# Managing the Consensus in Group Decision Making in an Unbalanced Fuzzy Linguistic Context with Incomplete Information

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**Abstract**

To solve group decision making problems we have to take in account different aspects. On the one hand, depending on the problem, we can deal with different types of information. In this way, most group decision making problems based on linguistic approaches use symmetrically and uniformly distributed linguistic term sets to express experts' opinions. However, there exist problems whose assessments need to be represented by means of unbalanced linguistic term sets, i.e., using term sets which are not uniformly and symmetrically distributed. On the other hand, there may be cases in which experts do not have an in-depth knowledge of the problem to be solved. In such cases, experts may not put their opinion forward about certain aspects of the problem and, as a result, they may present incomplete information. The aim of this paper is to present a consensus model to help experts in all phases of the consensus reaching process in group decision making problems in an unbalanced fuzzy linguistic context with incomplete information. As part of this consensus model, we propose an iterative procedure using consistency measures to estimate the incomplete information. In addition, the consistency measures are used together with consensus measures to guided the consensus model. The main novelty of this consensus model is that it supports the management of incomplete unbalanced fuzzy linguistic information and it allows to achieve consistent solutions with a great level of agreement.

**Keywords:** group decision making, unbalanced linguistic term set, incomplete information, consensus, consistency.

# 1   Introduction

The increasing complexity of the social-economic environment nowadays has caused that the decision making processes are being widely studied [15, 18]. Many organizations have moved from a single decision maker or

expert to a group of experts to accomplish this task successfully. A Group Decision Making (GDM) problem is usually understood as a decision problem which consists in finding the best alternative(s) from a set of feasible alternatives, $X = \{x_1, \ldots, x_n\}$, according to the preferences provided by a group of experts, $E = \{e_1, \ldots, e_m\}$, characterized by their experience and knowledge. To do this, experts have to express their preferences by means of a set of evaluations over the set of alternatives.

In this paper, we assume that experts use preference relations [8, 31, 47, 48], amongst other reasons, because they are a useful tool in the aggregation of experts preferences into group preference [8, 9, 10, 31, 32, 35, 45, 48] and focuses exclusively on two alternatives at a time, which facilitates experts when expressing their preferences. However, this way of providing preferences limits experts in their global perception of the alternatives and, as a consequence, the provided preferences could be not rational. Usually, rationality is related to consistency, which is associated with the *transitivity property*. Many properties have been suggested to model transitivity of a fuzzy preference relation [32]. One of these properties is the *additive consistency*, which, as shown in [32], can be seen as the parallel concept of Saaty's consistency property in the case of multiplicative preference relations [47]. Obviously, the consistent information, i.e., information which does not imply any kind of contradiction, is more relevant or important than information containing some contradictions. Thus, it would be of great importance to measure the level of consistency of each expert in the GDM problem.

In these problems, a difficulty that has to be addressed is the lack of information. As aforementioned, each expert has his/her own experience concerning the problem being studied, which also may imply a major drawback, that of an expert not having a perfect knowledge of the problem to be solved. Indeed, there may be cases where an expert would not be able to efficiently express any kind of preference degree between two or more of the available options. This may be due to an expert not possessing a precise or sufficient level of knowledge of part of the problem, or because that expert is unable to discriminate the degree to which some options are better than others. Experts in these situations would rather not guess those preference degrees and, as a consequence, they might provide incomplete information [1, 2, 4, 29, 30, 38, 39, 45, 49, 50]. Therefore, it would be of great importance to provide the experts with tools that allow them to express this lack of knowledge in their opinions.

Another important issue to bear in mind is the different types of information used by the experts to provide their opinions. Usually, many problems present quantitative aspects which can be assessed by means of precise numerical values [8, 30, 29, 37]. However, some problems present also qualitative aspects that are complex to assess by means of precise and exact values. In these cases, the fuzzy linguistic approach [19, 34, 41, 49, 50, 55, 56, 57] can be used to obtain a better solution. This is the case, for instance, when experts try to evaluate the "comfort" of a car, where linguistic terms like "good", "fair", "poor" are used [40]. Many of these problems use linguistic variables assessed in linguistic term sets whose terms are uniformly and symmetrically distributed, i.e., assuming the same discrimination levels on both sides of mid linguistic term. However, there exist problems that need to assess their variables with linguistic term sets that are not uniformly and symmetrically distributed [21, 33]. This type of linguistic term sets are called *unbalanced linguistic term sets* (see Fig. 1).
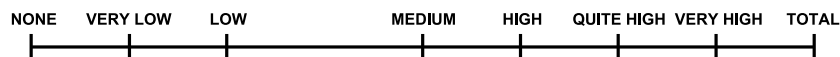


Figure 1: Example of an unbalanced linguistic term set of 8 labels

To solve GDM problems, the experts are faced by applying two processes before obtaining a final solution [22, 25, 31, 36, 37]: *the consensus process* and *the selection process* (see Fig. 2). The former consists in obtaining the maximum degree of consensus or agreement between the set of experts on the solution set of alternatives. Normally, the consensus process is guided by a human figure called moderator [7, 22, 25, 36], who is a person that does not participate in the discussion but monitors the agreement in each moment of the consensus process and is in charge of supervising and addressing the consensus process toward success, i.e., to achieve the maximum possible agreement and to reduce the number of experts outside of the consensus in each new consensus round. The latter refers to obtaining the solution set of alternatives from the opinions on the alternatives given by the experts. It involves two different steps [26, 46]: aggregation of individual opinions and exploitation of the collective opinion. Clearly, it is preferable that the set of experts achieves a great agreement among their opinions before applying the selection process and, therefore, in this paper we focus on the consensus process.
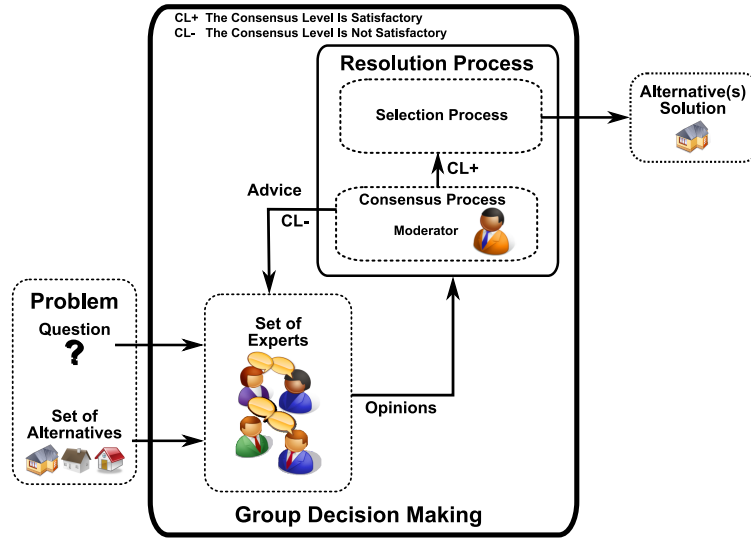


Figure 2: Resolution process of a GDM problem

A consensus process is defined as a dynamic and iterative group discussion process, coordinated by a moderator helping experts bring their opinions closer. If the consensus level is lower than a specified threshold, the moderator would urge experts to discuss their opinions further in an effort to bring them closer. On the contrary, when the consensus level is higher than the threshold, the moderator would apply the selection process in order to obtain the final consensus solution to the GDM problem. In this framework, an important question is how to substitute the actions of the moderator in the group discussion process in order to automatically model the whole consensus process. Some automatic consensus approaches have been proposed in [6, 29, 31, 34, 42]. Most of these consensus models use only consensus measures to control and guide the consensus process. However, if a consensus process is seen as a type of persuasion model [16], other criteria could be used to guide consensus reaching processes as, for example, the *cooperation* or *consistency criterion*. Some fuzzy consensus approaches based on both consistency and consensus measures can be found in [14, 17, 24, 29].

The aim of this paper is to present a consensus model to deal with GDM problems in which experts use incomplete unbalanced fuzzy linguistic preference relations to provide their preferences. This consensus model will not only be based on consensus measures but also on consistency measures. We use two kinds of consensus measures to guide the consensus reaching process, consensus degrees, which evaluate the agreement of all the

experts, and proximity measures, which evaluate the agreement between the experts' individual opinions and the group opinion. To compute them, first, all missing values are estimated using a consistency based estimation procedure. This estimation procedure is based on the Tanino's consistency principle and makes use of all the estimation possibilities that derive from it. In this approach, the computation of missing values in an expert's incomplete unbalanced fuzzy linguistic preference relation is done using only the preference values provided by that particular expert. By doing this, it is assured that the reconstruction of the incomplete unbalanced fuzzy linguistic preference relation is compatible with the rest of the information provided by that expert. Also, the main aim in the design of these approaches is to maintain or maximise the expert's global consistency, as it has been shown in [11]. Afterwards, some consistency measures for each expert are computed. Both consistency and consensus measures are used to design a feedback mechanism, and, in such a way, we substitute the actions of the moderator and give advice to the experts on how they should change and complete their opinions to obtain a solution with a high consensus degree (making experts' opinions closer).

The rest of the paper is set out as follows. Section 2 deals with the preliminaries necessary to develop our consensus model. In Section 3, the consensus model for GDM problems with incomplete unbalanced fuzzy linguistic information is presented. Section 4 shows a practical example to illustrate the application of the consensus model. Finally, some concluding remarks are pointed out in Section 5.

## 2 Preliminaries

In this section, we briefly present the tools necessary to design the consensus model, that is, the methodology used to manage unbalanced fuzzy linguistic information, the concept of incomplete unbalanced fuzzy linguistic preference relation, consistency measures and the consistency based procedure to estimate missing values.

### 2.1 Methodology to Manage Unbalanced Fuzzy Linguistic Information

To manage unbalanced fuzzy linguistic information, we propose a methodology similar to those proposed in [6, 21, 33]. This methodology is based on the transformation of the unbalanced fuzzy linguistic information in a *Linguistic Hierarchy (LH)* [28], which is the linguistic representation domain that allows us to develop comparison and combination processes of unbalanced fuzzy linguistic information.

A $LH$ is a set of levels, where each level represents a linguistic term set with different granularity from the remaining levels of the hierarchy. Each level is denoted as $l(t, n(t))$, where $t$ is a number indicating the level of the hierarchy, and $n(t)$ is the granularity of the linguistic term set of $t$. Then, a $LH$ can be defined as the union of all levels $t$.

Given a $LH$, we denote as $\mathcal{S}^{n(t)}$ the linguistic term set of $LH$ corresponding to the level $t$ of $LH$ characterized by a cardinality $n(t)$: $\mathcal{S}^{n(t)} = \{s_0^{n(t)}, \ldots, s_{n(t)-1}^{n(t)}\}$. Furthermore, the linguistic term set of the level $t+1$ is obtained from its predecessor as: $l(t, n(t)) \rightarrow l(t+1, 2 \cdot n(t) - 1)$.

The procedure to represent unbalanced fuzzy linguistic information presents the following steps:

1. Find a level $t^-$ of $LH$ to represent the subset of linguistic terms $\mathcal{S}_{un}^L$ on the left of the mid linguistic term of unbalanced fuzzy linguistic term set $\mathcal{S}_{un}$.

2. Find a level $t^+$ of $LH$ to represent the subset of linguistic terms $\mathcal{S}_{un}^R$ on the right of the mid linguistic term of $\mathcal{S}_{un}$.

3. Represent the mid term of $\mathcal{S}_{un}$ using the mid terms of the levels $t^-$ and $t^+$.

If there does not exist a level $t^-$ or $t^+$ in $LH$ to represent $\mathcal{S}_{un}^L$ or $\mathcal{S}_{un}^R$, respectively, then the procedure applies the following recursive algorithm, which is defined, in this case, assuming that there does not exist $t^-$, as it happens with the unbalanced fuzzy linguistic term set given in Fig. 1:

1. Represent $\mathcal{S}_{un}^L$:

    (a) Identify the mid term of $\mathcal{S}_{un}^L$, called $\mathcal{S}_{mid}^L$.

    (b) Find a level $t_2^-$ of the left sets of $LH^L$ to represent the left term subset of $\mathcal{S}_{un}^L$, where $LH^L$ represents the left part of $LH$.

    (c) Find a level $t_2^+$ of the right sets of $LH^L$ to represent the right term subset of $\mathcal{S}_{un}^L$.

    (d) Represent the mid term $\mathcal{S}_{mid}^L$ using the levels $t_2^-$ and $t_2^+$.

2. Find a level $t^+$ of $LH$ to represent the subset of linguistic terms $\mathcal{S}_{un}^R$.

3. Represent the mid term of $\mathcal{S}_{un}$ using the levels $t_2^+$ and $t^+$.

For example, applying this algorithm, the representation of the unbalanced fuzzy linguistic term set $\mathcal{S}_{un} = \{N, VL, L, M, H, QH, VH, T\}$, shown in Fig. 1, using a linguistic hierarchy $LH$ would be as it is shown in Fig. 3. In this example:

1. As there does not exist a level $t^-$ in $LH$ to represent the subset of linguistic terms $\mathcal{S}_{un}^L = \{N, VL, L\}$ on the left of the mid linguistic term of $\mathcal{S}_{un}$, which is $M$, we apply the above recursive algorithm:

    (a) Identify the mid term of $\mathcal{S}_{un}^L = \{N, VL, L\}$. In this example, $\mathcal{S}_{mid}^L = \{L\}$.

    (b) Find a level $t_2^-$ of the left sets of $LH^L$ to represent the left term subset of $\mathcal{S}_{un}^L$, where $LH^L = \{s_0^{n(1)}\} \bigcup \{s_0^{n(2)}, s_1^{n(2)}\} \bigcup \{s_0^{n(3)}, s_1^{n(3)}, s_2^{n(3)}, s_3^{n(3)}\}$ represents the left part of $LH$. In this case, $t_2^-$ is represented using the level 3.

    (c) Find a level $t_2^+$ of the right sets of $LH^L$ to represent the right term subset of $\mathcal{S}_{un}^L$. In this case, $t_2^+$ is represented using the level 2.

    (d) Represent the mid term $\mathcal{S}_{mid}^L = \{L\}$ using the levels $t_2^-$ and $t_2^+$, i.e., the levels 3 and 2 of $LH$.

2. The subset of linguistic terms $\mathcal{S}_{un}^R = \{H, QH, VH, T\}$ on the right of the mid linguistic term of $\mathcal{S}_{un}$ is represented using the level 3, i.e., $t^+ = 3$.

3. The mid term of $\mathcal{S}_{un}$, which is $M$, is represented using the mid terms of the levels $t^-$ and $t^+$, i.e., using the mid terms of the levels 2 and 3.

To operate with the linguistic information in $LH$, the 2-tuple fuzzy linguistic model [27] is used.

**Definition 2.1.** Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set and $\beta \in [0, g]$ a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to $\beta$ is obtained with the following function:
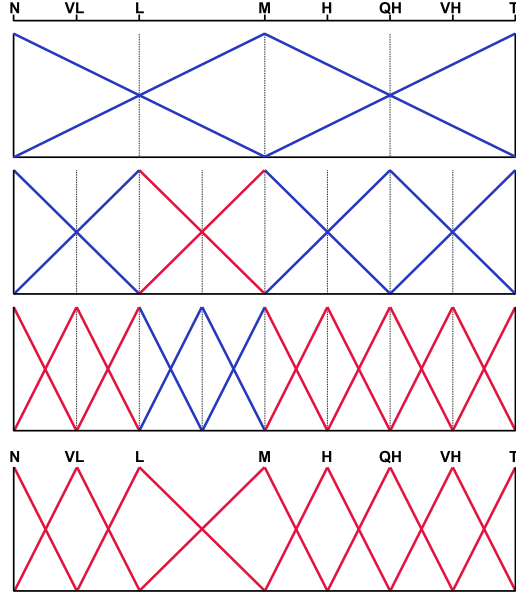
$$\Delta \colon [0, g] \longrightarrow S \times [-0.5, 0.5)$$

Figure 3: Representation for an unbalanced linguistic term set of 8 labels using a linguistic hierarchy

$$\Delta(\beta) = (s_i, \alpha), \ con \begin{cases} s_i, & i = round(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5), \end{cases} \tag{1}$$

where $round(\cdot)$ is the usual round operation, $s_i$ has the closest index label to "$\beta$", and "$\alpha$" is the value of the symbolic translation.

**Proposition 2.1.** *Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set and $(s_i, \alpha)$ be a 2-tuple. There is always a $\Delta^{-1}$ function such that from a 2-tuple it returns its equivalent numerical value $\beta \in [0, g]$.*

$$\Delta^{-1} \colon S \times [-0.5, 0.5) \longrightarrow [0, g]$$

$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta. \tag{2}$$

Finally, transformation functions between labels from different levels to make processes of computing with words in multigranular linguistic information contexts without loss of information were defined in [28].

**Definition 2.2.** [28] Let $LH = \bigcup_t l(t, n(t))$ be a linguistic hierarchy whose linguistic term sets are denoted as $\mathcal{S}^{n(t)} = \{s_0^{n(t)}, \ldots, s_{n(t)-1}^{n(t)}\}$, and let us consider the 2-tuple fuzzy linguistic representation. The transformation function from a linguistic label in level $t$ to a label in level $t'$ is defined as $TF_{t'}^t \colon l(t, n(t)) \longrightarrow l(t', n(t'))$ such that

$$TF_{t'}^t(s_i^{n(t)}, \alpha^{n(t)}) = \Delta_{t'} \left( \frac{\Delta_t^{-1}(s_i^{n(t)}, \alpha^{n(t)}) \cdot (n(t') - 1)}{n(t) - 1} \right). \tag{3}$$

## 2.2 Incomplete Unbalanced Fuzzy Linguistic Preference Relations

As aforementioned, among the different representation formats that experts may use to express their opinions, we assume that experts use preference relations because of their effectiveness as a tool for modelling decision processes and their utility and easiness of use when we want to aggregate experts' preferences into group ones [32, 35, 48]. A preference relation is defined as $P^h \subset X \times X$, where the value $\mu_{P^h}(x_i, x_k) = p_{ik}^h$ is interpreted as the preference degree of the alternative $x_i$ over $x_k$ for the expert $e_h$. According to the nature of the information

expressed for every pair of alternatives, there exist many different representation formats of preference relations [5, 8, 20, 34, 35, 43, 44, 47].

In this paper, we deal with GDM problems in an unbalanced fuzzy linguistic context, i.e., GDM problems where the experts $e_h$ express their preferences relations $P^h = (p_{ik}^h)$ on the set of alternatives $X$ using an unbalanced linguistic term set, $\mathcal{S}_{un} = \{s_0, \ldots, s_{mid}, \ldots, s_g\}$, which has a minimum label, called $s_0$, a maximum label, called $s_g$, and the remaining labels are non-uniformly and non-symmetrically distributed around the central one, called $s_{mid}$ (Fig. 1). Therefore, $p_{ik}^h \in \mathcal{S}_{un}$ represents the preference of alternative $x_i$ over alternative $x_k$ for the experts $e_h$ assessed on the unbalanced fuzzy linguistic term set $\mathcal{S}_{un}$.

**Definition 2.3.** An unbalanced fuzzy linguistic preference relation $P^h$ on a set of alternatives $X$ is characterized by a membership function:

$$\mu_{P^h} : X \times X \longrightarrow \mathcal{S}_{un}.$$

When cardinality of $X$ is small, the preference relation may be conveniently represented by a $n \times n$ matrix $P^h = (p_{ik}^h)$, being $p_{ik}^h = \mu_{P^h}(x_i, x_k)$, $\forall i, k \in \{1, \ldots, n\}$ and $p_{ik}^h \in \mathcal{S}_{un}$.

As aforementioned, missing information is a problem that needs to be addressed because it is not always possible for the experts to provide all the possible preference assessments on the set of alternatives. A missing value in an unbalanced linguistic preference relation is not equivalent to a lack of preference of one alternative over another. A missing value can be the result of the incapacity of an expert to quantify the degree of preference of one alternative over another. It must be clear then that when an expert $e_h$ is not able to express the particular value $p_{ik}^h$, because he/she does not have a clear idea of how better alternative $x_i$ is over alternative $x_k$, this does not mean that he/she prefers both options with the same intensity.

In order to model these situations, in the following definitions we express the concept of an incomplete unbalanced fuzzy linguistic preference relation:

**Definition 2.4.** A function $f : X \times Y$ is partial when not every element in the set $X$ necessarily maps to an element in the set $Y$. When every element from the set $X$ maps to one element of the set $Y$ then we have a total function.

**Definition 2.5.** An unbalanced fuzzy linguistic preference relation $P^h$ on a set of alternatives $X$ with a partial membership function is an incomplete unbalanced fuzzy linguistic preference relation.

Obviously, an unbalanced fuzzy linguistic preference relation is complete when its membership function is a total one. Clearly, definition (2.3) includes both definitions of complete and incomplete unbalanced fuzzy linguistic preference relations. However, as there is no risk of confusion between a complete and incomplete unbalanced fuzzy linguistic preference relation, in this paper we refer to the first type as simply unbalanced fuzzy linguistic preference relation.

## 2.3  Consistency Measures

In real GDM problems with preference relations, some properties about the preferences expressed by the experts are usually assumed desirable to avoid contradictions in their opinions, that is, to avoid inconsistent opinions. However, the previous definition of an unbalanced fuzzy linguistic preference relation does not imply any kind of consistency property. In fact, preference values of a preference relation can be contradictory. Obviously, an

inconsistent source of information is not as useful as a consistent one and, thus, it would be quite important to be able to *measure* the consistency of the information provided by experts for a particular problem.

One of these properties is the *transitivity property*, which represents the idea that the preference value obtained by directly two alternatives should be equal to or greater than the preference value between those two alternatives obtained using an indirect chain of alternatives. There are several possible characterizations for the transitivity property (see [32]). In this paper, we make use of the *additive transitivity property*, which can be seen for fuzzy preference relations as the parallel concept of Saaty's consistency property for multiplicative preference relations [47]. The mathematical formulation of the *additive transitivity* was given in [48]:

$$(p_{ij}^h - 0.5) + (p_{jk}^h - 0.5) = (p_{ik}^h - 0.5), \ \forall i,j,k \in \{1, \ldots, n\}. \tag{4}$$

In the case of an unbalanced fuzzy linguistic context, previously to carry out any computation task, we have to choose a level $t' \in \{t^-, t_2^-, t^+, t_2^+\}$, such that $n(t') = max\{n(t^-), n(t_2^-), n(t^+), n(t_2^+)\}$. Then, once a result is obtained, it is transformed to the correspondent level $t \in \{t^-, t_2^-, t^+, t_2^+\}$ by means of $TF_t^{t'}$ for expressing the result in the unbalanced fuzzy linguistic term set $\mathcal{S}_{un}$. In this way, the unbalanced fuzzy linguistic additive transitivity for unbalanced fuzzy linguistic preference relations is defined as:

$$
\begin{aligned}
TF_t^{t'}(\Delta_{t'}[(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ij}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}))) + (\Delta_{t'}^{-1}(TF_{t'}^t(p_{jk}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid})))]) = \\
TF_t^{t'}(\Delta_{t'}[(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ik}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid})))]), \quad \forall i,j,k \in \{1, \ldots, n\},
\end{aligned}
\tag{5}
$$

being $p_{ij}^h = (s_v^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, $p_{jk}^h = (s_w^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, $p_{ik}^h = (s_z^{n(t)}, \alpha_3)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, $s_{mid}$ is the mid term of $\mathcal{S}_{un}$ and $t' \in \{t^-, t_2^-, t^+, t_2^+\}$.

As in the case of additive transitivity, the unbalanced fuzzy linguistic additive transitivity implies unbalanced fuzzy linguistic additive reciprocity. Indeed, because $p_{ii}^h = (s_{mid}, 0)$, $\forall i$, if we make $k = i$ in (5), then we have: $TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ij}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(p_{ji}^h)))) = (s_g, 0)$, $\forall i,j \in \{1, \ldots, n\}$.

Expression (5) can be rewritten as:

$$p_{ik}^h = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ij}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(p_{jk}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0)))), \quad \forall i,j,k \in \{1, \ldots, n\}. \tag{6}$$

An unbalanced fuzzy linguistic preference relation will be considered "additive consistent" when for every three options in the problem, $x_i, x_j, x_k \in X$, their associated unbalanced fuzzy linguistic preference degrees, $p_{ij}^h, p_{jk}^h, p_{ik}^h$, fulfil (6). An additive consistent unbalanced fuzzy linguistic preference relation will be referred as consistent throughout the paper, as this is the only transitivity property we are considering.

Expression (6) can be used to calculate an estimated value of a preference degree using other preference degrees in an unbalanced fuzzy linguistic preference relation. Indeed, the preference value $p_{ik}^h$ $(i \neq k)$ can be estimated using an intermediate alternative $x_j$ in three different ways:

1. From $p_{ik}^h = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ij}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(p_{jk}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0))))$ we obtain the estimate

$$(cp_{ik}^h)^{j1} = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ij}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(p_{jk}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0)))). \tag{7}$$

2. From $p_{jk}^h = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ji}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(p_{ik}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0))))$ we obtain the estimate

$$(cp_{ik}^h)^{j2} = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{jk}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(p_{ji}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0)))). \tag{8}$$

3. From $p_{ij}^h = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ik}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(p_{kj}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0))))$ we obtain the estimate

$$(cp_{ik}^h)^{j3} = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ij}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(p_{kj}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0)))). \quad (9)$$

The overall estimated value $cp_{ik}^h$ of $p_{ik}^h$ is obtained as the average of all possible $(cp_{ik}^h)^{j1}$, $(cp_{ik}^h)^{j2}$ and $(cp_{ik}^h)^{j3}$ values:

$$cp_{ik}^h = TF_t^{t'}(\Delta_{t'}\left(\frac{\sum_{j=1; i \neq k \neq j}^n (\Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j1})) + \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j2})) + \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j3})))}{3(n-2)}\right)). \quad (10)$$

When the information provided is completely consistent, then $(cp_{ik}^h)^{jl} = p_{ik}^h$, $\forall j, l$. The error between a preference value and its estimated one is defined as follows.

**Definition 2.6.** The error between a preference value and its estimated one in $[0, 1]$ is computed as:

$$\varepsilon p_{ik}^h = \frac{|\Delta_{t'}^{-1}(TF_{t'}^t(cp_{ik}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(p_{ik}^h))|}{n(t') - 1}. \quad (11)$$

We should point out that in expressions (7), (8) and (9), we could find that the value of argument of the function $\Delta_{t'}$ could lie outside the interval $[0, n(t') - 1]$ [13, 30]. In order to avoid this problem, the following function is used on the arguments of $\Delta_{t'}$:

$$f(y) = \begin{cases} 0, \text{ if } y < 0 \\ n(t') - 1, \text{ if } y > n(t') - 1 \\ y, \text{ otherwise.} \end{cases} \quad (12)$$

Thus, it can be used to define the consistency level between the preference degree $p_{ik}^h$ and the rest of the preference values of the unbalanced fuzzy linguistic preference relation.

**Definition 2.7.** The consistency level associated to a preference value $p_{ik}^h$ is defined as:

$$cl_{ik}^h = 1 - \varepsilon p_{ik}^h. \quad (13)$$

When $cl_{ik}^h = 1$, then $\varepsilon p_{ik}^h = 0$ and there is no inconsistency at all. The lower the value of $cl_{ik}^h$, the higher the value of $\varepsilon p_{ik}^h$ and the more inconsistent is $p_{ik}^h$ with respect to the rest of information.

Easily, we can define the consistency measures for particular alternatives and for the whole unbalanced fuzzy linguistic preference relation.

**Definition 2.8.** The consistency measure, $cl_i^h \in [0, 1]$, associated to a particular alternative $x_i$ of an unbalanced fuzzy linguistic preference relation $P^h$ is defined as:

$$cl_i^h = \frac{\sum_{k=1; i \neq k}^n (cl_{ik}^h + cl_{ki}^h)}{2(n-1)}. \quad (14)$$

When $cl_i^h = 1$, all the preference values involving the alternative $x_i$ are fully consistent, otherwise, the lower $cl_i^h$, the more inconsistent these preference values are.

**Definition 2.9.** The consistency level, $cl^h \in [0, 1]$, of an unbalanced fuzzy linguistic preference relation $P^h$ is defined as follows:

$$cl^h = \frac{\sum_{i=1}^n cl_i^h}{n}. \quad (15)$$

When $cl^h = 1$, the unbalanced fuzzy linguistic preference relation $P^h$ is fully consistent, otherwise, the lower $cl^h$, the more inconsistent $P^h$.

When working with an incomplete unbalanced fuzzy linguistic preference relation, expression (10) cannot be used to obtain the estimate of a known preference value. If expert $e_h$ provides an incomplete unbalanced fuzzy linguistic preference relation $P^h$, the following sets can be defined [30, 29]:

$$
\begin{aligned}
A &= \{(i,j) \mid i,j \in \{1,\ldots,n\} \wedge i \neq j\} \\
MV^h &= \{(i,j) \in A \mid p_{ij}^h \text{ is unknown}\} \\
EV^h &= A \setminus MV^h \\
H_{ik}^{h1} &= \{j \neq i,k \mid (i,j),(j,k) \in EV^h\} \\
H_{ik}^{h2} &= \{j \neq i,k \mid (j,i),(j,k) \in EV^h\} \\
H_{ik}^{h3} &= \{j \neq i,k \mid (i,j),(k,j) \in EV^h\} \\
EV_i^h &= \{(a,b) \mid (a,b) \in EV^h \wedge (a = i \vee b = i)\},
\end{aligned}
\tag{16}
$$

$MV^h$ is the set of pairs of alternatives whose preference degrees are not given by expert $e_h$, $EV^h$ is the set of pairs of alternatives whose preference degrees are given by the expert $e_h$; $H_{ik}^{h1}$, $H_{ik}^{h2}$, $H_{ik}^{h3}$ are the sets of intermediate alternative $x_j$ ($j \neq i,k$) that can be used to estimate the preference value $p_{ik}^h$ ($i \neq k$) using (7)–(9), respectively; and $EV_i^h$ is the set of pairs of alternatives whose preference degrees involving the alternative $x_i$ are given by the expert $e_h$. Then, the estimated value of a particular preference degree $p_{ik}^h$ ($(i,k) \in EV^h$) can be calculated as [30, 29]:

$$
\text{if } (\#H_{ik}^{h1} + \#H_{ik}^{h2} + \#H_{ik}^{h3}) \neq 0 \Rightarrow
$$
$$
cp_{ik}^h = TF_t^{t'}\left(\Delta_{t'}\left(\frac{\sum_{j \in H_{ik}^{h1}} \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j1})) + \sum_{j \in H_{ik}^{h2}} \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j2})) + \sum_{j \in H_{ik}^{h3}} \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j3}))}{(\#H_{ik}^{h1} + \#H_{ik}^{h2} + \#H_{ik}^{h3})}\right)\right).
\tag{17}
$$

## 2.4 Estimation Procedure of Missing Values for Incomplete Unbalanced Fuzzy Linguistic Preference Relations

As we have already mentioned, missing information is a problem that has to be addressed because experts are not always able to provide preference degrees between every pair of possible alternatives. Therefore, it is necessary to estimate the missing values before the application of a consensus model or a selection model. To do that, we define an estimation procedure of missing values for incomplete unbalanced fuzzy linguistic preference relations. This procedure estimates missing information in an expert's incomplete unbalanced fuzzy linguistic preference relation using only the preference values provided by that particular expert. It is an iterative procedure that is designed using the expression (17). The procedure estimates missing information values by means of two different tasks: (A) *establish the elements that can be estimated in each iteration of the procedure*, and (B) *estimate a particular missing value.*

**A) Elements to be estimated in each iteration of the procedure**

Given an incomplete unbalanced fuzzy linguistic preference relation $P^h$, the subset of missing values $MV^h$ that can be estimated in step $t$ of our procedure is denoted by $EMV_t^h$ (*estimated missing values*) and defined as follows:

$$
EMV_t^h = \left\{(i,k) \in MV^h \setminus \bigcup_{l=0}^{t-1} EMV_l^h \mid i \neq k \wedge \exists j \in \{H_{ik}^{h1} \cup H_{ik}^{h2} \cup H_{ik}^{h3}\}\right\}
\tag{18}
$$

and $EMV_0^h = \emptyset$ (by definition). When $EMV_{maxIter}^h = \emptyset$, with $maxIter > 0$, the procedure will stop as there will not be any more missing values to be estimated. Furthermore, if $\bigcup_{l=0}^{maxIter} EMV_l^h = MV^h$, then all missing values are estimated and, consequently, the procedure is said to be successful in the completion of the incomplete unbalanced fuzzy linguistic preference relation.

**B) Estimate a particular missing value**

In order to estimate a particular value $p_{ik}^h$, with $(i,k) \in EMV_t^h$, the following function $estimate\_p(h,i,k)$ is proposed:

---

**function estimate_p(h,i,k)**

1) $(cp_{ik}^h)^1 = (s_0, 0)$, $(cp_{ik}^h)^2 = (s_0, 0)$, $(cp_{ik}^h)^3 = (s_0, 0)$, $\mathcal{K} = 0$

2) if $\#H_{ik}^{h1} \neq 0$, then $(cp_{ik}^h)^1 = TF_t^{t'}(\Delta_{t'}((\sum_{j \in H_{ik}^{h1}} \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j1})))/\#H_{ik}^{h1}))$, $\mathcal{K}++$

3) if $\#H_{ik}^{h2} \neq 0$, then $(cp_{ik}^h)^2 = TF_t^{t'}(\Delta_{t'}((\sum_{j \in H_{ik}^{h2}} \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j2})))/\#H_{ik}^{h2}))$, $\mathcal{K}++$

4) if $\#H_{ik}^{h3} \neq 0$, then $(cp_{ik}^h)^3 = TF_t^{t'}(\Delta_{t'}((\sum_{j \in H_{ik}^{h3}} \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j3})))/\#H_{ik}^{h3}))$, $\mathcal{K}++$

5) Calculate $cp_{ik}^h = TF_t^{t'}(\Delta_{t'}\left( \dfrac{\Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^1)) + \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^2)) + \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^3))}{\mathcal{K}} \right))$

**end function**

---

Then, the complete iterative estimation procedure is the following:

---

**ITERATIVE ESTIMATION PROCEDURE**

0. $EMV_0^h = \emptyset$

1. $t = 1$

2. while $EMV_t^h \neq \emptyset$ {

3.    for every $(i,k) \in EMV_t^h$ {

4.       estimate_p(h,i,k)

5.    }

6.    $t++$

7. }

---

Finally, the following proposition provides a sufficient condition that guarantees the success of this estimation procedure [30]:

**Proposition 2.2.** *An incomplete unbalanced fuzzy linguistic preference relation can be completed if a set of n-1 non-leading diagonal preferences, where each one of the alternatives is compared at least once, is known.*

# 3   A Consensus Approach to Model GDM Problems with Incomplete Unbalanced Fuzzy Linguistic Preference Relations

In this section, we present a consensus model for GDM problems where experts provide their preferences using incomplete unbalanced fuzzy linguistic preference relations. To solve GDM problems with this kind of preference relations, firstly, it is necessary to deal with the missing values [4, 29, 30, 38, 39]. The previous consistency based procedure of missing values allows us to measure the consistency levels of each expert. This consistency information is used in this section to propose a consensus model based not only on consensus criteria but also

on consistency criteria. We consider that both criteria are important to guide the consensus process in an incomplete decision framework. In such a way, we get that experts change their opinions toward agreement positions in a consistent way, which is desirable to achieve a consistent and consensus solution.

In GDM situations, the search for consistency often could lead to a reduction of the level of consensus, and viceversa. Therefore, whether to proceed from consistency to consensus or viceversa is a matter that has to be addressed. We have decided to proceed from consistency to consensus because, in GDM situations, consensus between experts is usually searched using the basic rationality principles that each expert presents. To simulate this, the consistency criteria is first applied in our model to fix the rationality of each expert and afterwards it searches to secure consensus and only thereafter consistency we could destroy the consensus in favor of the individual consistency and the main aim of our process, which is consensus, would be distorted.

The main characteristics of the proposed consensus model are the following:

- It is designed to guide the consensus process of incomplete unbalanced fuzzy linguistic GDM problems.

- It uses a consistency based procedure to calculate the incomplete unbalanced fuzzy linguistic information.

- It is based both consensus criteria and consistency criteria. The proposed consensus model is designed with the aim of obtaining the maximum possible consensus level while trying to achieve a high level of consistency in experts' preferences.

- A feedback mechanism is defined using the above criteria. It substitutes the moderator's actions, avoiding the possible subjectivity that he/she can introduce, and gives advice to the experts to find out the changes they need to make in their opinions to obtain a solution with certain consensus and consistency degrees simultaneously.

Although the main purpose of our consensus model is to support the experts throughout the consensus process, they are who decide whether or not to follow the advice generated by the consensus model. In any case, the consensus model considerably reduces the time associated with making the decision and, therefore, it extends the experts' ability to analyze the information involved in the decision making process. In particular, our consensus model develops its activity in five phases that will be described in further detail in the following subsections (see Figure 4): (1) computing missing information, (2) computing consistency measures, (3) computing consensus measures, (4) controlling the consistency/consensus state, and (5) feedback mechanism.

## 3.1 Computing Missing Information

In this first step, each incomplete unbalanced fuzzy linguistic preference relation is completed by means of the estimation procedure described in Section 2.4. Therefore, for each incomplete unbalanced fuzzy linguistic preference relation $P^h$, we obtain its corresponding complete unbalanced fuzzy linguistic preference relation $\bar{P}^h$.

## 3.2 Computing Consistency Measures

To compute consistency measures, first, for each $\bar{P}^h$ we compute its corresponding unbalanced fuzzy linguistic preference relation $CP^h = (cp_{ik}^h)$ according to expression (10). Second, we apply expressions (13)-(15) to $(\bar{P}^h, CP^h)$ $(\forall h)$ to compute the consistency measures $CL^h = (cl_{ik}^h)$, $cl_i^h$, $cl^h$, $\forall i, k \in \{1, \ldots, n\}$. Finally, we define a global consistency measure among all experts to control the global consistency situation.
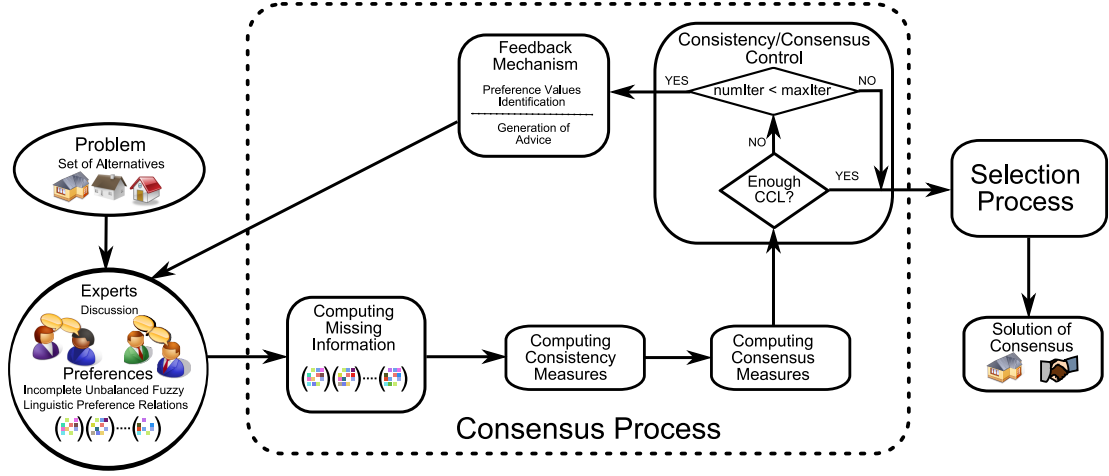
Figure 4: Scheme of consensus model

**Definition 3.1.** The global consistency measure is computed as follows:

$$CL = \frac{\sum_{h=1}^{m} cl^h}{m}.$$  (19)

## 3.3 Computing Consensus Measures

We compute several consensus measures for the different unbalanced fuzzy linguistic preference relations. In fact, as in [25, 29, 34], we compute two different kinds of measures: consensus degrees and proximity measures. Consensus degrees are used to measure the actual level of consensus in the process, while the proximity measures give information about how close to the collective solution every expert is. These measures are given on three different levels for a preference relation: pairs of alternatives, alternatives and relation. This measure structure will allow us to find out the consensus state of the process at different levels. For example, we will be able to identify which experts are close to the consensus solution, or in which alternatives the experts are having more trouble to reach consensus.

### 3.3.1 Consensus Degrees

For each pair of experts $(e_h, e_l)$ $(h = 1, \ldots, m - 1,\ l = h + 1, \ldots, m)$, a similarity matrix, $SM^{hl} = (sm_{ik}^{hl})$, is defined, where

$$sm_{ik}^{hl} = 1 - \frac{|\Delta_{t'}^{-1}(TF_{t'}^t(\bar{p}_{ik}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(\bar{p}_{ik}^l))|}{n(t') - 1},$$  (20)

being $\bar{p}_{ik}^h = (s_v^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, $\bar{p}_{ik}^l = (s_w^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, and $t' \in \{t^-, t_2^-, t^+, t_2^+\}$.

Then, a consensus matrix, $CM = (cm_{ik})$, is calculated by aggregating all the similarity matrices using the arithmetic mean as the aggregation function $\phi$:

$$cm_{ik} = \phi(sm_{ik}^{hl},\ h = 1, \ldots, m - 1,\ l = h + 1, \ldots, m).$$  (21)

Once the consensus matrix, $CM$, is computed, we proceed to calculate the consensus degrees at the three different levels:

1. **Level 1.** *Consensus degree on pairs of alternatives.* The consensus degree on a pair of alternatives $(x_i, x_k)$, called $cop^{lk}$, is defined to measure the consensus degree amongst all the experts on that pair

13

of alternatives. The closer $cop_{ik}$ to 1, the greater the agreement amongst all the experts on the pair of alternatives $(x_i, x_k)$. Thus, this measure is used to identify those pairs of alternatives with a poor level of consensus and is expressed by the element $(i, k)$ of the consensus matrix $CM$:

$$cop_{ik} = cm_{ik}; \ \forall \ i, k = 1, \ldots, n \land i \neq k. \tag{22}$$

2. **Level 2.** *Consensus degree on alternatives.* The consensus degree on an alternative $x_i$, called $ca_i$, is defined to measure the consensus degree amongst all the experts on that alternative:

$$ca_i = \frac{\sum_{k=1;k\neq i}^{n} (cop_{ik} + cop_{ki})}{2(n-1)}. \tag{23}$$

3. **Level 3.** *Consensus degree on the relation.* The consensus degree on the relation, called $cr$, is defined to measure the global consensus degree amongst all the experts' opinions and is used by the consensus model to control the consensus situation. It is calculated as the average of all the consensus degrees on the alternatives:

$$cr = \frac{\sum_{l=1}^{n} ca_i}{n}. \tag{24}$$

### 3.3.2 Proximity Measures

These measures evaluate the agreement between the individual experts' opinions and the group opinion. To compute them for each expert, we need to obtain the collective unbalanced fuzzy linguistic preference relation, $P^c = (p_{ik}^c)$, which summarizes preferences given by all the experts and is calculated by means of the aggregation of the set of individual unbalanced fuzzy linguistic preference relations $\{P^1, \ldots, P^m\}$. In this way, to obtain $P^c$ we use the unbalanced fuzzy linguistic version of an IOWA operator [12, 52, 53, 54], which uses both consensus and consistency criteria as inducing variable. Thus, we obtain each collective unbalanced fuzzy linguistic preference degree according to the most consistent and consensual individual unbalanced fuzzy linguistic preference degrees.

**Definition 3.2.** An IOWA operator of dimension $n$ is a function $\Phi_W : (\mathbb{R} \times \mathbb{R}) \to \mathbb{R}$, to which a weighting vector is associated, $W = (w_1, \ldots, w_n)$, with $w_i \in [0, 1]$, $\sum_i w_i = 1$, and it is defined to aggregate the set of second arguments of a list of $n$ 2-tuples $\{\langle u_1, p_1 \rangle, \ldots, \langle u_n, p_n \rangle\}$ according to the following expression:

$$\Phi_W(\langle u_1, p_1 \rangle, \ldots, \langle u_n, p_n \rangle) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)}, \tag{25}$$

being $\sigma$ a permutation of $\{1, \ldots, n\}$ such that $u_{\sigma(i)} \geq u_{\sigma(i+1)}, \forall i = 1, \ldots, n-1$, i.e., $\langle u_{\sigma(i)}, p_{\sigma(i)} \rangle$ is the 2-tuple with $u_{\sigma(i)}$ the i-th highest value in the set $\{u_1, \ldots, u_n\}$.

In the above definition, the reordering of the set of values to be aggregated, $\{p_1, \ldots, p_n\}$, is induced by the reordering of the set of values $\{u_1, \ldots, u_n\}$ associated with them, which is based upon their magnitude. Due to this use of the set of values $\{u_1, \ldots, u_n\}$, Yager and Filev called them the values of an order inducing variable $\{p_1, \ldots, p_n\}$ the values of the argument variable [12, 52, 53, 54].

A natural question in the definition of the IOWA operator is how to obtain the associated weighting vector. Following Yager's ideas on quantifier guided aggregation [51], we could compute the weighting vector of an IOWA operator using a linguistic quantifier $Q$ [58] as:

$$w_i = Q\left(\frac{\sum_{k=1}^{i} u_{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=1}^{i-1} u_{\sigma(k)}}{T}\right), \tag{26}$$

being $T = \sum_{k=1}^{n} u_k$ and $\sigma$ the permutation used to produce the ordering of the values to be aggregated.

Thus, to obtain each collective unbalanced fuzzy linguistic preference degree $p_{ik}^c$ according to the most consistent and consensual individual unbalanced fuzzy linguistic preference degrees, we propose to use an unbalanced fuzzy linguistic IOWA operator with the consistency/consensus values, $\{z_{ik}^1, z_{ik}^2, \ldots, z_{ik}^m\}$, as the values of the order inducing variable, i.e.,

$$p_{ik}^c = \Phi_W(\langle z_{ik}^1, \bar{p}_{ik}^1 \rangle, \ldots, \langle z_{ik}^m, \bar{p}_{ik}^m \rangle) = TF_t^{t'}(\Delta_{t'}(\sum_{h=1}^{m} w_h \cdot \Delta_{t'}^{-1}(TF_{t'}^t(\bar{p}_{ik}^{\sigma(h)})))), \tag{27}$$

where

- $\sigma$ is a permutation of $\{1, \ldots, m\}$ such that $z_{ik}^{\sigma(h)} \geq z_{ik}^{\sigma(h+1)}$, $\forall h = 1, \ldots, m-1$, i.e., $\langle z_{ik}^{\sigma(h)}, \bar{p}_{ik}^{\sigma(h)} \rangle$ is the 2-tuple with $z_{ik}^{\sigma(h)}$ the h-th highest value in the set $\{z_{ik}^1 \ldots, z_{ik}^m\}$;

- the weighting vector is computed according to the following expression:

$$w_h = Q\left(\frac{\sum_{j=1}^{h} z_{ik}^{\sigma(j)}}{T}\right) - Q\left(\frac{\sum_{j=1}^{h-1} z_{ik}^{\sigma(j)}}{T}\right), \tag{28}$$

with $T = \sum_{j=1}^{m} z_{ik}^j$;

- and the set of values of the inducing variable $\{z_{ik}^1, \ldots, z_{ik}^m\}$ are computed as

$$z_{ik}^h = (1 - \delta) \cdot cl_{ik}^h + \delta \cdot co_{ik}^h, \tag{29}$$

being $co_{ik}^h$ the consensus measure for the preference value $\bar{p}_{ik}^h$ and $\delta \in [0, 1]$ a parameter to control the weight of both consistency and consensus criteria in the inducing variable. Usually $\delta > 0.5$ will be used to give more importance to the consensus criterion. We should note that in our framework, each value $co_{ik}^h$ used to calculate $\{z_{ik}^1, \ldots, z_{ik}^m\}$ is defined as

$$co_{ik}^h = \frac{\sum_{l=h+1}^{n} sm_{ik}^{hl} + \sum_{l=1}^{h-1} sm_{ik}^{lh}}{n - 1}. \tag{30}$$

Once we have computed $P^c$, we can compute the proximity measures in each level of an unbalanced fuzzy linguistic preference relation.

1. **Level 1.** *Proximity measure on pairs of alternatives.* The proximity measure of an expert $e_h$ on a pair of alternatives $(x_i, x_k)$ to the group's one, called $pp_{ik}^h$, is calculated as:

$$pp_{ik}^h = 1 - \frac{\left|\Delta_{t'}^{-1}(TF_{t'}^t(\bar{p}_{ik}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(p_{ik}^c))\right|}{n(t') - 1}. \tag{31}$$

being $\bar{p}_{ik}^h = (s_v^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, $p_{ik}^c = (s_w^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, and $t' \in \{t^-, t_2^-, t^+, t_2^+\}$.

2. **Level 2.** *Proximity measure on alternatives.* The proximity measure of an expert $e_h$ on an alternative $x_i$ to the group's one, called $pa_i^h$, is calculated as follows:

$$pa_i^h = \frac{\sum_{k=1; k \neq i}^{n} (pp_{ik}^h + pp_{ki}^h)}{2(n - 1)}. \tag{32}$$

3. **Level 3.** *Proximity measure on the relation.* The proximity measure of an expert $e_h$ on his/her unbalanced fuzzy linguistic preference relation to the group's one, called $pr^h$, is calculated as the average of all proximity measures on the alternatives:

$$pr^h = \frac{\sum_{l=1}^{n} pa_i^h}{n}. \tag{33}$$

## 3.4 Controlling Consistency/Consensus State

The consistency/consensus state control process will be used to decide when the feedback mechanism should be applied to give advice to the experts or when the consensus reaching process has to come to an end. It should take into account both the consensus and consistency measures. To do that, we use a measure or level of satisfaction, called *consistenty/consensus level* (CCL) [29], which is used as a control parameter:

$$CCL = (1 - \delta) \cdot CL + \delta \cdot cr, \tag{34}$$

with $\delta$ the same value used in [36]. When CCL satisfies a minimum threshold value $\gamma \in [0, 1]$, then the consensus reaching process finishes and the selection process can be applied.

Additionally, the system should avoid stagnation, that is, situations in which consensus and consistency measures never reach an appropriate satisfaction value. To do so, a maximum number of iterations *maxIter* should be fixed and compared to the actual number of iterations of the consensus process *numIter*.

The consistency/consensus control routine is: first, the consistency/consensus level is checked against the minimum satisfaction threshold value. If CCL $> \gamma$, the consensus reaching process ends. Otherwise, it will check if the maximum number of iterations has been reached. If so, the consensus reaching process ends, if not it activates the feedback mechanism.

## 3.5 Feedback Mechanism

The feedback mechanism generates personalized advice to the experts according to the consistency and consensus criteria. It helps experts to change their preferences and to complete their missing values. This activity is carried out in two steps: (1) *Identification of the preference values* that should be changed, and (2) *generation of advice*.

1. **Identification of the preference values.** We must identify preference values provided by the experts that are contributing less to reach a high consistency/consensus state. To do that, we define set APS that contains 3-tuples $(h, i, k)$ symbolizing preference degrees $p_{ik}^h$ that should be changed because they affect badly to that consistency/consensus state. To compute APS, we apply a three step identification process that uses the proximity and consistency measures previously defined.

   (a) *Identification of experts.* We identify the set of experts EXPCH that should receive advice on how to change some of their preference values. The experts that should change their opinions are those whose preference relation level of satisfaction is lower than the satisfaction threshold $\gamma$, i.e.,

   $$EXCPCH = \{h \mid (1 - \delta) \cdot cl^h + \delta \cdot pr^h < \gamma\}. \tag{35}$$

   (b) *Identification of alternatives.* We identify the alternatives that the above experts should consider to change. This set of alternatives is denoted as ALT. To do this, we select the alternatives with a level of satisfaction lower than the satisfaction threshold $\gamma$, i.e.,

   $$ALT = \{(h, i) \mid e_h \in EXPCH \wedge (1 - \delta) \cdot cl_i^h + \delta \cdot pa_i^h < \gamma\}. \tag{36}$$

   (c) *Identification of pairs of alternatives.* Finally, we identify preference values for every alternative and expert $(x_i; e_h \mid (h, i) \in ALT)$ that should be changed according to their proximity and consistency measures on the pairs of alternatives, i.e.,

   $$APS = \{(h, i, k) \mid (h, i) \in ALT \wedge (1 - \delta) \cdot cl_{ik}^h + \delta \cdot pp_{ik}^h < \gamma\}. \tag{37}$$

Additionally, the feedback process must provide rules for missing preference values. To do so, it has to take into account in APS all missing values that were not provided by the experts, i.e.,

$$APS' = APS \cup \{(h, i, k) \mid p_{ik}^h \in MV_h\}. \tag{38}$$

2. **Generation of advice.** In this step, the feedback mechanism generates personalized recommendations to help the experts to change their preferences. These recommendations are based on easy recommendation rules that will not only tell the experts which preference values they should change, but will also provide them with particular values for each preference to reach a higher consistency/consensus state.

The new preference degree of alternatives $x_i$ over alternative $x_k$ to recommend to the expert $e_h$, $rp_{ik}^h$, is calculated as the following weighted average of the preference value $cp_{ik}^h$ and the collective preference value $p_{ik}^c$:

$$rp_{ik}^h = TF_t^{t'}(\Delta_{t'}((1 - \delta) \cdot \Delta_{t'}^{-1}(TF_{t'}^t(cp_{ik}^h)) + \delta \cdot \Delta_{t'}^{-1}(TF_{t'}^t(p_{ik}^c)))). \tag{39}$$

As previously mentioned, with $\delta > 0.5$, the consensus model leads the experts towards a consensus solution rather than towards an increase on their own consistency levels.

Finally, we should distinguish two cases:

- The recommendation is given because a preference value is far from the consistency/consensus state.

- The recommendation is given because the expert did not provide the preference value.

Therefore, $\forall (h, i, k) \in APS'$, the following hold:

(a) If $p_{ik}^h \in EV_h$, the recommendation generated for the expert $e_h$ is: "You should change your preference value $(i, k)$ to a value close to $rp_{ik}^h$".

(b) If $p_{ik}^h \in MV_h$, the recommendation generated for the expert $e_h$ is: "You should provide a value for $(i, k)$ close to $rp_{ik}^h$".

# 4  Example of Application

An investment company wants to invest a sum of money in the best industrial sector, from the set of four possible alternatives:

- Car industry: $x_1$.

- Food company: $x_2$.

- Computer company: $x_3$.

- Arms industry: $x_4$.

To do this, four consultancy departments within the company are requested to provide information:

- Risk analysis department: $e_1$.

- Growth analysis department: $e_2$.

- Social-political analysis department: $e_3$.

- Environmental impact analysis department: $e_4$.

Each department is directed by an expert who provides his/her preferences using the following unbalanced fuzzy linguistic term set $\mathcal{S}_{un} = \{N, VL, L, M, H, QH, VH, T\}$ (see Fig. 1 and Fig. 3). The incomplete unbalanced fuzzy linguistic preference relations provided by each one of the experts are:

$$
P^1 = \begin{pmatrix} - & H & QH & L \\ x & - & x & x \\ x & x & - & x \\ x & x & x & - \end{pmatrix} ; P^2 = \begin{pmatrix} - & H & H & VL \\ L & - & T & QH \\ VL & L & - & H \\ QH & N & L & - \end{pmatrix}
$$

$$
P^3 = \begin{pmatrix} - & x & L & x \\ x & - & x & L \\ T & x & - & M \\ x & QH & M & - \end{pmatrix} ; P^4 = \begin{pmatrix} - & VH & QH & M \\ VL & - & M & VH \\ L & M & - & VL \\ M & L & QH & - \end{pmatrix}
$$

The respective linguistic preference relations expressed in a 2-tuple linguistic representation model are the following:

$$
P^1 = \begin{pmatrix} - & (H,0) & (QH,0) & (L,0) \\ x & - & x & x \\ x & x & - & x \\ x & x & x & - \end{pmatrix} ; P^2 = \begin{pmatrix} - & (H,0) & (H,0) & (VL,0) \\ (L,0) & - & (T,0) & (QH,0) \\ (VL,0) & (L,0) & - & (H,0) \\ (QH,0) & (N,0) & (L,0) & - \end{pmatrix}
$$

$$
P^3 = \begin{pmatrix} - & x & (L,0) & x \\ x & - & x & (L,0) \\ (T,0) & x & - & (M,0) \\ x & (QH,0) & (M,0) & - \end{pmatrix} ; P^4 = \begin{pmatrix} - & (VH,0) & (QH,0) & (M,0) \\ (VL,0) & - & (M,0) & (VH,0) \\ (L,0) & (M,0) & - & (VL,0) \\ (M,0) & (L,0) & (QH,0) & - \end{pmatrix}
$$

**FIRST ROUND**

In the following, we show how to apply each step of the consensus model.

**A. Computing Missing Information:** Two given unbalanced fuzzy linguistic preference relations are incomplete $\{P^1, P^3\}$. As an example, we show how to complete $P^1$ using the estimation procedure described in Section 2.4:

**Step 1:** The set of elements that can be estimated are:

$$
EMV_1^1 = \{(2,3), (2,4), (3,2), (3,4), (4,2), (4,3)\}.
$$

After these elements have been estimated, we have:

$$
P^1 = \begin{pmatrix} - & (H,0) & (QH,0) & (L,0) \\ x & - & (H,0) & (VL,0) \\ x & (M,-0.5) & - & (N,0) \\ x & (VH,0) & (T,0) & - \end{pmatrix}.
$$

As an example, to estimate $p_{43}^1$ the procedure is as follows:

$H_{43}^{11} = \emptyset$.

$H_{43}^{12} = \{1\} \Rightarrow (cp_{43}^1)^2 = TF_t^{t'}(\Delta_t'(\Delta_{t'}^{-1}(TF_{t'}^t(cp_{43}^1)^{12}))) = TF_3^3(\Delta_3(\Delta_3^{-1}(TF_3^3(p_{13}^1)) - \Delta_3^{-1}(TF_3^3(p_{14}^1)) + \Delta_3^{-1}(TF_3^3(s_{mid}, 0)))) = TF_3^3(\Delta_3(\Delta_3^{-1}(TF_3^3(QH, 0)) - \Delta_3^{-1}(TF_3^3(L, 0)) + \Delta_3^{-1}(TF_3^3(M, 0)))) = TF_3^3(\Delta_3(\Delta_3^{-1}(QH, 0) - \Delta_3^{-1}(L, 0) + \Delta_3^{-1}(M, 0))) = TF_3^3(\Delta_3(6 - 2 + 4)) = (T, 0)$.

$H_{43}^{13} = \emptyset$.

$cp_{43}^1 = TF_3^3\left(\Delta_3\left(\dfrac{\Delta_3^{-1}(TF_3^3(cp_{43}^1)^2)}{1}\right)\right) = (T, 0)$.

**Step 2:** The set of elements that can be estimated are:

$$EMV_2^1 = \{(2, 1), (3, 1), (4, 1)\}.$$

After these elements have been estimated, we have the following complete unbalanced fuzzy linguistic preference relation:

$$\bar{P}^1 = \begin{pmatrix} - & (H, 0) & (QH, 0) & (L, 0) \\ (M, -0.5) & - & (H, 0) & (VL, 0) \\ (L, 0) & (M, -0.5) & - & (N, 0) \\ (QH, 0) & (VH, 0) & (T, 0) & - \end{pmatrix}.$$

For $P^3$, we get:

$$\bar{P}^3 = \begin{pmatrix} - & (M, -0.17) & (L, 0) & (L, 0.33) \\ (H, 0.33) & - & (L, 0) & (L, 0) \\ (T, 0) & (QH, 0) & - & (M, 0) \\ (VH, 0.33) & (QH, 0) & (M, 0) & - \end{pmatrix}.$$

**B. Computing Consistency Measures:** The corresponding unbalanced fuzzy linguistic preference relations, $CP^h$, for $P^1$, $P^2$, $P^3$ y $P^4$ are:

$$CP^1 = \begin{pmatrix} - & (H, 0) & (QH, 0) & (L, 0) \\ (M, -0.5) & - & (H, 0) & (VL, 0) \\ (L, 0) & (M, -0.5) & - & (N, 0) \\ (QH, 0) & (VH, 0) & (T, 0) & - \end{pmatrix}$$

$$CP^2 = \begin{pmatrix} - & (VL, -0.5) & (T, 0.33) & (T, -0.5) \\ (VH, 0) & - & (H, -0.5) & (M, -0.5) \\ (L, 0.17) & (VL, 0.25) & - & (L, -0.5) \\ (N, 0) & (L, 0.33) & (QH, 0) & - \end{pmatrix}$$

$$CP^3 = \begin{pmatrix} - & (M, -0.17) & (L, -0.05) & (VL, 0.33) \\ (H, 0) & - & (L, 0.14) & (L, 0.25) \\ (VH, -0.17) & (QH, 0.33) & - & (H, -0.33) \\ (VH, 0.17) & (QH, 0) & (M, 0) & - \end{pmatrix}$$

$$CP^4 = \begin{pmatrix} - & (M, 0) & (VH, -0.33) & (VH, -0.5) \\ (M, 0.33) & - & (QH, 0.17) & (VL, 0.17) \\ (VL, 0.17) & (L, 0) & - & (M, 0.33) \\ (L, -0.5) & (VH, -0.33) & (M, -0.08) & - \end{pmatrix}$$

The consistency measures for every pair of alternatives in the experts preferences are:

$$
CL^1 = \begin{pmatrix} - & 1.0 & 1.0 & 1.0 \\ 1.0 & - & 1.0 & 1.0 \\ 1.0 & 1.0 & - & 1.0 \\ 1.0 & 1.0 & 1.0 & - \end{pmatrix}; CL^2 = \begin{pmatrix} - & 0.44 & 0.62 & 0.19 \\ 0.37 & - & 0.56 & 0.62 \\ 0.83 & 0.90 & - & 0.56 \\ 0.25 & 0.67 & 0.5 & - \end{pmatrix}
$$

$$
CL^3 = \begin{pmatrix} - & 1.0 & 0.99 & 0.83 \\ 0.96 & - & 0.96 & 0.94 \\ 0.85 & 0.96 & - & 0.92 \\ 0.98 & 1.0 & 1.0 & - \end{pmatrix}; CL^4 = \begin{pmatrix} - & 0.96 & 0.92 & 0.69 \\ 0.58 & - & 0.73 & 0.27 \\ 0.90 & 0.75 & - & 0.70 \\ 0.69 & 0.42 & 0.73 & - \end{pmatrix}
$$

The consistency measure that each expert presents in his/her preferences are:

$$
cl^1 = 1.0 \quad cl^2 = 0.54 \quad cl^3 = 0.95 \quad cl^4 = 0.70.
$$

The global consistency level is:

$$
CL = \frac{1.0 + 0.54 + 0.95 + 0.70}{4} = 0.80.
$$

**C. Computing Consensus Measures:** We need to compute the six possible similarity matrices between every pair of different experts (not included for simplicity), and the collective one, which is:

$$
CM = \begin{pmatrix} - & 0.79 & 0.73 & 0.80 \\ 0.71 & - & 0.60 & 0.54 \\ 0.56 & 0.73 & - & 0.62 \\ 0.79 & 0.48 & 0.58 & - \end{pmatrix}.
$$

From $CM$, we obtain the following consensus degree on the relation:

$$
cr = 0.66.
$$

Computation of the collective unbalanced fuzzy linguistic preference relation:

1. To compute the proximity measures it is necessary to obtain the consistency/consensus values of the inducing variable of the unbalanced linguistic IOWA operator. To do so, first, we compute the consensus values matrices $co^h = (co_{ik}^h)$:

$$
co^1 = \begin{pmatrix} - & 0.86 & 0.79 & 0.85 \\ 0.78 & - & 0.70 & 0.50 \\ 0.71 & 0.79 & - & 0.58 \\ 0.86 & 0.45 & 0.50 & - \end{pmatrix}; co^2 = \begin{pmatrix} - & 0.86 & 0.79 & 0.76 \\ 0.77 & - & 0.46 & 0.58 \\ 0.62 & 0.71 & - & 0.58 \\ 0.86 & 0.37 & 0.50 & - \end{pmatrix}
$$

$$
co^3 = \begin{pmatrix} - & 0.75 & 0.54 & 0.85 \\ 0.58 & - & 0.54 & 0.58 \\ 0.21 & 0.62 & - & 0.66 \\ 0.75 & 0.54 & 0.67 & - \end{pmatrix}; co^4 = \begin{pmatrix} - & 0.69 & 0.79 & 0.73 \\ 0.69 & - & 0.71 & 0.50 \\ 0.71 & 0.79 & - & 0.66 \\ 0.69 & 0.54 & 0.67 & - \end{pmatrix}
$$

2. With values $co_{ik}^h$ and $cl_{ik}^h$, the inducing variable values for each expert, $z^h = (z_{ik}^h)$ (we assume that $\delta = 0.75$), are obtained:

$$z^1 = \begin{pmatrix} - & 0.89 & 0.84 & 0.89 \\ 0.83 & - & 0.77 & 0.62 \\ 0.78 & 0.84 & - & 0.68 \\ 0.89 & 0.59 & 0.62 & - \end{pmatrix} ; z^2 = \begin{pmatrix} - & 0.75 & 0.75 & 0.62 \\ 0.67 & - & 0.48 & 0.79 \\ 0.67 & 0.76 & - & 0.57 \\ 0.71 & 0.44 & 0.50 & - \end{pmatrix}$$

$$z^3 = \begin{pmatrix} - & 0.81 & 0.65 & 0.84 \\ 0.67 & - & 0.64 & 0.67 \\ 0.37 & 0.70 & - & 0.72 \\ 0.81 & 0.65 & 0.75 & - \end{pmatrix} ; z^4 = \begin{pmatrix} - & 0.76 & 0.82 & 0.72 \\ 0.66 & - & 0.71 & 0.44 \\ 0.76 & 0.78 & - & 0.67 \\ 0.69 & 0.51 & 0.68 & - \end{pmatrix}$$

3. Using the following fuzzy linguistic quantifier "most of", $Q(r) = r^{1/2}$, to compute the weighting vector of the unbalanced linguistic IOWA operator, the collective unbalanced fuzzy linguistic preference relation $P^c$ is:

$$P^c = \begin{pmatrix} - & (H, 0.02) & (H, 0.41) & (L, 0.11) \\ (L, 0.44) & - & (H, -0.36) & (H, -0.46) \\ (L, 0.16) & (M, -0.30) & - & (L, 0.41) \\ (QH, 0.04) & (H, -0.22) & (H, -0.18) & - \end{pmatrix}.$$

Computation of proximity measures:

1. The proximity measures on pairs of alternatives for each expert are:

$$pp^1 = \begin{pmatrix} - & 0.99 & 0.93 & 0.97 \\ 0.98 & - & 0.95 & 0.58 \\ 0.96 & 0.95 & - & 0.65 \\ 0.99 & 0.72 & 0.60 & - \end{pmatrix} ; pp^2 = \begin{pmatrix} - & 0.99 & 0.95 & 0.85 \\ 0.76 & - & 0.58 & 0.94 \\ 0.83 & 0.82 & - & 0.73 \\ 0.99 & 0.40 & 0.65 & - \end{pmatrix}$$

$$pp^3 = \begin{pmatrix} - & 0.83 & 0.57 & 0.94 \\ 0.69 & - & 0.67 & 0.68 \\ 0.29 & 0.67 & - & 0.85 \\ 0.84 & 0.85 & 0.90 & - \end{pmatrix} ; pp^4 = \begin{pmatrix} - & 0.75 & 0.93 & 0.78 \\ 0.76 & - & 0.92 & 0.69 \\ 0.96 & 0.92 & - & 0.77 \\ 0.74 & 0.65 & 0.85 & - \end{pmatrix}$$

2. The proximity measures on alternatives for each expert are:

$$pa^1 = (0.97 \quad 0.86 \quad 0.84 \quad 0.75).$$
$$pa^2 = (0.89 \quad 0.75 \quad 0.76 \quad 0.76).$$
$$pa^3 = (0.59 \quad 0.73 \quad 0.66 \quad 0.84).$$
$$pa^4 = (0.82 \quad 0.78 \quad 0.89 \quad 0.75).$$

3. The proximity measures on the relation for each expert are:

$$pr^1 = 0.85 \quad pr^2 = 0.79 \quad pr^3 = 0.70 \quad pr^4 = 0.81.$$

**D. Controlling Consistency/Consensus State:** We fix a minimum threshold value $\gamma = 0.75$. Because the consistency/consensus level at this moment is $CCL = (1 - 0.75) \cdot 0.80 + 0.75 \cdot 0.66 = 0.69$, then the consensus process applies the feedback mechanism.

**E. Feedback Mechanism:** The set of experts EXPCH that should receive advice on how to change some of their preference values is:

$$EXPCH = \{e_2\}.$$

The set of alternatives that the above experts should consider to change is:

$$ALT = \{(2, 2), (2, 3), (2, 4)\}.$$

The set of 3-tuples APS that experts should change is:

$$APS = \{(2, 2, 1), (2, 2, 3), (2, 3, 4), (2, 4, 2), (2, 4, 3)\}.$$

Taking into account all missing values not provided by the experts, the APS' set is:

$$APS' = \{(1, 2, 1), (1, 2, 3), (1, 2, 4), (1, 3, 1), (1, 3, 2), (1, 3, 4), (1, 4, 1), (1, 4, 2), (1, 4, 3), (2, 2, 1), (2, 2, 3), (2, 3, 4),$$
$$(2, 4, 2), (2, 4, 3), (3, 1, 2), (3, 1, 4), (3, 2, 1), (3, 2, 3), (3, 3, 2), (3, 4, 1)\}.$$

The recommendations for our example are as follows:

To expert $e_1 \Rightarrow$ You should provide a value for (2,1) close to $(L, 0.45)$.

To expert $e_1 \Rightarrow$ You should provide a value for (2,3) close to $(H, -0.27)$.

To expert $e_1 \Rightarrow$ You should provide a value for (2,4) close to $(M, -0.17)$.

To expert $e_1 \Rightarrow$ You should provide a value for (3,1) close to $(L, 0.12)$.

To expert $e_1 \Rightarrow$ You should provide a value for (3,2) close to $(M, -0.35)$.

To expert $e_1 \Rightarrow$ You should provide a value for (3,4) close to $(L, 0.05)$.

To expert $e_1 \Rightarrow$ You should provide a value for (4,1) close to $(QH, 0.03)$.

To expert $e_1 \Rightarrow$ You should provide a value for (4,2) close to $(H, 0.33)$.

To expert $e_1 \Rightarrow$ You should provide a value for (4,3) close to $(QH, -0.38)$.

To expert $e_2 \Rightarrow$ You should change your preference value for (2,1) to a value close to $(M, -0.05)$.

To expert $e_2 \Rightarrow$ You should change your preference value for (2,3) to a value close to $(H, -0.39)$.

To expert $e_2 \Rightarrow$ You should change your preference value for (3,4) to a value close to $(L, 0.24)$.

To expert $e_2 \Rightarrow$ You should change your preference value for (4,2) to a value close to $(H, 0.25)$.

To expert $e_2 \Rightarrow$ You should change your preference value for (4,3) to a value close to $(H, 0.11)$.

To expert $e_3 \Rightarrow$ You should change your preference value for (1,2) to a value close to $(H, -0.32)$.

To expert $e_3 \Rightarrow$ You should change your preference value for (1,4) to a value close to $(L, 0)$.

To expert $e_3 \Rightarrow$ You should change your preference value for (2,1) to a value close to $(M, -0.30)$.

To expert $e_3 \Rightarrow$ You should change your preference value for (2,3) to a value close to $(M, 0.05)$.

To expert $e_3 \Rightarrow$ You should change your preference value for (3,2) to a value close to $(M, 0.13)$.

To expert $e_3 \Rightarrow$ You should change your preference value for (4,1) to a value close to $(QH, 0.32)$.

**SECOND ROUND**

We assume that all the experts follow the recommendations they were given, which implies that the new unbalanced fuzzy linguistic preference relations for the second round of the consensus process are:

$$P^1 = \begin{pmatrix} - & (H,0) & (QH,0) & (L,0) \\ (L,0.45) & - & (H,-0.27) & (M,-0.17) \\ (L,0.12) & (M,-0.35) & - & (L,0.05) \\ (QH,0.03) & (H,0.33) & (QH,-0.38) & - \end{pmatrix} ; P^2 = \begin{pmatrix} - & (H,0) & (H,0) & (VL,0) \\ (M,-0.05) & - & (H,-0.39) & (QH,0) \\ (VL,0) & (L,0) & - & (L,0.24) \\ (QH,0) & (H,0.25) & (H,0.11) & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & (H,-0.32) & (L,0) & (L,0) \\ (M,-0.30) & - & (M,0.05) & (L,0) \\ (T,0) & (M,0.13) & - & (M,0) \\ (QH,0.32) & (QH,0) & (M,0) & - \end{pmatrix} ; P^4 = \begin{pmatrix} - & (VH,0) & (QH,0) & (M,0) \\ (VL,0) & - & (M,0) & (VH,0) \\ (L,0) & (M,0) & - & (VL,0) \\ (M,0) & (L,0) & (QH,0) & - \end{pmatrix}$$

Applying the same process (which will not be detailed here), we obtain the following global consistency and consensus levels:

$$CL = 0.77 \text{ and } cr = 0.79.$$

Obviously, the consistency level has decreased a little bit because the process gave more importance to the consensus criteria than the consistency one. However, the consensus level has increased. Finally, as the consistency/consensus level satisfies the minimum consensus threshold value, i.e.,

$$CCL = 0.78 > \gamma = 0.75,$$

the consensus reaching process ends. Then, a selection process [3, 30] would be applied to obtain the best industrial sector in which the investment company would invest a sum of money, according to the opinions expressed by the experts.

## 5   Concluding Remarks

In this paper, we have presented a model of consensus for GDM problems with incomplete unbalanced fuzzy linguistic information. It uses two different kinds of measures to guide the consensus reaching process, consistency and consensus measures, and applies a feedback mechanism to give personalized advice to the experts on how to change and complete their unbalanced fuzzy linguistic preference relations. As a consequence, this model allows us to achieve consistent and consensus solutions. In addition, the consensus model can be developed automatically without the participation of a human moderator.

In future works, it will be deployed into mobile and distributed GDM environments where the experts will be able to provide their preferences about the alternatives using devices as mobile phones and PDAs. On the other hand, we are studying the possibility to apply such GDM models in Web 2.0 frameworks because they could provide a useful tools to improve the collaboration among individuals. Finally, we also think that it would be interesting to research another alternative way to deal with unbalanced fuzzy linguistic information and their aggregation via the recently developed type-1 OWA operator [59].

## Acknowledgments

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