

ELEMENTARY SCHOOL STUDENTS´ APPROACHES TO SOLVING TRUE/FALSE NUMBER SENTENCES

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This paper focuses on eight-year old students' ways of approaching true/false number sentences. The data presented here belongs to a teaching experiment in which the use of relational thinking when solving number sentences was explicitly promoted. The study of the way of using this type of thinking and of students' structure of attention, allow us to make distinctions and to provide a description of students' different behaviours.

Keywords: Conceptual meta-strategies; Early algebra; Number sentences; Relational thinking.

Métodos de Alumnos de Educación Primaria para la Resolución de Sentencias Numéricas Verdaderas y Falsas

Este artículo se centra en las formas en que alumnos de ocho años abordan la resolución de sentencias numéricas verdaderas y falsas. Los datos que se presentan pertenecen a un experimento de enseñanza en el cual se promovió explícitamente el uso del pensamiento relacional en la resolución de sentencias numéricas. El estudio del modo en que es usado este tipo de pensamiento y de la estructura de la atención de los alumnos, nos permite distinguir y aportar una descripción de los diferentes comportamientos de los alumnos.

Términos clave: Meta-estrategias conceptuales; Early-algebra; Sentencias numéricas; Pensamiento relacional.

As a mathematics learner, when working with numeric or algebraic expressions, I have always enjoyed looking for ways to simplify the expressions before and during my manipulation of them. To me it is fun doing it and it also helps to save some work. From another point of view, as a mathematics teacher, it is quite frustrating when students embark on quite difficult computations before looking at the expressions they have to work on and getting a sense of their structure, so missing the opportunity to choose the best approach or to simplify the work to do

before starting to operate. Other researchers have commented on the occurrence of this type of event at the university level when working on other contexts such as integral calculus, by claiming that often students' mathematical knowledge seems to be only mechanical (Hejny, Jirotkova, & Kratochvilova, 2006).

TWO DIFFERENT APPROACHES

When working on true/false number sentences such as $257 - 34 = 257 - 30 - 4$ or $27 + 48 - 48 = 25$, whose design is based on some arithmetical properties, in general there are two different approaches to follow: Doing the computations on both sides and comparing both results, or looking to the whole sentence, appreciating its structure, and making use of relations between its elements as well as of knowledge of the structure of arithmetic to solve it (Carpenter, Franke, & Levi, 2003; Koehler, 2004; Molina & Ambrose, in press). Similarly when solving equations such as $\frac{1}{4} - \frac{x}{x-1} - x = 5 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$, students may proceed by operating on the variables and the numbers on each side as well as regrouping them or they may pay attention to its structure and appreciate that this equation is equivalent to $-x = 5$ by noticing that the expression $\frac{1}{4} - \frac{x}{x-1}$ is repeated in both sides (Hoch & Dreyfus, 2004).

We identify the first approach as what Hejny et al. (2006) call a "procedural meta-strategy" and the second one as a "conceptual meta-strategy". The main distinction between these two types of strategies is that the first one is based on the student activating some procedures in his/her mind after having identified the area to which the problem belongs, while, in the second one, the student creates a image of the problem in his/her mind as a whole, analyzes it to find its inner structure, and looks for some key elements or relations to construct a solving strategy. While the first process leads students to become more skilful in problems of the given type, the second one leads them towards a higher level of understanding of the situation in question (Hejny et al., 2006, p. 290).

Within the context of problem solving, the use of conceptual meta-strategies may be considered to be related to an element of heuristic competence referred as "internal monitor" (Mason, 1985) or "instructed manager" (Puig, 1996). This element include various capacities such as examining possible ways of approaching a problem before addressing its resolution, in order to make an informed choice of a solving strategy, or keeping an eye on calculations to make sure that they remain relevant to the question. However, the distinction we want to make when differencing conceptual and procedural meta-strategies, is not that the person does an informed choice of a strategy, but that s/he uses the mathematical structure of the situation (and mathematical relationships) when constructing the strategy.

NUMBER SENTENCES AND RELATIONAL THINKING

Our interest is focused on the use of these two types of strategies, or approaches, when working on solving number sentences, especially sentences whose design is based on some arithmetical properties. We choose this context because of its potential for integrating the learning of arithmetic and the development of algebraic thinking (Molina, 2006). Number sentences are frequently used to introduce students to equations by drawing a figure or a line instead of a variable, as in $_ + 4 = 5 + 7$ (Radford, 2000). Discussions about these equations and the properties that they may illustrate can help students learn arithmetic with understanding and develop a solid base for the later formal study of algebra by helping them to become aware of the structure underneath arithmetic (Carpenter et al., 2003; Hewitt, 1998; Kieran, 1992; Resnick, 1992). Carpenter et al. (2003) illustrate the potential of number sentences for working on the development of generalizations of arithmetic relations and their symbolic representation.

When students solve the sentences by using conceptual meta-strategies, we say that they are using *relational thinking* (a term introduced by Carpenter et al., 2003) or *analyzing expressions* (as expressed in Molina and Ambrose, in press), as their thinking makes use of relations between the elements in the sentence and relations which constitute the structure of arithmetic. Students who solved number sentences by using relational thinking (RT) employ their number sense and what Slavit (1999) called “operation sense” to consider arithmetic expressions from a structural perspective rather than simply a procedural one. When using relational thinking, sentences are considered as wholes instead of as processes to carry out step by step. For example, when considering the number sentence $8 + 4 = _ + 5$ some students notice that both expressions include addition and that one of the addends on the left side, 4, is one less than the addend on the other side, 5. Noticing this relation and having an (implicit or explicit) understanding of addition properties enable students to solve this problem without having to perform the computations 8 plus 4 and 12 minus 5.

Some previous studies have provided evidence that elementary students are capable of using relational thinking when solving number sentences, overcoming some issues such as the “lack of closure” as well as an operational understanding of the equal sign (Carpenter et al., 2003; Koehler, 2004; Molina, & Ambrose, in press; Molina, Castro, & Ambrose, 2006).

In this paper we focus on analyzing the students' ways of approaching true/false number sentences by distinguishing degrees in the use of relational thinking displayed, i.e. distinguishing more and less elaborated ways of using this type of thinking.

METHODOLOGY

We applied the “conjecture-driven research design”, which Confrey and Lachance (2000) propose for teaching experiments aiming to investigate new instructional strategies in classroom conditions and to analyze different approaches to the content and the pedagogy of a set of mathematical topics. Our research method shared the features of design experimentation identified by Cobb and his colleagues (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003).

We worked with a group of 26 eight-year old Spanish students during six sessions over a period of one year. In this paper we will focus on the data gathered on the last four sessions as the first two were directed to exploring and extending students’ understanding of the equal sign. The general aim of this research work was to study students’ thinking involved in solving number sentences, in the context of whole class activities and discussions. We analyzed the strategies that students used to solve the sentences, focusing on detecting evidences of use of relational thinking.

The tasks used were number sentences, mostly true/false number sentences¹ (e.g., $72 = 56 - 14$, $7 + 7 + 9 = 14 + 9$, $10 + 4 = 4 + 10$) which were proposed to the students in written activities, whole-class discussions and interviews. All the sentences used were based on some arithmetic property or principle (e.g., commutative property, inverse relation of addition and subtraction, compensation relation) and, therefore, could be solved by using relational thinking.

We did not promote the learning of specific relational strategies. Instead, we promoted the development of a habit of looking for relations, trying to help students to make explicit and apply the knowledge of structural properties which they had from their previous experience with arithmetic. Students’ use of relational thinking was favoured by encouraging them to look for different ways of solving the same sentence and to especially appreciate explanations based on relations.

The data here presented provide evidence of a diversity of approaches without paying attention to how frequently the students evidence each one. For further description and information about this teaching experiment see Molina (2006).

¹ In a previous study (Molina & Ambrose, 2006) we appreciated that true/false number sentences, unlike open number sentences, contribute to break the students’ computational mindset as the students don’t need to provide a numerical answer. Therefore, this type of sentences eases the consideration of number sentences as wholes and the use of relational thinking in its resolution.

STUDENTS' BEHAVIOURS WHEN SOLVING TRUE/FALSE NUMBER SENTENCES

We identified six different behaviours when attending at students' ways of solving the considered true/false sentences². In all of them, except for the first one, we identify some use of relational thinking.

Not-RT Behaviour

Students who display this behaviour solve each sentence by obtaining the numeric values of each side and comparing them. They do not provide any evidence of having noticed any relation or characteristic in the sentence apart from the numbers in it, the operations which combines them and the presence of the equal sign. For example, Irene displayed not-RT behaviour when solving the sentences gathered on Table 1. She computed the numeric values of each side by using the standard algorithms for addition and subtraction³.

Simple-RT Behaviour

Students displaying this behaviour solve some of the sentences by directly applying a known fact (an arithmetic law or principle) after having noticed a particular relation or characteristic in the sentence which led them to recall that fact. Students recognize in the sentence a particular case of a general fact that they know in an implicit or explicit way. Some students recall and use the properties of zero as identity element and the property " $a - a = 0$ ". We see this behaviour as a basic use of relational thinking. In other sentences they proceed as in behaviour not-RT.

For example, Jose Luis seemed to show this behaviour when solving the sentences $325 + 0 = 326$ and $24 - 24 = 0$: "It is false [why do you think is false?] because three hundreds and twenty-five plus zero is three hundreds and twenty-five, and three hundreds and twenty-six... is nothing"; "It is true because...because twenty-four minus twenty four is zero". We infer that he didn't do any computation as the numbers involved in the sentences do not allow easy mental computation and he provided his answers very quickly, without having done any writing. In other sentences he computed the numeric values of each side and compared them.

Sameness-RT Behaviour

The students displaying this behaviour solve some sentences without making any computations, by noticing some sameness between the numbers in the sentence.

² The examples provided have been translated from Spanish to English.

³ This student appreciates some structure in the sentence $18 - 7 = 7 - 18$ which she applies when solving the sentence $75 - 14 = 340$. However, she does not provide any evidence of having use relational thinking for solving any of the sentences in Table 1.

They apply the reflexive property of the equality relation, the commutative property of addition, an over-generalization of the commutative property to subtraction or an overgeneralization of the reflexive property (i.e. assuming that a sentence is true if and only if it contains repeated numbers, no matter their position). The relations that students appreciate are based on sameness or not-sameness. In other cases they may proceed as in the not-RT or simple-RT behaviours.

This behaviour shows a more elaborated use of relational thinking as it is not based on simply applying a known fact but on making flexible use of observed relations to get an answer.

Miguel showed this behaviour when solving the sentences $18 - 7 = 7 - 18$ and $75 + 23 = 23 + 75$, while solving other sentences by computing the numeric values of each side (see Table 1).

Table 1
Irene and Miguel's responses to some true/false number sentences

Number sentence	Students' answer	
	Irene	Miguel
$18 - 7 = 7 - 18$	False because $18 - 7 = 11$ and $7 - 18 = 19$. (She computes $7 - 18 = 19$ and $18 - 7 = 11$ by using the subtraction standard algorithm)	True because eighteen minus 7 and the other is the same, and if it is the same they are equal
$75 - 14 = 340$	False because $75 - 14 = 61$ and $14 - 75 = 49$ not 340. (She computes $75 - 14 = 61$ and $14 - 75 = 49$ by using the subtraction standard algorithm)	False because seventy-five minus 14 is not three hundreds and forty. (He computes $75 - 14 = 51$ by using the subtraction standard algorithm)
$17 - 12 = 16 - 11$	False because $16 - 11 = 05$. (She computes $17 - 12 = 05$ by using the subtraction standard algorithm)	True. (He writes vertically the numbers 17, 12, 16 and 11 with a minus sign, but he doesn't compute)
$6 + 4 + 18 = 10 + 18$	True because $6 + 4 + 18 = 28$ and $10 + 18 = 28$. (She computes $6 + 4 + 18 = 28$ and $10 + 18 = 28$ by using the addition standard algorithm)	True because 6 plus four plus eighteen is 28 and 10 plus eighteen is 28. (He computes $6 + 4 + 18 = 28$ and $10 + 18 = 28$ by using the standard algorithm)

Table 1
Irene and Miguel's responses to some true/false number sentences

Number sentence	Students' answer	
	Irene	Miguel
$75 + 23 = 23 + 75$	No response	True because they are the same and then it is the same

Within parenthesis we describe the students' computations done in the worksheet.

One-shot-RT, Frequent-RT and all-RT Behaviours

These three behaviours correspond to the cases when students solve some sentences by using relational thinking, making use of a pair of distinctions or relations such as sameness between the numbers in the sentence, differences of magnitude between those numbers, a number fact contained in the sentence, some numeric relations between the numbers or knowledge of the effect of operations on numbers. Table 2 shows some student's explanations which evidence the use of relational thinking based on pairs of the mentioned elements.

Table 2
Examples of students' explanations evidencing use of relational thinking

Elements at the base of the relational thinking used	Examples : Students' explanations
Sameness between numbers in the sentence and knowledge about the effect of operations	In $122 + 35 - 35 = 122$: "True because if we add 122 to 35 and we take it away, it is as if we don't add anything"
Number fact contained in the sentence and sameness between numbers	In $7 + 7 + 9 = 14 + 9$: "True. I did it by adding seven and seven.... which is fourteen. The same than there [right side]. Nine, the same than there [right side] too"
Numeric relations between numbers in the sentence and sameness between numbers	In $13 + 11 = 12 + 12$: "True because you subtract one to the twelve and you give it to the other twelve, and you get what it is there [left side]"
Differences of magnitude between numbers and knowledge about the effect of operations	In $75 - 14 = 340$: "False because 75 minus 14 is less, it can not be a bigger number"

Table 2

Examples of students' explanations evidencing use of relational thinking

Elements at the base of the relational thinking used	Examples : Students' explanations
Numeric relations between numbers in the sentence and knowledge about the effect of operations	In $11 - 6 = 10 - 5$: "True because if eleven is higher than ten and you subtract one more than five, you get the same"

We distinguish between the behaviours "one-shot-RT", "frequent-RT" and "all-RT", depending on the variety of ways in which students use relational thinking, according to the elements at the base of their thinking from those indicated in Table 2. These behaviours do not differ in the way students proceed when solving the sentences but in the diversity of ways of using relational thinking. In behaviour "one-shot-RT", students provide evidence of having used relational thinking based on just one of the above referred elements (but not just on sameness between numbers). "Frequent-RT" behaviour corresponds to those cases in which the students solve a variety of sentences by using relational thinking based on several but not all the elements in Table 2. The behaviour "all-RT" refers to the cases in which the child provides evidence of solving different sentences by using relational thinking based on all those elements.

In Figure 1 we display the relationships we identified between the type of evidences of use of relational thinking displayed by "students with different behaviour", i.e. students whose behaviour at a particular session where classified in different categories. The way of interpreting these relationships is as follows. Students displaying an all-RT behaviour also provide responses which are typical of sameness-RT and not-RT behaviours and, occasionally, simple-RT responses. In the case of the frequent-RT behaviour, some students' responses are of type not-RT and sometimes of type simple-RT and/or sameness-RT. Students displaying the behaviour one-shot-RT provide responses of type not-RT and, in some occasions, of type simple-RT. The behaviours sameness-RT and simple-RT, as we previously explained, include responses typical of the not-RT behaviour. Additionally, the behaviour sameness-RT may include some responses typical of the simple-RT behaviour.

An important relationship between the evidences of the use of relational thinking displayed in behaviors one-shot-RT, Frequent-RT and All-RT is also shown in Figure 1. Some of the responses typical of one-shot-RT also take place in behavior Frequent-RT and behavior All-RT includes responses typical of Frequent-RT.

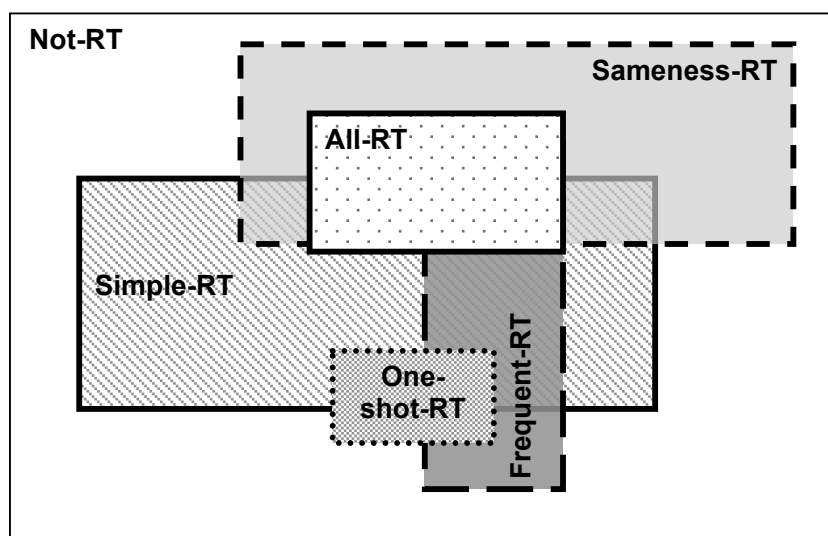


Figure 1. Relationships identified between the six behaviours

In all except the not-RT behaviour, we distinguish two different ways in which students invoke relational thinking. In some cases, they start by looking at the sentence, making distinctions and noticing some relationships which they used to solve the sentence, following what Hejny et al. called a “conceptual meta-strategy”. In other cases they start by computing some of the operations involved in the sentences and, during the computation process, they notice a special characteristic of the sentence or a relation between its elements which leads them to change their approach and to solve the sentence without computing the numeric values of both sides. This change of approach can be recognized in the following student’s answer to the sentence $51 + 51 = 50 + 52$: “True because as fifty-one plus fifty-one is one hundreds and two, but if you subtract [one] from fifty-one, fifty, you can add to fifty-one from the other, one more, and you get fifty-two... and you get fifty plus fifty-two”. In other cases we identify this change of approach by comparing the students’ notes with their explanation. In Molina, Castro, and Castro (forthcoming) we present a finer distinction between students’ particular strategies used for solving each sentence by differencing the role of computation as well as the role of relational thinking and moment of the solving process when it is applied. Relational thinking allows us to make a finer distinction within the duality procedural versus conceptual meta-strategies.

DISCUSSION

The different behaviours identified suggest a variety of ways in which students approach solving true/false number sentences whose design is based on arithmetical properties and relations. They display more and less elaborated ways of using relational thinking which correspond to more or less algebraic ways of ap-

proaching the work with these arithmetic expressions. The behaviors differ in the way students' pay attention to the sentences, the distinctions they make within them, the relations that they display evidence of noticing and the arithmetic knowledge that those distinctions and relations trigger in the students' mind.

These behaviors present a variety of different students' structures of attention when working with the sentences (Mason & Johnston-Wilder, 2004). Some students consider sentences and expressions with more than two terms in a global way (as a whole) and look across the equal sign as well as within each side to make distinctions and to establish relationships between elements. The way they use these relations is influenced by their awareness of the structure of the sentence (e.g., the equal sign differentiate two sides in the sentence) as well as their knowledge of arithmetic structure (e.g., inverse relation of addition and subtraction, commutative property of addition). Other students proceed to do the computations, apparently paying attention only to the numbers involved and the operations to perform on them, considering each side or even each operation separately. Other students' attention, while initially being placed on doing the computations, fluctuates between numbers, partial results and elements of the sentence. This fluctuation cues to become aware of characteristics or relations within the sentence not previously noticed.

Looking for evidence of the use of relational thinking in the students' solving strategies promotes appreciation of a range of strategies between procedural and conceptual meta-strategies. Considering these distinctions, the appreciated differences between the structures of students' attention provide a point of entrance to allow teachers to help students to develop more conceptual approaches. Some ways could be encouraging students to look at the sentence before computing and formulating questions which draw children's attention to the sameness or differences of some elements in the sentence as well as to look for relations between the terms. Discussions of students' various approaches in which the use of relational thinking is encouraged seem also to be effective.

Acknowledgments

I want to thank the members of the CERME 5 discussion group "Algebraic Thinking", especially the reviewers of an earlier version of this paper, as well as Rebecca Ambrose, for all their comments which contributed to the improvement of this paper.

This study has been developed within a Spanish national project of Research, Development and Innovation, identified by the code SEJ2006-09056, financed by the Spanish Ministry of Sciences and Technology and FEDER funds.

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This document was originally presented at the Fifth Conference of the European Society for Research in Mathematics Education (CERME 5), Larnaca, Cyprus, 2007.

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