TESIS DOCTORAL

SOME PROBLEMS ON PRESCRIBED MEAN CURVATURE AND KINEMATICS IN GENERAL RELATIVITY

Programa de doctorado FisyMat

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Resumen

La irrupción de la teoría de la Relatividad revolucionó la Física del siglo pasado al poner en pie de igualdad el espacio y el tiempo, dotando al conjunto de ambos de una geometría pseudoeuclídea. También nos revela que la gravedad puede ser descrita exitosamente a través de la curvatura del espaciotiempo.

Hoy en día, el marco matemático de la Relatividad General constituye una nueva rama de la Geometría (la Geometría Lorentziana), alcanzando un estatus similar al que ocupan las matemáticas de la Mecánica Clásica dentro de la Geometría Simpléctica. El creciente interés de la comunidad matemática por la Relatividad ha hecho surgir numerosos problemas analíticos y geométricos. Nuestra pretensión en este trabajo es estudiar algunos de ellos.

Esta tesis se estructura en dos grandes bloques. El primero de ellos (capítulos 3, 4 y 5) se ocupa del problema de la prescripción de la curvatura media. El tratamiento es mayoritariamente analítico, haciendo uso de teoremas clásicos de Teoría del Grado y métodos de aproximación-truncatura. Los resultados serán interpretados físicamente. En el segundo, varios conceptos bien conocidos de la Física clásica se transportan al ámbito relativista. Se formulan nuevas nociones y se plantean problemas en un lenguaje geométrico, usando argumentos analíticos cuando es necesario. Una vez más, Análisis y Geometría se funden para convertirse en una poderosa herramienta con la que formular y resolver problemas de índole física.

El problema de prescripción de la curvatura media será el hilo conductor de toda la memoria. Pese a estar organizada en dos partes bien diferenciadas, en ambas se estudiarán propiedades extrínsecas de ciertas subvariedades de un espaciotiempo. Concretamente, nos centraremos en las únicas subvariedades que son definidas y tales que su fibrado normal también lo es.

Comenzamos esta memoria estudiando el siguiente problema de Dirichlet.

Sea $B(R)$ la bola euclídea abierta, centrada en $0 \in \mathbb{R}^n$ y de radio R. Tomemos $I \subseteq \mathbb{R}$ un intervalo abierto con $0 \in I$, y sea $f \in C^\infty(I)$ positiva. Dada una función diferenciable radialmente simétrica $H : I \times B(R) \to \mathbb{R}$, nuestro objetivo será estudiar la existencia de soluciones positivas y radialmente simétricas del siguiente problema cuasilineal elíptico

$$\begin{gathered}
\operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - |\nabla u|^2}}\right) + \frac{f'(u)}{\sqrt{f(u)^2 - |\nabla u|^2}}\left(n + \frac{|\nabla u|^2}{f(u)^2}\right) = nH \quad \text{en} \quad B(R), \\
|\nabla u| < f(u), \qquad (1) \\
u = 0 \quad \text{en} \quad \partial B(R).
\end{gathered}$$

El planteamiento de esta EDP elíptica está motivado desde la geometría lorentziana. En concreto, se trata de un problema de prescripción de la curvatura media. Explícitamente, cada solución de (1) define un grafo espacial (la desigualdad en (1)) sobre una bola $B(R)$ de la fibra $\{0\} \times \mathbb{R}^n$ de un espaciotiempo de Friedmann-Lemaître-Robertson-Walker (FLRW), $\mathcal{M} = I \times_f \mathbb{R}^n$ (ver Capítulo 2 para más detalles) donde la función H prescribe la curvatura media.

Una hipersuperficie espacial en un espaciotiempo es una hipersuperficie en la que la métrica inducida por la métrica lorentziana del ambiente es riemanniana. Intuitivamente, una hipersuperficie espacial puede verse como el 'universo espacial' que observa una familia de observadores en un instante de su tiempo propio. En concreto, cada hipersuperficie espacial define una familia de *observadores normales*: cada geodésica en el espaciotiempo ambiente determinada por un punto de la hipersuperficie y el vector normal unitario que apunta al futuro en ese punto. La correspondiente función curvatura media mide *cómo estos observadores se alejan o se acercan respecto a uno dado, al promediar sobre todas las direcciones espaciales*. De hecho, estos observadores constituyen, localmente, las curvas integrales de un campo de observadores o 'reference frame' en el espaciotiempo y el signo de su divergencia (esto es, la medida de la expansión/contracción para los observadores del campo en el que están agrupados, [68, 78]) es el mismo que el signo de la función curvatura media. Nuestro interés reside en prescribir la función curvatura media en el caso en que los observadores instantáneos se alejen (esto es, midan expansión) en un espaciotiempo de FLRW.

Por otra parte, una hipersuperficie espacial conforma un subconjunto adecuado del espaciotiempo en donde el problema de valores iniciales asociado a las ecuaciones de la Relatividad General (ecuaciones de materia, ecuaciones de Maxwell y ecuaciones de Einstein) está bien planteado. El caso en el que la curvatura media es constante es relevante, especialmente cuando es idénticamente nula (esto es, para hipersuperficies maximales). Por un lado, cuando una hipersuperficie espacial tiene curvatura media nula, ésta puede constituir un buen conjunto inicial para el problema de Cauchy en Relatividad General [74]. Concretamente, Lichnerowicz probó que el problema de Cauchy con condiciones iniciales sobre una hipersuperficie maximal se reduce a una ecuación diferencial elíptica no lineal de segundo orden y un sistema de ecuaciones diferenciales lineales de primer orden, [3, 30, 61].

Incluso más, las hipersuperficies maximales poseen importancia en el análisis de la dinámica de un campo gravitatorio, o en el problema clásico de los n-cuerpos en el seno de un campo gravitatorio (véase, por ejemplo, [18] y referencias allí).

Por otro lado, cada hipersuperficie maximal puede describir, en algunos casos relevantes, la transición desde una fase expansiva a otra contractiva de un universo relativista. Es más, la existencia de una hipersuperficie de curvatura media constante (y en particular maximal) es necesaria para comprender la estructura de singularidades en el espacio de soluciones de la ecuación de Einstein. Un profundo conocimiento de estas hipersuperficies también es necesario en la prueba de la positividad de la masa gravitatoria. Poseen interés en Relatividad Numérica, donde las hipersuperficies maximales se usan para integrar en el tiempo. Todos estos aspectos físicos pueden ser consultados en [64] y referencias allí.

Geométricamente, las hipersuperficies espaciales con curvatura media constante en una variedad lorentziana (general) son los puntos críticos del funcional "área" bajo cierta "restricción de volumen" [16, 27, 28]. La existencia y unicidad de hipersuperficies espaciales de curvatura media constante son clásicos e importantes problemas geométricos (ver [17] y referencias allí dadas). Cheng y Yau, en su fecundo trabajo en el que se prueba la conjetura de Calabi-Bernstein para dimensión arbitraria, también introdujeron un nuevo tipo de problemas elípticos que han sido extendidos a espaciotiempos mucho más generales que el de Minkowski [16, 28, 75].

En los últimos años, numerosos investigadores han trabajado en el problema de la prescripción de la curvatura media en el ambiente riemanniano (especialmente en el espacio euclídeo) [50]. En el lorentziano, los esfuerzos se han centrado principalmente en el espaciotiempo de Minkowski. En este contexto, podemos destacar el celebrado "resultado de existencia universal" del problema Dirichlet, por parte de Bartnik y Simon [6] en 1982. Más recientemente, el interés se ha centrado en la existencia de soluciones positivas, usando una combinación de técnicas variacionales, teoría de puntos críticos, sub y supersoluciones y teoría del grado (ver por ejemplo [11–13, 31–33] y referencias allí citadas). Sin embargo, el problema de existencia de grafos espaciales de curvatura media prescrita en espaciotiempos de Friedmann-Lemaître-Robertson-Walker no ha sido considerado anteriormente. En este contexto, los problemas de unicidad sí han sido estudiados con más profundidad (ver por ejemplo [1],[19]).

Nuestro primer objetivo será tratar el problema de existencia mediante técnicas basadas en el teorema del punto fijo de Schauder (ver por ejemplo [42]). Antes de nada, hemos de notar que nuestros resultados no se siguen directamente de los obtenidos previamente cuando $\mathcal{M}$ es el espaciotiempo de Minkowski ([11] y referencias allí citadas). En realidad, la ecuación que aquí trataremos tendrá un término singular extra respecto a la considerada en el espaciotiempo de Minkowski. Sólo impondremos condiciones sobre la función de prescripción (no sobre la función de alabeo) que aseguran la simetría radial *a priori* de todas las (posibles) soluciones de la ecuación (1). En otras palabras, probaremos que *la simetría del dominio base 'se contagia a las soluciones'*. Para obtener este hecho, usaremos los resultados de B. Gidas, W. Ni and L. Nirenberg en [53] sobre la simetría de las soluciones de ciertas ecuaciones diferenciales no lineales. El método empleado por estos autores ya había sido utilizado por Alexandroff casi treinta años antes para probar con éxito que las esferas son las únicas hipersuperficies conexas, compactas y embebidas en el espacio euclídeo con curvatura media constante. Actualmente esta técnica se conoce como 'método de reflexión de Alexandroff', y su uso está muy extendido en el campo de las EDP's elípticas y el Análisis Geométrico. En nuestro caso, utilizaremos primero un argumento de truncatura expuesto en [31] para después aplicar los resultados de [53].

El primer resultado para el problema Dirichlet puede resumirse como sigue.

Teorema 3.3.5 *Sea $I \times_f \mathbb{R}^n$ un espaciotiempo de Friedmann-Lemaître-Robertson-Walker, y sea $B = B_0(R)$ la bola euclídea de radio R centrada en $0 \in \mathbb{R}^n$. Supongamos que $I_f(R) \subset I$, donde*

$$I_f(R) := \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds\, ,\, \int_{0}^{R} f(\varphi^{-1}(s))ds \right] \quad \text{y} \quad \varphi(t) = \int_0^t \frac{dt}{f(t)},$$

y supongamos que se satisface la siguiente desigualdad

$$\max_{\mathbb{R}^+ \cap I_f(R)} |f'| < \frac{1}{R}.$$

Para cada función de clase C^∞ radialmente simétrica $H : I \times \overline{B} \to \mathbb{R}$ que cumpla

$$H(t,r) \leq \frac{f'}{f}(t) \quad y \quad f'(t) \geq 0, \qquad \textit{para todo} \quad r \in]0,R[\ \ , \ \ t \in I_f(R),$$

existe un grafo espacial con función curvatura media H definido sobre $\overline{B}$, soportado sobre la hipersuperficie $t = 0$ y tocándolo sólo en el borde $\{0\} \times \partial B$, y formando un ángulo hiperbólico no nulo con $\partial_t := \partial/\partial t$. Además, si H es creciente en la segunda variable, tal grafo espacial debe ser radialmente simétrico.

La familia de espaciotiempos de Friedmann-Lemaître-Robertson-Walker sobre los que se aplica el resultado es muy amplia, y contiene espaciotiempos relevantes. Entre otros, incluye el espaciotiempo de Minkowski ($f = 1$, $I = \mathbb{R}$), el de Einstein-De Sitter ($I =]-t_0, +\infty[$, $f(t) = (t + t_0)^{2/3}$, con $t_0 > 0$), y el conocido como 'espaciotiempo de estado estable' ($I = \mathbb{R}$, $f(t) = e^t$), un cierto subconjunto abierto del espaciotiempo de De Sitter.

Notemos que en el teorema anterior impone una cota superior sobre el radio R del dominio base. Eliminar esta hipótesis requiere de un método diferente algo más sofisticado. Pasando a coordenadas polares, en la ecuación previa se hacen visibles dos singularidades: la primera ocurre en $r = 0$ y aparece frecuentemente en el centro de casi cualquier problema radialmente simétrico definido sobre una bola; la segunda no es estándar en la literatura relacionada, y aparece en la variable dependiente (ver el segundo término del lado izquierdo de la ecuación (3.5)). Para tratar la primera singularidad usaremos un método de aproximación mediante una familia de problemas 'truncados', un procedimiento bastante natural en este tipo de cuestiones (ver [70, Chapter 9] y referencias allí citadas), aunque en este contexto es esencialmente nuevo. Manipulando la segunda singularidad en esta sucesión de problemas aproximados, (ver el primer paso de la prueba del Teorema 3.5.1), obtenemos una sucesión de soluciones aproximadas. Para probar la convergencia de esta sucesión, el punto clave reside en una delicada estimación de cotas a priori sobre la derivada de las soluciones en la frontera (Proposición 3.4.2). Una vez resuelto el problema de Dirichlet, la existencia de soluciones enteras se obtiene por un argumento de extensión. Desde un punto de vista analítico, este problema se expresa mediante la siguiente ecuación cuasilineal elíptica,

$$\operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2-|\nabla u|^2}}\right)+\frac{f'(u)}{\sqrt{f(u)^2-|\nabla u|^2}}\left(n+\frac{|\nabla u|^2}{f(u)^2}\right)=nH(u,x),$$

$$|\nabla u|<f(u),$$

donde $f\in C^\infty(I)$ es una función positiva, I es un intervalo abierto en $\mathbb{R}$ con $0\in I$, $H: I\times\mathbb{R}^n\to\mathbb{R}$ es una función de clase C^∞ radialmente simétrica dada y u satisface $u(\mathbb{R}^n)\subset I$.

En comparación con el problema Dirichlet, el número de referencias dedicadas al estudio de grafos espaciales enteros en el espaciotiempo de Minkowski con curvatura constante o prescrita es apreciablemente menor. En este aspecto, el estudio de grafos espaciales enteros de curvatura media constante desarrollado en [84] está principalmente motivado por la destacada propiedad de Calabi-Bernstein en el caso maximal, es decir, cuando la curvatura media se anula. Calabi [21] mostró para $n\leq 4$, y más tarde Cheng y Yau [28] para todo n, que un grafo maximal entero en $\mathbb{L}^{n+1}$ debe ser un hiperplano espacial. Treibergs probó en $\mathbb{L}^{n+1}$ la existencia de grafos enteros de curvatura media constante bajo ciertas condiciones asintóticas. Más tarde, Bartnik y Simon [6, Th. 4.4] extendieron este resultado a una función curvatura media más general, pero son pocas las referencias que atienden al problema de prescripción de curvatura media para grafos enteros. En los últimos años, hasta donde sabemos, sólo [4, 15] tratan este problema usando una aproximación variacional para funciones curvatura media muy concretas. Este es el objetivo principal del Capítulo 3, cuyo resultado clave es el siguiente, mejorando el Teorema 3.3.7.

Teorema 3.5.2 *Sea $I\times_f\mathbb{R}^n$ un espaciotiempo de* FLRW, *y sea B una bola euclídea* *en $\mathbb{R}^n$ con radio R centrada en cero. Supongamos que $I_f(R) \subset I$. Entonces, para cada función de clase C^∞ radialmente simétrica $H : I \times \overline{B} \to \mathbb{R}$ que cumpla*

$$H(t,r) \leq \frac{f'}{f}(t) \quad y \quad f'(t) \geq 0, \qquad \textit{para todo} \quad r \in]0,R[\ ,\ \ t \in I_f(R),$$

existe un grafo espacial radialmente simétrico con función curvatura media H definida en $\overline{B}$, soportada sobre la hipersuperficie $t=0$, tocándola sólo en el borde $\{0\} \times \partial B$, y formando un ángulo hiperbólico no nulo con ∂_t. Además, si la función H es creciente en la segunda variable, cada grafo espacial satisfaciendo las hipótesis previas debe ser radialmente simétrico.

Este resultado de existencia permite enunciar el teorema central del Capítulo 3.

Teorema 3.6.1 *Sea $I \times_f \mathbb{R}^n$ un espaciotiempo de* FLRW*, y sea $R > 0$ tal que*

$$I_f(R) \subset I, \qquad \varphi^{-1}(\mathbb{R}^-) \subset I,$$

donde

$$I_f(R) := \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds\,,\ \int_0^R f(\varphi^{-1}(s))ds\right] \quad \text{y} \quad \varphi(t) = \int_0^t \frac{dt}{f(t)}.$$

Entonces, para cada función de clase C^∞ radialmente simétrica $H : I \times \mathbb{R}^n \to \mathbb{R}$ que cumpla

$$H(t,r) \leq \frac{f'}{f}(t) \quad y \quad f'(t) \geq 0, \qquad \textit{para todo} \quad r \in]0,R[\ ,\ \ t \in I_f(R),$$

existe un grafo espacial entero radialmente simétrico con función curvatura media H. Además, la hipersuperficie $t=0$ interseca al grafo en una esfera de radio R. En el caso particular en que $\inf I$ *es finito, el grafo entero se aproxima a un hiperplano.*

Nótese que este resultado puede particularizarse al simple pero importante caso

$H = 0$, proporcionando grafos enteros maximales en los espaciotiempos de FLRW del tipo $I \times_f \mathbb{R}^n$.

En el Capítulo 4, investigamos la existencia de soluciones de la siguiente ecuación de prescripción de la curvatura media,

$$\begin{gathered} \frac{1}{f}\,\mathrm{div}\left(\frac{f^2\,\nabla u}{\sqrt{1-f^2|\nabla u|^2}}\right) + \frac{\langle \nabla u\,,\,\nabla f\rangle}{\sqrt{1-f^2|\nabla u|^2}} = \frac{n}{f^2}\,H(x,u), \quad x \in M, \\ |\nabla u| < \frac{1}{f}, \end{gathered} \tag{2}$$

donde la variedad de Riemann M es o bien $\mathbb{R}^n$ o $\mathbb{R}^n \setminus \overline{B}_0(a)$, $a \geq 0$, dotada con una métrica radial

$$\langle\ ,\ \rangle = E^2(r)dr^2 + r^2 d\Theta^2,$$

siendo $E(r) > 0$, $d\Theta^2$ la métrica usual de la esfera $\mathbb{S}^{n-1}$. Finalmente, $f \in C^\infty(a, +\infty)$ es una función positiva y $H : M \times \mathbb{R} \to \mathbb{R}$ es una función de clase C^∞ radialmente simétrica dada.

El planteamiento de esta EDP está también motivado por un problema de prescripción de la curvatura media en geometría lorentziana. Explícitamente, cada solución de (2) define un grafo espacial en un espaciotiempo estático estándar, $\mathcal{M} := M \times_f I$, y la función H prescribe la curvatura media.

Por tanto, en el cuarto capítulo trataremos con grafos en espaciotiempos estáticos respecto de una familia de obsevadores para los cuales el universo espacial resulta ser 'siempre igual'. Hay muchos ejemplos relevantes de este tipo de espaciotiempos. Especialmente importantes son (además del espaciotiempo de Minkowski) los espaciotiempos de *Schwarzschild* y *Reissner-Nordström*. Ambos modelos relativistas describen un universo en los que sólo hay una masa esféricamente simétrica que no rota, como una estrella o un agujero negro. En el primer modelo la masa no tiene carga eléctrica, mientras que en el segundo está uniformemente cargada (en realidad, el espaciotiempo de Reissner-Nordström puede ser visto como una extensión del de Schwarzschild). En los dos hay *horizontes de sucesos* y una singularidad inevitable en el centro de la masa (ver [25, Chap. 3] y [25, Chap. 5] para detalles e interpretaciones físicas).

Hasta donde sabemos, el problema de existencia de grafos enteros de curvatura media prescrita con simetría radial en espaciotiempos estáticos no ha sido aún considerada previamente.

Los resultados más relevantes obtenidos en el Capítulo 4 puede resumirse en los siguientes teoremas.

Teorema 4.3.1 *Sea $\mathbb{R}^n \times_f \mathbb{R}$ un espaciotiempo estático estándar, provisto de una métrica esféricamente simétrica*

$$E^2(r)dr^2 + r^2 d\Theta^2 - f^2(r)dt^2,$$

y sea $H : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ una función continua radialmente simétrica. Entonces, existe un grafo espacial entero esféricamente simétrico con función curvatura media H. Además, para cada $R > 0$, el grafo puede ser elegido de manera que su intersección con la hipersuperficie $t = 0$ sea una esfera de radio R.

Teorema 4.3.2 *Sea $\mathcal{M}$ el espaciotiempo exterior de Schwarzschild o el espaciotiempo exterior de Reissner-Nordström con radio $a > 0$, y sea $H : \mathcal{M} \longrightarrow \mathbb{R}$ una función continua, acotada y esféricamente simétrica. Entonces, existe un grafo espacial entero esféricamente simétrico con curvatura media H que se aproxima al horizonte de sucesos cuando $r \to a$. Además, para cada $R > a$, el grafo puede ser escogido de tal manera que su intersección con la hipersuperficie $t = 0$ sea una esfera de radio R.*

Las pruebas de los teoremas previos están basado en los siguiente resultados de existencia de los problemas de Dirichlet asociados sobre una bola, también interesantes en sí mismos.

Teorema 4.2.1 *Sea $\mathbb{R}^n \times_f \mathbb{R}$ un espaciotiempo estático estándar, provisto de la métrica esféricamente simétrica*

$$E^2(r)dr^2 + r^2 d\Theta^2 - f^2(r)dt^2.$$

Sea $B = B_0(R)$ una bola euclídea de radio R centrada en $0 \in \mathbb{R}^n$, y sea $H : \overline{B} \times \mathbb{R} \to \mathbb{R}$ una función continua esféricamente simétrica. Entonces, existe un grafo espacial esféricamente simétrico con curvatura media H definido en $\overline{B}$ y soportado en el slice $t = 0$.

Teorema 4.2.13 *Sea $\mathcal{M}$ el espaciotiempo exterior de Schwarzschild o el espaciotiempo exterior de Reissner-Nordström (con radio a), y sea $H : \overline{A}(a, R) \times \mathbb{R} \to \mathbb{R}$ una función de clase C^∞, acotada y esféricamente simétrica, donde $\overline{A}(a, R)$ es el anillo cerrado en $\mathbb{R}^n$ $a \leq |x| \leq R$. Entonces, existe un grafo espacial esféricamente simétrico con función curvatura media H, que toca la hipersuperficie $t = 0$ en el borde $|x| = R$, y se aproxima al horizonte de sucesos cuando $|x| \to a$. Además, el grafo es radialmente decreciente en el anillo $\overline{A}(a, R)$ e interseca la hipersuperficie $t = 0$ sólo en el borde $|x| = R$.*

Una inspección de la prueba del Teorema 4.2.13 proporciona más detalles sobre la geometría de las soluciones. Uno de estos es que el ángulo hiperbólico entre el normal unitario del grafo y ∂_t puede ser prescrito. Más precisamente, existe χ_0 tal que para cada $\chi < \chi_0$, el grafo dado en el Teorema 4.2.13 puede ser escogido de tal manera que dicho ángulo hiperbólico es χ. Además el valor de χ_0 puede ser explícitamente calculado (ver la desigualdad (4.22)).

En el siguiente capítulo afrontamos el problema de la prescripción de las curvaturas medias de orden superior. Dada una hipersuperficie orientable en $\mathbb{R}^{n+1} \equiv \mathbb{R}^{n+1}_a$, $a = 0$, o hipersuperficie espacial en $\mathbb{L}^{n+1} \equiv \mathbb{R}^{n+1}_a$, $a = 1$, las k-ésimas curvaturas medias son invariantes geométricos que codifican toda la geometría extrínseca de la hipersuperficie. Desde un punto de vista algebraico, cada una de estas funciones se corresponde con un coeficiente del polinomio característico del operador de Weingarten correspondiente a un campo de vectores unitario normal ($a = 0$) o a un campo de vectores temporal unitario normal apuntando al futuro ($a = 1$). En realidad, cada k-curvatura media es un tipo de promedio de las curvaturas principales de la hipersuperficie. En particular, la primera curvatura media corresponde con la usual curvatura media si $a = 0$ o su opuesta si $a = 1$, la segunda curvatura media coincide, salvo un factor constante, con la curvatura escalar, y la n-ésima curvatura media es la curvatura de Gauss-Kronecker si $a = 0$ o $(-1)^{n+1}$ veces la curvatura de Gauss-Kronecker si $a = 1$. Cada k-ésima curvatura media tiene una naturaleza variacional [71], y, en el caso riemanniano, las hipersuperfices de k-curvatura media constante han sido extensamente estudiadas ([54], [77] por ejemplo). Desde una perspectiva física, las k-ésimas curvaturas medias juegan un papel importante en Relatividad General. La k-sima curvatura media mide, intuitivamente, la evolución temporal hacia el futuro (o hacia el pasado) del universo espacial que representa la hipersuperficie espacial considerada (ver Nota 2.2.1).

Previamente hemos descrito el trabajo que ha sido realizado en el problema de la prescripción de la curvatura media. Respecto al problema Dirichlet asociado a la prescripción de la curvatura escalar, podemos destacar [20], en el contexto euclídeo. Por otra parte, Bayard probó la existencia de hipersuperficies espaciales enteras con curvatura escalar prescrita en el espacio de Minkowski [7], usando algunos trabajos previos en el problema Dirichlet ([8] y [85] y referencias allí citadas). Gerhardt [52] también obtuvo resultados importantes para espaciotiempos más generales. Finalmente, la curvatura de Gauss-Kronecker también ha sido bastante bien estudiada en ambos ambientes. En el espacio euclídeo, Wang [87] prescribió la curvatura de Gauss-Kronecker de una hipersuperficie convexa. En el espacio de Minkowski, destacamos el trabajo de Li [60] para curvatura de Gauss constante y el de Delanoë [43], en el que prueba la existencia de hipersuperficies espaciales enteras asintóticas a un cono de luz con curvatura de Gauss-Kronecker prescrita.

Excepto en la última década, poca ha sido la atención prestada al problema de prescripción de la función k-ésima curvatura media cuando $3 \leq k < n$. Uno de los primeros trabajos en esta dirección fue realizado por Ivochkina (ver [57] y referencias allí citadas). Más recientemente han surgido varias contribuciones (por ejemplo [55], [86]), especialmente en el problema Dirichlet. El estudio suele enfocarse generalmente hacia la búsqueda de cotas a priori de la norma del operador forma, suponiendo que las soluciones son k-estables para asegurar la elipticidad de los operadores diferenciales involucrados (ver [86] para más detalles). Luego se impone alguna dependencia especial a la función de prescripción para obtener resultados parciales. En este quinto capítulo proporcionaremos varios resultados sobre este problema abierto, suponiendo que la función de prescripción dada es rotacionalmente simétrica respecto de una recta, en el caso euclídeo, o un observador inercial γ (es decir, una recta temporal unitaria apuntando al futuro) en el caso de Minkowski. Probaremos la existencia de grafos rotacionalmente simétricos con k-ésima curvatura media prescrita asociados al problema de Dirichlet cuando el dominio es una bola n-dimensional, usando un adecuado operador de punto fijo. Finalmente, probaremos que tales grafos se pueden extender a todo el espacio, proporcionando también información sobre su unicidad.

Usaremos las usuales coordenadas cilíndricas (t, r, Θ) en $\mathbb{R}_a^{n+1}$ asociadas γ, es decir, $t \in \mathbb{R}$ es el parámetro de γ, $r \in \mathbb{R}^+$ es la distancia radial a γ y $\Theta = (\theta_1, ..., \theta_{n-1})$ son las coordenadas esféricas estándar de la esfera unitaria $(n-1)$-dimensional, $\mathbb{S}^{n-1}$. Supondremos que las funciones de prescripción H_k son radialmetne simétricas respecto a γ. Por tanto, es natural considerar $H_k(t, x) = H_k(t, r)$ donde r denota la distancia de $x \in \mathbb{R}^n$ a γ.

Debido a la notable diferencia entre la geometría de los espacios euclídeo y de Minkowski, en la literatura relacionada uno puede encontrar una clara distinción entre dos grandes grupos de artículos, dependiendo de si el ambiente es euclídeo o lorentziano. Sin embargo, nosotros aquí presentamos los resultados en ambos contextos porque, aunque los teoremas son distintos, el tratamiento matemático es similar.

A continuación resumimos los principales resultados sobre grafos espaciales enteros en el ambiente lorentziano. El primer teorema es un tipo de “resultado de existencia universal” cuando k es impar.

Teorema 5.4.1 *Sea* $H_k : \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$*, con* k *un entero impar positivo, una función continua rotacionalmente simétrica con respecto de un observador inercial* γ *de* $\mathbb{L}^{n+1}$*. Entonces, para cada* $R > 0$*, existe al menos un grafo espacial entero, rotacionalmente simétrico respecto a* γ*, cuya* k*-ésima curvatura media es igual a* H_k *y tal que interseca el hiperplano ortogonal a* γ *en* $\gamma(0)$ *en una* $(n-1)$*-esfera de radio* R *centrada en* $\gamma(0)$*. Además, si* H_k *es no decreciente con respecto al tiempo propio de* γ*, entonces el grafo espacial es único.*

Para k par, tenemos que introducir una restricción natural en la curvatura, como muestra el siguiente resultado.

Teorema 5.4.2 *Sea* $H_k : \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$*, con* k *un entero par positivo, una función continua tal que*

$$\int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \geq 0 \quad \text{para todo} \quad r \in \mathbb{R}^+, \quad \text{y} \quad v \in C^1,\ |v'| < 1, \tag{3}$$

y que es rotacionalmente simétrica respecto de un observador inercial γ *de* $\mathbb{L}^{n+1}$*. Si* $H_k(0,\cdot) \not\equiv 0$*, para cada* $R > 0$*, entonces existen al menos dos grafos espaciales enteros rotacionalmente simétricos diferentes cuya* k*-ésima curvatura es igual a* H_k *intersecando al hiperplano ortogonal a* γ *en* $\gamma(0)$ *en una* $(n-1)$*-esfera con radio* R *centrada en* $\gamma(0)$*. La curva del perfil radial de uno de ellos es creciente y la del otro decreciente. Además, la condición* (3) *es necesaria para la existencia de tales grafos.*

Por otra parte, para el caso de grafos enteros en $\mathbb{R}^{n+1}$ no podemos esperar un resultado de tipo 'universal' como el anterior. Si queremos obtener un grafo rotacionalmente simétrico con k-curvatura media prescrita que interseque al hiperplano ortogonal en una n-esfera de radio R, es necesario introducir una cota adicional en la función de prescripción, como en el siguiente resultado.

Teorema 5.4.3 *Sea $H_k : \mathbb{R}^{n+1} = \mathbb{R} \times \Pi \longrightarrow \mathbb{R}$, con k un entero impar positivo, una función continua rotacionalmente simétrica respecto de una línea orientada γ, ortogonal a Π. Para cada $R > 0$, supongamos que hay algún $\alpha \in (0, R^{-k})$, satisfaciendo*

$$|H_k(t,r)| \le \alpha \qquad \text{para todo} \qquad r \in [0,R]\,, \quad t \in [-R\,\beta, R\,\beta], \tag{4}$$

donde $\beta := \dfrac{R\alpha^{1/k}}{\sqrt{1-R^2\alpha^{2/k}}}$ *y*

$$0 \le \int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds < \frac{r^{n-k}}{n}, \qquad \text{para todo} \qquad r > R \quad \text{y} \quad v \in C^1. \tag{5}$$

Entonces, existe al menos un grafo entero, rotacionalmente simétrico respecto a γ, con k-ésima curvatura media igual a H_k e intersecando al hiperplano Π en una $(n-1)$-esfera de radio R centrada en $\gamma(0)$. Además, si H_k no es decreciente a lo largo de la línea γ, entonces el grafo es único.

En la memoria veremos que la condición (4) es bastante natural, en particular es necesaria cuando la función de prescripción H_k es constante. El último resultado del capítulo considera el caso de k par en el espacio euclídeo.

Teorema 5.4.4 *Sea $H_k : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, con k un entero par positivo, una función continua rotacionalmente simétrica respecto de una línea γ. Para cada $R > 0$, su-*

pongamos que hay algún $0 < \alpha < R^{-k}$*, satisfaciendo* (4)*,* (5) *y*

$$\int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \geq 0, \qquad \text{para todo} \qquad r \in [0, R] \quad \text{y} \quad v \in B_{R\beta,\beta},$$

Entonces, si $H_k(0, \cdot) \not\equiv 0$*, existen al menos dos grafos enteros rotacionalmente simétricos diferentes con* k*-ésima curvatura media igual a* H_k *e intersecando al hiperplano ortogonal a* γ *en* $\gamma(0)$ *en una* $(n-1)$*-esfera de radio* R *centrada en* $\gamma(0)$*. Además, la curva del perfil radial de uno de ellos es creciente y la del otro decreciente.*

Los últimos tres capítulos, de fuerte motivación física, están orientados introducir y analizar las nociones relativistas de movimiento uniformemente acelerado, rectilíneo y circular uniforme, conceptos que por otra parte son bien conocidos en el contexto de la mecánica clásica.

En una primera impresión, el movimiento uniformememte acelerado puede parecer imposible en Relatividad, debido a la existencia de una cota superior para las velocidades dada por la velocidad de la luz. En mecánica newtoniana, se dice que una partícula está acelerada cuando cualquier familia de observadores inerciales mide una *aceleración relativa* de la partícula respecto a ellos. Pero no es necesario fijar un sistema de referencia para definir la aceleración de una partícula (en realidad, en Relatividad General la noción de observador inercial no está definida). En el ambiente prerrelativista, la aceleración de una partícula puede detectarse mediante un *acelerómetro*. Intuitivamente, podemos imaginar un acelerómetro como una esfera (tridimensional) en cuyo centro colocamos un pequeño objeto redondo que está unido mediante cuerdas elásticas a (idealmente todos) los puntos de la superficie esférica. Así, un observador en caída libre provisto de tal acelerómetro observará que la bola permanece inmóvil en el centro de la esfera. Por el contrario, estará acelerado siempre que la vea descentrada. Este argumento intuitivo sugiere que un movimiento uniformemente acelerado podrá ser detectado mediante un desplazamiento constante del pequeño objeto redondo del acelerómetro. Y esta idea puede ser usada independientemente de si el espaciotiempo es relativista o no. Nuestro primer problema consiste en dar rigor matemático a la afirmación "el acelerómetro marca un valor constante".

La noción de movimiento uniformemente acelerado en Relatividad General ha sido discutida muchas veces en los últimos cincuenta años [47]. El trabajo pionero de Rindler [72] fue motivado, en parte, por algunos aspectos de lanzamientos de cohetes intergalácticos, usando las fórmulas del movimiento hiperbólico en Relatividad Especial [62].

La relación entre el movimiento uniformemente acelerado y las circunferencias lorentzianas (ciertas hipérbolas euclídeas) en el espaciotiempo de Minkowski expuesta en [73] (ver también [66, Sec. 6.2]) ya fue descrita por Rindler en 1960 [72] para definir lo que él denominó movimiento hiperbólico en Relatividad General, extendiendo la noción de movimiento uniformemente acelerado ya existente en el espaciotiempo de Minkowski (de acuerdo con una de las dos nociones propuestas pro Marder pocos años antes [63]). Pese al tiempo transcurrido desde el artículo seminal de Rindler, el asunto sigue siendo de interés actual ([47], [48] y referencias allí citadas. Ver también [65]).

El primer objetivo del Capítulo 6 es estudiar el movimiento uniformemente acelerado en Relatividad General, desde el enfoque de la geometría lorentziana. Para ello, recordemos que una *particula* de masa $m > 0$ en una espaciotiempo $(M, \langle\, ,\, \rangle)$ es una curva $\gamma : I \longrightarrow M$, tal que su velocidad $\gamma\,'$ satisface $\langle \gamma\,', \gamma\,' \rangle = -m^2$ y señala al futuro. Un *observador* es una partícula de masa $m = 1$. La derivada covariante de $\gamma\,'$, $\dfrac{D\gamma\,'}{dt}$, es su aceleración (propia) que puede verse como la traducción matemática de los valores medidos por un acelerómetro como el mencionado anteriormente. Intuitivamente, la partícula obedece un movimiento uniformemente acelerado si su aceleración permanece inalterada. Matemáticamente necesitamos una *conexión a lo largo de* γ que permita comparar direcciones espaciales en diferentes instantes de la vida de γ. En Relatividad General esta conexión es conocida como la *conexión de*

Fermi-Walker de γ (ver sección 2.3 para más detalles). Entonces, usando la correspondiente derivada covariante de Fermi-Walker $\frac{\widehat{D}}{dt}$, diremos que una partícula obedece un movimiento *uniformemente acelerado* (UA) si satisface la siguiente ecuación diferencial de tercer orden,

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right) = 0,$$

es decir, si la aceleración del observador γ es Fermi-Walker paralela (constante para él) a lo largo de su línea de vida.

Hay muchas situaciones físicas en las que aparecen los movimientos UA. Por ejemplo, cuando colocamos una partícula cargada eléctricamente $(\gamma(t), m, q)$ en presencia de un campo electromagnético F, la dinámica de la partícula viene completamente descrita por la bien conocida ecuación de la fuerza de Lorentz (ver [78] por ejemplo),

$$m\frac{D\gamma'}{dt} = q\widetilde{F}(\gamma'),$$

donde $\widetilde{F}$ es el campo tensorial de tipo (1,1) métricamente equivalente a la 2-forma cerrada F ($\widetilde{F}$ tiene la misma información física que F). El campo vectorial $\widetilde{F}(\gamma')$ a lo largo de γ es llamado campo eléctrico relativo a γ, [78, p. 75]. La conexión de Fermi-Walker de γ nos capacita para decir que γ percibe un campo eléctrico "constante" si

$$\frac{\widehat{D}}{dt}\left(\widetilde{F}(\gamma')\right) = 0.$$

Entonces, si una partícula γ se mueve en presencia de un campo electromagnético F y su campo eléctrico relativo satisface la ecuación anterior, γ obedece un movimiento uniformemente acelerado.

En la sección 6.2 expondremos en detalle cómo los observadores UA pueden ser vistos como circunferencias lorentzianas en un espaciotiempo general (Proposición 6.2.2). En particular, la afirmación (d) de este resultado corresponde con la noción propuesta originalmente por Rindler en [72]. Además, caracterizaremos los espaciotiempos estáticos estándar como aquellos que admiten un campo de observadores uniformemente acelerado, rígido y localmente sincronizable (Teorema 6.2.4).

Por otra parte, los observadores UA pueden ser tratados como proyecciones en el espaciotiempo de las curvas integrales de un campo de vectores definido sobre un cierto fibrado. Usando este campo vectorial, analizamos la completitud de los observadores UA inextensibles, buscando hipótesis geométricas que aseguren que tales observadores no desaparecen del espaciotiempo en un tiempo propio finito (la ausencia de singularidades de este tipo). Concretamente, en un espaciotiempo compacto que admita un campo de vectores temporal conforme y cerrado, cualquier observador UA inextensible es completo (Teorema 6.3.5).

Retornando a la imagen intuitiva del acelerómetro, si la bola se mueve a lo largo de un radio, un observador podría pensar que se mueve obedeciendo un movimiento rectilíneo. Esta simple idea nos lleva a la definición de movimiento rectilíneo en Relatividad General que, hasta donde sabemos, ha sido estudiado pocas veces y sólo en el contexto de la Relatividad Especial [47], [48].

Siguiendo esta línea, en el Capítulo 7 daremos una definición rigurosa de la afirmación "la aceleración propia no cambia su dirección", introduciendo de esta manera, el movimiento rectilíneo (brevemente, movimiento UD) en Relatividad General.

Usando la correspondiente derivada de Fermi-Walker $\frac{\widehat{D}}{dt}$, diremos que una partícula estrictamente acelerada obedece un *movimiento rectilíneo* (UD) si verifica la siguiente ecuación diferencial (Definición 1),

$$\frac{\widehat{D}}{dt}\left(\left|\frac{D\gamma'}{dt}\right|^{-1}\frac{D\gamma'}{dt}\right)=0,$$

es decir, si el vector aceleración normalizado del observador γ es Fermi-Walker paralelo a lo largo de su línea de vida. Nótese que si un observador γ obedece un movimiento uniformemente acelerado y la constante $\left|\frac{D\gamma'}{dt}\right|$ es positiva, entonces obedece un movimiento UD, sin embargo, la clase de observadores UD es mucho más amplia que la de los uniformemente acelerados. En realidad, si un observador en el espaciotiempo de Minkowski $n(\geq 2)$-dimensional $\mathbb{L}^n$ permanece en un plano lorentziano, entonces obedece un movimiento rectilíneo. Recíprocamente, cada observador UD en $\mathbb{L}^n$, γ, está contenido en un plano lorentziano determinado por el punto $\gamma(0)$, la aceleración inicial y la 4-velocidad inicial. Más generalmente, cada observador UD en un espaciotiempo de curvatura seccional constante debe estar contenido en una subvariedad lorentziana 2-dimensional totalmente geodésica.

Más ampliamente, introduciremos la noción de movimiento rectilíneo a trozos (Definición 2). Un observador UD a trozos es esencialmente un observador que puede cambiar su dirección sólo cuando su acelerómetro marca cero. Cada uno de estos observadores aparece naturalmente como una solución de una ODE, (7.4), más general que la fórmula (1.7). También establecemos y resolvemos completamente el problema de encontrar un observador UD, γ, con aceleración escalar $\left|\frac{D\gamma'}{dt}\right|$ prescrita, obteniendo una integral primera explícita (Teorema 7.1.1). A continuación, caracterizaremos geométricamente a los observadores UD a trozos (Proposición 7.2.2(c)). Nótese que la prescripción de la aceleración escalar de los observadores UD puede verse como un problema de prescripción de la curvatura media (Proposición 7.2.2(e)). Entre las caracterizaciones geométricas de los movimientos rectilíneos a trozos de la Proposición 7.2.2 (extendiendo [39], [72]), cabe destacar que un observador obedece un movimiento rectilíneo a trozos sí y sólo sí su desarrollable , en el sentido de [59, Sect. III.4], en el espacio tangente al espaciotiempo, en cualquiera de sus puntos, es una curva plana a trozos (Proposición 7.2.2(c)).

Al igual que los observadores uniformemente acelerados, los rectilíneos son caracterizados como curvas obtenidas por la proyección sobre el espaciotiempo de las curvas integrales de un campo vectorial sobre cierto fibrado (Lema 7.3.2). Mediante este campo, analizamos la completitud de los observadores UD inextensibles cuando el espaciotiempo posee cierta simetría conforme. Concretamente, mostramos que un observador UD en un espaciotiempo que admite un campo de vectores temporal conforme y cerrado (como en un espaciotiempo de Robertson-Walker Generalizado) puede ser extendido si está contenido en un subconjunto compacto del espaciotiempo (Teorema 7.3.5). Finalmente, trataremos el problema de extendibilidad cuando el espaciotiempo ambiente pertenece a una clase de espaciotiempos pp-wave: los espaciotiempos Plane Wave. Ahora la herramienta clave será la integral primera explícita obtenida en Teorema 7.1.1. Así, probamos que cada trayectoria inextensible de un observador UD con aceleración prescrita a en un espaciotiempo Plane Wave es completa (Teorema 7.4.3).

Finalmente, en el Capítulo 8, estudiamos el movimiento circular uniforme. Éste ha sido ampliamente estudiado en Relatividad Especial (ver, por ejemplo, [46]). Su estudio ha sido motivado por numerosos fenómenos físicos y paradojas, relacionados a veces con la precesión de Thomas, y su interés aún sigue vigente (ver, por ejemplo, [80] para una introducción intuitiva).

El movimiento circular uniforme se introduce habitualmente fijando una familia de observadores inerciales, y considerando a uno de ellos como el 'centro'. Así, suele decirse que un observador describe un movimiento circular respecto al 'centro' si la trayectoria medida por dicha familia de observadores inerciales es una circunferencia y la velocidad angular es constante para ellos. También se han hecho otras aproximaciones a partir de las ecuaciones de Frenet [47], [48].

Varios ejemplos de movimiento circular ya han sido considerados previamene en modelos relativistas relevantes como los espaciotiempos de Schwarzschild, Reissner-Nordström y Kerr, todos ellos con alguna simetría rotacional [66, Ch. 25]. Cada uno de estos espaciotiempos posee una familia especial de observadores que juega un papel similar al de los observadores inerciales en el espaciotiempo de Minkowski. Un observador situado en el centro de una estrella (o un agujero negro), y en reposo respecto a ella, será considerado el centro de las trayectorias circulares. El observador en movimiento circular describirá un circunferencia respecto a la familia inercial considerada y la velocidad angular medida por ésta será constante. El análisis de este tipo de movimientos es de reconocido interés físico y tecnológico, pues se corresponde con el movimiento de algunos satélites, planetas o estrellas (ver, por ejemplo, [49]).

Nuestro interés aquí se centra en primer lugar en la introducción de una definición de movimiento circular uniforme en un espaciotiempo general involucrando cantidades observables medibles por el propio observador. En otras palabras, daremos una definición 'intrínseca' de movimiento circular uniforme, sin considerar ninguna familia externa de observadores sincronizados que juzgue si el movimiento es circular o no. Obsérvese además que la existencia de una tal familia de observadores no está garantizada en un espaciotiempo genérico. Por supuesto, nuestra definición coincidirá con la noción estándar de los casos especiales previamente citados. Para determinar el estado cinemático inherente a un observador nos centraremos en su aceleración propia.

Un observador es capaz de detectar si está rotando o no por medio de un giróscopo o un acelerómetro como el comentado anteriormente. Intuitivamente, si un observador comprueba que la bola del acelerómetro describe una rotación uniforme plana, entonces pensará que obedece un movimiento circular uniforme.

En primer lugar, establecemos la noción de movimiento 'plano' en un espaciotiempo arbitrario. Clásicamente, se dice que un observador se 'mueve en un plano' cuando la proyección de su trayectoria espaciotemporal sobre el espacio euclídeo absoluto está contenida en un plano. Equivalentemente, cuando su aceleración propia permanece siempre en el mismo plano. Esta última caracterización puede ser extendida a cualquier espaciotiempo, relativista o no. Obviamente, como ocurre en mecánica clásica, un movimiento circular uniforme debe ser también un movimiento plano. El sutil problema que surge en Relatividad General reside en dar sentido a la frase '*permanece en el mismo plano*'. Nuevamente, esta tarea puede realizarse con precisión y rigor mediante la conexión de Fermi-Walker de cada observador.

En efecto, supongamos que un observador γ obedece un movimiento plano. Un primer requisito necesario para que un observador obedezca un movimiento *circular uniforme* (UC) es que el módulo de su aceleración propia se mantenga constante, es decir,

$$\left|\frac{D\gamma\,'}{dt}\right|^2 = \text{constante}.$$

Usando la correspondiente derivada covariante de Fermi-Walker $\dfrac{\widehat{D}}{dt}$, si una partícula obedece un movimiento UC, se requerirá también que,

$$\left|\frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right)\right|^2 = \text{constante},$$

es decir, el módulo del cambio de su aceleración debe de permanecer inalterado. Tras estas consideraciones, llegamos a la noción principal expuesta en la Definición 6.

Por otra parte, los movimientos UC aparecen naturalmente en muchas situaciones físicas. Continuando con la notación anterior, consideremos en el espacio de Minkowski $\mathbb{L}^4$ el campo electromagnético,

$$F = 2B_0\, dx \wedge dy,$$

donde $B_0 \in (0, \infty)$ y (t, x, y, z) son las coordenadas estándar de $\mathbb{L}^4$. La familia de observadores inerciales ∂_t mide un campo magnético uniforme de módulo B_0 y señalando hacia ∂_z (y un campo eléctrico nulo) para este campo electromagnético F. Ahora bien, la partícula γ obedecerá un movimiento UC y su trayectoria puede expresarse como [78, Proposición 3.8.2],

$$\gamma(\tau) \,=\, p \,+\, \Big(m\tau\,\sqrt{1 + R^2 w^2}, R\cos(wm\tau + \vartheta), R\sin(wm\tau + \vartheta), 0\Big)\,,$$

con $w = \frac{qB_0}{m} \in \mathbb{R}$, $p \in \mathbb{L}^4$, $R > 0$ y $\vartheta \in \mathbb{R}$, siempre que la velocidad inicial de la partícula respecto a la familia de observadores inerciales permanezca en el plano xy [78, p. 88].

Además de presentar la noción de movimiento UC (Definición 6), analizamos con detalle el correspondiente sistema diferencial y el problema de valores iniciales asociado. Posteriormente, vemos que un observador UC puede ser geométricamente identificado con una hélice lorentziana en un espaciotiempo general (Proposición 8.2.4) y utilizaremos este resultado para caracterizar los observadores UC como soluciones de una ecuación diferencial de cuarto orden (Proposición 8.19). Análogamente a los observadores UA y UD, los UC son las proyecciones en el espaciotiempo de las curvas integrales de cierto campo de vectores definido sobre un fibrado de tipo Stiefel. Nuevamente, usaremos esta identificación para estudiar la completitud de los observadores UC inextensibles (Lema 8.3.3). Así por ejemplo obtenemos que todo observador UC en un espaciotiempo de Robertson-Walker Generalizado cuya línea de vida permanezca en un subconjunto compacto del espaciotiempo 'vivirá para siempre' (Teorema 8.3.4). Finalmente, probamos la completitud de cualquier observador UC en un espaciotiempo Plane Wave, haciendo uso de técnicas analíticas (Teorema 8.4.3), demostrando la ausencia de este tipo de singularidades.

Los resultados de esta memoria aparecen en las siguientes referencias de la bibliografía [10], [36], [37], [38], [39], [40] y [41].

Chapter 1

Introduction

The ground-breaking discovery of the theory of General Relativity revealed that gravity can be successfully described by treating space and time on the same footing by means of a pseudo-Euclidean geometry. Nowadays, the mathematical framework of General Relativity can be regarded as a branch of Geometry (Lorentzian Geometry), in a similar sense like mathematics of Classical Mechanics is a branch of Symplectic Geometry. The growing interest to the Relativity from the mathematical community gave rise to the emergence of a great number of analytical and geometrical problems. We will study some of them along this work.

This thesis may be structured into two blocks. In the first one (Chapters 3, 4 and 5) we deal with the problem of the mean curvature prescription. The treatment is mostly analytical, making use of classical degree theorems and approximation methods. Physical interpretation of the results is also given. Respect to the second part (Chapters 6, 7 and 8), we introduce and analyse in detail several concepts in the relativistic framework, which already had a well-known classical formulation. Geometric approach is used to formulate definitions and problems to solve, although sometimes analytical arguments are also required. In fact, it would be more accurate to claim that geometric analysis is the mathematical background to investigate on such physical problems. This memory is another example showing that the current separation between different branches of Mathematics disappears when are applied to model the nature.

The mean curvature prescription problem will be the thread of this memory. Despite being organized in two well differenced parts, in both extrinsic properties of certain submanifolds of a spacetime will be studied. Concretely, we will center on the unique definite submanifolds such that its normal bundle is also definite.

We start this memory studying the following Dirichlet boundary problem.

Let $B(R)$ be the Euclidean ball, centered at $0 \in \mathbb{R}^n$ with radius R. Let $I \subseteq \mathbb{R}$ be an open interval with $0 \in I$, and let $f \in C^\infty(I)$ be a positive function. For a given smooth radially symmetric function $H : I \times B(R) \to \mathbb{R}$, we study the existence of positive, radial solutions of the following quasilinear elliptic problem

$$\begin{aligned} \operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - |\nabla u|^2}}\right) + \frac{f'(u)}{\sqrt{f(u)^2 - |\nabla u|^2}}\left(n + \frac{|\nabla u|^2}{f(u)^2}\right) &= nH \quad \text{in} \quad B(R), \\ |\nabla u| &< f(u), \\ u = 0 \quad \text{in} \quad \partial B(R). & \end{aligned} \tag{1.1}$$

The approach to this PDE is motivated from Lorentzian Geometry, specifically from the problem of the mean curvature prescription. Explicitly, every solution of (1.1) defines a spacelike graph on a ball of the fiber of the Friedman-Lemaître-Robertson-Walker (FLRW) spacetime, $\mathcal{M} = I \times_f \mathbb{R}^n$ (see next chapter for details) where the function H prescribes the mean curvature.

A spacelike hypersurface in a spacetime is a hypersurface which inherits a Riemannian metric from the ambient Lorentzian one. Intuitively, a spacelike hypersurface is the spatial universe at one instant of proper time of a family of observers. In fact, such a hypersurface defines the family of *normal observers*: each geodesic in the ambient spacetime determined from a point of the spacelike hypersurface and the future pointing unit normal vector at this point. The corresponding mean curvature function measures *how these observers get away or come together with respect to a given one*. Indeed, these observers can be locally collected as the integral curves of a reference frame in spacetime and the sign of its divergence (i.e., the measure of expansion/contraction for the observers in the reference frame, [68, 78]) is the same of the sign of the mean curvature function. We are interested here in prescribing the mean curvature function for the case these observers get away in a FLRW cosmological model.

On the other hand, a spacelike hypersurface is a suitable subset in spacetime where the initial value problem for each of the classical equations in General Relativity (matter equations, Maxwell equations and Einstein equations) is well posed. The case of constant mean curvature is relevant, specially when it vanishes (i.e., the maximal case). On the one hand, they can constitute an initial set for a Cauchy problem [74]. Specifically, Lichnerowicz proved that a Cauchy problem with initial conditions on a maximal hypersurface reduces to a second-order non-linear elliptic differential equation and a first-order linear differential system [3, 30, 61]. Moreover, these hypersurfaces are important in order to analyse the dynamics of a gravitational field or the classical n-body problem in a gravitational field (see, for instance, [18] and references therein).

Each maximal hypersurface can describe, in some relevant cases, the transition between an expanding and contracting phase of a relativistic universe. Moreover, the existence of a constant mean curvature spacelike (maximal in particular) hypersurface is necessary for the study of the structure of singularities in the space of solutions to the Einstein equations. The deep understanding of this kind of hypersurfaces is also essential to prove the positivity of the gravitational mass. In Numerical Relativity, maximal hypersurfaces are used to integrate forward in time. All these physical aspects can be found in [64] and references therein.

Geometrically, spacelike hypersurfaces with constant mean curvature in a Lorentzian manifold appear as the critical points of the "area" functional under certain "volume constraint" [16, 27, 28]. The existence of spacelike hypersurfaces with constant mean curvature constitutes a classical and important problem (see [17] and references therein). It has been useful to prove satisfactory uniqueness results. Among the uniqueness results, the seminal paper by Cheng and Yau [28] where the proof of the Calabi-Bernstein conjecture for any n-dimensional Minkowski spacetime was given, also introduced a new type of elliptic problem which have been developed in several different spacetimes, see for instance [16, 28, 75].

In the latter years, many researchers dealt with the prescribed mean curvature problem in Riemannian ambient (especially in Euclidean space). In the Lorentzian ambient, efforts have mainly focused in the Minkowski spacetime. In this context, it is remarkable the celebrated paper of Bartnik and Simon [6] in 1982, where a kind of "universal existence result" is proved for the Dirichlet problem. More recently, the interest is focused on the existence of positive solutions, by using a combination of variational techniques, critical point theory, sub-supersolutions and topological degree (see for instance [11–13, 31–33] and the references therein). Up to our knowledge, the problem of the existence of prescribed mean curvature spacelike graphs for FLRW spacetimes has not been considered before. In this context, the uniqueness problem for constant mean curvature has been studied in more depth (see for instance [1], [19]).

The first aim consists in by using an approach based on the Schauder fixed point Theorem (see for instance [42]) to deal with the existence problem. It should be noted that our results do not follow directly from the obtained ones previously when $\mathcal{M}$ is Minkowski spacetime ([11] and references therein). In fact, we will deal here with an equation with an extra singular term with respect to the considered in Minkowski spacetime. Besides, we give conditions only on the prescription function which ensure *a priori* radial symmetry of all the (possible) positive solutions of the equation (3.3). In other words, we will prove that *the symmetry of the base domain 'spreads to solutions'*. In order to do that, we will take advantage of the results obtained in 1979 by B. Gidas, W. Ni and L. Nirenberg in [53] about the symmetry of the solutions of certain nonlinear differential equations. The method used by the three authors was previously invented by Alexandroff almost thirty years before, when he proved successfully that the round spheres are the only connected, compact hypersurfaces embedded in the Euclidean space with constant mean curvature. Indeed, currently this technique is known as 'Alexandroff reflection method' and its use is very extended in the field of the elliptic PDE and Geometric Analysis. In our case, we are able to use a truncature argument exposed in [31] and then to apply the results of [53].

The first finding about the associated Dirichlet problem can be summarized as follows.

Theorem 3.3.5 *Let $I \times_f \mathbb{R}^n$ be a Friedmann-Lemaître-Robertson-Walker spacetime, and let $B = B_0(R)$ be the Euclidean ball with radius R centred at $0 \in \mathbb{R}^n$. Assume $I_f(R) \subset I$, where*

$$I_f(R) := \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds \, , \, \int_0^R f(\varphi^{-1}(s))ds \right] \quad \text{and} \quad \varphi(t) = \int_0^t \frac{dt}{f(t)},$$

and suppose that the following inequality holds

$$\max_{\mathbb{R}^+ \cap I_f(R)} |f'| < \frac{1}{R}.$$

For each radially symmetric smooth function $H : I \times \overline{B} \to \mathbb{R}$ such that

$$H(t,r) \leq \frac{f'}{f}(t) \quad \textit{and} \quad f'(t) \geq 0, \qquad \textit{for any} \quad r \in]0,R[\ , \ t \in I_f(R),$$

there exists a spacelike graph with mean curvature function H defined on $\overline{B}$, supported on the slice $t = 0$, only touching it on the boundary $\{0\} \times \partial B$, and forming a non-zero hyperbolic angle with ∂_t. Moreover, if H is increasing in the second variable, such a spacelike graph must be radially symmetric.

The family of FLRW spacetimes where the result applies is very wide, and contains relevant relativistic spacetimes. Indeed, it includes the Minkowski spacetime ($f = 1$, $I = \mathbb{R}$), the Einstein-De Sitter spacetime ($I =]-t_0, +\infty[$, $f(t) = (t + t_0)^{2/3}$, with $t_0 > 0$), and the steady state spacetime ($I = \mathbb{R}$, $f(t) = e^t$), which may be seen as an open subset of the De Sitter spacetime.

However, a suitable bound for the radius R is required. To remove such assumption, we have to use a different method to achieve the proof. While the previous result relies on a basic application of Schauder's fixed point theorem, now we need a more sophisticated approach. When passing to polar coordinates, we arrive to a problem with a double singularity: the first singularity on the independent variable at $r = 0$ (it is the usual singularity that appears at the origin in any radially symmetric problem defined on a ball), the second singularity is not of a standard type on the related literature since it involves on the dependent variable (see the second term of the left-hand side of equation (3.5)). To handle the first singularity, we use an approximation method through a family of truncated problems, which is a classical approach for radial problems defined on a ball (see for example [70, Chapter 9] and references therein), although in this context it is essentially new. On this sequence of approximated problems, the second singularity is handled by an accurate manipulation of the equation (see the first step of the proof of Theorem 3.5.1) that leads to a sequence of approximated solutions. The key point to prove the convergence of this sequence is a delicate estimate of an a priori bound for the derivative of the solutions on the boundary (see Proposition 3.4.2). Once the Dirichlet problem is solved, the existence of a entire solution is obtained by an extension argument of the Dirichlet solution. From an analytical point of view, this problem is expressed by means of the following quasilinear elliptic equation

$$\operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - |\nabla u|^2}}\right) + \frac{f'(u)}{\sqrt{f(u)^2 - |\nabla u|^2}}\left(n + \frac{|\nabla u|^2}{f(u)^2}\right) = nH(u,x), \tag{E1}$$
$$|\nabla u| < f(u), \tag{E2}$$

where $f \in C^\infty(I)$ is a positive function, I is an open interval in $\mathbb{R}$ with $0 \in I$,

$H : I \times \mathbb{R}^n \to \mathbb{R}$ is a given smooth radially symmetric function and u satisfies $u(\mathbb{R}^n) \subset I$.

In comparison with the Dirichlet problem, the number of references devoted to the study of entire spacelike graphs in the Minkowski spacetime with constant or prescribed mean curvature is appreciably lower. In this setting, the study of entire constant mean curvature spacelike graphs developed in [84] is motivated by the remarkable Calabi-Bernstein property in the maximal case, i.e., when mean curvature identically vanishes. Namely, Calabi [21] showed for $n \leq 4$, and latter Cheng and Yau [28] for all n, that an entire maximal graph in $\mathbb{L}^{n+1}$ must be a spacelike hyperplane. Treibergs proved the existence of entire spacelike graphs of constant mean curvature in $\mathbb{L}^{n+1}$ with certain asymptotic conditions. Bartnik and Simon [6, Th. 4.4] extended later this result to a more general mean curvature function, but related references concerning the prescribed mean curvature problem for entire spacelike graphs are rare. Up to our knowledge, only [4, 15] treat this problem by using a variational approach for very concrete prescribed mean curvature. On the other hand, it is natural to wonder for the existence problem of prescribed mean curvature entire spacelike graphs with radial symmetry in spacetimes where they are expected, like in certain FLRW spacetimes. This is the main aim of the chapter, whose key point is the following result, improving Theorem 3.3.7.

Theorem 3.5.2 *Let $I \times_f \mathbb{R}^n$ be a* FLRW *spacetime, and let B be the Euclidean ball in $\mathbb{R}^n$ with radius R centered at zero. Assume that $I_f(R) \subset I$. Then, for each radially symmetric smooth function $H : I \times \overline{B} \to \mathbb{R}$ such that*

$$H(t, r) \leq \frac{f'}{f}(t) \quad \textit{and} \quad f'(t) \geq 0, \qquad \textit{for any} \quad r \in]0, R[\ , \ t \in I_f(R),$$

there exists a radially symmetric spacelike graph with mean curvature function H defined on $\overline{B}$, supported on the spacelike slice $t = 0$, only touching it on the boundary $\{0\} \times \partial B$, and defining a non-zero hyperbolic angle with ∂_t. Moreover, if the function H is increasing in the second variable, every spacelike graph satisfying the previous *assumptions must be radially symmetric.*

This existence result let us to state the main theorem of Chapter 3.

Theorem 3.6.1 *Let $I \times_f \mathbb{R}^n$ be a* FLRW *spacetime, and let $R > 0$ be such that*

$$I_f(R) \subset I, \qquad \varphi^{-1}(\mathbb{R}^-) \subset I,$$

where

$$I_f(R) := \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds\,,\ \int_0^R f(\varphi^{-1}(s))ds \right] \quad \text{and} \quad \varphi(t) = \int_0^t \frac{dt}{f(t)}.$$

Then, for each radially symmetric smooth function $H : I \times \mathbb{R}^n \to \mathbb{R}$ such that

$$H(t,r) \leq \frac{f'}{f}(t) \quad \textit{and} \quad f'(t) \geq 0, \qquad \textit{for any} \quad r \in]0, R[\ ,\ \ t \in I_f(R),$$

there exists an entire radially symmetric spacelike graph with mean curvature function H. In addition, the spacelike slice $t = 0$ intersects the graph in a sphere with radius R. In the particular case that $\inf I$ *is finite, the entire spacelike graph approaches to an hyperplane.*

Note that this result specializes to the particular but important case $H = 0$, providing entire maximal graphs in the FLRW spacetime $I \times_f \mathbb{R}^n$.

Chapter 4 deals with the existence of solutions of the following prescribed mean curvature spacelike graph equation

$$\begin{gathered}\frac{1}{f}\operatorname{div}\left(\frac{f^2\,\nabla u}{\sqrt{1-f^2|\nabla u|^2}}\right)+\frac{\langle \nabla u\,,\,\nabla f\rangle}{\sqrt{1-f^2|\nabla u|^2}}=\frac{n}{f^2}\,H(x,u),\quad x\in M,\\ |\nabla u|<\frac{1}{f},\end{gathered}\tag{1.2}$$

where the Riemannian manifold M is either $\mathbb{R}^n$ or $\mathbb{R}^n\setminus\overline{B}_0(a)$, $a\geq 0$, endowed with a radial metric

$$\langle\ ,\ \rangle = E^2(r)dr^2+r^2d\Theta^2,$$

being $E(r)>0$, $d\Theta^2$ the usual metric of the sphere $\mathbb{S}^{n-1}$, $f\in C^\infty(a,+\infty)$ is a positive function and $H:M\times\mathbb{R}\to\mathbb{R}$ is a given smooth radially symmetric function.

The study of this PDE is also motivated from the mean curvature prescription in Lorentzian Geometry. Explicitly, every solution of (1.2) defines a spacelike graph in a standard static spacetime, $\mathcal{M}:=M\times_f I$, and the function H prescribes the corresponding mean curvature.

Hence, in Chapter 4, we consider spacelike graphs in static spacetimes with respect to a family of observers, (spatial universe always looks the same for them). There are many relevant examples of this kind of spacetimes. Of special importance are (in addition to Minkowski spacetime) *Schwarzschild* and *Reissner-Nordström* spacetimes. Both relativistic models describe a universe where there is only a spherically symmetric non-rotating mass, representing a star or a black hole. In the first model, the mass has no electric charge, while in the second it is uniformly charged (in fact, Reissner-Nordström spacetime may be seen as an extension of Schwarzschild one). They have one or two *event horizons* and an inevitable singularity at the center of the mass (see [25, Chap. 3] and [25, Chap. 5] for details and physical interpretation).

From the analysis of the related bibliography, it seems that the problem of the existence of prescribed mean curvature entire graphs with radial symmetry for static spacetimes has not been considered before.

The main findings of Chapter 4 can be summarized as follows.

Theorem 4.3.1 *Let $\mathbb{R}^n \times_f \mathbb{R}$ be a standard static spacetime, endowed with the spherically symmetric Lorentzian metric*

$$E^2(r)dr^2 + r^2 d\Theta^2 - f^2(r)dt^2,$$

and let $H : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a radially symmetric continuous function. Then, there exists a spherically symmetric entire spacelike graph with mean curvature function H. Moreover, for each $R > 0$, the graph may be chosen such that its intersection with $t = 0$ is a sphere of radius R.

Theorem 4.3.2 *Let $\mathcal{M}$ be either the Schwarzschild exterior spacetime or the Reissner-Nordström exterior spacetime with exterior radius $a > 0$, and let $H : \mathcal{M} \longrightarrow \mathbb{R}$ be a spherically symmetric and bounded continuous function. Then, there exists a spherically symmetric entire spacelike graph with mean curvature function H that approaches the event horizon as $r \to a$. Moreover, for each $R > a$, the graph may be chosen such that its intersection with $t = 0$ is a sphere of radius R.*

The proofs of the previous theorems are based on the following two existence results for the associated Dirichlet problems on a ball, which are interesting by themselves.

Theorem 4.2.1 *Let $\mathbb{R}^n \times_f \mathbb{R}$ be a standard static spacetime, endowed with the spherically symmetric Lorentzian metric*

$$E^2(r)dr^2 + r^2 d\Theta^2 - f^2(r)dt^2.$$

Let $B = B_0(R)$ be the Euclidean ball with radius R centered at $0 \in \mathbb{R}^n$, and let $H : \overline{B} \times \mathbb{R} \to \mathbb{R}$ be a spherically symmetric continuous function. Then, there exists a spherically symmetric spacelike graph with mean curvature function H defined on $\overline{B}$ and supported on the slice $t = 0$.

Theorem 4.2.13 *Let $\mathcal{M}$ be either the Schwarzschild exterior spacetime or the Reissner-Nordström exterior spacetime (with radius a), and let $H : \overline{A}(a, R) \times \mathbb{R} \to \mathbb{R}$ be a spherically symmetric and bounded smooth function, where $\overline{A}(a, R)$ is the closed annulus $a \leq |x| \leq R$. Then, there exists a spherically symmetric spacelike graph with mean curvature function H, which touches the slice $t = 0$ on the boundary $|x| = R$, and approaches the event horizon as $|x| \to a$. Moreover, the graph is radially decreasing on the annulus $\overline{A}(a, R)$ and it intersects the slice $t = 0$ only at the boundary $|x| = R$.*

In Theorem 4.2.13, we have chosen the previous statement for the sake of simplicity. An inspection of its proof will provide more details about the geometry of the solutions. One of the main features is that the hyperbolic angle of the graph with the slice $t = 0$ can be prescribed. More precisely, there exists χ_0 such that for every $\chi < \chi_0$, the spacelike graph given in Theorem 4.2.13 can be chosen so that the hyperbolic angle with the slice $t = 0$ is precisely χ. The value of χ_0 can be explicitly calculated (see the inequality (4.22)).

Next chapter is devoted to the higher mean curvature prescription problem. For a two sided hypersurface in $\mathbb{R}^{n+1} \equiv \mathbb{R}^{n+1}_a$, $a = 0$, or a spacelike hypersurface in $\mathbb{L}^{n+1} \equiv \mathbb{R}^{n+1}_a$, $a = 1$, the k-th mean curvatures are geometric invariants which encode the geometry of the hypersurface. From an algebraic point of view, each one of these functions corresponds to a coefficient of the characteristic polynomial of the shape operator corresponding to a unit normal vector field ($a = 0$) or to a unit timelike normal vector field pointing to future ($a = 1$). In fact, each k-th mean curvature is a sort of average of the principal curvatures of the hypersurface. In particular, the 1-th mean curvature corresponds with the usual mean curvature if $a = 0$ or its opposite if $a = 1$, the 2-th mean curvature is, up to a constant factor, the scalar curvature, and the n-th mean curvature is the Gauss-Kronecker curvature if $a = 0$ and $(-1)^{n+1}$ times the Gauss-Kronecker curvature if $a = 1$. Each k-th mean curvature appears in a variational problem [71], and in the Riemannian case the constant k-th mean curvature has been extensively studied ([54], [77] for instance). From a physical perspective, the k-th mean curvatures have a relevant role in General Relativity. The k-th mean curvature function intuitively measures the time evolution of the physical space of the initial instant towards the future or towards past of the spatial universe (see Remark 2.2.1).

Previously, we have already commented the work done on the prescribed mean curvature problem. Respect to the prescription of the scalar curvature, we refer to [20] in the Euclidean context. On the other hand, Bayard proved the existence of prescribed scalar entire spacelike hypersurfaces in Minkowski spacetime [7], making use of other previous works on the Dirichlet problem ([8] and [85] and references therein) and Gerhardt [52] obtained important results when more general ambient spacetimes are considered. Finally, the Gauss-Kronecker curvature has been also quite well studied in both settings. In Euclidean space, Wang [87] prescribed the Gauss-Kronecker curvature of a convex hypersurface. In Minkowski spacetime, we highlight the work of Li [60] on constant Gauss curvature and Delanoè [43], in which the existence of entire spacelike hypersurfaces asymptotic to a lightcone with prescribed Gauss-Kronecker curvature function is proved.

Up to the last decade, little attention was paid to hypersurfaces with prescribed k-th mean curvature when $3 \leq k < n$. One of the first works in this direction was done by Ivochkina (see [57] and references therein). More recently, several contributions (for instance, [86], [55]) especially on the Dirichlet problem have been done. However, the general problem is still open in both settings. The study has usually been mainly focused in the search of some *a priori* bounds on the norm of the shape operator, assuming that solutions are k-stable to ensure the ellipticity of some involved differential operators (see [86] for more details). Then, some special dependence in the prescription function is imposed in order to obtain partial results.

In this chapter, we provide several results on this open problem, assuming that the prescription function is rotationally symmetric respect to a unit parametrized line or an inertial observer γ (i.e., a unit timelike parametrized line pointing to the future) in $\mathbb{R}^{n+1}_a$ with $a = 0$ or $a = 1$, respectively. We prove the existence of rotationally symmetric graphs with prescribed k-th mean curvature of the associated Dirichlet problem when the domain is an n-dimensional ball, by using a suitable fixed point operator. Besides, we prove that such graph can be extended to the whole space, providing some information about uniqueness as well.

We will use the usual cylindrical coordinates (t, r, Θ) in $\mathbb{R}^{n+1}_a$ associated to γ, namely, $t \in \mathbb{R}$ is the parameter of γ, $r \in \mathbb{R}^+$ is the radial distance to γ and $\Theta = (\theta_1, ..., \theta_{n-1})$ are the standard spherical coordinates of the $(n-1)$-dimensional unit round sphere $\mathbb{S}^{n-1}$. The prescription functions H_k will be assumed to be radially symmetric with respect to γ. Therefore, it is natural to consider $H_k(t, x) = H_k(t, r)$ where r denotes the distance of $x \in \mathbb{R}^n$ to γ.

Although hypersurfaces in Euclidean space and spacelike hypersurface in Minkowski spacetime have a different geometry, we have decided here to show the results of both contexts in a single chapter because, even if the results are different, the mathematical treatment is similar.

Below, we summarize the main results on entire graphs in Minkowski spacetime. The first theorem may be named as a "universal existence result" when k is odd.

Theorem 5.4.1 *Let $H_k : \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$, with k an odd positive integer, be a continuous function which is rotationally symmetric with respect to an inertial observer γ of $\mathbb{L}^{n+1}$. Then, for each $R > 0$, there exists at least an entire spacelike graph, rotationally symmetric respect to γ, whose k-th mean curvature equals to H_k and such that it intersects the spacelike hyperplane orthogonally to γ at $\gamma(0)$ in an $(n-1)$-sphere with radius R centered at $\gamma(0)$. In addition, if H_k is non decreasing with respect to the proper time of γ, then the spacelike graph is unique.*

For k even, we have to introduce a natural restriction on the curvature, as it is shown in the next result.

Theorem 5.4.2 *Let $H_k : \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$, with k an even positive integer, be a continuous function such that*

$$\int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \geq 0 \quad \text{for all} \quad r \in \mathbb{R}^+, \quad \text{and} \quad v \in C^1,\ |v'| < 1, \tag{1.3}$$

and which is rotationally symmetric with respect to an inertial observer γ of $\mathbb{L}^{n+1}$. If $H_k(0,\cdot) \not\equiv 0$, for each $R > 0$, then there exists at least two different rotationally symmetric entire spacelike graphs whose k-th mean curvature equals to H_k and such that it intersects the spacelike hyperplane orthogonally to γ at $\gamma(0)$ in an $(n-1)$-sphere with radius R centered in $\gamma(0)$. Moreover, the radial profile curve of one of them is increasing and the other one is decreasing. Besides, condition (1.3) *is necessary for the existence of such spacelike graphs.*

On the other hand, in the case of entire graphs in $\mathbb{R}^{n+1}$ we cannot expect a kind of universal result like Theorem 5.4.1. If we want to obtain a rotationally symmetric graph with prescribed k-curvature that intersects the orthogonal hyperplane in a n-sphere with radius R, it is necessary to introduce an additional bound for the size of the prescribed curvature, as the following result shows.

Theorem 5.4.3 *Let $H_k : \mathbb{R}^{n+1} = \mathbb{R} \times \Pi \longrightarrow \mathbb{R}$, where Pi is a hyperplane in $\mathbb{R}^{n+1}$ and k an odd positive integer, be a continuous function which is rotationally symmetric respect to an oriented line γ, orthogonal to Π. Given a fixed $R > 0$, assume there is some $\alpha \in (0, R^{-k})$, satisfying*

$$|H_k(t,r)| \leq \alpha \qquad \text{for all} \qquad r \in [0,R]\,, \quad t \in [-R\,\beta, R\,\beta], \tag{1.4}$$

where $\beta := \dfrac{R\alpha^{1/k}}{\sqrt{1-R^2\alpha^{2/k}}}$, *and*

$$0 \leq \int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds < \frac{r^{n-k}}{n}, \qquad \text{for all} \qquad r > R \quad \text{and} \quad v \in C^1. \tag{1.5}$$

Then, there exists at least an entire graph, rotationally symmetric respect to γ*, whose k-th mean curvature equals to* H_k *and such that it intersects the hyperplane* Π *in an* $(n-1)$*-sphere with radius* R *centered at* $\gamma(0)$*. In addition, if* H_k *is non decreasing along the line* γ*, then the graph is unique.*

It should be noted that the assumption (1.4) is quite natural, in particular it is necessary when the prescription function H_k is a constant. The last result considers the case of k even in the Euclidean space.

Theorem 5.4.4 *Let* $H_k : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$*, with* k *an even positive integer, be a continuous function which is rotationally symmetric respect to a line* γ*. For each* $R > 0$*, assume there is some* $0 < \alpha < R^{-k}$*, satisfying* (1.4)*,* (1.5) *and*

$$\int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \geq 0, \qquad \text{for all} \qquad r \in [0, R] \quad \text{and} \quad v \in B_{R\beta,\beta},$$

where $B_{R\beta,\beta} = \{v \in C^1 : \|v\|_\infty < R\beta,\ \|v'\|_\infty < \beta\}$*. Then, if* $H_k(0, \cdot) \not\equiv 0$*, there exists at least two different entire graphs, rotationally symmetric, whose k-th mean curvatures equal to* H_k *and such that they intersect the hyperplane orthogonally to* γ *at* $\gamma(0)$ *in an* $(n-1)$*-sphere with radius* R *centered at* $\gamma(0)$*. Moreover, the radial profile curve of one of them is increasing and the other one is decreasing.*

The last three chapters, with a strong physical motivation, are oriented to analyse and introduce the relativistic notions of uniformly accelerated, unchanged direction and uniform circular motion, concepts which are already well known in the Classical Mechanics context.

Uniformly accelerated motions seem to be impossible in Relativity from a naive first impression. A deep reason is the existence of an upper bound of the light speed. In Newtonian Mechanics, a particle is said to be accelerated when *inertial observers* measure a (nonzero) relative acceleration (of the particle) respect to them. However, it is not necessary the use of a fixed reference frame to define the acceleration of a particle. In fact, in General Relativity the notion of inertial observer has no sense. In a non-Relativistic setting, a particle may be detected as accelerated by using an *accelerometer*. An accelerometer may be intuitively thought as a sphere in whose center there is a small round object which is supported on elastic radii of the sphere surface. If a free falling observer carries such a accelerometer, then it will notice that the small round object remains just at the center. Whereas it will be displaced if the observer obeys an accelerate motion. This argument suggests that a uniformly accelerated motion may be detected from a constant displacement of the small round object. This idea has the advantage that may be used independently if the spacetime where the observer lies is relativistic or not. Now, we need to provide rigour to the assertion "the accelerometer marks a constant value".

The notion of uniformly accelerated motion in General Relativity has been discussed many times over the last 50 years [47], [48] and references therein (see also [65]). In the pioneering work by Rindler [72], it was motivated in part by some aspects of intergalactic rocket travel by use of the special relativistic formulas for hyperbolic motion in [62]. The relation between uniformly accelerated motion and Lorentzian circles in Minkowski spacetime described in [73] (see also [66, Sec. 6.2]) was used by Rindler in [72] to define what he named hyperbolic motion in General Relativity, extending the uniformly accelerated motion in Minkowski spacetime (in agreement with one of two proposed notions made by Marder a few years before [63]).

The first aim of Chapter 6 is to study uniformly accelerated motion in General Relativity. Our approach differs of previous ones and lies in the realm of modern Lorentzian geometry. In order to do that, recall that a *particle* of mass $m > 0$ in a spacetime $(M, \langle\, , \rangle)$ is a curve $\gamma : I \longrightarrow M$, such that its velocity γ' satisfies

$\langle\gamma', \gamma'\rangle = -m^2$ and points to the future. A particle with $m = 1$ is called an *observer*. The covariant derivative of γ', $\dfrac{D\gamma'}{dt}$, is its (proper) acceleration, which may be seen as a mathematical translation of the values which measures an accelerometer as named above. Intuitively, the particle obeys a uniformly accelerated motion if its acceleration remains unchanged. Mathematically, we need a *connection along* γ which permits to compare spatial directions at different instants of the life of γ. In General Relativity this connection is known as the *Fermi-Walker connection* of γ (see Section 2.3 for more details). Thus, using the corresponding Fermi-Walker covariant derivative $\dfrac{\widehat{D}}{dt}$, we will say that a particle obeys a *uniformly accelerated* (UA) motion if the following third-order differential equation is fulfilled,

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right) = 0, \tag{1.6}$$

i.e., if the acceleration of the observer γ is Fermi-Walker parallel along its world line.

There are several physical situations where UA motions appear. For instance, when an electric charged particle (γ, m, q) is considered, in presence of an electromagnetic field F, the dynamics of the particle is fully described by the well-known Lorentz force equation,

$$m\frac{D\gamma'}{dt} = q\widetilde{F}(\gamma'),$$

where $\widetilde{F}$ is the (1,1)-tensor field metrically equivalent to the closed 2-form F. The vector field $\widetilde{F}(\gamma')$ along γ is called the electric field relative to γ, [78, p. 75]. The Fermi-Walker connection of γ allows to define that γ perceives a “constant” electric vector field if

$$\frac{\widehat{D}}{dt}\left(\widetilde{F}(\gamma')\right) = 0.$$

Thus, if a particle γ is moving in presence of an electromagnetic field F and its relative electric vector field satisfies previous equation, then γ obeys a UA motion.

In Section 6.2 we expose in detail how UA observers can be seen as Lorentzian circles in any general spacetime (Proposition 6.2.2), in particular, assertion (d) in this result corresponds to the original notion proposed by Rindler in [72]. Moreover, static standard spacetimes are characterized as those 1-connected and geodesically complete Lorentzian manifolds which admit a rigid and locally sincronizable reference frame whose integral curves are UA observers (Theorem 6.2.4).

Next we characterize UA observers as the projection on the spacetime of the integral curves of a vector field defined on a certain fiber bundle over the spacetime. Using this vector field, the completeness of inextensible UA motions is analysed in the search of geometric assumptions which assure that inextensible UA observers do not disappear in a finite proper time (in particular, the absence of certain timelike singularities). In particular, any inextensible UA observer is complete under the assumption of compactness of the spacetime and that it admits a conformal and closed timelike vector field (Theorem 6.3.5).

Coming back to the intuitive notion of an accelerometer, if the ball moves along a radius, the observer may though that its motion obeys a rectilinear trajectory. This simple idea leads us to the definition of "rectilinear motion" in Relativity, which has been discussed a few times as far as we know [47], [48]. The first aim of Chapter 7 is to provide a rigorous notion for the assertion "the proper acceleration does not change its direction". Thus, unchanged direction motion in General Relativity is introduced and studied. Using the corresponding Fermi-Walker covariant derivative $\frac{\widehat{D}}{dt}$, we will say that a particle with nowhere zero acceleration obeys an *unchanged direction* (UD) motion if the following third-order differential equation is fulfilled (Definition 1),

$$\frac{\widehat{D}}{dt}\left(\left|\frac{D\gamma'}{dt}\right|^{-1}\frac{D\gamma'}{dt}\right)=0, \tag{1.7}$$

i.e., if the normalized acceleration vector field of the observer γ is Fermi-Walker parallel along its world line. Note that an observer γ obeys a UA motion and the constant $\left|\frac{D\gamma'}{dt}\right|$ is positive then it obeys a UD motion, however the class of UD observers is much bigger that the one of UA observers. In fact, if an observer in $n(\geq 2)$-dimensional Minkowski spacetime $\mathbb{L}^n$ lies in a Lorentzian plane, then it obeys a UD motion. Conversely, each UD observer γ in $\mathbb{L}^n$ is contained in a Lorentzian plane determined by the initial point $\gamma(0)$, the initial acceleration and the initial 4-velocity. More generally, every UD observer in a spacetime of constant sectional curvature must be contained in a totally geodesic 2-dimensional Lorentzian submanifold.

More generally, we will introduce the notion of piecewise UD motion (Definition 2). A piecewise UD observer is essentially an observer which may change its direction only at the instant when its accelerometer marks zero. Each piecewise UD observer naturally appears as a solution of an ODE, (7.4), much more general than equation (1.7). Now we can assert that a free falling observer obeys trivially a piecewise UD motion although it cannot be considered as a UD observer. Also the problem of finding a UD observer γ prescribing its scalar acceleration $\left|\dfrac{D\gamma'}{dt}\right|$ is suitably stated and completely solved, obtaining an explicit first integral of a UD motion with prescribed acceleration (Theorem 7.1.1). Piecewise UD observers are geometrically characterized later (Proposition 7.2.2(c)). Our approach here fits within the mean curvature prescription problem for a definite submanifold in spacetime (Proposition 7.2.2(e)). Among the geometric characterizations of piecewise UD motions in Proposition 7.2.2 (widely extending [39], [72]), it is remarkable that an observer obeys a piecewise UD motion if and only if its development, in the sense of [59, Sect. III.4], in the tangent space to spacetime at any of its points is a piecewise planar curve (Proposition 7.2.2(c)).

UD observers are characterized (as previously UA observers) as the curves obtained by projection on the spacetime of the integral curves of a certain vector field defined on a certain fiber bundle over the spacetime (Lemma 7.3.2). Using this vector field, the completeness of inextensible UD motions is analysed when the ambient spacetime has a certain conformal symmetry. Concretely, it is shown that a UD observer defined on a finite interval in spacetime which admits a conformal and closed timelike vector field (in particular, in a Generalized Robertson-Walker spacetime) can be extended if it is contained in a compact subset of spacetime (Theorem 7.3.5). Finally, we deals with the problem of extensibility of UD observers but this time in an important subclass of pp-wave spacetimes: the so-called plane wave spacetimes. Now the key tool is the explicit first integral of a UD observer obtained in Theorem 7.1.1. Thus, it is proved that every inextensible UD trajectory with prescribed acceleration a in a Plane Wave spacetime must be complete (Theorem 7.4.3).

Finally, in Chapter 8 we study the uniform circular motion. It has been widely studied in Special Relativity (see for instance [46]). Relevant physical phenomena and paradox, usually related with the Thomas precession, have motivated its interest until now (see, for instance [80] for an intuitive introduction). The usual approach consists in setting a family of inertial observers, one of them is considered 'the center'. Thus, an observer is said that describes a uniform circular motion respect to the fixed 'center' if the trajectory measured by that family of inertial observers is circular and its angular velocity is constant for them. Others viewpoints have been done from the Frenet equations [47], [48].

Specific motions which may be seen as very particular cases of uniform circular motions have been previously considered in relevant relativistic models as Schwarzschild, Reissner-Nordström and Kerr spacetimes, all of them with some rotational symmetry [66, Ch. 25]. Each of these spacetimes has a remarkable family of observers, with a similar role that the inertial observers in Minkowski spacetime. An observer that is placed on the star or the black hole (according the case) and at rest with respect to it, is considered the center of the circular trajectories. The uniform circular observer describes a circle with respect to the special fixed family of synchronized observers and the angular velocity measured is constant. The analysis of this kind of motions has a recognized physical and technological interest because they correspond with the orbits of some satellites, planets and stars (see, for instance, [49]).

We are interested here in introducing a definition of uniform circular motion in a general spacetime involving only the physical observable quantities measured by the own observer. In other words, we give an 'intrinsic' definition of uniform circular motion, without considering an external family of synchronized observers for which the observer has a circular trajectory or not. Notice in addition, that the existence of such a family of observers is not guaranteed in a general spacetime. Of course, our definition will agree with the standard notion in previously quoted cases. In order to determine the inherent kinematic state of an observer we will focus as before on its proper acceleration.

Intuitively, an observer is able to detect if it is rotating by using a gyroscope or an *accelerometer*, as previously commented. Thus, it would be natural that if an observer checks that the small ball describes a plane uniform rotation, then it believes that it obeys a uniform circular motion. The first challenge we have to face is to state a notion of 'planar' motion in any spacetime. Classically, a motion is said to be planar when the projection of its spacetime trajectory on the absolute Euclidean space is contained in a plane. Equivalently, its proper acceleration is contained in the same plane at any instant. This alternative notion may be extended to any spacetime. Obviously, as it happens in Classical Mechanics, a uniform circular motion should be a planar motion. The subtle problem in Relativity consists in to give sense to the sentence *the same plane forever*. This is well done again making use of the Fermi-Walker connection of each observer.

Assume the particle γ obeys a planar motion. In order to arrive to a suitable notion of uniform circular motion, we will require that the modulus of its acceleration remains unchanged, i.e.,

$$\left|\frac{D\gamma'}{dt}\right|^2 = \text{constant}.$$

Using the corresponding Fermi-Walker covariant derivative $\frac{\widehat{D}}{dt}$, if a particle obeys a uniform circular (UC) motion, it is also necessary,

$$\left|\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)\right|^2 = \text{constant},$$

i.e., the modulus of the change of its acceleration should be constant. Thus, we will arrive to the notion of the uniform circular motion in Definition 6.

It should be pointed that UC motions also appear naturally in any spacetime. For instance, they arise from a dynamical point of view. With the same notation as before, let us consider in Minkowski spacetime $\mathbb{L}^4$ the electromagnetic field,

$$F = 2B_0\, dx \wedge dy,$$

where $B_0 \in (0, \infty)$ and (t, x, y, z) are the standard coordinates of $\mathbb{L}^4$. The family of inertial observers ∂_t measures a uniform magnetic field with modulus B_0 and pointing towards ∂_z (and zero electric field) for this electromagnetic field F. Now, if a particle (γ, m, q) obeys the Lorentz force equation, then it obeys a UC motion and its trajectory is expressed as [78, Prop. 3.8.2],

$$\gamma(\tau) \;=\; p \,+\, \left(\sqrt{1 + R^2 w^2}\, m\tau, R\cos(wm\tau + \vartheta), R\sin(wm\tau + \vartheta), 0\right),$$

with $w = \frac{qB_0}{m} \in \mathbb{R}$, $p \in \mathbb{L}^4$, $R > 0$ and $\vartheta \in \mathbb{R}$, whenever the initial velocity of the particle with respect to the family of inertial observers lies in the plane xy [78, p. 88].

The notion of UC observer is analysed in detail by means of the corresponding differential system and the associated initial value problem. Next we prove how a UC observer can be seen as a Lorentzian helix in any general spacetime (Proposition 8.2.4). We use this result to characterize each UC observer as a solution of a fourth-order differential equation (Proposition 8.19). Analogously to UA and UD observers, UC ones may be characterized geometrically as the projection on the spacetime of the integral curves of a certain vector field defined on a suitable fiber bundle over the spacetime. Using this vector field, the completeness of inextensible UC motions is analysed in the search of geometric assumptions which assure that inextensible UC observers do not disappear in a finite proper time (Lemma 8.3.3). Thus we obtain that, any UC observer defined on a finite interval in a Generalized Robertson-Walker spacetime can be extended whenever its image lies in a compact subset of spacetime

(Theorem 8.3.4). The chapter ends with the proof of the completeness of inextensible UC observers in a Plane Wave spacetime, making use of more analytical techniques (Theorem 8.4.3). Both results may be seem as giving the absence of this kind of singularities.

The results of this memory appear in the following references of the bibliography [10], [36], [37], [38], [39], [40] and [41].

Chapter 2

Preliminaries

This chapter is devoted to expose the geometrical and analytical background that will come in handy throughout this memory.

Firstly, we introduce some notation. Along this memory the signature of a Lorentzian metric is $(-,+,...,+)$. $\mathbb{R}^{n+1}_a$, $a=0,1$, will denote, for $a=0$, the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ endowed with its standard Riemannian metric $\langle\, ,\, \rangle = \sum_{i=1}^{n+1} dx_i^2$ and, for $a=1$, the $(n+1)$-dimensional Lorentzian spacetime $\mathbb{L}^{n+1}$ endowed with its standard Lorentzian metric $\langle\, ,\, \rangle = -dx_1^2 + \sum_{j=2}^{n+1} dx_j^2$ and with the time orientation defined by $\partial/\partial x_1$.

2.1 Relevant ambient spacetimes

In this section, we present the main ambient spacetimes where we will usually work along this memory.

2.1.1 Friedmann-Lemaître-Robertson-Walker spacetimes

We introduce the ambient spacetimes considered in Chapter 3 where our spacelike graphs are embedded. We consider the Euclidean space $(\mathbb{R}^n, \langle\ ,\ \rangle)$, and let $I \subseteq \mathbb{R}$ be a open interval in $\mathbb{R}$ with the metric $-dt^2$. Throughout this memory we will denote by $(\mathcal{M}, g)$ the $(n+1)$-dimensional product manifold $I \times \mathbb{R}^n$ endowed the Lorentzian metric

$$g = \pi_I^*(-dt^2) + f^2(\pi_I)\pi_F^*(\langle\,,\rangle) \equiv -dt^2 + f^2(t)\langle\ ,\ \rangle\,, \tag{2.1}$$

where $f > 0$ is a smooth function on I, and π_I and π_F denote the projections onto I and $\mathbb{R}^n$ respectively. Thus, $(\mathcal{M}, g)$ is a Lorentzian warped product with base, I fiber $\mathbb{R}^n$ and warping function f, and we will denote it by $I \times_f M$. We will refer $\mathcal{M}$ as a (flat fiber) Friedman-Lemaître-Robertson-Walker (FLRW) spacetime.

Given an n-dimensional manifold S, an immersion $\phi : S \to \mathcal{M}$ is said to be *spacelike* if the Lorentzian metric given by (2.1) induces, via ϕ, a Riemannian metric g_S on S. In this case, S is called a spacelike hypersurface.

Observe that the vector field $\partial_t := \partial/\partial t \in \mathfrak{X}(\mathcal{M})$ is timelike and unit which determines a time-orientation on $\mathcal{M}$. Thus, if $\phi : S \longrightarrow \mathcal{M}$ is a (connected) spacelike hypersurface in $\mathcal{M}$, the time-orientability of $\mathcal{M}$ allows us to define $N \in \mathfrak{X}^\perp(S)$ as the only globally defined, unit timelike vector field normal to S in the same time-orientation of ∂_t.

There is a remarkable family of spacelike hypersurfaces in the FLRW spacetime $\mathcal{M}$. Namely, the level hypersurfaces of projection function t. They are also called *spacelike slices*. Each spacelike slice $t = t_0$ is umbilical and its mean curvature is $f'(t_0)/f(t_0)$.

Among the spacelike hypersurfaces, the spacelike graphs on domains of the fiber $\mathbb{R}^n$, appear in a natural way. We will denote by Σ_u the graph defined from $u \in C^\infty(U)$ such that $u(U) \subseteq I$, i.e., $\Sigma_u = \{(x, u(x)) \ :\ x \in U\}$. The spacelike condition is

expressed as follows
$$|\nabla u| < f(u) \qquad \text{in} \quad U. \tag{2.2}$$

For a spacelike graph Σ_u, the unit timelike normal vector field in the same time orientation of ∂_t it is given by

$$N = \frac{f(u)}{\sqrt{f(u) - |\nabla u|^2}} \left(\frac{1}{f^2(u)} \nabla u + \partial_t \right),$$

The corresponding mean curvature associated to N, is defined by

$$\operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - |\nabla u|^2}} \right) + \frac{f'(u)}{\sqrt{f(u)^2 - |\nabla u|^2}} \left(n + \frac{|\nabla u|^2}{f(u)^2} \right),$$

which can be seen as a quasilinear elliptic operator Q, because of (2.2). Hence, our prescription problem is translated into the equation

$$Q(u)(x) = nH(u, x). \tag{2.3}$$

2.1.2 Static spacetimes

In a spacetime $\mathcal{M}$, a vector field Q which is unit timelike and future pointing is called a *reference frame* [78, Def. 2.31]. Each integral curve of Q represents an observer in $\mathcal{M}$. So, the choice of a reference frame in a spacetime gives a distinguished family of observers in $\mathcal{M}$. A spacetime is said to be *static* relative to a reference frame Q if Q is irrotational (i.e., $\operatorname{curl}(Q)(X, Y) = 0$ for any $X, Y \in Q^\perp$) and there is $f \in C^\infty(\mathcal{M})$ such that the vector field fQ is Killing. Observe that Q is irrotational if and only if the distribution $Q^\perp$ is integrable [68, Prop. 12.30]. Therefore, given an event $p \in \mathcal{M}$ there exists a unique (inextensible) spacelike hypersurface $\mathcal{S}$ of $\mathcal{M}$ such that $p \in \mathcal{S}$ and $T_q\mathcal{S} = Q_q^\perp$, for all $q \in \mathcal{S}$ (in fact, $\mathcal{S}$ is an inextensible leaf of the foliation $Q^\perp$ through $p \in \mathcal{M}$). The spacelike hypersurface $\mathcal{S}$ may be interpreted as the spatial universe of each observer in Q which intersects $\mathcal{S}$ at an instant of its proper time.

On the other hand, any (local) flow $\{\phi_t\}$ of the Killing vector field fQ consists of (local) isometries of $\mathcal{M}$ which preserve restspaces of observers in q, i.e., if S is an integral leaf of the foliation $Q^\perp$ through $p \in \mathcal{M}$, then $\phi_t(S)$ is an integral leaf of $Q^\perp$ through $\phi_t(p)$. Therefore, the spatial universe always looks the same for the observers in Q at least locally. Geometrically, for each $p \in \mathcal{M}$ there exists an open neighbourhood U of p such that $\phi : S \times I \longrightarrow U$, $(q,t) \mapsto \phi_t(q)$ is a diffeomorphism, where S is a leaf of $Q^\perp$. Using now that fQ is Killing we have that $f\big(\phi(q,t)\big)$ is independent of t. Moreover, if g denotes the Lorentzian metric of $\mathcal{M}$, then on $S \times I$ we can write

$$\phi^* g = \pi_S^* g_S - h_S(\pi_S)^2 \pi_I^* dt^2,$$

where π_S, π_I are the projections onto S and I respectively, g_S denotes the Riemannian metric on S obtained by restriction of g and $h_S \in C^\infty(S)$ satisfies $h_S(q) = f\big(\phi(q,t)\big) > 0$ for all $q \in S$. Note that in the spacetime $(S \times I\,, \phi^* g)$ the role of the Killing vector field fQ is played by the coordinate vector field ∂_t, in fact $\phi_*(\partial_t) = fQ$.

Motivated by the last observation, a spacetime $\mathcal{M}$ is said to be a *standard static spacetime* if it is a warped product $M \times_f I$ where (M, g) is a Riemannian manifold, I is an open interval of $\mathbb{R}$ endowed with the metric $-dt^2$, and $f : M \longrightarrow \mathbb{R}^+$ is a smooth function. In other words, $\mathcal{M} = M \times I$ with the metric

$$\overline{g} := \pi_M^* g + f^2(\pi_M)\pi_I^*(-dt^2) \equiv g - f^2(x)dt^2. \tag{2.4}$$

Observe that the vector field ∂_t is timelike which determines the time-orientation on $\mathcal{M}$. Thus, if $\phi : S \longrightarrow \mathcal{M}$ is a (connected) spacelike hypersurface in $\mathcal{M}$, the time-orientability of $\mathcal{M}$ allows us to construct a globally defined, unit, timelike vector field, N, normal to S in the same time-orientation of ∂_t.

There is a remarkable family of spacelike hypersurfaces in the standard static spacetime $\mathcal{M}$. Namely, the so called spacelike slices $\{t = t_0\}$, which are totally geodesic and, therefore, its mean curvature function vanishes.

Among the spacelike hypersurfaces, the spacelike graphs on domains of the base M appear in a natural way. We will denote by $\Sigma_u(U)$ the graph of $u \in C^\infty(U)$ such that $u(U) \subseteq I$, i.e.,

$$\Sigma_u(U) = \{(x, u(x)) \ : \ x \in U\}.$$

From (2.1) the induced metric on $\Sigma_u(U)$ is given on U by $g - f^2(x)du^2$. Therefore, it is spacelike whenever the following inequality holds

$$|\nabla u| < \frac{1}{f} \qquad \text{on} \quad U. \tag{2.5}$$

2.1.3 PP-waves spacetimes

PP-waves type spacetimes are a physically relevant family of spacetimes. Considering such a spacetime as an exact solution of Einstein's equations, it can model radiation (electromagnetic or gravitational) moving at the speed of light. The recent interest on PP-wave type spacetime may be explained, on the oe had, by its classical geometrical properties, on the other by its applications to String Theory as well as the possibilities of direct detection of gravitational waves. Historically, the study of gravitational waves goes back to Einstein [44] but the standard exact model was already introduced by Brinkmann in order to determine Einstein spaces which can be improperly mapped conformally on some Einstein one. The original definition by Brinkmann says that a PP-wave spacetime is any spacetime whose metric tensor can be locally described, with respect to suitable coordinates, in the form

$$g = H(u, x, y)\, du^2 + 2dudv + dx^2 + dy^2,$$

where H is any smooth function. Nowadays, PP-wave means any spacetime which admits a parallel global lightlike vector field.

A (four dimensional) *Plane Wave* is a spacetime $(\mathbb{R}^4, g)$ which admits a coordinate system (u, v, x, y) such that the Lorentzian metric may be expressed as follows,

$$g = H(u, x, y)\, du^2 + 2dudv + dx^2 + dy^2,$$

where $H(u, x, y)$ is a quadratic function in the coordinates x and y with coefficients depending on u (see [22] and references therein), that is,

$$H(u, x, y) = A(u)x^2 + B(u)y^2 + C(u)xy + D(u)x + E(u)y + F(u). \tag{2.6}$$

The coordinates are known as a Brinkmann coordinate system of $(\mathbb{R}^4, g)$. The mentioned parallel global lightlike is ∂_v.

In these coordinates, the Christoffel symbols of g are easily computed as follows,

$$\Gamma^1_{i,j} = 0 \qquad \text{for all } \; i, j = 1, \ldots, 4, \tag{2.7}$$

$$\Gamma^2_{1,1} = \frac{1}{2}\frac{\partial H}{\partial u},\; \Gamma^2_{1,3} = \Gamma^2_{3,1} = \frac{1}{2}\frac{\partial H}{\partial x},\; \Gamma^2_{1,4} = \Gamma^2_{4,1} = \frac{1}{2}\frac{\partial H}{\partial y}, \tag{2.8}$$

$$\Gamma^3_{1,1} = -\frac{1}{2}\frac{\partial H}{\partial x},\; \Gamma^4_{1,1} = -\frac{1}{2}\frac{\partial H}{\partial y}, \tag{2.9}$$

and the remaining symbols are zero.

2.2 Higher mean curvatures

In this section we expose the notion of higher mean curvatures and its relation with the usual mean curvature. It will be essential to face the prescription problem presented in Chapter 5.

Let us consider a smooth immersion $\varphi : \Sigma \longrightarrow \mathbb{R}^{n+1}_a$ of an n-dimensional manifold Σ in $\mathbb{R}^{n+1}_a$, which is two sided if $a = 0$ and spacelike (i.e., the induced metric via φ is Riemannian) if $a = 1$. Assume N is a unit normal vector field along φ, which we choose pointing to the future if $a = 1$. The shape operator of Σ relative to N, is defined by

$$A(X) = -\nabla_X N,$$

where $X \in T_p\Sigma$, $p \in \Sigma$, and ∇ denote the Levi-Civita connection of $\mathbb{R}^{n+1}_a$ which is given by

$$\nabla_X N = (X(N_1), ..., X(N_{n+1}))\,,$$

where $N = (N_1, ..., N_{n+1})$, contemplated as a map from Σ to $\mathbb{R}^{n+1}_a$. The linear operator A of $T_p\Sigma$, $p \in \Sigma$, is self-adjoint with respect to the induced metric. Its eigenvalues $\kappa_1(p), \kappa_2(p), ..., \kappa_n(p)$ are called the principal curvatures of the hypersurface. Consider the the characteristic polynomial of A,

$$\det(tI - A) = \sum_{k=0}^{n} c_k\, t^{n-k} = \prod_{i=1}^{n}(t - \kappa_i),$$

where we put $c_0 = 1$. It is not difficult to see that

$$\begin{aligned} c_1 &= -\sum_{i=1}^{n} \kappa_i \\ c_k &= (-1)^k \sum_{i_1<\ldots<i_k} \kappa_{i_1} \cdots \kappa_{i_k}, \quad 2 \leq k \leq n. \end{aligned}$$

The k-th mean curvature S_k of Σ is defined as follows,

$$S_k = \frac{(-1)^{k(a+1)}}{\binom{n}{k}}\, c_k,$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$. For instance, when $k = 1$, we get $S_1 = \frac{(-1)^{a+1}}{n}\, c_1 = \frac{(-1)^a}{n}\,\mathrm{trace}(A)$, the usual mean curvature of Σ. Moreover, S_2 is, up to a constant, the scalar curvature of Σ and, when $k = n$, we recover the Gauss-Kronecker curvature $S_n = (-1)^{an}\det(A)$ of Σ. It is interesting to note that k-th mean curvatures are in fact intrinsic geometric invariants of the hypersurfaces when k is even. Precisely, the parity of k plays an important role in the treatment of the equations, as it will be shown in next sections.

Remark 2.2.1. *For the case of a spacelike hypersurface Σ in a (general) spacetime*

M, the geometric information contained in a k-th mean curvature S_k can be locally propagated to the future and the past in M and, then, physically interpreted. In fact, for each $p_0 \in \Sigma$ there exist an open neighbourhood U of p_0 in M and a reference frame Q on U such that $Q_p = N_p$ for all $p \in \Sigma \cap U$. The operator field $X \mapsto -\nabla_X Q$, $X \in T_qM$, $q \in U$, may be restricted to $Q^\perp$ providing with $A_Q : Q^\perp \to Q^\perp$ on U. Note that A_Q equals to the shape operator corresponding to N on $\Sigma \cap U$. On U, consider the $n+1$ smooth functions $\overline{S}_0$, $\overline{S}_1$,...,$\overline{S}_n$ defined by

$$\overline{S}_k = \frac{1}{\binom{n}{k}}\, c_k,$$

where the function c_k is defined as previously from

$$\det\left(tI - A_Q\right) = \sum_{k=0}^{n} c_k\, t^{n-k}.$$

Each $\overline{S}_k$ is a relative quantity for the observers in Q and it equals to the k-th mean curvature S_k on $\Sigma \cap U$. Assume $S_k > 0$ (resp. $S_k < 0$) at some $p \in \Sigma \cap U$. Then $\overline{S}_k > 0$ (resp. $\overline{S}_k < 0$) near p in U. In particular, if the mean curvature H satisfies $H > 0$ (resp. $H < 0$) at p then $div(Q) > 0$ (resp. $div(Q) < 0$) near p in U, i.e., for the observers in Q the universe is expanding (resp. contracting) near p in U.

2.3 The Fermi-Walker connection

Finally, we present the mathematical tool to formulate rigorously the physical concepts of Chapters 6, 7 and 8.

Let $(M, \langle\, ,\, \rangle)$ be an $n(\geq 2)-$ dimensional spacetime, endowed with a fixed time orientation. As usual, we will refer the points of M as events and we will consider an observer in M as a (smooth) curve $\gamma : I \longrightarrow M$, I an open interval of the real line $\mathbb{R}$, such that $\langle \gamma'(t), \gamma'(t)\rangle = -1$ and $\gamma'(t)$ is future pointing for any proper time

t of γ. At each event $\gamma(t)$ the tangent space $T_{\gamma(t)}M$ splits as

$$T_{\gamma(t)}M = T_t \oplus R_t,$$

where $T_t = \mathrm{Span}\{\gamma'(t)\}$ and $R_t = T_t^{\perp}$. Endowed with the restriction of $\langle\, ,\, \rangle$, R_t is a spacelike hyperplane of $T_{\gamma(t)}M$. It is interpreted as the instantaneous physical space observed by γ at t. Clearly, the observer γ is able to compare spatial directions at t. In order to compare $v_1 \in R_{t_1}$ with $v_2 \in R_{t_2}$, $t_1 < t_2$ and $|v_1| = |v_2|$, the observer γ could use, as a first attempt, the parallel transport along γ defined by the Levi-Civita covariant derivative along γ,

$$P^{\gamma}_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_2)}M.$$

Unfortunately, this linear isometry satisfies $P^{\gamma}_{t_1,t_2}(R_{t_1}) = R_{t_2}$ if γ is free falling (i.e., γ is a geodesic) but this property does not remain true for any general observer. In order to solve this difficulty, recall that the Levi-Civita connection ∇ of M induces a connection along $\gamma : I \longrightarrow M$ such that the corresponding covariant derivative is the well-known covariant derivative of vector fields $Y \in \mathfrak{X}(\gamma)$ (the vector fields along γ), namely, $\dfrac{DY}{dt} = \nabla_{\partial_t} Y \in \mathfrak{X}(\gamma)$.

Now, for each $Y \in \mathfrak{X}(\gamma)$ put Y_t^T, Y_t^R the orthogonal projections of Y_t on T_t and R_t, respectively, i.e., $Y_t^T = -\langle Y_t, \gamma'(t)\rangle\, \gamma'(t)$ and $Y_t^R = Y_t - Y_t^T$. In this way, we define $Y^T, Y^R \in \mathfrak{X}(\gamma)$. We have, [78, Prop. 2.2.1],

Proposition 2.3.1. *There exists a unique connection $\widehat{\nabla}$ along γ such that*

$$\widehat{\nabla}_X Y = \left(\nabla_X Y^T\right)^T + \left(\nabla_X Y^R\right)^R,$$

for any $X \in \mathfrak{X}(I)$ *and* $Y \in \mathfrak{X}(\gamma)$.

This connection $\widehat{\nabla}$ is called the *Fermi-Walker connection of* γ. It shows the suggestive property that if $Y \in \mathfrak{X}(\gamma)$ satisfies $Y = Y^R$ (i.e., Y_t may be observed by γ at any t) then $\left(\widehat{\nabla}_X Y\right)_t \in R_t$ for any t.

Denote by $\widehat{D}/dt$ the covariant derivative corresponding to $\widehat{\nabla}$. Then, we have [78, Prop. 2.2.2],

$$\frac{\widehat{D}Y}{dt} = \frac{DY}{dt} + \langle \gamma', Y\rangle \frac{D\gamma'}{dt} - \langle \frac{D\gamma'}{dt}, Y\rangle \gamma', \tag{2.10}$$

for any $Y \in \mathfrak{X}(\gamma)$. Note that $\dfrac{\widehat{D}}{dt} = \dfrac{D}{dt}$ if and only if γ is free falling.

Associated to the Fermi-Walker connection on γ there exist a parallel transport

$$\widehat{P}^{\gamma}_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_1)}M,$$

which is a lineal isometry and satisfies $\widehat{P}^{\gamma}_{t_1,t_2}(R_{t_1}) = R_{t_2}$. Therefore, given $v_1 \in R_{t_1}$ and $v_2 \in R_{t_2}$, with $t_1 < t_2$ and $|v_1| = |v_2|$, the observer γ may consider $\widehat{P}^{\gamma}_{t_1,t_2}(v_1)$ instead v_1, with the advantage to wonder if $\widehat{P}^{\gamma}_{t_1,t_2}(v_1)$ is equal to v_2 or not (compare with [66, Sec. 6.5]).

The acceleration $\dfrac{D\gamma'}{dt}$ satisfies $\dfrac{D\gamma'}{dt}(t) \in R_t$, for any t. Therefore, it may be observed by γ whereas the velocity γ' is not observable by γ.

Chapter 3

Existence and extendibility of prescribed mean curvature function graphs in FLRW spacetimes

In this chapter we face the prescribed mean curvature function problem in a Friedmann-Lemaître-Robertson-Walker spacetime. We will concentrate our attention on a particular kind of spacelike hypersurfaces; the rotationally symmetric spacelike graphs. We will provide several existence results on the Dirichlet problem and we will analyse when these hypersurfaces can be extended as an entire graph satisfying the given prescription.

The results of this chapter are picked up in [37] and [10].

3.1 The prescription mean curvature problem in FLRW spacetimes

Our prescription problem may be formulated, as we exposed in the previous chapter (equation 2.3) by means of the following quasilinear elliptic equation,

$$Q(u)(x) = \operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - |\nabla u|^2}}\right) + \frac{f'(u)}{\sqrt{f(u)^2 - |\nabla u|^2}}\left(n + \frac{|\nabla u|^2}{f(u)^2}\right) = nH(u,x). \tag{3.1}$$

Note that Q is a quasilinear elliptic operator defined only on smooth functions which satisfy (2.2). In order to face our problem, the first step is to perform a suitable variable change in (2.3) to make it easier. Indeed, consider

$$v = \varphi(u), \qquad \text{where} \qquad \varphi(t) = \int_0^t \frac{ds}{f(s)}.$$

Clearly, φ is a diffeomorphism from I to another open interval J in $\mathbb{R}$. Consequently, it follows that $\nabla v = \dfrac{1}{f(u)}\nabla u$. Therefore, $|\nabla u| < f(u)$ holds if and only if $|\nabla v| < 1$. It is clear that u is radially symmetric if and only if v is also radially symmetric.

Taking into account

$$\begin{aligned}
\operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - |\nabla u|^2}}\right) &= \operatorname{div}\left(\frac{\nabla v}{f(u)\sqrt{1 - |\nabla v|^2}}\right) \\
&= \frac{1}{f(u)}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \left\langle \nabla \frac{1}{f(u)}, \frac{\nabla v}{\sqrt{1-|\nabla v|^2}} \right\rangle \\
&= \frac{1}{f(u)}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) - \left\langle \frac{f'(u)}{f(u)^2}\nabla u, \frac{\nabla v}{\sqrt{1-|\nabla v|^2}} \right\rangle
\end{aligned}$$

$$= \frac{1}{f(u)}\mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) - \frac{f'(u)}{f(u)\sqrt{1-|\nabla v|^2}}|\nabla v|^2,$$

our equation is transformed in

$$\mathcal{Q}(v) := \mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \frac{nf'(\varphi^{-1}(v))}{\sqrt{1-|\nabla v|^2}} = nf(\varphi^{-1}(v))H(\varphi^{-1}(v),x). \qquad (3.2)$$

Actually, the previous variable change is equivalent to consider the following conformal map

$$\begin{array}{rcl} \varphi \times Id : I \times_f \mathbb{R}^n & \longrightarrow & (J \times \mathbb{R}^n, -ds^2 + g) \\ (t,p) & \longmapsto & (\varphi(t), p), \end{array}$$

which has conformal factor $\dfrac{1}{f(t)}$. The Lorentzian product spacetime is in fact an open subset of Lorentz-Minkowski $\mathbb{L}^{n+1}$. In $\mathbb{L}^{n+1}$, the mean curvature function of the spacelike graph by v is

$$\frac{1}{n}\mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right).$$

From now on, we will deal with equation (3.2), under the conditions $|\nabla v| < 1$ on a ball B, centered in 0 of radius R, and $v = 0$ on ∂B. From the boundedness of the lenght of the gradient of v (the spacelike condition) it follows that $|v| < R$ on $\overline{B}$, i.e., the image of v lies in the interval $[-R, R]$ or, equivalently, the image of the original function $u = \varphi^{-1}(v)$ is contained in $\varphi^{-1}([-R, R])$. Hence, we have an a priori upper bound of the spacelike graph. Thus, the first assumption on the interval I in our FLRW spacetime is

(A1) $\quad [-R, R] \subset \varphi(I)$, i.e.,

$$I_f(R) := \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds,\ \int_0^R f(\varphi^{-1}(s))ds\right] \subset I.$$

Basically, (A1) means that the interval I must be big enough to contain the highest or lowest possible spacelike graph.

Summarizing, in the following sections we will take care of the problem

$$\begin{gathered}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)+\frac{nf'(\varphi^{-1}(v))}{\sqrt{1-|\nabla v|^2}}=nf(\varphi^{-1}(v))H(\varphi^{-1}(v),x)\quad \text{in}\quad B,\\ v=0\quad\text{in}\quad \partial B.\end{gathered}\tag{3.3}$$

We may observe that the last term in the left-hand side goes to infinity when $|\nabla v|$ approaches to 1. The main difficulty of the problem comes from this singularity of the gradient. For nonlinearities not depending on the gradient, we mentioned in the Introduction that Bartnik and Simon proved a kind of general existence result, later generalized to continuous nonlinearities with possible dependence on the gradient in [11, Th. 2.1]. The presence of the singular term prevents from a direct application of such results.

3.2 A priori radial symmetry of positive solutions

The aim of this section is to provide sufficient conditions on the prescription function to ensure that any eventual positive solution of (3.3) must be radially symmetric. In fact, we can state the following result.

Theorem 3.2.1. *Let $I\times_f\mathbb{R}^n$ be a FLRW spacetime, and let $B=B_0(R)$ be the Euclidean ball with radius R centred at $0\in\mathbb{R}^n$.. For each smooth radially symmetric function $H:I\times[0,R]\to\mathbb{R}$, $H=H(t,r)$, radially increasing in the second variable and which satisfies $H(0,r)\le\dfrac{f'(0)}{f(0)}$ in ∂B, any positive solution v of equation* (3.3) *is radially symmetric. Moreover, $\dfrac{\partial v}{\partial r}<0$ holds in ∂B.*

Remark 3.2.2. Geometrically, the last assertion means that the hyperbolic angle between the unit normal vector field N and ∂_t is nowhere zero at the points of the graph corresponding to $\{0\} \times \partial B$.

In order to use the Strong Maximum Principle (see for instance [58]) to derive a suitable Alexandroff reflection method, it is required that the involved differential operator is defined on $C^2(\overline{B}(R))$, and it must be uniformly elliptic. To this aim, we apply to our operator (3.3) a truncature argument first used in [31] for the Lorentz-Minkowski operator.

First of all, we rewrite our operator $\mathcal{Q}$ as

$$\mathcal{Q}(v) = \operatorname{div}\big(h(|\nabla v|^2)\nabla v\big) + nh(|\nabla v|^2)f'(\varphi^{-1}(v)),$$

where $h(s) := \dfrac{1}{\sqrt{1-s}}$.

Fix $v \in C^2\big(\overline{B}(R)\big)$ a positive solution of (3.3), and let $m := \max_{\overline{B}(R)}|\nabla v| < 1$. We define the truncated function $\overline{h}$,

$$\overline{h}(s) = \begin{cases} h(s) & \text{if} \quad s \leq m^2, \\ \alpha(s) & \text{if} \quad m^2 < s < 1, \\ c & \text{if} \quad s \geq 1, \end{cases}$$

where the function $\alpha : \mathbb{R} \to \mathbb{R}^+$ and the constant c are such that $\overline{h} \in C^\infty(\mathbb{R})$ is increasing and positive. Observe that both $\overline{h}$ and $\overline{h}'$ are bounded on all $\mathbb{R}$. We introduce a new operator, denoted by $\mathcal{Q}_v$, as follows,

$$w \longmapsto \mathcal{Q}_v(w) = \operatorname{div}\big(\overline{h}(|\nabla w|^2)\nabla w) + n\overline{h}(w)f'(\varphi^{-1}(w)), \tag{3.4}$$

where $w \in C^2(\overline{B}(R))$. Note that $\mathcal{Q}_v(w) = \mathcal{Q}(w)$ whenever $|\nabla w| \leq |\nabla v|$. It is not difficult to compute the principal symbol of $\mathcal{Q}_v$ (see [58, chap. 1]) and to prove that $\mathcal{Q}_v$ is uniformly elliptic.

Now the Strong Maximum Principle may be applied to $\mathcal{Q}_v$ and then, the proof of Theorem 3.2.1 follows from [53, Cor. 1].

For the sake of completeness, a sketch of the proof is included in the Appendix.

Under these considerations, passing to polar coordinates, the equation (3.3) is reduced to the following ODE with mixed boundary conditions

$$\begin{gathered} \frac{1}{r^{n-1}}\left(r^{n-1}\phi(v')\right)' + \frac{nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} = nH(\varphi^{-1}(v), r)f(\varphi^{-1}(v)) \qquad \text{in} \quad]0,R[, \\ v'(0) = 0 = v(R), \end{gathered} \tag{3.5}$$

where $\phi(s) := \dfrac{s}{\sqrt{1-s^2}}$. By a *solution* we understand a function $v \in C^2]0,R[\cap C^1[0,R]$ with $|v'| < 1$ on $]0,R[$ and satisfying the above mixed boundary value problem. From now on, we will work with this equation.

3.3 First existence result of Dirichlet problem

We have just proved that, under some conditions, every eventual positive solution v of problem (3.3) must be radially symmetric. The purpose of this section is provide sufficient conditions for the existence of such radially symmetric solutions.

We fix some notation which will be used in the rest of the section. Let C be the Banach space of the real continuous functions in $[0,R]$, with the maximum norm, and C^1 the space of continuously differentiable functions with its usual norm $\|v\| = \|v\|_\infty + \|v'\|_\infty$. We write $B_{\rho,\gamma} = \{v \in C^1 : \|v\|_\infty < \rho, \|v'\|_\infty < \gamma\}$.

Our first step is to associate a *fixed point operator* $\mathcal{N}$ to problem (3.5). We start by defining

$$S : C \longrightarrow C^1,$$

$$S(v)(r) = \frac{1}{r^{n-1}} \int_0^r t^{n-1} v(t)dt \quad (r \in (0, R]), \quad S(v)(0) = 0,$$

$$K : C^1 \longrightarrow C^1,$$

$$K(v)(r) = \int_r^R v(t)dt.$$

An easy checking shows that, for each $h \in C$, the mixed problem

$$\left(r^{n-1}\phi(v')\right)' + r^{n-1}h = 0, \quad v'(0) = v(R) = 0,$$

has a unique solution given by

$$v = K \circ \phi^{-1} \circ S(h).$$

Now, we consider the Nemytskii operator

$$N_F : B_{\alpha,1} \subset C^1 \longrightarrow C, \quad N_F(v) = F(\cdot, v, v'),$$

where $F : [0, R] \times \varphi(I) \times (-1, 1) \to \mathbb{R}$ is given by

$$F(r, s, t) = -nH(\varphi^{-1}(s), r)f(\varphi^{-1}(s)) + \frac{n(f' \circ \varphi^{-1})(s)}{\sqrt{1 - t^2}}.$$

Obviously, N_F is continuous and $N_F(\overline{B}_{\rho,\gamma})$ is a bounded subset of C for all $\rho > 0$ and $0 \leq \gamma < 1$. Moreover, from the compactness of K we deduce the compactness of $K \circ \phi^{-1} \circ S : C \to C^1$ in $\overline{B}_{\rho,\gamma}$ for all $\rho > 0$ and $0 \leq \gamma < 1$. In this way, solving the problem (3.5) is equivalent to find the fixed points of $\widehat{\mathcal{N}}$.

Lemma 3.3.1. *A function $v \in C^1$ is a solution of equation* (3.5) *if and only if v is* *a fixed point of the nonlinear compact operator*

$$\widehat{\mathcal{N}} : B_{\rho,\gamma} \subset C^1 \longrightarrow C^1, \quad \widehat{\mathcal{N}} = K \circ \phi^{-1} \circ S \circ N_F. \tag{3.6}$$

Remark 3.3.2. Note that the image of the operator $\widehat{\mathcal{N}}$ is contained in $C^2[0, R]$, so the fixed points (solutions of the equation (3.5)) will be of class C^2. Moreover, using the regularity theorem for elliptic nonlinear operators, (see [58, Chapter 4]) we conclude that, if the prescription function H is of class C^∞, then the solutions will also be infinitely derivable.

Note that fixed points of $\widehat{\mathcal{N}}$ always verify the restrictions $v'(0) = v(R) = 0$. Therefore, we will consider the Banach subspace of C^1 which satisfy these boundary conditions. Our aim is to search a suitable subset to apply the Schauder point fixed theorem. Let us define the set

$$\mathcal{B}(\gamma) = \{v \in \overline{B}_{R,\gamma} : \, v'(0) = 0 = v(R)\}.$$

Since the graph associated to v is spacelike, i.e., $\|v'\|_\infty < 1$, we deduce that $\|v\|_\infty < R$. So, the image of v is in $[-R, R]$ or, equivalently, the image of $u = \varphi^{-1}(v)$ is contained in $\varphi^{-1}([-R, R])$. Hence, this observation gives us a height bound of the spacelike graphs. In order to restrict the operator $\widehat{\mathcal{N}}$ to $\mathcal{B}(\gamma)$, we impose the first assumption on the interval I in our FLRW spacetime

(A1) $[-R, R] \subset \varphi(I)$, i.e., $I_f(R) := \Big[-\int_{-R}^{0} f\big(\varphi^{-1}(s)\big)ds \, , \, \int_0^R f\big(\varphi^{-1}(s)\big)ds\Big] \subset I$.

Basically, (A1) says that the interval I must be sufficiently big to contain the highest or lowest possible spacelike graph.

Now, the compact operator $\widehat{\mathcal{N}}$ restricted to $\mathcal{B}(\gamma)$ will be denoted by $\mathcal{N} : \mathcal{B}(\gamma) \rightarrow$

C^1. It is possible write it explicitly as follows

$$\mathcal{N}(v)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{n-1}}\int_0^s \tau^{n-1}F(\tau, v, v')d\tau\right] ds.$$

By using that $\phi^{-1}(\mathbb{R}) = (-1, 1)$, one gets

$$\|\mathcal{N}(v)\|_\infty < R \quad \text{for all} \quad v \in \mathcal{B}(\gamma). \tag{3.7}$$

On the other hand, deriving $\mathcal{N}(v)$

$$\mathcal{N}'(v)(s) = -\phi^{-1}\left[\frac{1}{s^{n-1}}\int_0^s \tau^{n-1}F(\tau, v, v')d\tau\right].$$

Then, taking into account that ϕ is an odd and increasing homeomorphism, we have

$$\begin{aligned} \|\mathcal{N}'(v)\|_\infty &\leq \phi^{-1}\left(\max_{[0,R]}\left[\left(h^* + \frac{g*}{\sqrt{1-\gamma^2}}\right)\frac{n}{s^{n-1}}\textstyle\int_0^s \tau^{n-1}d\tau\right]\right) = \\ &= \phi^{-1}\left(\left[h^* + \frac{g*}{\sqrt{1-\gamma^2}}\right]R\right), \end{aligned} \tag{3.8}$$

for every $v \in \mathcal{B}(\gamma)$, where we have defined

$$h^* = \max\{|H(r, \varphi^{-1}(s))f(\varphi^{-1}(s))| : \; r \in [0, R], \; s \in [-R, R]\},$$

$$g^* = \max\{|(f' \circ \varphi^{-1})(s)| : \; s \in [-R, R]\}.$$

At this point, the second assumption on the warping function f is imposed.

(A2) The absolute value of the expansion, f', along the temporal interval $I_f(R)$ is lower than $\dfrac{1}{R}$.

This is equivalent to say that $\left|(f' \circ \varphi^{-1})(s)\right| = \left|\dfrac{(f \circ \varphi^{-1})'}{f \circ \varphi^{-1}}(s)\right| < \dfrac{1}{R}$ for all $s \in$

$[-R, R]$, or more simply $g^* < \frac{1}{R}$. Using this hypothesis, we can take a $\gamma \in (0,1)$ sufficiently close to 1 such that

$$R\left[h^* + \frac{g^*}{\sqrt{1-\gamma^2}}\right] \leq \phi(\gamma).$$

Introducing this inequality in (3.8),

$$\|\mathcal{N}'(v)\|_\infty \leq \gamma.$$

This last inequality, together with (3.7), implies that $\mathcal{N}(\mathcal{B}(\gamma)) \subset \mathcal{B}(\gamma)$. Since $\mathcal{B}(\gamma)$ is contractible to a point, and $\mathcal{N}$ is a continuous and compact operator, the Schauder Point Fixed theorem applies, leading to the following result.

Proposition 3.3.3. *Assume* (A1) *and* (A2). *Then, problem* (3.3) *has at least one radially symmetric solution.*

Note that the solution given in previous result is not necessarily positive. To assure the positivity of the solutions we need an additional condition.

Proposition 3.3.4. *Assume that*

$$\text{(A3)} \quad H(t,r) \leq \frac{f'}{f}(t) \quad \text{and} \quad f'(t) \geq 0, \qquad \text{for any} \quad r \in]0, R[\ ,\ \ t \in I_f(R),$$

Then, any v not identically zero solution of (3.5) *verifies $v > 0$ on* $[0, R)$.

Proof. First, note that condition (A3) implies that F is nonnegative in $[0, R] \times [-R, R] \times [0, \gamma]$. From the equality

$$v'(r) = -\phi^{-1}\left[\frac{n}{r^{n-1}}\int_0^r \tau^{n-1}F(\tau, v, v')d\tau\right], \tag{3.9}$$

and taking into account that ϕ is an odd increasing diffeomorphism, we deduce that v is decreasing. Since $v(R) = 0$, we have $v \geq 0$ on $[0, R]$. v is a solution identically zero if and only if $H(0,r) = \dfrac{f'}{f}(0)$ for all $r \in [0,R]$. If v does not vanished identically, then $v(0) > 0$ and there exists $r_0 \in (0,R)$ where $v'(r_0) < 0$. Then, from (3.9) we get

$$\int_0^{r_0} \tau^{n-1} F(\tau, v, v')d\tau > 0.$$

Since $F(\tau, v, v') \geq 0$ for all $\tau \in [0,R]$, this implies

$$\int_0^{r} \tau^{n-1} F(\tau, v, v')d\tau > 0 \qquad \text{for all } r > r_0.$$

From (3.9), we get $v'(r) < 0$ on $[r_0, R]$ and therefore, we conclude that $v > 0$ on $[0,R)$. □

Summarizing in a more geometric perspective, we can state the following result.

Theorem 3.3.5. *Let $I \times_f \mathbb{R}^n$ be a Friedmann-Lemaître-Robertson-Walker spacetime and $B = B(R)$ the Euclidean ball centred in $0 \in \mathbb{R}^n$ with radius R. Assume that $I_f(R) \subset I$, where*

$$I_f(R) = \left[-\int_{-R}^{0} f\big(\varphi^{-1}(s)\big)ds, \int_0^R f\big(\varphi^{-1}(s)\big)ds \right].$$

Let $H : I \times \overline{B} \to \mathbb{R}$ be a smooth radially symmetric function. Suppose that the following inequality holds

$$\max_{I_f(R)} |f'| < \frac{1}{R}.$$

Then, there exists at least one spacelike graph defined on $\overline{B}$ with mean curvature function H, supported on the slice $\{t = 0\}$. Moreover, if

$$H(t,r) \leq \frac{f'}{f}(t) \quad \text{and} \quad f'(t) \geq 0, \qquad \text{for any} \quad r \in]0,R[\ , \ t \in I_f(R),$$

then the graph is either a slice or is above of $\{t = 0\}$ and only touches it on the boundary $\{0\} \times \partial B$.

Remark 3.3.6. *Observe that in the previous result, if (A3) is assumed from the beginning, every eventual solution is a priori positive, therefore condition (A2) can be weakened to* $\mathbb{R}^+ \cap I_f(R) = \left[0 \, , \, \int_0^R f\big(\varphi^{-1}(s)\big) ds\right]$.

Thus, as a direct consequence of Theorems 3.2.1 and 3.3.5, we may enunciate the following one.

Theorem 3.3.7. *Let* $I \times_f \mathbb{R}^n$ *be a Friedmann-Lemaître-Robertson-Walker spacetime, and let* $B = B_0(R)$ *be the Euclidean ball with radius* R *centred at* $0 \in \mathbb{R}^n$*. Assume* $I_f(R) \subset I$*, where*

$$I_f(R) := \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds \, , \int_0^R f(\varphi^{-1}(s))ds\right] \quad \text{and} \quad \varphi(t) = \int_0^t \frac{dt}{f(t)},$$

and suppose that the following inequality holds

$$\max_{\mathbb{R}^+ \cap I_f(R)} |f'| < \frac{1}{R}.$$

For each radially symmetric smooth function $H : I \times \overline{B} \to \mathbb{R}$ *such that*

$$H(t,r) \leq \frac{f'}{f}(t) \quad \textit{and} \quad f'(t) \geq 0, \qquad \textit{for any} \quad r \in]0, R[\ , \ t \in I_f(R),$$

there exists a spacelike graph with mean curvature function H *defined on* $\overline{B}$*, supported on the slice* $t = 0$ *and only touching it on the boundary* $\{0\} \times \partial B$*, and forming a non-zero hyperbolic angle with* ∂_t*. Moreover, if* H *is increasing in the second variable, such a spacelike graph must be radially symmetric.*

3.4 Strictly spacelike character and bounds on the derivative of the solutions

Graphs which are solution of the equation (E) are spacelike on the open ball. However, there could exist solutions which are of light type on the boundary, ∂B. The following lemma ensures a priori that each possible solution v of (3.5) is spacelike on the boundary too, i.e., $|v'| < 1$ on $[0, R]$.

Lemma 3.4.1. *Let $v \in C^2[0, R]$ be a solution of* (3.5). *Then $|v'| < 1$ on $[0, R]$.*

Proof. On $[0, R[$ the solution satisfies $|v'| < 1$. We only have to prove the inequality at $r = R$. Suppose that there exists $\{r_k\} \subset]0, R[$ converging to R, such that

$$\lim_{k\to\infty} |v'(r_k)| = 1 \qquad \text{and} \qquad \lim_{k\to\infty} |\phi\big(v'(r_k)\big)| = \infty.$$

For $k \in \mathbb{N}$ sufficiently large, one has for $r = r_k$,

$$\frac{1}{r^{n-1}}\left(r^{n-1}\phi(v')\right)' + \frac{nf'(\varphi^{-1}(v))}{v'}\,\phi(v') = nH(\varphi^{-1}(v), r)f(\varphi^{-1}(v)),$$

implying

$$\frac{\left[r^{n-1}\phi\big(v'\big)\right]'}{\left[r^{n-1}\phi\big(v'\big)\right]} = n\left(\frac{H(\varphi^{-1}(v), r)f(\varphi^{-1}(v))}{\phi(v')} - \frac{f'(\varphi^{-1}(v))}{v'}\right).$$

Let $\overline{r} \in]0, R[$ be such that $|v'(\tau)| > 0$ for any $\tau \in]\overline{r}, R[$. Integrating the last equality, we have

$$\log\left|r_k^{n-1}\phi\big(v'(r_k)\big)\right| - \log\left|\overline{r}^{n-1}\phi\big(v'(\overline{r})\big)\right| =$$

$$n\int_{\overline{r}}^{r_k}\left(\frac{H(\varphi^{-1}(v), r)f(\varphi^{-1}(v))}{\phi(v')} - \frac{f'(\varphi^{-1}(v))}{v'}\right)dr.$$

Taking limits, we check that left member tends to infinity while the right one is finite. Therefore, we deduce that $|\phi(v')|$ is bounded and, consequently, $\|v'\|_\infty$ must be strictly lower than 1. $\square$

In the next result, we provide an a priori bound of the derivative of the solutions on the boundary R. This fact will play a key role later.

Proposition 3.4.2. *There exists $0 < \gamma < 1$ such that for any $\varepsilon \in [0,1]$, one has that any $u \in C^2[R/2, R]$ with $u(R) = 0$ and satisfying on $]R/2, R[$ the equation*

$$\frac{1}{(r+\varepsilon)^{n-1}} \big((r+\varepsilon)^{n-1}\phi(u')\big)' + \frac{nf'(\varphi^{-1}(u))}{\sqrt{1-u'^2}} = nH(\varphi^{-1}(u), r) f(\varphi^{-1}(u)),$$

satisfies $|u'(R)| < \gamma$.

Proof. Let $w^+ : [R/2, R] \longrightarrow \mathbb{R}$ be given by

$$w^+(r) = \int_0^{R-r} \frac{1}{\sqrt{1+\beta(t)}}\, dt,$$

where $\beta(t) = \alpha\, e^{\lambda t}$, with α and λ constants which will be specified later. This type of function was used by Gerhardt in [51] for a similar purpose (see formula (2.9) therein).

Clearly, for all $r \in [R/2, R]$,

$$|(w^+)'(r)| = \frac{1}{\sqrt{1+\beta(R-r)}} < 1.$$

Now, let u be satisfying the hypothesis and consider the elliptic operator depending on u

$$Q_u(v)(r) := -\frac{1}{(r+\varepsilon)^{n-1}} \left[(r+\varepsilon)^{n-1}\phi(v')\right]' - \frac{nf'\big(\varphi^{-1}(u)\big)}{\sqrt{1-v'^2}}.$$

It follows that

$$Q_u(w^+)(r) = \frac{1}{\sqrt{\beta(R-r)}}\left[\frac{n-1}{r+\varepsilon} + \frac{\lambda}{2} - nf'\big(\varphi^{-1}(u)\big)\sqrt{1+\alpha\, e^{\lambda(R-r)}}\right].$$

Using that $|u| < R/2$ on $[R/2, R]$, we can choose $\lambda > 0$ sufficiently large and $\alpha > 0$ sufficiently small which do not depend on u and $\varepsilon \in [0,1]$ such that

$$\frac{\lambda}{2} + \frac{n-1}{r+\varepsilon} - nf'\big(\varphi^{-1}(u)\big)\sqrt{1+\alpha\, e^{\lambda(R-r)}} > 0,$$

on $[R/2, R]$. Because of $\varepsilon \in [0,1]$, note that α and λ can be chosen independently of ε. In fact, the choice only depends on functions f and H. Hence, making α smaller if necessary, we can get

$$Q_u(w^+) \geq \max\Big\{ -nf(t)H(t,r) \ : \ r \in [R/2, R],\ t \in [-R/2, R/2] \Big\},$$

implying that

$$Q_u(w^+) \ \geq \ Q_u(u).$$

We have two situations. In the first one $w^+(R/2) \geq u(R/2)$ and in the second $w^+(R/2) < u(R/2)$. Assume that we are in the second case and take

$$K = \max_{[\mathrm{R/2,R}]} |u'|.$$

Observe that $K < 1$ by Lemma 3.4.1. Then, there exists $r_0 \in]R/2, R[$ satisfying

$$r_0 - \frac{R}{2} > \frac{KR}{2}.$$

So, we can consider $\alpha_u < \alpha$ such that

$$\left[\frac{(r_0 - \frac{R}{2})^2}{(u(R/2) - w^+(r_0))^2} - 1\right] e^{-\lambda R/2} \ > \ \alpha_u \ > \ 0.$$

It follows that, considering the function on $[0, R/2]$ given by

$$\overline{\alpha}(s) = \begin{cases} \alpha & \text{if} \qquad s \le R - r_1, \\ h(s) & \text{if} \quad R - r_1 \le s \le R - r_0, \\ \alpha_u & \text{if} \quad R - r_0 < s \le R/2, \end{cases}$$

where $r_1 \in]r_0, R[$, h is a decreasing function that makes $\overline{\alpha}$ differentiable, and

$$w_u^+(r) := \int_0^{R-r} \frac{1}{\sqrt{1 + \overline{\alpha}(t)\, e^{\lambda t}}}\, dt, \qquad r \in [R/2, R],$$

one has that $w_u^+(R/2) \ge u(R/2)$. By a simple computation,

$$Q_u(w_u^+) \ge Q_u(w^+).$$

Hence, it follows that $v = w^+$ or $v = w_u^+$ is an upper-solution of the original equation on $[R/2, R]$, that is,

$$\begin{aligned} Q_u(v) &\ge Q_u(u) \\ v(R) &= u(R) = 0, \\ v(R/2) &\ge u(R/2). \end{aligned}$$

Therefore, from Maximum Principle (see the Comparison Principle in [58, Theorem 4.4]) we conclude that

$$v(r) \ge u(r), \qquad r \in [R/2, R].$$

Since $v(R) = u(R)$ and taking into account that $v'(R)$ does not depend on u and ε, we deduce that

$$u'(R) \ge v'(R) =: \gamma^+ > -1, \quad |\gamma^+| < 1.$$

Analogously, taking

$$w^-(r) := -\int_0^{R-r} \frac{1}{\sqrt{1 + \widehat{\beta}(t)}}\, dt,$$

where $\widehat{\beta}(t) = \widehat{\alpha}\, e^{\widehat{\lambda} t}$, we have

$$u'(R) \le v'(R) =: \gamma^- < 1, \quad |\gamma^-| < 1,$$

where $v = w^-$ or $v = w_u^-$

Consequently, taking $\gamma := \max\{|\gamma^+|, |\gamma^-|\}$, we conclude that

$$|u'(R)| < \gamma < 1.$$

□

3.5 Second existence result of Dirichlet problem

In this section we give sufficient conditions for the existence of positive and radially symmetric solutions of problem (3.5), but deleting the assumption imposed on the radius of the domain (A2). In certain sense, we will improve the result of Theorems 3.3.7 and 3.3.5 (see Remark 3.5.3 for comparing both results).

Throughout the section $C[0, R]$ denotes the Banach space of the real continuous functions in $[0, R]$, endowed with the maximum norm, and $C^1[a, b]$ the Banach space of continuously differentiable functions in $[a, b]$ endowed with the usual norm.

Our strategy consists on a truncation of the singular term, obtaining a family of problems tending to the original one, that can be solved through degree techniques. Then, we take the limit of the solutions of the truncated equations, and we have to prove that this limit is really a solution of the singular problem. Some arguments in our proof come from [70, Chapter 9] (see also the references therein), nevertheless the computations are essentially different because [70] only considers the case of a regular ϕ-laplacian defined on the whole real line, whereas in our case the ϕ-laplacian is singular.

The main existence result goes as follows.

Theorem 3.5.1. *If* (A1) *and* (A3) *hold true, then there exists at least one positive solution of problem* (3.5)*.*

Proof. The proof is organized in three steps.

- First step: *Truncation*

First of all, we embed the initial problem in to the family of mixed boundary value problems

$$\frac{1}{(r+\varepsilon)^{n-1}}\left((r+\varepsilon)^{n-1}\phi(v')\right)' + \frac{nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} = nH(\varphi^{-1}(v),r)f(\varphi^{-1}(v)), \tag{3.10}$$
$$v'(0) = 0 = v(R),$$

where $\varepsilon \in [0,1]$. Expanding the left member of the truncated equation and multiplying by $\sqrt{1-v'^2}$, we get

$$\frac{v''}{1-v'^2} = -(n-1)\,\frac{v'}{r+\varepsilon} + nf(\varphi^{-1}(v))\,H(v,r)\sqrt{1-v'^2} - nf'(\varphi^{-1}(v)). \tag{3.11}$$

Since

$$\frac{1}{1-v'^2} = \frac{1}{2}\left(\frac{1}{1+v'} + \frac{1}{1-v'}\right),$$

we may rewrite the previous expression as follows

$$\begin{aligned}\left[\frac{1}{2}\log\left(\frac{1+v'}{1-v'}\right)\right]' &= -(n-1)\,\frac{v'}{r+\varepsilon} + \\ &+ nH(\varphi^{-1}(v),r)f(\varphi^{-1}(v))\sqrt{1-v'^2} - nf'(\varphi^{-1}(v)).\end{aligned}$$

We define

$$\psi \,:\,]-1,1[\longrightarrow \mathbb{R}, \qquad \psi(s) = \frac{1}{2}\log\left(\frac{1+s}{1-s}\right),$$

which is an increasing diffeomorphism satisfying $\psi(0) = 0$. So, we have transformed the initial family of ϕ-Laplacians problems into the following ψ-Laplacians equations

$$\begin{gathered}\big(\psi(v')\big)' = -(n-1)\,\frac{v'}{r+\varepsilon} + nH(\varphi^{-1}(v),r)f(\varphi^{-1}(v))\sqrt{1-v'^2} - nf'(\varphi^{-1}(v))\\ v'(0)=0=v(R),\end{gathered}$$

Note that our problem, corresponding to $\varepsilon = 0$, has now a singular term in zero, but the singularity on the derivative has disappeared.

We denote by

$$G\ :]0,R]\times[-R,R]\times[-1,1]\longrightarrow\mathbb{R}$$

$$G(r,s,y) := -(n-1)\frac{y}{r} + nH(\varphi^{-1}(s),r)f(\varphi^{-1}(s))\sqrt{1-y^2} - nf'(\varphi^{-1}(s)),$$

and we define the family of functions depending on $\varepsilon > 0$,

$$G_\varepsilon : [0,R]\times[-R,R]\times[-1,1]\longrightarrow\mathbb{R}$$

$$G_\varepsilon(r,s,y) = -(n-1)\frac{y}{r+\varepsilon} + nH(\varphi^{-1}(s),r)f(\varphi^{-1}(s))\sqrt{1-y^2} - nf'(\varphi^{-1}(s)).$$

One clearly has

$$G_\varepsilon \to G \qquad \text{pointwise.}$$

On the other hand, for each $\varepsilon > 0$,

$$|G_\varepsilon| \le \frac{n-1}{\varepsilon} + nf^*H^* + nf'^* =: \Lambda,$$

where

$$f^* = \max_{[-R,R]} f, \quad f'^* = \max_{[-R,R]}|f'| \quad \text{and}$$

$$H^* = \max\Big\{\,|H(\varphi^{-1}(s),r)| \quad : \quad r\in[0,R], \quad s\in[-R,R]\,\Big\}.$$

From [14], for any $\varepsilon > 0$, the problem

$$\big(\psi(v')\big)' = G_\varepsilon\big(r, v, v'\big), \quad v'(0) = 0 = v(R),$$

has at least one solution $v_\varepsilon \in C^\infty[0, R]$. This is an immediate consequence of Schauder's fixed Point Theorem.

- Second step: *Convergence of* v_ε

Firstly, because $||v_\varepsilon||_\infty < R$ and $||v'_\varepsilon||_\infty < 1$, using Ascoli-Arzela Theorem, passing if necessary to a subsequence, there exists $v \in C[0, R]$ such that

$$||v - v_\varepsilon||_\infty \to 0.$$

Note that

$$v(R) = 0.$$

Consider $0 < a \leq R$. Looking to the expanded problem, we have for any $r \in [a, R]$,

$$|v''_\varepsilon(r)| \leq \frac{(n-1)}{a} + nf^*H^* + nf'^*,$$

implying that the family $\{v'_\varepsilon\}_{\varepsilon>0}$ is equicontinuous on $[a, R]$. Since $||v'_\varepsilon||_\infty < 1$, it follows from the Ascoli-Arzela Theorem that there exists $w \in C[a, R]$ such that

$$v'_\varepsilon \to w \qquad \text{in} \quad C[a, R].$$

It follows that $v \in C^1[a, R]$ and $\{v_\varepsilon\}$ converges to v in $C^1[a, R]$.

- Third step: *The limit is a solution*

Clearly, from the previous steps we deduce that

$$\lim_{\varepsilon\to 0^+} G_\varepsilon(r, v_\varepsilon(r), v'_\varepsilon(r)) = G(r, v(r), v'(r)) \qquad \text{for each} \quad r \in]0, R].$$

Now, choose an arbitrary $r \in]0, R[$, and notice that

$$\big(\psi(v_\varepsilon')\big)' = G_\varepsilon(\tau, v_\varepsilon, v_\varepsilon') \quad \text{in} \quad [r, R].$$

Integrating between r and R, we infer that

$$\psi\big(v_\varepsilon'(R)\big) - \psi\big(v_\varepsilon'(r)\big) = \int_r^R G_\varepsilon(\tau, v_\varepsilon(\tau), v_\varepsilon'(\tau))\, d\tau.$$

Then, the Lebesgue Dominated Convergence Theorem and Proposition 3.4.2 imply that $|v'| < 1$ on $]0, R]$ and

$$\psi\big(v'(R)\big) - \psi\big(v'(r)\big) = \int_r^R G(\tau, v(\tau), v'(\tau))\, d\tau, \quad r \in]0, R].$$

It follows that

$$\big(\psi(v')\big)' = G(r, v, v') \quad \text{in} \quad]0, R]. \tag{3.12}$$

Moreover,

$$\int_0^R G_\varepsilon\big(\tau, v_\varepsilon(\tau), v_\varepsilon'(\tau)\big)\, d\tau = \psi\big(v_\varepsilon'(R)\big).$$

Making use of the Proposition 3.4.2, there exists $\gamma \in (0, 1)$ such that

$$|\psi\big(v_\varepsilon'(R)\big)| < |\psi(\gamma)| \qquad \text{for all} \quad \varepsilon > 0.$$

Then, we rewrite

$$G_\varepsilon(r, s, t) = -(n-1)\, \frac{t}{r+\varepsilon} + g(r, s, t),$$

where

$$g(r, s, t) := nH(\varphi^{-1}(s), r) f(\varphi^{-1}(s)) \sqrt{1 - t^2} - n f'(\varphi^{-1}(s)).$$

It is clear that the function $r \longmapsto g(r, v_\varepsilon(r), v_\varepsilon'(r))$ is integrable on $[0, R]$. Moreover, we have

$$|g(r, v_\varepsilon(r), v_\varepsilon'(r))| < n f^* H^* + n f'^* =: K \qquad \text{for any} \quad \varepsilon > 0.$$

Hence,

$$(n-1)\Big|\int_0^R \frac{v_\varepsilon'(\tau)}{\tau+\varepsilon}\,d\tau\Big| < RK + |\psi(\gamma)|.$$

On the other hand, from (3.10), we get

$$v_\varepsilon'(r) = -\phi^{-1}\left[\frac{n}{(r+\varepsilon)^{n-1}}\int_0^r (\tau+\varepsilon)^{n-1}F(\tau, v_\varepsilon(\tau), v_\varepsilon'(\tau))\,d\tau\right],$$

where

$$F(r,s,t) := H(\varphi^{-1}(s), r) f(\varphi^{-1}(s)) - \frac{f'(\varphi^{-1}(s))}{\sqrt{1-t^2}}.$$

Now, using (A3), one has that the integrand is positive and, therefore, v_ε' is non-positive for all $\varepsilon > 0$. Thus,

$$(n-1)\int_0^R \frac{|v_\varepsilon'(\tau)|}{\tau+\varepsilon}\,d\tau = (n-1)\Big|\int_0^R \frac{v_\varepsilon'(\tau)}{\tau+\varepsilon}\,d\tau\Big| < RK + |\psi(\gamma)|. \tag{3.13}$$

We deduce that, $\Big\{-(n-1)\dfrac{v_\varepsilon'(r)}{r+\varepsilon}\Big\}_{\varepsilon>0}$ is a set of positive integrable functions, satisfying (3.13) and pointwise convergent to the function $-(n-1)\dfrac{v'(r)}{r}$. Applying Fatou Lemma, we conclude that the limit is integrable on $[0,R]$ and

$$r \longmapsto G(r, v(r), v'(r)) \qquad \text{is integrable on} \quad [0,R].$$

Now we are in a position to prove that $\lim\limits_{\mathrm{r}\to 0} v'(r) = 0$. From integrability of $r \longmapsto \dfrac{v'(r)}{r}$, it is clear that, if the limit exists, it should be 0. So, it suffices to prove the existence of $\lim\limits_{\mathrm{r}\to 0} v'(r)$. From (3.12), integrating from r to R, we obtain

$$\psi\big(v'(r)\big) = \psi\big(v'(R)\big) - \int_r^R G\big(\tau, v(\tau), v'(\tau)\big)d\tau.$$

Since $\tau \longmapsto G\big(\tau, v(\tau), v'(\tau)\big)$ is integrable on $[0,R]$, the limit of the right member exists when r tends to 0. Therefore, by using that ψ is a diffeomorphism, we deduce the existence of $\lim\limits_{\mathrm{r}\to 0} v'(r)$. The proof is done.

□

As a direct consequence of Theorems 3.2.1 and 3.5.1, we can enunciate.

Theorem 3.5.2. *Let* $I \times_f \mathbb{R}^n$ *be a* FLRW *spacetime, and let* B *be the Euclidean ball in* $\mathbb{R}^n$ *with radius* R *centered at zero. Assume that* $I_f(R) \subset I$. *Then, for each radially symmetric smooth function* $H : I \times \overline{B} \to \mathbb{R}$ *such that*

$$H(t,r) \leq \frac{f'}{f}(t) \quad \text{and} \quad f'(t) \geq 0, \qquad \text{for any} \quad r \in]0,R[\ , \ \ t \in I_f(R),$$

there exists a radially symmetric spacelike graph with mean curvature function H *defined on* $\overline{B}$, *supported on the spacelike slice* $t = 0$ *and only touching it on the boundary* $\{0\} \times \partial B$ *and defining a non-zero hyperbolic angle with* ∂_t. *Moreover, if the function* H *is increasing in the second variable, every spacelike graph satisfying the previous assumptions must be radially symmetric.*

Remark 3.5.3. *The result of Theorem 3.5.2 is not exactly an improvement of Theorem 3.3.5, in spite of assumption on the radius of the domain is deleted. Observe that hypothesis* $(A3)$ *in Theorem 3.5.2 is necessary to prove the existence of a radially symmetric spacelike graph solving the prescription problem, while in Theorem 3.3.5* $(A3)$ *only appears in order to assure (together with other assumptions) the rotational symmetry of the solutions.*

3.6 Existence of entire graphs

we are in a position to study the existence of entire spacelike graphs with prescribed mean curvature in a FLRW spacetime. The main result of this chapter is picked up in the following theorem.

Theorem 3.6.1. *Let $I \times_f \mathbb{R}^n$ be a* FLRW *spacetime, and let $R > 0$ be such that*

$$I_f(R) \subset I, \qquad \varphi^{-1}(\mathbb{R}^-) \subset I,$$

where

$$I_f(R) := \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds \,,\, \int_0^R f(\varphi^{-1}(s))ds \right] \quad \text{and} \quad \varphi(t) = \int_0^t \frac{dt}{f(t)}.$$

Then, for each radially symmetric smooth function $H : I \times \mathbb{R}^n \to \mathbb{R}$ such that

$$H(t,r) \leq \frac{f'}{f}(t) \quad \textit{and} \quad f'(t) \geq 0, \qquad \textit{for any} \quad r \in]0,R[\ , \ \ t \in I_f(R),$$

there exists an entire radially symmetric spacelike graph with mean curvature function H. In addition, the spacelike slice $t = 0$ intersects the graph in a sphere with radius R. In the particular case that $\inf I$ *is finite, the entire graph approaches to an hyperplane.*

Proof. To prove Theorem 3.6.1, once R is fixed, Theorem 3.5.1 provides a solution v of problem (3.5). Then, it suffices to guarantee that v can be continued until $+\infty$ as a strictly decreasing solution. First, we can rewrite equation (3.11), with $\epsilon = 0$, as a system of two ordinary differential equations of first order

$$v' = z$$
$$z' = (1-z^2)\left(-(n-1)\,\frac{z}{r} + n(f(\varphi^{-1}(v))\,H\big(\varphi^{-1}(v),r\big)\sqrt{1-z^2} - n(f'(\varphi^{-1}(v))\right),$$

which we can abbreviate

$$\begin{bmatrix} v' \\ z' \end{bmatrix} = \mathcal{F}\big(r,(v,z)\big),$$

where $\mathcal{F} : \mathbb{R}^+ \times J \times]-1\,,1[\longrightarrow \mathbb{R}^2$.

Let $[0,b[$ be the maximal interval of definition of v. Suppose that $b < +\infty$. By the standard prolongability theorem of ordinary differential equations (see for instance [83, Section 2.5]), we have that the graph $\big\{\big(r,v(r),v'(r)\big) \ : \ r \in [R/2,b[\big\}$ goes out of any compact subset of $\mathbb{R}^+ \times J \times]-1,1[$. However $|v(r)| < b$ then, since $\mathbb{R}^- \subset J$ and v is decreasing, we know that $v(r) \in [-b, R]$. Moreover, by Lemma 3.4.1, $|v'(r)| < \rho < 1$. Therefore, the graph can not go out of the compact subset $[R/2\,, b] \times [-b, R] \times [-\rho, \rho]$ contained in the domain of $\mathcal{F}$. This is a contradiction, then $b = +\infty$.

From $\mathbb{R}^- \subset \varphi(I)$ we have that $f(t)$ tends to 0 when t goes to $\inf I$. Then u' tends to 0 and, taking into account that u is strictly decreasing, we obtain the conclusion.

□

3.7 Final remarks and applications

It should be pointed out that the assumptions of the main result have a reasonable physical interpretation. In fact, the inequality $f'(t) \geq 0$ means that the divergence in the spacetime $I \times_f \mathbb{R}^n$ of the reference frame ∂_t is nonnegative, which indicates that the comoving observers are on average spreading apart [78, p.121] and so, for these observers, the universe is really expanding whenever $f'(t) > 0$. On the other hand, the inequality $H(t, r) \leq (f'/f)(t)$ expresses an above control of the prescription function by the Hubble function f'/f of the spacetime $I \times_f \mathbb{R}^n$.

Note that previous inequality is not a comparison assumption between extrinsic quantities of two spacelike hypersurfaces of $\mathcal{M}$ (the right member corresponds to a spacelike slice which changes when changes the point at the graph). This kind of inequality has been used to characterize the spacelike slices of some $I \times_f \mathbb{R}^n$ when $n = 2$ [75].

Moreover, the family of FLRW spacetimes where the result may be applied is very wide, and it contains relevant relativistic spacetimes. Indeed, it includes the Lorentz-Minkowski spacetime ($f = 1$, $I = \mathbb{R}$), the Einstein-De Sitter spacetime (

$I =]-t_0, +\infty[$, $f(t) = (t+t_0)^{2/3}$, with $t_0 > 0$), and the steady state spacetime ($I = \mathbb{R}$, $f(t) = e^t$), which is an open subset of the De Sitter spacetime.

Computing the interval $I_f(R)$ in the two previous cases, we obtain respectively,

$$]-\infty\,,\,-\log(1-R)[\quad \text{and} \quad \Big]-t_0 + (t_0^{1/3} - R/3)^3\,,\,(R/3 + t_0^{1/3}) - t_0\Big[\,,$$

and for the interval $J = \varphi(I)$,

$$]-\infty\,,\,1[\quad \text{and} \quad \Big]-3t_0^{1/3}\,,\,\infty\Big[\,.$$

Observe that we can ensure the existence of radially symmetric spacelike graphs with prescribed mean curvature (under the hypotheses of Theorem 3.5.2) on a ball when the radius R is less than 1 and $3t_0^{1/3}$ respectively.

Finally, note that for the steady state spacetime such a graph can be extended to the whole fiber $\mathbb{R}^n$, because $\int_{-\infty}^{0} e^{-s}ds = \infty$. It is very easy to construct explicit examples of FLRW spacetimes leading to entire graphs tending to a hyperplane. For instance, $I =]-t_0\,,+\infty[$ and $f(t) = (t+t_0)^\alpha$, with $t_0 > 0$ and $\alpha \geq 1$.

Chapter 4

Prescription of the mean curvature in static spacetimes

In this chapter we study the mean curvature prescription problem in static spacetimes, paying special attention to the physically relevant ones. The content is organized as follows. Section 4.1 is devoted to the construction of the elliptic differential equation to be studied along the chapter. In Section 4.2 we prove Theorems 4.2.1 and 4.2.13 by using classical arguments from Fixed Point Theory. Once the problem is written as fixed point problem for a suitable operator, the proof of Theorem 4.2.1 follows from a basic application of Schauder Fixed Point Theorem. Meanwhile, the proof of Theorem 4.2.13 is more involved because the metric associated to the Schwarzschild or Reissner-Nordström exterior spacetime presents a singularity. Such a singularity implies that the coefficients in the corresponding differential equation are singular. This technical problem is solved by a suitable transformation and an approximation argument. Finally, Section 4.3 is devoted to the proofs of Theorems 4.3.1 and 4.3.2, consisting on a simple prolongability argument on the solutions of the Dirichlet problem.

This chapter is based on [36].

4.1 Obtaining the equation

In this section we construct the equation which satisfies a smooth function $u : U \subset M \longrightarrow \mathbb{R}$ defining a spacelike graph $\Sigma_u(U)$ in $\mathcal{M}$ with mean curvature function H.

From now on, we denote by
$$g' = \frac{1}{f^2}\, g - dt^2,$$
and we will point with a superscript $'$ the geometric quantities related with metric g' (∇', H', etc). Note that the Lorentzian metric (2.4) may be now written as $\overline{g} = f^2 g'$. Denote by $\overline{\nabla}$ the Levi-Civita connection and the gradient operator associated to $\overline{g}$.

The extrinsic geometry of $\Sigma_u(U)$ in $\mathcal{M}$ is completely codified by the shape operator $\overline{A}$, which is defined by
$$\overline{A}(X) = -\overline{\nabla}_X \overline{N},$$
for all $X \in \mathfrak{X}\big(\Sigma_u\big)$, where $\overline{N}$ is the unit normal vector field on $\Sigma_u(U)$ in the time orientation of $\mathcal{M}$.

Observe that if N' is the unit normal vector field on $\Sigma_u(U)$ on $(M \times I, g')$, then $N' = f\overline{N}$.

Our strategy now is to use the well-known relation between the Levi-Civita connections of two conformal metrics to get the following equation which gives the shape operator $\overline{A}$ from the shape operator A' of $\Sigma_u(U)$ in $(M \times I, g')$. Making use of the cited formulae,

$$\overline{A}(X) = fA'(X) - g'(\nabla' f, N')X - 2g'(\nabla' f, X)N' + g'(X, N')\nabla' f.$$

Taking traces in both members we obtain

$$nH = nfH' + (n+1)g'(\nabla' f, N').$$

It is not difficult to compute that the unit future pointing vector field N' is

$$N' = \frac{1}{\sqrt{1 - f^2|\nabla u|^2}}\big(f^2\nabla u + \partial_t\big),$$

where ∇ and $|\cdot|$ are the gradient operator and the modulus associated to the metric g on M.

Therefore, we obtain the relation

$$H = fH' + \frac{n+1}{n}\frac{f^2 g(\nabla u, \nabla f)}{\sqrt{1 - f^2|\nabla u|^2}}. \tag{4.1}$$

Now, we have to compute H'. In order to do that, consider the Riemannian metric $g^* := \dfrac{1}{f^2}g$ on M. It is well-known that the mean curvature function H' may be expressed by

$$H' = \frac{1}{n}\mathrm{div}^*\left(\frac{\nabla^* u}{\sqrt{1 - |\nabla^* u|^{*2}}}\right),$$

where $\nabla^* u$ means the g*-gradient of u, $|\nabla^* u|^{*2} = g^*(\nabla^* u, \nabla^* u)$ and div^* represents the divergence operator corresponding to g^*.

On the other hand, taking into account the formulae which relates the divergence operators associated to two conformal metrics, we have

$$nH' = \mathrm{div}\left(\frac{f^2\nabla u}{\sqrt{1 - f^2|\nabla u|^2}}\right) - \frac{nf}{\sqrt{1 - f^2|\nabla u|^2}}g(\nabla u, \nabla f)\,. \tag{4.2}$$

From the last equality and equation (4.1) we conclude

$$\frac{1}{f}\mathrm{div}\left(\frac{f^2\nabla u}{\sqrt{1 - f^2|\nabla u|^2}}\right) + \frac{g(\nabla u\,,\,\nabla f)}{\sqrt{1 - f^2|\nabla u|^2}} = \frac{n}{f^2}H(x,u), \quad x \in U. \tag{4.3}$$

Note that (1.2) is obtained as a particular case of (4.3). In fact, we will consider from now on that M is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \overline{B}_0(a)$, where $B_0(a)$ is the euclidean ball centered at 0 with radius $a \geq 0$. As it is shown later, this choice cover the most important explamples from a physical point of view.

We will consider in M a spherically symmetric metric, with the following form

$$g \equiv \langle\, , \rangle = E^2(r)dr^2 + r^2 d\Theta^2. \tag{4.4}$$

According to [68, Chap. 13], the spacetime $\mathcal{M}$ is spherically symmetric in the sense that for any $\psi \in O(3)$, the map $(t, x) \mapsto \big(t, \psi(x)\big)$ is an isometry of $\overline{g}$. Thus, endowed with $\overline{g}$, $\mathcal{M}$ becomes a spherically symmetric static spacetime (see [68, pag. 365] for more details).

In this setting, it is natural to consider a spacelike graph which inherits the symmetry assumption of $\mathcal{M}$, so we will assume $u(x) \equiv u(r)$ and $f(x) \equiv f(r)$. In this case, we clearly have $\nabla u = (u'/E^2)\, \partial_r$ and $\nabla f = (f'/E^2)\, \partial_r$ and equation (1.2) may be rewritten as the following ODE

$$\begin{cases} \dfrac{1}{r^{n-1}E(r)} \left(\dfrac{r^{n-1}E(r)(f/E)^2 u'}{\sqrt{1-(f/E)^2 u'^2}} \right)' + \dfrac{1}{E^2} \dfrac{f'(r)f(r)u'}{\sqrt{1-(f/E)^2 u'^2}} = \dfrac{n}{f} H(r,u) & \text{in} \quad (a, +\infty), \\ |u'| < E/f & \text{in} \quad (a, +\infty). \end{cases} \tag{4.5}$$

4.2 Existence results of the associated Dirichlet problem

Our ultimate objective is to prove the existence of entire radial spacelike graphs. To this aim, we will first study a Dirichlet problem that may be interesting by itself. Thus, in this section, we deal with the Dirichlet problem on $U = B_0(R)$, in the case of $M = \mathbb{R}^n$, or $U = B_0(R) \setminus \overline{B}_0(a)$, with $0 < a < R$, if we consider $M = \mathbb{R}^n \setminus \overline{B}_0(a)$. In both cases, we impose the condition $u(R) = 0$.

Before of distinguishing the two cases, the following considerations are pertinent. Let us define the functional spaces

$$\widehat{C}^1[a,R] := \left\{u \in C^1[a,R] \;:\; u(R) = 0\right\},$$

and

$$\widehat{C}^1(a,R] := \left\{u \in C^1(a,R] \;:\; u(R) = 0\right\},$$

and the linear operator

$$\begin{aligned} T : \widehat{C}^1[a,R] &\longrightarrow \widehat{C}^1(a,R] \\ T[u](r) &:= -\int_r^R \frac{E}{f}(s)\, u'(s) ds. \end{aligned} \tag{4.6}$$

This operator is inyective but, in general, is not onto on $\widehat{C}^1(a,R]$. From now on, we consider solutions which belong to the image of T. Calling W to this image, the restriction $T : \widehat{C}^1[a,R] \longrightarrow W \subset \widehat{C}^1(a,R]$ is invertible and the inverse is given by

$$T^{-1}[u](r) = -\int_r^R \frac{f}{E}(s)\, u'(s) ds.$$

Now, introducing the change of variable

$$v(r) = T^{-1}[u](r), \tag{4.7}$$

equation (4.5) is rewritten as

$$\begin{aligned} &(r^{n-1}\phi(v'))' + 2\frac{f'}{f}(r)\,(r^{n-1}\phi(v')) = nr^{n-1}\frac{E}{f^2}(r)H(r,T[v]) \qquad \text{in} \quad (a,R), \\ &|v'| < 1 \qquad \text{in} \quad (a,R), \end{aligned} \tag{4.8}$$

where $\phi(s) = \dfrac{s}{\sqrt{1-s^2}}$.

In the following, we distinguish two cases according to the behaviour of the limits of $f(r)$ and $E(r)$ when $r \to a^+$. In the first case (regular case below), $\lim_{r\to a^+} f(r)$ and $\lim_{r\to a^+} E(r)$ are finite and positive, and the problem for $M = \mathbb{R}^n$ will be essentially the same that for $M = \mathbb{R}^n \setminus \overline{B}_0(0)$. In the second case (singular case), $\lim_{r\to a^+} f(r) = 0$ and $\lim_{r\to a^+} E(r) = +\infty$.

4.2.1 The regular case.

In the first situation under study, we assume that $a \geq 0$ and the following hypotheses

(A1) f, $E : [a, +\infty) \longrightarrow \mathbb{R}^+$ are continuous functions.

(A2) $H(r, u)$ is continuous in $[a, R] \times \mathbb{R}$.

The first important result is contained in the following theorem.

Theorem 4.2.1. *Let $\mathbb{R}^n \times_f \mathbb{R}$ be a standard static spacetime, endowed with the spherically symmetric metric*

$$E^2(r)dr^2 + r^2 d\Theta^2 - f^2(r)dt^2.$$

Let $B = B_0(R)$ be the Euclidean ball with radius R centered at $0 \in \mathbb{R}^n$, and let

$H : \overline{B} \times \mathbb{R} \to \mathbb{R}$ be a spherically symmetric continuous function. Then, there exists a spherically symmetric spacelike graph with mean curvature function H defined on $\overline{B}$ and supported on the slice $t = 0$.

This theorem is a direct consequence of the following result.

Theorem 4.2.2. *Assume (A1) and (A2). Then, the problem (4.8) has at least one radially symmetric solution v such that $v'(a) = 0$, $v(R) = 0$.*

The proof uses a fixed point argument. Taking

$$w(r) := r^{n-1}\phi\big(v'(r)\big),$$

equation (4.8) is transformed into

$$\begin{aligned} w'(r) + 2\,\frac{f'}{f}(r)w(r) = nr^{n-1}\frac{E}{f^2}(r)H(r, T[v]) \quad \text{in} \quad (a, R), \\ w(0) = 0. \end{aligned} \tag{4.9}$$

Observe that condition $|v'| < 1$ is necessary to have w well-defined. Recall that, from the variation of constants formula, the linear equation

$$\begin{aligned} w'(r) + h(r)w(r) = \varphi(r) \quad \text{in} \quad (a, R), \\ w(0) = 0, \end{aligned} \tag{4.10}$$

with $h, \varphi \in C^1[a, R]$, has a unique solution given by

$$w(r) = \int_a^r \varphi(t)e^{-\int_t^r h(s)ds}dt.$$

In the case of (4.9),

$$e^{-\int_t^r h(s)ds} = \frac{f^2(t)}{f^2(r)},$$

hence,

$$r^{n-1}\phi(v') = \int_a^r \varphi(t)\frac{f^2(t)}{f^2(r)}dt.$$

After some easy computations, it turns out that a solution of problem (4.9) must verify

$$v = \mathcal{A}[v], \tag{4.11}$$

where $\mathcal{A} : C^1[a, R] \longrightarrow C^1[a, R]$ is defined as

$$\mathcal{A}[v](r) := -\int_r^R \phi^{-1}\left[\frac{1}{\tau^{n-1}f^2(\tau)}\int_a^\tau nt^{n-1}E(t)H(t, T[v])dt\right]d\tau.$$

This operator can be written as

$$\mathcal{A} = K \circ \phi^{-1} \circ S \circ N_H,$$

where

$$\begin{gathered} S : C[a, R] \longrightarrow C[a, R] \\ S[v](r) = \frac{1}{r^{n-1}}\int_a^r t^{n-1}E(t)v(t)dt \quad (r \in (a, R]), \quad S[v](a) = 0, \end{gathered} \tag{4.12}$$

$$\begin{gathered} K : C[a, R] \longrightarrow C^1[a, R] \\ K[v](r) = -\int_r^R v(t)dt. \end{gathered} \tag{4.13}$$

and N_H is the Nemytskii operator associated to H,

$$N_H : C^1[a, R] \longrightarrow C[a, R], \quad N_H[v] = H(\cdot, T[v]). \tag{4.14}$$

From assumptions $(A1)$, $(A2)$, S and N_H are continuous and, from the compactness of K, we deduce that $\mathcal{A}$ is a continuous and compact operator in the Banach space $C^1[a, R]$ (endowed with its usual norm $\|v\| = \|v\|_\infty + \|v'\|_\infty$). In conclusion, we can state the following lemma.

Lemma 4.2.3. *A function $v \in C^1[a, R]$ is a solution of equation* (4.9) *if and only if v is a fixed point of the nonlinear compact continuous operator $\mathcal{A}$.*

Remark 4.2.4. *Note that the image of the operator $\mathcal{A}$ is contained in $C^2[a,R]$, so the fixed points (solutions of (4.9)) will be of class C^2. Moreover, using the regularity theorem for elliptic nonlinear operators, (see [58, Chap. 4]) we conclude that, if the prescription function H is of class C^∞, then the solutions will also be infinitely derivable.*

Observe that fixed points of $\mathcal{A}$ always verify the boundary conditions $v'(a) = 0$ and $v(R) = 0$. Define the set

$$\mathcal{B} := \{v \in C^1[a,R] \ : \ \|v\|_\infty < R - a, \ \|v'\|_\infty < 1\}.$$

By using that $\phi^{-1}(\mathbb{R}) = (-1\,,1)$, one gets

$$\|\mathcal{A}(v)\|_\infty < R - a \quad \text{and} \quad \|\big(\mathcal{A}(v)\big)'\|_\infty < 1 \qquad \text{for all} \quad v \in \mathcal{B}.$$

These inequalities implies that $\mathcal{A}(\overline{\mathcal{B}}) \subset \mathcal{B}$. Since $\overline{\mathcal{B}}$ is contractible to a point, and $\mathcal{A}$ is a continuous and compact operator, the Schauder Fixed Point Theorem applies, finishing the proof of Theorem 4.2.2.

Remark 4.2.5. *The proof of Theorem 4.2.1 follows immediately by taking $a = 0$. With the same arguments, we could impose a different constraint in the derivative of v, for instance, to prescribe a fixed value of $v'(a)$ or $v'(R)$. By simplicity, we have just considered the condition $v'(a) = 0$, which is the most important example.*

Remark 4.2.6. *If $H \leq 0$, from the fixed point formulation (4.11), we easily deduce that v is decreasing and positive in $[a,R]$. The same conclusion is reached for u. Since the slices $\{t = t_0\}$ are totally geodesic, the hypothesis $H \leq 0$ is interpreted by saying that the mean curvature prescription function is less than the mean curvature of the slices.*

4.2.2 The singular case.

The main motivation of this chapter is to study the Schwarzschild and Reissner-Nordström spacetimes, which play a central role in General Relativity (see for instance [25], [68], [78]). The *Schwarzschild exterior spacetime* models the exterior region of a spacetime where there is only a spherically symmetric non-rotating star without charge. Such a spacetime is defined by the metric

$$E^2(r)dr^2 + r^2 d\Theta^2 - f^2(r)dt^2$$

where

$$f(r) = \sqrt{1 - \frac{2m}{r}} \qquad \text{and} \qquad E(r) = \frac{1}{\sqrt{1 - \frac{2m}{r}}}.$$

Here, m is interpreted as the mass of a star (or black hole) in certain unit system. The value of the radius $r = 2m$ is known as *Schwarzschild radius*. When this radius is bigger than the radius of the star, we are in presence of a *Schwarzschild black hole*.

A generalization of the latter example is the *Reissner-Nordström exterior spacetime*, in which the mass has non-zero electric charge. In this case, we have

$$f(r) = \sqrt{1 - \frac{2m}{r} + \frac{r_Q^2}{r^2}} \qquad \text{and} \qquad E(r) = \frac{1}{\sqrt{1 - \frac{2m}{r} + \frac{r_Q^2}{r^2}}},$$

where $r_Q > 0$ is a characteristic length relative to the charge Q of the mass. Our interest lies in the region where $r > m + \sqrt{m^2 - r_Q^2}$, i.e., outside of the exterior event horizon (recall that, in this spacetime, there are two horizons, in the physical and realistic case $m > r_Q$).

Our idea is to treat both spacetimes in the same way. To this aim, let us consider $a > 0$ and continuous functions $f, E : (a, +\infty) \longrightarrow \mathbb{R}^+$, $H : [a, R] \times \mathbb{R} \to \mathbb{R}$ such that

(B1) $\lim_{r\to a^+} f(r) = 0$ and $\lim_{r\to a^+} E(r) = +\infty$, but $E(r)$ is integrable in $[a, R]$.

(B2) H is bounded in $[a, R] \times \mathbb{R}$, i.e., there exists a constant $C > 0$ such that $|H(r, t)| < C$ for any $r \in [a, R]$ and $t \in \mathbb{R}$.

In this context, the most geometrically natural and physically relevant condition on u is $\lim_{r\to a^+} u(r) = +\infty$ (and, therefore, $\lim_{r\to a^+} u'(r) = -\infty$). Physically, it may be interpreted as our spacelike graph tends to the *event horizon* (see again [25], [68], [78]). From the mathematical standpoint, we are looking for blow-up solutions of equation (4.5). Let us see how such condition can be guaranteed.

Lemma 4.2.7. *Let us assume the condition*

$(B3)$ *E/f is not integrable on $[a, R]$.*

Then,

$$\lim_{r\to a^+} v'(r) = -1, \tag{4.15}$$

implies

$$\lim_{r\to a^+} u(r) = +\infty.$$

Proof. From the transformation (4.7),we obtain

$$u(r) = \int_r^R \frac{E(s)}{f(s)} v'(s) ds,$$

and the result is trivial in view of $(B3)$.

□

As a first step, we are going to fix the hyperbolic angle between our graph and the observers in the reference frame $\frac{1}{f}\partial_t$ (called Schwarzschild observers). Analytically, this is equivalent to fix the value of $u'(R)$. Therefore, we are interested in proving the existence of solutions of the following problem,

$$\begin{aligned} (r^{n-1}\phi(v'))' + 2\,\frac{f'}{f}(r)\,(r^{n-1}\phi(v')) = nr^{n-1}\frac{E}{f^2}(r)H(r,Tv) \qquad \text{in} \quad (a,R), \\ v(R) = 0, \quad v'(R) = k, \\ \lim_{r\to a^+} v'(r) = -1, \end{aligned} \tag{4.16}$$

where k is a constant, $|k| < 1$.

The main result of this section (Theorem 4.2.13) will be a direct consequence of the following theorem.

Theorem 4.2.8. *Let us assume* $(B1)$, $(B2)$ *and* $(B3)$. *Then, there exists* $k_0 < 0$ *such that, for any* $-1 < k \leq k_0$, *problem* (4.16) *has at least one solution.*

In order to prove this theorem, we consider an intermediate proposition by adding a technical assumption to be deleted later.

Proposition 4.2.9. *Let us assume* $(B1) - (B3)$ *and the additional condition*

$(B4)$ *H has compact support contained in* $[a,R] \times [-j,j]$, *for some natural* j.

Then, there exists $k_0 < 0$ *such that, for any* $-1 < k \leq k_0$, *problem* (4.16) *has at least one solution.*

Proof. By defining $w(r) := r^{n-1}\phi\big(v'(r)\big)$, problem (4.16) is transformed into

$$\begin{aligned} w'(r) + 2\,\frac{f'}{f}(r)w(r) &= nr^{n-1}\frac{E}{f^2}(r)H(r,Tv) \qquad \text{in} \quad (a,R),\\ |v'| &< 1 \qquad\qquad\qquad\qquad\qquad\quad\; \text{in} \quad (a,R), \\ w(R) &= -A/f^2(R),\\ \lim_{r\to a^+} w(r) &= -\infty, \end{aligned} \tag{4.17}$$

where $A = -R^{n-1}\phi(k)f^2(R)$.

Let us consider the linear problem

$$w'(r) + 2\,\frac{f'}{f}(r)w(r) = \varphi(r) \qquad \text{in} \quad (a,R), \tag{4.18}$$

$$w(R) = -A/f^2(R), \tag{4.19}$$

$$\lim_{r\to a^+} w(r) = -\infty, \tag{4.20}$$

where φ is an arbitrary continuous function defined on (a,R). Applying the variation of constants formula, the unique solution of the initial value problem (4.18)-(4.19) is

$$w(r) = -\frac{A}{f^2(r)} - \frac{1}{f^2(r)}\int_r^R \varphi(s)f^2(s)ds.$$

by using $(B1)$ and $(B2)$, the limit condition (4.20) is satisfied if A is chosen such that

$$A > n\,C\int_a^R r^{n-1}E(r)dr. \tag{4.21}$$

The relation between k and the hyperbolic angle $\chi \in \mathbb{R}$ between the Schwarzschild observers and the normal vector field $\overline{N}$ (see Section 3.2) is given by

$$\sinh(\chi) = \phi(k).$$

Since by definition $A = -R^{n-1}\phi(k)f^2(R)$, condition (4.21) holds if and only if

$$\phi(k) = \sinh(\chi) < -\frac{n\,C}{f^2(R)}\frac{1}{R^{n-1}}\int_a^R r^{n-1}E(r)dr, \tag{4.22}$$

or equivalently, $k < k_0$ where

$$k_0 := \phi^{-1}\left(-\frac{n\,C}{f^2(R)}\frac{1}{R^{n-1}}\int_a^R r^{n-1}E(r)dr\right).$$

In this way, we define the nonlinear operator $\mathcal{N} : X \longrightarrow X$,

$$\mathcal{N}[v](r) := \int_r^R \phi^{-1}\left[\frac{1}{\tau^{n-1}f^2(\tau)}\left(A + \int_\tau^R nt^{n-1}E(t)H(t,T[v])dt\right)\right]d\tau, \tag{4.23}$$

where

$$X = \{v \in C^1[a,R] \;:\; v(R) = 0, v'(a) = -1\}.$$

A function $v \in C^1[a,R]$ is a solution of problem (4.16) if and only if v is a fixed point of the nonlinear operator $\mathcal{N}$.

Lemma 4.2.10. *Assume* $(B1)-(B4)$. *Then,* $\mathcal{N}$ *is a compact and continuous nonlinear operator.*

Proof of the lemma.

Let us write

$$\mathcal{N} = K \circ \mathcal{S} \circ N_H,$$

where the operators K and N_H are defined in the previous subsection by (4.13) and (4.14) respectively, and the operator $\mathcal{S} : C^1[a,R] \longrightarrow C^1[a,R]$ has the expression

$$\mathcal{S}[v](r) := \phi^{-1}\left[\frac{1}{r^{n-1}f^2(r)}\left(A + \int_r^R nt^{n-1}E(t)v(t)dt\right)\right]. \tag{4.24}$$

The operator K is continuous and compact, so we only have to verify the continuity of the operator $\mathcal{N}$.

The first step is to prove that $\mathcal{S}$ is continuous. By $(B2)$, the image of N_H is bounded, then it is sufficient to prove that $\mathcal{S}$ is continuous on a certain closed ball B_ρ of $C^1[a,R]$, with arbitrary radius ρ. So, let $\{v_k\}$ be a sequence converging to v uniformly in B_ρ. The objective is to see that $\mathcal{S}[v_k]$ converges uniformly to $\mathcal{S}[v)]$.

Denote by $g_k(r) := \int_r^R nt^{n-1}E(t)v_k(t)dt$. Since

$$|g_k(r) - g_l(r)| \leq \|v_k - v_l\|_\infty \int_a^R nt^{n-1}E(t)dt,$$

from the uniform convergence of $\{v_k\}$ and Cauchy criterion, we deduce that $\{g_k\}$ is also uniformly convergent. The Dominated Convergence Theorem ensures that the limit is

$$g := \int_r^R nt^{n-1}E(t)v(t)dt.$$

Let us denote $\varrho = \rho \int_a^R nt^{n-1}E(t)dt$ and $x_k(r) := \big(r, g_k(r)\big)$. With this notation, we can write

$$\mathcal{S}[v_k](r) = F\big(x_k(r)\big),$$

where $F : [a,R] \times [-\varrho, \varrho] \longrightarrow \mathbb{R}$ is a continuous function. From the uniform convergence of v_k and g_k, we deduce the uniform convergence of $\{x_k\}$ to (id, g) (for any fixed norm in $\mathbb{R}^3$). Hence, since F is uniformly continuous, because of compactness of $[a,R] \times [-\varrho, \varrho]$, we conclude that $\mathcal{S}[v_k]$ converges uniformly to $\mathcal{S}[v]$.

It remains to prove the continuity of Nemytskii operator N_H. At this point, the hypothesis $(B4)$ is crucial. Note that the boundedness of H is not enough (for instance, $H(r,s) = \sin(s)$).

Let $\{v_k\} \subset X$ be a sequence which converges to $v \in X$ (in the usual C^1-norm). We have to prove that $H(r, T[v_k](r)) \longrightarrow H(r, T[v](r))$ uniformly on $[a,R]$. The uniform convergence on any compact set in $(a,R]$ follows from applying Ascoli-Arzela

Theorem, once it is observed that the derivative of $\phi^{-1}(s) = \frac{s}{\sqrt{1+s^2}}$ is a bounded function. On the other hand, $v'(a) = v_k'(a) = -1$ and condition $(B3)$ implies that

$$\lim_{r\to a^+} T[v_k](r) = \lim_{r\to a^+} T[v](r) = +\infty.$$

By condition $(B4)$, this means that

$$\lim_{r\to a^+} H(r, T[v_k](r)) = \lim_{r\to a^+} H(r, T[v](r)) = 0.$$

Therefore, the pointwise convergence at $r = a$ is ensured.

From $(B1)$ and using that $v'(a) = -1$, we may take $\overline{r} \in (a, R)$ such that, for any $r \in [a, \overline{r}]$, $|-1 - v'(r)| < 1/3$ and the following inequality holds

$$-\int_r^R \frac{E}{f}(t)v'(t)dt > j + 1. \tag{4.25}$$

Taking $0 < \varepsilon < \min\{\frac{1}{3}, -1/\int_{\overline{r}}^R \frac{E}{f}(t)dt\}$ there exists k_0 such that, for all $k > k_0$, $\|v_k' - v'\|_\infty < \varepsilon$, hence we have

$$-\int_{\overline{r}}^R \frac{E}{f}(t)v_k'(t)dt > j + 1 - \varepsilon \int_{\overline{r}}^R \frac{E}{f}(t)dt.$$

Since $v_k' < 0$ on $[a, \overline{r}]$, we obtain

$$-\int_r^R \frac{E}{f}(t)v_k'(t)dt > j.$$

As a consequence of $(B4)$, from the latter inequality and (4.25) we conclude that

$$H(r, Tv_k) = H(r, Tv) = 0 \quad \text{on} \quad [a, \overline{r}].$$

Thus, the uniform convergence is trivial on the compact set $[a, \overline{r}]$ and the proof is finished.

$\square$

Now, fixed an hyperbolic angle χ satisfying (4.22) (so, fixed the corresponding k), one gets that the image of $\mathcal{N}$ is contained in the closed and convex set $\overline{\mathcal{D}_k}$ defined by

$$\overline{\mathcal{D}} = \{v \in X \,:\, v'(R) = k,\ \|v\|_\infty \leq (R-a),\ \|v'\|_\infty \leq 1\}.$$

Then, a basic application of the Schauder Fixed Point theorem finishes the proof of Proposition 4.2.9.

□

Now, as a final step for the proof of Theorem 4.2.8, we are going to remove assumption $(B4)$ by means of a truncation argument. Let $h_j : \mathbb{R} \longrightarrow [0,1]$ be a smooth function such that is equal to 1 on $[-j+1, j-1]$ and vanishes outside of the interval $(-j, j)$, $j > 1$. Then, we construct the sequence of functions

$$H_j : [a,R] \times \mathbb{R} \longrightarrow \mathbb{R}, \qquad H_j(r,s) := H(r,s)\, h_j(s),$$

which converges pointwise to the function H. Note that each H_n satisfies the assumption $(B4)$. Therefore, by using Proposition 4.2.9, we have a sequence $\{v_j\}_{j=1}^{\infty}$ of fixed points of the nonlinear operators

$$\mathcal{N}_j[v](r) := \int_r^R \phi^{-1}\left[\frac{1}{\tau^{n-1}f^2(\tau)}\left(A + \int_\tau^R nt^{n-1}E(t)H_j(t,Tv)dt\right)\right] d\tau, \tag{4.26}$$

i.e.,

$$v_j = \mathcal{N}_j[v_j]. \tag{4.27}$$

It is immediate that $\|v_j\|_\infty < R-a$ and $\|v_j'\|_\infty \leq 1$. Thus, from Ascoli-Arzela Theorem, there exists a subsequence and a function $v \in C[a,R]$ such that

$$\{v_j\} \longrightarrow v \qquad \text{uniformly on } [a,R].$$

Choose an arbitrary closed interval $[c,d] \subset (a,R]$. Performing into (4.26)-(4.27) and using again that the derivative of $\phi^{-1}(s)$ is a bounded function, it is easy to check that there exists a constant L, depending only on the interval $[c,d]$, such that $\|v_j''\|_\infty < L$. Hence, the family $\{v_j'\}_{j=1}^\infty$ is equicontinuous on $[c,d]$. Since $|v_j'| < 1$ we can apply the Ascoli-Arzela Theorem and conclude that there exists a continuous $v \in C^1[c,d]$ such that

$$\{v_j'\} \longrightarrow v' \qquad \text{in} \quad C^1[c,d].$$

In order to prove Theorem 4.2.8, we only have to see that v is a fixed point of the nonlinear operator defined by (4.23). Taking limits in (4.27) and using the C^1-convergence of $\{v_j\}_{j=1}^\infty$ on compacts in $(a,R]$ we get

$$v(r) = \mathcal{N}[v](r) \qquad r \in (a,R].$$

Moreover, $v(a) = \mathcal{N}[v](a)$ holds because the function $r \longmapsto nr^{n-1}E(r)H\big(r, Tv(r)\big)$ is integrable on $[a,R]$ (although it is not assured the existence of limit of this function when $r \to a$). Therefore, v is a fixed point of $\mathcal{N}$, or equivalently, v is a solution of problem (4.16).

To conclude the proof of Theorem 4.2.8, we only have to observe that any choice of χ satisfying (4.22) implies that the solution as a fixed point of the non linear operator $\mathcal{N}$ is a decreasing and non-negative function in $(a,R]$.

Remark 4.2.11. *In the particular case of a constant mean curvature H, the operator $\mathcal{N}$ provides an explicit integral expression of a radially symmetric spacelike graph with constant mean curvature tending to the event horizon. In particular, we obtain maximal graphs different from the slices.*

Remark 4.2.12. *Imposing $\lim_{r\to a^+} v'(r) = +1$, instead of (4.15), we obtain the existence of non-positive and increasing solutions in (a,R) which approach the event horizon in the past of the Schwarzschild observers. The arguments of the proof remain unchaged.*

Now we may enunciate the announced important theorem.

Theorem 4.2.13. *Let $\mathcal{M}$ be either the Schwarzschild exterior spacetime or the Reissner-Nordström exterior spacetime (with radius a), and let $H : \overline{A}(a,R) \times \mathbb{R} \to \mathbb{R}$ be a spherically symmetric and bounded smooth function, where $\overline{A}(a,R)$ is the closed annulus $a \leq |x| \leq R$. Then, there exists a spherically symmetric spacelike graph with mean curvature function H, which touches the slice $t = 0$ on the boundary $|x| = R$, and approaches the event horizon as $|x| \to a$. Moreover, the graph is radially decreasing on the annulus $\overline{A}(a,R)$ and it intersects the slice $t = 0$ only at the boundary $|x| = R$.*

4.3 Extendibility result and entire graphs

Finally, we will prove that any graph on a ball obtained in the previous section can be extended on $\mathbb{R}^n$ or $\mathbb{R}^n \setminus B_0(a)$, depending on the case. Explicitly, we want to prove the following two theorems.

Theorem 4.3.1. *Let $\mathbb{R}^n \times_f \mathbb{R}$ be a standard static spacetime, endowed with the spherically symmetric metric*

$$E^2(r)dr^2 + r^2 d\Theta^2 - f^2(r)dt^2,$$

and let $H : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a radially symmetric continuous function. Then, there exists a spherically symmetric entire spacelike graph with mean curvature function H. Moreover, for each $R > 0$, the graph may be chosen such that its intersection with $t = 0$ is a sphere of radius R.

Theorem 4.3.2. *Let $\mathcal{M}$ be either the Schwarzschild exterior spacetime or the Reissner-Nordström exterior spacetime with exterior radius $a > 0$, and let $H : \mathcal{M} \longrightarrow \mathbb{R}$ be a spherically symmetric and bounded continuous function. Then, there exists a spherically symmetric entire spacelike graph with mean curvature function H that approaches the event horizon as $r \to a$. Moreover, for each $R > a$, the graph may be* *chosen such that its intersection with $t = 0$ is a sphere of radius R.*

In order prove them, we need the following lemma.

Lemma 4.3.3. *Let $v \in C^2[a, b]$ be a solution of* (4.8)*. Then $|v'| < 1$ on $(a, b]$.*

Proof. On (a, b) the solutions verify $|v'| < 1$. We only have to prove the inequality at $r = b$. Suppose that

$$\lim_{r \to b^-} |v'(r)| = 1 \qquad \text{and} \qquad \lim_{r \to b^-} |\phi\big(v'(r)\big)| = \infty.$$

For r sufficiently close to b it is easy to see

$$\frac{\big[r^{n-1}\phi\big(v'(r)\big)\big]'}{\big[r^{n-1}\phi\big(v'(r)\big)\big]} = -2\frac{f'}{f}(r) + n\frac{E}{f^2}(r)\frac{H\big(r, T(v(r))\big)}{\phi\big(v'(r)\big)}.$$

Let $\overline{r} \in (a, b)$ be such that $|v'(\tau)| > 0$ for any $\tau \in (\overline{r}, b)$. Integrating the last equality, we have

$$\log\big|r^{n-1}\phi\big(v'(r)\big)\big| - \log\big|\overline{r}^{n-1}\phi\big(v'(\overline{r})\big)\big| = \int_{\overline{r}}^{r} \left(-2\frac{f'}{f}(\tau) + n\frac{E}{f^2}(\tau)\frac{H\big(\tau, T(v(\tau))\big)}{\phi\big(v'(\tau)\big)}\right) d\tau.$$

Taking limits, $r \to b^-$, we check that the left member tends to infinity while the right one is finite. Therefore, we deduce that $|\phi(v')|$ is bounded and, consequently, $\|v'\|_\infty$ must be strictly lower than 1. □

To prove Theorem 4.3.1, once R is fixed, Theorems 4.2.1 and 4.2.13 provide a solution v of problem (4.8). Then, it suffices to guarantee that v can be continued until $+\infty$ as a solution. First, we rewrite equation (4.8), as a system of two ordinary differential equations of first order

$$\begin{aligned} v' &= z \\ z' &= (1 - z^2)\left(-(n-1)\frac{z}{r} - 2\frac{f'}{f}(r)z + n\frac{E}{f^2}(r)\sqrt{1 - z^2}H(r, Tv)\right), \end{aligned}$$

which we can abbreviate as

$$\begin{bmatrix} v' \\ z' \end{bmatrix} = \mathcal{F}\big(r,(v,z)\big),$$

where $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R} \times (-1\,,1) \longrightarrow \mathbb{R}^2$.

Let (a,b) be the maximal interval of definition of v. Suppose that $b < +\infty$. By the standard prolongability theorem of ordinary differential equations (see for instance [83, Section 2.5]), we have that the graph $\{\big(r, v(r), v'(r)\big) \ : \ r \in [a + (R-a)/2, b)\}$ goes out of any compact subset of $\mathbb{R}^+ \times \mathbb{R} \times (-1,1)$. However $|v(r)| < b$ then we know that $v(r) \in [-b, R]$. Moreover, by Lemma 4.3.3, $|v'(r)| < \rho < 1$. Therefore, the graph can not go out of the compact subset $[a+(R-a)/2\,, b] \times [-b, R] \times [-\rho, \rho]$ contained in the domain of $\mathcal{F}$. This is a contradiction and then $b = +\infty$.

Chapter 5

Prescription of the higher order curvature functions

The prescribed k-th mean curvature problem in $\mathbb{R}_a^{n+1}$ consists in finding, for a given prescription function H_k, a (embedded) hypersurface if $a = 0$ or a spacelike hypersurface if $a = 1$, Σ in $\mathbb{R}_a^{n+1}$ which satisfies

$$S_k(p) = H_k(p) \qquad \text{for all} \quad p \in \Sigma. \tag{5.1}$$

We will focus here the problem as follows. Results of this chapter may be found in [38].

Consider a line γ in $\mathbb{R}_a^{n+1}$ (as defined before) and put Π the hyperplane through $p = \gamma(0)$ and orthogonal to $\mathbb{R}_a^{n+1}$. We will look for Σ as a graph if $a = 0$ or a spacelike graph if $a = 1$ for a suitable function v defined on Π, i.e., $\Sigma = \{(v(x), x) \,:\, x \in \Pi\} \subset \mathbb{R} \times \mathbb{R}^n$. If the prescription function H_k were assumed rotationally symmetric with respect to γ, then it would be natural to assume v also has the same symmetry, i.e., $v(x) = v(r)$ where $r = r(x)$ is the distance in Π from x to $\gamma(0)$. On the other hand, using a cylindrical coordinate system (t, r, Θ), $\Theta = (\theta_1, ..., \theta_{n-1})$, as before of $\mathbb{R}_a^{n+1}$, the metric of $\mathbb{R}_a^{n+1}$ may be expressed as

$$\langle\,,\,\rangle = \epsilon\, dt^2 + dr^2 + r^{n-1} d\Theta^2, \qquad \Theta = (\theta_1, \cdots, \theta_{n-1}),$$

where $\epsilon = (-1)^a$ and $d\Theta^2$ the standard Riemannian metric on the unit round sphere $\mathbb{S}^{n-1}$. With respect to the coordinate frame $\{\partial_t, \partial_r, \partial_{\theta_1}, ..., \partial_{\theta_{n-1}}\}$, the unit normal vector field N along Σ in $\mathbb{R}_a^{n+1}$ is given by

$$N = \frac{\partial_t - \epsilon v' \partial_r}{\sqrt{1+\epsilon\, v'^2}}.$$

where v', the derivative of $v = v(r)$, satisfies $|v'| < 1$ if $a = 1$.

The value of the principal curvatures is given by the following lemma.

Lemma 5.0.4. *For each fixed point $(v(r), r) \in \Sigma$, the vectors ∂_{θ_i} and $v'\partial_t + \partial_r$ are eigenvectors of the shape operator A, with eigenvalues*

$$\kappa_1(v,r) = \cdots = \kappa_{n-1}(v,r) = \frac{\epsilon\, v'}{r\sqrt{1+\epsilon\, v'^2}}, \qquad \text{and} \qquad \kappa_n(v,r) = \frac{\epsilon\, v''}{(1+\epsilon\, v'^2)^{3/2}}.$$

Proof. In order to check that ∂_{θ_i} are $n-1$ different eigenvalues of A, we compute

$$A(\partial_{\theta_i}) = -\nabla_{\partial_{\theta_i}} N = -\frac{1}{\sqrt{1+\epsilon\, v'^2}} \left[\nabla_{\partial_{\theta_i}} \partial_t - \epsilon\, v' \nabla_{\partial_{\theta_i}} \partial_r\right] = \frac{\epsilon\, v'}{\sqrt{1+\epsilon\, v'^2}}\, \Gamma^{\theta_i}_{\theta_i r} \partial_{\theta_i}.$$

In the last step, we have taken into account that the only non-zero Christoffel symbols involving the angles θ_i are $\Gamma^{\theta_i}_{\theta_i r} = \Gamma^{\theta_i}_{r\theta_i} = 1/r$, and $\Gamma^{r}_{\theta_i\theta_i} = -r$, and the result follows directly. The last eigenvector is obtained by imposing the orthogonality respect to N and the rest of the eigenvectors ∂_{θ_i}. □

Making some computations, the differential operators S_k associated to the k-curvature of rotationally symmetric graphs in M, $1 \leq k \leq n$, can be written as follows,

$$S_k^+ : \{v \in C^2(\mathbb{R}^+) \,:\, v'(0) = 0\} \longrightarrow \mathbb{R},$$

$$S_k^+[v](r) = \begin{cases} \dfrac{1}{n\, r^{n-1}} \left(r^{n-k}\psi^k(v')\right)' & \text{in} \quad (0,\infty), \\ 0 & \text{in} \quad r = 0, \end{cases}$$

where $\psi(s) := \dfrac{s}{\sqrt{1+s^2}}$ in Euclidean space, and

$$S_k^- : \{v \in C^2(\mathbb{R}^+) \,:\, v'(0) = 0,\, |v'| < 1\} \longrightarrow \mathbb{R},$$

$$S_k^-[v](r) = \begin{cases} \dfrac{1}{n\, r^{n-1}} \left(r^{n-k}\phi^k(v')\right)' & \text{in} \quad (0,\infty), \\ 0 & \text{in} \quad r = 0, \end{cases}$$

where $\phi(s) := \dfrac{s}{\sqrt{1-s^2}}$ in Minkowski spacetime.

Then, our aim is to prove the existence of solutions of the equations

$$S_k^\pm[v](r) = H_k(v(r), r) \qquad r \in \mathbb{R}^+, \tag{5.2}$$

for a given function $H_k : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$.

Note that, in general, these differential operators are not elliptic. Although we are interested in the existence of entire graphs, we also will deal with graphs defined over a ball $B_{\gamma(0)}(R) \subset \Pi$, with Dirichlet boundary conditions. This problem is not only a previous step to face the main purpose, but it has its own interest. Next two sections are devoted to this aim.

5.1 Existence results of the Dirichlet problem in Minkowski spacetime.

Associated to equation (5.1), in the Minkowski ambient, we can consider the corresponding Dirichlet problem on a ball with radius R contained in Π. Passing to polar coordinates, we get the following boundary value problem,

$$\begin{aligned} \left(r^{n-k}\phi^k(v')\right)' &= n\, r^{n-1}\, H_k(v(r), r) \qquad && \text{in} \quad (0,R), \\ |v'| &< 1 && \text{in} \quad (0,R), \\ v'(0) &= 0 = v(R), && \end{aligned} \tag{5.3}$$

where $\phi(s) := \dfrac{s}{\sqrt{1-s^2}}$ and $1 \le k \le n$.

It is easy to compute the profile of the rotationally symmetric graphs with constant k-th mean curvature. If H_k is constant (non negative if k is even), we can integrate directly equation (5.3) in order to obtain an hyperboloid,

$$v(r) = \sqrt{R^2 + H_k^{-2/k}} - \sqrt{r^2 + H_k^{-2/k}},$$

for each $1 \le k \le n$. On the other hand, if k is even and $H_k < 0$, it is easy to realize that (5.3) has no solutions. By this reason, for a more general prescription of the curvature we need to distinguish two cases, depending if k is an even or odd natural number.

We fix some notation which will be used in the rest of the section. Let C be the Banach space of the real continuous functions in $[0,R]$, with the maximum norm, and C^1 the space of continuously differentiable functions with its usual norm $\|v\| = \|v\|_\infty + \|v'\|_\infty$. We write $B_{R,1} = \{v \in C^1 : \|v\|_\infty < R, \, \|v'\|_\infty < 1\}$.

5.1.1 Case 1: k odd.

In this case, the existence problem is a straightforward application of the results exposed in [11] (see Proposition 2.4 therein), taking into account that $(\phi)^k : (-1,1) \longrightarrow \mathbb{R}$ is also an increasing homeomorphism such that $\phi(0) = 0$. The result is enunciated as follows.

Proposition 5.1.1. *Let be k odd. Let $B_0(R)$ be an Euclidean ball centered at 0 with radius R contained in a spacelike hyperplane $\Pi \subset \mathbb{L}^{n+1}$ orthogonal to a inertial observers vector field. For every rotationally symmetric (in the second argument) and continuous function $H_k : [-R, R] \times B_0(R) \subset \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$, there exists at least one rotationally symmetric spacelike graph with k-curvature equal to H_k such that its boundary is in the hyperplane Π.*

5.1.2 Case 2: k even.

When k is even, ϕ^k is not a homeomorphisms between $(-1,1)$ and $\mathbb{R}$ and then, the arguments of [11] must be modified.

First of all, from equation (5.3) we have that

$$\big[\phi(v')\big]^k(r) = \frac{n}{r^{n-k}} \int_0^r s^{n-1} H_k\big(v(s), s\big) ds. \tag{5.4}$$

Hence, since k is an even number, we have that the previous integral term is non negative. Then, it is quite natural to impose the following condition on the mean k-curvature prescription function,

$$\int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \geq 0 \qquad \text{for all} \qquad r \in [0, R]\,, \quad v \in B_{R,1}. \tag{5.5}$$

Note that condition (5.16) implies in particular condition (5.5) for any $R > 0$. Our first step is to construct a *fixed point operator* $\mathcal{A}$ such that its fixed points are solutions of to problem (5.3). We start by defining

$$K : C^1 \longrightarrow C^1,$$
$$K(v)(r) = \int_r^R v(t)dt,$$

$$S : C \longrightarrow C^1,$$
$$S(v)(r) = \frac{n}{r^{n-k}} \int_0^r t^{n-1} v(t)dt \quad (r \in (0, R]), \quad S(v)(0) = 0.$$

Besides, consider the Nemytskii operator associated to H_k,

$$N_{H_k} : B_{R,1} \subset C^1 \longrightarrow C, \quad N_{H_k}(v) = H_k(\cdot, v).$$

Obviously, N_{H_k} is continuous and $N_F(\overline{B}_{R,1})$ is a bounded subset of C. Finally, we define the operator

$$\mathcal{A} : \overline{B}_{R,1} \subset C^1 \longrightarrow C^1, \quad \mathcal{A} = K \circ (\phi^{-1})^{1/k} \circ S \circ N_F, \tag{5.6}$$

where $(\phi^{-1})^{1/k} : \mathbb{R}^+ \longrightarrow [0, 1)$ means the (positive) k-root composed with the inverse of ϕ, i.e., $(\phi^{-1})^{1/k}(s) = \phi^{-1}(s^{1/k})$. Note that $\mathcal{A}$ is well-defined thanks to condition (5.5).

Note that $\mathcal{A}$ is a composition of continuous operators, hence it is continuous. Moreover, from the compactness of K, $\mathcal{A}$ is a compact and continuous operator. Note that the image of the operator $\mathcal{A}$ is contained in $C^2[0, R]$, so the fixed points (solutions of the equation (5.3)) will be of class C^2.

Fixed points of $\mathcal{A}$ always verify the restrictions $v'(0) = v(R) = 0$, in consequence we can consider the Banach subspace $\widehat{C}^1 \subset C^1$ of the functions that satisfy these boundary conditions. Let us define the set

$$\widehat{B}_{R,1} = \{v \in \overline{B}_{R,1} : v'(0) = 0 = v(R)\}.$$

A straightforward checking shows that if a function $v \in \widehat{C}^1$ is a fixed point of the nonlinear compact operator (5.6), then v is a solution of equation (5.3).

More explicitly, operator $\mathcal{A}$ can be written as

$$\mathcal{A}(v)(r) = -\int_r^R \phi^{-1}\left[\left(\frac{n}{s^{n-k}}\int_0^s \tau^{n-1}H_k(\tau, v(\tau))d\tau\right)^{1/k}\right] ds,$$

and its derivative is

$$\big(\mathcal{A}(v)\big)'(r) = \phi^{-1}\left[\left(\frac{n}{r^{n-k}}\int_0^r \tau^{n-1}H_k(\tau, v(\tau))d\tau\right)^{1/k}\right].$$

By using that $\phi^{-1}(\mathbb{R}^+) = [0,1)$, one gets

$$\|\big(\mathcal{A}(v)\big)'\|_\infty < 1 \quad \text{and} \quad \|\mathcal{A}(v)\|_\infty < R \quad \text{for all} \quad v \in B_{R,1}. \tag{5.7}$$

Such inequalities imply that $\mathcal{A}(\widehat{B}_{R,1}) \subset \widehat{B}_{R,1}$. Since $\widehat{B}_{R,1}$ is closed and contractible to a point, and $\mathcal{A}$ (restricted to $\widehat{B}_{R,1}$) is a continuous and compact operator, the Schauder Point Fixed theorem applies, leading to the following result.

Proposition 5.1.2. *Assume condition* (5.5) *over the prescription function* H_k. *Then, problem* (5.3) *has at least one radially symmetric solution.*

Note that the solution given in previous result satisfies

$$v'(r) = \phi^{-1}\left[\left(\frac{n}{r^{n-k}}\int_0^r \tau^{n-1}H_k(\tau, v(\tau))d\tau\right)^{1/k}\right] \geq 0,$$

then, v is increasing and negative.

Nevertheless, it is possible to obtain a second solution of (5.3) by taking the negative k-root in equality (5.4) and proceeding in the same way. In this second case, the solution is decreasing and positive. Moreover, one solution is not the symmetric respect to the hyperplane Π of the other one, except when $H_k(t,r) = H_k(-t,r)$ for all $t \in [-R,R]$ and $r \in [0,R]$.

Summarizing, we have proved the following result.

Proposition 5.1.3. *Let be k even. Let $B_0(R)$ be an Euclidean ball centered at 0 with radius R contained in a spacelike hyperplane $\Pi \subset \mathbb{L}^{n+1}$ orthogonal to a inertial observers vector field. For every rotationally symmetric and continuous function $H_k : [-R,R] \times B_0(R) \subset \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$, satisfying (5.5) such that $H_k(0,\cdot) \not\equiv 0$, there exist at least two different rotationally symmetric spacelike graphs with k-curvature equal to H_k such that its boundary is in the hyperplane Π. One is above and the other one below the hyperplane Π.*

5.2 Existence results of the Dirichlet problem in Euclidean space.

In the Euclidean ambient, the prescribed k-curvature equation for a rotationally symmetric graph $\Sigma_v \subset \mathbb{R}^n$ with Dirichlet boundary conditions is written as

$$\begin{aligned} \left(r^{n-k}\psi^k(v')\right)' &= n\, r^{n-1}\, H_k(v(r),r) \qquad \text{in} \quad (0,R), \\ v'(0) &= 0 = v(R), \end{aligned} \tag{5.8}$$

where $\psi(s) := \dfrac{s}{\sqrt{1+s^2}}$ and $1 \leq k \leq n$.

From $\psi(\mathbb{R}) = (-1,1)$, we immediately note that mean k-curvature function along the graph must satisfy the inequality

$$|\int_0^r s^{n-1} H_k\big(v(s),s\big)\,ds| < \frac{r^{n-k}}{n} \qquad \text{for all} \qquad r \in [0,R]. \tag{5.9}$$

Analogously to the Minkowski case, if H_k is a constant (non negative if k is even) satisfying $H_k \leq R^{-k}$, a straight integration of (**??**) gives

$$v(r) = \sqrt{H_k^{-2/k} - R^2} - \sqrt{H_k^{-2/k} - r^2},$$

for each $1 \leq k \leq n$. On the other hand,for $H_k > R^{-k}$ the inequality (5.9) means that (5.8) has no solution. This fact suggests that, contrarily to the Minkovski case, in order to find solutions one has to impose a restriction on the size of the prescribed curvature function. This is a common feature of the Euclidean ambient. The following results are based on the analysis perfomed in [11].

5.2.1 Case 1: k odd.

The following result for k odd is proved by adapting the proof of [11, Proposition 2.5] applied to (5.8), due to the fact that $\psi^k : \mathbb{R} \longrightarrow (-1,1)$ is an increasing homeomorphism and $\psi(0) = 0$. The result is picked up in the following theorem.

Theorem 5.2.1. *Let $B_0(R)$ be an Euclidean ball centered at 0 with radius R contained in a hyperplane $\Pi \subset \mathbb{R}^{n+1}$, and let $H_k : \mathbb{R} \times B_0(R) \subset \mathbb{R}^{n+1} = \mathbb{R} \times \Pi \longrightarrow \mathbb{R}$ (k odd) be a rotationally symmetric and continuous function such that, for some*

$0 < \alpha < R^{-k}$, *satisfies*

$$|H_k(t,r)| \leq \alpha \qquad \text{for all} \qquad r \in [0,R]\,, \quad t \in [-R\,\beta\,, R\,\beta],$$

where $\beta := \psi^{-1}(R\alpha^{1/k})$*. Then, there exists at least one rotationally symmetric graph with k-curvature equal to* H_k *such that its boundary is in the hyperplane* Π*.*

Proof. Denote by $\Omega_\alpha := [-R\,\beta\,, R\,\beta]$. We show that

$$\mathcal{B}(\Omega_\alpha) \subset \Omega_\alpha, \tag{5.10}$$

where $\mathcal{B}$ is the same operator than $\mathcal{A}$ but replacing ϕ by ψ. Let $u \in \Omega_\alpha$ and $v = \mathcal{B}(u)$. By using the assumption $|H_k| \leq \alpha$, we have

$$\big|\big(\psi(v'(r))\big)^k\big| = \big|\frac{n}{r^{n-k}}\int_0^r t^{n-1}H_k(u(t),t)dt\big| \leq \alpha R^k$$

for all $r \in (0,R]$, and the hypothesis $\alpha < R^{-k}$ ensures that the image is less than 1.

Since $\psi^k(v'(0)) = 0$, and $\psi^k : \mathbb{R} \longrightarrow (-1,1)$ is an homeomorphism, it follows that

$$v'(r) \in [-\psi^{-1}(R\alpha^{1/k}), \psi^{-1}(R\alpha^{1/k})].$$

Therefore, $v(r) \in \Omega_\alpha$, and (5.10) is proved. Now, using the fact that Ω_α is a closed convex set in $\widehat{C}^1$ invariant by the compact operator $\mathcal{B}$, from the Schauder fixed point theorem, we conclude that there exists $u \in \Omega_\alpha$ such that $\mathcal{B}(u) = u$, which is a solution of the initial Dirichlet problem. □

5.2.2 Case 2: k even.

Our aim here consist in to construct a fixed point operator associated to the equation (5.8). In order to do this, we take $0 < \alpha < R^{-k}$ and we name $\beta := \psi^{-1}(R\alpha^{1/k})$.

As in Minkowski setting, we need to impose some restriction on the prescription function H_k,

$$\int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \geq 0 \qquad \text{for all} \qquad r \in [0, R]\,, \quad v \in B_{R\beta,\beta}. \tag{5.11}$$

In addition, as in the case with k odd, we introduce a boundedness assumption on H_k,

$$|H_k(t, r)| \leq \alpha \qquad \text{for all} \qquad r \in [0, R]\,, \quad t \in [-R\,\beta, R\,\beta]. \tag{5.12}$$

Now, we can proceed in the same way that in the Minkowski setting, using the same notation, and define the operator

$$\mathcal{B} : \overline{B}_{R\beta,\beta} \subset C^1 \longrightarrow C^1, \quad \mathcal{B} = K \circ (\psi^{-1})^{1/k} \circ S \circ N_F, \tag{5.13}$$

where $(\psi^{-1})^{1/k} : [0,1) \longrightarrow \mathbb{R}^+$ is defined by $(\psi^{-1})^{1/k}(s) := \psi^{-1}(+s^{1/k})$.

Explicitly,

$$\mathcal{B}(v)(r) = -\int_r^R \psi^{-1}\left[\left(\frac{n}{s^{n-k}} \int_0^s \tau^{n-1} H_k(\tau, v(\tau)) d\tau\right)^{1/k}\right] ds,$$

Note that $\mathcal{B}$ is well defined due to (5.11) and (5.17). Now, restricting $\mathcal{B}$ to the subset $\widehat{B}_{R\beta,\beta} := \{v \in B_{R\beta,\beta} \ : \ v'(0) = v(R) = 0\}$.

Again, if a function $v \in \widehat{C}^1$ is a fixed point of the nonlinear compact operator (5.13), then v is a solution of equation (5.8). Following the arguments of the Minkowski setting, with k even, we conclude that there exist one increasing solution and other decreasing solution. We may enunciate this result,

Proposition 5.2.2. *Let $B_0(R)$ be an Euclidean ball centered at 0 with radius R contained in a hyperplane $\Pi \subset \mathbb{R}^{n+1}$. For every rotationally symmetric and continuous function $H_k : \mathbb{R} \times B_0(R) \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, ($k$ even), satisfying (5.11) and (5.17) such that $H_k(0,\cdot) \not\equiv 0$, there exists at least two different rotationally symmetric graph with k-curvature equal to H_k such that its boundary is in the hyperplane Π. One is above and the other one below the hyperplane Π.*

5.3 Uniqueness results

It is possible to ensure the uniqueness of certain rotationally symmetric solutions of equation (5.3) under some hypothesis on the prescription function. As before, these results depend again of the parity of k, but the treatment will be the same in the Minkowski and the Euclidean cases. Therefore, we will denote both, $\dfrac{s}{\sqrt{1-s^2}}$ and $\dfrac{s}{\sqrt{1+s^2}}$, by $\chi(s)$.

Proposition 5.3.1. (*Case k odd*). *If $H_k(\cdot, r)$ is a non decreasing prescription function for each fixed $r \in [0,R]$, then equation*

$$\begin{aligned} \left(r^{n-k}\chi^k(v')\right)' &= n\, r^{n-1}\, H_k(v(r),r) \qquad \text{in} \quad (0,R), \\ v'(0) &= 0 = v(R), \end{aligned} \tag{5.14}$$

has at most one solution.

Proof. Suppose that u and v are different solutions of equation (5.3). Since $u(R) = v(R) = 0$, the set $F = \{r \in [0,R] : u'(r) \neq v'(r)\}$ has positive measure. Multiplying by $(u-v)$ the identity

$$\left[r^{n-k}\left(\chi^k(u') - \chi^k(v')\right)\right]' = nr^{n-1}\left[H_k(u(r),r) - H_k(v(r),r)\right]$$

and integrating over $[0, R]$, and using the boundary conditions we have

$$\begin{aligned} & -\int_F \left[\chi^k(u'(r)) - \chi^k(v'(r))\right] \left[u'(r) - v'(r)\right] r^{n-k} dr \\ = n & \int_0^R r^{n-1} \left[H_k(u(r), r) - H_k(v(r), r)\right] \left[u(r) - v(r)\right] dr. \end{aligned} \tag{5.15}$$

From the increasing character of χ^k, the first term is strictly negative, while the second one is non negative due to the increasing assumption over H_k. This is a contradiction and the result follows. □

Now we deal with the case with k even. The proof of the following proposition is similar to the previous one, but taking into account that χ^k is only increasing on the positive real numbers in which it is defined.

Proposition 5.3.2. (*Case k even*).

- *If $H_k(\cdot, r)$ is a non decreasing prescription function for each fixed $r \in [0, R]$, then equation* (5.14) *has at most one increasing solution.*
- *If $H_k(\cdot, r)$ is a non increasing prescription function for each fixed $r \in [0, R]$, then equation* (5.14) *has at most one decreasing solution.*

5.4 Extendibility of the solutions as entire graphs.

Now we are in a position to enunciate the main results of the chapter, related with the extendibility of the solutions of previous sections as entire graph. Depending on the parity of k, we have the following two theorems in Minkowski spacetime.

Theorem 5.4.1. *Let $H_k : \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$, with k an odd positive integer, be a continuous function which is rotationally symmetric with respect to an inertial observer*

γ of $\mathbb{L}^{n+1}$. Then, for each $R > 0$, there exists at least an entire spacelike graph, rotationally symmetric respect to γ, whose k-th mean curvature equals to H_k and such that it intersects the hyperplane orthogonal to γ at $\gamma(0)$ in an $(n-1)$-sphere with radius R centered at $\gamma(0)$. In addition, if H_k is non decreasing with respect to the proper time of γ, then the spacelike graph is unique.

Theorem 5.4.2. *Let $H_k : \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$, with k an even positive integer, be a continuous function such that*

$$\int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \geq 0 \quad \text{for all} \quad r \in \mathbb{R}^+, \quad \text{and} \quad v \in C^1,\ |v'| < 1, \tag{5.16}$$

and which is rotationally symmetric with respect to an inertial observer γ of $\mathbb{L}^{n+1}$. If $H_k(0,\cdot) \not\equiv 0$, for each $R > 0$, then there exists at least two different entire spacelike graphs and rotationally symmetric whose k-th mean curvature equals to H_k and such that it intersects the hyperplane orthogonal to γ at $\gamma(0)$ in an $(n-1)$-sphere with radius R centered in $\gamma(0)$. Moreover, the radial profile curve of one of them is increasing and the other one is decreasing. Besides, condition (5.16) is necessary for the existence of such graphs.

Respect to the Euclidean space, we also may enunciate the following two results.

Theorem 5.4.3. *Let $H_k : \mathbb{R}^{n+1} = \mathbb{R} \times \Pi \longrightarrow \mathbb{R}$, with k an odd positive integer, be a continuous function which is rotationally symmetric respect to an oriented line γ, orthogonal to Π. Given a fixed $R > 0$, assume there is some $\alpha \in (0, R^{-k})$, satisfying*

$$|H_k(t,r)| \leq \alpha \qquad \text{for all} \qquad r \in [0,R]\,, \quad t \in [-R\,\beta, R\,\beta], \tag{5.17}$$

where $\beta := \dfrac{R\alpha^{1/k}}{\sqrt{1 - R^2\alpha^{2/k}}}$, and

$$0 \leq \int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds < \frac{r^{n-k}}{n}, \qquad \text{for all} \qquad r > R \quad \text{and} \quad v \in C^1. \tag{5.18}$$

Then, there exists at least an entire graph, rotationally symmetric respect to γ, whose

k-th mean curvature equals to H_k and such that it intersects the hyperplane Π in an $(n-1)$-sphere with radius R centered in $\gamma(0)$. In addition, if H_k is non decreasing along the line γ, then the graph is unique.

Theorem 5.4.4. *Let $H_k : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, with k an even positive integer, be a continuous function which is rotationally symmetric respect to a line γ. For each $R > 0$, assume there is some $0 < \alpha < R^{-k}$, satisfying (5.17), (5.18) and*

$$\int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \geq 0, \qquad \text{for all} \qquad r \in [0, R] \quad \text{and} \quad v \in B_{R\beta,\beta},$$

being $B_{R\beta,\beta} = \{v \in C^1 : \|v\|_\infty < R\beta,\ \|v'\|_\infty < \beta\}$. Then, if $H_k(0, \cdot) \not\equiv 0$, there exists at least two different entire graphs, rotationally symmetric, whose k-th mean curvatures equal to H_k and such that they intersect the hyperplane orthogonal to γ in $\gamma(0)$ in an $(n-1)$-sphere with radius R centered in $\gamma(0)$. Moreover, the radial profile curve of one of them is increasing and the other one is decreasing.

In order to prove Theorems 5.4.1–5.4.4, it suffices to guarantee that every solution v, given by Theorems 5.1.1, 5.1.3, 5.2.1, 5.2.2, once R is fixed, can be continued until $+\infty$ as a solution of equations (5.2). We need the following lemma.

Lemma 5.4.5. *Every solution $v \in C^2[0, \varrho]$ of (5.3) verifies that $|v'| < 1$ on $[0, \varrho]$. Analogously, each solution $v \in C^2[0, \varrho]$ of (5.8) satisfies that $|v'| < +\infty$ on $[0, \varrho]$.*

Proof. From (5.14), we have

$$v'(r) = \chi^{-1}\left[\left(\frac{n}{r^{n-k}} \int_0^r \tau^{n-1} H_k(\tau, v(\tau)) d\tau\right)^{1/k}\right],$$

and, taking into account (5.18) the result follows immediately. □

Remark 5.4.6. *Graphs defined by the solution of Equation (5.3) are spacelike on the open ball. However, there could exist solutions which are of light type on the* *boundary, ∂B. The previous lemma ensures a priori that each possible solution v of* (5.3) *is spacelike on the boundary too.*

The rest of the proof does not depend on the ambient space (Euclidean or Minkowski), thus we follow with the notation of the previous section. However, we have to distinguish odd and even cases again.

First, assume that k is odd. Let v be a solution of equation (5.14), and let $[0,b[$ be the maximal interval of definition of v. Suppose that $b<+\infty$. We can rewrite equation (5.14) as a system of two ordinary differential equations of first order

$$\begin{aligned} v'(r) &= \chi^{-1}\big[\big(\frac{z(r)}{r^{n-k}}\big)^{1/k}\big] \\ z'(r) &= n\, r^{n-1} H(v(r),r), \end{aligned}$$

which we can abbreviate

$$\begin{bmatrix} v' \\ z' \end{bmatrix} = \mathcal{F}\big(r,v,z\big),$$

where $\mathcal{F}:\mathbb{R}^+\times\mathbb{R}\times J\longrightarrow\mathbb{R}^2$, and J is $\mathbb{R}$ or $(-b^{n-k},b^{n-k})$ if the ambient is Minkowski or Euclidean space respectively.

By the standard prolongability theorem of ordinary differential equations (see for instance [83, Section 2.5]), we have that the graph $\big\{\big(r,v(r),z(r)\big) \;:\; r\in [R/2,b[\big\}$ goes out of any compact subset of $\mathbb{R}^+\times\mathbb{R}\times J$. However, by Lemma 5.4.5, $|v'(r)|<\rho$ (of course, ρ depends on the chosen solution v), then $|v(r)|<b\rho$. Therefore, the graph can not go out of the compact subset $[R/2\,,b]\times[-b\rho,b\rho]\times[-b^{n-k}\chi^k(\rho),b^{n-k}\chi^k(\rho)]$ contained in the domain of $\mathcal{F}$. This is a contradiction, then $b=+\infty$.

If k is even, we know that at least there exist one increasing and one decreasing solutions of equation (5.14). For instance, let v be a increasing solution (the argument of the proof is similar for a decreasing solution), and let $b<+\infty$ its maximal interval of definition. In this way, $v'(r)>0$ for all $r\in(0,R]$. Moreover, if condition (5.16) (or (5.18) in the Euclidean setting) holds, then extension of v will be also increasing on $(0, b)$. Hence, v and z, where $z(r) := r^{n-k}\chi^k(v'(r))$, verify the ODE system (5.19), taking the positive k-root in the first equation. From this point, the proof continues being the same that case k odd, and we deduce that v can be extended to $+\infty$ as an increasing solution of (5.2).

Chapter 6

Uniformly accelerated motion in General Relativity

6.1 The concept of uniformly accelerated motion

The family of UA observers in the Lorentz-Minkowski spacetime $\mathbb{L}^n$ was completely determined long time ago [73] (see [47] and references therein for a historical approach). It consists of timelike geodesics and Lorentzian circles. For instance, in $\mathbb{L}^2$, using the usual coordinates (x,t), the UA observer $\gamma(\tau) = \big(x(\tau), t(\tau)\big)$ throughout $(0,0)$ with zero velocity relative to certain family of inertial observers (the integral curves of vector field ∂_t) and proper acceleration a is given by,

$$x(\tau) = \frac{c^2}{a}\left[\cosh\left(\frac{a\tau}{c}\right) - 1\right], \qquad t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right),$$

where $\tau \in \mathbb{R}$ is the proper time of γ, and c is the light speed in vacuum.

The wordline of γ described by the inertial observer is given by (see, for instance,

[73]),

$$x(t) = \frac{c^2}{a}\left[\sqrt{1+\left(\frac{at}{c}\right)^2} - 1\right],$$

which reduce to the classical one $x(t) \approx \frac{1}{2}at^2$. In fact, according to the spirit of [78, Prop. 0.2.1], γ may be thought, in certain sense, as the *relativistic trajectory associated to the Newtonian one* $x(t)$. Also note that the *radius* of this Lorentzian circle is $1/a$, so if the acceleration approached to zero, then γ becomes a unit timelike geodesic.

Now, taking into account the Fermi-Walker formalism provided in Preliminaries, we are in a position to give rigorously the notion of UA observer [39] [34] or [35]. With the same notation used there, an observer $\gamma : I \longrightarrow M$ is said to obey a uniformly accelerated motion if

$$\widehat{P}^{\gamma}_{t_1,t_2}\left(\frac{D\gamma'}{dt}(t_1)\right) = \frac{D\gamma'}{dt}(t_2), \tag{6.1}$$

for any $t_1, t_2 \in I$ with $t_1 < t_2$, equivalently, if the equation (1.6) holds everywhere on I, i.e., $\dfrac{D\gamma'}{dt}$ is Fermi-Walker parallel along γ. Clearly, if γ is free falling, then it is a UA observer.

Since we deal with a third-order ordinary differential equation, the following initial value problem has a unique local solution,

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right) = 0, \tag{6.2}$$

$$\gamma(0) = p, \quad \gamma'(0) = v, \quad \frac{D\gamma'}{dt}(0) = w,$$

where $p \in M$ and $v, w \in T_pM$ such that $|v|^2 = -1$, $\langle v, w\rangle = 0$, $|w|^2 = a^2$, and a is a positive constant.

For any observer we have a conservation result as a consequence of the slightly more general lemma,

Lemma 6.1.1. *Let σ be a curve in M, defined on an open interval $I \subset \mathbb{R}$, which satisfies the equation*

$$\frac{D^2\sigma'}{dt^2} = \left|\frac{D\sigma'}{dt}\right|^2 \sigma' - \left\langle \sigma', \frac{D\sigma'}{dt} \right\rangle \frac{D\sigma'}{dt}. \tag{6.3}$$

Then, $\left|\frac{D\sigma'}{dt}\right|^2(t)$ is constant on I.

Proof. Multiplying (6.3) by $\frac{D\sigma'}{dt}$, we directly obtain

$$\left\langle \frac{D^2\sigma'}{dt^2}, \frac{D\sigma'}{dt} \right\rangle = \frac{1}{2}\frac{d}{dt}\left\langle \frac{D\sigma'}{dt}, \frac{D\sigma'}{dt} \right\rangle = 0,$$

and the proof is done. □

Remark 6.1.2. ***(a)*** *Observe that no assumption is made on the spacetime in previous result. On the other hand, the constant a has a geometrical meaning for a UA observer in terms of its Frenet-Serret formulas (see next section).* ***(b)*** *The family of the UA observers lies into a bigger family of observers which has shown to be relevant in the study of the global geometry of spacetimes, the so called bounded acceleration observers. Recall that [9, Def. 6.6] an observer $\gamma : I \longrightarrow M$ is said to have bounded acceleration if there exists a constant $B > 0$ such that $\left|\frac{D\gamma'}{dt}\right| \leq B$ for all $t \in I$.* ***(c)*** *On the other hand, note that if a UA observer is not free falling, then $\left(1\Big/\left|\frac{D\gamma'}{dt}\right|\right)\frac{D\gamma'}{dt}$ is also Fermi-Walker parallel along γ.*

Taking into account formula (2.10), an observer γ satisfies equation (1.6) if and only if

$$\frac{D^2\gamma'}{dt^2} = \left\langle \frac{D\gamma'}{dt}, \frac{D\gamma'}{dt} \right\rangle \gamma', \tag{6.4}$$

which is a third order equation. Alternatively, γ satisfies (6.4) if and only if $\dfrac{D^2\gamma'}{dt^2}(t)$ is collinear to $\gamma'(t)$ at any $t \in I$.

Example 6.1.3. ***(a)*** *In any Generalized Robertson-Walker (GRW) spacetime, each integral curve of the coordinate reference frame is trivially a UA observer.* ***(b)*** *Consider now a static standard spacetime $M = S \times I$, with metric $\langle\, ,\rangle = g_S - h^2 dt^2$, where g_S is a Riemannian metric on S, $h \in C^\infty(S)$, $h > 0$ and I an open interval of the real line $\mathbb{R}$. Let $\gamma = \gamma(s)$ be any integral curve of the reference frame $Q = \dfrac{1}{h}\,\partial_t$. A direct computation gives*

$$\frac{D\gamma'}{ds} = \frac{\nabla h}{h} \circ \gamma.$$

On the other hand, taking into account [68, Prop 7.35], we get

$$\frac{D^2\gamma'}{ds^2} = \frac{|\nabla h|^2}{h^2}\,\gamma'.$$

From the two previous formulas and (6.4) it follows that γ is a UA observer.

6.2 UA motion and Lorentzian circles

Consider a UA observer $\gamma : I \longrightarrow M$ with $a = \left|\frac{D\gamma'}{dt}\right| > 0$ (constant from Lemma 6.1.1) and put $e_1(t) = \gamma'(t)$, $e_2(t) = \dfrac{1}{a}\dfrac{D\gamma'}{dt}(t)$. Then, from (6.4) we have

$$\frac{De_1}{dt} = a e_2(t),$$
$$\frac{De_2}{dt} = a e_1(t).$$

Conversely, assume this system holds true for an observer γ with $a > 0$ constant. Then, equation (6.4) also holds true. In other words, a (non free falling) UA observer may be seen as a *Lorentzian circle* of constant curvature a and identically zero torsion (see [56]).

Remark 6.2.1. *Circles in a Riemannian manifold were studied by Nomizu and Yano in [67] in order to characterize umbilical submanifolds with parallel mean curvature vector field in an arbitrary Riemannian manifold. They described a circle by a third-order differential equation similar to the previous equation (6.4). The results in [67] were extended to the Lorentzian case by Ikawa in [56].*

The previous results can be summarized as follows,

Proposition 6.2.2. *For any observer $\gamma : I \longrightarrow M$, the following assertions are equivalent:*

(a) γ *is a UA observer.*

(b) γ *is a solution of third-order differential equation* (6.4).

(c) γ *is a Lorentzian circle or it is free falling.*

(d) γ *has constant curvature and the remaining curvatures equal to zero.*

(e) γ*, viewed as an isometric immersion from $(I, -dt^2)$ to M, is totally umbilical with parallel mean curvature vector.*

Now we are in a position to get a converse to previous Example 6.1.3 (b).

Proposition 6.2.3. *Consider $M = S \times I$ with a Lorentzian metric of the type $\langle\, ,\, \rangle = g_S - f^2 dt^2$ where $f \in C^\infty(M)$, $f > 0$. Assume each integral curve of the reference frame $\frac{1}{f}\,\partial_t$ is a* UA *observer. Then, there exist $h \in C^\infty(S)$, $h > 0$ and $\phi \in C^\infty(I)$, $\phi > 0$, such that*

$$f(x,t) = h(x)\phi(t),$$

for all $(x,t) \in M$. Therefore, M is a standard static spacetime with $\langle\,,\,\rangle = g_S - h^2 ds^2$ and $ds = \phi\, dt$.

Proof. We have that the mean curvature vector field of each submanifold $\{x_0\} \times I$, $x_0 \in S$, is

$$-\frac{1}{f}\,\nabla f - \frac{f'}{f^3}\,\partial_t.$$

On the other hand, our assumption means that $-\frac{1}{f}\,\nabla f - \frac{f'}{f^3}\,\partial_t$ is parallel (as a normal vector field to $\{x_0\} \times I$). Now, making use of [26, Prop. 1.2(3)], we get that $f = f(x,t)$ is the product of two positive functions $h = h(x)$ and $\phi = \phi(t)$ on S and I, respectively. □

We end the section with the statement of a characterization of standard static spacetimes in terms of the existence of certain reference frame whose integral curves are UA observers. Before we need to recall some notions to be used later.

A reference frame Q in a spacetime M is said to be *locally sincronizable* if $Q^b \wedge dQ^b = 0$ where Q^b is the 1-form on M metrically equivalent to Q [78, p. 53]. Equivalently, Q is locally sincronizable if and only if the distribution $Q^\perp$ is integrable or, if and only if $\langle \nabla_X Q, Y\rangle = \langle X, \nabla_Y Q\rangle$ for any $X, Y \in Q^\perp$, [68, Prop. 12.30].

On the other hand, a reference frame Q is said to be *rigid* if $\langle \nabla_X Q, Y\rangle + \langle X, \nabla_Y Q\rangle = 0$ for all $X, Y \in Q^\perp$, [78, p. 56].

Theorem 6.2.4. *Let M be a simply connected and geodesically complete spacetime. If M admits a rigid and locally sincronizable reference frame Q such that any integral curve of Q is a* UA *observer, then M is a static standard spacetime.*

Proof. Since Q is also assumed to be rigid, each leaf of the foliation $R = Q^\perp$ is in fact totally geodesic. Therefore, any inextensible leaf of R is geodesically complete (with respect to the induced Riemannian metric). On the other hand, any leaf of $T = \operatorname{Span}\{Q\}$ is totally umbilical since T is 1-dimensional. Even more, if any integral curve of Q is a UA observer, then the leaves of T are extrinsic spheres. The conclusion follows now from [69, Cor. 1]. $\square$

6.3 Completeness of the inextensible UA trajectories

This section is devoted to the study of the completeness of the inextensible solutions of equation (6.4). First of all, we are going to relate the solutions of equation (6.4) with the integral curves of a certain vector field on a Stiefel bundle type on M (compare with [59, p. 6]).

Given a Lorentzian linear space E and $a \in \mathbb{R}$, $a > 0$, denote by $V^a_{n,2}(E)$ the (n,2)-Stiefel manifold over E, defined by

$$V^a_{n,2}(E) = \left\{(v,w) \in E \times E \ : \ |v|^2 = -1,\, |w|^2 = a^2, \ \ \langle v, w\rangle = 0\right\}.$$

The (n,2)-Stiefel bundle over the spacetime M is then defined as follows,

$$V^a_{n,2}(M) = \bigcup_{\mathrm{p}\in \mathrm{M}} \{p\} \times V^a_{n,2}(T_pM).$$

Note that $V^a_{n,2}(M)$ is a bundle on M with dimension $3(n-1)$ and fiber diffeomorphic to $\mathbb{S}_a\big(\mathbb{H}^{n-1}\big)$, the spherical fiber bundle on the hyperbolic space $(n-1)$-dimensional and fiber $\mathbb{S}^{n-2}$ of radius a.

First we construct a vector field $G \in \mathfrak{X}\big(V^a_{n,2}(M)\big)$, which is the key tool in the study of completeness,

Lemma 6.3.1. *Let $\sigma : I \longrightarrow M$ be a curve satisfying* (6.3) *with initial conditions*

$$\sigma'(0) = v, \qquad \frac{D\sigma'}{dt}(0) = w,$$

where v is a unitary timelike vector and w is orthogonal to v. Then, we have $|\sigma'(t)|^2 = -1$ for all $t \in I$, and therefore, $\big\langle \sigma'(t), \frac{D\sigma'}{dt}(t) \big\rangle = 0$ holds everywhere on I.

Proof. Multiplying (6.3) by σ', and after easy computations, we arrive to the following ordinary differential equation

$$\frac{1}{2}\, x'' + \frac{1}{4}\, (x')^2 - a^2 x = a^2,$$

where $a := \big| \frac{D\sigma'}{dt} \big|$ is constant, and $x(t) := |\sigma'(t)|^2$. From the assumption, we know that $x(t)$ satisfying the initial conditions

$$x(0) = -1,$$

and

$$x'(0) = 2 \left\langle \sigma'(0)\, , \frac{D\sigma'}{dt}(0) \right\rangle = 2\langle v, w \rangle = 0.$$

Since $x(t) = -1$ is a solution of this initial value problem, the result is a direct consequence of the existence and uniqueness of solutions to second order differential equations. □

Now, we are in a position to define the announced vector field G. Let (p, v, w) be a point of $V^a_{n,2}(M)$, and $f \in C^\infty\big(V^a_{n,2}(M)\big)$. Let σ be the unique inextensible curve solution of (6.3) satisfying the initial conditions

$$\sigma(0) = p, \quad \sigma'(0) = v, \quad \frac{D\sigma'}{dt}(0) = w.$$

So, we define

$$G_{(p,v,w)}(f) := \frac{d}{dt}\Big|_{t=0} f\Big(\sigma(t), \sigma'(t), \frac{D\sigma'}{dt}(t)\Big).$$

From Lemma 6.1.1 and Lemma 6.3.1, we have $\big(\sigma(t), \sigma'(t), \dfrac{D\sigma'}{dt}(t)\big) \in V^a_{n,2}(M)$ and G is well defined.

The following result follows easily,

Lemma 6.3.2. *There exists a unique vector field G on $V^a_{n,2}(M)$ such that the curves $t \longmapsto \big(\gamma(t), \gamma'(t), \dfrac{D\gamma'}{dt}(t)\big)$ are the integral curves of G, for any solution γ of equation* (6.2).

Once defined G, we will look for assumptions which assert its completeness.

Recall that an integral curve α of a vector field defined on some interval $[0, b)$, $b < +\infty$, can be extended to b (as an integral curve) if and only if there exists a sequence $\{t_n\}_n$, $t_n \nearrow b$, such that $\{\alpha(t_n)\}_n$ converges (see for instance [68, Lemma 1.56]). The following technical result directly follows from this fact and Lemma 6.3.2.

Lemma 6.3.3. *Let $\gamma : [0, b) \longrightarrow M$ be a solution of equation* (6.2) *with $0 < b < \infty$. The curve γ can be extended to b as a solution of* (6.3) *if and only if there exists a sequence $\big\{\gamma(t_n), \gamma'(t_n), \dfrac{D\gamma'}{dt}(t_n)\big\}_n$ which is convergent in $V^a_{n,2}(M)$.*

Although we know that $|\gamma'(t)|^2 = -1$ from Lemma 6.3.1, this is not enough to apply Lemma 6.3.3 even in the geometrically relevant case of M compact. The reason is similar to the possible geodesic incompleteness of a compact Lorentzian manifold (see for instance [68, Example 7.16]).

However, it is relevant that if a compact Lorentzian manifold admits a timelike conformal vector field, then it must be geodesically complete [76]. Therefore, from a geometric viewpoint, it is natural to assume the existence of such infinitesimal conformal symmetry to deal with the extendibility of the solutions of (6.2).

Recall that a vector field K on M is called *conformal* if the Lie derivative of the metric with respect to K satisfies

$$L_K\langle\,,\rangle = 2h\langle\,,\rangle, \tag{6.5}$$

for some $h \in C^\infty(M)$, equivalently, the local flows of K are conformal maps. In particular, if holds (6.5) with $h = 0$, K is called a *Killing* vector field.

Note that for any curve $\gamma : I \longrightarrow M$, the relation (6.5) implies

$$\frac{d}{dt}\langle K, \gamma'\rangle = \Big\langle K, \frac{D\gamma'}{dt}\Big\rangle + h(\gamma)|\,\gamma'|^2. \tag{6.6}$$

On the other hand, if a vector field K satisfies

$$\nabla_X K = hX \qquad \text{for} \quad \text{all} \qquad X \in \mathfrak{X}(M), \tag{6.7}$$

then clearly we get (6.5). Moreover, for the 1-form K^b metrically equivalent to K, we have

$$dK^b(X,Y) = \langle \nabla_X K, Y\rangle - \langle \nabla_Y K, X\rangle = 0,$$

for all $X, Y \in \mathfrak{X}(M)$, i.e., K^b is closed. We will call to K which satisfies (6.7) a *conformal and closed* vector field. A Lorentzian manifold which admits a timelike conformal and closed vector field is locally a Generalized Roberson-Walker spacetime [26], [79].

The following result, inspired from [24, Lemma 9], will be decisive to assure that the image of the curve in $V^a_{n,2}(M)$, associated to a UA observer γ, is contained in a compact subset.

Lemma 6.3.4. *Let M be a spacetime and let Q be a unitary timelike vector field. If $\gamma : I \longrightarrow M$ is a solution of* (6.2) *such that $\gamma(I)$ lies in a compact subset of M and*

$\langle Q, \gamma'\rangle$ is bounded on I, then the image of $t \longmapsto \big(\gamma(t), \gamma'(t), \dfrac{D\gamma'}{dt}\big)$ is contained in a compact subset of $V^a_{n,2}(M)$ where a is the constant $|\frac{D\gamma'}{dt}|$.

Proof. Consider the 1-form Q^b metrically equivalent to Q and the associated Riemannian metric $g_R := \langle\ ,\ \rangle + 2\,Q^b \otimes Q^b$. We have,

$$g_R(\gamma', \gamma') = \langle \gamma', \gamma'\rangle + 2\,\langle Q, \gamma'\rangle^2,$$

which, by hypothesis, is bounded on I. Hence, there exists a constant $c > 0$ such that

$$\big(\gamma(I), \gamma'(I), \frac{D\gamma'}{dt}(I)\big) \subset C, \qquad C := \big\{(p, v, w) \in V^a_{n,2}(M) \,:\, p \in C_1, \quad g_R(v, v) \leq c\big\},$$

where C_1 is a compact set on M such that $\gamma(I) \subset C_1$. Hence, C is a compact in $V^a_{n,2}(M)$. □

Now, we are in a position to state the following completeness result (compare with [24, Th. 1] and [23, Th. 1]),

Theorem 6.3.5. *Let M be a spacetime which admits a timelike conformal and closed vector field K. If $\mathrm{Inf}_M \sqrt{-\langle K, K\rangle} > 0$ then, each solution $\gamma : I \longrightarrow M$ of (6.2) such that $\gamma(I)$ lies in a compact subset of M can be extended.*

Proof. Let $I = [0, b)$, $0 < b < +\infty$, be the domain of a solution γ of equation (6.2). Derivating (6.6), it follows

$$\frac{d^2}{dt^2}\langle K, \gamma'\rangle = \langle \frac{DK}{dt}, \frac{D\gamma'}{dt}\rangle + \langle K, \frac{D^2\gamma'}{dt^2}\rangle - \frac{d}{dt}(h \circ \gamma).$$

The first right term vanishes because K is conformal and closed,

$$\left\langle \frac{DK}{dt}, \frac{D\gamma'}{dt} \right\rangle = h(\gamma)\,\langle \gamma', \frac{D\gamma'}{dt}\rangle = 0.$$

On the other hand, the second right term equals to $a^2\langle K, \gamma'\rangle$. Thus, the function $t \mapsto \langle K, \gamma'\rangle$ satisfies the following differential equation,

$$\frac{d^2}{dt^2}\langle K, \gamma'\rangle - a^2\langle K, \gamma'\rangle = (h \circ \gamma)'(t). \tag{6.8}$$

Using now that $\gamma(I)$ is contained in a compact of M, the function $h\circ\gamma$ is bounded on I. Moreover, since I is assumed bounded, using (6.8) there exists a constant $c_1 > 0$ such that

$$|\langle K, \gamma'\rangle| < c_1. \tag{6.9}$$

Now, if we put $Q := \dfrac{K}{|K|}$, where $|K|^2 = -\langle K, K\rangle > 0$, then Q is a unitary timelike vector field such that, by (6.9),

$$|\langle Q, \gamma'\rangle| \leq m\, c_1 \qquad \text{on} \quad I,$$

where $m = \mathrm{Sup}_M |K|^{-1} < \infty$. The proof ends making use of Lemmas 6.3.3 and 6.3.4. $\square$

Remark 6.3.6. *Note that the previous theorem implies the following result of mathematical interest: Let M be a compact spacetime which admits a timelike conformal and closed vector field K. Then, each inextensible solution of* (6.2) *must be complete. Note that the Lorentzian universal covering of M inherits the completeness of inextensible UA observers form the same fact on M.*

Example 6.3.7. *Let $f \in C^\infty(\mathbb{R})$ be a positive periodic function and let (N, g) be a compact Riemannian manifold. The GRW spacetime $\mathbb{R}\times_f N$ is a Lorentzian covering manifold of the compact spacetime $\mathbb{S}^1 \times_{\widetilde{f}} N$ where $\widetilde{f}$ is the induced function from f on $\mathbb{S}^1$. The result in [76] may be applied to $\mathbb{S}^1 \times_{\widetilde{f}} N$ with $K = \widetilde{f}Q$, which is timelike, conformal and closed [79], where Q is the vector field on $\mathbb{S}^1$ induced from ∂_θ. Thus, we have that any inextensible UA observer in the spacetime $\mathbb{R}\times_f N$ must be complete.*

Chapter 7

Unchanged direction motion in General Relativity

7.1 The relativistic notion of unchanged direction motion

Once exposed the Fermi-Walker machinery, we are in a position to give accurately the notion of unchanged direction observer.

Definition 1. *An observer $\gamma : I \longrightarrow M$ is said to obey an unchanged direction (UD) motion if*

$$\widehat{P}^{\gamma}_{t_1,t_2}\left(\frac{D\gamma'}{dt}(t_1)\right) = \lambda(t_1,t_2)\,\frac{D\gamma'}{dt}(t_2), \tag{7.1}$$

for a certain proportional factor λ and for any $t_1, t_2 \in I$ with $t_1 < t_2$.

Clearly, if an observer γ is a free falling then it obeys a UD motion. More generally, a uniformly accelerated (UA) observer [39] satisfies (7.1) with $\lambda = 0$. Thus, it obeys a UD motion. Of course, the family of UD observers is much bigger than the one of the UA observers.

Note that, if $\dfrac{D\gamma'}{dt}(t) \neq 0$ for all $t \in I$, Definition 1 is equivalent to say that the normalized acceleration, $\left|\dfrac{D\gamma'}{dt}\right|^{-1} \dfrac{D\gamma'}{dt}$, is Fermi-Walker parallel along γ. Taking into account that the Leibniz rule holds true for the Fermi-Walker covariant derivative,

$$\frac{\widehat{D}}{dt}\langle Y_1 \,, Y_2\rangle = \Big\langle \frac{\widehat{D}Y_1}{dt}, \, Y_2\Big\rangle + \Big\langle Y_1 \,, \frac{\widehat{D}Y_2}{dt}\Big\rangle \,, \tag{7.2}$$

for any $Y_1, Y_2 \in \mathfrak{X}(\gamma)$. From (7.1) we arrive to the following expression,

$$\Big|\frac{D\gamma'}{dt}\Big|^2 \frac{\widehat{D}}{dt}\Big(\frac{D\gamma'}{dt}\Big) = \Big\langle \frac{\widehat{D}}{dt}\Big(\frac{D\gamma'}{dt}\Big), \frac{D\gamma'}{dt}\Big\rangle \, \frac{D\gamma'}{dt} \, . \tag{7.3}$$

We observe that this equation is well defined for every observer, not only for those with acceleration nonzero everywhere. By using (2.10), last formula can be equivalently expressed as follows,

$$\Big|\frac{D\gamma'}{dt}\Big|^2 \frac{D^2\gamma'}{dt^2} = \frac{1}{2}\frac{d}{dt}\Big|\frac{D\gamma'}{dt}\Big|^2 \frac{D\gamma'}{dt} + \Big|\frac{D\gamma'}{dt}\Big|^4 \gamma' \, . \tag{7.4}$$

Note that if γ is a UA observer, then $\Big|\dfrac{D\gamma'}{dt}\Big|^2 = a^2$. If γ is not free falling, then a a positive constant, and (7.4) reduces to

$$\frac{D^2\gamma'}{dt^2} = a^2 \, \gamma' ,$$

which is just the equation defining a UA motion [39].

However, a solution of equation (7.4) does not describe a UD observer in general. In fact, a solution γ of equation (7.4) is a UD observer whenever $\dfrac{D\gamma'}{dt} \neq 0$ everywhere on the domain I of γ. On the other hand, when the acceleration vector field vanishes identically on a subinterval J of I, then equation (7.4) is automatically satisfied on J and γ is a free falling on J until it eventually returns to be accelerated out of J in a possibly different direction. Thus we introduce the following notion.

Definition 2. *An observer $\gamma : I \longrightarrow M$ is said to obey a piecewise unchanged direction motion if γ satisfies equation (7.4).*

From an analytical point of view, the Cauchy problem associated to equation (7.4) does not have a unique solution in general. If fact, in a local coordinate system, equation (7.4) gives a system of ordinary differential equations which cannot be written in normal form. Hence, the classical Picard-Lindelöf theorem cannot be applied, and the existence and uniqueness of the solutions are not a priori guaranteed. We will see before that although existence is true, there is not uniqueness in general.

Now we will state the prescription acceleration problem as follows. Let $a : I \longrightarrow \mathbb{R}$ be a smooth function (the prescribed acceleration function) and consider the initial value problem

$$\left|\frac{D\gamma'}{dt}\right|^2 \frac{D^2\gamma'}{dt^2} = \frac{1}{2}\frac{d}{dt}\left|\frac{D\gamma'}{dt}\right|^2 \frac{D\gamma'}{dt} + \left|\frac{D\gamma'}{dt}\right|^4 \gamma',$$

$$\gamma(0) = p, \quad \gamma'(0) = v, \quad \frac{D\gamma'}{dt}(0) = a(0)\, w,$$

where $p \in M$ and $v, w \in T_pM$ such that $|v|^2 = -1$, $\langle v, w\rangle = 0$ and $|w|^2 = 1$. Thus, $a^2(t)$ will prescribe the square modulus of the proper acceleration vector field. The sign of $a(t)$ indicates if the sense of the acceleration is the same or the opposite respect to the initial one, i.e., if $a(t)$ has and $a(0)$ have the same sign then γ observes that its accelerometer points at the proper time t in the same sense that in the initial instant.

Using $a^2(t) = \left|\dfrac{D\gamma'}{dt}\right|^2(t)$ in equation (7.4), we get,

$$a^2(t)\,\frac{D^2\gamma'}{dt^2} = \frac{1}{2}\frac{d}{dt}(a^2(t))\frac{D\gamma'}{dt} + a^4(t)\gamma'. \tag{7.5}$$

If the observer is always accelerated, i.e., if the prescription function $a(t) \neq 0$ for all $t \in I$, the last equation reduces to

$$\frac{D^2\gamma'}{dt^2} = \frac{a'(t)}{a(t)}\frac{D\gamma'}{dt} + a^2(t)\gamma'. \tag{7.6}$$

Conversely, if an observer γ satisfies (7.6) with $a(t) \neq 0$ for all $t \in I$, then its acceleration satisfies $\left|\dfrac{D\gamma'}{dt}\right|^2(t) = a^2(t)$. In order to prove this, if we multiply both members of equation (7.6) by γ', then

$$\left\langle \gamma'(t)\,, \frac{D^2\gamma'}{dt^2}(t)\right\rangle = -a^2(t),$$

and, since γ is an observer, we get the announced result.

Note that if $a(t) \neq 0$ everywhere on I, the initial value problem associated to equation (7.6) has a unique local solution. The lack of uniqueness of the initial value problem associated to equation (7.4) is now clear. In fact, take two different prescription functions with the same initial value. The solutions of the Cauchy problem corresponding to (7.6) are two different solutions of (7.4).

However, if the prescribed acceleration $a(t)$ vanishes at some instant, the uniqueness of solutions of (7.5) is not guaranteed. Moreover, as commented before, γ can be a solution of (7.5), even although it is not a UD observer (only a piecewise UD observer). It is necessary to add some additional assumption in (7.5) to assure that each solution is a unique UD observer.

Let $U(t)$ be the unitary acceleration, defined only at the instant t at which $a(t) \neq 0$,

$$U(t) = \left|\frac{D\gamma'}{dt}\right|^{-1}\frac{D\gamma'}{dt}.$$

Put $J = \{t \in I \ : \ a(t) \neq 0\}$ and for each $t_0 \in I$ let us consider

$$t_0^* = \sup\{t \in I, \ \ t < t_0, \ \ t \in J\},$$

and the "extended unitary acceleration", U^*, defined on the whole interval I as follows,

$$U^*(t) = \begin{cases} U(t) & \text{if} \quad t \in J, \\ P^{\gamma}_{t^*,\,t}\left(\displaystyle\lim_{s\in J,\, s\to t^{*-}} U(s)\right) & \text{if} \quad t \notin J. \end{cases}$$

Definition 3. *An observer $\gamma : I \longrightarrow M$ is said to be an* UD observer with prescribed acceleration $a : I \longrightarrow \mathbb{R}$*, if it satisfies (7.5) and*

$$\lim_{t\to t_0^-} U^*(t) = \pm\left(\lim_{t\to t_0^+} U^*(t)\right), \tag{7.7}$$

for all $t_0 \in I$ such that $a(t_0) = 0$.

In particular, for a prescription function $a(t)$ which only vanishes in a discrete subset $T \subset I$, a solution γ of (7.5) is a UD observer which prescribed acceleration a if and only if

$$\lim_{t\to t_0^-} U(t) = \pm \lim_{t\to t_0^+} U(t),$$

where $t \in I \setminus T$ and $t_0 \in T$.

Note that at any instant where the limits have opposite sign, the observer changes the sense of his acceleration, but not the direction.

The following result will be useful tool in order to study the completeness of the inextensible trajectories in Section 7.4.

Theorem 7.1.1. *Let $a : I \longrightarrow \mathbb{R}$ be a smooth function and $v, w \in T_pM$ such that $|v|^2 = -1$, $|w|^2 = 1$ and $\langle v, w\rangle = 0$. The 4-velocity of the unique UD observer γ satisfying the initial conditions*

$$\gamma(0) = p, \quad \gamma'(0) = v, \quad \frac{D\gamma'}{dt}(0) = a(0)\, w, \tag{7.8}$$

is given by

$$\gamma'(t) = \cosh\big(V(t)\big)\, L(t) + \sinh\big(V(t)\big)\, M(t), \tag{7.9}$$

where

$$V(t) = \int_0^t a(s)ds,$$

and L, M are two Levi-Civita parallel vector fields along γ with $L(0) = v$ and $M(0) = a(0)\, w$.

Proof. First, if γ satisfies (7.9) and $a(t) \neq 0$ everywhere in I, then it is a solution of (7.6). From assumptions on L and M, after easy computations we get

$$\frac{\widehat{D}L}{dt}(t) = -a(t)\, M(t) \qquad \text{and} \qquad \frac{\widehat{D}M}{dt}(t) = -a(t)\, L(t).$$

Also note

$$U(t) = \left|\frac{D\gamma'}{dt}\right|^{-1}\frac{D\gamma'}{dt} = \operatorname{sign}\big(a(t)\big)\, \big(\sinh(V)L + \cosh(V)M\big), \tag{7.10}$$

Therefore, from these identities, we conclude $\dfrac{\widehat{D}U}{dt} = 0$.

We analyse now the case when $a(t)$ vanishes at some instant. Let $t_0 \in I$ be such that $a(t_0) = 0$. We have,

$$U^*(t_0) = \operatorname{sign}\left(\lim_{t \to t_0^{*-}} \frac{a(t)}{|a(t)|}\right) \left[\sinh\big(V(t_0^*)\big) L(t_0) + \cosh\big(V(t_0^*)\big) M(t_0)\right].$$

Since $V(t_0^*) = V(t_0)$, condition (7.7) is satisfied, and γ is an UD observer with prescribed acceleration $a(t)$.

$\square$

By using the Levi-Civita parallel transport, we can express (7.9) as the following first order integro-differential equation,

$$\begin{aligned} \gamma'(t) &= \cosh\big(V(t)\big)\, P^{\gamma}_{0,t}(v) + \sinh\big(V(t)\big)\, a(0)\, P^{\gamma}_{0,t}(w), \\ |v|^2 &= -1, \quad |w|^2 = 1, \quad \langle v, w\rangle = 0. \end{aligned} \tag{7.11}$$

7.2 UD motion from a geometric viewpoint

Now we proceed to find the Frenet equations which satisfies (and in fact redefines) a UD observer in the particular case of nowhere zero acceleration.

Consider a UD observer $\gamma : I \longrightarrow M$ with $\left|\dfrac{D\gamma'}{dt}\right| > 0$ everywhere on I. In this case, the two following vector fields along γ are well-defined,

$$e_1(t) = \gamma'(t) \qquad \text{and} \qquad e_2(t) = \left|\frac{D\gamma'}{dt}\right|^{-1} \frac{D\gamma'}{dt}(t).$$

Then, from (2.10) and (7.1) we have

$$\frac{De_1}{dt} = \left|\frac{D\gamma'}{dt}\right| e_2(t), \tag{7.12}$$

$$\frac{De_2}{dt} = \left|\frac{D\gamma'}{dt}\right| e_1(t). \tag{7.13}$$

In particular, if $\left|\frac{D\gamma'}{dt}\right|^2 = a^2$, with a constant, the observer γ obeys a uniformly accelerated motion [39] and these equations define a Lorentzian circle [56], [67].

Conversely, assume this system holds true for an observer γ. Then, equation (7.1) also is satisfied. In other words, a UD observer may be seen as a unit timelike curve with (first) curvature $\left|\frac{D\gamma'}{dt}\right|$ and identically zero torsion and the rest of curvatures. From the reduction of codimension Erbacher theorem (see [45]), we conclude that if the spacetime has constant sectional curvature, then a UD observer is contained in a 2-dimensional totally geodesic Lorentzian submanifold.

We next characterize a piecewise UD observer from the point of view of its development curve [59, Sect. III.4]. We will say that a curve in an affine space is *piecewise planar* if its torsion, whenever is defined, is zero. Thus, inspired from [59, Prop. III.6.2], we get,

Proposition 7.2.1. *An observer $\gamma : I \longrightarrow M$ obeys a piecewise UD motion if and only if its development $\overline{\gamma}$ in the tangent space $T_{\gamma(t_0)}M$ is a piecewise planar curve for any $t_0 \in I$.*

Proof. Put

$$\overline{X}(t) = P^{\gamma}_{t,0}(\gamma'(t)).$$

where $P^{\gamma}_{0,t}$ is the Levi-Civita parallel displacement of tangent vectors along γ from $\gamma(t)$ to $p = \gamma(0)$. Recall that the development $\overline{\gamma}$ is the unique curve in T_pM starting in the origin of T_pM such that its tangent vector $\overline{X}(t)$ is parallel to $\gamma'(t)$ (in the usual sense).

By simplicity of notation, we suppose that $t_0 = 0$. First, we assume that γ is an UD observer and let $\overline{\gamma}(t)$ its development. Since $P^{\gamma}_{t,0} : T_{\gamma(t)}M \longrightarrow T_pM$ is a linear isometry, we have

$$\frac{d\overline{X}(t)}{dt} = \lim_{h\to 0}\frac{\overline{X}(t+h) - X^*(t)}{h} = \lim_{h\to 0}\frac{\overline{X}(t+h) - \overline{X}(t)}{h} =$$

$$= P^{\gamma}_{t,0}\left(\lim_{h\to 0}\frac{P^{\gamma}_{t+h,t}X(t+h) - X(t)}{h}\right) = P^{\gamma}_{t,0}\big(\frac{D\gamma'}{dt}\big).$$

Making use of this identity, (7.12) implies that $\left|\dfrac{D\gamma'}{dt}\right|(t)$ is the first curvature of the development. Thus, at any instant where the acceleration of γ does not vanish, the normal vector of $\overline{\gamma}$ is $\overline{Y}(t) = P^{\gamma}_{t,0}\big(U(t)\big)$ and therefore

$$\frac{d\overline{Y}}{dt}(t) = P^{\gamma}_{t,0}\big(\frac{DU}{dt}(t)\big).$$

From (7.13), we deduce that the torsion of $\overline{\gamma}$ is zero. Therefore, γ is a piecewise planar curve.

Conversely, assume the development $\overline{\gamma}$ is a planar curve in the tangent of a point p. Then

$$\frac{d\overline{X}}{dt}(t) = \left|\frac{D\gamma'}{dt}\right|(t)\,\overline{Y}(t) \qquad \text{and} \qquad \frac{d\overline{Y}}{dt}(t) = \left|\frac{D\gamma'}{dt}\right|(t)\,\overline{X}(t),$$

are satisfied. Since $P^{\gamma}_{t,0}$ is an isometry between $T_{\gamma(t)}M$ and T_pM, from these equations we obtain (7.12) and (7.13).

□

The previous results can be summarized as follows,

Proposition 7.2.2. *For any observer $\gamma : I \longrightarrow M$ with nowhere zero acceleration the following assertions are equivalent:*

(a) *γ is a piecewise UD observer.*

(b) *γ is a solution of third-order differential equation* (7.4)*.*

(c) *The development of γ is a piecewise planar curve in the tangent space of every point.*

(d) *γ has all the curvatures equal to zero except (possibly) the first one.*

(e) *γ, viewed as an isometric immersion from $(I, -dt^2)$ to M, is (totally umbilical) with parallel normalized mean curvature vector whenever it is defined.*

7.3 Completeness of the inextensible UD trajectories in spacetimes with some symmetries

This section is devoted to the study of the completeness of the inextensible solutions of equation (7.6), i.e., the UD equation with never zero prescribed acceleration. Here we assume the prescription function a is smooth, positive and defined on $\mathbb{R}^+$.

First of all, we are going to relate the solutions of equation (7.6) with the integral curves of a certain vector field on a Stiefel type bundle on M (compare with [59, p. 6]).

Given a Lorentzian linear space E, denote by $V_{n,2}(E)$ the (n,2)-Stiefel manifold over E, defined by

$$V_{n,2}(E) = \big\{(v, w) \in E \times E \;:\; |v|^2 = -1, \;\; \langle v, w\rangle = 0\big\}.$$

The (n,2)-Stiefel bundle over the spacetime M is then defined as follows,

$$V_{n,2}(M) = \bigcup_{\mathrm{p}\in\mathrm{M}} \{p\} \times V_{n,2}(T_pM).$$

Note that $V_{n,2}(M)$ is a bundle on M with dimension 3n-2 and fiber diffeomorphic to the tangent fiber bundle of the $(n-1)$-dimensional hyperbolic space.

Now we construct a vector field $G \in \mathfrak{X}\big(V_{n,2}(M)\big)$, which is the key tool in the study of completeness that follows,

Lemma 7.3.1. *Let $\sigma : I \longrightarrow M$ be a curve satisfying the following initial value problem*

$$\frac{D^2\sigma'}{dt^2} = \left[\frac{a'(t)}{a(t)} - \Big\langle \sigma', \frac{D\sigma'}{dt}\Big\rangle\right] \frac{D\sigma'}{dt} + a^2(t)\,\sigma'. \tag{7.14}$$
$$\sigma(0) = p, \quad \sigma'(0) = v, \quad \frac{D\sigma'}{dt}(0) = w,$$

where v is a future pointing unit timelike tangent vector and w is orthogonal to v. Then σ is an observer, and $\left|\dfrac{D\sigma'(t)}{dt}\right|^2 = a^2(t)$ holds everywhere on I.

Proof. Multiplying (7.14) by σ' and $\dfrac{D\sigma'}{dt}$ we obtain two ordinary differential equations which, after easy computations, can written as follows,

$$
\begin{aligned}
\frac{1}{2}\, x''(t) - y(t) &= a^2(t)x(t) + \frac{1}{2}\left[\frac{a'(t)}{a(t)} - \frac{1}{2}x'(t)\right] x'(t), \\
\frac{1}{2}\, y'(t) &= \frac{1}{2}\left(a^2(t) - y(t)\right) x'(t) + \frac{a'(t)}{a(t)}\, y(t),
\end{aligned}
$$

where $x(t) := |\sigma\,'(t)|^2$ and $y(t) := \left|\dfrac{D\sigma\,'(t)}{dt}\right|^2$. From the assumption, we know that $x(t)$ and $y(t)$ satisfy the initial conditions

$$
\begin{aligned}
x(0) &= -1, \\
x'(0) &= 2\left\langle \sigma\,'(0)\,, \frac{D\sigma\,'}{dt}(0)\right\rangle = 2\langle v, w\rangle = 0, \\
y(0) &= |w|^2 = a^2(0).
\end{aligned}
$$

Since $x(t) = -1$ and $y(t) = a^2(t)$ are solutions of the previous initial value problem, the result is a direct consequence of the existence and uniqueness of solutions to second order differential equations. □

Observe that solutions of (7.6) under the initial conditions (7.8) are obviously solutions of the problem (7.14). In the previous result we have proved that the converse is true. The advantage now is that (7.14) is a real initial value problem.

Now, we are in a position to construct the announced vector field G. Let (p, v, w) be a point of $V_{n,2}(M)$, and $f \in C^\infty\big(V_{n,2}(M)\big)$. Let σ be the unique inextensible curve solution of (7.14) satisfying the initial conditions

$$
\sigma(0) = p, \quad \sigma\,'(0) = v, \quad \frac{D\sigma\,'}{dt}(0) = w,
$$

for $(p, v, w) \in V_{n,2}(M)$. Define

$$
G_{(p,v,w)}(f) := \frac{d}{dt}\Big|_{t=0} f\Big(\sigma(t), \sigma\,'(t), \frac{D\sigma\,'}{dt}(t)\Big).
$$

From Lemma 7.3.1, we know $\left(\sigma(t), \sigma'(t), \dfrac{D\sigma'}{dt}(t)\right) \in V_{n,2}(M)$ and the following result easily follows,

Lemma 7.3.2. *There exists a unique vector field G on $V_{n,2}(M)$ such that its integral curves are $t \longmapsto \left(\gamma(t), \gamma'(t), \dfrac{D\gamma'}{dt}(t)\right)$, where γ is a solution γ of equation* (7.6)*.*

Once defined G, we will look for assumptions which assert its completeness (as a vector field). Recall that an integral curve α of a vector field defined on some interval $[0, b)$, $b < +\infty$, can be extended to b (as an integral curve) if and only if there exists a sequence $\{t_n\}_n$, $t_n \nearrow b$, such that $\{\alpha(t_n)\}_n$ converges (see for instance [68, Lemma 1.56]). The following technical result directly follows from this fact and Lemma 7.3.2,

Lemma 7.3.3. *Let $\gamma : [0, b) \longrightarrow M$ be a solution of equation* (7.6) *with $0 < b < \infty$. The curve γ can be extended to b as a solution of* (7.14) *if and only if there exists a sequence $\{t_n\}_n$, $t_n \nearrow b$ such that $\left\{\gamma(t_n), \gamma'(t_n), \dfrac{D\gamma'}{dt}(t_n)\right\}_n$ is convergent in $V_{n,2}(M)$.*

Although, from Lemma 7.3.1, we have $|\gamma'(t)|^2 = -1$, this is not enough to apply Lemma 7.3.3 even in the geometrically relevant case of M compact. It is necessary to ensure that the image of the curve in $V_{n,2}(M)$, associated to a UD observer γ, is contained in a compact subset.

Lemma 7.3.4. *Let M be a spacetime and let Q be a unit timelike vector field on M. If $\gamma : I \longrightarrow M$ is a solution of* (7.6) *such that $\gamma(I)$ lies in a compact subset of M and $\langle Q, \gamma' \rangle$ is bounded on I, then the image of $t \longmapsto \left(\gamma(t), \gamma'(t), \dfrac{D\gamma'}{dt}\right)$ is contained in a compact subset of $V_{n,2}(M)$.*

Proof. Consider the 1-form Q^b metrically equivalent to Q. Now we can construct on M a Riemannian metric $g_R := \langle \, , \, \rangle + 2\, Q^b \otimes Q^b$. We have,

$$g_R(\gamma\,', \gamma\,') = \langle \gamma\,', \gamma\,' \rangle + 2\, \langle Q, \gamma\,' \rangle^2,$$

which, from the assumptions, is bounded on I. Hence, taking into account that

$$\Big| \frac{D\gamma\,'}{dt} \Big|^2 \leq \max_{t \in [0,b]} a^2(t),$$

there exists a constant $c > 0$ such that

$$\Big(\gamma, \gamma\,', \frac{D\gamma\,'}{dt} \Big)(I) \subset C, \qquad C := \big\{ (p, v, w) \in V_{n,2}(M) \, : \, p \in C_1, \quad g_R(v, v) \leq c \big\},$$

where C_1 is a compact set on M such that $\gamma(I) \subset C_1$. Hence, C is a compact in $V_{n,2}(M)$.

$\square$

Now, we are in a position to state the following extensibility result (compare with [24, Th. 1] and [23, Th. 1]),

Theorem 7.3.5. *Let M be a spacetime which admits a timelike conformal and closed vector field K. Suppose that* $\mathrm{Inf}_M \sqrt{-\langle K, K \rangle} > 0$ *and a positive prescription function defined on* $\mathbb{R}^+$ *is given. Then, each solution* $\gamma : I \longrightarrow M$ *of* (7.6) *such that* $\gamma(I)$ *lies in a compact subset of* M *can be extended.*

Proof. Let $I = [0, b)$, $0 < b < +\infty$, be the domain of a solution γ of equation (7.6). From (6.6), it follows

$$\frac{d^2}{dt^2} \langle K, \gamma\,' \rangle = \Big\langle \frac{DK}{dt}, \frac{D\gamma\,'}{dt} \Big\rangle + \Big\langle K, \frac{D^2\gamma\,'}{dt^2} \Big\rangle - \frac{d}{dt}(h \circ \gamma).$$

Now, note that the first right term vanishes because K is conformal and closed,

$$\left\langle \frac{DK}{dt}, \frac{D\gamma'}{dt} \right\rangle = h(\gamma) \left\langle \gamma', \frac{D\gamma'}{dt} \right\rangle = 0.$$

On the other hand, for the second right term we have,

$$\begin{aligned} \left\langle K, \frac{D^2\gamma'}{dt^2} \right\rangle &= a^2(t)\langle K, \gamma' \rangle + \frac{a'(t)}{a(t)} \left\langle \frac{D\gamma'}{dt}, K \right\rangle \\ &= a^2(t)\langle K, \gamma' \rangle + \frac{a'(t)}{a(t)} \left(\frac{d}{dt} \langle K, \gamma' \rangle + h \circ \gamma \right). \end{aligned}$$

Thus, the function $t \mapsto \langle K, \gamma' \rangle$ satisfies the following differential equation,

$$\frac{d^2}{dt^2}\langle K, \gamma' \rangle - \frac{a'(t)}{a(t)} \frac{d}{dt}\langle K, \gamma' \rangle - a^2(t)\langle K, \gamma' \rangle = \frac{a'(t)}{a(t)}(h \circ \gamma) - (h \circ \gamma)'(t). \tag{7.15}$$

Using now that $\gamma(I)$ is contained in a compact of M, the function $h \circ \gamma$ is bounded on I. Moreover, since I is assumed bounded, from (7.15) we have a constant $c_1 > 0$ such that

$$|\langle K, \gamma' \rangle| \leq c_1. \tag{7.16}$$

Now, if we put $Q := \dfrac{K}{|K|}$, where $|K|^2 = -\langle K, K \rangle > 0$, then Q is a unit timelike vector field. Now, from (7.16) we obtain,

$$|\langle Q, \gamma' \rangle| \leq m\, c_1 \qquad \text{on} \quad I,$$

where $m = \mathrm{Sup}_M |K|^{-1} < \infty$. The proof ends making use of Lemmas 7.3.3 and 7.3.4. □

Remark 7.3.6. *The previous result gives the following result of mathematical interest: Let M be a compact spacetime which admits a timelike conformal and closed vector field K. Then, each inextensible solution of* (7.6) *must be complete.*

Example 7.3.7. *Let $f \in C^\infty(\mathbb{R})$ be a periodic positive function and let (N, g) be a* *compact Riemannian manifold. Consider the Generalized Robertson-Walker spacetime $\mathbb{R} \times_f N$, i.e., the warped product with base $(\mathbb{R}, -dt^2)$, fiber (N, g) and warping function f. The Lorentzian manifold $\mathbb{R} \times_f N$ is a Lorentzian covering of the compact spacetime $\mathbb{S}^1 \times_{\widetilde{f}} N$, where $\widetilde{f}$ denotes the induced function from f on $\mathbb{S}^1$. The result in [76] may be applied to $\mathbb{S}^1 \times_{\widetilde{f}} N$ with $K = \widetilde{f}Q$, which is timelike, conformal and closed, where Q is the vector field on $\mathbb{S}^1$ naturally induced from ∂_θ. As a practical application of Theorem 7.3.5, we get that any inextensible UD observer with prescribed acceleration $a : \mathbb{R} \to \mathbb{R}^+$ in the spacetime $\mathbb{R} \times_f N$ must be complete.*

7.4 Completeness of the inextensible UD trajectories in a Plane Wave spacetime

Let us consider a spacetime M which admits a global chart with coordinates $\big(x_1, \cdots, x_n\big)$. In these coordinates, we can write equation (7.11) as follows,

$$
\begin{aligned}
\gamma_k'(t) = &\quad \cosh\big(V(t)\big)\, L_k(t) + \sinh\big(V(t)\big)\, M_k(t),\\
L_k'(t) = &\quad -\textstyle\sum_{i,j}\left[\Gamma_{ij}^k \cosh\big(V(t)\big)\, L_i(t)\, L_j(t) + \Gamma_{ij}^k \sinh\big(V(t)\big)\, M_i(t)\, L_j(t)\right], \qquad (7.17)\\
M_k'(t) = &\quad -\textstyle\sum_{i,j}\left[\Gamma_{ij}^k \sinh\big(V(t)\big)\, M_i(t)\, M_j(t) + \Gamma_{ij}^k \cosh\big(V(t)\big)\, M_i(t)\, L_j(t)\right],\\
&\gamma_k(0) = p_k, \quad L_k(0) = v_k \quad M_k(0) = w_k.
\end{aligned}
$$

Here, v_k and w_k are the coordinates of the vectors v and w respectively, and satisfy

$$\sum_{i,j} v_i v_j g_{ij}(0) = -1, \qquad \text{and} \qquad \sum_{i,j} v_i w_j g_{ij}(0) = 0,$$

being $g_{ij}(0)$ the coefficients of the metric in the point $\gamma(0)$ in these coordinates. Moreover, all the Christoffel symbols are evaluated on γ, i.e., $\Gamma_{ij}^k(t) := \Gamma_{ij}^k(\gamma(t))$.

Now, let us consider a Plane Wave spacetime $(\mathbb{R}^4, g)$ (see 2.1.3), and a UD observer $\gamma : I \to \mathbb{R}^4$ satisfying the initial conditions as previously,

$$\gamma(0) = p, \quad \gamma'(0) = v, \quad \frac{D\gamma'}{dt}(0) = a(0)\, w.$$

Our objective is to prove that such trajectory is extensible to the whole real line, i.e., that the maximal interval of definition of γ is $I = \mathbb{R}$. Making use of Proposition 7.1.1, we can write

$$\gamma'(t) = \cosh\big(V(t)\big)\, L(t) + \sinh\big(V(t)\big)\, M(t),$$

where $L, M : I \to \mathbb{R}^4$ are solutions of system (7.17) with initial conditions $L(0) = v$ and $M(0) = a(0)w$. Denote by (L_1, L_2, L_3, L_4) and (M_1, M_2, M_3, M_4) the respective coordinates of L and M. We have the following simple but important fact,

Lemma 7.4.1. *The first components of L and M satisfy*

$$L_1(t) = v_1, \quad M_1(t) = a(0)w_1, \quad \textit{for all} \quad t.$$

Proof. It trivially follows from (2.7) and (7.17) that $L_1' = M_1' = 0$. Therefore, L_1 and M_1 are constants and equal to the respective initial condition.

□

Of course, a direct consequence of the latter result is

$$\gamma_1'(t) = v_1 \cosh\big(V(t)\big) + a(0)w_1 \sinh\big(V(t)\big),$$

which provides with the following explicit expression for the first component of γ,

$$\gamma_1(t) = v_1 \int_0^t \cosh\big(V(s)\big) ds + a(0)w_1 \int_0^t \sinh\big(V(t)\big) ds + p_1. \tag{7.18}$$

Lemma 7.4.2. *The functions L_3, M_3, L_4, M_4 are prolongable to the whole real line as solutions of system* (7.17).

Proof. A first observation is that from (2.9), the equations from (7.17) for $k = 3, 4$ are

$$\begin{aligned} L_k'(t) = & \quad -\Gamma_{11}^k(\gamma(t)) \left[\cosh\big(V(t)\big)\, L_1(t)^2 + \sinh\big(V(t)\big)\, M_1(t)\, L_1(t)\right], \\ M_k'(t) = & \quad -\Gamma_{11}^k(\gamma(t)) \left[\sinh\big(V(t)\big)\, M_1(t)^2 + \cosh\big(V(t)\big)\, M_1(t)\, L_1(t)\right], \end{aligned}$$

and as a consequence of Lemma 7.4.1,

$$\begin{aligned} L_k'(t) = & \quad -\Gamma_{11}^k(\gamma(t)) \left[v_1^2 \cosh\big(V(t)\big) + a(0) v_1 w_1 \sinh\big(V(t)\big)\right], \\ M_k'(t) = & \quad -\Gamma_{11}^k(\gamma(t)) \left[a(0)^2 w_1^2 \sinh\big(V(t)\big) + a(0) v_1 w_1 \cosh\big(V(t)\big)\right], \end{aligned}$$

for $k = 3, 4$. To simplify the writing, we define the functions

$$\begin{aligned} f(t) = & \quad v_1^2 \cosh\big(V(t)\big) + a(0) v_1 w_1 \sinh\big(V(t)\big), \\ g(t) = & \quad a(0)^2 w_1^2 \sinh\big(V(t)\big) + a(0) v_1 w_1 \cosh\big(V(t)\big). \end{aligned}$$

Thus,

$$\begin{aligned} L_k'(t) = & \quad -f(t)\Gamma_{11}^k(\gamma(t)), \\ M_k'(t) = & \quad -g(t)\Gamma_{11}^k(\gamma(t)). \end{aligned} \tag{7.19}$$

Considering that H is defined by (2.6), we have

$$\Gamma_{11}^3(\gamma) = -\frac{1}{2}\frac{\partial H}{\partial x}(\gamma(t)) = 2A(\gamma_1)\gamma_3 + C(\gamma_1)\gamma_4 + D(\gamma_1),$$

and

$$\Gamma_{11}^4(\gamma) = -\frac{\partial H}{\partial y}(\gamma(t)) = 2B(\gamma_1)\gamma_4 + C(\gamma_1)\gamma_3 + D(\gamma_1),$$

where $\gamma_1(t)$ is explicitly given by (7.18). Since

$$\gamma_k(t) = \int_0^t \left[\cosh\big(V(s)\big)\, L_3(s) + \sinh\big(V(s)\big)\, M_k(s)\right] ds + p_k,$$

then system (7.19) (with $k = 3, 4$) can be seen as a integro-differential system of four equations. To pass to a standard system of differential equations, we define the new variables

$$\mathcal{L}_k(t) = \int_0^t \cosh\big(V(s)\big)\, L_3(s)ds, \qquad \mathcal{M}_k(t) = \int_0^t \sinh\big(V(s)\big)\, M_k(s)ds.$$

For the new variables,

$$\begin{aligned}
\mathcal{L}_k'' &= a(t)\tanh\big(V(s)\big)\mathcal{L}_k' - f(t)\cosh\big(V(s)\big)\Gamma^k_{11}(\gamma(t)) \\
\mathcal{M}_k'' &= a(t)\tanh\big(V(s)\big)\mathcal{M}_k' - g(t)\sinh\big(V(s)\big)\Gamma^k_{11}(\gamma(t))
\end{aligned}$$

and introducing the specific formulas for $\Gamma^3_{11}, \Gamma^4_{11}$ computed before, we arrive to

$$\begin{aligned}
\mathcal{L}_3'' = {} & a(t)\tanh\big(V(t)\big)\mathcal{L}_3' + \tfrac{1}{2}f(t)\cosh\big(V(t)\big)\left[2A(\gamma_1)\left[\mathcal{L}_3 + \mathcal{M}_3 + p_3\right]\right. \\
& \left. + C(\gamma_1)\left[\mathcal{L}_4 + \mathcal{M}_4 + p_4\right] + D(\gamma_1)\right] \\
\mathcal{M}_3'' = {} & a(t)\coth\big(V(t)\big)\mathcal{M}_k' + \tfrac{1}{2}g(t)\sinh\big(V(t)\big)\left[2A(\gamma_1)\left[\mathcal{L}_3 + \mathcal{M}_3 + p_3\right]\right. \\
& \left. + C(\gamma_1)\left[\mathcal{L}_4 + \mathcal{M}_4 + p_4\right] + D(\gamma_1)\right] \\
\mathcal{L}_4'' = {} & a(t)\tanh\big(V(t)\big)\mathcal{L}_4' + \tfrac{1}{2}f(t)\cosh\big(V(t)\big)\left[2B(\gamma_1)\left[\mathcal{L}_4 + \mathcal{M}_4 + p_4\right]\right. \\
& \left. + C(\gamma_1)\left[\mathcal{L}_3 + \mathcal{M}_3 + p_3\right] + E(\gamma_1)\right] \\
\mathcal{M}_4'' = {} & a(t)\coth\big(V(t)\big)\mathcal{M}_4' + \tfrac{1}{2}g(t)\sinh\big(V(t)\big)\left[2B(\gamma_1)\left[\mathcal{L}_4 + \mathcal{M}_4 + p_4\right]\right. \\
& \left. + C(\gamma_1)\left[\mathcal{L}_3 + \mathcal{M}_3 + p_3\right] + E(\gamma_1)\right]
\end{aligned}$$

with $\gamma_1(t)$ given by (7.18). This is a linear system of second order differential equations on the involved variables, and can be easily transformed into a first order system $x' = A(t)x$ of order 8. Now, the basic theory of linear systems states that every solution has the whole real line as a maximal interval of definition, closing the proof.

$\square$

Up to now, we have that L_1, L_3, L_4 (resp. M_1, M_3, M_4) are defined on the whole $\mathbb{R}$. It remains to prove the completeness of $L_3(t)$ (resp. $M_3(t)$). The equations (7.17) for L_2 is

$$L_2'(t) = -\sum_{i,j} \left[\Gamma_{ij}^2 \cosh\big(V(t)\big)\, L_i(t)\, L_j(t) + \Gamma_{ij}^2 \sinh\big(V(t)\big)\, M_i(t)\, L_j(t)\right]$$

but note that $\Gamma_{ij}^2 = 0$ if $i = 2$ or $j = 2$, and moreover H does not depend on the second variable. This implies that the right-hand side part of the latter equation depends on functions $L_k(t), M_k(t)$ (k=1,3,4), which we have proved that are globally defined, but not on L_2, M_2. Thus, $L_2'(t)$ is defined for every t, and a simple integration leads to the conclusion. An analogous argument serves for $M_2(t)$.

Previous results are picked up in the following theorem.

Theorem 7.4.3. *Let M be a Plane Wave spacetime and $a : \mathbb{R} \longrightarrow M$ a positive smooth function. Every inextensible UD trajectory with prescribed acceleration a is complete.*

Chapter 8

Uniform circular motion in General Relativity

8.1 The relativistic uniformly circular motion

Firstly, we expose the important concept of 'planar motion' to make precise when an observer intuitively considers that it is moving along a plane. Physically, an observer will say that its motion is planar when the small ball of its accelerometer moves along a constant plane. In the mathematical translation of this intuitive idea, the main difficulty lies in what is the meaning of a 'constant plane' relative to the observer. For this purpose we will use again the Fermi-Walker connection.

Definition 4. *An observer $\gamma : I \longrightarrow M$ obeys a* planar motion *if for some $t_0 \in I$, there exists an observable plane $\Pi_{t_0} \subset R_{t_0} \subset T_{\gamma(t_0)}M$, such that*

$$\widehat{P}^{\gamma}_{t,t_0}\left(\frac{D\gamma'}{dt}(t)\right) \in \Pi_{t_0} \tag{8.1}$$

for any $t \in I$.

Intuitively, $\frac{D\gamma'}{dt}$ corresponds with the displacement of the small ball of the accelerometer, and $\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)$ may be seen as the velocity of the ball. Whenever both vectors are linearly independent, both directions define the observable 2-plane Π_{t_0}.

As a direct consequence of the definition, from the equality

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t_0) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\widehat{P}^{\gamma}_{t_0+\varepsilon,t_0}\left(\frac{D\gamma'}{dt}(t_0+\varepsilon)\right) - \frac{D\gamma'}{dt}(t_0)\right],$$

the vector $\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t_0)$ is also in Π_{t_0}. In fact, if γ is not in an unchanged direction motion at a neighbourhood of the instant t_0, [40], the plane Π_{t_0} is generated by the proper acceleration of the observer and the variation which it measures, i.e.,

$$\Pi_{t_0} = \operatorname{span}\left\{\frac{D\gamma}{dt}(t_0)\,,\ \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t_0)\right\}.$$

In this case, we may define the following family of 2-planes along γ

$$\Pi_t := \operatorname{span}\left\{\widehat{P}^{\gamma}_{t_0,t}\left(\frac{D\gamma}{dt}(t_0)\right)\,,\ \widehat{P}^{\gamma}_{t_0,t}\left(\frac{\widehat{D}}{dt}\left(\frac{D\gamma}{dt}\right)(t_0)\right)\right\} \subset R_{t_0}. \tag{8.2}$$

Observe that this family of planes is Fermi-Walker parallel in the sense of the following definition,

Definition 5. *Given an observer $\gamma : I \longrightarrow M$ in the spacetime M, a family of planes along γ, $\{\Pi_t\}_{t\in I}$, is said to be Fermi-Walker parallel if for any $t_1, t_2 \in I$ and for any vector $v \in \Pi_{t_1}$, the following relation holds*

$$\widehat{P}^{\gamma}_{t_1,t_2}(v) \in \Pi_{t_2}\,.$$

In addition, the previous family of planes (8.2) satisfies the following property,

Lemma 8.1.1. *For any* $t, t_1 \in I$, *we have*

$$\widehat{P}^{\gamma}_{t,t_1}\left(\frac{D\gamma'}{dt}(t)\right) \in \Pi_{t_1}.$$

Proof. Taking the inverse mapping of $\widehat{P}^{\gamma}_{t,t_0}$ in (8.1), we have that there exist $a, b \in \mathbb{R}$ such that

$$\frac{D\gamma}{dt}(t) = a\,\widehat{P}^{\gamma}_{t_0,t}\left(\frac{D\gamma}{dt}(t_0)\right) + b\,\widehat{P}^{\gamma}_{t_0,t}\left(\frac{\widehat{D}}{dt}\left(\frac{D\gamma}{dt}\right)(t_0)\right).$$

Now, the desired relation follows by taking $\widehat{P}^{\gamma}_{t,t_1}$ in both members of the previous equality. $\square$

Note that the family $\{\Pi_t\}_{t\in I}$ satisfies the previous property, but it is not unique in general (a generically planar motion may be a free falling motion from some instant). However, if the observer γ is not an unchanged direction observer at every i nstant [40], i.e., if $\{\frac{D\gamma}{dt}(t), \frac{\widehat{D}}{dt}\left(\frac{D\gamma}{dt}\right)(t)\}$ are linearly independent for any $t \in I$, then the only family of 2-planes satisfying Lemma 8.1.1 is $\{\Pi_t\}_{t\in I}$.

Now we will introduce a uniform circular (UC) motion as a very particular case of planar motion. Intuitively, a UC observer will see that the small ball of its accelerometer is rotating with constant angular velocity, describing a circular trajectory. Hence, the velocity of the ball for the observer, $\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)$, will have a constant modulus. Motivated by these intuitive ideas, we are in a position to give an accurately definition.

Definition 6. *An observer* $\gamma : I \longrightarrow M$ *which satisfies a planar motion is said to* *obey a UC motion if*

$$\left|\frac{D\gamma'}{dt}\right|^2 = a^2 \qquad \text{and} \qquad \left|\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)\right|^2 = a^2 w^2, \tag{8.3}$$

where the constants a, w *satisfy* $a, w > 0$ *and* $a < w$.

Here a is the modulus of the acceleration, and w corresponds with the angular velocity which the observer perceives. Therefore, taking into account the classical relation between the radius R, the angular velocity w and the centripetal acceleration a on a circular motion,

$$a = w^2 R,$$

a UC observer will measure a uniform rotation with frequency $\frac{w}{2\pi}$ and 'radius' equal to $R := a/w^2$. We point out that the quantity R does not represent a real observable distance in general. It is only the radius of the trajectory which the UC observer assumes, using the classical intuition, from the evolution of its acceleration. Moreover, the velocity of the ball of the accelerometer measured by the UC observer equals to $wR = \frac{a}{w}$, and this quantity has to satisfy $\frac{a}{w} < 1$, where 1 the (normalized)light speed. This comment justifies the assumptions on the observable quantities a and w.

Remark 8.1.2. *Note that if* $a = 0$, *we recover the definition of a free falling observer. Moreover, when* $a > 0$ *and* $w = 0$, *we retrieve the definition of a uniform accelerated observer [39]. Observe that, in this case, the circular trajectory measured by the observer has a infinite 'radius', i.e., the observer obeys a rectilinear motion. From now on, we will only consider the proper case in which* a *and* w *are strictly positive, i.e., when the observer obeys a strict UC motion.*

Naturally, in order to determine a UC observer trajectory, it is necessary to know the initial observable 2-plane, the initial spin sense and the initial values of the position, 4-velocity and proper acceleration. In an $n(\geq 3)$-dimensional spacetime, the initial 2-plane can be determined by means of $n-3$ observable directions $u_4, \cdots, u_n \in$

R_{t_0}, orthogonal to the initial acceleration $\frac{D\gamma'}{dt}(0)$. So, the vector $\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0)$ will point towards the unique observable direction which is orthogonal to $\frac{D\gamma'}{dt}(0)$ and $u_4, \cdots, u_n$. From (8.3), the modulus of the vector $\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0)$ is also known, and it is equal to aw. However, the initial spin sense is needed to determine the sense of that vector.

The initial plane Π_0 is given by

$$\Pi_0 = \operatorname{span}\left\{\frac{D\gamma}{dt}(0), \frac{\widehat{D}}{dt}\left(\frac{D\gamma}{dt}\right)(0)\right\}.$$

We consider the following unit vectors, related with the initial values of the problem,

$$u_1 = \gamma'(0), \quad u_2 = \frac{1}{a}\frac{D\gamma'}{dt}(0), \quad u_3 = \frac{1}{aw}\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0),$$

and, let us denote by

$$e_4(t), \cdots, e_n(t)$$

the Fermi-Walker parallel vector fields along γ satisfying the initial conditions $e_i(0) = u_i$, for each $4 \leq i \leq n$. Now, the family of Fermi-Walker parallel planes (8.2), corresponding to the UC observer γ is given by

$$\Pi_t = \operatorname{span}\{\widehat{P}^{\gamma}_{0,t}(u_2),\ \widehat{P}^{\gamma}_{0,t}(u_3)\} = \left(\operatorname{span}\{\widehat{P}^{\gamma}_{0,t}(u_1), e_4(t), \cdots, e_n(t)\}\right)^{\perp} \subset R_{t_0}.$$

Remark 8.1.3. *Note that, in the physically relevant case $n = 4$, we have*

$$e_4(t) = \frac{1}{a^2 w}\frac{D\gamma'}{dt} \times \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right),$$

where $\times$ denotes the natural cross product defined in R_t.

From the previous discussion we can state the following initial value equations for a UC observer with 'frequency' $\frac{w}{2\pi}$ and 'radius' $R = \frac{a}{w^2}$, as follows

$$\langle \gamma' , \gamma' \rangle = -1, \tag{8.4}$$

$$\left| \frac{D\gamma'}{dt} \right|^2 = a^2, \tag{8.5}$$

$$\left| \frac{\widehat{D}}{dt} \left(\frac{D\gamma'}{dt} \right) \right|^2 = a^2 w^2, \tag{8.6}$$

$$\frac{\widehat{D} e_i}{dt} = 0 \quad \text{for} \quad 4 \leq i \leq n, \tag{8.7}$$

$$\left\langle \frac{D\gamma'}{dt} , e_i \right\rangle = 0 \quad \text{for} \quad 4 \leq i \leq n, \tag{8.8}$$

under the initial conditions

$$\gamma(0) = p, \quad \gamma'(0) = u_1, \quad \frac{D\gamma'}{dt}(0) = a\, u_2, \quad \frac{\widehat{D}}{dt} \left(\frac{D\gamma'}{dt} \right)(0) = aw\, u_3, \tag{8.9}$$
$$e_i(0) = u_i \quad \text{for} \quad 4 \leq i \leq n,$$

where, p is an event in the n-dimensional spacetime M.

Note that (8.8) automatically implies

$$\left\langle \frac{\widehat{D}}{dt} \left(\frac{D\gamma'}{dt} \right)(t) , e_i(t) \right\rangle = 0 \quad \text{for} \quad 4 \leq i \leq n, \quad t \in I.$$

We point out that the local existence and uniqueness of this initial problem is not yet guaranteed because it is not possible write it in the normal form (therefore the classical Picard-Lindelöf theorem can not be applied). On the other hand, note that one of the initial conditions is imposed to the third derivative although the system is of third order. However, in spite of previous difficulties, we will prove in the following section that the initial problem (8.4)-(8.9) has a unique inextensible solution.

At this point, we remark that it is not easy to hand with Definition 6, although it has a clear physical meaning. Therefore, we expose in the following result and in the next section two mathematically easier characterizations.

Proposition 8.1.4. *Let M be an n-dimensional spacetime, $n \geq 3$, and let a, w be two positive constants, $w > a$. Let us consider u_1, u_2, $u_3 \in T_pM$ three orthogonal vectors such that $|u_1|^2 = -1$ and $|u_2|^2 = |u_3|^2 = 1$, and consider also $n-3$ vectors $\{u_4, \cdots, u_n\} \subset T_pM$ such that $\{u_1, u_2, u_3, u_4, \cdots, u_n\}$ is an orthonormal basis of T_pM. Then, the velocity of the only UC observer γ satisfying the initial conditions*

$$\gamma(0) = p, \quad \gamma'(0) = u_1, \quad \frac{D\gamma'}{dt}(0) = a\, u_2, \quad \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0) = aw\, u_3, \tag{8.10}$$

is given by the expression

$$\gamma'(t) = \frac{w}{\sqrt{w^2-a^2}}\, L(t) + \frac{a}{\sqrt{w^2-a^2}} \left[\cos\left(\sqrt{w^2-a^2}\, t\right) M(t) + \sin\left(\sqrt{w^2-a^2}\, t\right) N(t)\right], \tag{8.11}$$

being L, M, N three unit (Levi-Civita) parallel vector fields along γ satisfying

$$L(0) = \frac{1}{\sqrt{w^2-a^2}}(wu_1 + au_3),$$

$$M(0) = \frac{-1}{\sqrt{w^2-a^2}}(au_1 + wu_3),$$

$$N(0) = u_2,$$

$$\langle L(0)\,, u_i\rangle = \langle M(0)\,, u_i\rangle = \langle N(0)\,, u_i\rangle = 0 \quad \text{for} \quad 4 \leq i \leq n.$$

Proof. First, we check that if γ verifies (8.11) then it is a solution of (8.4)-(8.8).

From assumptions on L, M and N, a direct computation gives us

$$\frac{D\gamma'}{dt}(t) = a\left(-\sin\left(\sqrt{w^2-a^2}\,t\right)M(t) + \cos\left(\sqrt{w^2-a^2}\,t\right)N(t)\right).$$

Then, by using the orthogonality of M and N, equation (8.5) is automatically satisfied. Analogously, after easy computations we get

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right) = -\frac{a^2 w}{\sqrt{w^2-a^2}}L - \frac{a w^2}{\sqrt{w^2-a^2}}\left(\cos\left(\sqrt{w^2-a^2}\,t\right)M + \sin\left(\sqrt{w^2-a^2}\,t\right)N\right),$$

and equation (8.6) holds. The equations (8.7) and (8.8) are satisfied because of assumptions on L, M and N and their initial relations with the vectors $u_4, \cdots, u_n$. The initial conditions are straightforward satisfied.

In order to prove the converse, we only must take into account the uniqueness of inextensible solutions of initial value problem (8.4)-(8.8). □

Now, by using the Levi-Civita parallel transport, we can express (8.11) as the following first order integro-differential equation,

$$\gamma'(t) = \frac{1}{\sqrt{w^2-a^2}}\left[w\,P^{\gamma}_{0,t}(v_1) + a\,\cos\left(\sqrt{w^2-a^2}\,t\right)\,P^{\gamma}_{0,t}(v_2) + a\,\sin\left(\sqrt{w^2-a^2}\,t\right)\,P^{\gamma}_{0,t}(u_2)\right],$$
$$|u_1|^2 = -1, \qquad |u_2|^2 = |u_3|^2 = 1, \qquad \langle u_1,\,u_2\rangle = \langle u_1,\,u_3\rangle = \langle u_2,\,u_3\rangle = 0.$$

being $v_1 = \dfrac{wu_1 + au_3}{\sqrt{w^2-a^2}}$ and $v_2 = -\dfrac{au_1 + wu_3}{\sqrt{w^2-a^2}}$.

These result will be used in the analytical study of the completeness of inextensible trajectories which we will deal in Section 8.4.

Example 8.1.5. *Consider the 3-dimensional Minkowski spacetime $M = \mathbb{L}^3$, endowed with its standard coordinate system (t, x, y). Note that in $\mathbb{L}^3$ every motion must be planar. Consider the UC observer with frequency $\frac{w}{2\pi}$ and 'radius' equal to $\frac{a}{w^2}$ and satisfying the initial conditions*

$$\gamma(0) = (0,\, \frac{a}{w^2},\, 0), \quad \gamma\,'(0) = \big(\frac{w}{\sqrt{w^2-a^2}},\, 0,\, \frac{a}{\sqrt{w^2-a^2}}\big),$$

$$\frac{D\gamma\,'}{d\tau}(0) = (0,\, -a,\, 0), \quad \frac{\widehat{D}}{d\tau}\Big(\frac{D\gamma\,'}{d\tau}\Big)(0) = \big(\frac{a^2 w}{\sqrt{w^2-a^2}},\, 0,\, \frac{a w^2}{\sqrt{w^2-a^2}}\big),$$

is given by the expression,

$$\gamma(\tau) = \big(t(\tau),\, x(\tau),\, y(\tau)\big),$$

where

$$t(\tau) = \frac{a\tau}{\sqrt{w^2-a^2}}, \quad x(\tau) = \frac{a}{w^2-a^2}\cos\big(\sqrt{w^2-a^2}\,\tau\big), \quad y(\tau) = \frac{a}{w^2-a^2}\sin\big(\sqrt{w^2-a^2}\,\tau\big).$$

According to the trajectory obtained, we may interpret that the UC observer measures an acceleration with constant modulus equal to a, and an angular velocity w. Moreover, it measures $R = \frac{a}{w^2}$ for its radius. However, each member of the family of inertial observers which measures an initial velocity of the UC observer equal to $\frac{a}{\sqrt{w^2-a^2}}$, observes that it describes a UC motion with acceleration a but with lower angular velocity, equal to $\sqrt{w^2-a^2}$. Moreover, the radius of the observed trajectory by these inertial observers is $\frac{a}{w^2-a^2}$.

8.2 UC motion as a Lorentzian helix

In this section we analyse the UC motion from a more geometric viewpoint, with the aim of relating it with the well known notion of Lorentzian helix and, by the way, of characterizing geometrically it. First, we proceed to find the Frenet equations associated to each UC observer.

Let $\gamma : I \longrightarrow M$ be a UC observer with angular velocity w and radius $R = \frac{a}{w^2}$. We define the following three vector fields along γ, which are orthonormal from equations (8.3),

$$\begin{aligned} e_1(t) &= \gamma'(t), \\ e_2(t) &= \frac{1}{a}\frac{D\gamma'}{dt}(t), \\ e_3(t) &= \frac{1}{aw}\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t). \end{aligned}$$

Let $\{u_4, \cdots, u_n\}$ be $n-3$ vectors in $T_{\gamma(0)}M$ such that $\{e_1(0), e_2(0), e_3(0), u_4, \cdots, u_n\}$ is an orthogonal basis of $T_{\gamma(0)}M$. Now, consider the Fermi-Walker parallel vector fields along γ starting at u_i,

$$e_i(t) = \widehat{P}^{\gamma}_{0,t}(u_i), \quad \text{for} \quad 4 \leq i \leq n.$$

Since a UC motion is planar, the 2-plane Π_t, given in (8.2), is orthogonal to the subspace generated by $\{\widehat{P}^{\gamma}_{0,t}(u_i)\}_{4\leq i\leq n}$. Consequently, the vector fields $\{e_i(t)\}_{1\leq i\leq n}$ are orthonormal for every instant $t \in I$.

Now, we are in a position to obtain the Frenet equations. A direct computation gives

$$\frac{De_1}{dt} = a\, e_2.$$

On the other hand,

$$\frac{De_2}{dt} = \frac{1}{a}\left[\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right) + a^2\gamma'\right] = a\, e_1 + w\, e_3.$$

Taking into account that γ is a UC observer, we obtain

$$\left\langle \frac{De_3}{dt}, e_1 \right\rangle = \frac{1}{aw} \left\langle \frac{\widehat{D}}{dt}\left(\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)\right), \gamma' \right\rangle = 0,$$

and

$$\left\langle \frac{De_3}{dt}, e_2 \right\rangle = \frac{1}{a^2 w}\left[\frac{d}{dt}\left\langle \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right), \frac{D\gamma'}{dt}\right\rangle - \left|\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)\right|^2\right] = -w.$$

Hence, we get

$$\frac{De_3}{dt} = \frac{1}{aw}\frac{\widehat{D}}{dt}\left(\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)\right) = -w\, e_2.$$

Finally, for $4 \leq i \leq n$,

$$\frac{De_i}{dt} = \frac{\widehat{D}e_i}{dt} - \langle \gamma', e_i\rangle \frac{D\gamma'}{dt} + \langle \frac{D\gamma'}{dt}, e_i\rangle \gamma' = 0.$$

Summarizing, the Frenet equations corresponding to a UC observer are

$$\frac{De_1}{dt} = a\, e_2, \tag{8.12}$$

$$\frac{De_2}{dt} = a\, e_1 + w\, e_3, \tag{8.13}$$

$$\frac{De_3}{dt} = -w\, e_2, \tag{8.14}$$

$$\frac{De_i}{dt} = 0 \quad \text{for} \quad 4 \leq i \leq n. \tag{8.15}$$

Therefore, we get that a UC observer can be described by a Lorentzian helix, i.e., a unit timelike curve with constant curvature and torsion, and with other higher order curvatures identically zero (see for instance [56]).

Conversely, assume the Frenet system of equations (8.12)-(8.15) holds true for a curve γ with the initial conditions (8.10). The first Frenet equation gives no information. The second one can be rewritten as follows

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right) = \frac{D^2\gamma\,'}{dt^2} - a^2\,\gamma\,'.$$

From the relation between Fermi-Walker and Levi-Civita covariant derivations (2.10) we obtain

$$\left(\left|\frac{D\gamma\,'}{dt}\right|^2 - a^2\right)\,\gamma\,' = \langle\gamma\,',\frac{D\gamma'}{dt}\rangle\frac{D\gamma\,'}{dt}.$$

Multiplying this expression by $\frac{D\gamma\,'}{dt}$ we get,

$$|\gamma\,'|^2 = -1 \qquad \text{and} \qquad \left|\frac{D\gamma\,'}{dt}\right|^2 = a^2,$$

which are just the first two equations of the system (8.4)-(8.8). On the other hand, from (8.14) we obtain the fourth-order equation,

$$\frac{D}{dt}\left[\frac{D^2\gamma\,'}{dt^2} + (w^2 - a^2)\gamma\,'\right] = 0.$$

From the second Frenet equation we know $\langle\frac{D\gamma'}{dt}\,,\gamma'\rangle = 0$, and thus we conclude that $\left|\frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right)\right|^2$ is constant along γ, and, the from initial conditions (8.10), we arrive to the second equation in (8.3). Finally, from definition of $e_i(t)$ we deduce equations (8.7). As a consequence,

$$0 = \frac{\widehat{D}e_i}{dt} = \langle e_i\,,\gamma\,'\rangle\frac{D\gamma\,'}{dt} - \left\langle\frac{D\gamma\,'}{dt}\,e_i\right\rangle\gamma\,',$$

and we obtain equations (8.8).

The previous results can be summarized as follows,

Proposition 8.2.1. *An observer $\gamma : I \longrightarrow M$ obeys a UC motion with angular velocity w and radius $R = \frac{a}{w^2}$ if and only if γ is a unit timelike Lorentzian helix, i.e., a unit timelike curve with constant curvature, equal to a, and constant torsion, equal to w, while the rest of higher order curvatures are identically zero.*

Remark 8.2.2. *We point out that we arrived to Definition 6 from purely physical arguments which are mathematically expressed by means of the Fermi-Walker covariant derivative of the moving particle. As an intermediate step, we have considered planar motions (Definition 4) in a general spacetime. After describing Definition 6 in terms of the differential system (8.4)-(8.8), we get to the geometric characterization given in Proposition 8.2.1 which is just the so-called notion of constant rotational acceleration notion in [47] and [48].*

Remark 8.2.3. *We note that when the spacetime has constant sectional curvature the codimension of a UC observer can be reduced (see [45]). Therefore, each UC observer lies, in this case, in a* 3*-dimensional totally geodesic Lorentzian submanifold.*

Now, observe that the Frenet equations (8.12) constitute a fourth order differential system for γ. The corresponding initial value problem can be stated as follows,

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right) = \frac{D^2\gamma'}{dt^2} - a^2\,\gamma', \tag{8.16}$$

$$\frac{D}{dt}\left[\frac{D^2\gamma'}{dt^2} + (w^2 - a^2)\gamma'\right] = 0, \tag{8.17}$$

$$\gamma(0) = p, \quad \gamma'(0) = u_1, \quad \frac{D\gamma'}{dt}(0) = a\,u_2, \quad \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0) = aw\,u_3, \tag{8.18}$$

where $u_1, u_2, u_3 \in T_pM$ are orthonormal and satisfy $|u_1|^2 = -1$ and $|u_2|^2 = |u_3|^2 = 1$.

The existence and uniqueness of this problem are not immediate because it involves two differential equations. However, we are going to show that this initial problem is equivalent to the following one, clearly with a unique inextensible solution.

$$\frac{D}{dt}\left[\frac{D^2\gamma'}{dt^2}+\langle\gamma',\frac{D\gamma'}{dt}\rangle\frac{D\gamma'}{dt}+\left(w^2-\left|\frac{D\gamma'}{dt}\right|^2\right)\gamma'\right]=0, \tag{8.19}$$

$$\gamma(0)=p,\quad \gamma'(0)=u_1,\quad \frac{D\gamma'}{dt}(0)=a\,u_2,\quad \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0)=aw\,u_3, \tag{8.20}$$

where $u_1, u_2, u_3 \in T_pM$ satisfy the previous conditions.

Taking into account (2.10), it is clear that a solution of (8.16)-(8.18) is a solution of (8.19)-(8.20). For the converse, we only have to prove that $|\gamma'|^2$ and $\left|\frac{D\gamma'}{dt}\right|^2$ are constant along γ.

In order to do that put $x(t) := |\gamma'|^2$ and $y(t) := \left|\frac{D\gamma'}{dt}\right|^2$. Then, the initial values are written

$$x(0)=-1,\quad x'(0)=x''(0)=0,\quad y(0)=a^2,\quad y'(0)=y''(0)=0. \tag{8.21}$$

Multiplying equation (8.19) by γ' and $\frac{D\gamma'}{dt}$, we obtain respectively,

$$2x'''-\frac{1}{2}y''+\left[4x'^2+(w^2-y)x\right]-\frac{1}{2}y'-2w^2x'=0, \tag{8.22}$$

and

$$\left|\frac{D^2\gamma'}{dt^2}\right|^2=\frac{1}{2}y''+f(x',y,y'), \tag{8.23}$$

where we put $f(x',y,y')=2x''y-x'y'+w^2y-y^2$.

On the other hand, multiplying (8.19) by $\frac{D^2\gamma'}{dt^2}$ and using (8.23), we get

$$\frac{1}{4}y''' + \frac{1}{2}\left(f(x', y, y')\right)' - x''y' + x'y'' + \frac{1}{2}yy' + x'y'' + x'f(x', y, y') - \frac{w^2}{2}y' = 0. \quad (8.24)$$

So, equations (8.22) and (8.24), together with the initial conditions (8.21), have a unique solution. Since $x(t) = -1$ and $y(t) = a^2$ satisfy this initial value problem, we get the announced conclusion.

Previous results are picked up in the following result,

Proposition 8.2.4. *The three following assertions are equivalent:*

(a) *The curve γ in M is solution of* (8.4)-(8.9).

(b) *The curve γ in M is solution of* (8.16)-(8.18).

(c) *The curve γ in M is* (*the unique*) *solution of* (8.19)-(8.20).

8.3 Completeness of the inextensible UC trajectories in spacetimes with symmetries

This section is devoted to the study of the completeness of the inextensible UC observers. First of all, we are going to relate the solutions of equation (8.19) with the integral curves of a certain vector field on a Stiefel bundle type on M (compare with [59, p. 6]).

Given a Lorentzian linear space E and $a, w \in \mathbb{R}$, $w > a > 0$, denote by $V^{a,w}_{n,3}(E)$

the (n,3)-Stiefel manifold over E, defined by

$$V_{n,3}^{a,w}(E) = \big\{(v_1, v_2, v_3) \in E^3 \ : \ |v_1|^2 = -1,\ |v_2|^2 = a^2,\ |v_3|^2 = a^2w^2,\ \ \langle v_i, v_j\rangle = 0,\ i \neq j\big\}.$$

The (n,3)-Stiefel bundle over the spacetime M is then defined as follows,

$$V_{n,3}^{a,w}(M) = \bigcup_{\mathrm{p\in M}} \{p\} \times V_{n,3}^{a,w}(T_pM).$$

First we construct a vector field G on the differentiable manifold $V_{n,3}^{a,w}(M)\big)$, which is the key tool in the study of completeness. Let (p, u_1, au_2, awu_3) be a point of $V_{n,3}^{a,w}(M)$, and $f \in C^\infty\big(V_{n,3}^{a,w}(M)\big)$. Let σ be the unique inextensible curve solution of (8.19) satisfying the initial conditions

$$\sigma(0) = p, \quad \sigma'(0) = u_1, \quad \frac{D\sigma'}{dt}(0) = au_2, \quad \frac{\widehat{D}}{dt}(\frac{D\sigma'}{dt})(0) = awu_3.$$

We define

$$G_{(p,u_1,au_2,awu_3)}(f) := \frac{d}{dt}\Big|_{t=0} f\Big(\sigma(t), \sigma'(t), \frac{D\sigma'}{dt}(t), \frac{\widehat{D}}{dt}\Big(\frac{D\sigma'}{dt}\Big)(t)\Big).$$

From results of the previous section, we have

$$\left(\sigma(t), \sigma'(t), \frac{D\sigma'}{dt}(t), \frac{\widehat{D}}{dt}\left(\frac{D\sigma'}{dt}\right)(t)\right) \in V_{n,3}^{a,w}(M),$$

G is well defined and the following result follows easily,

Lemma 8.3.1. *There exists a unique vector field G on $V_{n,3}^{a,w}(M)$ such that its integral curves are $t \longmapsto \Big(\gamma(t), \gamma'(t), \frac{D\gamma'}{dt}(t), \frac{\widehat{D}}{dt}\big(\frac{D\gamma'}{dt}\big)(t)\Big)$ where γ is any solution of equation (8.19).*

Once defined G, we will look for assumptions which assert its completeness (as a vector field). We first recall that an integral curve α of a vector field defined on some interval $[0,b)$, $b < +\infty$, can be extended to b (as an integral curve) if and only if there exists a sequence $\{t_n\}_n$, $t_n \nearrow b$, such that $\{\alpha(t_n)\}_n$ converges (see for instance [68, Lemma 1.56]). Now, the following technical result directly follows from this fact and Lemma 8.3.1.

Lemma 8.3.2. *Let* $\gamma : [0,b) \longrightarrow M$ *be a solution of equation* (8.19) *with* $0 < b < \infty$. *The curve* γ *can be extended to* b *as a solution of* (8.19) *if and only if there exists a sequence* $\left\{\gamma(t_n), \gamma'(t_n), \frac{D\gamma'}{dt}(t_n), \frac{\widehat{D}}{dt}\big(\frac{D\gamma'}{dt}\big)(t_n)\right\}_n$ *which is convergent in* $V^{a,w}_{n,3}(M)$.

Even in the geometrically relevant case of M compact, $|\gamma'(t)|^2 = -1$ is not enough to apply Lemma 8.3.2. By the same reason that the cases considered in the two previous chapters, it is natural to assume the existence of such infinitesimal conformal symmetry to deal with the extendibility of the solutions of (8.19)-(8.20). The following result, will be decisive to assure that the image of the curve in $V^{a,w}_{n,3}(M)$, associated to a UC observer γ, is contained in a compact subset.

Lemma 8.3.3. *Let* M *be a spacetime and let* Q *be a unit timelike vector field. If* $\gamma : I \longrightarrow M$ *is a solution of* (8.19)-(8.20) *such that* $\gamma(I)$ *lies in a compact subset of* M *and* $\langle Q, \gamma'\rangle$ *is bounded on* I, *then the image of* $t \longmapsto \Big(\gamma(t), \gamma'(t), \frac{D\gamma'}{dt}(t), \frac{\widehat{D}}{dt}\big(\frac{D\sigma'}{dt}\big)(t)\Big)$ *is contained in a compact subset of* $V^{a,w}_{n,3}(M)$.

Proof. Consider the 1-form Q^b metrically equivalent to Q and the associated Riemannian metric $g_R := \langle\, ,\, \rangle + 2\, Q^b \otimes Q^b$. Clearly, we have,

$$g_R(\gamma', \gamma') = \langle \gamma', \gamma'\rangle + 2\,\langle Q, \gamma'\rangle^2,$$

which, by hypothesis, is bounded on I. Hence, there exists a constant $c > 0$ such that

$$\Big(\gamma(I), \gamma'(I), \frac{D\gamma'}{dt}(I), \frac{\widehat{D}}{dt}\big(\frac{D\sigma'}{dt}\big)(I)\Big) \;\subset\; C,$$

with

$$C := \big\{(p, u_1, au_2, awu_3) \in V^{a,w}_{n,3}(M)\,:\, p \in C_1, \quad g_R(v,v) \leq c\big\},$$

where $C_1 \subset M$ is compact and $\gamma(I) \subset C_1$. Hence, C is a compact subset in $V^{a,w}_{n,3}(M)$.
□

Now, we are in a position to state the following completeness result (compare with [24, Th. 1] and [23, Th. 1]),

Theorem 8.3.4. *Let M be a spacetime which admits a timelike conformal and closed vector field K. If $\mathrm{Inf}_M\sqrt{-\langle K, K\rangle} > 0$ then, each solution $\gamma : I \longrightarrow M$ of* (8.19)-(8.20) *such that $\gamma(I)$ lies in a compact subset of M can be extended.*

Proof. Let $I = [0, b)$, $0 < b < +\infty$, be the domain of a solution γ of equation (8.19)-(8.20). Multiplying γ' by the vector field K and making use of the representation (8.1.4, we obtain,

$$\langle K\,,\gamma'\rangle = \frac{w}{\sqrt{w^2-a^2}}\langle K\,,L\rangle + \frac{a}{\sqrt{w^2-a^2}}\left[\cos\big(\sqrt{w^2-a^2}\,t\big)\,\langle K\,,M\rangle + \sin\big(\sqrt{w^2-a^2}\,t\big)\,\langle K\,,N\rangle\right].$$

On the other hand, taking into account that L is Levi-Civita parallel and (6.7),

$$\frac{d}{dt}\langle K\,,L\rangle = \left\langle \frac{DK}{dt}\,,L\right\rangle = h\langle\gamma'\,,L\rangle\,(h\circ\gamma) = -\frac{w(h\circ\gamma)}{\sqrt{w^2-a^2}}.$$

Analogously,

$$\frac{d}{dt}\langle K\,,M\rangle = \frac{a(h\circ\gamma)}{\sqrt{w^2-a^2}}\cos\big(\sqrt{w^2-a^2}\,t\big),$$

and

$$\frac{d}{dt}\langle K\,,N\rangle = \frac{a(h\circ\gamma)}{\sqrt{w^2-a^2}}\sin\big(\sqrt{w^2-a^2}\,t\big).$$

Using now that $\gamma(I)$ is contained in a compact of M, the function $h\circ\gamma$ is bounded on I. Therefore, since I is assumed bounded, the functions $\langle K\,,L\rangle$, $\langle K\,,M\rangle$ and $\langle K\,,N\rangle$ are also bounded on I and, as a consequence, there exists a constant $c_1>0$ such that

$$|\langle K,\gamma'\rangle|<c_1. \tag{8.25}$$

Now, if we put $Q:=\dfrac{K}{|K|}$, where $|K|^2=-\langle K,K\rangle>0$, then Q is a unit timelike vector field which , making use of (8.25), satisfies

$$|\langle Q,\gamma'\rangle|\leq m\,c_1 \qquad \text{on} \quad I,$$

where $m=\mathrm{Sup}_M|K|^{-1}<\infty$. Now, Lemmas 8.3.2 and 8.3.3 are called to end the proof. □

Remark 8.3.5. *The previous theorem implies the following result of mathematical interest: If a compact spacetime M admits a timelike conformal and closed vector field K, then each inextensible solution of* (8.19)-(8.20) *must be complete. Note that the Lorentzian universal covering of M inherits the completeness of inextensible UC observers from the same fact on M.*

8.4 Completeness of UC trajectories in a Plane Wave spacetime

In this section, we study the completeness of the inextensible UC trajectories with positive prescribed acceleration, but working in a more analytical way.

Let us consider a spacetime M admitting a global coordinate system $\big(x_0,x_1,\cdots,x_n\big)$. In these coordinates, we can write equation (8.12) as follows

$$
\begin{aligned}
\gamma'_k(t) &= \frac{w}{\sqrt{w^2-a^2}}\, L_k(t) + \frac{a}{\sqrt{w^2-a^2}}\left[\cos\left(\sqrt{w^2-a^2}\,t\right) M_k(t) + \sin\left(\sqrt{w^2-a^2}\,t\right) N_k(t)\right], \\
L'_k(t) &= \sum_{i,j} \frac{-\Gamma_{ij}^k}{\sqrt{w^2-a^2}}\left[w L_i L_j + a\cos\left(\sqrt{w^2-a^2}t\right) L_i\, M_j + a\sin\left(\sqrt{w^2-a^2}t\right) L_i\, N_j\right], \\
M'_k(t) &= \sum_{i,j} \frac{-\Gamma_{ij}^k}{\sqrt{w^2-a^2}}\left[w M_i L_j + a\cos\left(\sqrt{w^2-a^2}t\right) M_i\, M_j + a\sin\left(\sqrt{w^2-a^2}t\right) M_i\, N_j\right] \quad (8.26)\\
N'_k(t) &= \sum_{i,j} \frac{-\Gamma_{ij}^k}{\sqrt{w^2-a^2}}\left[w N_i L_j + a\cos\left(\sqrt{w^2-a^2}t\right) N_i\, M_j + a\sin\left(\sqrt{w^2-a^2}t\right) N_i\, N_j\right], \\
\gamma_k(0) &= p_k,\ L_k(0) = \tfrac{1}{\sqrt{w^2-a^2}}(w u_{1k} + a u_{3k}),\ M_k(0) = \tfrac{-1}{\sqrt{w^2-a^2}}(a u_{1k} + w u_{3k}),\ N_k(0) = u_{2k}.
\end{aligned}
$$

Here, u_{1k}, u_{2k} and u_{3k} are the coordinates of the vectors u_1, u_2 and u_3 respectively, and satisfy

$$
\sum_{i,j} u_{1i}\, u_{1j}\, g_{ij}(0) = -1, \quad \sum_{i,j} u_{2i}\, u_{2j}\, g_{ij}(0) = \sum_{i,j} u_{2i}\, u_{2j}\, g_{ij}(0) = 1,
$$

$$
\sum_{i,j} u_{1i}\, u_{2j}\, g_{ij}(0) = \sum_{i,j} u_{1i}\, u_{3j}\, g_{ij}(0) = \sum_{i,j} u_{2i}\, u_{3j}\, g_{ij}(0) = 0,
$$

being $g_{ij}(0)$ the coefficients of the metric at $\gamma(0)$ in these coordinates. Moreover, all the Christoffel symbols are evaluated on γ.

Now, let us consider a Plane Wave spacetime $((\mathbb{R}^4, g)$ and a UC observer $\gamma : I \to \mathbb{R}^4$ satisfying as before the initial conditions

$$
\gamma(0) = p, \quad \gamma'(0) = u_1, \quad \frac{D\gamma'}{dt}(0) = a\, u_2, \quad \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0) = aw\, u_3.
$$

Our final objective is to prove that such trajectory is extensible to the whole real line, i.e., that the maximal interval of definition of γ is $I = \mathbb{R}$.

By Proposition 8.1.4, we can write

$$\gamma'(t) = \frac{w}{\sqrt{w^2 - a^2}}\, L(t) + \frac{a}{\sqrt{w^2 - a^2}} \left[\cos\left(\sqrt{w^2 - a^2}\, t\right) M(t) + \sin\left(\sqrt{w^2 - a^2}\, t\right) N(t)\right]$$

where $L, M, N : I \longrightarrow \mathbb{R}^4$ are solutions of the system (8.26) with initial conditions

$$L(0) = \frac{1}{\sqrt{w^2 - a^2}}(wu_1 + au_3), \quad M(0) = \frac{-1}{\sqrt{w^2 - a^2}}(au_1 + wu_3), \quad N(0) = u_2.$$

Writing in coordinates

$$L = (L_1, L_2, L_3, L_4), \quad M = (M_1, M_2, M_3, M_4), \quad N = (N_1, N_2, N_3, N_4),$$

we have a simple but important fact.

Lemma 8.4.1. *The first components of L, M and N are constant with values*

$$L_1 = \frac{1}{\sqrt{w^2 - a^2}}(wu_{11} + au_{31}), \quad M_1 = \frac{-1}{\sqrt{w^2 - a^2}}(au_{11} + wu_{31}), \quad N_1 = u_{21}$$

Proof. It follows trivially from (8.26) and (2.7) that $L_1' = M_1' = N_1' = 0$, then L_1, M_1, N_1 are constants and equal to the respective initial condition. □

A direct consequence of the latter lemma is that

$$\gamma_1{}'(t) = \frac{w}{\sqrt{w^2 - a^2}}\, L_1 + \frac{a}{\sqrt{w^2 - a^2}} \left[\cos\left(\sqrt{w^2 - a^2}\, t\right) M_1 + \sin\left(\sqrt{w^2 - a^2}\, t\right) N_1\right]$$

and we have an explicit expression for $\gamma_1(t)$ as

$$\gamma_1(t) = p_1 + \frac{w}{\sqrt{w^2 - a^2}}\, L_1 + \tag{8.27}$$

$$+\frac{a}{\sqrt{w^2 - a^2}} \left[M_1 \int_0^t \cos\left(\sqrt{w^2 - a^2}\, t\right) dt + N_1 \int_0^t \sin\left(\sqrt{w^2 - a^2}\, t\right) dt\right].$$

Lemma 8.4.2. *As solutions of system* (8.26)*, the functions* $L_3, M_3, N_3, L_4, M_4, N_4$ *are extensible to the whole real line.*

Proof. The equations from (8.26) for $k = 3, 4$ are

$$\begin{aligned} L_k'(t) &= \frac{-\Gamma_{11}^k}{\sqrt{w^2-a^2}}\left[wL_1^2 + a\cos\left(\sqrt{w^2-a^2}t\right) L_1\, M_1 + a\sin\left(\sqrt{w^2-a^2}t\right) L_1\, N_1\right], \\ M_k'(t) &= \frac{-\Gamma_{11}^k}{\sqrt{w^2-a^2}}\left[wM_1L_1 + a\cos\left(\sqrt{w^2-a^2}t\right) M_1^2 + a\sin\left(\sqrt{w^2-a^2}t\right) M_1\, N_1\right], \\ N_k'(t) &= \frac{-\Gamma_{11}^k}{\sqrt{w^2-a^2}}\left[wN_1L_1 + a\cos\left(\sqrt{w^2-a^2}t\right) N_1\, M_1 + a\sin\left(\sqrt{w^2-a^2}t\right) N_1^2\right]. \end{aligned}$$

Note that the expressions between brackets are trigonometric functions. For convenience we define

$$\begin{aligned} f(t) &:= \frac{1}{\sqrt{w^2-a^2}}\left[wL_1^2 + a\cos\left(\sqrt{w^2-a^2}t\right) L_1\, M_1 + a\sin\left(\sqrt{w^2-a^2}t\right) L_1\, N_1\right], \\ g(t) &:= \frac{1}{\sqrt{w^2-a^2}}\left[wM_1L_1 + a\cos\left(\sqrt{w^2-a^2}t\right) M_1^2 + a\sin\left(\sqrt{w^2-a^2}t\right) M_1\, N_1\right], \\ h(t) &:= \frac{1}{\sqrt{w^2-a^2}}\left[wN_1L_1 + a\cos\left(\sqrt{w^2-a^2}t\right) N_1\, M_1 + a\sin\left(\sqrt{w^2-a^2}t\right) N_1^2\right], \end{aligned}$$

Then, the previous system is written as

$$\begin{aligned} L_k'(t) &= -f(t)\Gamma_{11}^k\left(\gamma(t)\right), \\ M_k'(t) &= -g(t)\Gamma_{11}^k\left(\gamma(t)\right), \\ N_k'(t) &= -h(t)\Gamma_{11}^k\left(\gamma(t)\right). \end{aligned} \tag{8.28}$$

The key point is to analyse the particular form of the Christoffel symbols $\Gamma_{11}^k(\gamma(t))$, $k = 3, 4$. Considering that H is defined by (2.6), we have

$$\Gamma_{11}^3(\gamma) = -\frac{1}{2}\frac{\partial H}{\partial x}(\gamma(t)) = 2A(\gamma_1)\gamma_3 + C(\gamma_1)\gamma_4 + D(\gamma_1),$$

and

$$\Gamma_{11}^4(\gamma) = -\frac{\partial H}{\partial y}(\gamma(t)) = 2B(\gamma_1)\gamma_4 + C(\gamma_1)\gamma_3 + D(\gamma_1),$$

where $\gamma_1(t)$ is explicitly given by (8.27). Since

$$\gamma_k(t) = p_k \quad + \int_0^t \left[\frac{w}{\sqrt{w^2-a^2}}\, L_k(s) + \frac{a}{\sqrt{w^2-a^2}} \left[\cos\left(\sqrt{w^2-a^2}\, t\right) M_k(s) + \sin\left(\sqrt{w^2-a^2}\, t\right) N_k(s)\right]\right] ds,$$

then system (8.28) (with $k = 3, 4$) can be seen as an integro-differential system of six equations. To pass to a standard system of differential equations, we define the new variables

$$\mathcal{L}_k(t) = \frac{w}{\sqrt{w^2-a^2}} \int_0^t L_k(s) ds,$$

$$\mathcal{M}_k(t) = \frac{a}{\sqrt{w^2-a^2}} \int_0^t \cos\left(\sqrt{w^2-a^2}\, s\right) M_k(s) ds,$$

$$\mathcal{N}_k(t) = \frac{a}{\sqrt{w^2-a^2}} \int_0^t \sin\left(\sqrt{w^2-a^2}\, s\right) N_k(s) ds,$$

for $k = 3, 4$. With the new variables,

$$\begin{aligned} \mathcal{L}_k'(t) &= \frac{w}{\sqrt{w^2-a^2}}\, L_k(t), \\ \mathcal{M}_k'(t) &= \frac{a}{\sqrt{w^2-a^2}} \cos\left(\sqrt{w^2-a^2}\, t\right) M_k(t), \\ \mathcal{N}_k'(t) &= \frac{a}{\sqrt{w^2-a^2}} \sin\left(\sqrt{w^2-a^2}\, t\right) N_k(t), \end{aligned} \tag{8.29}$$

for $k = 3, 4$. Besides,

$$\gamma_k(t) = \mathcal{L}_k + \mathcal{M}_k + \mathcal{N}_k + p_k \qquad (k = 3, 4).$$

Recall that $\gamma_1(t)$ is known explicitly, see (8.27). Therefore, attending to the expression of the Christoffel symbols computed before, equations (8.28) are linear on the variables $\mathcal{L}_k$, $\mathcal{M}_k$, $\mathcal{N}_k$. Summing up, equations (8.28)-(8.29) compose a linear system of 12 equations on the involved variables L_k, M_k, N_k, $\mathcal{L}_k$, $\mathcal{M}_k$, $\mathcal{N}_k$ ($k = 3, 4$). The basic theory of linear systems states that any solution of a linear system is globally defined on the whole real line, closing the proof. □

Up to now, we have proved that L_k, M_k, N_k with $k = 1, 3, 4$ are defined on the whole $\mathbb{R}$. To finish the proof, it remains to prove the completeness of $L_2(t), M_2(t), N_2(t)$. The equations (8.26) for L_2 is

$$L_2'(t) = \sum_{i,j} \frac{-\Gamma_{ij}^2}{\sqrt{w^2 - a^2}} \left[wL_i L_j + a\cos\left(\sqrt{w^2 - a^2}t\right) L_i\, M_j + a\sin\left(\sqrt{w^2 - a^2}t\right) L_i\, N_j \right],$$

but note that $\Gamma_{ij}^2 = 0$ if $i = 2$ or $j = 2$, and moreover H does not depend on the second variable. This implies that the right-hand side part of the latter equation depends only on the functions $L_k(t), M_k(t), N_k(t)$ (k=1,3,4), which we have proved that are globally defined, but not on L_2, M_2, N_2. Thus, $L_2'(t)$ is defined for every t, and a simple integration leads to the conclusion. An analogous argument serves for $M_2(t), N_2(t)$.

Theorem 8.4.3. *Every UC inextensible trajectory in a Plane Wave spacetime admitting a global Brinkmann chart is complete.*

Conclusions and future research

In this thesis we have dealt with several physical and mathematical problems arising in the theory of Relativity. It has been structured in two parts. In the first one (Chapters 3, 4 and 5) we have faced the mean curvature prescription problem on different sort of spacetimes. The second one (Chapters 6, 7 and 8) has been devoted to introduce and analyse in detail several concepts in the relativistic framework which already had a well known classical formulation.

In Chapter 3, we first attend the mean curvature function prescription problem in Friedamnn-Lemaître-Robertson-Walker spacetimes of flat fiber with Dirichlet boundary conditions on an Euclidean ball. To face it, we develop two different procedures according to the technique used.

The first technique (Theorem 3.3.5) states sufficient conditions on the warping function and the radius of the domain under which a spacelike graph with prescribed mean curvature exists. Adding some physical hypothesis on the prescription function, the second technique assures the existence of radially symmetric solutions of the prescription problem but deleting the assumption imposed on the size of the domain (Theorem 3.5.2). This improvement allows us to extend such a solution as an entire rotationally symmetric spacelike graph satisfying an initially given prescription (Theorem 3.6.1).

In Chapter 4, new existence results of rotationally symmetric spacelike graphs have been obtained in a quite different kind of spacetimes, the static spacetimes. The singularities arising in this context force us to impose some assumptions on the ambient spacetime but sufficiently weak to include the physically relevant Schwarzschild and Reissner-Nordström spacetimes (Theorems 4.3.2 and 4.2.13). In both cases, the entire spacelike graph asymptotically approaches the event horizon. Spacelike graphs of constant mean curvature remain as a particular situation in the existence results, obtaining explicit expressions for the solutions. The proof of the results is based on the analysis of the associated homogeneous Dirichlet problem on an Euclidean ball together with the obtaining of a suitable bound for the length of the gradient of a solution which permits the prolongability to the whole space.

The wider prescription problem of the higher mean curvatures is analysed in Chapter 5, in the framework of the Minkowski spacetime. The Euclidean space is also considered by its recognized geometrical interest. As a previous step, we analyse the associated homogeneous Dirichlet problem on a ball, and then we prove the possibility to extend the solutions (Theorems 5.4.1–5.4.4). Some uniqueness results are also given.

Under a physical motivation, in Chapter 6, the notion of a uniformly accelerated motion of an observer in a general spacetime is analysed in detail. Such an observer may be seen as a Lorentzian circle, providing a new characterization of a static standard spacetime. The trajectories of uniformly accelerated observers are seen as the projection on the spacetime of the integral curves of a vector field defined on a certain fiber bundle over the spacetime. Using this tool, we find geometric assumptions to ensure that an inextensible uniformly accelerated observer does not disappear in a finite proper time.

In the next chapter, we introduce the notion of unchanged direction (UD) motion in General Relativity, extending widely the concept of uniformly accelerated motion. Such an observer is geometrically characterized as a pointing future unit timelike curve with all its curvatures identically zero up to the first one. The initial value problem when the acceleration of the motion prescribed is analysed. It is also studied the completeness of inextensible UD motions, that can be physically interpreted saying that observers which obey a UD motion live forever. For certain spacetimes with relevant symmetries that includes the Generalized Robertson-Walker spacetimes, a geometric approach leads to the completeness. On the other hand, a more analytical approach allows us to prove completeness of inextensible UD motions in a relevant family of pp-wave spacetimes, the Plane Wave spacetimes.

In the last chapter, the notion of uniform circular motion in a general spacetime is introduced in Chapter 8 as a particular case of a planar motion. Geometrically, an observer which obeys a uniform circular motion is characterized as a Lorentzian helix. The completeness of its inextensible trajectories is studied in Generalized Robertson-Walker spacetimes and in a Plane Wave spacetime. The results may be physically interpreted saying that, under reasonable assumptions, a uniformly circular observer lives forever in these spacetimes, providing the absence of these kind of singularities.

Many open problems still remain to be solved. Respect to the mean curvature prescription problem, the existence of spacelike graphs may be studied in a more general environment, for instance, in Friedmann-Lemaître-Robertson-Walker spacetimes with non flat fiber, or in the wider family of Generalized Robertson-Walker spacetimes. In these cases, the *a priori* rotational symmetry is not proved (except when the fiber is an Euclidean space), and neither Alexandroff reflection method nor any result of Gidas, Ni and Nirenberg paper can not be applied directly. It would also be desirable to improve some conditions (as condition ($A3$) in Proposition 3.3.4 and subsequent theorems) or remove some assumptions (like the boundedness of the prescription function in Theorem 4.3.2) on our existence results. The Dirichlet problem on non rotationally symmetric domains is also a challenge to study in the future.

The open problems we will deal with in future do not restrict to Lorentzian Geometry. In Riemannian Geometry, the warped product manifolds are a family of paramount importance. Several results of this memory can be extended to the Riemannian ambient imposing an ‘artificial’ assumptions on the height of the graphs.

The challenge would consist in deleting such hypothesis.

Other interesting research line points to consider general spacelike hypersurfaces, not only spacelike graphs. Every hypersurface is locally a graph, hence, the techniques here presented are potentially applicable to general hypersurfaces. Even, they may be useful to face the mean prescription vector field problem of submanifolds with codimension bigger than one. These ones have also a great physical interest, especially in the study of trapped surfaces in black holes [81].

By other hand, it would be interesting to deal with the higher mean curvature prescription in more general space, not only in the Euclidean and Minkowski spaces, as we have done in this work. Up to the dealt spaces here the resulting equations are not generically elliptic and most of the usual techniques used to solve the mean curvature prescription are not directly applicable.

As for the future work related with the second block, one interesting question is about the (possible) variational nature of the uniformly accelerated, unchanged direction and uniformly circular motion equations. A variational setting give us more tools in order to study the completeness of their inextensible trajectories, improving the exposed results here. Anyway, we are interested in to sharp the assumption of compactness imposed on the spacetime, because of its incompatibility with the causality principle. Also the a little restrictive existence a conformal and closed vector field should be weakened.

To sum up, this work opens the door to new different research lines which can provide new interesting problems and which can help us to understand better the role and physical meaning of the Geometry in General Relativity.

Appendix

We will expose here a proof of Theorem 3.2.1. We denote by $\Omega := B(R)$. Let begin by choosing a unit vector γ in $\mathbb{R}^n$ and, for each $\lambda \in \mathbb{R}$ let $T_\lambda \subset \mathbb{R}^n$ be the hyperplane define by $\langle \gamma, p \rangle = \lambda$. We may suppose that $\gamma = (1, 0, \cdots, 0)$, choosing a suitable coordinate system. Being Ω bounded, we have that T_λ is disjoint with $\overline{\Omega}$ for $\lambda = \lambda_0$ big enough. Now, we continuously move the hyperplane towards Ω, maintaining the same normal direction, i.e., decreasing λ until T_λ starts to intersect to $\overline{\Omega}$. At this moment (when $\lambda = R$), T_λ will start to divide Ω into two open subregions. We will denote by $\Sigma(\lambda)$ the part of Ω which is at the same side of T_λ that T_{λ_0}. We will write $\pi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ for the reflection respect T_λ, $\Sigma'(\lambda)$, and put $x^\lambda := \pi_\lambda(x)$, for all $x \in \Omega$, and $\Sigma'(\lambda) := \pi_\lambda(\Sigma(\lambda))$.

Note that we are only reflecting in the base of the graph, instead of reflecting the hypersurface entire. Now, every reflection respect to T_λ in the domain base provides another one in $\mathcal{M}$ with respect an 'hyperplane' $\widehat{T_\lambda}$ (the normal hyperplane to γ in $\mathcal{M}$), which is also an isometry of the warped metric (2.1). Roughly, the main idea consists in moving λ and reflecting the graph respect these 'hyperplanes' until one of the following situations happens (see the pictures below)

(a) The first contact point holds at an interior point of the graph.

(b) The reflection hyperplane cuts the graph orthogonally at some point. This is, in fact, the limit case of the previous one.

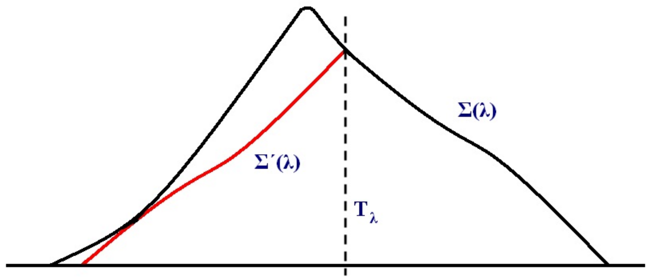

Figure 8.1: Case (a)

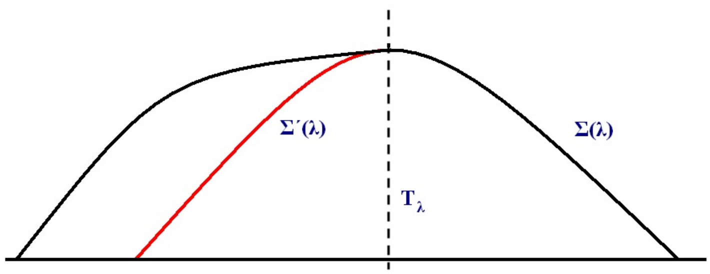

Figure 8.2: Case (b)

(c) The first contact point holds at the boundary of the graph.

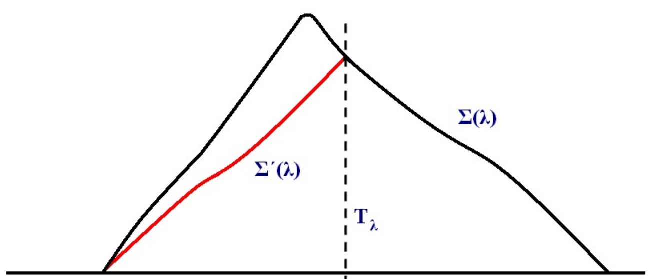

Figure 8.3: Case (c)

Note that situation (c) only happens when $\lambda = 0$. We want to see that the part reflected portion coincides with the part not reflected one in the three cases. In that way, we prove that the graph is symmetric with respect to the normal hyperplane to γ. Taking into account that γ was arbitrarily chosen, we conclude that graph is radially symmetric.

Before, we have to prove that it is possible reflecting the graph avoiding cases (a) and (b) whenever λ is taken close enough to R, i.e., *the method boots well.* This question is trivial if the hypersurface is assume compact and without boundary, because the reflection hyperplane is tangent at any of the first contact points. But this is not the present case. Therefore we have to prove the following technical result.

We will consider now that Ω is a general bounded domain in $\mathbb{R}^n$, and ν is the unit outward normal vector field to $\partial\Omega$.

Lemma 8.4.4. *Let $v \in C^2(\Omega)$ be satisfying* (3.3), *and let $x_0 \in \partial\Omega$ with $\nu_1(x_0) > 0$, where ν_1 is the firs component of ν. Suppose that*

$$H(0,x) \leq \frac{f'(0)}{f(0)} \quad \forall x \in \partial\Omega \qquad \text{or} \qquad H(0,x) > \frac{f'(0)}{f(0)} \qquad \text{in} \quad \overline{\Omega}. \tag{8.30}$$

Then there exists $\delta > 0$ such that $v_{x_1} := \dfrac{\partial v}{\partial x_1} < 0$ in $\{x \in \Omega \ : \ |x - x_0| < \delta\}$.

Proof. First, as $v > 0$ in Ω, necessarily $v_\nu := \dfrac{\partial v}{\partial \nu} \leq 0$ in $\partial\Omega$ and, therefore, $v_{x_1} \leq 0$ in a neighbourhood of x_0 in $\partial\Omega$. By for reductio ad absurdum, we suppose that the conclusion is false. In this case, we have a sequence $\{x^j\} \to x_0$, with $v_{x_1}(x^j) \geq 0$. As for each j, the straight line through x^j in the x_1 direction cuts $\partial\Omega$ at a point $\widehat{x}^j$ such that $u_{x_1}(\widehat{x}^j) \leq 0$. From the mean value theorem, there is another point y^j in the segment $[x^j, \widehat{x}^j]$, at which v_{x_1} vanishes. Using the same result one more time, a point z^j, is obtained in $[y^j, y^{j+1}]$, at which $v_{x_1 x_1}$ vanishes. As $\{z^j\} \to x_0$, we have $v_{x_1}(x_0) = 0$ and $v_{x_1 x_1}(x_0) = 0$, from a continuity argument. Since $v|_{\partial\Omega} = 0$, and the x_1 direction is transversal to $\partial\Omega$ at x_0, it follows that

$$\nabla v|_{x_0} = 0.$$

On the other hand, v is also a solution of the equation which results from replacing in (3.3) the operator $\mathcal{Q}$ by the built in (3.4), and we had denoted by $\mathcal{Q}_v$.

We recall the well-known result (see [58] for instance) which asserts that, given two functions u_1 and u_2 and a nonlinear, second order differential operator Q, there exists a linear operator L such that

$$Q(u_1) - Q(u_2) = L(u_1 - u_2). \tag{8.31}$$

Applying this theorem to the functions v and 0, and the operator $\mathcal{Q}_v$, we deduce that an linear operator L_v exits such that

$$\mathcal{Q}_v(v) - \mathcal{Q}_v(0) = L_v(v) = n\left(f'(0) + f(\varphi^{-1}(v))H(v,x)\right).$$

If we call $-g(x,v)$ to the right term, the function v satisfies

$$L_v(v) + g(x,v) = 0.$$

From the assumption (8.30) we know that $g(x,0)$ is positive or strictly negative for all $x \in \Omega$. We are going to consider each of the two cases separately.

First we suppose that $g(x,0) \geq 0$ for all $x \in \Omega$. Then,

$$L_v(v) + g(x,v) - g(x,0) \leq 0,$$

and, making use of the mean value theorem, there exist a function $c(x)$ such that

$$L_v(v) + c(x)v \leq 0.$$

Now we are in a position to apply Hopf-Serrin lemma to the function $-v$, obtaining $v_\nu(x_0) < 0$, and therefore $v_{x_1}(x_0) < 0$, which contradicts our assumption.

Now assume $g(x,0) < 0$. We denote by $\{E_i\}_{i=1}^n$ the standard orthonormal basis at x_0 in $\mathbb{R}^n$. Take next an orthonormal basis $\{e_i\}_{i=1}^{n-1}$ of $T_{x_0}\partial\Omega$, which, with joint to ν, form an orthonormal basis of $T_{x_0}\Omega$. We have

$$E_i = \sum_{k=1}^{n-1} \langle E_i, e_k \rangle e_k + \nu_i \, \nu,$$

where $\nu_i = \langle \nu, \nu_i \rangle$. Therefore,

$$v_{ij} = \langle \nabla_{E_i} \nabla v, E_j \rangle = \sum_{k=1}^{n-1} \sum_{l=1}^{n-1} \langle E_j, e_k \rangle \langle E_i, e_l \rangle \langle \nabla_{e_l} \nabla v, e_k \rangle +$$

$$\sum_{k=1}^{n-1} \langle E_j, e_k \rangle \nu_i \langle \nabla_\nu \nabla v, e_k \rangle + \sum_{l=1}^{n-1} \langle E_i, e_l \rangle \nu_j \langle \nabla_\nu \nabla v, e_l \rangle + \nu_i \nu_j \langle \nabla_\nu \nabla v, \nu \rangle.$$

Next, we are going to see that the second and the third term of the previous expression vanish. In order to do that, it suffices to prove

$$\nabla^2 v(\nu, e_k) = 0 \quad \text{for all} \quad k = 1, ..., n-1.$$

By reductio ad absurdum, we suppose that $\nabla^2 v(\nu, e_i) \neq 0$ for some i. Since $v > 0$ in Ω, and it is of class C^2 in $\overline{\Omega}$, performing a Taylor expansion at x_0 we get

$$0 > v\big(x_0 + t(\delta_i e_i - \nu)\big) =$$

$$= v(x_0) + t \langle \nabla v|_{x_0}, \delta_i e_i - \nu \rangle + \frac{t^2}{2} \nabla^2 v|_{x_0} (\delta_i e_i - \nu, \delta_i e_i - \nu) + R(t)$$

$$= t^2 \{ -\delta_i \nabla^2 v|_{x_0} (e_i, \nu) + \frac{1}{2} \nabla^2 v|_{x_0} (\nu, \nu) \} + R(t),$$

where $\delta_i \in \mathbb{R}^+$ and $\lim_{t \to 0} \dfrac{R(t)}{t^2} = 0$.

Choosing $\delta_i = \dfrac{|\nabla^2 v|_{x_0}(\nu, \nu)| + 1}{\nabla^2 v|_{x_0}(e_i, \nu)}$, we obtain,

$$0 > v(x_0 + t(\delta_i e_i - \nu)) \leq -t^2 + R(t) < 0,$$

for $t > 0$ sufficiently small, which gives a contradiction.

Therefore, we have proved that,

$$v_{ij} = \delta \nu_i \nu_j$$

where $\delta := \langle \nabla_\nu \nabla v, \nu \rangle$, at x_0. Using now that $v_i(x_0) = 0$ and $v(x_0) = 0$, v_{ij} satisfies the following equation

$$\sum_{i,j} a_{ij}(x_0) v_{ij} = -g(x_0, 0),$$

at x_0, where a_{ij} denote the second order coefficients of the linear operator L_v. From the two previous formula we obtain that

$$\delta = -\frac{g(x_0, 0)}{\sum_{i,j} a_{ij}(x_0) \nu_i \nu_j} > 0.$$

Note that the denominator is strictly positive because of the ellipticity of L_v. In particular, when $i = j = 1$:

$$v_{11} = \delta \nu_1^2 > 0,$$

which contradicts $v_{11}(x_0) = 0$. $\square$

Coming back to our situation, i.e., $\Omega = B(R)$ we will prove in the following result that if one of the critical cases (a) or (b) holds, then the reflected piece equals to the non reflected one and moreover, $\langle \nabla v \,, \gamma \rangle < 0$ in $\Sigma(0)$.

Lemma 8.4.5. *Let $v \in C^2(\overline{\Omega})$ be satisfying* (3.3)*, and assume f and H satisfy* (8.30)*. If, for some λ in $[0\,, R)$ we have*

$$\begin{aligned} &H(t, \cdot) \quad \textit{is radially monotonically increasing}, \\ &v_{x_1} \le 0, \quad v \le v \circ \pi_\lambda \quad \textit{and} \quad v \not\equiv v \circ \pi_\lambda \quad \textit{in} \quad \Sigma(\lambda). \end{aligned} \tag{8.32}$$

Then, $v(x) < v(x^\lambda)$ in $\Sigma(\lambda)$ and $v_{x_1}(x) < 0$ in $\Omega \cap T_\lambda$.

Proof. For each $x \in \Sigma'(\lambda)$ (recall $\pi_\lambda(\Sigma(\lambda))\Sigma(\lambda))$ we define $w(x) := v(x^\lambda)$. Using $v_{x_1}(x^\lambda) \le 0$, we have $w_{x_1}(x) \ge 0$. From (3.3) , we can check that w satisfies the following equation:

$$\mathcal{Q}(w)\mid_x = n\overline{H}(w, x^\lambda),$$

where $\overline{H}(v, x) := H(v, x)f(\varphi^{-1}(v))$. Analogously we get

$$\overline{H}(v, x) := H(v, x)f(\varphi^{-1}(v)). \tag{8.33}$$

Since $x \in \Sigma'(\lambda)$, we have that $(x^\lambda)_1 \geq x_1$, and therefore, using the radial growth of $H(t, \cdot)$, we obtain that $H(w, x^\lambda) \leq H(w, x)$ and hence $\overline{H}(w, x^\lambda) \leq \overline{H}(w, x)$.

From (8.33) and making use again of (8.31), we get

$$\begin{aligned}
0 &= \mathcal{Q}_v(w) - n\overline{H}(w, x^\lambda) - \big(\mathcal{Q}_v(v) - n\overline{H}(v, x)\big) \\
&\leq \mathcal{Q}_v(w) - \mathcal{Q}_v(v) + n\overline{H}(v, x) - n\overline{H}(w, x^\lambda) \\
&\leq L_v(w - v) + n\left(\overline{H}(v, x) - \overline{H}(w, x^\lambda)\right) \\
&\leq L_v(w - v) + n\left(\overline{H}(v, x) - \overline{H}(w, x)\right).
\end{aligned}$$

for any $x \in \Sigma'(\lambda)$. By assumption we have $z(x) := w(x) - v(x) \leq 0$ and z is not identically zero. Moreover, from the mean value theorem, there exists $c(x)$ such that

$$L_v(z) + c(x)z \geq 0.$$

Since $z = 0$ in $T_\lambda \cap \Omega$, from the Maximum Principle and the Hopf-Serrin Lemma it follows $z < 0$ in $\Sigma'(\lambda)$ and $z_{x_1} > 0$ in T_λ. We conclude the proof observing that in T_λ we have $z_{x_1} = w_{x_1} - v_{x_1} = -2v_{x_1}$. □

Finally, we will show that the first 'critical' situation is (c) when $\lambda = 0$. Moreover, if (a) or (b) is reached, then graph is symmetric with respect to the hyperplane $\widehat{T}_0$.

In fact we have

Lemma 8.4.6. *Let v be a solution of the problem* (3.3)*, with H satisfying* (8.30) *and*

(8.32). *Then, for each* $\lambda \in (0, R)$, *we have*

$$v_{x_1} < 0 \quad \textit{and} \quad v(x) < v(x^\lambda) \quad \textit{for} \quad x \in \Sigma(\lambda), \tag{8.34}$$

therefore $v_{x_1} < 0$ *in* $\Sigma(0)$. *Moreover, if* $v_{x_1} = 0$ *at some point of* $\Omega \cap T_\lambda$, *then, necessarily* v *is symmetric with respect to the hyperplane* T_{λ_1} *and*

$$\Omega = \Sigma(\lambda_1) \cup \Sigma'(\lambda_1) \cup (T_\lambda \cap \Omega).$$

The proof follows analogously that [53, Theorem 2.1] and hence is omitted.

Now, we are in a position to prove Theorem (3.2.1).

The first part of the previous lemma asserts that if $x_1 > 0$ then $v_{x_1} < 0$. Since $v \in C^1(\overline{\Omega})$, we deduce that $v_{x_1} = 0$ in $x_1 = 0$. From the last conclusion of the previous lemma, we get that v is symmetric in x_1. As the x_1 direction is arbitrary, we conclude that v must be radially symmetric and $0 < r < R$. $\square$

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