be uses to prove Theorem 0.4.4, which allows us to consider the case in which Σ^+ is contractible. Since (H3) does not hold, we cannot apply the first existence result. The key idea comes from [90], where an improvement of the Moser-Trudinger inequality involving the order of singularities is proved. In a certain sense, if $u \in J_\lambda^{-L}$, then $\frac{\tilde{K}^+ u \chi_{\Sigma^+}}{\int_\Sigma \tilde{K}^+ e^u}$ concentrate around a point of Σ^+ with the exception of those p_i such that $8\pi(1+\alpha_i) > \lambda$. We can conclude the existence of solutions since $\overline{\Sigma^+} \setminus \{p_i : 8\pi(1+\alpha_i) > \lambda\}$ is not contractible.

Let us introduce some notation related to the connected components of Σ^+, which will be made of use along this chapter. We will denote by A_i the non-contractible connected components of Σ^+ and C_h be the contractible ones, $i = 1, \ldots, N$, $h = 1, \ldots, M$ and $N, M \in \mathbb{N} \cup \{0\}$, $N + M = N^+$. Obviously,

$$\Sigma^+ = \coprod_{i=1}^{N} A_i \amalg \coprod_{h=1}^{M} C_h.$$

Recall that a *bouquet of* g *circles* is a set $B^g = \bigcup_{j=1}^{g} S'_j$ where S'_j are simple closed curves verifying that $S_i' \cap S_j' = \{q\}$. If A_i has genus g_i and b_i boundaries, it is well known that A_i can be retracted to an inner bouquet B^{g_i}, where $g_i = 2g_i + b_i - 1$. Instead, C_h is homotopically equivalent to any point $y_h \in C_h$. Therefore

$$\Sigma^+ \simeq \coprod_{i=1}^{N} B^{g_i} \amalg \{y_1, \dots, y_M\}, \quad \text{with} \quad g_i = 2g_i + b_i - 1 \quad \text{for} \quad i = 1, \dots, N. \tag{3.31}$$

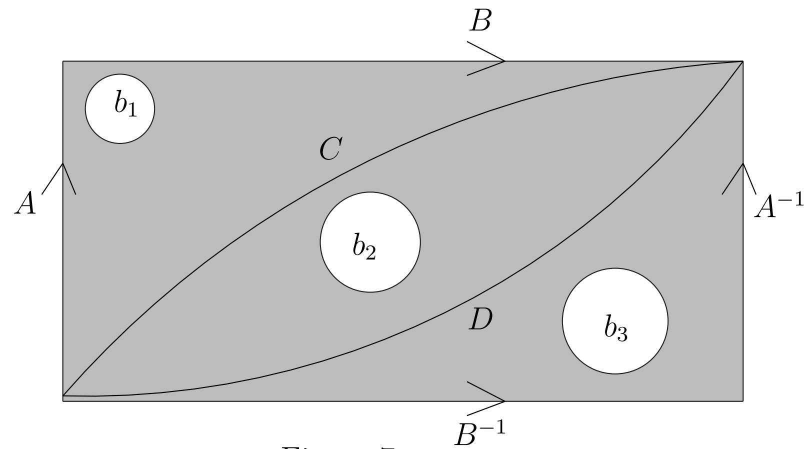

Figure 7

In Figure 7 we represent a 2-torus via its fundamental polygon, with three disks removed b_1, b_2, b_3. This set can be retracted to the boquet $ABCD$ of four loops.

3.2.1 A topological description of the low sublevels of J_λ

The first case

The first result allows us to project continuously functions u with a low energy level onto the set of formal barycenters on a union of bouquets and a simplex contained in Σ^+, namely

$$Z = \coprod_{i=1}^{N} B^{g_i} \amalg Y_M \subset \Sigma^+ \setminus \{p_1, \cdots, p_\ell\} \quad \text{and} \quad Y_M = \{y_1, \dots, y_M\}, \tag{3.32}$$

where $B^{g_i} \subset A_i$ is a bouquet of g_i circles, with g_i defined in (3.31), and $y_h \in C_h$.

Proposition 3.2.1. *Let $\lambda \in (8k\pi, 8\pi(k+1))$, $k \in \mathbb{N}$, and assume (H1), (H2). Then for $L > 0$ sufficiently large there exists a continuous projection*

$$\Psi : J_\lambda^{-L} \longrightarrow Bar_k(Z).$$

Moreover, if $\frac{\tilde{K}^+ e^{u_n}}{\int_\Sigma \tilde{K}^+ e^{u_n} dV_g} \rightharpoonup \sigma$, for some $\sigma \in Bar_k(Z)$, then $\Psi(u_n) \to \sigma$.

Remark 3.2.2. *Under assumption (H3), the topological set $Bar_k(Z)$ is not contractible, defined in (3.32). In case $N = 0$, $Bar_k(Z)$ is the $(k-1)$-skeleton of $(M-1)$-symplex, which is non-contractible if $k < M$ (see Exercise 16 in Section 2.2 of [64]).*

Let us consider the set $S = \{s_1, s_2, s_3, s_4\} \subset \Sigma$ to clarify the latter statement. We can represent s_i as the vertices of a tetrahedron in $\mathbb{R}^3$. Obviously, the set $Bar_1(S)$ corresponds to the four vertices, $Bar_2(S)$ to the six straight edges and $Bar_3(S)$ to the four triangular faces; whereas $Bar_4(S)$ is the whole symplex. Therefore, $Bar_k(S)$ is not contractible if $k < 4$ and it is contractible if $k \geq 4$, as the last remark states.

Before proving Proposition 3.2.1, let us introduce an extra lemma to construct a continuous projection from the low subvelels of J_λ into the barycenters of order k on Σ^+.

Lemma 3.2.3. *Under the assumptions of Proposition 3.2.1, for $L > 0$ there exists a continuous projection*

$$\tilde{\Psi} : J_\lambda^{-L} \longrightarrow Bar_k(\overline{\Sigma^+}),$$

such that $\frac{\tilde{K}^+ e^{u_n}}{\int_\Sigma \tilde{K}^+ e^{u_n} dV_g} \rightharpoonup \sigma$, for some $\sigma \in Bar_k(\overline{\Sigma^+})$, then $\tilde{\Psi}(u_n) \to \sigma$.

Proof. This lemma is proved in the spirit of [53], but following closely the approach of [10].

Claim: If $J_\lambda(u_n) \to -\infty$, up to a subsequence,

$$\sigma_n := \frac{\tilde{K}^+ e^{u_n}}{\int_\Sigma \tilde{K}^+ e^{u_n} dV_g} \rightharpoonup \sigma \in Bar_k(\overline{\Sigma^+}).$$

Suppose by contradiction that there exist $k+1$ points $x_1, \dots, x_{k+1} \subset supp(\sigma)$. For $r > 0$ such that $B_{x_i}(2r) \cap B_{x_j}(2r) = \emptyset$ for $i \neq j$. Therefore, there exists $\varepsilon > 0$ such that $\sigma(B_{x_i}(2r)) > 2\varepsilon$. As a consequence, $\sigma_n(B_{x_i}(r)) \geq \varepsilon$, and we can apply Lemma 1.1.4, which violates the hypothesis that $J_\lambda(u_n)$ diverges negatively.

By the claim, given a neighborhood V of $Bar_k(\Sigma^+)$ in the weak topology of measures, there exists $L_0 > O$ large enough such that if $L > L_0$, then

$$\frac{\tilde{K}^+ e^u}{\int_\Sigma \tilde{K}^+ e^u \, dV_g} \in V, \quad \forall u \in J_\lambda^{-L}. \tag{3.33}$$

In the appendix of [10], it is proved that $Bar_k(\overline{\Sigma^+})$ is a Euclidean Neighborhood Retract. Observe that the σ-weak topology of measures is metrizable on bounded sets, see Theorem 3.28 of [16]. By Lemma E.1 of [14], there exists V a neighborhood of $Bar_k(\overline{\Sigma^+})$ in the weak topology of measures, and a continuous retraction $\mathcal{X} : V \to Bar_k(\overline{\Sigma^+})$. Finally, by (3.33), we define $\tilde{\Psi}$ as

$$\begin{array}{rccccc} \tilde{\Psi}: & J_\lambda^{-L} \longrightarrow & V \longrightarrow & Bar_k(\overline{\Sigma^+}) \\ & u \longmapsto & \frac{\tilde{K}^+ e^u}{\int_\Sigma \tilde{K}^+ e^u, dV_g} \longmapsto & \sum_{i=1}^k t_i \delta_{x_i}. \end{array}$$

□

Proof of Proposition 3.2.1. Observe that we can retract continuously $\overline{A_i}$ onto B^{g_i} and $\overline{C_h}$ onto a single point $y_h \in C_h$. Consequently, we can define the retraction

$$r : \overline{\Sigma^+} \longrightarrow Z. \tag{3.34}$$

We are now in conditions to define the map Ψ as the composition of $\tilde{\Psi}$, defined in Lemma 3.2.3, with the function $r^* : Bar_k(\overline{\Sigma^+}) \longrightarrow Bar_k(Z)$, the pushforward induced by the retraction r, then

$$\begin{array}{rccc} \Psi: & J_\lambda^{-L} & \longrightarrow & Bar_k(Z) \\ & u & \longmapsto & \sum_i s_i \delta_{x_i}, \end{array}$$

where the values s_i are defined by $\tilde{\Psi}$.

□

On the other hand, for $\lambda \in (8k\pi, 8\pi(k+1))$, $k \in \mathbb{N}$, we consider test functions *concentrated* in at most k points of Z with arbitrary low energy. For $b > 0$ small enough, we consider a smooth non-decreasing cut-off function $\chi_b : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\chi_b(t) = \begin{cases} t & \text{for } t \in [0, b], \\ 2b & \text{for } t \geq 2b. \end{cases} \tag{3.35}$$

For $\mu > 0$ and $\sigma = \sum_{i=1}^{k} t_i \delta_{x_i} \in Bar_k(Z)$, we define

$$\phi_{\mu,\sigma} : \Sigma \to \mathbb{R} \qquad \phi_{\mu,\sigma}(x) = \log \sum t_i \left(\frac{\mu}{1 + (\mu \chi_b(d(x, x_i)))^2} \right)^2,$$
$$\varphi_{\mu,\sigma}(x) = \phi_{\mu,\sigma}(x) - \int_\Sigma \phi_{\mu,\sigma}\, dV_g. \tag{3.36}$$

Lemma 3.2.4. *Let $\lambda \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$. Then*

(i) *given $L > 0$ there exists $\mu(L) > 0$ such that for $\mu \geq \mu(L)$, $\varphi_{\mu,\sigma} \in \bar{X}$, where $\bar{X}$ is defined in (3.5), and $J_\lambda(\varphi_{\mu,\sigma}) < -L$ for any $\sigma \in Bar_k(Z)$;*

(ii) *for any $\sigma \in Bar_k(Z)$*

$$\frac{\tilde{K}^+ e^{\varphi_{\mu,\sigma}}}{\int_{\Sigma^+} \tilde{K}^+ e^{\varphi_{\mu,\sigma}} dV_g} \rightharpoonup \sigma \qquad \text{as } \mu \to +\infty.$$

Proof. See Appendix. □

The second case

We now deal with the case in which Σ^+ has only simply connected component and $N^+ \leq k$; however, we are restricted to $\lambda \in (8\pi, 16\pi)$. Indeed in this situation (H4) is not satisfied and so Proposition 3.2.1 does not provide a map from J_λ^{-L} onto a non-contractible set, see Remark 3.2.2. As we showed before, functions at low energy level will concentrate around one point of $\overline{\Sigma^+}$. Here we will show that, roughly speaking, it avoids the singular points p_i with $8\pi(1 + \alpha_i) > \lambda$, namely the points contained in Θ_λ defined in (3.2). Hence, $\mathcal{Z}$ will be $\Sigma^+ \setminus \Theta_\lambda$, which is not contractible. For that, a suitable barycenter map is needed:

Proposition 3.2.5. *Let $\lambda \in (8\pi, 16\pi)$, assume (H1), (H2) hold and $C_1 > 2$ is a constant, then there exist $\tau > 0$, $L_0 > 0$ and a continuous map*

$$\beta : J_\lambda^{-L_0} \to \overline{\Sigma^+}, \tag{3.37}$$

satisfying the following property: for any $u \in J_\lambda^{-L_0}$ there exist $\bar{\sigma} > 0$ and $\bar{y} \in \Sigma$ such that $d(\bar{y}, \beta(u)) < 5C_1\bar{\sigma}$ and

$$\int_{B_{\bar{y}}(\bar{\sigma})\cap\Sigma^+} \tilde{K}e^u \, dV_g = \int_{\Sigma^+\setminus B_{\bar{y}}(C_1\bar{\sigma})} \tilde{K}e^u \, dV_g \geq \tau \int_{\Sigma^+} \tilde{K}e^u \, dV_g. \tag{3.38}$$

Remark 3.2.6. *Proposition 3.2.5 is an adapted version of Proposition 3.1 of [90]. However, it is worth to point out that, even though the original proposition holds true also on a manifold with boundary, we cannot apply directly such result because our functional J_λ is defined on functions in $H^1(\Sigma)$ and not in $H^1(\Sigma^+)$. However we will follow the arguments of [90] modifying them in order to handle the fact that $\tilde{K}$ changes sign; so Σ^- has positive measure and Σ^+ is not necessarily connected.*

Proof. Let us define

$$\mathcal{A}_0 = \{f \in L^1(\Sigma) \mid f(x) \geq 0 \text{ a.e.}, \int_\Sigma f \, dV_g = 1\},$$

$$\sigma : \Sigma \times \mathcal{A}_0 \longrightarrow (0, +\infty),$$

where $\sigma = \sigma(x, f)$ is such that

$$\int_{B_x(\sigma)} f \, dV_g = \int_{\Sigma\setminus B_x(C_1\sigma)} f \, dV_g.$$

Notice that the value $\sigma(x, f)$ is *not* uniquely determined.

Now let us define $T : \Sigma \times \mathcal{A}_0 \longrightarrow (0, +\infty)$ by

$$T(x, f) = \int_{B_x(\sigma(x,f))} f \, dV_g.$$

Notice that $T(x, f)$ does not depend on σ and it is uniquely determined.

Step 0: T is continuous.

Let us suppose by contradiction that there exist $(x_n, f_n) \in \Sigma \times \mathcal{A}_0$ such that

$$(x_n, f_n) \to (x, f) \in \Sigma \times \mathcal{A}_0 \quad \text{but} \quad |T(x_n, f_n) - T(x, f)| \not\to 0 \quad \text{as } n \to +\infty.$$

Being $0 < \sigma(x_n, f_n) < \frac{1}{2}\,\mathrm{diam}(\Sigma)$, up to a subsequence $\sigma(x_n, f_n) \to \sigma_\infty$, as $n \to +\infty$.

Now if $\sigma_\infty = \sigma(x, f)$, then

$$\mathrm{meas}(B_{x_n}(\sigma(x_n, f_n)) \,\triangle\, B_x(\sigma(x, f))) \to 0 \qquad \text{as } n \to +\infty, \tag{3.39}$$

and so by the convergence of f_n to f in L^1 we have

$$\begin{aligned} |T(x, f) - T(x_n, f_n)| \;\le\; & \int\limits_{B_x(\sigma(x,f))\setminus B_{x_n}(\sigma(x_n,f_n))} f\, dV_g + \int\limits_{B_{x_n}(\sigma(x_n,f_n))\setminus B_x(\sigma(x,f))} f_n\, dV_g \\ & + \int\limits_{B_{x_n}(\sigma(x_n,f_n))\cap B_x(\sigma(x,f))} |f_n - f|\, dV_g \overset{n\to+\infty}{\longrightarrow} 0, \end{aligned} \tag{3.40}$$

which gives the desired contradiction.

On the other hand if $\sigma_\infty > \sigma(x, f)$, then for n sufficiently large

$$\begin{gathered} B_x(\sigma(x, f)) \subset B_{x_n}(\sigma(x_n, f_n)) \\ \Sigma \setminus B_x(C_1\sigma(x, f)) \supset \Sigma \setminus B_{x_n}(C_1\sigma(x_n, f_n)). \end{gathered} \tag{3.41}$$

Then for n sufficiently large

$$|T(x, f) - T(x_n, f_n)| \le \int\limits_{B_{x_n}(\sigma(x_n,f_n))} |f_n - f|\, dV_g + \int\limits_{B_{x_n}(\sigma(x_n,f_n))\setminus B_x(\sigma(x,f))} f\, dV_g$$

and so in turn by the convergence of f_n to f we have that

$$\liminf_{n\to+\infty} \int\limits_{B_{x_n}(\sigma(x_n,f_n))\setminus B_x(\sigma(x,f))} f\, dV_g > 0. \tag{3.42}$$

By the definition of σ, the convergence of f_n to f and (3.41) we get

$$
\begin{aligned}
\int\limits_{B_x(\sigma(x,f))} f\, dV_g &= \int\limits_{\Sigma\setminus B_x(C_1(\sigma(x,f)))} f\, dV_g = \int\limits_{\Sigma\setminus B_x(C_1(\sigma(x,f)))} f_n\, dV_g + o(1) \\
&\geq \int\limits_{\Sigma\setminus B_{x_n}(C_1(\sigma(x_n,f_n)))} f_n\, dV_g + o(1) = \int\limits_{B_{x_n}(\sigma(x_n,f_n))} f_n\, dV_g + o(1) \\
&= \int\limits_{B_{x_n}(\sigma(x_n,f_n))} (f_n - f)\, dV_g + \int\limits_{B_{x_n}(\sigma(x_n,f_n))\setminus B_x(\sigma(x,f))} f\, dV_g \\
&\quad + \int\limits_{B_x(\sigma(x,f))} f\, dV_g + o(1) \\
&\geq \int\limits_{B_{x_n}(\sigma(x_n,f_n))\setminus B_x(\sigma(x,f))} f\, dV_g + \int\limits_{B_x(\sigma(x,f))} f\, dV_g + o(1)
\end{aligned}
$$

which, combined with (3.42), gives the desired contradiction.

At last, the case $\sigma_\infty < \sigma(x,f)$ can be treated exactly as the latter case, just reversing the roles of $B_{x_n}(\sigma(x_n,f_n))$ and $B_x(\sigma(x,f))$.

Step 1: There exists $\tau > 0$ such that $\max_{x\in\Sigma} T(x,f) > 2\tau$ for all $f \in \mathcal{A}_0$.

Let us introduce

$$\mathcal{A} = \{h \in L^1(\Sigma),\, h(x) > 0 \text{ a.e.}, \int_\Sigma h\, dV_g = 1\}.$$

It is easy to see that $\mathcal{A}$ is dense in $\mathcal{A}_0$.

We claim that there exists $\tilde\tau > 0$ such that $\max_{x\in\Sigma} T(x,f) > 2\tilde\tau$ for all $f \in \mathcal{A}$. So, Step 1 follows from these facts and Step 0. Indeed, fix $f \in \mathcal{A}_0$ and let $\{h_n\} \subset \mathcal{A}$ such that $h_n \to f$ in $L^1(\Sigma)$ and let $x_n \in \Sigma$ such that $T(x_n,h_n) = \max_{x\in\Sigma} T(x,h_n)$, then $T(x_n,h_n) > 2\tilde\tau$. Up to a subsequence $x_n \to x_0 \in \Sigma$ as $n \to +\infty$ and so, by the continuity of T, $T(x_n,h_n) \to T(x_0,f) \geq 2\tilde\tau$. The thesis follows taking $\tau = \frac{\tilde\tau}{2}$.

In order to show the claim, let us take $f \in \mathcal{A}$ and $x_0 \in \Sigma$ s.t. $T(x_0,f) = \max_{x\in\Sigma} T(x,f)$, and fix $x \in A(\sigma(x_0,f), C_1\sigma(x_0,f))$. We state the following inequalities

$$d(x,x_0)+C_1\sigma(x,f)\geq C_1\sigma(x_0,f), \tag{3.43}$$

$$d(x,x_0)-C_1\sigma(x,f)\leq \sigma(x_0,f). \tag{3.44}$$

Suppose by contradiction that for some $\varepsilon>0$ such that $d(x,x_0)+C_1\sigma(x,f)<C_1\sigma(x_0,f)-2\varepsilon$; by the triangular inequality, $B_x(C_1\sigma(x,f))\subset B_{x_0}(C_1\sigma(x_0,f)-2\varepsilon)$. Taking into account the definition of σ, then $B_{x_0}(C_1\sigma(x_0,f))\neq\Sigma$. Therefore, $A_{x_0}(C_1\sigma(x_0,f)-2\varepsilon, C_1\sigma(x_0,f))$ is non-empty. So,

$$\begin{aligned}T(x,f)=\int_{B_x(\sigma(x,f))} f\,dV_g &= \int_{\Sigma\setminus B_x(C_1\sigma(x,f))} f\,dV_g\\ &> \int_{\Sigma\setminus B_{x_0}(C_1\sigma(x_0,f))} f\,dV_g = T(x_0,f),\end{aligned}$$

contradicting the definition of x_0. Therefore, (3.43) is proved.

Now, suppose by contradiction that for some $\varepsilon>0$ such that $d(x,x_0)-C_1\sigma(x,f)>\sigma(x_0,f)+2\varepsilon$. This implies that $B_{x_0}(\sigma(x_0,f)+2\varepsilon)\subset(\Sigma\setminus B_{x_0}(C_1\sigma(x,f)))$. As before, the set $A_{x_0}(\sigma(x_0,f),\sigma(x_0,f)+2\varepsilon)$ is not empty. Again, we have that

$$T(x,f)=\int_{B_x(\sigma(x,f))} f\,dV_g=\int_{\Sigma\setminus B_x(C_1\sigma(x,f))} f\,dV_g>\int_{B_{x_0}(\sigma(x_0,f))} f\,dV_g=T(x_0,f),$$

which is a contradiction that proves (3.44).

Substracting (3.44) from (3.43), we can deduce $\sigma(x,f)\geq\frac{C_1-1}{2C_1}\sigma(x_0,f)\geq\frac{1}{4}\sigma(x_0,f)$ for every $x\in A_{x_0}(\sigma(x_0,f),C_1\sigma(x_0,f))$. For a given $C_1>2$, there exists $k=k(C_1)$ such that $A_y(\sigma,C_1\sigma)\subset\cup_{i=1}^k B_{x_i}(\frac{1}{4}\sigma)$ for every $\sigma>0$ and any $y\in\Sigma$, were $x_i\in A_y(\sigma,C_1\sigma)$.

In this situation, we obtain

$$\int_{A_{x_0}(\sigma(x_0,f),C_1\sigma(x_0,f))} f\,dV_g\leq\sum_{i=1}^k\int_{B_{x_i}(\sigma(x_i,f))} f\,dV_g=\sum_{i=1}^k T(x_i,f)\leq kT(x_0,f).$$

On the other hand,

$$\int_{B_{x_0}(\sigma(x_0,f))} f\,dV_g = \int_{\Sigma\setminus B_{x_0}(C_1\sigma(x_0,f))} f\,dV_g = T(x_0,f).$$

Since $\int_\Sigma f\,dV_g = 1$, it is immediate that $T(x_0,f) \geq \frac{1}{k+2}$, which completes the proof of this step.

Step 2: Let us define

$$S(f) = \{x \in \Sigma \mid T(x,f) \geq \tau\}.$$

By Step 0 and Step 1 $S(f)$ is a non empty compact set for any $f \in \mathcal{A}_0$.

Let us define also

$$\bar{\sigma}(f) = \sup_{x\in S(f)} \sigma(x,f).$$

Let us prove that even if σ is not continuous, up to eventually redefine $\sigma(\cdot,f)$ in a point, there exists

$$\bar{y} \in S(f) \quad \text{such that} \quad \sigma(\bar{y},f) = \bar{\sigma}.$$

Indeed let $\{x_n\} \subset S(f)$ such that $\sigma(x_n,f) \to \bar{\sigma}(f)$, then since $S(f)$ is compact, up to a subsequence, $x_n \to \bar{y} \in S(f)$. Thus

$$\int_{B_{x_n}(\sigma(x_n,f))} f\,dV_g = \int_{\Sigma\setminus B_{x_n}(C_1\sigma(x_n,f))} f\,dV_g$$

and so

$$\int_{B_{\bar{y}}(\bar{\sigma}(f))} f\,dV_g = \int_{\Sigma\setminus B_{\bar{y}}(C_1\bar{\sigma}(f))} f\,dV_g.$$

Now if $\sigma(\bar{y},f) < \bar{\sigma}(f)$ we can redefine $\sigma(\cdot,f)$ at $\bar{y}$ as $\sigma(\bar{y},f) = \bar{\sigma}(f)$, and the proof of our claim is completed. Clearly this modification does not affect the previous steps.

For $u \in X$, take $f \equiv f_u = \frac{\tilde{K}^+ e^u}{\int_\Sigma \tilde{K}^+ e^u\,dV_g}$.

Step 3: For any $\varepsilon > 0$ there exists $L_0 > 0$ large enough such that $\operatorname{diam} S(f) \leq (C_1+1)\bar{\sigma} < \varepsilon$ for all $u \in J_\lambda^{-L_0}$, $L \geq L_0$.

By definition of $\bar{\sigma}$, $S(f)$ and Σ^+

$$\int_{B_{\bar{y}}(\bar{\sigma})\cap\Sigma^+} \tilde{K}e^u\,dV_g \geq \tau\int_{\Sigma^+} \tilde{K}e^u\,dV_g \geq \tau\int_{\Sigma} \tilde{K}e^u\,dV_g \quad \text{and}$$

$$\int_{\Sigma^+\setminus B_{\bar{y}}(C_1\bar{\sigma})} \tilde{K}e^u\,dV_g \geq \tau\int_{\Sigma^+} \tilde{K}e^u\,dV_g \geq \tau\int_{\Sigma} \tilde{K}e^u\,dV_g.$$

Then Proposition 1.1.4 implies that $\bar{\sigma} \to 0$, as $L \to +\infty$, uniformly for $u \in J_\lambda^{-L}$. Thus we can choose $L_0 > 0$ such that $\bar{\sigma} < \min\left\{\frac{\varepsilon}{C_1+1}, \frac{\min_i(\operatorname{diam} D_i)}{6}\right\}$ for any $u \in J_\lambda^{-L}$, where D_i are the connected components of Σ^+.

Now take x, $y \in S(f)$, where $f = \frac{\tilde{K}^+e^u}{\int_\Sigma \tilde{K}e^u\,dV_g}$, $u \in J_\lambda^{-L_0}$, we claim that

$$d(x,y) \leq C_1\max\{\sigma(x,f),\sigma(y,f)\} + \min\{\sigma(x,f),\sigma(y,f)\}. \tag{3.45}$$

Let us prove (3.45). Let us suppose by contradiction that $B_x(C_1(\sigma(x,f))) \cap B_y(\sigma(y,f)+\varepsilon) = \emptyset$ for some $\varepsilon > 0$. Clearly we can take $\varepsilon < \frac{\min_i(\operatorname{diam} D_i)}{6}$ and such that $B_y(\sigma(y,f)+\varepsilon)$ does not exhaust the whole Σ^+. Let us now show that $A_y(\sigma(y,f),\sigma(y,f)+\varepsilon)\cap\Sigma^+$ is a nonempty open set.
Let us prove first that there exists $z \in \partial B_y(\sigma(y,f)+\varepsilon)\cap\Sigma^+$.

By contradiction we suppose that $\partial B_y(\sigma(y,f)+\varepsilon)\cap\Sigma^+ = \emptyset$.

Since $\int_{B_y(\sigma(y,f))\cap\Sigma^+} f\,dV_g > 0$, $B_y(\sigma(y,f)+\varepsilon)\cap\Sigma^+ \neq \emptyset$, so $D_i \subset B_y(\sigma(y,f)+\varepsilon)$ for some i. This would imply that $\min_i(\operatorname{diam}(D_i)) < 2(\sigma(y,f)+\varepsilon) \leq 2\bar{\sigma}+2\varepsilon < \frac{2}{3}\min_i(\operatorname{diam}(D_i))$ which is impossible.

Next, being Σ^+ open, $B_z(\varepsilon)\cap A_y(\sigma(y,f),\sigma(y,f)+\varepsilon)\cap\Sigma^+$ is a nonempty open set. Then

$$\begin{aligned}\int_{B_x(\sigma(x,f))\cap\Sigma^+} \tilde{K}e^u\,dV_g &= \int_{\Sigma^+\setminus B_x(C_1\sigma(x,f))} \tilde{K}e^u\,dV_g \\ &\geq \int_{B_y(\sigma(y,f)+\epsilon)\cap\Sigma^+} \tilde{K}e^u\,dV_g > \int_{B_y(\sigma(y,f))\cap\Sigma^+} \tilde{K}e^u\,dV_g.\end{aligned}$$

By interchanging the roles of x and y, we would also obtain the reverse inequality. This contradiction proves (3.45).

Then by (3.45) and the definition of $\bar{\sigma}$ we have $d(x,y) \leq (C_1+1)\bar{\sigma}$ for any given $x, y \in S(f)$.

Step 4: Definition of β and conclusion.

We consider Σ isometrically embedded in $\mathbb{R}^N$ and we define

$$\eta : J_\lambda^{-L_0} \to \mathbb{R}^N,\ \eta(u) = \frac{\int_\Sigma [T(x,f)-\tau]^+ x\, dV_g}{\int_\Sigma [T(x,f)-\tau]^+ dV_g} \quad \text{where } f \equiv f_u = \frac{\tilde{K}^+ e^u}{\int_\Sigma \tilde{K}^+ e^u\, dV_g}.$$

Notice that in the above terms the integrands vanish outside $S(f)$.

From now on, for $r > 0$, according to our notation we will denote by $(\Sigma^+)^r = \{x \in \Sigma \,|\, d(x,\Sigma^+) < r\}$. Clearly, $B_{\bar{y}}(\bar{\sigma}) \cap \Sigma^+ \neq \emptyset$, namely

$$\bar{y} \in (\Sigma^+)^{\bar{\sigma}}, \tag{3.46}$$

moreover by Step 3 $\mathrm{diam}(S(f)) \leq (C_1+1)\bar{\sigma}$ and therefore being $\bar{y} \in S(f)$

$$S(f) \subset (\Sigma^+)^{(C_1+2)\bar{\sigma}} \qquad \text{and} \qquad S(f) \subset \bar{B}_{\bar{y}}^{\mathbb{R}^N}((C_1+1)\bar{\sigma}).$$

Being $\eta(u)$ a barycenter of a function supported in $S(f)$, we have

$$|\eta(u) - \bar{y}| \leq (C_1+1)\bar{\sigma}. \tag{3.47}$$

Let $U \supset \Sigma$, $U \subset \mathbb{R}^N$ an open tubular neighborhood of Σ, and $P : U \to \Sigma$ an orthogonal projection onto Σ. Moreover by Step 3 there exists $L_0 > 0$ sufficiently large such that $\eta(u) \in U$ for any $u \in J_\lambda^{-L_0}$. Thus we can define

$$\tilde{\beta} : J_\lambda^{-L_0} \to \Sigma \qquad\qquad \tilde{\beta}(u) = P \circ \eta(u).$$

Next, we claim that, eventually for a larger L_0,

$$d(\bar{y}, \tilde{\beta}(u)) \leq 2C_1\bar{\sigma}. \tag{3.48}$$

Let $T_{\bar{y}}(\Sigma)$ be the tangent space to Σ at $\bar{y}$. For any $x \in S(f) \subset B_{\bar{y}}^{\mathbb{R}^N}((C_1+1)\bar{\sigma})$, we have that

$$\min\{|\bar{y} + y - x| : y \in T_{\bar{y}}(\Sigma)\} \leq C\bar{\sigma}^2,$$

where C depends only on the C^2 regularity of Σ. Since $\eta(u)$ is a barycenter of a function supported in $S(f)$, it is clear that

$$\min\{|\bar{y}+y-\eta(u)| : y \in T_{\bar{y}}(\Sigma)\} \leq C\bar{\sigma}^2.$$

By taking a larger L_0, if necessary, by Step 3 $\bar{\sigma}$ is small enough such that

$$|\tilde{\beta}(u)-\eta(u)| = \min_{x\in\Sigma}|\eta(u)-x| \leq 2C\bar{\sigma}^2 \leq \bar{\sigma}. \tag{3.49}$$

Since $C_1 > 2$, let $\nu = \frac{2C_1}{C_1+2} > 1$, again, by Step 3 we can take L_0 large enough such that $\bar{\sigma}$ satisfies that for $x, y \in \Sigma$, if $|x-y| \leq (C_1+2)\bar{\sigma}$, then $d(x,y) \leq \nu|x-y|$. This together with (3.47) and (3.49) proves (3.48).

Combining (3.46) and (3.48) we obtain that

$$d(\tilde{\beta}(u), \overline{\Sigma^+}) < (2C_1+1)\bar{\sigma}. \tag{3.50}$$

Besides by the regularity of $\partial\Sigma^+$ there exists $\gamma > 0$ and a continuous projection $\pi : (\Sigma^+)^\gamma \to \overline{\Sigma^+}$ such that

$$\pi_{|\overline{\Sigma^+}} = Id_{|\overline{\Sigma^+}} \qquad \text{and} \qquad d(x,\pi(x)) = d(x,\overline{\Sigma^+}). \tag{3.51}$$

Again for $L_0 > 0$ large enough $2(C_1+1)\bar{\sigma} < \gamma$ and so, by (3.50), $\tilde{\beta}(J_\lambda^{-L_0}) \subset (\Sigma^+)^\gamma$. Then we can define $\beta : J_\lambda^{-L_0} \to \overline{\Sigma^+}$ as

$$\beta(u) = \pi \circ \tilde{\beta}(u).$$

At last by (3.48), (3.51), (3.50) and $C_1 > 2$ we have

$$\begin{aligned} d(\bar{y},\beta(u)) &\leq d(\bar{y},\tilde{\beta}(u)) + d(\tilde{\beta}(u),\pi\circ\tilde{\beta}(u)) \\ &\leq 2C_1\bar{\sigma} + d(\tilde{\beta}(u),\overline{\Sigma^+}) \\ &\leq (4C_1+1)\bar{\sigma} < 5C_1\bar{\sigma}. \end{aligned}$$

□

Remark 3.2.7. *With the above construction, if* $f_n = \frac{\tilde{K}^+e^{u_n}}{\int_\Sigma \tilde{K}^+e^{u_n}dV_g} \rightharpoonup \delta_x$ *for some*

$x \in \overline{\Sigma^+}$ then one also has $\beta(u_n) \to x$.

Proof. Let us take $\overline{\sigma}_n = \overline{\sigma}(f_n)$ and $x_n \in S(f_n)$. Up to subsequence, we have that $\overline{\sigma}_n \to \overline{\sigma}_0$ $x_n \to x_0$.

In order to prove the remark, it suffices to check that $\overline{\sigma}_0 = 0$ and $x_0 = x$. Suppose by contradiction, that $\overline{\sigma}_0 > 0$. By construction, then

$$\int_{B_{x_0}(\overline{\sigma}/2)\cap\Sigma^+} f_n \, dV_g > \tau, \quad \int_{\Sigma^+\setminus B_{x_0}((C_1-1)\overline{\sigma})} f_n \, dV_g > \tau,$$

which contradicts the hypothesis.

Now, suppose that $x_0 \neq x$, then $0 < \delta < d(x_0, x)/2$. Since $\overline{\sigma}_n \to 0$, it holds

$$\int_{B_{x_0}(\delta)\cap\Sigma^+} f_n \, dV_g > \tau,$$

which is a new contradiction. □

Next we show that the functional J_λ is bounded from below on the functions in $\beta^{-1}(\Theta_\lambda)$, where β is the map constructed in Proposition 3.2.5 and Θ_λ is defined in (3.2).

Proposition 3.2.8. *Let $\alpha_1, \ldots, \alpha_m > 0$ and $\lambda \in (8\pi, 16\pi)$. Assume (H1) and (H2), then there exist $C_1 > 0$ sufficiently large, $L_0 > 0$, $\tau > 0$ such that Proposition 3.2.5 applies and there exists $L > L_0$ such that $J_\lambda(u) > -L$ for any $u \in J_\lambda^{-L_0}$ satisfying that $\beta(u) = p_i \in \Theta_\lambda$.*

Proof. We will follow very closely the proof of Proposition 4.1 in [90], adapting it to our different definition of β.

Let $\varepsilon > 0$ to be fixed later depending only on λ and a universal constant C_0. In turn let $C_1 > 4$ large enough so that $\varepsilon^{-1} + 1 < \log_4 C_1$ and let $L_0 > 0$ and $\tau > 0$ such that Proposition 3.2.5 applies.

Let us suppose by contradiction that there exists a sequence $u_n \in X$ such that $J_\lambda(u_n) \to -\infty$ and $\beta(u_n) = p_i \in \Theta_\lambda$ as $n \to +\infty$. Clearly, we can assume without loss of generality that $\int_\Sigma u_n dV_g = 0$.

Let $\bar{y}_n \in \Sigma$, $\bar{\sigma}_n > 0$ be as in Proposition 3.2.5, such that $d(\bar{y}_n, p_i) < 5C_1\bar{\sigma}_n$. It is easy to see, applying Proposition 1.1.4 that $\bar{\sigma}_n \to 0$. Consequently, by virtue of

(H2), for n large enough $\bar{y}_n \in \Sigma^+$. Then we fix $\delta > 0$, smaller than the injectivity radius and such that $B_{\bar{y}_n}(\delta) \subset \Sigma^+$ for any n sufficiently large, and we choose

$$N \in \mathbb{N} \qquad \text{such that} \qquad \varepsilon^{-1} < N < \log_4 C_1. \tag{3.52}$$

Since $\bar{\sigma}_n \to 0$ we have that for n sufficiently large $C_1\bar{\sigma}_n < \delta$ and so

$$\cup_{m=1}^N A_{\bar{y}_n}(4^{m-1}\bar{\sigma}_n, 4^m\bar{\sigma}_n) \subset A_{\bar{y}_n}(\bar{\sigma}_n, C_1\bar{\sigma}_n) \subset B_{\bar{y}_n}(\delta).$$

Then there exists $s_n \in [2\bar{\sigma}_n, \frac{C_1}{2}\bar{\sigma}_n]$ such that

$$\int_{A_{\bar{y}_n}(\frac{s_n}{2},2s_n)} |\nabla u_n|^2 \, dV_g \leq \frac{1}{N} \int_{B_{\bar{y}_n}(\delta)} |\nabla u_n|^2 \, dV_g. \tag{3.53}$$

From now on, in order to simplify the notation, we drop the dependence on n. Let us define

$$\mathcal{D}_1 = \int_{B_{\bar{y}}(s)} |\nabla u|^2 dV_g, \qquad \mathcal{D}_2 = \int_{\Sigma \setminus B_{\bar{y}}(s)} |\nabla u|^2 dV_g, \qquad \mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2.$$

The proof proceeds in three steps.

Step 1: We apply Proposition 1.1.3 to a convenient dilation of u given by

$$v(x) = u(sx + \bar{y}).$$

We have

$$\int_{B_{\bar{y}}(s)} |\nabla u|^2 dV_g = \int_{B_0(1)} |\nabla v|^2 dV_g, \qquad ⨍_{B_{\bar{y}}(s)} u dV_g = ⨍_{B_0(1)} v dV_g,$$

$$\begin{aligned}
\int_{B_{\bar{y}}(\frac{s}{2})\cap\Sigma^+} \tilde{K}e^u dV_g &\leq C \int_{B_{\bar{y}}(\frac{s}{2})\cap\Sigma^+} |x - p_i|^{2\alpha_i} e^u \, dV_g \\
&\leq Cs^{2\alpha_i} \int_{B_{\bar{y}}(\frac{s}{2})\cap\Sigma^+} e^u dV_g \leq Cs^{2\alpha_i+2} \int_{B_0(\frac{1}{2})} e^v dV_g.
\end{aligned}$$

In the above computations we have used that $|\bar{y} - p_i| \leq Cs$. Then, recalling that by definition of τ (see Proposition 3.2.5)

$$\int_{B_{\bar{y}}(\frac{s}{2})\cap\Sigma^+} \tilde{K}e^u dV_g \geq \tau \int_{\Sigma^+} \tilde{K}e^u dV_g \geq \tau \int_{\Sigma} \tilde{K}e^u dV_g$$

and applying Proposition 1.1.3 to v (with $\tilde{H}=1$) we get

$$\begin{aligned} \log \int_{\Sigma} \tilde{K}e^u \, dV_g &\leq C + 2(1+\alpha_i)\log s + \log \int_{B_0(\frac{1}{2})} e^v \, dV_g \\ &\leq C + 2(1+\alpha_i)\log s + \frac{1}{16\pi - \varepsilon}\int_{B_0(1)} |\nabla v|^2 \, dV_g + \fint_{B_0(1)} v \, dV_g \\ &= C + 2(1+\alpha_i)\log s + \frac{1}{16\pi-\varepsilon}\mathcal{D}_1 + \fint_{B_{\bar{y}}(s)} u \, dV_g. \end{aligned} \tag{3.54}$$

Step 2: Exactly as in Proposition 4.1 of [90], we estimate $\fint_{\partial B_{\bar{y}}(s)} u \, dV_g$. By the trace embedding $\tilde{u} = u - \fint_{B_{\bar{y}}(s)} u \, dV_g \in L^1(\partial B_{\bar{y}}(s))$ and thanks to the Poincaré-Wirtinger inequality we get

$$\left| \fint_{\partial B_{\bar{y}}(s)} \tilde{u} \, dx \right| \leq C\|\tilde{u}\|_{H^1} \leq C\left(\int_{B_{\bar{y}}(s)} |\nabla u|^2 \, dV_g\right)^{\frac{1}{2}}.$$

Therefore,

$$\left| \fint_{\partial B_{\bar{y}}(s)} u \, dx - \fint_{B_{\bar{y}}(s)} u \, dV_g \right| \leq C\left(\int_{B_{\bar{y}}(s)} |\nabla u|^2 \, dV_g\right)^{\frac{1}{2}} \leq \varepsilon \mathcal{D}_1 + C'. \tag{3.55}$$

Now notice that, since the above inequality is invariant under dilation, the constant C is independent of s and hence C' depends only on ε.

Step 3: By virtue of the fact that $\tilde{K}(x) \sim \mathrm{d}(x,p_i)^{2\alpha_i}$ near p_i, and $|x-p_i| \leq C|x-\bar{y}|$ in $\Sigma^+ \setminus B_{\bar{y}}(s)$, we get the following estimate

$$
\begin{aligned}
\int_{\Sigma^+\setminus B_{\bar y}(s)} \tilde K e^u \, dV_g = & \int_{\Sigma^+\setminus B_{\bar y}(s)} \frac{\tilde K(x)}{|x-\bar y|^{2\alpha_i}} |x-\bar y|^{2\alpha_i} e^u \, dV_g \leq \\
\frac{C}{s^{2\alpha_i}} \int_{\Sigma^+\setminus B_{\bar y}(s)} e^{\hat v} \, dV_g \leq & \frac{C}{s^{2\alpha_i}} \int_{\Sigma} e^{\hat v} \, dV_g,
\end{aligned}
\tag{3.56}
$$

where $\hat v(x) = \hat u(x) + 4\alpha_i w(x)$,

$$
w(x) = \begin{cases} \log s & x \in B_{\bar y}(s), \\ \log|x-\bar y| & x \in A_{\bar y}(s,\delta), \\ \log \delta & \Sigma \setminus B_{\bar y}(\delta), \end{cases} \qquad \begin{cases} -\Delta_g \hat u = 0 & x \in B_{\bar y}(s), \\ \hat u(x) = u(x) & x \notin B_{\bar y}(s). \end{cases}
$$

In order to apply the Moser-Trudinger inequality to $\hat v$ we observe that

$$
\fint_\Sigma \hat v \, dV_g \leq C + \fint_\Sigma \hat u. \tag{3.57}
$$

Since $\fint_\Sigma u \, dV_g = 0$ and $\hat u - u$ is compactly supported in $B_{\bar y}(s)$,

$$
\left| \fint_\Sigma \hat u \, dV_g \right| = \left| \fint_\Sigma (\hat u - u) \right| \leq C \left(\int_{B_{\bar y}(s)} |\nabla \hat u - \nabla u|^2 \, dV_g \right)^{\frac12} \leq \varepsilon \mathcal{D} + C_\varepsilon. \tag{3.58}
$$

We now estimate using (3.53) and (3.52) the Dirichlet energy

$$
\int_{B_{\bar y}(s)} |\nabla \hat v|^2 \, dV_g = \int_{B_{\bar y}(s)} |\nabla \hat u|^2 \, dV_g \leq C_0 \int_{A_{\bar y}(\frac{s}{2}, 2s)} |\nabla u|^2 \, dV_g \leq C_0 \varepsilon \mathcal{D}, \tag{3.59}
$$

where C_0 is independent on the radius s, since the inequality is dilation invariant.

On the other hand integrating by parts we obtain

$$\begin{aligned}\int_{\Sigma\setminus B_{\bar y}(s)}|\nabla\hat v|^2\,dV_g &= \int_{\Sigma\setminus B_{\bar y}(s)}|\nabla\hat u|^2\,dV_g + 16\alpha_i^2\int_{\Sigma\setminus B_{\bar y}(s)}\frac{1}{|x-\bar y|^2}dV_g\\ &\quad +8\alpha_i\int_{\Sigma\setminus B_{\bar y}(s)}\nabla u\cdot\nabla(\log|x-\bar y|)\,dV_g \\ &\leq \mathcal{D}_2-32\pi\alpha_i^2\log s-16\pi\alpha_i\fint_{\partial B_{\bar y}(s)}u\,dV_g+C.\end{aligned}\tag{3.60}$$

Finally applying to $\hat v$ the Moser-Trudinger inequality, Proposition 1.1.2, and in turn (3.59), (3.60), (3.57), (3.58) we get

$$\begin{aligned}\log\int_\Sigma e^{\hat v}\,dV_g &\leq \frac{1}{16\pi}\int_{B_{\bar y}(s)}|\nabla\hat v|^2\,dV_g+\frac{1}{16\pi}\int_{\Sigma\setminus B_{\bar y}(s)}|\nabla\hat v|^2\,dV_g+\fint_\Sigma\hat v\,dV_g+C\\ &\leq \frac{C_0\varepsilon\mathcal{D}}{16\pi}+\frac{\mathcal{D}_2}{16\pi}-2\alpha_i^2\log s-\alpha_i\fint_{\partial B_{\bar y}(s)}u\,dV_g+\varepsilon\mathcal{D}+C.\end{aligned}\tag{3.61}$$

Now, recalling that $B_{\bar y}(s)\subset B_{\bar y}(C_1\bar\sigma)$, the definition of $\bar y$ (see Proposition 3.2.5), (3.56) and (3.61) we have that

$$\begin{aligned}\log\int_\Sigma\tilde K e^u\,dV_g &\leq \log\int_{\Sigma^+}\tilde K e^u\,dV_g\leq\log\left(\frac{1}{\tau}\int_{\Sigma^+\setminus B_{\bar y}(s)}\tilde K e^u\,dV_g\right)\\ &\leq -2\alpha_i(1+\alpha_i)\log s+C_0\varepsilon\mathcal{D}+\frac{\mathcal{D}_2}{16\pi}-\alpha_i\fint_{\partial B_{\bar y}(s)}u\,dV_g+C.\end{aligned}\tag{3.62}$$

At last, adding (3.54) (multiplied by α_i) to (3.62) and using (3.55) and the assumption $\alpha_i\leq 1$ we have

$$(\alpha_i+1)\log\int_\Sigma\tilde K e^u\,dV_g\leq\left(\frac{1}{16\pi-\varepsilon}+C_0\varepsilon\right)\mathcal{D}+C,$$

so plugging this estimate in the functional we derive that

$$J_\lambda(u)\geq\left(\frac{1}{2}-\lambda\left(\frac{1}{(16\pi-\varepsilon)(\alpha_i+1)}+\frac{C_0}{\alpha_i+1}\varepsilon\right)\right)\int_\Sigma|\nabla u_n|\,dV_g-C.$$

In order to conclude it suffices to take ε small enough, depending only on λ and C_0

(C_0 is a universal constant), such that $\left(\frac{1}{2} - \lambda\left(\frac{1}{(16\pi-\varepsilon)(\alpha_i+1)} + \frac{C_0}{\alpha_i+1}\varepsilon\right)\right) > 0$. Indeed, recalling that we were working with a sequence u_n, we get $J_\lambda(u_n) \geq -C$ which leads to the desired contradiction. □

Next, we can apply the last proposition to map continuously functions u with a low energy level onto the set of formal barycenters on a union of bouquets and a simplex contained in $\Sigma^+ \setminus \Theta_\lambda$. Let us define the set

$$\Theta_{\lambda,A_i} = \Theta_\lambda \cap A_i \quad \text{for } i = 1, \dots, N, \qquad \Theta_{\lambda,C_h} = \Theta_\lambda \cap C_h \quad \text{for } h = 1, \dots, M, \tag{3.63}$$

where Θ_λ is defined in (3.2), and let us assume that, up to reordering, $M_\lambda \in \{1, \dots, M\}$ is such that $\Theta_{\lambda,C_h} \neq \emptyset$ if $h \in \{1, \dots, M_\lambda\}$ and $\Theta_{\lambda,C_h} = \emptyset$ if $h \in \{M_\lambda + 1, \dots, M\}$.

And let us introduce

$$\tilde{Z} = \coprod_{i=1}^{N} B^{g_i+|\Theta_{\lambda,A_i}|} \amalg \coprod_{h=1}^{M_\lambda} B^{|\Theta_{\lambda,C_h}|} \amalg \hat{Y}_{M_\lambda}, \tag{3.64}$$

$B^{g_i+|\Theta_{\lambda,A_i}|} \subset A_i$, $B^{|\Theta_{\lambda,C_h}|} \subset C_h$ are bouquets of $g_i + |\Theta_{\lambda,A_i}|$ and $|\Theta_{\lambda,C_h}|$ circles respectively, with g_i defined in (3.31), for $i = 1, \dots, N$ and $h = 1, \dots, M_\lambda$, and $\hat{Y}_{M_\lambda} = \coprod_{h=M_\lambda+1}^{M} \{y_h\}$ with $y_h \in Y_h$ for $h = M_\lambda + 1, \dots, M$.

Proposition 3.2.9. *Let $\lambda \in (8\pi, 16\pi)$ and assume (H1), (H2). Then for $L > 0$ sufficiently large there exists a continuous projection*

$$\Psi : J_\lambda^{-L} \to Bar_1(\tilde{Z}) \cong \tilde{Z}.$$

Moreover, if

$$\frac{\tilde{K}^+ e^{u_n}}{\int_\Sigma \tilde{K}^+ e^{u_n} dV_g} \rightharpoonup \sigma, \qquad \text{for some } \sigma \in Bar_1(\tilde{Z}),$$

then $\Psi(u_n) \to \sigma$.

Remark 3.2.10. *Observe that $\tilde{Z}$ is not contractible if and only if either $N \geq 1$ or $M_\lambda > 1$, i.e. if (H3) holds, or if $\Theta_\lambda \neq \emptyset$, namely (H4) holds.*

Proof. By Proposition 3.2.8, we have constructed a continuous projection

$$\beta : J_\lambda^{-L} \to \overline{\Sigma^+} \setminus \Theta_\lambda,$$

with the property that if $\frac{\tilde{K}^+ e^{u_n}}{\int_\Sigma \tilde{K}^+ e^{u_n}\, dV_g} \rightharpoonup \delta_x$ for some $x \in \overline{\Sigma^+} \setminus \Theta_\lambda$ then $\beta(u_n) \to x$.

As we did with Σ^+, we can rewrite $\overline{\Sigma^+} \setminus \Theta_\lambda$ as

$$\coprod_{i=1}^{N} A_i' \amalg \coprod_{h=1}^{M_\lambda} C_h' \amalg \coprod_{h=M_\lambda+1}^{M} C_h,$$

where $A_i' = A_i \setminus \Theta_{\lambda,A_i}$ and $C_h' = C_h \setminus \Theta_{\lambda,C_h}$.

The sets A_i' can be retracted to an inner bouquet $B^{g_i+|\Theta_{\lambda,A_i}|} \subset A_i'$ and C_h' to $B^{|\Theta_{\lambda,C_h}|} \subset C_h'$, in a similar way to the proof of Proposition 3.2.1, we can define a retraction

$$r : \overline{\Sigma^+} \setminus \Theta_\lambda \longrightarrow \tilde{Z}.$$

Finally, we can define Ψ as the composition of β with the pushforward $r* : Bar_1(\overline{\Sigma^+} \setminus \Theta_\lambda) \longrightarrow Bar_1(\tilde{Z})$, then

$$\begin{array}{rccc} \Psi : & J_\lambda^{-L} & \longrightarrow & Bar_1(\tilde{Z}) \\ & u & \longmapsto & \delta_x. \end{array}$$

Since r is a retraction, the second part of the proposition is satisfied.

□

Next, for $\lambda \in (8\pi, 16\pi)$, we introduce appropriate test functions that will allow us to map a compact subset $\tilde{Z}$ of $\Sigma^+ \setminus \Theta_\lambda$ into low sublevels of J_λ.

Let $\tilde{\alpha} = \max_{n \le \ell \,|\, p_n \notin \Theta_\lambda} \alpha_n$ or $\tilde{\alpha} = 0$ if $\Theta_\lambda = \{p_1, \ldots, p_\ell\}$ or $\ell = 0$. Fix $\alpha \in (\tilde{\alpha}, \frac{\lambda}{8\pi} - 1)$, $\mu > 0$ and $z \in \tilde{Z}$, we define

$$\tilde{\phi}_{\mu,z} : \Sigma \to \mathbb{R}, \quad \tilde{\phi}_{\mu,z}(x) = 2\log\left(\frac{\mu^{1+\alpha}}{1 + (\mu\chi_b(d(x,z)))^{2(1+\alpha)}}\right),$$

$$\tilde{\varphi}_{\mu,z}(x) = \tilde{\phi}_{\mu,z}(x) - \int_\Sigma \tilde{\phi}_{\mu,z}\, dV_g. \tag{3.65}$$

where χ_b is defined in (3.35).

Since $\tilde{Z}$ is a compact subset of $\Sigma^+ \setminus \Theta_\lambda$, we can claim the following lemma.

Lemma 3.2.11. *Let $\lambda \in (8\pi, 16\pi)$ and let $\tilde{Z}$ be a compact subset of $\Sigma^+ \setminus \Theta_\lambda$, then*

(i) *given any $L > 0$, there exist a small b and a large $\mu(L)$ such that, for any $\mu \geq \mu(L)$, $\tilde{\varphi}_{\mu,z} \in \bar{X}$ and $J_\lambda(\tilde{\varphi}_{\mu,z}) < -L$ for any $z \in \tilde{Z}$;*

(ii) *for any $z \in \tilde{Z}$,*

$$\frac{\tilde{K}^+ e^{\tilde{\varphi}_{\mu,z}}}{\int_\Sigma \tilde{K}^+ e^{\tilde{\varphi}_{\mu,z}} dV_g} \rightharpoonup \delta_p \qquad \textit{as } \mu \to +\infty.$$

Proof. See Appendix. □

3.2.2 Proof of Theorems 0.4.3,0.4.4

The compactness result stated in Theorem 0.4.2, combined with the arguments in [87], allows us to prove the next alternative bypassing the Palais-Smale condition, which is not known for the functional J_λ.

Lemma 3.2.12. *Let $\lambda \notin \Lambda_\ell$ and assume (H1), (H2). If J_λ has no critical levels inside some interval $[a, b]$, then J_λ^a is a deformation retract of J_λ^b.*

Remark 3.2.13. *Actually the deformation lemma in [87] is originally proved for the regular case and for K positive, but it adapts in a straightforward way to the singular one, even for K sign changing.*

Indeed, in the latter case since J_λ decreases along the flow and $J_\lambda(u) \to +\infty$ as u approaches the boundary of $\bar{X}$, we have that $\bar{X}$ is positively invariant under the flow.

In turn, since Theorem 0.4.2 implies that the functional J_λ stays uniformly bounded on the solutions of (3.1), the above Lemma can be used to show that it is possible to retract the whole space $\bar{X}$ onto a high sublevel J_λ^b (see [89, Corollary 2.8], also for this issue minor changes are required).

Lemma 3.2.14. *Let $\lambda \notin \Lambda_\ell$ and assume (H1), (H2). If $b > 0$ is sufficiently large, the sublevel J_λ^b is a retract of $\bar{X}$ and hence is contractible.*

In order to show the contractibility of $\bar{X}$ it suffices to fix $v \in \bar{X}$ we can construct a map $g : \bar{X} \times [0,1] \to \bar{X}$, defined as $g([u,t]) = \log(tv + (1-t)u)$. In this way, we can retract $\bar{X}$ to the function v.

Let us introduce some notations in order to unify the proofs of Theorem 0.4.3 and 0.4.4. Define

$$\mathcal{Z} = \begin{cases} Bar_k(Z) & \text{if } \lambda \in (8\pi k, 8\pi(k+1)),\ k \geq 2, \\ Bar_1(\tilde{Z}) & \text{if } \lambda \in (8\pi, 16\pi), \end{cases}$$

where Z and $\tilde{Z}$ are defined in (3.32) and in (3.64) respectively.

Moreover, we set

$$\omega \in \begin{cases} Bar_k(Z) & \text{if } \lambda \in (8\pi k, 8\pi(k+1)),\ k \geq 2, \\ Bar_1(\tilde{Z}) & \text{if } \lambda \in (8\pi, 16\pi), \end{cases}$$

$$\Phi_\mu(\omega) = \begin{cases} \varphi_{\mu,\sigma} & \text{if } \lambda \in (8\pi k, 8\pi(k+1)),\ k \geq 2, \\ \tilde{\varphi}_{\mu,z} & \text{if } \lambda \in (8\pi, 16\pi), \end{cases}$$

where $\varphi_{\mu,\sigma}$ is defined in (3.36) and $\varphi_{\mu,z}$ in (3.65).

We have already defined a couple of maps such that

$$\mathcal{Z} \xrightarrow{\Phi_\mu} J^{-L} \xrightarrow{\Psi} \mathcal{Z},$$

is homotopically equivalent to the identity. Recall that Ψ is defined in Proposition 3.2.1 for $k \geq 2$ and Proposition 3.2.9 for $k = 1$.

Proposition 3.2.15. *If $\alpha_1, \ldots, \alpha_\ell > 0$ and under assumptions (H1), (H2) and (H3) or (H4), for any $\lambda \in (8k\pi, 8(k+1)\pi)$, then the composition $\Psi \circ \Phi_\mu|_{\mathcal{Z}}$ is homotopically equivalent to the identity map. Moreover, $\Phi_\mu(\mathcal{Z})$ is not contractible in J_λ^{-L} for L sufficiently large.*

Proof. Let us introduce the homotopy

$$\begin{aligned} H : [0,1] \times \mathcal{Z} &\longrightarrow \mathcal{Z} \\ (t, \omega) &\longmapsto H(t,\omega) = \Psi \circ \Phi_{\mu(t)}(\omega), \end{aligned}$$

where $\mu(0) = \mu$ for some $\mu > 0$ large enough and $\mu(t)$ is an increasing continuous function with $\mu(t) \to +\infty$ as $t \to 1$.

Combining Lemma 3.2.4 with Proposition 3.2.1 or Lemma 3.2.11 with Proposition 3.2.9 we obtain that

$$H(t,\omega) \longrightarrow \omega \qquad \text{as } t \to +\infty,$$

so H realizes the desired homotopy equivalence.

In turn, by virtue of assumption (H3) or (H4), $\mathcal{Z}$ is not contractible, see Remark 3.2.2 and Remark 3.2.10. The above assertion implies easily that $\Phi_\mu(\mathcal{Z})$ is also not contractible. □

This, together with Lemma 3.2.12 and Lemma 3.2.14, allows us to conclude the proofs of Theorem 0.4.3 and Theorem 0.4.4.

3.3 Two multiplicity results

In this section we present two results which deal with the multiplicity of solutions for the singular mean field problem. More precisely, we are able to estimate the number of solutions under the setting of the existence theorems. These results are valid under a nondegeneracy assumption on the solutions of (3.1). Due to a transversality argument, for a generic choice of K and g, any solution of (3.1) is non degenerate. In that way, we show that the Morse inequalities can be applied to estimate the number of solutions of (3.1). These inequalities, together with the topological description of the sublevels of J_λ, motivates the computation of the Betti numbers (namely, the dimension of the homology groups) of the spaces $Bar_k(Z), Bar_k(\tilde{Z})$ introduced in the Section 3.2. Therefore, we will see how the number of solutions for the mean field problem depends strongly on the topology of Σ^+ and the singularities p_i's.

Let us define $\mathcal{M}$ as the space of all Riemannian metrics on Σ equipped with the $\mathcal{C}^{2,\alpha}$ norm and

$$\mathcal{K}_\ell = \left\{ K : \Sigma \to \mathbb{R} : \begin{array}{l} K \text{ satisfies (H1), (H2)} \\ K(p_i) > 0 \text{ for } i \leq \ell,\ K(p_i) < 0 \text{ for } i \geq \ell + 1 \end{array} \right\}, \tag{3.66}$$

also equipped with the $C^{2,\alpha}$ norm.

Theorem 3.3.1. *Let $\ell \in \{0, \dots, m\}$ and let us assume that $\alpha_1, \dots, \alpha_\ell > 0$. If $\lambda \in (8k\pi, 8(k+1)\pi) \setminus \Lambda_\ell$, $k \in \mathbb{N}$, then for a generic choice of function K and metric g (namely for (K, g) in an open and dense subset of $\mathcal{K}_\ell \times \mathcal{M}$), then*

$$\#\{\textit{solutions of } (3.1)\} \geq \sum_{q \geq 0} d_q,$$

where if $\quad k + 1 - M \leq N$, then

$$d_q = \begin{cases} \binom{N+M-1}{N+M-p} \displaystyle\sum_{\substack{a_1 + \dots + a_N = k-p+1 \\ a_i \geq 0}} s_{a_1,g_1} \dots s_{a_N,g_N} & \textit{if } \ q = 2k - p \ \ (1 \leq p \leq k+1), \\ 0 & \textit{otherwise;} \end{cases}$$

while if $\quad k + 1 - M \geq N$, then

$$d_q = \begin{cases} \binom{N+M-1}{N+M-p} \displaystyle\sum_{\substack{a_1 + \dots + a_N = k-p+1 \\ a_i \geq 0}} s_{a_1,g_1} \dots s_{a_N,g_N} & \textit{if } \ q = 2k - p \ \ (1 \leq p \leq N), \\ \binom{N+M-s}{M-s} \displaystyle\sum_{\substack{a_1 + \dots + a_N = k-N-s+1 \\ a_i \geq 0}} s_{a_1,g_1} \dots s_{a_N,g_N} & \textit{if } \ q = 2k - N - s \ \ (1 \leq s \leq M), \\ 0 & \textit{otherwise;} \end{cases}$$

with $s_{a,g} = \binom{a+g-1}{g-1}$ and g_i defined in (3.31).

Moreover we adopt the following convention: if $N = 0$

$$\sum_{\substack{a_1+\ldots+a_N=h \\ a_i\geq 0}} s_{a_1,g_1}\ldots s_{a_N,g_N} = \begin{cases} 1 & \text{if } h=0, \\ 0 & \text{if } h\neq 0. \end{cases}$$

Notice that if $k+1-M=N$ the two formulas coincide.

We point out that $\#\{\text{solutions of } (3.1)\} \to +\infty$ as $N^+ = N+M \to +\infty$. Moreover, observe that if $K>0$ then $N^+=1$ and the above formula coincides with that given by Bartolucci, De Marchis and Malchiodi in [5].

The above result gives no information if Σ^+ has trivial topology; however, our second multiplicity result can be applied also in this case.

Theorem 3.3.2. *Let $\ell \in \{0,\ldots,m\}$ and let us assume that $\alpha_1,\ldots,\alpha_\ell > 0$. If $\lambda \in (8\pi, 16\pi)\setminus\Lambda_\ell$, then for a generic choice of function K and metric g (namely for (K,g) in an open and dense set of $\mathcal{K}_\ell\times\mathcal{M}$), then*

$$\#\{\text{solutions of } (3.1)\} \geq N^+ - 1 + \sum_{i=1}^{N} g_i + |\Theta_\lambda|,$$

where the set Θ_λ is defined in (3.2) and g_i in (3.31).

3.3.1 Morse inequalities for J_λ

The aim of this subsection is to prove a Morse-theoretical result for J_λ, which will be crucial to get the multiplicity estimates of Theorem 3.3.1 and Theorem 3.3.2.

Proposition 3.3.3. *Let $\ell \in \{0,\ldots,m\}$ and let us assume $\alpha_1,\ldots,\alpha_\ell > 0$. If $\lambda \in (8\pi,+\infty)\setminus\Lambda_\ell$, then for a generic choice of the function K, g (namely for (K,g) in an open and dense subset of $\mathcal{K}_\ell\times\mathcal{M}$)*

$$\text{any solution of } (3.1) \text{ is non degenerate,}$$

where to emphasize the dependence on K and g we write

$$\bar{X}_{K,g} = \{u \in H^1_g(\Sigma) \,:\, \int_\Sigma u dV_g = 0, \; \int_\Sigma K e^{-h_m} e^u dV_g > 0\}$$

with h_m defined in (0.8).

Proof. Let us fix $(\bar{K}, \bar{g}) \in \mathcal{K}_\ell \times \mathcal{M}$.

Next, we introduce the space $\mathcal{S}$ of all $C^{2,\alpha}$ symmetric matrices on Σ. $\mathcal{S}$ is a Banach space endowed with the $C^{2,\alpha}$ norm,

The set $\mathcal{M}$ of all $C^{2,\alpha}$ Riemannian metrics on Σ is an open subset of $\mathcal{S}$.

It is easy to verify that for small $\delta > 0$, and any $g \in \mathcal{G}_\delta := \{g \in \mathcal{S} \,:\, \|g\|_2 < \delta\}$, $\bar{g} + g$ is a Riemannian metric and the sets $H^1_{\bar{g}+g}(\Sigma)$, $L^2_{\bar{g}+g}(\Sigma)$, $L^1_{\bar{g}+g}(\Sigma)$ coincide respectively with $H^1_g(\Sigma)$, $L^2_g(\Sigma)$, $L^1_g(\Sigma)$ and the two norms are equivalent.

Being $\bar{K} \in \mathcal{K}_\ell$, it satisfies (H1), (H2). Thus, it is not hard to see that for $\delta > 0$ small enough $\bar{K} + K$ satisfies (H1), (H2) for any $K \in \mathcal{H}_\delta := \{h \in C^{2,\alpha}(\Sigma) \,:\, \|h\|_{C^{2,\alpha}(\Sigma)} < \delta\}$.

Furthermore, by Theorem 0.4.2, it suffices to take a smaller δ so that there exists $R > 0$ such that for any $(K, g) \in \mathcal{H}_\delta \times \mathcal{G}_\delta$ all the critical points (with zero mean value) of $I_{\lambda, \bar{K}+K, \bar{g}+g}$ are contained in the ball $\mathcal{B} = B_0(R) \subset H^1_{\bar{g}}(\Sigma)$.

Finally, let us introduce the map $F : \mathcal{H}_\delta \times \mathcal{G}_\delta \times \mathcal{B} \to \bar{X}_{K,g}$ as

$$F(K, g, u) = S_g^{-1}(\tilde{F}_g(K, S_g(u))), \tag{3.67}$$

where

$$\begin{array}{llll} \tilde{F}_g : & H_\delta \times \bar{X}_{K,\tilde{g}+g} & \longrightarrow & \bar{X}_{K,\tilde{g}+g} \\ & (K, w) & \mapsto & w - A_g\left(\lambda \frac{(\bar{K}+K)e^{-h_m}e^w}{\int_\Sigma (\bar{K}+K)e^{-h_m}e^w dV_{\tilde{g}+g}} - \frac{\lambda}{\int_\Sigma dV_{\tilde{g}+g}} + w\right). \end{array}$$

Here $S_g : \bar{X}_{K,g} \to \bar{X}_{K,\tilde{g}+g}$ defined as $S_g(u) = u - \fint u \, dV_{\tilde{g}+g}$; whereas A_g is the linear operator such that

$$(A_g u, v)_{H^1_{\tilde{g}+g}} = (u, v)_{L^2_{\tilde{g}+g}(\Sigma)} \quad \forall v \in H^1_{\tilde{g}+g}, \forall u \in L^2_g(\Sigma).$$

Indeed, A_g is the adjoint operator $i^*_{g+\tilde{g}}$ of the compact embedding $i_{g+\tilde{g}} : \bar{X}_{K,\tilde{g}+g} \to$

$L^2_{\tilde{g}+g}(\Sigma)$. By integrating by parts, one can see that the main term of the explicit expression of A_g is the inverse of the laplacian operator. Since the operator A_g is of class C^1, then the map F also is.

It is not difficult to check that u is a critical point of $J_{\lambda,K+\tilde{K},g+\tilde{h}}$ if and only if $(K,g,u) \in \mathcal{H}_\delta \times \mathcal{G}_\delta \times \mathcal{B}$ are such that $F(K,g,u) = 0$.

Once δ is fixed, by the computations of [46], the following results hold:

i) For any $(K,g) \in \mathcal{H}_\delta \times \mathcal{G}_\delta$ and $u \in \mathcal{B}$, the map $u \mapsto F(K,g,u)$ is Frendholm of index 0.

ii) The set

$$\{u \in \mathcal{B} : F(K_0, g_0, u) = 0,\ (K_0, g_0) \text{ belongs to a compact subset of } \mathcal{H}_\delta \times \mathcal{G}_\delta\}$$

is relatively compact in $\mathcal{B}$.

iii) Given $(K_0, g_0, u_0) \in \mathcal{H}_\delta \times \mathcal{G}_\delta \times \mathcal{B}$ such that $F(K_0, g_0, u_0) = 0$ and for any $v \in \bar{X}_{K,g}$ there exists $(K_v, g_v, u_v) \in \mathcal{C}^{2,\alpha}(\Sigma) \times \mathcal{M} \times \bar{X}_{K,g}$ such that

$$\tilde{F}'_{(K,g)}(K_0, g_0, u_0)[K_v, g_v] + \tilde{F}'_u(K_0, g_0, u_0)[u_v] = v.$$

Now, let us recall the following transversality theorem, given in [106] for instance.

Theorem 3.3.4. *Let X, Y, Z be three real Banach spaces and let $U \subset X$, $V \subset Y$ be open subsets. Let $F : V \times U \to Z$ be a map of class $\mathcal{C}^k$ with $k \geq 1$ such that it hold the folling statemnts*

1. *for any $y \in V$, $F(y,\cdot) : x \mapsto F(y,x)$ is a Fredholm map of index l with $l \leq k$;*

2. *z_0 is a regular value of F, that is the operator $F'(y_0, x_0) : Y \times X \to Z$ onto at any point (x_0, y_0) such that $F(y_0, x_0) = z_0$;*

3. *the set of $x \in U$ such that $F(y_0, x_0) = z_0$ with y in a compact set of V is relatively compact in U.*

Then the set $\{y \in V : z_0$ is a regular value of $F(y,\cdot)\}$ is a dense open subset of V.

By i)-iii), if we take as F the map defined (3.67), and set $X = Z = \bar{X}_{K,g}, Y = \mathcal{M} \times \mathcal{C}^{2,\alpha}(\Sigma), V = H_\delta \times \mathcal{G}_\delta, U = \mathcal{B}$ and $z_0 = 0$, hypothesis of Theorem 3.3.4 are satisfied. Therefore, we deduce that the following set is an open and dense subset of $\mathcal{H}_\delta \times \mathcal{G}_\delta$

$$\left\{ (K,g) \in \mathcal{H}_\delta \times \mathcal{G}_\delta \ : \ \begin{array}{l} \text{any } u \in J^b_{\lambda,\bar{K}+K,\bar{g}+g} \text{ solution of the equation} \\ -\Delta_{\bar{g}+g} u = \lambda \left(\frac{(\bar{K}+K)e^{-h_m}e^u}{\int_\Sigma (\bar{K}+K)e^{-h_m}e^u dV_{\bar{g}+g}} - \frac{1}{\int_\Sigma dV_{\bar{g}+g}} \right) \quad \text{is non degenerate} \end{array} \right\}.$$

Since this holds for any choice of $(\bar{K}, \bar{g})$ the thesis follows.

□

As recalled in the previous section we do not know whether J_λ satisfies the (PS) condition or not, thus Theorem 1.2.1 cannot be directly applied. However, as already pointed out in [5], the (PS)-condition can be replaced by the request that appropriate deformation lemmas hold for the functional.

In particular, a flow defined by Malchiodi in [89] allows to adapt to J_λ the classical deformation lemmas [19, Lemma 3.2 and Theorem 3.2] needed so that Theorem 1.2.1 can be applied for $H = \bar{X}$ and $I = J_\lambda$. It is worth to point out that, even if the flow is defined for K positive, arguing as in Remark 3.2.13 it is not hard to check that the same construction applies also in the sign changing case.

In conclusion, under the assumptions of Theorem 3.3.1, let a, b be regular values of J_λ and assume that all the critical points in $\{a \leq J_\lambda \leq b\}$ are non degenerate, then

$$\#\{\text{critical points of } J_\lambda \text{ in } \{a \leq J_\lambda \leq b\}\} \geq \sum_{q \geq 0} \dim(H_q(J^b_\lambda, J^a_\lambda)), \tag{3.68}$$

for any $q \geq 0$.

By the topological description of the sublevels of J_λ^{-L}, we can estimate this homology groups (see subsetcion 0.4.3).

Proposition 3.3.5. *Let $\lambda \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$, and assume (H1), (H2).*

If $b > 0$ is such that J^b_λ is contractible, then there exists $L > 0$ sufficiently large so that

$$\#\{\textit{solutions of } (3.1)\ \} \geq \sum_{q\geq 0} \dim(H_{q+1}(J_\lambda^b, J_\lambda^{-L})) \geq \sum_{q\geq 0} \dim(\tilde{H}_q(Bar_k(Z))),$$

where Z is defined in (3.32).

Proof. By assumption J_λ^b is contractible thus, from the exactness of the homology sequence,

$$\ldots \to \tilde{H}_{q+1}(J_\lambda^{-L}) \to \tilde{H}_{q+1}(J_\lambda^b) \to H_{q+1}(J_\lambda^b, J_\lambda^{-L}) \to \tilde{H}_q(J_\lambda^{-L}) \to \ldots$$

we derive that

$$\begin{aligned} &H_{q+1}(J_\lambda^b, J_\lambda^{-L}) \cong \tilde{H}_q(J_\lambda^{-L}), &&\text{for any } q \geq 0,\\ &H_0(J_\lambda^b, J_\lambda^{-L}) = 0. \end{aligned}$$

Since $\Psi_* \circ \Phi_* = \mathrm{Id}_{|H_*(Bar_k(Z))}$, and so

$$\dim(\tilde{H}_q(J_\lambda^{-L})) \geq \dim(\tilde{H}_q(Bar_k(Z))).$$

The inequality (3.68) completes the proof. □

Proposition 3.3.6. *Let $\lambda \in (8\pi, 16\pi)$ and assume (H1), (H2). If $b > 0$ is such that J_λ^b is contractibe, then there exists $L > 0$ sufficiently large so that*

$$\#\{\textit{solutions of } (3.1)\ \} \geq \sum_{q\geq 0} \dim(H_{q+1}(J_\lambda^b, J_\lambda^{-L})) \geq \sum_{q\geq 0} \dim(\tilde{H}_q(Bar_1(\tilde{Z})))$$

where $\tilde{Z}$ is defined in (3.64).

Proof. The proof is completely analogous to the one of the previous proposition, where $\tilde{Z}$ and $\varphi_{\mu,z}$ play the role of Z and $\varphi_{\mu,\sigma}$ respectively, while Proposition 3.2.9 and Lemma 3.2.11 must be applied instead of Proposition 3.2.1 and Lemma 3.2.4. □

3.3.2 Proof of the multiplicity results

This subsection is dedicated to prove Theorem 3.3.1 and Theorem 3.3.2. In view of Proposition 3.3.5 and Proposition 3.3.6, we just need to compute the dimension of $Bar_k(Z)$, $Bar_1(\tilde{Z})$ introduced in the previous section.

We set

$$d_q(k,N,M) = \dim(\tilde{H}_q(Bar_k(Z))), \tag{3.69}$$

with the convention that $d_q(k,N,M) = 0$ if $q < 0$.

Proposition 3.3.7. *Let $k \in \mathbb{N}$, $N, M \in \mathbb{N} \cup \{0\}$, with $N + M \geq 1$, and $q \in \mathbb{N} \cup \{0\}$, then*

if $\quad k + 1 - M \leq N$,

$d_q(k,N,M) =$

$$\begin{cases} \binom{N+M-1}{N+M-p} \displaystyle\sum_{\substack{a_1+\ldots+a_N = k-p+1 \\ a_i \geq 0}} s_{a_1,g_1} \ldots s_{a_N,g_N} & \textit{if } \; q = 2k-p \;\; (1 \leq p \leq k+1), \\ 0 & \textit{otherwise;} \end{cases}$$

if $\quad k + 1 - M \geq N$,

$d_q(k,N,M) =$

$$\begin{cases} \binom{N+M-1}{N+M-p} \displaystyle\sum_{\substack{a_1+\ldots+a_N = k-p+1 \\ a_i \geq 0}} s_{a_1,g_1} \ldots s_{a_N,g_N} & \textit{if } \; q = 2k-p \;\; (1 \leq p \leq N), \\ \binom{N+M-s}{M-s} \displaystyle\sum_{\substack{a_1+\ldots+a_N = k-N-s+1 \\ a_i \geq 0}} s_{a_1,g_1} \ldots s_{a_N,g_N} & \textit{if } \; q = 2k-N-s \;\; (1 \leq s \leq M), \\ 0 & \textit{otherwise;} \end{cases}$$

where $s_{a,g} = \binom{a+g-1}{g-1}$ and g_i is defined in (3.31).

Moreover we adopt the following convention: if $N = 0$

$$\sum_{\substack{a_1+\ldots+a_N = h \\ a_i \geq 0}} s_{a_1,g_1} \ldots s_{a_N,g_N} = \begin{cases} 1 & \textit{if } h = 0, \\ 0 & \textit{if } h \neq 0. \end{cases}$$

Notice that if $k+1-M=N$ the two formulas coincide.

Proof.

Step 1. The thesis holds true if $k=1$ or $N=0$. If $k=1$, $Bar_1(Z)\cong Z$ and so by direct computation we have:

$$d_q(1,N,M)=\begin{cases}\sum_{i=1}^{N} g_i\ (=\sum_{i=1}^{N} s_{1,g_i}) & \text{if } \ q=1,\\ N+M-1 & \text{if } \ q=0,\\ 0 & \text{otherwise.}\end{cases}$$

If $N=0$, $Bar_k(Z_{0,M})$ is the $(k-1)$-skeleton of a $(M-1)$-symplex and so the following formula holds

$$d_q(k,0,M)=\begin{cases}\binom{M-1}{k} & \text{if } \ q=k-1,\\ 0 & \text{otherwise,}\end{cases}$$

where we adopt the convention that $\binom{a}{b}=0$ if $a<b$.

Step 2. The thesis holds true if $M=0$ for any $k\geq 2$, $N\geq 1$: that is,

$$d_q(k,N,0)=\begin{cases}\binom{N-1}{N-p}\displaystyle\sum_{\substack{a_1+\ldots+a_N=k-p+1\\ a_i\geq 0}} s_{a_1,g_1}\ldots s_{a_N,g_N} & \text{if } \ q=2k-p \ \ (1\leq p\leq\min\{k+1,N\}),\\ 0 & \text{otherwise.}\end{cases}\tag{3.70}$$

We will demonstrate (3.70) by induction on N, for any fixed $k\geq 2$.

If $N=0$, the formula holds by Step 1. Now, assume by induction that (3.70) holds true for a certain N and let us show its validity for $N+1$. Being

$$X_{N+1}=X_N\amalg B^{g_{N+1}},$$

by Proposition 1.2.2, (1.30) and (1.32) we get

$$d_q(k, N+1, 0) =$$
$$d_q(k, N, 0) + d_{q-1}(k-1, N, 0) + \dim(\tilde{H}_q(Bar_k(B^{g_{N+1}}))) + \dim(\tilde{H}_{q-1}(Bar_{k-1}(B^{g_{N+1}}))) +$$
$$\sum_{\ell=1}^{k-1} \dim(\tilde{H}_q(Bar_{k-\ell}(X_N) * Bar_\ell(B^{g_{N+1}}))) + \sum_{\ell=2}^{k-1} \dim(\tilde{H}_q(\Sigma Bar_{k-\ell}(X_N) * Bar_{\ell-1}(B^{g_{N+1}}))), \tag{3.71}$$

Let us compute all the terms in (3.71).

The first two can be obtained using the inductive assumption. Next, again by the computations in [5, Proposition 3.2], we know that

$$\dim(\tilde{H}_q(Bar_k(B^{g_{N+1}}))) = \begin{cases} s_{a_{N+1}, g_{N+1}} & \text{if } \ q = 2k-1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.72}$$

and so

$$\dim(\tilde{H}_{q-1}(Bar_{k-1}(B^{g_{N+1}}))) = \begin{cases} s_{a_{N+1}, g_{N+1}} & \text{if } \ q = 2k-2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.73}$$

Moreover, by (1.33), using (3.72) and the inductive assumption we have that

$$\sum_{\ell=1}^{k-1} \dim(\tilde{H}_q(Bar_{k-\ell}(X_N) * Bar_\ell(B^{g_{N+1}}))) =$$
$$= \begin{cases} \binom{N-1}{N-p} \displaystyle\sum_{\substack{a_1 + \ldots + a_N + \ell = k-p+1 \\ a_i \geq 0, \ \ell \geq 1}} s_{a_1, g_1} \ldots s_{a_N, g_N} \, s_{a_{N+1}, \ell} & \text{if } \ q = 2k-p \ \ (1 \leq p \leq N), \\ 0 & \text{otherwise,} \end{cases} \tag{3.74}$$

and

$$\sum_{\ell=2}^{k-1}\dim(\tilde{H}_q(\Sigma Bar_{k-\ell}(X_N)*Bar_{\ell-1}(B^{g_{N+1}})))=$$

$$\begin{cases}\displaystyle\binom{N-1}{N-p+1}\sum_{\substack{a_1+\ldots+a_N+\ell=k-p+1\\ a_i\geq 0,\ \ell\geq 2}} s_{a_1,g_1}\ldots s_{a_N,g_N}\, s_{a_{N+1},\ell-1} & \text{if } \ q=2k-p\ \ (2\leq p\leq N+1),\\ 0 & \text{otherwise.}\end{cases} \tag{3.75}$$

In conclusion, combining (3.71), (3.72), (3.73), (3.74) and (3.75) we obtain that

$d_q(k,N+1,0)=$

$$\begin{cases}\displaystyle\binom{N}{N+1-p}\sum_{\substack{a_1+\ldots+a_{N+1}=k-p+1\\ a_i\geq 0}} s_{a_1,g_1}\ldots s_{a_{N+1},g_{N+1}} & \text{if } \ q=2k-p\ \ (1\leq p\leq \min k+1,N+1),\\ 0 & \text{otherwise,}\end{cases}$$

so (3.70) holds true for $N+1$ and this completes the proof of (3.70).

Step 3. Conclusion.

We will prove the formula by induction on M, with $k\geq 2$ and $N\geq 1$ fixed.

If $M=0$ the thesis is true by *Step* 2. Now, let us suppose that (3.70) holds for M and we prove that then it is also true for $M+1$. Being

$$Z_{N,M+1}=Z\amalg\{y_{M+1}\},$$

and $\tilde{H}_*(Bar_k(\{y_{M+1}\}))=0$, by (1.30) and (1.32) we get

$$d_q(k,N,M+1)=d_q(k,N,M)+d_{q-1}(k-1,N,M).$$

Hence by the inductive assumption we can compute $d_q(k,N,M+1)$, obtaining that

if $\quad k+1-(M+1)\leq N,$

$d_q(k,N,M+1)=$

$$\begin{cases}\displaystyle\binom{N+(M+1)-1}{N+(M+1)-p}\sum_{\substack{a_1+\ldots+a_N=k-p+1\\ a_i\geq 0}} s_{a_1,g_1}\ldots s_{a_N,g_N} & \text{if } \ q=2k-p\ \ (1\leq p\leq k+1),\\ 0 & \text{otherwise;}\end{cases}$$

if $\quad k+1-(M+1) \geq N,$

$$d_q(k,N,M+1) =$$

$$\begin{cases} \binom{N+(M+1)-1}{N+(M+1)-p} \displaystyle\sum_{\substack{a_1+\ldots+a_N=k-p+1 \\ a_i \geq 0}} s_{a_1,g_1}\ldots s_{a_N,g_N} & \text{if } \ q=2k-p \ \ (1 \leq p \leq N), \\ \binom{N+(M+1)-s}{(M+1)-s} \displaystyle\sum_{\substack{a_1+\ldots+a_N=k-N-s+1 \\ a_i \geq 0}} s_{a_1,g_1}\ldots s_{a_N,g_N} & \text{if } \ q=2k-N-s \ \ (1 \leq s \leq M+1), \\ 0 & \text{otherwise.} \end{cases}$$

So the formula holds for $M+1$ and this concludes the proof. □

Lemma 3.3.8. *Let N, $M \in \mathbb{N} \cup \{0\}$, $N+M \geq 1$ and let $\tilde{Z}$ be the set defined in* (3.64)*, then*

$$\dim(\tilde{H}_q(Bar_1(\tilde{Z}))) = \begin{cases} N+M-1 & q=0, \\ \sum_{i=1}^N g_i + |\Theta_\lambda| & q=1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Being $Bar_1(\tilde{Z}) \cong \tilde{Z}$ it is immediate to see that

$$\dim(\tilde{H}_q(\tilde{Z})) = \begin{cases} N+M-1 & q=0, \\ \sum_{i=1}^N (g_i + |\Theta_{\lambda,A_i}|) + \sum_{h=1}^M |\Theta_{\lambda,C_h}| & q=1, \\ 0 & \text{otherwise,} \end{cases}$$

hence the thesis follows observing that

$$\sum_{i=1}^N (g_i + |\Theta_{\lambda,A_i}|) + \sum_{h=1}^M |\Theta_{\lambda,C_h}| = |\Theta_\lambda|,$$

where Θ_{λ,A_i} and Θ_{λ,C_h} are defined in (3.63). □

Proof of Theorem 3.3.1 is immediate from Proposition 3.3.3 and Proposition 3.3.5, which allow us to use the computation of Proposition 3.3.7; whereas Theorem 3.3.2 is deduced from Proposition 3.3.6 and Lemma 3.3.8 instead.

3.4 Proof of the non-existence theorem

In this section we will prove the existence of a family of functions K for which problem (3.1) does not admit a solution. First of all, let us introduce a non-existence result for the problem

$$-\Delta u = R(x)e^u \quad \text{in } \mathbb{R}^2, \tag{3.76}$$

where R is a sign changing function. In [30], Chen and Li deduce that there is no solution for the *regular* Nirenberg problem (equation (0.4) with $\lambda = 4\pi$ and $\Sigma = \mathbb{S}^2$) if K is axially symmetric, sign changing and monotone in the region where K is positive. In order to do it, the authors performed a stereographic projection to transform the equation (0.4) into (3.76). In particular, for that case, the solutions behave at infinity as

$$u \sim -4\log|x|, \tag{3.77}$$

and that R has a finite limit as $|x| \to +\infty$.

Actually, we will show that this approach, with proper modifications, allows us to deal with solutions of (3.76) under the less restrictive assumption

$$u \sim -\eta\log|x|, \tag{3.78}$$

for some $\eta > 4$ and if the function R satisfies

$$\lim_{|x|\to+\infty} R(x)|x|^{4-\eta} \in (0, +\infty). \tag{3.79}$$

Let $r_0 > 0$ and $R \in C^0_{rad}(\mathbb{R}^2)$, we assume that

$$R \text{ is positive and non-increasing for } r < r_0, \text{ negative for } r > r_0. \tag{3.80}$$

Theorem 3.4.1. *Assume that $R \in C^0_{rad}(\mathbb{R}^2)$ is a bounded function verifying (3.80). Then there is no solution for the problem (3.76) such that (3.78) holds.*

The key point to derive this generalized result is to modify properly Lemma 2.1

in [30], taking into account the new asymptotic behavior. Applying properly the moving spheres method, we can obtain the following estimate.

Lemma 3.4.2. *Let $R \in C^0_{rad}(\mathbb{R}^2)$ be a bounded function satisfying (3.80). Without loss of generality, we can assume $r_0 = 1$. Let u be a solution of (3.76) such that (3.78) holds, then*

$$u(\mu x) > u\left(\frac{\mu x}{|x|^2}\right) - \eta \log|x| \quad \forall x \in B_0(1), \quad 0 < \mu \leq 1. \tag{3.81}$$

Proof. **Step 1:** We claim that (3.81) is true for $\mu = 1$.

Let $v(x) = u\left(\frac{x}{|x|^2}\right) - \eta \log|x|$, then v verifies

$$-\Delta v = |x|^{\eta-4} R\left(\frac{1}{|x|}\right) e^v.$$

By (3.80), $\Delta u < 0$ and $\Delta v \geq 0$ in $B_0(1)$, then $-\Delta(u - v) > 0$ in $B_0(1)$. Since $u = v$ in $\partial B_0(1)$, then

$$u > v \quad \text{in } B_0(1)$$

by using the maximum principle.

Step 2: At this point, we move $\partial B_0(\mu)$ towards $\mu = 0$. Let $u_\mu(x) = u(\mu x) + 2\log\mu$ and $v_\mu(x) = u_\mu\left(\frac{x}{|x|^2}\right) - \eta \log|x|$, then

$$-\Delta u_\mu = R(\mu|x|)e^{u_\mu}, \quad -\Delta v_\mu = |x|^{\eta-4} R\left(\frac{\mu}{|x|}\right) e^{v_\mu}.$$

Taking the auxiliary function $w_\mu = u_\mu - v_\mu$, we obtain that

$$\Delta w_\mu + |x|^{\eta-4} R\left(\frac{\mu}{|x|}\right) e^{\phi_\mu(x)} w_\mu(x) = \left[R\left(\frac{\mu}{|x|}\right)|x|^{\eta-4} - R(\mu|x|)\right] e^{u_\mu(x)} \tag{3.82}$$

for $x \in B_0(1)$ where ϕ_μ is a function between $u_\mu(x)$ and $v_\mu(x)$. Observe that by (3.80), we have that

$$R\left(\frac{\mu}{|x|}\right)|x|^{\eta-4} - R(\mu|x|) \leq 0, \quad \text{for } |x| \leq 1 \text{ and } \mu \leq 1.$$

Therefore

$$\Delta w_\mu + C_\mu(x) w_\mu \leq 0, \tag{3.83}$$

where $C_\mu(x)$ is a bounded function if μ is bounded away from 0. Moreover the strict inequality in (3.83) holds somewhere. Thus, applying the strong maximum principle, to get (3.81) it is enough to show that

$$w_\mu(x) \geq 0 \quad \text{in } B_0(1). \tag{3.84}$$

From Step 1 (3.84) is true for $\mu = 1$. Next, we decrease μ. By contradiction, suppose that there exists $\mu_0 > 0$ such that (3.84) is true for $\mu \geq \mu_0$ and fails for $\mu < \mu_0$. For $\mu = \mu_0$ we can use the strong maximum principle and then the Hopf lemma in (3.82) to obtain that

$$w_{\mu_0} > 0 \ \text{ in } B_0(1) \quad \text{and} \quad \frac{\partial w_{\mu_0}}{\partial r} < 0 \ \text{ on } \partial B_0(1).$$

In addition, by the minimality of μ_0 for any sequence $\mu_n \nearrow \mu_0$ there exists $x_n \in B_0(1)$ verifying $w_{\mu_n}(x_n) < 0$. This, combined with the fact that $w_{\mu_n} = 0$ on $\partial B_0(1)$, implies that there exists some y_n on the segment connecting x_n and $\frac{x_n}{|x_n|}$ so that $\frac{\partial w_{\mu_n}}{\partial r}(y_n) > 0$. Up to a subsequence $x_n \to x_0 \in \overline{B_0(1)}$ with $w_{\mu_0}(x_0) \leq 0$, so $x_0 \in \partial B_0(1)$ and $y_n \to x_0$. Thus $\frac{\partial w_{\mu_0}}{\partial r}(x_0) \geq 0$ and we get the desired contradiction. Therefore (3.84) holds for any $\mu \in (0, 1]$. □

In this way, the proof of Theorem 3.4.1 is a direct consequence of the previous estimate.

Proof of Theorem 3.4.1. Applying Lemma 3.4.2 we get (3.81). Letting $\mu \to 0$ in (3.81) we obtain that $\log |x| > 0$ for $|x| < 1$ which is a contradiction. □

Now we are ready to prove our non-existence result applying the previous Theorem. Without loss of generality, suppose that $p = (0, 0, 1) \in (\mathbb{S}^2)^+ \setminus p_1$. Let us introduce the set $\mathcal{F} \subset C^0(\mathbb{S}^2)$ defined as

$$\mathcal{F} = \left\{ F \in C^0(\mathbb{S}^2) : \begin{array}{l} F \text{ is sign changing, rotationally symmetric} \\ \text{with respect to } (0,0,1), \text{ monotone in the region} \\ \text{where it is positive and } F(-p) = \max\limits_{\mathbb{S}^2} F \end{array} \right\}. \tag{3.85}$$

Given a function $F \in \mathcal{F}$, defined on (3.85), the strategy is to construct a function K_F defined in $\mathbb{S}^2$ such that the stereographic projection of $\tilde{K} = Ke^{-h_1}$ in $\mathbb{R}^2$ is a radial function satisfying the monotonicity condition (3.80). In this situation, Theorem 3.4.1 applies.

Let us transform problem (3.1) into (3.76). Let π be the stereographic projection from $\mathbb{S}^2 \setminus \{(0,0,1)\}$ to $\mathbb{R}^2$ defined by

$$\pi(x_1, x_2, x_3) = (y_1, y_2), \quad y_i = \frac{x_i}{1 - x_3}, \quad i = 1, 2. \tag{3.86}$$

The inverse map $\pi^{-1} : \mathbb{R}^2 \mapsto \mathbb{S}^2 \setminus \{p\}$ is

$$\pi^{-1}(y_1, y_2) = \frac{1}{1 + |y|^2}(2y_1, 2y_2, |y|^2 - 1). \tag{3.87}$$

For any function ψ on $\mathbb{S}^2$

$$\int_{\mathbb{S}^2} \psi(x) dV_g = \int_{\mathbb{R}^2} \psi(P^{-1}(y)) \frac{4}{(1 + |y|^2)^2} dy.$$

Let u be a solution of (3.1), we introduce the following variable change

$$v(y) = u(\pi^{-1}(y)) + \frac{\lambda}{8\pi} \log \left(\frac{4}{(1 + |y|^2)^2} \right), \tag{3.88}$$

then v verifies

$$-\Delta v = \tilde{K}(\pi^{-1}(y)) f_\lambda(y) e^v \qquad \text{in } \mathbb{R}^2, \tag{3.89}$$

with asymptotic growth at infinity

$$v \sim -\frac{\lambda}{2\pi} \log |y|, \tag{3.90}$$

where

$$f_\lambda(y) = \lambda \left(\frac{4}{(1 + |y|^2)^2} \right)^{1 - \frac{\lambda}{8\pi}}, \tag{3.91}$$

with $\frac{\lambda}{2\pi} > 4$ and let us set $R(y) = \tilde{K}(\pi^{-1}(y))$.

Notice that the assumptions (H1) and (H2) on K guarantee that R satisfies (H1) and (H2) in $\mathbb{R}^2$. Besides, by our choice of p

$$\lim_{|y|\to+\infty} R(y)f_\lambda(y)|y|^{4-\eta} \in (0,+\infty).$$

Proof of Theorem 0.4.6. Let us fix a function $F = F(\varphi)$ in $\mathcal{F}_p$ expressed in spherical coordinates, where without loss of generality we can suppose $p = (0,0,1)$. Let h be the regular part of the function h_1, introduced in (0.8) and define

$$K_F(\varphi) = F(\varphi)e^{h(\varphi)}g_\lambda(\pi(\varphi)) \quad \text{with } \varphi \in (0,\pi]\ , \quad K_F(0) < 0, \tag{3.92}$$

where $\pi : (0,\pi] \to \mathbb{R}^2$ is the stereographic projection of $\mathbb{S}^2$ into $\mathbb{R}^2$ and $g_\lambda(y) = \left(\frac{4}{(1+|y|^2)^2}\right)^{\frac{\lambda}{8\pi}-1}$ for $y \in \mathbb{R}^2$. By (3.92) we have that

$$\tilde{K}_F(\varphi) = K_F(\varphi)e^{-h_1(\varphi)} = F(\varphi)\varphi^{2\alpha}g_\lambda(\pi(\varphi)) \quad \text{with } \varphi \in (0,\pi]\ ,$$

where $\log(\varphi)^{2\alpha}$ corresponds to the singular part of $-h_1$ in spherical coordinates.

Now, as before, by means of the stereographic projection we transform (3.1) into

$$-\Delta v = \hat{K}_{F,\lambda}\, e^v \qquad \text{in } \mathbb{R}^2,$$

where v satisfies (3.90) and

$$\hat{K}_{F,\lambda}(y) = \tilde{K}_F(\pi^{-1}(y))\lambda g_\lambda^{-1}(y) = \lambda F(\pi^{-1}(y))(\pi^{-1}(y))^{2\alpha}$$

is bounded and verifies condition (3.80), being $F \in \mathcal{F}_p$.

To conclude it suffices to apply Theorem 3.4.1 with $R(y) = \hat{K}_{F,\lambda}(y)$. □

3.5 Final remarks and open problems

Remark 3.5.1. *Prompted by our work, D'Aprile, De Marchis and Ianni have recently constructed solutions for the problem* (3.1) *in the sign changing case via perturbative methods. More precisely, the construction assumes that K satisfies (H1),* *(H2), (H3) and* $\lambda = 8k\pi - \varepsilon$ *for some small* $\varepsilon > 0$*, which depends on the function* K *and* Σ*. In addition, another construction is given under the hypotheses of our Theorem 0.4.4 for* $\lambda \in (16\pi - \varepsilon, 16\pi)$*. Morever, this method allows the authors to derive a completely new existence result if (H1), (H2), (H4) hold and* $\lambda = 16\pi + \varepsilon$*, extending the result of Theorem 0.4.4.*

On the other hand, if Σ^+ *is contractible and* $m = 1$*,* $\ell = 0$*, we have showed that for a family of functions* K *the singular mean field equation does not admit a solution. Neverthless, if* $\lambda = 8k\pi - \varepsilon$*, the problem is solvable assuming conditions on the convexitiy and concavity of* K *around local maximas and minimas. For more details, see [44].*

Open Problem 3.5.2. *In this way, it seems to be reasonable to ask if the variational argument could be applied to cover the two new existence results stated in [44], which our theorems are not able to cover, i.e. to analyze the sharpness of the hypotheses (H3) and (H4) for the existence of solutions for problem* (3.1)*.*

Open Problem 3.5.3. *In view of our non-existence result, for axially symmetric function* K*, the fact that* $\tilde{K}'(\varphi)$ *changes sign in* Σ^+ *is necessary for the solvability of* (3.1)*. A natural question is to determine if that condition is also sufficient, as Xu and Yang show for the regular Nirenberg problem (see [119]).*

Remark 3.5.4. *From our compactness result, the sequence* $f_k = \tilde{K}e^{u_k}$ *turns out to be bounded in* $L^1(\Sigma)$*, even in the event of blow-up. It was already known that if* u_n *blows–up, and* f_k *is bounded in* $L^1(\Sigma)$*, then there exists a finite blow–up set with a minimal mass under less restrictive assumptions on* K*, see Chapter 5 of [112]. However, these results do not give any location of the possible blow-up points nor any precise information about the quantization.*

It is worth to point out that the estimates given by Lemma 3.1.3 uses strongly the fact that the problem is defined on a surface without boundary. Since the condition f_k *is bounded in* $L^1(\Sigma)$ *derives strongly from these estimates, our result seems to depend on the fact that* Σ *does not have boundary.*

Open Problem 3.5.5. *A future study could be to establish an analogous compactness theorem for surfaces with boundary. For instance, one can consider the BVP*

$$\begin{cases} -\Delta_g u_n = \tilde{K}_n(x)e^{u_n} - f_n & \text{in } \Omega, \\ B(u_n) = 0 & \text{on } \partial\Omega, \end{cases}$$

where B corresponds to either Dirichlet or Neumann boundary conditions. The difficulties to bypass seem to be clear: we do not have any control on the integrals in all the surface and the moving planes method needs to be adapated if $\Gamma \cap \partial\Omega \neq \emptyset$.

Open Problem 3.5.6. *An interesting question is to extend our result for the case in which conical singularities with negative order are included $\alpha_i < 0$. As we have mentioned, Carlotto and Malchiodi have studied the problem for positive potentials K and all negative orders, [20, 21]. They prove that J_λ^{-L} inherits the topology of the following set of formal barycenters*

$$Bar_{k,\lambda}(\Sigma) = \left\{ \sum_{i=1}^{k} t_i \delta_{x_i} \,:\, t_i \in [0,1],\, \sum_{i=1}^{k} t_i = 1,\, x_i \in \Sigma \text{ and } 8\pi \sum_{i=1}^{k} (1+\alpha_i) < \lambda \right\},$$

where $\alpha_i = 0$ if x_i is a regular point. The problem to determine whether $Bar_{k,\lambda}(\Sigma)$ is contractible or not is really delicate. In that sense, some necessary conditions for the contractibility of $Bar_{k,\lambda}(\Sigma)$ are given in [20].

We point out that our compactness theorem can be of use in this case, because it is valid for every $\alpha_i > -1$.

Chapter 4

Conclusions and future perspectives

In this thesis we have dealt with some mean field problems of Liouville type. As we have mentioned, these problems arise in differential geometry and current physical theories. In particular, our contribution is focused on the sign changing potential case, a case which has been little studied. By means of variational methods, in Chapter 2 we have studied the existence of solutions for the mean field equation in a subdomain of the sphere; whereas in Chapter 3 has been considered the presence of conical singularities on compact surfaces.

From our study, we have obtained the following main conclusions:

- ***The crucial role of the topology of the positive components.*** We have determined the existence of critical points of the energy functionals through a topological characterization of their low sublevels. In some cases, we have seen that the low sublevels inherits the topology of a non-contractible compact set. More precisely, in Chapter 2 we showed that the topology of I_λ^{-L} dominates the non-trivial topology of $(\partial\Omega)^+$; whereas J_λ^{-L} has a richer topology than the topology of the barycenters on a compact set $Z \subset \Sigma^+$, see Section 3.2. Under our assumptions, $(\partial\Omega)^+$ and $Bar_k(Z)$ were not contractible and this implied the existence of solutions via suitable variational methods.

  As we can observe, the approach used in this thesis reveals that the existence of solutions for this kind of problems is strongly linked to the topology of the region where K is strictly positive. This crucial idea allowed us to consider accurate assumptions on the sign of K and, jointly with the non-existence result (Theorem 0.4.6), to say that our results are somehow sharp.

This idea has been confirmed by the multiplicity results for (0.15). More precisely, the more involved the topology of Σ^+ is (namely, the more number of connected components, boundaries or holes), the more number of solutions the problem has.

- ***The validity of the compactness result.*** A property that guarantees the compactness of solutions completed our existence results. The problem considered in Chapter 2 has taken profit of energy estimates to deduce compactness. However, in Section 3.1, the compactness of solutions has been presented as a consequence of a new blow–up alternative in spirit of Brezis-Merle or Bartolucci-Tarantello, Theorem 0.4.2.

  Our proof has corrected different unclear deductions and introduced new techniques with respect to previous results, [31–33]. It is strongly inspired in the pioneer work [31] (see also [33]), but the adaptation of these results to the 2-dimensional case is not straightforward. A first attempt is given in [32], but it is restricted to the sphere and contains a gap due to the fact that u is not bounded from below. In our problem the solutions may change sign and the lack of control of their lower bound becomes a serious obstacle. Here, the Kato inequality has appeared as an efficient tool and has allowed us to control the oscillation of the negative part of the solutions.

- ***A new research door.*** The sign changing potential case has opened a new door on the study of Liouville type equations. In order to attack other problems or study qualitative properties, it is required to clarify fundamental issues, such as compactness and multiplicity. Indeed, prompted by our work, D'Aprile, De Marchis and Ianni have recently considered the sign changing case by means of perturbative methods. They are able to show existence and asymptotic behavior of solutions which cover some of our results.

  We hope to obtain extensions of the results presented in this thesis. Some open problems have already been suggested in subsection 2.2 and subsection 3.5 concerning mainly (0.43) and (0.15) or similar versions. However, let us now comment different Liouville-type problems of interest to be studied in the future.

4.1 Gaussian-geodesic prescription problem.

If we consider a Liouville problem in a surface with boundary, then boundary conditions are in order. Homogeneous Dirichlet and Neumann boundary conditions have already been considered in the literature. However, motivated by its geometric meaning, in this proposal we consider a nonlinear boundary condition. It would be interesting to find out if such problem has also a motivation from a physical point of view.

Indeed, our aim is to prescribe not only the Gaussian curvature in Σ, but also the geodesic curvature on $\partial\Sigma$. More precisely, given a metric $\tilde{g} = ge^v$, if $K_g, K_{\tilde{g}}$ are the Gaussian curvatures and $h_g, h_{\tilde{g}}$ the geodesic curvatures of $\partial\Sigma$, relative to these metrics, then v satisfies the boundary value problem

$$\begin{cases} -\Delta_g v + 2K_g = 2K_{\tilde{g}} e^v & \text{in } \Sigma, \\ \frac{\partial v}{\partial n} + 2h_g = 2h_{\tilde{g}} e^{v/2} & \text{on } \partial\Sigma. \end{cases} \tag{4.1}$$

Some versions of this problem have been studied in the literature. For instance, if either $K_{\tilde{g}}$ or $h_{\tilde{g}}$ are equal to 0, some results are available, see for instance [24, 79, 85] and the references therein.

The case of constants $K_{\tilde{g}}$, $h_{\tilde{g}}$ has also been considered. For instance, Brendle ([15]) uses a parabolic flow to show that this problem admits always a solution for some constant curvatures. By using complex analysis techniques, explicit expressions for the solutions and the exact values of the constants are determined if Σ is a disk or an annulus, see [63, 68]. However, the non-constant case has not been much considered. As far as we know, the only available works are [39, 62], which give some partial results.

The case of a disk is particularly challenging, since can be seen as a version of the Nirenberg problem. Also here, the main difficulties are due to the non-compact effect of the group of conformal transformations of the disk.

Integrating (4.1) and applying the Gauss-Bonnet theorem, one obtains

$$\int_\Sigma K_{\tilde{g}} e^v + \int_{\partial\Sigma} h_{\tilde{g}} e^{v/2} = 2\pi\chi(\Sigma).$$

If the curvatures have different sign, for instance, we do not have any control on the boundedness of the integral terms, due to a possible compensation between both terms. In this situation both terms are competing, and the study of their interaction seems a challenging question. Because of that, the study of compactness of solutions needs arguments different from the usual ones. The presence of infinite mass blowing-up solutions would also be an interesting question; this presence is confirmed by the explicit computations given in [63, 68].

In addition, it is not clear what is the natural variational setting for that problem. To find a reasonable energy functional and study its geometrical properties is one of the main tasks of this proposal.

One could also consider the presence of conical singularities in the model. Here everything is to be done, departing from singular versiones of the Moser-Trudinger inequalities with boundary terms.

4.2 Sinh-Gordon and Tzitzéica equations.

Other interesting problem is the following mean field equation

$$-\Delta_g u = \lambda_1 \left(\frac{h_1 e^u}{\int_\Sigma h_1 e^u} - \frac{1}{|\Sigma|} \right) - \alpha\lambda_2 \left(\frac{h_2 e^{-\alpha u}}{\int_\Sigma h_2 e^{-\alpha u}} - \frac{1}{|\Sigma|} \right) \quad \text{in} \quad \Sigma,$$

where α, λ_1, λ_2 are positive parameters. This equation arises in statistical mechanics and studies the 2D-turbulence, as proposed Onsager in [102], and also in Geometry. The problem is the so-called sinh-Gordon equation for $\alpha = 1$, which is related to constant mean curvature surfaces, and *Tzitzéica* equation for $\alpha = 2$, which appears in the context of affine geometry. Under the positivity assumption on h_1, h_2, the existence of solutions by means of variational methods and the compactness of solutions has been studied recently for some cases, ([66, 67]). See also [60, 103, 105]. One could propose to study these problems under the sign changing condition, taking into account the interaction between the nodal regions of the two potentials which makes more difficult the compactness question.

4.3 **Blowing–up solutions.** Let us recall that in Section 3.1 we have been able to exclude blowing-up of solutions at the nodal set $\Gamma = \{x \in \Sigma : \ K(x) = 0\}$, provided that ∇K does not vanish in that set. As already commented, this assumption is not just technical: from [13, 50] we get the existence of solutions blowing up at a point p with $K(p) = 0$, $\nabla K(p) = 0$. Being more specific, in those papers the point p corresponds to a nondegenerate local maximum of K.

However, the asymptotic behavior of the blowing-up solutions at 0 values of K is not completely clear. In [13] Borer, Galimberti and Struwe show, via a blow-up analysis, that there are at most two possible asymptotic profile of the solutions. Del Pino and Román ([50]) are able to construct blowing up solutions of the first type, but it is not clear if the second type can be realized. In the latter case, one ends up with the entire problem:

$$-\Delta w = (2 + HessK(p)(x,x))e^w \quad \text{in} \quad \mathbb{R}^2,$$

whose solutions seem not to be known. The study of this entire problem would be the first step to be made.

Another question of interest is to find out if the bubbling phenomenon appears (either of first or second type) for points p with $K(p) = 0$, $\nabla K(p) = 0$, different from local maxima. In order to build blowing-up solutions the singular perturbation method could be of help, but its application to this problem seems far from trivial, as can be seen from [50].

4.4 **Liouville-type systems.**

Finally, let us focus on some Liouville-type systems. In particular, the Toda system has been extensively studied. Consider the problem

$$\begin{cases} -\Delta_g u = 2\lambda_1 \left(\frac{h_1 e^u}{\int_\Sigma h_1 e^u} - \frac{1}{|\Sigma|} \right) - \lambda_2 \left(\frac{h_2 e^v}{\int_\Sigma h_2 e^v} - \frac{1}{|\Sigma|} \right) & \text{in} \quad \Sigma, \\ -\Delta_g v = 2\lambda_2 \left(\frac{h_2 e^v}{\int_\Sigma h_2 e^v} - \frac{1}{|\Sigma|} \right) - \lambda_1 \left(\frac{h_1 e^u}{\int_\Sigma h_1 e^u} - \frac{1}{|\Sigma|} \right) & \text{in} \quad \Sigma. \end{cases}$$

This problem arises in Geometry and from the non-abelian Chern-Simons theory (see [54, 112]). Again, the known results deal with the case of positive potentials. For instance, since this system admits a variational structure, a very general existence result is stated in [10]. As we have done throughout this thesis, the approach is based on the study of the toology of the low energy sublevels. The compactness question has been treated for the *regular* Toda system in [69]. In this case different asymptotic behavior of blowing–up solutions are possible, see [11, 42, 43, 69, 82, 83, 98], to cite a few. The case of sign changing potentials h_1, h_2 is completely open. Everything it to be done, from the compactness question to the description of the topology of the energy sublevels.

Appendix

In this appendix we include the proof of some asymptotic estimates, which have been omitted before in order to make the reading more fluent.

Concretely, we prove Lemma 2.1.2, Lemma 3.2.4 and Lemma 3.2.11 which show that the energy functionals I_λ and J_λ are not bounded from below for large values of λ. In order to do it, we evaluate the integral terms at the test functions with arbitrary low energy level considered in Chapter 2 and Chapter 3.

Let us first recall the definition of our test functions. For the case studied in Chapter 2, $\Omega \subset \mathbb{S}^2$, consider the test function *concentrated* in a point p of $(\partial\Omega)^+$

$$\varphi_{\mu,p} : \Omega \to \mathbb{R}, \qquad \varphi_{\mu,p}(x) = \log\left(\frac{\mu}{1+\mu^2 d(x,p)^2}\right)^2.$$

For $b > 0$ small enough, let $\chi_b : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth non-decreasing cut-off function such that

$$\chi_b(t) = \begin{cases} t & \text{for } t \in [0,b], \\ 2b & \text{for } t \geq 2b. \end{cases}$$

In the second case, Chapter 3, let Z defined in (3.32), we consider test functions *concentrated* in at most k points of Z. For $\mu > 0$ and $\sigma = \sum_{i=1}^k t_i \delta_{x_i} \in Bar_k(Z)$, we define

$$\phi_{\mu,\sigma} : \Sigma \to \mathbb{R} \qquad \phi_{\mu,\sigma}(x) = \log \sum t_i \left(\frac{\mu}{1+(\mu\chi_b(d(x,x_i)))^2}\right)^2,$$

$$\varphi_{\mu,\sigma}(x) = \phi_{\mu,\sigma}(x) - \int_\Sigma \phi_{\mu,\sigma}\, dV_g.$$

At last, we introduce a test function *concentrated* in one point of the compact set $\tilde{Z}$ defined in (3.64), where Θ_λ is defined in (0.50). Let $\tilde{\alpha} = \max_{n\leq \ell \,|\, p_n \notin \Theta_\lambda} \alpha_n$ or $\tilde{\alpha} = 0$ if $\Theta_\lambda = \{p_1, \dots, p_\ell\}$ or $\ell = 0$. For any $\alpha \in (\tilde{\alpha}, \frac{\lambda}{8\pi} - 1)$, $\mu > 0$ and $z \in \tilde{Z}$, we define

$$\tilde{\phi}_{\mu,z} : \Sigma \to \mathbb{R}, \qquad \tilde{\phi}_{\mu,z}(x) = 2\log\left(\frac{\mu^{1+\alpha}}{1 + (\mu\chi_b(d(x,z)))^{2(1+\alpha)}}\right),$$

$$\tilde{\varphi}_{\mu,z}(x) = \tilde{\phi}_{\mu,z}(x) - \int_\Sigma \tilde{\phi}_{\mu,z}\, dV_g.$$

Let us claim that the following estimates hold

$$\int_\Omega |\nabla\varphi_{\mu,p}|^2\, dV_{g_0} \leq 16\pi \log\mu + O(1); \tag{A.1}$$

$$\int_\Omega \varphi_{\mu,p}\, dV_{g_0} = -\lambda\log\mu + O(1); \tag{A.2}$$

$$\int_\Omega K(x)e^{\varphi_{\mu,p}}\, dV_{g_0} = O(1). \tag{A.3}$$

Proof of Lemma 2.1.2. Combining the above estimates, we obtain that

$$I_\lambda(\varphi_{\mu,p}) \leq 2(4\pi - \lambda)\log\mu + O(1).$$

Since $\lambda \in (4\pi, 8\pi)$, for any $L > 0$, there exits $\mu(L) > 0$ such that $I_\lambda(\varphi_{\mu,p}) < -L$ for every $\mu > \mu(L)$ as it was desired. □

Analogously, we claim that

$$\int_\Sigma |\nabla\phi_{\mu,\sigma}|^2\, dV_g \leq (32k\pi + o_b(1))\log\mu + C_b; \tag{A.4}$$

$$\int_\Sigma \phi_{\mu,\sigma}\, dV_g = -2|\Sigma|\log\mu + O(|\log b|) + O(b^2\log\mu); \tag{A.5}$$

$$\int_\Sigma \tilde{K}(x)e^{\phi_{\mu,\sigma}}\, dV_g = O(1). \tag{A.6}$$

Proof of Lemma 3.2.4. It suffices to apply (A.4), (A.5) and (A.6) to obtain that

$$J_\lambda(\varphi_{\mu,\sigma}) = J_\lambda(\phi_{\mu,\sigma}) \leq 2(8k\pi - \lambda)\log\mu + O(|\log b|) + O(b^2|\log\mu|) + O(1).$$

The last inequality indicates that for every $L > 0$, there exists a value $\mu(L)$ such that $J_\lambda(\varphi_{\mu,\sigma}) < -L$ for $\mu > \mu(L)$ and $\lambda \in (8k\pi, 8(k+1)\pi)$ with $k \in \mathbb{N}$ as we wanted to prove (i).

Now, consider the function

$$\sigma_{\mu,x_i}(x) = \left(\frac{\mu}{1 + (\mu\chi_b(d(x,x_i)))^2}\right)^2, \qquad \text{with } x \in \Sigma,$$

where $x_i \in \Sigma$. It is easy to show that

$$\frac{\sigma_{\mu,x_i}}{\int_\Sigma \sigma_{\mu,x_i} dV_g} \rightharpoonup \delta_{x_i},$$

as $\mu \to +\infty$. Since $e^{\phi_{\mu,\sigma}} = \sum_{i=1}^k t_i\sigma_{\mu,x_i}$ and $\int_\Sigma \sigma_{\mu,x_i}\, dV_g = O(1)$ (as shows (A.5)), statement (ii) is verified.

□

Now, let us state that

$$\int_\Sigma |\nabla\tilde{\phi}_{\mu,z}|^2\, dV_g \leq (32(1+\alpha)^2\pi + o_b(1))\log\mu + C_b; \tag{A.7}$$

$$\int_\Sigma \tilde{\phi}_{\mu,z}\, dV_g = -2(1+\alpha)|\Sigma|\log\mu + O(|\log b|) + O(b^2\log\mu); \tag{A.8}$$

$$\int_\Sigma \tilde{K}(x)e^{\tilde{\phi}_{\mu,z}}\, dV_g \geq \frac{1}{C_b}\mu^{2(\alpha-\alpha_i)} + o_\mu(1), \tag{A.9}$$

where α_i is the order of a singularity $p_i \notin \Theta_\lambda$. If $p_i \in \Theta_\lambda$, $\alpha_i = 0$. Observe that, $\alpha > \alpha_i$ for all $i = 1, \ldots, m$ by the definition of α.

Proof of the statement (i) of Lemma 3.2.11. Employing (A.7), (A.8) and (A.9), it follows that

$$J_\lambda(\tilde\varphi_{\mu,z}) = J_\lambda(\tilde\phi_{\mu,z}) \le 16(1+\alpha)^2\pi\log\mu - 2\lambda(1+\alpha)\log\mu - 2\lambda(\alpha-\alpha_i)\log\mu + O(|\log b|)$$
$$+ O(b^2\log\mu) = 2\left[(1+\alpha)(8\pi(1+\alpha)-\lambda) + \lambda(\alpha_i-\alpha)\right]\log\mu + O(|\log b|) + O(b^2\log\mu).$$

Since $\lambda > 8\pi(1+\alpha_i)$ and $\alpha > \alpha_i$, the statement (i) has been proved. □

Proof of claims (A.1)-(A.9). Given $\sigma = \sum_{i=1}^k t_i\delta_{x_i} \in Bar_k(Z)$, $z \in \tilde Z$, $p \in (\partial\Omega)^+$, let us fix $b > 0$ small enough such that

- K is strictly positive in $B_p(b) \cap \overline{\Omega}$;
- $B_{x_i}(2b) \cap B_{y_j}(2b) = \emptyset$ and $B_{x_i}(2b) \subset \Sigma^+ \setminus \{p_1,\ldots,p_m\}$ for any $i,j \in \{1,\ldots,k\}$ with $i \neq j$;
- $B_z(2b) \subset \Sigma^+ \setminus \Theta_\lambda$.

Now, applying the inequality $|\nabla d(x,z)^2| \le 2d(x,z)$, we have that

$$\begin{aligned}|\nabla\tilde\phi_{\mu,z}(x)| &= 2(1+\alpha)\mu^{2(1+\alpha)}\frac{d(x,z)^{2\alpha}|\nabla d(x,z)^2)|}{1+(\mu d(x,z))^{2(1+\alpha)}}\\ &\le 4(1+\alpha)\mu^{2(1+\alpha)}\frac{d(x,z)^{1+2\alpha}}{1+(\mu d(x,z))^{2(1+\alpha)}} \qquad \text{for every } x \in B_z(b).\end{aligned}$$

Employing this last inequality, we use normal coordinates centered at z and the estimate

$$dV_g = (1+o_b(1))dx, \quad d(z,x) = (1+o_b(1))|x|, \text{ for } x \in B_z(b),$$

to calculate

$$\begin{aligned}\int_{B_z(b)}|\nabla\tilde\phi_{\mu,z}|^2\,dV_g &\le 16(1+\alpha)^2\mu^{4(1+\alpha)}\int_{B_0(b)}(1+o_b(1))\frac{|x|^{2+4\alpha}\,dx}{(1+(\mu|x|)^{2(1+\alpha)})^2}\\ &= 16(1+\alpha)\pi(1+o_b(1))\log(1+(\mu b)^{2(1+\alpha)}) + \frac{(\mu b)^{2(1+\alpha)}}{1+(\mu b)^{2(1+\alpha)}}. \qquad \text{(A.10)}\end{aligned}$$

Then

$$|\nabla \tilde{\phi}_{\mu,z}(x)| = 2\mu^{2(1+\alpha)} \frac{|\nabla \chi_b^{2(1+\alpha)}(d(x,z))|}{1+(\mu\chi_b(d(x,z))^{2(1+\alpha)}}$$
$$= 4(1+\alpha)\mu^{2(1+\alpha)} \frac{\chi_b^{1+2\alpha}(d(x,z))|\chi_b'(d(x,z)|}{1+(\mu\chi_b(d(x,z))^{2(1+\alpha)}} \leq \frac{C\mu^{2(1+\alpha)}b^{1+2\alpha}}{1+(\mu b)^{2(1+\alpha)}} \leq \frac{C}{b}.$$

Consequently,

$$\int_{B_z(2b)\setminus B_z(b)} |\nabla \tilde{\phi}_{\mu,z}|^2 \, dV_g \leq Cb. \tag{A.11}$$

Since $\nabla \tilde{\phi}_{\mu,z}$ vanishes in $\Sigma \setminus B_z(2b)$, (A.10) and (A.11) implies (A.7).

Repeating the previous argument for $\phi_{\mu,\sigma}$ and considering $B_{x_i}(b)$ instead of $B_z(b)$ in (A.10) and $B_{x_i}(2b) \setminus B_{x_i}(b)$ instead of $B_z(2b) \setminus B_z(b)$ for every $i = 1, \dots, k$, we obtain the same estimates with $\alpha = 0$ which allow us to show (A.4).

Next, notice that

$$\tilde{\phi}_{\mu,z} = \log \frac{\mu^{2(1+\alpha)}}{(1+(2\mu b)^{2(1+\alpha)})^2} \quad \text{in} \quad \Sigma \setminus B_z(2b),$$

and

$$\log \frac{\mu^{2(1+\alpha)}}{(1+(2\mu b)^{2(1+\alpha)})^2} \leq \tilde{\phi}_{\mu,z} \leq \log \mu^{2(1+\alpha)} \quad \text{in} \quad B_z(2b).$$

If we write

$$\int_\Sigma \tilde{\phi}_{\mu,z} \, dV_g = |\Sigma| \log \frac{\mu^{2(1+\alpha)}}{(1+(2\mu b)^{2(1+\alpha)})^2} + \int_{B_z(2b)} \left(\tilde{\phi}_{\mu,z} - \log \frac{\mu^{2(1+\alpha)}}{(1+(2\mu b)^{2(1+\alpha)})^2} \right) dV_g,$$

we can conclude that

$$\int_\Sigma \tilde{\phi}_{\mu,z} \, dV_g = |\Sigma| \log \frac{\mu^{2(1+\alpha)}}{(1+(2\mu b)^{2(1+\alpha)})^2} + O\left(b^2 \log(1+(2\mu b)^{2(1+\alpha)}) \right),$$

which proves directly (A.8). Now, if one repeats the same reasoning, considering

$\alpha = 0$ and $\phi_{\mu,\sigma}$ instead of $\tilde{\phi}_{\mu,z}$, then

$$\int_\Sigma \phi_{\mu,\sigma}\, dV_g = |\Sigma| \log \frac{\mu^2}{(1+(2\mu b)^2)^2} + O\left(b^2 \log(1+(2\mu b)^2)\right).$$

This estimate implies immediately (A.5).

In order to compute the exponential terms, notice that $\tilde{K}$ is strictly positive and bounded in $B_{x_i}(2b)$ for any $i = 1, \dots, k$, so there exists a positive constant C_b such that

$$\frac{1}{C_b}\int_{B_{x_i}(2b)} e^{\phi_{\mu,\sigma}}\, dV_g \le \int_{B_{x_i}(2b)} \tilde{K} e^{\phi_{\mu,\sigma}}\, dV_g \le C_b \int_{B_{x_i}(2b)} e^{\phi_{\mu,\sigma}}\, dV_g.$$

As before, one can use normal coordinates centered at x_i to obtain

$$\int_{B_{x_i}(b)} \left(\frac{\mu}{1+(\mu\chi_b(d(x,x_i)))^2}\right)^2 dV_g = \int_{B_0(b)} (1+o_b(1))\frac{\mu^2\, dx}{(1+\mu^2|x|^2)^2} = (1+o_b(1))\pi,$$

and by a change of variables

$$\int_{B_0(b)} \frac{\mu^2\, dx}{(1+\mu^2|x|^2)^2} = 2\pi \int_0^{\mu b} \frac{t}{(1+t^2)^2} = \pi.$$

Observe that this result coincides with the quantization of a entire solution of the Liouville problem (0.1) under a finite curvature condition, namely the area of the 2-sphere. Now, we can conclude that

$$\int_{B_{x_i}(b)} \left(\frac{\mu}{1+(\mu\chi_b(d(x,x_i)))^2}\right)^2 dV_g = (1+o_b(1))\pi \tag{A.12}$$

Observe that

$$\frac{\mu^2}{(1+4\mu^2 b^2)^2} \le e^{\phi_{\mu,\sigma}} \le \frac{\mu^2}{(1+\mu^2 b^2)^2} \quad \text{in} \quad \Sigma \setminus \cup_{i=1}^k B_{x_i}(b),$$

which implies

$$\frac{\mu^2}{(1+4\mu^2 b^2)^2}\int_{\Sigma\setminus\cup_{i=1}^k B_{x_i}(b)} \tilde{K}\,dV_g \le \int_{\Sigma\setminus\cup_{i=1}^k B_{x_i}(b)} \tilde{K}e^{\phi_{\mu,\sigma}}\,dV_g \le \frac{\mu^2}{(1+\mu^2 b^2)^2}\int_{\Sigma\setminus\cup_{i=1}^k B_{x_i}(b)} \tilde{K} dV_g.$$

Taking into account that $\tilde{K}\in L^1(\Sigma)$, the last inequality proves

$$\int_{\Sigma\setminus\cup_{i=1}^k B_{x_i}(b)} \tilde{K}e^{\phi_{\mu,\sigma}}\,dV_g = o_\mu(1). \tag{A.13}$$

So, from (A.12) and (A.13), (A.6) holds.

On the other hand, if one chooses $\sigma=\delta_p$ with $p\in\partial\Omega$ and $\Sigma=\mathbb{S}^2$, it is obvious that

$$\varphi_{\mu,p}(x) = \phi_{\mu,\sigma}(x) + 2\log\mu \qquad \text{if } x\in B_p(b).$$

Indeed, $\nabla\varphi_{\mu,p}(x) = \nabla\phi_{\mu,\delta_p}(x)$ for $x\in B_p(b)$, then by straightforward calculations

$$\int_{B_p(b)} |\nabla\varphi_{\mu,p}|^2 \le 32\pi(1+o_b(1))\log\mu + O(1)).$$

Since $\varphi_{\mu,p}$ is independent of b and $|\nabla\varphi_{\mu,p}|<C$ in $\Omega\setminus B_p(b)$, (A.1) is a consequence of the previous estimate by the smoothness of $\partial\Omega$.

In addition, we have proved that

$$\int_{B_p(b)} \varphi_{\mu,p}\,dV_g = -2\lambda\log\mu,$$

and

$$\int_{B_p(b)} e^{\varphi_{\mu,p}}\,dV_g = O(1),$$

so (A.2) and (A.3) are directly proved due to $\varphi_{\mu,p}$ is bounded in $\Omega\setminus B_p(b)$.

Finally, we deal with the proof of (A.9). First, suppose that $B_z(b)\cap\{p_1,\dots,p_\ell\}=\emptyset$, so there exists $C_b>0$ such that

$$\int_{B_z(b)} \tilde{K} e^{\tilde{\phi}_{\mu,z}}\, dV_g \geq \frac{1}{C_b} \int_{B_z(b)} \frac{\mu^{2(1+\alpha)}}{(1+(\mu d(x,z))^{2(1+\alpha)})^2}\, dV_g.$$

Taking normal coordinates we have

$$\int_{B_0(b)} (1+o_b(1)) \frac{\mu^{2(1+\alpha)}\, dx}{(1+(\mu|x|)^{2(1+\alpha)})^2} \geq (1+o_b(1)) \frac{\mu^{2(1+\alpha)}}{(1+(\mu b)^{2(1+\alpha)})^2} \geq (1+o_b(1)) \frac{1}{C} \mu^{2\alpha}. \tag{A.14}$$

Now, suppose that $|z-p_i| < b^2$ where $p_i \in \{p_1, \ldots, p_m\} \setminus \Theta_\lambda$, we should estimate

$$\int_{B_{p_i}(b)} \tilde{K} \frac{\mu^{2(1+\alpha)}\, dV_g}{(1+(\mu d(x,z))^{2(1+\alpha)})^2} \geq \frac{1}{C_b} \int_{B_{p_i}(b)} d(x,p_i)^{2\alpha_i} \frac{\mu^{2(1+\alpha)}\, dV_g}{(1+(\mu d(x,z))^{2(1+\alpha)})^2}.$$

We compute

$$\int_{B_{p_i}(b)} |x-p_i|^{2\alpha_i} \frac{\mu^{2(1+\alpha)}}{(1+(\mu|x-z|)^{2(1+\alpha)})^2}\, dx = \int_{B_0(b)} |x|^{2\alpha_i} \frac{\mu^{2(1+\alpha)}}{(1+(\mu|x-z|)^{2(1+\alpha)})^2}\, dx.$$

Let us divide the domain into the tree sets

$$B_1 = \left\{ |x| < \sqrt{b}|z| \right\}, \quad B_2 = \left\{ \sqrt{b}|z| < |x| \leq \frac{1}{\sqrt{b}}|z| \right\}, \quad B_3 = \left\{ \frac{1}{\sqrt{b}}|z| < |x| \leq b \right\}.$$

Since $|x-z| = (1+o_b(1))|z|$ in B_1, then

$$\begin{aligned} \int_{B_1} |x|^{2\alpha_i} \frac{\mu^{2(1+\alpha)}\, dx}{(1+(\mu|x-z|)^{2(1+\alpha)})^2} &\geq \frac{1}{C_b} \frac{\mu^{2(1+\alpha)}}{(1+(\mu|z|)^{2(1+\alpha)})^2} \int_{B_0(\sqrt{b}|z|)} |x|^{2\alpha_i}\, dx \\ &\geq \frac{1}{C_b} \frac{\mu^{2(1+\alpha)}|z|^{2\alpha_i+2}}{(1+(\mu|z|)^{2(1+\alpha)})^2}. \end{aligned}$$

Since $0 < \sqrt{b}|z| < |x| \leq \frac{1}{\sqrt{b}}|z|$ in B_2, we can multiply by $|z|$ and apply a change of variables to obtain

$$\int_{B_2} \frac{\mu^{2(1+\alpha)}|x|^{2\alpha_i}}{(1+(\mu|x-z|)^{2(1+\alpha)})^2}\,dx \geq \frac{1}{C_b}\int_{\mu(B_2-z)} \frac{\mu^{2\alpha}|z|^{2\alpha_i}}{(1+|y|^{2(1+\alpha)})^2}dy$$
$$\geq \frac{1}{C_b}\mu^{2\alpha}|z|^{2\alpha_i}\int_0^{\frac{\mu|z|}{C_b}} \frac{t\,dt}{(1+t^{1+\alpha})^2}.$$

Therefore

$$\int_{B_2} \frac{\mu^{2(1+\alpha)}|x|^{2\alpha_i}}{(1+(\mu|x-z|)^{2(1+\alpha)})^2}\,dx \geq \frac{1}{C_b}\mu^{2\alpha}|z|^{2\alpha_i}\frac{\mu^2|z|^2}{1+\mu^2|z|^2}.$$

Note that $|x-z|^2 = (1+o_b(1))|x|^2$ in B_3, through a change of variables we are left with

$$\int_{B_3} \frac{\mu^{2(1+\alpha)}|x|^{2\alpha_i}}{(1+(\mu|x-z|)^{2(1+\alpha)})^2}\,dx \geq \frac{1}{C_b}\int_{B_3} \frac{\mu^{2(1+\alpha)}|x|^{2\alpha_i}}{(1+(\mu|x|)^{2(1+\alpha)})^2}\,dx$$
$$\geq \frac{1}{C_b}\mu^{2(\alpha-\alpha_i)}\int_{C_b\mu|z|}^{\mu/C_b} \frac{t^{2\alpha_i+1}}{(1+t^{2(1+\alpha)})^2}\,dt.$$

In this way, we obtain that

$$\int_{B_3} \frac{\mu^{2(1+\alpha)}|x|^{2\alpha_i}}{(1+(\mu|x-z|)^{2(1+\alpha)})^2}\,dx \geq \frac{1}{C_b}\mu^{2(\alpha-\alpha_i)}\frac{1}{(1+(\mu|z|)^{2+4\alpha-2\alpha_i})^2}.$$

Finally, plugging the above inequalities, we obtain that

$$\int_{B_{p_i}(b)} |x-p_i|^{2\alpha_i}\frac{\mu^{2(1+\alpha)}}{(1+(\mu|x-z|)^{2(1+\alpha)})^2}\,dx$$
$$\geq \frac{1}{C_b}\left(\frac{\mu^{2(1+\alpha)}|z|^{2\alpha_i+2}}{(1+(\mu|z|)^{2(1+\alpha)})^2} + \frac{\mu^{2(\alpha-\alpha_i)}}{(1+(\mu|z|)^{2+4\alpha-2\alpha_i})^2}\right) \geq \frac{\mu^{2(\alpha-\alpha_i)}}{C_b}. \tag{A.15}$$

Since,

$$\frac{\mu^{2(1+\alpha)}}{(1+(2\mu b)^{2(1+\alpha)})^2} \leq e^{\tilde{\phi}_{\mu,z}} \leq \frac{\mu^{2(1+\alpha)}}{(1+(\mu b)^{2(1+\alpha)})^2} \quad \text{in} \quad \Sigma\setminus B_z(b),$$

and $\tilde{K}\in L^1(\Sigma)$, then

$$\int_{\Sigma \setminus B_z(b)} \tilde{K} e^{\tilde{\phi}_{\mu,z}} \, dV_g = o_\mu(1). \tag{A.16}$$

The estimate (A.9) is a direct consequence from (A.14), (A.15) and (A.16). □

Proof of the statement (ii) of Lemma 3.2.11. For every $\varepsilon > 0$ and $z \in \tilde{Z}$, by (A.16), it is shown that

$$\frac{\int_{\Sigma \setminus B_z(\varepsilon)} \tilde{K} e^{\tilde{\phi}_{\mu,z}} \, dV_g}{\int_\Sigma \tilde{K} e^{\tilde{\phi}_{\mu,z}} \, dV_g} = \frac{\int_{\Sigma \setminus B_z(\varepsilon)} \tilde{K} e^{\tilde{\varphi}_{\mu,z}} \, dV_g}{\int_\Sigma \tilde{K} e^{\tilde{\varphi}_{\mu,z}} \, dV_g} \to 0,$$

as $\mu \to +\infty$. This fact concludes the proof. □

Resumen

Esta tesis se centra en el estudio de ecuaciones de tipo Liouville en superficies compactas. En concreto, nuestro trabajo está focalizado en tres objetos de análisis fundamentales en el campo de las ecuaciones en derivadas parciales: existencia, multiplicidad y compacidad de soluciones.

El interés en la ecuación de Liouville data del siglo XIX a través del trabajo del propio Liouville, [84], el cual clasifica las soluciones del problema

$$-\Delta u = 2e^u \quad \text{en} \quad \mathbb{R}^2.$$

Este tipo de ecuaciones despertaron mucho interés en los años 70 debido a su significado geométrico. Sea (Σ, g) una superficie Σ equipada con una cierta métrica g y $\tilde{g}$ una métrica conforme a g sobre Σ, es decir $\tilde{g} = ge^v$. Si K_g, $K_{\tilde{g}}$ son las curvaturas Gaussianas relativas a dichas métricas, entonces el logaritmo del factor conforme satisface la ecuación

$$-\Delta_g v + 2K_g = 2K(x)e^v \quad \text{en} \quad \Sigma. \tag{R.1}$$

Aquí Δ_g denota el operador de Laplace-Beltrami en (Σ, g).

Por otro lado, el clásico *Teorema de Uniformización* asegura que toda superficie de Riemann simplemente conexa es conformemente equivalente a una de las tres superficies de Riemann: el disco unidad abierto, el plano complejo o la esfera de Riemann. Como consecuencia, se puede concluir que toda superficie compacta orientable admite una métrica con curvatura Gaussiana constante. Por tanto, podemos asumir de ahora en adelante que K_g es constante.

En este punto, uno se puede formular la siguente pregunta: *dada una función K*

definida en Σ, existe alguna deformación de la métrica g tal que K se convierta en la curvatura de la nueva métrica? Dicho problema es conocido como el *problema de la curvatura Gaussiana prescrita*, y se reduce al estudio de la existencia de soluciones de la ecuación (R.1). Esta cuestión fue propuesta por Kazdan y Warner para superficies arbitrarias en [72] y por Nirenberg en el caso especial de la esfera.

Integrando la ecuación (R.1) y teniendo en cuenta el *Teorema de Gauss-Bonnet*, se obtiene que

$$\lambda = 2\int_\Sigma K_g \, dV_g = 2\int_\Sigma K e^v \, dV_g = 4\pi\chi(\Sigma), \tag{R.2}$$

donde $\chi(\Sigma)$ es la característica de Euler de Σ. Mediante esta fórmula, podemos ver cómo la topología de Σ da condiciones necesarias para la elección de la función K. De hecho el signo de la función K en al menos un punto de Σ está prescrito por $\chi(\Sigma)$.

Si $\lambda \neq 0$, el problema (R.1) se puede reformular como sigue

$$-\Delta_g u = \lambda\left(\frac{Ke^u}{\int_\Sigma Ke^u dV_g} - \frac{1}{|\Sigma|}\right) \quad \text{in} \quad \Sigma. \tag{R.3}$$

Este problema es usualmente denominado como *ecuación de campo medio* de tipo Liouville.

En un trabajo precursor [115], Troyanov propone la construcción de métricas conformes con curvatura de Gauss prescrita sobre superficies con singularidades cónicas. Este problema se puede considerar el análogo *singular* al problema de la curvatura Gaussiana prescrita, discutido previamente. Se dice que una métrica $\tilde{g}$ definida en Σ admite una singularidad cónica de orden $\alpha > -1$ en $p \in \Sigma$, si

$$\tilde{g} \sim |x-p|^{2\alpha} g \;\; \text{cuando} \;\; x \to p.$$

En otras palabras, Σ admite un cono con vértice p de ángulo total $\vartheta = 2\pi(1+\alpha)$. Entonces, el problema de prescribir una curvatura K y una serie de singularidades cónicas $p_1, \dots, p_m \in \Sigma$ con órdenes $\alpha_1, \dots, \alpha_m \in (-1, +\infty)$ equivale a resolver la siguiente ecuación

$$-\Delta_g v + 2K_g = 2K(x)e^v - 4\pi \sum_{j=1}^{m} \alpha_j \delta_{p_j} \quad \text{en} \quad \Sigma, \tag{R.4}$$

donde δ_{p_j} denota una delta de Dirac sobre el punto $p_j \in \Sigma$. Además, integrando (R.4) y teniendo en cuenta la *fórmula de Gauss-Bonnet*

$$\lambda := 4\pi\chi(\Sigma) + 4\pi \sum_{j=1}^{m} \alpha_j = 2 \int_\Sigma K e^v dV_g.$$

Al asumir que K_g es constante, utilizando el cambio de variable

$$u = v + h_m$$

se puede rescribir (R.4) como

$$-\Delta_g u = \lambda \left(\frac{\tilde{K} e^u}{\int_\Sigma \tilde{K} e^u dV_g} - \frac{1}{|\Sigma|} \right) \quad \text{en} \quad \Sigma, \tag{R.5}$$

donde λ está definida en (R.2) y $\tilde{K}$

$$\tilde{K} = K e^{-h_m}, \qquad \text{con} \qquad h_m(x) = 4\pi \sum_{j=1}^{m} \alpha_j G(x, p_j), \tag{R.6}$$

en el que $G(x, y)$ es la función de Green asociada a Δ_g, ver (0.7) para más detalles.

Se tiene que

$$\tilde{K}(x) \simeq d(x, p_j)^{2\alpha_j} K(x) \qquad \text{cerca de } p_j.$$

Obviamente, K y $\tilde{K}$ tienen el mismo signo en $\Sigma \setminus \{p_1, \ldots, p_m\}$.

Este problema es conocido como *ecuación de campo medio singular* de tipo Liouville. La principal ventaja de la última formulación es que esta admite estructura variacional y permite la búsqueda de soluciones de (R.5) como puntos críticos del siguiente funcional de energía

$$J_\lambda(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 dV_g + \frac{\lambda}{|\Sigma|} \int_\Sigma u \, dV_g - \lambda \log \int_\Sigma \tilde{K} e^u dV_g, \tag{R.7}$$

definido en el dominio

$$X = \left\{ u \in H^1(\Sigma) : \int_\Sigma \tilde{K} e^u \, dV_g > 0 \right\}. \tag{R.8}$$

Durante los últimos años, la relevancia de esta clase de ecuaciones ha experimentado un gran crecimiento por su conexión con actuales teorías en Física. Por citar algunas, las ecuaciones de tipo campo medio aparecen en el estudio de configuraciones de tipo vórtice en la teoría **Electroweak de Glashow-Salam-Weinberg** en régimen autodual. Referimos al lector a [9, 35, 76, 112, 112, 113, 120] para obtener una amplia descripción del modelo y varios resultados en este contexto. Este tipo de problemas surgen también en la **teoría de Chern-Simons-Higgs**. Como se discute en [54], las teorías de Chern–Simons son relevantes en el estudio de varios fenómenos físicos tales como la superconductividad crítica de alta temperatura, el efecto Hall cuántico o la teoría conforme de campos. En concreto, los problemas de campo medio aparecen en la búsqueda de vórtices periódicos autoduales de tipo Chern-Simons.

En algunos procesos biológicos también emergen las ecuaciones de tipo Liouville. Como muestra, la quimiotaxis es el fenómeno en que un grupo de organismos (células o bacterias) se mueve de acuerdo a la presencia de ciertos químicos. Las ecuaciones de **tipo reacción-difusión**, como el modelo de Keller-Segel, son apropiados para investigar esta clase de procesos. En concreto se puede ver que las soluciones estacionarias de dicho modelo dan lugar a una ecuación con no-linealidades exponenciales en un dominio con condiciones de frontera Neumann. Ver [57, 74, 99, 107] para ampliar esta motivación. Por último, las ecuaciones de campo medio surgen en el **estudio del comportamiento turbulento del flujo de Euler con vórtices**. Una forma conocida de analizar la turbulencia estacionaria es la de hacer tender el número de vórtices a infinito. En esta situación, el límite del modelo es un problema de tipo Liouville definido en un dominio con condiciones Dirichlet. Para una rigurosa derivación del modelo, ver [18, 75].

Objetivos

Dentro del análisis de la ecuación de campo medio, esta tesis presenta diversos resultados que consideran el caso en que la función K puede cambiar de signo. La ausencia de restricciones sobre el signo abre un gran número de problemas a estudiar. Hasta donde se sabe, esta situación no ha sido prácticamente considerada con anterioridad. Por esta razón, las cuestiones que son analizadas en este trabajo son algunas de las más fundamentales en el estudio clásico de EDPs: existencia, multiplicidad y compacidad de soluciones. Cabe reseñar que desde un punto de vista geométrico no hay razón para exigir que K sea estrictamente positiva.

Esta tesis contiene los primeros estudios sobre el caso de cambio de signo para superficies singulares con un número arbitrario de puntos cónicos, [47, 48]. En particular, este problema estaba propuesto en Remark 2.8 de [5], el cual señala que *las dificultades son heredadas de la falta de resultados de concentración-compacidad-cuantización.*

Nuestro estudio ha comenzado a generar un interés real. Motivado por nuestro trabajo, esta situación ha sido recientemente tratada por D'Aprile, De Marchis and Ianni usando métodos perturbativos, [44].

Primer problema: El problema de campo medio en un subdominio de la esfera

Sea Ω un subdominio de $\mathbb{S}^2$ con la métrica usual, en primer lugar se estudiará la existencia de soluciones del problema

$$\begin{cases} -\Delta_{g_0} u + 2 = 2K(x)e^u & \text{en } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{sobre } \partial\Omega, \end{cases} \tag{R.9}$$

donde K es una función continua definida en $\overline{\Omega}$.

Es importante observar que con esta condición de frontera, (R.9) no es invariante bajo transformaciones conformes.

Este problema ha sido estudiado en [22, 61, 79, 118]. Integrando la ecuación (R.9) se obtiene que

$$\lambda = 2|\Omega| = 2 \int_{\Omega} K(x)e^u dV_{g_0}. \tag{R.10}$$

En particular, no existe solución si K es negativa. De ahora en adelante asumiremos que

(A1) $K(x) > 0$ para algún $x \in \overline{\Omega}$.

Más aún, (R.10) permite que (R.9) puede ser reescrita de la forma

$$\begin{cases} -\Delta_{g_0} u = \lambda \left(\frac{Ke^u}{\int_\Omega Ke^u dV_{g_0}} - \frac{1}{|\Omega|} \right) & \text{en } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{sobre } \partial\Omega. \end{cases} \tag{R.11}$$

El problema (R.11) es la ecuación de Euler-Lagrange del funcional de energía

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dV_{g_0} + 2 \int_\Omega u \, dV_{g_0} - \lambda \log \int_\Omega Ke^u \, dV_{g_0},$$

definido en el dominio

$$X = \left\{ u \in H^1(\Omega) : \int_\Omega Ke^u \, dV_{g_0} > 0 \right\}.$$

Observa que la hipótesis (A1) implica que X es no vacío. Al igual que el problema (R.11), el funcional I_λ es invariante por adición de constantes.

En [22] se demuestra que I_λ está acotado inferiormente y es coercivo si $\lambda < 4\pi$, i.e. $|\Omega| < 2\pi$). De este modo, se obtiene solución por un argumento de minimización. El caso $\lambda = 4\pi$ es crítico, I_λ permanece acotado inferiomente pero deja de ser coercivio. De hecho, el problema puede presentar pérdida de compacidad debido a la presencia de soluciones que *explotan*.

En esta tesis se considera el caso $\lambda \in (4\pi, 8\pi)$, es decir $|\Omega| > 2\pi$. Para este problema, la condición

(Q1) $K(x) < 0$ para todo $x \in \partial\Omega$,

ya fue considerada por [61]. Asumiendo (Q1), se puede ver que I_λ está acotado inferiormente y es coercivo. De nuevo, mediante un argumento de minimización se puede encontrar una solución.

En cambio, si $K(x) > 0$ en algún punto $x \in \partial\Omega$, entonces I_λ no está acotada inferiormente. Para encontar puntos críticos de tipo silla, los argumentos de tipo min-max emergen como una técnica adecuada para tratar el problema. Un primer resultado en esta dirección fue dado en [118], donde la existencia de soluciones para (R.9) es demostrada bajo la hipótesis

(Q2) $\partial\Omega$ es disconexo y $K(x) > 0$ para todo $x \in \partial\Omega$.

En esta tesis se extenderá los resultados de existencia de [61] y [118] mediante una única condición general, en concreto,

(A2) $K(x) \neq 0$ para todo $x \in \partial\Omega$.

Teorema 4.0.1. *Asume (A1) y (A2). Si Ω es subdominio suave de $\mathbb{S}^2$ tal que $|\Omega| \in (2\pi, 4\pi)$, entonces el problema (R.9) admite una solución.*

Cabe resaltar que nuestra suposición (A2) contiene a (Q1) y (Q2) como casos particulares. Además, nuestra prueba corrige algunos errores presentes en la prueba de [118], como se explica en el Capitulo 2.

Este resultado ha sido incluido en la publicación [86].

Segundo problema: El problema singular de campo medio en superficies compactas

Ahora nos trasladamos al análisis del problema (R.5). En concreto, nuestras aportaciones tratan el caso en que K es una función que cambia de signo. En este trabajo, se dan nuevos resultados de existencia y multiplicidad genérica por medio de métodos variacionales. Para obtener dichos resultados, es imprescindible establecer un resultado que determine la compacidad de las soluciones del problema, presentado en lo sucesivo. Además, se incluye un resultado de no existencia de soluciones cuando no se satisfacen los supuestos dados. En cierto sentido, podemos decir que nuestras hipótesis son en cierto modo precisas.

Se introduce la primera hipótesis sobre K:

(H1) K es una función $C^{2,\alpha}$ que cambia de signo con $\nabla K(x) \neq 0$ para todo $x \in \Sigma$ donde $K(x) = 0$.

Se definen los conjuntos

$$\Sigma^+ = \{x \in \Sigma :\ K(x) > 0\}, \quad \Sigma^- = \{x \in \Sigma :\ K(x) < 0\}, \quad \Gamma = \{x \in \Sigma :\ K(x) = 0\}.$$

Observa que la suposición (H1) implica que el conjunto de curvas nodales Γ es regular y que

$$N^+ = \#\{\text{componentes conexas de } \Sigma^+\} < +\infty. \tag{R.12}$$

En lo que sigue se asume que

(H2) $p_j \notin \Gamma$ para todo $j \in \{1, \dots, m\}$.

Es posible reordenar las singularidades de tal forma que

$$p_j \in \Sigma^+ \text{ para } j \in \{1 \dots, \ell\}, \quad p_j \in \Sigma^- \text{ para } j \in \{\ell+1, \dots, m\}. \tag{R.13}$$

Ahora se define el conjunto de valores críticos como

$$\Lambda_\ell = \left\{ 8\pi r + \sum_{j=1}^{\ell} 8\pi(1+\alpha_j)n_j : \ r \in \mathbb{N} \cup \{0\}, n_j \in \{0,1\} \right\} \setminus \{0\}. \tag{R.14}$$

Si K es una función positiva, el conjunto de soluciones del problema (R.5) es compacto para $\lambda \notin \Lambda_m$, ver [9, 17, 78]. En el próximo teorema se obtiene una conclusión análoga en nuestro ambiente.

Teorema 4.0.2. *Se asume que $\alpha_1, \dots, \alpha_m > -1$ y sea K_n una sucesión de funciones con $K_n \to K$ en sentido $C^{2,\alpha}$, donde K verifica (H1), (H2). Sea u_n una sucesión de soluciones del problema*

$$-\Delta_g u_n = \tilde{K}_n e^{u_n} - f_n \quad en \quad \Sigma, \tag{R.15}$$

con $f_n \to f$ en sentido $C^{0,\alpha}$ y $\tilde{K}_n = K_n e^{-h_m}$ con h_m dado en (R.6). Entonces, salvo subsucesiones, se verifica la siguiente alternativa:

1. *o u_n es uniformemente acotado en $L^\infty(\Sigma)$;*
2. *o bien u_n diverge a $-\infty$ uniformemente;*
3. *o bien existe un conjunto finito $S = \{q_1, \dots, q_r\} \subset \Sigma^+$ de puntos de blow–up.*

   En tal caso, $u_n \to -\infty$ en conjuntos compactos de $\Sigma \backslash S$ y $\tilde{K}_n e^{u_n} \rightharpoonup \sum_{i=1}^r \beta(q_i)\delta_{q_i}$ en el sentido débil de las medidas donde $\beta(q_i) = 8\pi$ if $q_i \notin \{p_1, \dots, p_m\}$ y $\beta(q_i) = 8(1+\alpha_j)\pi$ si $q_i = p_j$ para cualquier $1 \leq j \dots \ell$.

   En particular, $\lim_{n\to+\infty} \int_\Sigma \tilde{K}_n e^{u_n} \in \Lambda_\ell$, definido en (R.14).

Nótese que la ecuación (R.5) se puede escribir en la forma (R.15) por adición de una constante apropiada a $u_n = u$, con $K_n = K$ y $f_n = \frac{\lambda}{|\Sigma|}$.

Por lo que respecta a la existencia y multiplicidad de soluciones, nos restringiremos al caso de **órdenes positivos** α_j. Nuestras demostraciones emplean herramientas propias de los métodos variacionales. De hecho, el problema (R.5) es la ecuación de Euler-Lagrange del funcional de energía de (R.7).

Si $\lambda < 8\pi$, entonces J_λ es coercivo y se puede encontrar solución como mínimo de J_λ, mientras que J_λ no está acotado inferiormente si $\lambda > 8\pi$. En esta tesis se considerará este último caso.

Antes de establecer nuestro resultado de existencia, es preciso añadir una hipótesis sobre K:

(H3) $N^+ > k$ o Σ^+ tiene una componente conexa no simplemente conexa,

donde N^+ está definida en (R.12).

Teorema 4.0.3. *Sea* $\alpha_1, \dots, \alpha_\ell > 0$*, con* ℓ *definido en* (R.13) *y* $\lambda \in (8k\pi, 8(k+1)\pi) \setminus \Lambda_\ell$*. Si (H1), (H2), (H3) se satisfacen, entonces* (R.5) *admite una solución.*

Obsérvese que si Σ^+ tiene topología trivial, entonces el Teorema 4.0.3 no se puede aplicar. Sin embargo, siguiendo las ideas de [90], el cual considera potenciales positivos, podemos aportar también un resultado que cubra este caso. Para ello, se define el conjunto

$$\Theta_\lambda = \{p_j \in \Sigma^+ : \ \lambda < 8\pi(1+\alpha_j)\},$$

y se introduce la hipótesis

(H4) $\Theta_\lambda \neq \emptyset$.

Teorema 4.0.4. *Sea* $\alpha_1, \dots, \alpha_\ell \in (0,1]$*, donde* ℓ *está definido en* (R.13) *y* $\lambda \in (8\pi, 16\pi) \setminus \Lambda_\ell$*. Si (H1), (H2), (H4) se satisfacen, entonces* (R.5) *admite una solución.*

Remark 4.0.1. *Existen muchos tipos de aplicaciones de estos resultados al problema geométrico comentado previamente. Sólo por mostrar un ejemplo, considera el* *problema de buscar una métrica conforme en* $\Sigma = \mathbb{T}^2$ *con curvatura Gaussiana* K *y un punto cónico de orden* α*. Asumimos que las suposiciones (H1), (H2) se satisfacen. Entonces Teorema 4.0.3 implica que el problema admite solución si y sólo si se verifican alguna de estas dos condiciones*

1. $\alpha \in (k, k+1)$ *con* $k \in \mathbb{N}$ *y* Σ^+ *tiene más de* k *componentes conexas;*

2. $\alpha \in (k, k+1)$ *con* $k \in \mathbb{N}$ *y* Σ^+ *tiene una componente no simplemente conexa.*

Consideremos el mismo problema pero con m *puntos cónicos, todos de orden* α*. Entonces el Teorema 4.0.4 implica que el problema geométrico es resoluble si* $1 < m\,\alpha < 1 + \alpha$ *y al menos uno de los puntos cónicos está situado en* Σ^+*.*

Muchos otros ejemplos podrían ser construidos.

En nuestro próximo resultado para $\Sigma = \mathbb{S}^2$, se presenta una clase de funciones K para la cual (R.5) no admite solución. De hecho, estas funciones satisfacen (H1) y (H2) pero no (H3), ni tampoco (H4). Para hacer más claro el enunciado del teorema, no entraremos en detalles sobre la definción de K.

Teorema 4.0.5. *Sea* $p \in \mathbb{S}^2$ *y* $\alpha > 0$ *con* $m = 1$, $p_1 = p$, $\alpha_1 = \alpha$ *y* $\tilde{K} = e^{-h_1} K$, *entonces existe una familia de funciones* K *tal que (H1) y (H2) se cumplen pero la ecuación* (R.5) *no admite solución para* $\lambda \in (8\pi, +\infty)$*,*

Como consecuencia del Teorema 4.0.5, la función K no puede ser la curvatura Gaussiana de $\mathbb{S}^2$ para ninguna métrica con una singularidad de orden $\alpha > 0$.

Finalmente, se presentan dos resultados de multiplicidad para elecciones genéricas del par (K, g), los cuales cubren los casos estudiados por nuestros resultados de existencia. Intuitivamente, estos resultados muestran que el número de soluciones crece conforme la topologia de Σ^+ se vuelve más complicada. Como el enunciado de estos resultados requiere introducir notación extra, no enunciaremos los mismos en este resumen. Emplazamos al lector a la Sección 3.3.

Los resultados descritos anteriormente se han incluido en las publicaciones [47], [48].

Metodología: estrategia de las demostraciones

Como se ha comentado anteriormente, nuestras pruebas utilizan argumentos de tipo min-max para demostrar existencia de puntos críticos asociados a los respectivos funcionales de energía. A continuación se presenta un esquema que provee una estrategia común de las demostraciones e introduce algunos hechos conococidos que engloban a los argumentos empleados.

- ***Existencia:*** Respecto a los teoremas de existencia Theorems 4.0.1, 4.0.3 y 4.0.4, se siguen las bases de la teoría de Morse, la que, de forma intuitiva, asegura que la topología de los subniveles de energía no varía si no existen puntos críticos. Sea E un subconjunto abierto en un espacio de Hilbert y $\mathcal{F} \in \mathcal{C}^1(E, \mathbb{R})$, se denota como subnivel

  $$\mathcal{F}^a = \{e \in E : \mathcal{F}(e) \leq a\},$$

  donde $a \in \mathbb{R}$. Se puede decir por tanto que una variación de la topología de los subniveles implica la existencia de un punto crítico. Así pues, nuestro primer objetivo es obtener una descripción topológica precisa de los subniveles de I_λ y J_λ. Se verá que los subniveles de energía bajos tienen topología no trivial, mientras que los altos son triviales, lo que confirma la existencia de un cambio de topología entre niveles altos y bajos.

  En cierto sentido, funciones a nivel de energía baja tienden a concentrarse alrededor de un número finito de puntos. Esta configuración de puntos se utiliza para estudiar la topología de $\mathcal{F}^{-L}$ con $L > 0$ suficientemente grande. De esta forma, el objetivo principal es encontrar un espacio topológico no compacto $\mathcal{Z}$, para describir la topología de $\mathcal{F}^{-L}$. A continuación se construye una función continua Ψ que proyecta $\mathcal{F}^{-L}$ en $\mathcal{Z}$ y otra Φ en sentido contrario, tal que la composición

  $$\mathcal{Z} \xrightarrow{\Phi} \mathcal{F}^{-L} \xrightarrow{\Psi} \mathcal{Z} \tag{R.16}$$

  sea homotópicamente equivalente a la identidad en $\mathcal{Z}$. De esta forma se dice que la topología de $\mathcal{F}^{-L}$ es más rica que la de $\mathcal{Z}$. De hecho, la no contractibilidad de $\mathcal{Z}$ implica que $\Phi(\mathcal{Z})$ es no contraible en $\mathcal{F}^{-L}$.

  El cambio de topología de los subniveles implica la existencia de una sucesión de Palais-Smale $\{u_n\}$, es decir $I_\lambda(u_n) \to c_\lambda > -\infty$ y $I'_\lambda(u_n) \to 0$, donde

c_λ es el valor min-max, ver [4] por ejemplo. Sin embargo, este hecho no implica directamente la existencia de puntos críticos. Esta dificultad puede ser solucionada por el comúnmente conocido como truco de monotonía de Struwe, [109], el cual garantiza acotación, y por tanto convergencia, de la sucesión de Palais-Smale para casi todo valor del parámetro λ. Para extender la existencia de puntos críticos a todos los valores del intervalo, se precisa de una propiedad de compacidad. Este objetivo se introduce a continuación.

- ***Compacidad:*** Teniendo en cuenta lo anteriormente comentado, uno se enfrenta al siguente problema: dado u_n una sucesión de soluciones de (R.11) o (R.5) para $\lambda = \lambda_n \to \lambda_0$, es uniformemente acotada?

  Esta cuestión fue estudiada en [17, 78] para el problema regular, y en [8, 9] para la ecuación con vórtices, siempre con la suposición de potenciales $K(x)$ positivos. La condición sobre el signo de K no es sólo una cuestión técnica, como se puede deducir de recientes ejemplos de soluciones que explotan dados en [13, 50]. Estas soluciones se concentran en máximos locales de K al nivel 0, una situación que, a priori, podría reproducirse en nuestros problemas. Sin embargo, la suposición sobre dónde y cómo K cambia de signo, (A2) and (H1) respectivamente, nos permite descartar este fenómeno.

  Para el primer problema podemos concluir compacidad por medio de estimas de energía. Este argumento parece completamente nuevo para este tipo de problemas, pero no puede ser interpretado como un resultado completo de compacidad debido a la suposición extra sobre el nivel de energía. De hecho, este argumento está restringido a la forma específica del problema (R.11) y no funciona para problemas más generales.

  Para estudiar la cuestión de la compacidad para (R.3), se adopta una estrategia diferente. En primer lugar se deriva una estima integral uniforme en subconjuntos de Σ^+ o Σ^-, la cual permite obtener estimas a priori en la región $\{x \in \Sigma : \; K(x) < -\delta\}$, para un $\delta > 0$ pequeño. Después, a través del método de *moving-plane* se obtiene una comparación entre valores de u sobre los dos lados de la curva Γ. Esto, junto a la estima integral mencionada anteriormente, implica acotación uniforme en un entorno de Γ. Para concluir, se pueden aplicar los resultados de blow–up en la región $\{x \in \Sigma : \; K(x) > \delta\}$ para obtener las cuantizaciones exactas y como consecuencia el criterio de compacidad.

- ***Multiplicidad:*** Las estimas del número de soluciones son válidas bajo la suposición de no degeneración de soluciones. Un argumento de *transversalidad*, ver [106] por ejemplo, garantiza que para una elección genérica de (K, g), las soluciones del problema (R.5) son no degeneradas. De forma precisa, (g, K) está en un conjunto abierto y denso de $\mathcal{M}^2 \times C^{2,\alpha}(\Sigma)$, donde $\mathcal{M}^2$ denota el espacio de todas las métricas de Riemann $C^{2,\alpha}$ en Σ equipado con la norma $C^{2,\alpha}$.

  Bajo estas condiciones, podemos utilizar las desigualdades débiles de Morse, las que, junto al cálculo de los grupos de homología de un par, nos permite probar que

$$\#\{\text{puntos críticos de } I_\lambda \text{ en } \{a \le I_\lambda \le b\} \;\} \ge \sum_{q \ge 0} dim \left(H_q(I_\lambda^b, I_\lambda^a) \right).$$

  La fórmula anterior sugiere estudiar de forma rigurosa la homología de los subniveles de energía altos y bajos. Entoneces, se puede hacer uso de la descripción topológica de los subniveles dados en la parte de existencia. De hecho, por (R.16),

$$\sum_{q \ge 0} dim \left(H_q(I_\lambda^b, I_\lambda^a) \right) \ge \sum_{q \ge 0} dim(H_q(\mathcal{Z}).$$

  Para calcular los números de Betti de $\mathcal{Z}$, se necesita hacer uso de algunas herramientas de topología algebraica. En particular, una de las dificultades principales es estudiar los grupos de homología de $\mathcal{Z}$, el cual será el conjunto de baricentros de una unión disjunta. Esta dificultad se resuelve a través de una fórmula que conecta la homología de los baricentros sobre la unión disjunta a la homología de los baricentros sobre cada uno de los espacios disjuntos.

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