Projective modules over certain

Non–Commutative Polynomial Rings

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por

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El Director. El Doctorando.

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To my parents

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Introduction

In [19, 1955], J. P. Serre raised the question of whether any finitely generated projective module over the ring of commutative polynomials $k[X_1, \ldots, X_n]$ over a field k is free. After almost 20 years of attempts, Suslin [26] and Quillen [14] independently (and using different methods) obtained an affirmative answer to Serre's question.

The situation in the non-commutative case is completely different: there were constructed counterexamples, for instance, *stably free non-free* modules in several classes of noncommutative rings.

Here we will study the category of finitely generated projective modules $\mathrm{proj}(R)$, the group $K_0(R)$ of stably isomorphism classes and the monoid $V(R)$ of the isomorphism classes of these modules over some non-commutative noetherian domains. In particular we will study polynomial rings over noncommutative division rings and Weyl algebras over a field or more generally over noncommutative division ring of characteristic zero.

In Section (2), the general structure of stably free R-modules, their characterizations and their direct sums will be studied. Indeed stably free right R-modules are equivalence to the right invertible rectangular matrices over R. In addition a characterization when the stably free modules are free will be given by two versions, module and matrix versions.

In Section (3), if D is a division ring with center the field k,and $A = D[X_1, \ldots, X_n]$, $B = k[X_1, \ldots, X_n]$ the polynomial rings over commutative indeterminates $X_1, \ldots, X_n$. Then B is the center of A.

We will show the following result concerns the two–sided ideals of A.

Let I be a two–sided ideal of A, then $I = A(I \cap B)$, i. e., I is generated by a Groebner basis constituted of central elements.

In Section (4), as a consequence of the following result due to Stafford: [23, Theorem 2.9]:

Every finitely generated projective right A–module is either free or isomorphic to a non–free projective right ideal of A.

We compute the K_0 group of A, it is isomorphic to $\mathbb{Z}$. We give examples of non-free stably free R-modules, and we found its two generators as a right ideal of R. Therefore the monoid $V(R)$ fails to satisfy separative cancelation.

Also we study an examples of non-free projective right ideals $I = (y^2+1)A + (y+j)(x-ti)A$ of $\mathbb{H}[X,Y]$, where $\mathbb{H}$ is the quaternion division ring, which due to R. G. Swan. In this case simple criterion were given for I to be free. Moreover under some conditions, these right ideals were classified up to isomorphism. As a consequence of this classification we can show that: There are an infinite number of isomorphism classes of such modules over $A = \mathbb{H}[X,Y]$

In section (5), also as a consequence of the following results due to Stafford: [20, Theorem 2.2]:

All finitely generated projective right $A_n(D)$-modules are stably free.

and [22, Theorem 3.6(b)]:

Every finitely generated projective right $A_n(k)$-module is either free or isomorphic to a non–free projective right ideal of A.

we compute the K_0 group of the $n-th$ Weyl algebra over a commutative field and over noncommutative division ring. In the two cases it has shown that $K_0 \cong \mathbb{Z}$. We give an example of stably free non-free right ideal of $A_1(k)$.

In section (6), in the classification of projective right ideals of $A_1(k)$. R. Cannings and M. P. Holland established in [5] a bijection correspondence between primary decomposable subspaces of $R = k[t]$ and projective right ideals I of $A_1(k)$ which have non-trivial intersection with $k[t]$.

[5, Theorem 0.5] (Bijective correspondence theorem):

$$\Gamma : V \mapsto \mathfrak{D}(R,V) \quad , \quad \Gamma^{-1} : I \mapsto I \star 1.$$

Indeed this bijection had been founded only when the field k is algebraically closed field and of characteristic zero.

M. K. Kouakou and A. Tchoudjem in [9], generalized the definition of primary decomposable subspaces of $k[t]$ when k is any field of characteristic zero, particulary for $\mathbb{Q}$ and $\mathbb{R}$, and it has shown that R. Cannings and M. P. Holland correspondence theorem holds. Thus projective right ideals of $A_1(\mathbb{Q})$, $A_1(\mathbb{R})$ are also described by this theorem.

In this section, we will reanalyze the main theory of this classification and describe the isomorphism classes of the projective right ideals of A_1 and we restrict this theorem for some particular classes of these ideals, as a consequence we get the following result:

Reducible polynomials with the same degree and same roots correspond to the same isomorphic class of projective right ideals of $A_1(k)$.

In Section (7), we get the following result that have both $A_n(k)$ and $A = D[X_1, \ldots, X_n]$:

They are 3-Hermite rings and the have stable range rank 2.

In addition, we give a description of the isomorphism classes of the non-free stably free right A-modules of rank 1 depending on two crucial facts, each projective module generated by two elements and looking to them as an A-submodules of A^2. We determine these isomorphism classes by a structure of matrices given in the end of this section.

In Section (8), we show that the monoid $V(A)$ has only the two trivial archimedean components, therefore it has only the two trivial prime ideals, and hence A has only the two trivial trace ideals A and $\{0\}$.

Contents

1 Projective modules

Let R be a ring. A right R–module P is **projective** if the functor $\mathrm{Hom}_R(P,-) : \boldsymbol{Mod}\text{–}R \longrightarrow \mathcal{A}b$ is exact, or equivalently, if for every epimorphism $\varepsilon : M \to M''$ any map $f : P \to M''$ can be lifted to M, i.e., there exists a map $f' : P \to M$ such that $f = \varepsilon \circ f'$.

$$\begin{array}{ccccc} & & P & & \\ & \overset{f'}{\swarrow} & \downarrow f & & \\ M & \xrightarrow{\ \varepsilon\ } & M'' & \longrightarrow & 0 \end{array}$$

Projective right R–modules are also characterized as direct summand of free right modules.

Projective modules carry a lot of information on the ring R and its modules. Indeed, every right R–module M is a epimorphic image of a free, hence projective, right module P_0. Thus many properties of M may be deduced from the short exact sequence

$$0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where K_1 is the kernel of the epimorphism $P_0 \to M$. Now to study M it is sufficient to study P_0 and K_1. But for K_1 we can find a similar construction:

$$0 \longrightarrow K_2 \longrightarrow P_1 \longrightarrow K_1 \longrightarrow 0.$$

Following in this way we find that all information about M is in the exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0,$$

i.e., in a projective resolution of M.

For that reason to know as must as possible on the structure and behavior of projective modules is interesting to know more on the ring R itself. We observe that direct sums of projective modules are projective modules. Thus we have a class of modules: the class of projective right R–modules, which is closed under direct sums. The first question is: which kind of information provides this class?

First may we observe that in order to have a set instead a proper class of modules, we may consider the isomorphism equivalence relation, i.e., P_1 and P_2 are related if there is an isomorphism, $g : P_1 \cong P_2$. Hence in the quotient set of projective modules up to this equivalence relation, say $\mathcal{M}(R)$, we have a monoid structure induced by the direct sum of modules.

Again the monoid $\mathcal{M}(R)$ carries a information on R. For instance if R is a division ring, this monoid contains a submonoid isomorphic to the additive monoid $\mathbb{N}$ of all natural numbers. Indeed this submonoid is the monoid of all cosets of finitely generated projective right R–modules. This is because every projective right R–module is free.

When we consider a different ring, for instance $R[X]$, the polynomial ring over a division ring R, we get the same result as every finitely generated projective right $R[X]$–module is free. When we consider a general ring R, this submonoid is not necessarily isomorphic to $\mathbb{N}$, even it may exist non-free finitely generated right R–modules; this is the case of $R[X, Y]$, the polynomial ring in two indeterminates over some a division ring R.

The monoid $\mathcal{M}(R)$ may be studied through a group if we define in $\mathcal{M}(R) \times \mathcal{M}(R)$ the equivalence relation

$$(a_1, b_1) \sim (a_2, b_2), \text{ if there exists } c \in \mathcal{M} \text{ such that } a_1 + b_2 + c = a_2 + b_1 + c.$$

and call $G(R) := (\mathcal{M}(R) \times \mathcal{M}(R))/\sim$. In $G(R)$ we define a binary inner operation as follows

$$[(a_1, b_1)] + [(a_2, b_2)] = [(a_1 + a_2, b_1 + b_2)],$$

hence $(G(R).+)$ is an abelian group.

The group $G(R)$ is of null interest, for instance if we consider an arbitrary element $[(a, b)]$, such that a and b are classes of free right R–modules, and define c, free right R–module of infinite dimension and bigger that the dimensions of a and b, we get the relationship $0 + b + c = 0 + a + c$, hence $[(a, b)] = [(0, 0)]$. In particular the submonoid generated by the class of R goes to zero in $G(R)$.

To overlap this we restrict ourselves to consider only finitely generated projective right R–modules. If we denote by $\mathcal{V}(R)$ the corresponding monoid of cosets, the associated group is the Grothendieck group of R and is denoted by $K_0(R)$.

In the particular case of a finitely generated projective right R–module P we always may find a split short exact sequence

$$0 \longrightarrow R^n \xrightarrow{f} R^m \longrightarrow P \longrightarrow 0.$$

Hence P is an epimorphic image of a finitely generated free R–module and its kernel is also a finitely generated free R–module. In particular P is a finitely presented right R–module and the map f, may be described, after fixing bases in R^n and R^m, by a $m \times n$-matrix.

We represent by n the coset of R^n in $\mathcal{V}(R)$, then:

(1) It may be $[R^n] = [R^m]$ and $n \neq m$. For instance we may consider the ring $R = \mathrm{End}_K(V)$, being K a field and V a vector space with numerable dimension. In this case it happens that $R^2 \cong R$. Rings satisfying $n = m$ whenever $[R^n] = [R^m]$ are called **invariant basis number rings** or **IBN rings**. Every non trivial commutative ring is IBN, as is every right noetherian ring.

(2) For any $a \in \mathcal{V}(R)$ there exists $b \in \mathcal{V}(R)$ and n such that $a+b = n$. Of particular interest are those a such that there exists m such that $a + m = n$. Them will be studied in Section (2).

2 Stably Free Modules

Let R be any ring, we will study the general structure of stably free right R-modules and their direct sums, showing their equivalence to the right invertible rectangular matrices over R. Indeed, as we will see in the following, stably free modules play an important role in K_0-theory.

A right R-module P is **stably free of type** m with $(0 \leq m < \infty)$ if $P \oplus R^m$ is free. A module is **stably free** if it is stably free of type m for some $m \in \mathbb{N}$. (Stably free modules are, of course, projective.)

Thus, a projective right R-module P is stably free of type m and rank $= n - m$ if $P \oplus R^m \cong R^n$, i.e., if, and only if,

$$P \cong \mathrm{Ker}(R^n \xrightarrow{f} R^m)$$

for some suitable epimorphism f, which automatically splits.

If M is the $m \times n$–matrix associated with f, then M is right invertible, i.e., there exist an $n \times m$ matrix N such that $MN = \mathrm{Id}_m$. Conversely any right invertible $m \times n$-matrix M defines a finitely generated stably free right R-module P of type m, namely the **solution space** of M,

$$P = \left\{ \alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid M\alpha = 0 \right\}.$$

In this way the study of finitely generated stably free right R- modules is equivalent to the study of right invertible rectangular matrices over R.

Proposition. 2.1.
The kernel P of an epimorphism $f : R^n \to R^m$ is free if, and only if, f can be lifted to an isomorphism $f' : R^n \to R^m \oplus R^r$ for some r, such that $pr_1 \circ f' = f$.

Proof. Suppose P is free,then there exists an r such that $g : P \to R^r$ is an isomorphism. We can write $R^n = Q \oplus P$ in such a way that the restriction of f to Q gives an isomorphism $f_0 : Q \to R^m$. Then $f_0 \oplus g : R^n \to R^m \oplus R^r$ clearly gives the desired isomorphism.

Conversely suppose that an isomorphism f' with $pr_1 \circ f' = f$ exists. Then $P = \mathrm{Ker}(f) \cong \mathrm{Ker}(pr_1) = R^r$ is free. □

Let M denotes the $m \times n$–matrix corresponding to f and let N the $(m+r) \times n$–matrix corresponding to f', if f' exists. The condition: $pr_1 \circ f' = f$, says that M is the sub-matrix of N, consisting of its first m rows. The condition: $\acute{f}$ is an isomorphism, says that N is a, not necessarily square, right and left invertible matrix, i.e., there exists another matrix N' of size $n \times (m+r)$ such that

$$NN' = \mathrm{Id}_{m+r}, \quad N'N = \mathrm{Id}_n .$$

The following is the matrix theoretic version of the above proposition.

Proposition. 2.2.
For any right invertible $m \times n$–matrix M, $m < n$, the (stably free) solution space of M is free if, and only if, M can be completed to an invertible matrix by adding a suitable number of new rows.

Let R be a commutative ring, $n \in \mathbb{N}$ is in the **general linear range** of R provided that $P \oplus R \cong R^{n+1}$ implies that $P \cong R^n$. If $n - m$ in the general linear range of R, then $0 \longrightarrow P \longrightarrow R^n \overset{\phi}{\to} R^m \longrightarrow 0$, for $m < n$, implies that $P \cong R^{n-m}$. Now we will extend this fact to non-commutative rings.

Lemma. 2.3. (M. Gabel)
Let $\phi : R^n \to R^m$ be an epimorphism and let $P = \mathrm{Ker}(\phi)$. If there exists a basis $\{d_1, \ldots, d_n\}$ of R^n such that R^m can be generated by $\{\phi(d_1), \ldots, \phi(d_k)\}$, for some $k \leq n - m$, then $P \cong R^{n-m}$.

Proof. Let $F = \sum_{i=1}^{k} Rd_i \cong R^k$. Since $\{\phi(d_1), \ldots, \phi(d_k)\}$ generates R^m, then $P + F = R^n$. If K denotes the kernel of $\phi_{|P} : P \to R^n$, then $K = P \cap F$. Since $k \leq n - m$, then $n - m = k + s$ for some s. We have two exact sequences.

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\phi} R^m \longrightarrow 0,$$

$$0 \longrightarrow K \longrightarrow P \longrightarrow P/K \longrightarrow 0,$$

where $P/K \cong P/(P \cap F) \cong (P + F)/F = R^n/F \cong R^{n-k} \cong R^{m+s}$. Thus,

$$P \cong K \oplus R^{m+s} \cong K \oplus R^m \oplus R^s \cong F \oplus R^s \cong R^{k+s} \cong R^{n-m}$$

as claimed. □

Indeed this lemma characterizes when stably free modules are free. The following is the matrix theoretic version.

Lemma. 2.4. ([6, lemma 3.2])
Let $\phi : R^n \to R^m$ be an epimorphism and $P = \mathrm{Ker}(\phi)$. Let $A = [B, D]$ be the matrix representing ϕ with respect to the standard basis of R^n and R^m. If there exists $U \in GL_m(R)$ and $V \in GL_n(R)$ such that $UAV = [B_{m\times k}, 0]$ for some $k \leq n - m$, then $P \cong R^{n-m}$.

Lemma. 2.5. (Whitehead´s Lemma for Rectangular Matrices)
Let $A \in M_{m,n}(R)$, $B \in M_{s,t}(R)$. Assume B has a right inverse. Let $A = (M, V)$ with $M \in M_{m,s}(R)$, $V \in M_{m,n-s}(R)$. Let P, Q denotes the solution space of A, B, respectively. Then $P \oplus Q$ is isomorphic to the solution space of (MB, V).

PROOF. Let $C \in M_{t,s}(R)$ with $BC = I_s$, $Q \oplus P$ is the solution space of $\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$. By elementary transformations we have:

$$\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} B & 0 & 0 \\ 0 & M & V \end{pmatrix} \mapsto \begin{pmatrix} B & -BC & 0 \\ 0 & M & V \end{pmatrix}$$
$$= \begin{pmatrix} B & -I_s & 0 \\ 0 & M & V \end{pmatrix} \mapsto \begin{pmatrix} B & -I_s & 0 \\ MB & 0 & V \end{pmatrix} \mapsto \begin{pmatrix} 0 & -I_s & 0 \\ MB & 0 & V \end{pmatrix}.$$

It is clear that the last matrix has solution space isomorphic to that of (MB, V). □

Corollary. 2.6.
Let $A \in M_{m,n}(R)$, $B \in M_{n,t}(R)$ such that B has a right inverse. Then the solution space of AB is isomorphic to the direct sum of the solution spaces of A and B.

Thus we can see that the study of stably free modules is equivalent to study of right invertible rectangular matrices, such that the direct sum of stably free modules corresponds to the product of right invertible rectangular matrices, and in general two stably free modules are isomorphic if, and only if, they corresponds to two matrices with the same solution space.

3 Polynomials over Division Rings

Let D be a division ring with center the field k, and let $A = D[X_1, \ldots, X_n]$, $B = k[X_1, \ldots, X_n]$ be the polynomial rings over commutative indeterminates $X_1, \ldots, X_n$. Then B is the center of A.

Let $F \in A$, we may write F as the sum of its monomials in this way $F = \sum_\alpha d_\alpha X^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and the $d_\alpha \in D$ are almost all zero. In order to write F in a unique way as a sum of monomials it is enough to introduce a well founded order in $\mathbb{N}^n$ which compatible with the addition in $\mathbb{N}^n$ and such that $0 \leq \alpha$ for every $\alpha \in \mathbb{N}^n$.

If we denote by $\preceq$ such a well founded order then every $F \neq 0$ can be written $F = d_{\alpha^1} X^{\alpha^1} + \cdots + d_{\alpha^t} X^{\alpha^t}$ being $\alpha^1 \succ \ldots \succ \alpha^t$ with $d_{\alpha^i} \neq 0$ for any $i = 1, \ldots, n$.

As a consequence this expression is unique in the sense that if $F = d_{\beta^1} X^{\beta^1} + \cdots + d_{\beta^s} X^{\beta^s}$ satisfies the same properties, then $s = t$, $\alpha^i = \beta^i$ for $i = 1, \ldots, t$ and $d_{\alpha^i} = d_{\beta^i}$ for $i = 1, \ldots, t$.

We call α^1 the **exponent** of F, it is represented by $\exp(F)$, d_{α^1} the **leader coefficient** of F, it is represented by $\mathrm{lc}(F)$ and $\{\alpha^1, \ldots, \alpha^t\}$ the **Newton diagram** of F, it is represented by $\mathcal{N}(F)$.

Observe that if $F, G \in A$ are nonzero, then $\exp(FG) = \exp(F) + \exp(G)$ and $\mathrm{lc}(FG) = \mathrm{lc}(F)\,\mathrm{lc}(G)$. If $F = 0$, we can extend to it the definition of exponent and leader coefficient, but it is not useful at this moment.

Let $F \in A$ and $\{G_1, \ldots, G_t\} \subseteq A$, we may perform the **division** of F by $\{G_1, \ldots, G_t\}$ as follows. First we define

$$\Delta^1 = \exp(G_1) + \mathbb{N}^n,$$

and

$$\Delta^i = \exp(G_i) + \mathbb{N}^n \setminus \cup_{j=1}^{i-1} \Delta^j, \text{ if } i = 2, \ldots, t.$$

Finally we define

$$\overline{\Delta} = \mathbb{N}^n \setminus \cup_{j=1}^{t} \Delta^j.$$

Thus $\{\Delta^1, \ldots, \Delta^t, \overline{\Delta}\}$ is a partition of $\mathbb{N}^n$.

We define $F_0 = F$ and continuous as follows: for any $h \geq 1$,

$$F_h = F_{h-1} - \mathrm{lc}(F_{h-1})\,\mathrm{lc}(F_{j_h})^{-1} G_{j_h} X^{\exp(F_{h-1})-\exp(G_{j_h})}$$

where $G_{j_h} \in \{G_1, \ldots, G_t\}$, whenever $\exp(F_{h-1}) \in \Delta^{j_h}$ or

$$F_h = F_{h-1} - \mathrm{lc}(F_{h-1}) X^{\exp(F_{h-1})},$$

whenever $\exp(F_{h-1}) \in \overline{\Delta}$.

This process finishes as $\mathbb{N}^n$ is well founded with respect to $\preceq$.

If we have a division algorithm, then we may have a Groebner basis theory, and, as consequence, it is easy to prove that every two–sided ideal I has a finite Groebner basis as a right A–module. See [4] and [8].

Theorem. 3.1.
Let I be a two–sided ideal of A, then $I = A(I \cap B)$. As a consequence I is generated by a Groebner basis constituted of central elements.

PROOF. Indeed, let $\mathbb{G} = \{G_1, \ldots, G_t\}$ a system of generators of I as left A–module. We may assume that $\mathbb{G}$ satisfies the following properties:

(1) $\{G_1, \ldots, G_r\}$ is a system of generators (Groebner basis) of $I \cap B$ as B–module and $\{G_{r+1}, \ldots, G_t\} \subseteq I \setminus A(I \cap B)$.

(2) All G_i are monic polynomials.

(3) $\exp(G_1) < \cdots < \exp(G_r)$ and $\exp(G_{r+1}) < \ldots < \exp(G_t)$.

(4) $\exp(G_1) = \min(\exp(I \cap B))$, $\exp(G_{r+1}) = \min(\{\exp(F) \mid F \in I \setminus A(I \cap B)\})$ and $\exp(G_1) < \exp(G_{r+1})$.

Proof of (4). For any $0 \neq d \in D$, we have $\exp(dG_{r+1} - G_{r+1}d) < \exp(G_{r+1})$ and $dG_{r+1} - G_{r+1}d \in A(I \cap B)$. Therefore $\exp(G_1) \leq \exp(dG_{r+1} - G_{r+1}d) < \exp(G_{r+1})$ and $\exp(G_1) < \exp(G_{r+1})$. △

Let us assume $r < t$, then the set $\mathcal{Y} = \{\exp(f) \mid f \in I \setminus A(I \cap B)\}$ is nonempty as $\exp(G_{r+1}) \in \mathcal{Y}$. By assumption $\exp(G_{r+1}) = \min(\mathcal{Y})$. For any $0 \neq d \in D$ we obtain

$\exp(dG_{r+1} - G_{r+1}d) < \exp(G_{r+1})$ hence $dG_{r+1} - G_{r+1}d \in A(I \cap B)$. If the division of G_{r+1} by $\{G_1, \ldots, G_r\}$ is

$$G_{r+1} = \sum_{i=1}^{r} Q_i G_i + R, \quad Q_i \in A,$$

then either $\mathcal{N}(R) \subseteq \overline{\Delta}$ or $R = 0$. If $R \neq 0$, then $\exp(R) \leq \exp(G_{r+1})$. We may assume that R is monic. For any $0 \neq d \in D$, we obtain $\exp(dR - Rd) < \exp(R)$ hence $dR - Rd \in A(I \cap B)$; since $\mathcal{N}(R) \subseteq \overline{\Delta}$, then $dR - Rd = 0$, i,e, $R \in \text{Cen}(A) = B$. Hence and $G_{r+1} \in A(I \cap B)$, which is a contradiction. □

4 Projective modules and $K_0(A)$

Now we compute $K_0(A)$, the Grothendieck group of $A = D[X_1, \dots, X_n]$. In the case of B we have $K_0(B) = \mathbb{Z}$ as a consequence of Quillen–Suslin's theorem. Therefore every finitely generated projective B–module is free. In the case of A, we have the following result due to Stafford.

Theorem. 4.1. ([23, Theorem 2.9])
Every finitely generated projective right A–module is either free or isomorphic to a non–free projective right ideal of A whenever the centre of A is infinite. If $A = D[X, Y]$ condition on the centre can be removed.

In addition, in some cases, we may find non–free projective right ideals in A.

Example. 4.2.
Let $\mathbb{H}$ be the quaternion division ring. The center of $\mathbb{H}$ is $\mathbb{R}$. Indeed, $\mathbb{H}$ is a 4–dimensional $\mathbb{R}$–vector space generated, as an algebra, by i, j satisfying: $i^2 = j^2 = (ij)^2 = -1$ and $ij = -ji$. If we take $A = \mathbb{H}[X, Y]$, hence $B = \mathbb{R}[X, Y]$. We will find in A a non–free finitely generated projective right ideal.

Proof. Observe $A = \mathbb{H}[X, Y]$ is an IBN ring, it is enough to consider the evaluation map $\varepsilon_{0,0} : \mathbb{H}[X, Y] \longrightarrow \mathbb{H}$ defined $\varepsilon_{0,0}(X) = 0 = \varepsilon_{0,0}(Y)$. It is a right noetherian ring as a consequence of Hilbert basis theorem. The map $f : A^2 \longrightarrow A$, defined $f(\lambda.\mu) = (X + j)\lambda - (Y + i)\mu$, is an A–linear map. In addition it is surjective as $f(Y + i, X + j) = ij - ji = -2ij$ is an invertible element in A. Let $K = \text{Ker}(f)$, i.e.,

$$K = \text{Ker}(f) = \{(\lambda, \mu) \mid (X + j)\lambda - (Y + i)\mu = 0\}.$$

Since $Y + i$ is a non–zero divisor in A we find an isomorphic map $K = \text{Ker}(f) \longrightarrow A$, defined $(\lambda, \mu) \mapsto \lambda$. The image is the right ideal $J = \{\lambda \in A \mid (X+j)\lambda \in (Y + i)A\}$. Then the short exact sequence

$$0 \longrightarrow \text{Ker}(f) \longrightarrow A^2 \longrightarrow A \longrightarrow 0,$$

splits and we have $A^2 \cong \text{Ker}(f) \oplus \text{Im}(f) \cong K \oplus A \cong J \oplus A$. We claim J is non–free. Otherwise, since A is IBN then $J \cong A$; in this case there exists $Z \in J$ such that $J = ZA$. We have:

$$(X + j)(X - j)(Y + i) = (X^2 - j^2)(Y + i) = (X^2 + 1)(Y + i) = (Y + i)(X^2 + 1),$$
$$(X + j)(Y^2 + 1) = (Y^2 + 1)(X + j) = (Y + i)(Y - i)(X + j).$$

Hence $(X-j)(Y+i) \in J$ and Z has Y–degree less or equal than 1. If it is equal to 1, let $Z = F_1Y + F_0$, where $F_1, F_0 \in \mathbb{H}[X]$. Since $Y^2+1 \in J$, there are $G_1, G_0 \in \mathbb{H}[X]$ such that

$$Y^2 + 1 = (F_1Y + F_0)(YG_1 + G_0) = F_1G_1Y^2 + (F_0G_1 + F_1G_0)Y + F_0G_0.$$

Hence $F_1, F_0 \in \mathbb{H}$, and we may assume $F_1 = 1$. Otherwise $(X-j)(Y+i) \in J$, hence there is $G \in \mathbb{H}[X]$ such that

$$(X - j)(Y + i) = (Y + F_0)G.$$

Thus $X - j = G$ and $(X - j)i = F_0G$. Therefore $(X - j)i = F_0(X - j)$, and we obtain $F_0 = i$ and $-ji = -F_0j$. This implies $-ij = -ji$, which is a contradiction.

As a consequence J is non–free. □

This example is an extension of:

Theorem. 4.3. ([13, Proposition 1])
Let D be a non commutative division ring. Then $A = D[X, Y]$ contains a non-free projective right ideal P such that $P \oplus A \cong A^2$.

The basic construction idea in this case is the following: let $a, b \in A$ be such that the additive commutator $c = ab - ba \in U(A)$, (the group of units of A). For any central elements $x, y \in A$, we define $\phi : A^2 \to A$ by the rectangular matrix $(x + a, y + b)$, i.e., $\phi(e_1) = x + a$ and $\phi(e_2) = y + b$ for the unit vectors e_1, e_2.The map ϕ is onto since

$$\phi\begin{pmatrix} y+b \\ -(x+a) \end{pmatrix} = (x+a, y+b)\begin{pmatrix} y+b \\ -(x+a) \end{pmatrix} = ab - ba = c$$

is invertible. Thus the solution space $P = P(x + a, y + b) = \mathrm{Ker}(\phi)$ is stably free of type 1 (since the splitting of ϕ leads to $P \oplus A \cong A^2$). Under suitable assumption on $a, b, x, y \in A$ it can be shown that P is isomorphic to a non free right ideal J of A.

Let us assume that $x + a$ and $y + b$ are not 0-divisors in A. Then the second coordinate projection $\pi_2 : A^2 \to A$ maps P isomorphically onto the the right ideal

$$J = \{\beta \in A |\ (y + b)\beta \in (x + a)A\}$$

and left multiplication by $y + b$ defines an isomorphism from J onto the very nicely expressed right ideal $(x + a)A \cap (y + b)A$.

Since $P \oplus R \cong R^2$, then J has two generators f_1, f_2. We will find them.

Lemma. 4.4.
$J = f_1A + f_2A$, where $f_1 = (x+a)c^{-1}(x+a)$ and $f_2 = 1 + (x+a)c^{-1}(y+b)$.

Proof. The epimorphism ϕ splits by the map $\psi : R \to R^2$ defined by

$$\psi(1) = \begin{pmatrix} (y+b)c^{-1} \\ -(x+a)c^{-1} \end{pmatrix}.$$

Thus P is the image of the projection $Id - \psi\phi$ on R^2. So

$$\begin{aligned} f_1 &= \pi_2(Id - \psi\varphi)(e_1) = \pi_2(e_1 - \psi\varphi(e_1)) = \pi_2(e_1 - \psi(x+a)) \\ &= \pi_2((1,0) - ((x+b)c^{-1}(y+a), -(x+a)c^{-1}(x+a))) = (x+a)c^{-1}(x+a), \end{aligned}$$

and

$$\begin{aligned} f_2 &= \pi_2(Id - \psi\varphi)(e_2) = \pi_2(e_2 - \psi\varphi(e_2)) = \pi_2(e_2 - \psi(y+b)) \\ &= \pi_2((0,1) - ((y+b)c^{-1}(y+b), -(x+a)c^{-1}(y+b))) = 1 + (x+a)c^{-1}(y+b). \end{aligned}$$

□

In the special case when $a, b \in D$ the set of non-zero commutators $c = [a,b] = ab - ba$ produces non-free stably free right ideals $I = (f_1, f_2)A$ by the same way. In addition I is non–free if, and only if, $c \neq 0$.

Thus for the module $P = P(x+a, y+b)$ in the previous theorem, for instance, we have automatically $P \oplus P \cong A^2$, $P \oplus P \oplus P \cong A^3$, etc., provided that D has infinite center. In this case, one say that proj(A) fails to satisfy separative cancelation, in that we have modules P, Q in proj(A) (here $Q = A$) such that

$$P \oplus P \cong P \oplus Q \cong Q \oplus Q,$$

but $P \ncong Q$.

Although we have the non–cancelation in proj(A), we will show that the group $K_0(A)$ is "trivial".

Corollary. 4.5.
$K_0(A) \cong \mathbb{Z}$.

Proof. By Grothendieck's theorem [1, Theorem XII.3.1] every finitely generated projective right A-module P is stably free and hence by Stafford's result [23, Theorem 2.9], $P \cong A^n \oplus I$, for some projective non-free right ideal I of A with stable rank 1. Hence there is an isomorphism $I \oplus A \cong A^2$. Therefore, as a consequence, we obtain $K_0(A) \cong \mathbb{Z}$. □

For that reason in order to study the ideal structure of A it is of interest to develop a particular study of the monoid $\mathcal{V}(A)$ of isomorphism classes of right ideals. It contains more information than $K_0(A)$ on the structure of A.

In the problem of classification of stably free non-free right ideals of A, we will see the following case which was studied by R. G. Swan. In particular examples of stably free non-free right ideals over $\mathbb{H}[X, Y]$, where $\mathbb{H}$ is the quaternion division ring, are obtained.

Let $A = \mathbb{H}[X, Y]$, where $\mathbb{H}$ is the quaternion division ring. The center of $\mathbb{H}$ is $\mathbb{R}$, then the center of $A = \mathbb{H}[X, Y]$ is $B = \mathbb{R}[X, Y]$. Let $t \in \mathbb{R}^\star$ and consider the stably free module $P = P(X + ti, Y + j)$ over A. We will assume that $Y^2 + 1$ is regular in A so that $Y + j$ is regular and, therefore the $\pi_2 : P \to A$ maps P isomorphically onto the right ideal $I = (Y^2 + 1)A + (X - ti)(Y + j)A$. Now we will give two simple criterion for I to be principal.

Lemma. 4.6. **([27, Lemma 6.2])**
Assume that $1 + Y^2$ is regular. Let $f \in A$. Then $I = fA$ if, and only if, $f \in I$ and $f\overline{f} = u(1 + Y^2)$ for some $u \in B^\star$.

Corollary. 4.7. **([27, Corollary 6.3])**
Assume that $1 + Y^2$ is regular. Let $f \in A$. Then $I = fA$ if, and only if, $\overline{f}f = u(1 + Y^2)$ for some $u \in B^\star$ and $(X - ti)(Y + j) \equiv 0 \pmod{fA}$.

Theorem. 4.8. **([27, Theorem 7.1])**
Suppose that for an infinite set S of real numbers such that $U(A/A(Y - s)) = \mathbb{H}^\star$ for any $s \in S$. Then the stably free A-modules $P(X + ti, Y + j)$ with $t \neq 0$ are all non-free and $P(X + ri, Y + j) \cong P(X + ti, Y + j)$ if and only if $r = \pm t$.

As a consequence of this theorem we have:

Corollary. 4.9.
There are infinitely many isomorphism classes of such modules over $A = \mathbb{H}[X, Y]$.

However, the classification of all stably free non-free right ideals of this ring is still open.

Question. 4.10.
For instance when the non-free projective right ideals $P = P(X + a_1, Y + b_1)$ *and* $Q = Q(X + a_2, Y + b_2)$ *of* $A = D[X, Y]$ *are isomorphic for any noncommutative division ring* D*?*

5 K_0 of Weyl Algebras

Let D be a division ring of characteristic zero, then the n-th **Weyl Algebra** $A_n(D)$ is the associative D-algebra with 1 generated by the $2n$ elements $x_1, \dots, x_n, y_1, \dots, y_n$ with relations $[x_i, x_j] = [y_i, y_j] = 0$ and $[x_i, y_j] = \delta_{ij}$ (the Kronecker's delta), where $[a, b] = ab - ba$. It is well known that $A_n(D)$ is simple right noetherian domain.

For $n = 1$, we have the **first Weyl algebra** $A_1(D)$. It is the associative D-algebra generated by elements x and y subject to the relation $xy - yx = 1$. This algebra has been much studied in the case when D is a field of characteristic zero.

The existence of non-free stably free right ideals for $R = A_1(k)$, where k is a field of characteristic zero, was first pointed out by Webber [28]. In fact, Webber proved that all right ideals in R are stably free, also it has been shown that, any right ideal can be generated by two elements, and that any projective right module is either free or isomorphic to a right ideal, while Rinehart [16] had noted earlier that R is not a principal right ideal domain.

We will give an example of stably free non-free left ideal of $A_1(k)$.

Example. 5.1.
The left ideal I of $R = A_1(k)$ generated by y^2 and $1 + xy$ is non-free.

PROOF. Assume, on the contrary, that $I = Rf$, where $f \in R$. We think of f as a polynomial of the form $\sum_i f_i(y)x^i$, so we can define $\deg_x(f)$ to be $\max\{i |\ f_i \neq 0\}$. Write $y^2 = gf$, $xy + 1 = hf$ where $g, h \in R$. Since $\deg_x(y^2) = 0$ and f cannot be a constant, we must have $f = ay$ or $f = ay^2$ for some $a \in k$. But then $xy + 1 = hf \in Ry$, and hence $1 \in Ry$, which is a contradiction. □

The results of the first Weyl algebra were generalized by Stafford. He has shown that the same results are true for $A_n(k)$, where the structure of finitely generated projective modules over $A_n(k)$ over a commutative field k or a non-commutative division ring D of characteristic 0 has been studied in separate series of his papers.

Theorem. 5.2. ([21, Theorem 2.2])
All finitely generated projective right $A_n(D)$-modules are stably free.

Theorem. 5.3. ([22, Theorem 3.6(b)])
Any finitely generated projective right $R = A_n(k)$-module is either free or isomorphic

to a right ideal.

Corollary. 5.4.
$K_0(A_n(k)) \cong \mathbb{Z}$.

Proof. Let P be finitely generated projective right $R = A_n(k)$-module, then $P \cong R^n \oplus J$, for some stably free non-free right ideal of R such that $J \oplus R \cong R^2$, following with this we get $K_0(R) \cong \mathbb{Z}$. □

Now for non-commutative division ring D,we have:

Theorem. 5.5. ([20, corollary 6.4])
Any finitely generated projective right $R = A_n(D)$-module of rank ≥ 5 is free.

In addition Stafford conjectured that this bound could be reduced to 2.

In this case also we can compute the K_0 group, since $A_n(D)$ is stably free and since it is left noetherian, so it is IBN, this implies that $K_0(R) \cong \mathbb{Z}$.

6 Projective right ideals of $A_1(k)$

In the classification of projective right ideals of $A_1(k)$. R. Cannings and M. P. Holland established in [5] a bijective correspondence between primary decomposable subspaces of $R = k[t]$ and projective right ideals I of $A_1(k)$ which have non-trivial intersection with $k[t]$. Indeed this bijection had been founded only when the field k is algebraically closed field of characteristic zero.

M. K. Kouakou and A. Tchoudjem in [9], generalized the definition of primary decomposable subspaces of $k[t]$ when k is any field of characteristic zero. Particulary for $\mathbb{Q}$, $\mathbb{R}$, and it has shown that R. Cannings and M. P. Holland correspondence theorem holds. Thus projective right ideals of $A_1(\mathbb{Q})$, $A_1(\mathbb{R})$ are also described by this theorem.

Now we will reanalyze the main theory of this classification and describe the isomorphism classes of the projective right ideals of A_1.

First we will introduce the relation between $A_1(k)$ and differential operators. Let $A_1(k) = k[t, \partial]$. A_1 contains the subring $R = k[t]$ and $S = k[\partial]$. It is well known that A_1 is an integral domain, two sided noetherian and since the characteristic of k is zero, A_1 is hereditary that every right ideal is projective. In particular, A_1 has a quotient division ring denoted by Q_1.

Q_1 contains the subrings $D = k(t)[\partial]$ and $D = k(\partial)[t]$. The elements of D are k-linear endomorphisms of $k(t)$.Precisely if, $d = a_n\partial^n + ... + a_1\partial + a_0$ where $a_i \in k(t)$ and $h \in k(t)$, then

$$d(h) = a_n h^{(n)} + ... + a_1 h^1 + a_0 h$$

where $h^{(i)}$ denote the $i - th$ derivative of h.

For V and W two vector subspaces of $k(t)$, we set:

$$\mathfrak{D}(V, W) = \{d \in k(t)[\partial] : d(V) \subset W\}$$

$\mathfrak{D}(V, W)$ is called the set of differential operators from V to W. Note that $\mathfrak{D}(R, V)$ is an A_1-right submodule of Q_1 and $\mathfrak{D}(V, R)$ is an A_1-left submodule of Q_1. If $V \subseteq R$, one note that $\mathfrak{D}(R, V)$ is a right ideal of A_1. When $V = R$,then$\mathfrak{D}(R, R) = A_1$.

If I is a right ideal of A_1, we set $I \star 1 = \{d(1), d \in I\}$. Clearly $I \star 1$ is a vector subspace of $k[t]$ and $I \subseteq \mathfrak{D}(R, I \star 1)$.

The following theorem of Stafford is the first step in the classification of right ideals of the first Weyl algebra A_1

Theorem. 6.1. ([25, lemma 4.2])
Let I be a non-zero right ideal of A_1, then there exist $x, e \in Q_1$ such that :

(1) $xI \subset A_1$ and $xI \cap k[t] \neq \{0\}$

(2) $eI \subset A_1$ and $eI \cap k[\partial] \neq \{0\}$

Corollary. 6.2.
Every non-zero right ideal I of A_1 is isomorphic to another right ideal J with a non-trivial intersection with $k[t]$.

PROOF. As a consequence of (1) in the previous theorem, we can choose $J = xI$, for some $x \in Q_1$, therefore $I \cong J$ and $J \cap k[t] \neq \{0\}$. □

We denote $\mathfrak{I}_t$ the set of right ideals I of A_1 such that $I \cap k[t] \neq \{0\}$.

Now let C be algebraically closed field of characteristic zero. Cannings and Holland have defined **primary decomposable subspaces** of $C[t]$ as finite intersections of **primary subspaces** which are vector subspaces of $C[t]$ containing a power of a maximal ideal M of $C[t]$. Since C is algebraically closed field, maximal ideals of $C[t]$ are generated by one polynomial of degree one: $M = (t - \lambda)C[t]$. So, a vector subspace V is primary decomposable subspaces if $V = \bigcap_{i=1}^{n} V_i$, where each V_i contains a power of a maximal ideal M_i of $C[t]$.

They have established the nice well-known bijective correspondence between primary decomposable subspaces of $C[t]$ and $\mathfrak{I}_t$ by:

Theorem. 6.3. ([5, Theorem 0.5] (Bijective correspondence theorem))

$$\Gamma : V \mapsto \mathfrak{D}(R, V) \quad , \quad \Gamma^{-1} : I \mapsto I \star 1.$$

The proof of Cannings and Holland´s theorem one can see in [5].

Now we will give some definitions: Let $b, h \in R = k[t]$ and V a k-subspace of $k[t]$. We set

$$O(b) = \{a \in R : a' \in bR\} \quad \textit{and} \quad O(b, h) = \{a \in R : a' + ah \in bR\}$$

where a' denote the formal derivative of a.

$$S(V) = \{a \in R : aV \subseteq V\} \quad and \quad C(R,V) = \{a \in R : aR \subseteq V\}$$

It is clear that $O(b)$ and $S(V)$ are k-subalgebras of $k[t]$. The set C(R,V)is an ideal of R contained in both $S(V)$ and V.

Definition. 6.4.
A k-vector subspace V of $k[t]$ is said to be **primary decomposable** if $S(V)$ contains a k-subalgebra $O(b)$ of $k[t]$, with $b \neq 0$.

Example. 6.5.
$O(b) \subseteq S(O(b,h))$ in particular $O(b,h)$ is primary decomposable subspace when $b \neq 0$.

Proof. Let $a \in O(b)$, then $a' \in bR$. We need to show that $a \in S(o(b,h))$, i.e., $aO(b,h) \subseteq O(b,h)$. Let $x \in aO(b,h)$, then $x = ay$ for some $y \in O(b,h)$, so $y' + yh \in bR$. Now $x' + xh = ay' + ya' + ayh = a(y' + yh) + ya' \in bR$. Therefore $x \in O(b,h)$. □

Now we will show that classical primary decomposable subspaces are primary decomposable in the new way.

Lemma. 6.6.
Let k be a field of characteristic zero and $\lambda_1, \dots, \lambda_n$ finite distinct elements of k. Suppose that $V_1, \dots, V_n$ are k-vector subspaces of $k[t]$, and each V_i contains $(t-\lambda_i)^{r_i}k[t]$ for some $r_i \in \mathbb{N}^\star$. Then

$$O((t-\lambda_1)^{r_1-1}, \dots, (t-\lambda_n)^{r_n-1}) \subseteq S(\bigcap_{i=1}^{n} V_i).$$

Proof. We have $O((t-\lambda_i)^{r_i-1}) = k + (t-\lambda_i)^{r_i}k[t]$. And since $O(lcm(a,b)) = O(a) \cap O(b)$, then one has

$$O((t-\lambda_1)^{r_1-1}, \dots, (t-\lambda_n)^{r_n-1}) = \bigcap_{i=1}^{n} O((t-\lambda_i)^{r_i-1} = \bigcap_{i=1}^{n} k + (t-\lambda_i)^{r_i}k[t]$$
$$\subseteq \bigcap_{i=1}^{n} V_i \subseteq S(\bigcap_{i=1}^{n} V_i).$$

□

As a consequence of this lemma we have:

Corollary. 6.7.
In the above hypothesis of lemma (6.6.), let $V = \bigcap_{i=1}^{n} V_i$*. If* $q \in C(R, V)$*, then* $O(q) \in S(V)$*.*

PROOF. If $q \in bk[t]$, then $O(q) \subseteq O(b)$. Let $b = (t-\lambda_1)^{r_1}, \ldots, (t-\lambda_n)^{r_n}$. In the above hypothesis, we have

$$C(R, V) = \bigcap_{i=1}^{n} C(R, V_i) = \bigcap_{i=1}^{n} (t-\lambda_i)^{r_i} k[t] = (\prod_{i=1}^{n} (t-\lambda_i)^{r_i}) k[t] = bk[t].$$

Since $b \in (t-\lambda_1)^{r_1-1}, \ldots, (t-\lambda_n)^{r_n-1} k[t] = b_0 k[t]$, then $O(b_0) V_i \subseteq V_i$ for all i, so $O(b_0) \subseteq S(V)$ and $O(q) \subseteq O(b) \subseteq O(b_0)$. □

The two definitions are the same when k is algebraically closed field of characteristic zero.

Proposition. 6.8. ([9, Lemma 4])
Let k be algebraically closed field of characteristic zero and V be a k-vector subspace of $k[t]$ such that $S(V)$ contains a k-subalgebra $O(b)$ where $b \neq 0$. Then V is a finite intersection of subspaces which contains a power of a maximal ideal of $k[t]$

Now we can see that the Bijective correspondence theorem is still true according to the new definition of the primary decomposable subspaces of $k[t]$.

Indeed Cannings and Holland´s theorem use the Lemma (6.9.), which holds even the field is just of characteristic zero, and Proposition (6.10.), of M. K. Kouakou and A. Tchoudjem which has been shown according to the new definition; also the converse of this proposition can be given, but with more conditions added. Hence we obtain a characterization of primary decomposable subspaces of $k[t]$.

Lemma. 6.9. ([5, corollary 3.5])
Let $I \in \mathfrak{I}_t$ and $V = I \star 1$*. Then* $I = D(R, V)$*.*

Proposition. 6.10. ([9, Proposision 7])
Let k be field of characteristic zero and V a k-vector subspace of $k[t]$ such that $S(V)$ contains a k-subalgebra $O(b)$. Then $\mathfrak{D}(R, V) \star 1 = V$.

Theorem. 6.11. ([9, Theorem 8])
Let k be a field of characteristic zero and V a k-vector subspace of $k[t]$ such that: $C(R,V) = qk[t]$ with $q \neq 0$ and $\mathfrak{D}(R,V) \star 1 = V$. Then $S(V)$ contains some k-subalgebra $O(b)$ with $b \neq 0$.

We give an example of a subspace of $k[t]$ that is not primary decomposable.

Example. 6.12.
Suppose the field k is of characteristic zero and one can find $q \in k[t]$ such that: q is irreducible and $deg(q) \geq 2$. Then the vector subspace $V = k + qk[t]$ is not primary decomposable.

Proof. Since q is irreducible, then it can be shown by a direct calculations that the right ideal qA_1 is maximal. Clearly one has $qA_1 \subseteq D(R,V)$, and $D(R,V) \neq A_1$ since $1 \notin D(R,V)$. So one has $qA_1 = D(R,V)$. Suppose V is primary decomposable. Applying $R = k[t]$ on the both sides, we get $V = qk[t]$, hence V is not primary decomposable. □

Lemma. 6.13.
Let k be field of characteristic zero, V and W be primary decomposable subspaces of $k[t]$.

(1) $V + W$ and $V \cap W$ are primary decomposable subspaces.

(2) If $q \in k(t)$ such that $qV \subseteq k[t]$, the qV is primary decomposable subspace.

Proof. (1). It is clear that $O(ab) \subseteq O(a) \cap O(b)$ for all $a, b \in k[t]$.So the result is a consequence of this.

(2). As V is primary decomposable subspaces,then $O(b) \subseteq S(V)$ for some $b \neq 0$.That is $O(b)V \subseteq V$,hence $qO(b)V \subseteq qV$,implies $O(b)qV \subseteq qV$.Therefore $O(b) \subseteq S(qV)$. □

Theorem. 6.14.
The set of primary decomposable subspaces of $k[t]$ ordered by inclusion form a Lattice.

Proof. It is clear that it is a partially ordered set and by (1) in the above lemma we have $V + W$ and $V \cap W$ are primary decomposable subspaces moreover they are respectively the **join** and the **meet** for any primary decomposable subspaces V and W of $k[t]$. $\square$

It is well known that the ring of differential operators on R is defined inductively $\mathfrak{D}(R) = \bigcup_{i=0}^{\infty} \mathfrak{D}^i(R)$,where $\mathfrak{D}^0(R) = \mathrm{End}_R(R) \subseteq End_k(R)$ and $\mathfrak{D}^i(R) = \{\theta \in \mathrm{End}_k(R) : [\theta, R] \subseteq \mathfrak{D}^{i-1}(R)\}$, for $i \geq 1$. By convention $\mathfrak{D}^i(R) = 0$ if $i < 0$ and we will identify $R = \mathfrak{D}^0(R)$.

Now we are ready to classify the projective right ideals of A_1 up to isomorphism. For that, we introduce an equivalence relation on $\mathrm{Decomp}(R)$, the set of the primary decomposable subspace of $k[t]$. Let $V, W \in \mathrm{Decomp}(R)$.Then $V \sim W$ if, and only if, $W = qV$,for some $0 \neq q \in k(t)$. Denote the equivalence class of V by $[V]$.

Theorem. 6.15. (Classification projective right ideals)
There is a bijection $\mathrm{Decomp}(R)/\sim \longrightarrow \mathfrak{I}_t/\cong$ *defined by* $[V] \mapsto [\mathfrak{D}(R, V)]$.

Proof. Let ϕ be the given map. It well defined if $0 \neq q \in k(t)$ and $V, qV \in \mathrm{Decomp}(R)$, then

$$\mathfrak{D}(R, qV) = q\mathfrak{D}(R, V) \cong \mathfrak{D}(R, V).$$

That ϕ is surjective follows from Proposition (6.10.). Suppose that $\mathfrak{D}(R, V) = \mathfrak{D}(R, W)$ for some $V, W \in \mathrm{Decomp}(R)$. Then there exist $0 \neq q \in Q_1$ with $q\mathfrak{D}(R, V)] = \mathfrak{D}(R, W)]$. Multiplying on the right by $D = k(t)[\partial] = \mathfrak{D}^1(k(t))$ we see that $q \in D$, say $q \in \mathfrak{D}^n(k(t))\backslash\mathfrak{D}^{n-1}(k(t))$. This means that if $0 \neq \theta \in q\mathfrak{D}(R, V)$ then $\theta \in D\backslash\mathfrak{D}^{n-1}(k(t))$. But $\mathfrak{D}(R, W) \cap R \neq 0$. Thus $n = 0$ and $q \in k(t)$. Evaluating at $R = k[t]$, we have $qV = W$. $\square$

Finally one can note that the correspondence theorem [5, Theorem 0.5], holds for $\mathfrak{D}[k[t_1, ..., t_n] = A_n(k)$. But the classification theorem doesn't hold for $A_n(k)$ when $n \geq 2$. This is because $A_1(k)$ is hereditary that every right ideal is projective, while $A_n(k)$ is not. So the problem of classification of projective right ideals of $A_n(k)$ up to isomorphism still open and not treated completely.

Now we will study a particular case of this work. The set of right projective ideals of $A_1(k)$ that appears as a consequence of Example (6.12.), namely the right ideals that correspond to the primary decomposable subspaces $V = k + qk[t]$ of $k[t]$.

Let k be a field. A polynomial $f \in k[x]$ is associated of a polynomial $g \in k[x]$ if $f = cg$ for some nonzero $c \in k$. A nonconstant polynomial $p \in k[x]$ is said to be irreducible if its only divisors are its associates and the nonzero constants polynomials (i.e., the invertible elements of $k[x]$). A nonconstant polynomial that is not irreducible is said to be reducible.

As a consequence of Example (6.12.), we have the following:

Corollary. 6.16.
For all $q \in k[t]$,if $V = k + qk[t]$ is primary decomposable subspace of $k[t]$, then either q is reducible or $deg(q) < 2$.

We define on $k[t]$ the relation $p \preceq q$ if and only if $p|q$ (p divides q).It is clear that $k[t]$ with this order is a partially ordered set **poset**.

Also we can define on $k[t]$ the following binary operations. For any $p, q \in k[t]$.

$$p \wedge q = \gcd(p, q), \qquad \text{(a greatest common divisor), and}$$

$$p \vee q = \operatorname{lcm}(p, q), \qquad \text{(a least common multiple).}$$

One can see that $k[t]$, with these two binary operations, form a lattice.

On the other hand we have $S = \{V_q |\ V_q = k + qk[t],\ q \in k[t]\}$ is a set of subspaces of $k[t]$. Clearly S is a **poset** if its elements are ordered by inclusion. Moreover S will be a lattice under the following two binary operations.

$$V_p \wedge V_q = k + lcm(p, q)k[t],$$

$$V_p \vee V_q = k + gcd(p, q)k[t].$$

Let $A = \{q \in k[t] : q \text{ is reducible or } deg(q) < 2\}$. Then $A_V = \{V_q |\ V_q = k+qk[t],\ q \in A\}$ is a set of primary decomposable subspaces of $k[t]$. Our aim is to classify the right projective ideals that associated to this subspaces.

Proposition. 6.17.
$q|p$ if and only if $V_p \subseteq V_q$

Proof. Suppose $q|p$, then there exist $f \in k[t]$ such that $qf = p$. Therefore

$$V_p = k + pk[t] = k + qfk[t] \subseteq k + qk[t] = V_q.$$

Conversely, suppose $k + pk[t] \subseteq k + qk[t]$. It is clear that $p \in k + qk[t]$, so $p = \beta + qf$ for some $\beta \in k$ and $f \in k[t]$. Moreover for all $\alpha \in k$ and $g \in k[t]$, we have

$\alpha + pg \in k + qk[t]$, but $\alpha + pg = \alpha + (\beta + qf)g = \alpha + \beta g + qfg$. Therefore $\beta g \in k + qk[t]$, so $\beta g = \gamma + qh$ for some $\gamma \in k$ and $h \in k[t]$. Now if $\beta \neq 0$,then $g = \frac{\alpha}{\beta} + q\frac{h}{\beta}$, but this is for all $g \in k[t]$,which is not possible, thus we must have $\beta = 0$ and therefore $p = qf$. □

In the general problem of classification of projective right ideals of $A_1(k)$, an equivalence relation was defined on the primary decomposable subspaces of $k[t]$ as the following: for any $V, W \in \mathrm{Decomp}(R)$, $V \sim W$ if and only if $W = \alpha V$,for some $0 \neq \alpha \in k(t)$. We can restrict this relation to the set A_V in order to get the classification of the corresponding projective right ideals up to isomorphism.

Lemma. 6.18.
$V_p \sim V_q$ if, and only if, $V_p = \alpha V_q$, for some $\alpha \in k^\star$.

Corollary. 6.19.
(1) $\alpha V_p = V_{\alpha p}$, for all $\alpha \in k^\star$.

(2) $V_p \sim V_q$ if and only if $p = \delta q$ for some $\delta \in k^\star$.

Proof. (1). $\alpha V_p = \alpha(k + pk[t]) = \alpha k + \alpha pk[t] \subseteq k + \alpha pk[t] = V_{\alpha p}$. On the other hand we have $V_{\alpha p} = k + \alpha pk[t] \subseteq \alpha k + \alpha pk[t] = \alpha V_p$.

(2). $V_p \sim V_q \iff V_p = \alpha V_q$, for some $\alpha \in k^\star \iff V_p = V_{\alpha q} \iff p|\alpha q$ and $\alpha q|p$ $\iff pg = \alpha q$ for some $g \in k[t]$ and $\alpha qf = p$ for some $f \in k[t]$. We get $gf = 1$, so $f, g \in k^\star$. □

One can see that there is a 1-1 correspondence between A and A_V defined by $q \mapsto V_q = k + qk[t]$ and hence by Cannings and Holland´s theorem there is a 1-1 correspondence between A and $\{\mathfrak{D}(R, V_q) : q \in A\}$ defined by $q \mapsto \mathfrak{D}(R, V_q)$

Corollary. 6.20.
$\mathfrak{D}(R, V_p) \cong \mathfrak{D}(R, V_q)$ if, and only if, $p = \delta q$ for some $\delta \in k^\star$.

Proof.
$$\mathfrak{D}(R, V_p) \cong \mathfrak{D}(R, V_q) \iff V_p \sim V_q \iff p = \delta q$$
for some $\delta \in k^\star$ □

This means that reducible polynomials with the same degree and same roots correspond to the same isomorphic class of projective right ideals of $A_1(k)$.

7 The monoid $V(A)$

In this section we will consider the ring A of polynomials over non commutative division ring $D[X_1, \ldots, X_n]$ and the n-th Weyl Algebra $A_n(k)$ where k is a field of characteristic zero. Since in each case we have the same structure of finitely generated projective modules and stably free modules of stable rank > 1 are free, the structure of the monoid $V(A)$ could be studied by the same way.

In fact a general structure of the monoid $V(A)$ of isomorphism classes of finitely generated right projective A-modules can be given as the following.

$$V(A) \cong \mathbb{N} \cup CI(A),$$

where $CI(A) = \{[I] |\ [I] + [A] = 2,\ [I] \neq [A]\}$ is the set of all isomorphism classes of the non-free right ideals of A.

Lemma. 7.1.
For any two elements $[I], [J] \in CI(A)$,*we have* $[I] + [J] = 2$.

PROOF. We have $I \oplus A \cong A^2$ and $J \oplus A \cong A^2$. It follows that $I \oplus J \oplus A^2 \cong A^4$, but $I \oplus J$ is stably free module of rank 2, then it is free and so we get $I \oplus J \cong A^2$. □

Let R be any ring; a stably free projective R-module P such that $P \oplus R^r \cong R^m$ has an m-element generating set, so it is called stably m-free. If every stably m-free module over R is free of unique rank, for all $m \leq n$ then R is said to be n-**Hermite**. If R is n-Hermite, for all n, then every stably free module is free, of unique rank, in other words, R is **Hermite**.

A row $(a_1, ..., a_n)$ over a ring R is called **right unimodular** if $a_1R + \cdots + a_nR = R$. If $(a_1, ..., a_n)$ is a right unimodular n-row over a ring R then we say that $(a_1, \ldots, a_n)$ is **reducible** if there exists an $(n-1)$-row $(b_1, \ldots, b_{n-1})$ such that the $(n-1)$-row $(a_1 + a_nb_1, \ldots, a_{n-1} + a_nb_{n-1})$ is right unimodular. A ring R is said to have **stable range** n if n is the least positive integer such that every right unimodular $(n+1)$-row is reducible. This number n will be denoted by $s.r.(R)$.

A ring which satisfies the following properties is called a (right) Hermite ring.

Proposition. 7.2.
For any ring R, the following statements are equivalent.

(1) Any finitely generated stably free right R-module is free.

(2) Any finitely generated stably free right R-module of type 1 is free.

(3) Any right unimodular row over R can be completed to an invertible matrix (by adding a suitable number of new rows).

PROOF. (2) $\Leftrightarrow$ (3). It follows from Proposition (2.2.).

(1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (1). We prove the result by induction on m. This is clear for $m = 1$ by (2). Assume the result for $m-1$, and let P be finitely generated stably free of type m. Then $P \oplus R^m \cong R^n$, i.e., $(P \oplus R^{m-1}) \oplus R \cong R^n$. This implies $P \oplus R^{m-1}$ is stably free of type 1. By (2) it is free. Thus $P \oplus R^{m-1} \cong R^k$ for some k. By induction P is free. □

A right unimodular row which satisfies condition (3) above is called **completable**.

More generally we can give a characterization for the right $(n+1)$–Hermite rings

Corollary. 7.3.
The following properties of a ring R are equivalent:

(1) Any finitely generated stably free right R-module of rank $\geq n$ is free.

(2) Any finitely generated stably free right R-module of rank $\geq n$ and of type 1 is free.

(3) Any unimodular row of length $\geq n+1$ is completable.

Example. 7.4.
$D[X_1, \ldots, X_n]$ and $A_n(k)$ are 3-Hermite rings.

The set of all (right) unimodular rows of length n with entries in R is denoted by $Um_n(R)$.

The group $GL_n(R)$ of invertible $n \times n$ square matrices over R acts on the set $Um_n(R)$ of unimodular rows in the following natural manner: If $v \in Um_n(R)$, $\sigma \in GL_n(R)$, then $v \mapsto v\sigma$.

Note that if $w \in M_{1,r}(R)$ is such that $vw^t = 1$, then $v\sigma(w(\sigma^{-1})^t)^t = 1$, and so $v\sigma \in Um_n(R)$. Thus, the above map defines an action of $GL_n(R)$ on $Um_n(R)$.

If $v' = v\sigma$ for some $\sigma \in GL_n(R)$, then we write this as $v \sim v'$ or $v \sim_{GL_n(R)} v'$.

Proposition. 7.5. ([15, proposition 3.3.1])
The orbits of $Um_n(R)$ under the $GL_n(R)$-action are in one to one correspondence with the isomorphism classes of right R modules P for which $P \oplus R \cong R^n$. Under this correspondence orbit of $(1, 0, \dots, 0)$ corresponds to the free module R^{n-1}.

Corollary. 7.6.
Let $(b_1, \dots, b_n) \in Um_n(R)$. The following statements are equivalent:

(1) $(b_1, \dots, b_n)$ is completable;

(2) $P(b_1, \dots, b_n) \cong R^{n-1}$.

(3) $(b_1, \dots, b_n) \sim (1, 0, \dots, 0)$

Proof. (2) $\Leftrightarrow$ (3). It follows from Proposition (7.5.)

(1) $\Rightarrow$ (3). $(b_1, \dots, b_n) \in Um_n(R)$ is completable to an invertible matrix $M' \in GL_n(R)$. If $M'M = I_n$, then $e_1M'M = (b_1, \dots, b_n)M = e_1I_n = e_1$, i.e., $(b_1, \dots, b_n) \sim e_1$.

(3) $\Rightarrow$ (1). Suppose $(b_1, \dots, b_n) = (1, 0, \dots, 0)M$. Then M is a completion of $(b_1, \dots, b_n)$ to a square invertible matrix. □

Proposition. 7.7.
If R is a right(left) n-Hermite ring then R has stable range $\leq n$.

Proof. Let $a_1R + ... + a_nR + a_{n+1}R = R$. Since R is a right n-Hermite ring, $(a_1, \dots, a_n)P = (d, 0, ..., 0)$ for some $d \in R$, $P = (p_{ij}) \in GL_n(R)$.

Let $P^{-1} = (\alpha_{ij}) \in GL_n(R)$.We claim that

$$(a_1 + a_{n+1}\alpha_{n1}) + ... + (a_n + a_{n+1}\alpha_{nn})$$

is a right unimodular row. We have $(a_1 + a_{n+1}\alpha_{n1})p_{1n} + \cdots + (a_n + a_{n+1}\alpha_{nn})p_{nn} = a_1p_{1n} + \cdots + a_np_{nn} + a_{n+1}(\alpha_{n1}p_{1n} + \cdots + \alpha_{nn}p_{nn}) = 0 + a_{n+1}.1 = a_{n+1}$ and $(a_1 +$

$a_{n+1}\alpha_{n1})p_{11} + \cdots + (a_n + a_{n+1}\alpha_{nn})p_{n1} = a_1 p_{11} + \cdots + a_n p_{n1} + a_{n+1}(\alpha_{n1}p_{11} + \cdots + \alpha_{nn}p_{n1}) = d + a_{n+1}.0 = d$. Therefore,

$$a_{n+1}, d \in (a_1 + a_{n+1}\alpha_{n1})R + \cdots + (a_n + a_{n+1}\alpha_{nn})R.$$

Since $(a_1, \ldots, a_n)P = (d, 0, \ldots, 0)$, we obtain $a_1R + \cdots + a_nR = dR$.

On the other hand, we have $a_1R + ... + a_nR + a_{n+1}R = R$ then $dR + a_{n+1}R = R$.Since $a_{n+1}, d \in (a_1 + a_{n+1}\alpha_{n1})R + ... + (a_n + a_{n+1}\alpha_{nn})R$,we see that $(a_1 + a_{n+1}\alpha_{n1})R + ... + (a_n + a_{n+1}\alpha_{nn})R = R$ and $s.r.(R) = n$. □

Corollary. 7.8.
Any stably free right R-module M with $rank(M) \geq s.r(R)$ is free

Example. 7.9.
$s.r.(D[X_1, \ldots, X_n]) = 2$,and $s.r.(A_n(k)) = 2$

Thus the right invertible 1×2 matrices over A cannot be completable in general. Only the matrices that associated to free modules are. So to determine the structure of the monoid $V(A)$ we need to study the isomorphism classes of the stably free non free modules of *rank* 1.

We will determine the isomorphism classes of the these modules depending on two crucial facts, each projective module generated by two elements and looking to them as R-submodules of R^2.

let I and J are two non-free stably free R-modules of stable *rank* 1, then we have $I \oplus R \cong R^2$ and $J \oplus R \cong R^2$, so I and J are generated by two elements.

Suppose $I = \langle g_1, g_2 \rangle \subseteq R^2 = \langle e_1, e_2 \rangle$, then

$$g_1 = d_{11}e_1 + d_{12}e_2 = \begin{pmatrix} d_{11} \, d_{12} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$g_2 = d_{21}e_1 + d_{22}e_2 = \begin{pmatrix} d_{21} \, d_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

let $x \in I$, then $x = a_1g_1 + a_2g_2 = \begin{pmatrix} a_1 \, a_2 \end{pmatrix} \begin{pmatrix} d_{11} \, d_{12} \\ d_{21} \, d_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$.

Similarly if $J = \langle h_1, h_2 \rangle \subseteq A^2 = \langle e_1, e_2 \rangle$, then

$$h_1 = f_{11}e_1 + f_{12}e_2 = \begin{pmatrix} f_{11} \, f_{12} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$h_2 = f_{21}e_1 + f_{22}e_2 = \begin{pmatrix} f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

and for $y \in J$, $y = b_1h_1 + b_2h_2 = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$.

Suppose $I \cong J$,then there exist a left R-isomorphism $\alpha : I \to J$ where $\alpha(g_i) = a_{i1}h_1 + a_{i2}h_2$,so

$$\alpha(x) = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \alpha \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

Also there is a a left A-isomorphism $\beta : J \to I$ where $\beta(h_i) = b_{i1}g_1 + b_{i2}g_2$, and so

$$\beta(y) = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \beta \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$= \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

Now we have: $\alpha\beta(y) = y$ where $y = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

$$y = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \xrightarrow{\beta} \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \xrightarrow{\alpha} \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} b_1 & b_2 \end{pmatrix} F = \begin{pmatrix} b_1 & b_2 \end{pmatrix} BAF. \tag{1}$$

Similarly $\beta\alpha(x) = x$ implies

$$\begin{pmatrix} a_1 & a_2 \end{pmatrix} D = \begin{pmatrix} a_1 & a_2 \end{pmatrix} ABD. \tag{2}$$

Where,

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}, D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

As a result of this calculation we have:

Theorem. 7.10.
$I \cong J$ if and only if there exist a structure of matrices satisfying equations (1) and (2).

Now an equivalent conditions under which every finitely generated stably free module of positive rank over an associative ring is power-free has given by the following result due to Chen.

A stably free (finitely generated) right R-module P of rank s is **power–free** if there exists some $n \in \mathbb{N}$ such that $P^n = R^{ns}$

For any R epimorphism $\sigma : R^m \longrightarrow R^n$, we have a matrix $A \in M_{n\times m}(R)$ corresponding to σ . Let $A = (a_{ij}) \in M_{n\times m}(R)$, and let $I_s = diag(1, \dots, 1) \in M_s(R)$. We use the Kronecker product $A \otimes I_s$ to stand for the matrix $(a_{ij}I_s) \in M_{ns\times ms}(R)$.

Theorem. 7.11. ([6, Theorem 2.1])
Let R be a ring. Then the following are equivalent:

(1) Every finitely generated stably free right R-module of positive rank is power-free.

(2) For any right invertible rectangular matrix (a_{ij}), there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.

Since the stably free right A modules are power-free, then we can get the following result concerns to the completion of the right invertible rectangular matrices over A. Indeed s can be chosen such that $s \geq 2$

Corollary. 7.12.
Let (a_{ij}) be a right invertible rectangular matrix over a A. Then there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.

Proof. Since every finitely generated stably free right A-module of *rank* ≥ 2 is free, and the non-free stably free right A-modules P with *rank* 1 have the relation

$P \oplus P \cong R^2$. So every finitely generated stably free right A-module of positive rank is power-free. In view of Theorem (7.11.), we get the result. □

For instance, the right invertible rectangular matrices (a_{ij}) over A correspond to the non-free stably free right A-modules P with the relation $P \oplus P \cong R^2$ can not be completed to an invertible matrix, while the matrices $(a_{ij} I_s)$ can be completed to an invertible matrix for $s \geq 2$.

8 More about the structure of $V(A)$

We will introduce the basic properties and a general structure of the monoid $V(R)$ for any ring R, and we will apply one of the main results for $D[X_1, \dots, X_n]$ and $A_n(k)$. In particular we can get information about the ring R comes from the monoid $V(R)$.

Let M be a commutative monoid, we denote by $U(M)$ the set of all elements $a \in M$ with an opposite $-a \in M$, and we say that M is **reduced** if $U(M) = 0$.

There is a natural pre-order (reflexive and transitive relation) on any commutative additive monoid M, called the algebraic pre-order on M, defined by $x \leq y$ if there exist $z \in M$ such that $x + z = y$. An element u of M is an **order–unit** if for every $x \in M$ there exists an integer $n \geq 0$ such that $x \leq nu$.

Theorem. 8.1. ([2, Theorem 6.2 and 6.4] and [[3, p.315]])
A monoid M is isomorphic to the monoid $V(R)$ for some ring R if, and only if, it is a commutative monoid that is reduced and has an order-unit.

$V(R)$ describes the behavior of direct-sum decomposition of finitely generated projective R-modules up to isomorphism,in the sense that to every decomposition of a projective module $A_R \in \text{proj}(R)$ as a direct sum of finitely many submodules there corresponds a decomposition of element $\langle A_R \rangle$ of the monoid $V(R)$ as a sum of elements of $V(R)$, and two direct-sum decomposition of A_R are isomorphic in the sense of the Krull-Schmidt theorem if and only if they correspond to the same sum decomposition of $\langle A_R \rangle$ in the monoid $V(R)$ up to the order of summands.

A submonoid N of a commutative monoid M is said to be **divisor–closed** if $x \in M$, $y \in N$ and $x \leq y$ in M implies $x \in N$.For each $x \in M$ we denote by $[x]$ the smallest divisor-closed submonoid of M containing x. It is the set of all $y \in M$ with $y \leq nx$ for some $n \geq 0$.The order units of M are exactly the elements $u \in M$ such that $M = [u]$.

A commutative semigroup S is **archimedean** if for every pair (x, y) of elements of S there exist a positive integer n with $x \leq ny$. More generally, let M be a commutative monoid. For $x, y \in M$, define $x \asymp y$ if there exist positive integers n and m such that $x \leq ny$ and $y \leq mx$. Thus $x \asymp y$ if, and only if, $[x] = [y]$.

The relation $\asymp$ is the least congruence relation on M such that every element in the quotient monoid $M/\asymp$ is idempotent.The equivalence classes of M modulo $\asymp$ are additively closed subsets of M, called the **archimedean components** of M.

Let R be a ring. For any subclass $\mathcal{U}$ of proj(R), the ideal $Tr_R(\mathcal{U})$ will denote the **trace** of $\mathcal{U}$ in R, that is, the sum of all images $f(A_R)$ where A_R ranges in the modules $A_R \in \mathcal{U}$ and f ranges in the homomorphisms from A_R into R_R.

If $\mathcal{U}$ has a unique elements A_R, we write $Tr_R(A_R)$ instead of $Tr_R(\mathcal{U})$. The trace $Tr_R(\mathcal{U})$ is characterized as the smallest two-sided ideal I of R such that $A_R I = A_R$ for every $A_R \in \mathcal{U}$.

We call **trace ideals** of R all two-sided ideals of R equal to $Tr_R(\mathcal{U})$ for some subclass $\mathcal{U}$ of proj(R),**finitely generated trace ideals** the ideals equal to $Tr_R(\mathcal{U})$ for some finite subset $\mathcal{U}$ of $proj(R)$, and **maximal trace ideals** the trace ideals of R that are maximal in the set of all proper trace ideals of R partially ordered by set inclusion.Every trace ideal contained in a maximal trace ideal.

We will denote the set of all finitely generated trace ideals of R and trace ideals of R by $\mathcal{T}_{fg}(R)$ and $\mathcal{T}(R)$ respectively.

A **prime ideal** of a commutative monoid M is a proper subset P of M such that, for any $x, y \in M$ we have $x + y \in P$ if, and only if, either $x \in P$ or $y \in P$. The empty set Φ is the smallest prime ideal of every commutative monoid.

Finally we denote $add(M_R)$ to the subcategory of $Mod - R$ whose objects are all R-modules isomorphic to direct summand of finite direct sums M_R^n of copies of M.

Proposition. 8.2. ([7, Proposision 1.1])
Let R be a ring and $A_R, B_R \in$ proj(R). The following conditions are equivalent:

(1) $\langle A_R \rangle$ and $\langle B_R \rangle$ belongs to the same archimedean component of $V(R)$.

(2) $add(A_R) = add(B_R)$.

(3) $Tr(A_R) = Tr(B_R)$.

Thus there is a one-to-one correspondence between:

1. the set of all archimedean components of $V(R)$.

2. the set $\mathcal{T}_{fg}(R)$ of all finitely generated trace ideals of R.

3. The set of all prime ideals P of $V(R)$ of the type $P = V(R) \setminus (\langle A_R \rangle)$ for some $\langle A_R \rangle \in V(R)$.

Corollary. 8.3.
The monoid $V(A)$ has only the two trivial archimedean components.

PROOF. For the ring A we have seen that $V(A) \cong \mathbb{N} \cup \{[I] | \ I \oplus R \cong R^2\}$, it is clear that all the non-zero elements of $V(A)$ belong to the same archimedean component,hence it has exactly two archimedean components, one with only the zero element, and the other with all non-zero, and then by the correspondence $V(A)$ has only the two trivial prime ideals 0 and $V(R) \setminus \{0\}$, and therefore A has only the two trivial trace ideals A and $\{0\}$. □

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