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Manejando Información Incompleta en Problemas de Toma de Decisiones en Grupo en Contexto Difuso

Tesis Doctoral

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María Raquel Ureña Pérez

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La memoria titulada "*Manejando información incompleta en problemas de toma de decisiones en grupo en contexto difuso*", que presenta Dña. Mª Raquel Ureña Pérez para optar al grado de doctor, ha sido realizada dentro del Máster Oficial de Doctorado "*Ingeniería de Computadores y Redes*" del Departamento de Ciencias de la Computación e Inteligencia Artificial de la Universidad de Granada bajo la dirección de los doctores D. Enrique Herrera Viedma y D. Francisco Chiclana Parrilla.

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Table of Contents

I. PhD dissertation **1**

1. Introducción 1
1. Introduction 7
2. Preliminaries 10
   - 2.1. Group decision making processes 10
   - 2.2. Preference Relations in Decision Making 14
     - 2.2.1. Numeric Preferences 15
       - 2.2.1.a. Crisp Preference Relation 16
       - 2.2.1.b. Additive Preference Relation 16
       - 2.2.1.c. Multiplicative Preference Relation 17
       - 2.2.1.d. Interval Valued Preference Relation 18
       - 2.2.1.e. Intuitionistic Preference Relation 19
     - 2.2.2. Linguistic Preferences 21
       - 2.2.2.a. LPR based on cardinal representation 22
       - 2.2.2.b. LPR based on ordinal representation 22
   - 2.3. Consistency in decision making 24
   - 2.4. Granular information 28
   - 2.5. Incomplete information 29
3. Justification 31
4. Objectives 33

5. Joint discussion of results . . . 35
5.1. Analyzing the main approaches to deal with incomplete information for different types of preference relations. . 35
5.1.1. Estimation approaches in GDM situations . . 36
5.1.2. Approaches dealing with total ignorance situations . . . 41
5.2. Developing new group decision making approaches under highly uncertain contexts . . . 43
5.2.1. Confidence-consistency decision making approaches in highly uncertain contexts . . . 43
5.2.2. Consensus processes where uncertainty is modeled via information granularity . . . 49
5.3. Developing new software tools to deal with Group decision making processes with incomplete information . . . 57
5.3.1. System architecture . . . 58
5.3.2. Graphical evolution of the decision making process . . . 59
6. Conclusiones . . . 63
6.1. Análisis de las principales metodologías de toma de decisión en grupo que trabajan con información incompleta para distitos tipos de relaciones de preferencia . . . 63
6.2. Toma de decisión en grupo en contextos de alta incertidumbre . . . 64
6.2.1. Trabajando con relaciones de preferencia intuicionistas incompletas . . . 64
6.2.2. Empleando información granular en procesos de consenso bajo incertidumbre . . . 65
6.3. Herramientas software para trabajar con procesos de toma de decisión en grupo con información incompleta . . 65
6. Concluding remarks . . . 67
6.1. Analyzing the main approaches to deal with incomplete information for different types of preference relations . . 67
6.2. Group decision making under highly uncertain contexts 67

6.2.1. Dealing with incomplete intuitionistic fuzzy preference relations 67

6.2.2. Using multigranular information in consensus process under uncertainty 68

6.3. Software tools to deal with Group decision making processes with incomplete information 69

7. Future work 71

7.1. Development of new GDM methodologies dealing with incomplete information modeled with other promising types of preference relations 71

7.2. Dealing with Linguistic Incomplete Preference relations modeled as Interval-Type-2-Fuzzy-Sets 71

7.3. Validation frameworks for GDM approaches with incomplete preferences 72

II. Publications: Published, Accepted and Submitted Papers 75

1. Analyzing the main approaches to deal with incomplete information for different types of preference relations. 75

2. New group decision making approaches under highly uncertain contexts 109

2.1. Confidence-consistency driven group decision making approach in uncertainty environments 109

2.2. Consensus processes where uncertainty is modeled via information granularity 133

3. Developing new software tools to deal with Group decision making processes with incomplete information 151

Bibliografía 185

Table of Figures

1. Classical GDM approach 11
2. GDM approach when 30
3. Classification of the different approaches to deal with missing information in DM. 37
4. Consistency degrees versus α for the fuzzy preference relations PR^1–PR^4. 52
5. Fitness function f in successive PSO generations for selected values of α (here $\delta = 0{,}75$). 53
6. Plots of Q_1 and Q_2 versus α for selected values of δ. 54
7. Plots of Q_1 and Q_2 versus δ for selected values of α. 54
8. Architecture of the developed framework and the main interactions between modules. 59

Part I. PhD dissertation

1. Introducción

La toma de decisión consiste en un proceso mental y cognitivo de seleccionar la mejor y más lógica opción de entre todas las disponibles y es una tarea permanente en la rutina de los seres humanos. De hecho, tomamos decisiones cuyo nivel de complejidad va desde nimiedades tales cómo elegir el tipo de café que queremos tomar a decisiones mucho más complejas y transcendentales cómo la compra de una vivienda, o la mejor inversión a realizar. De hecho, el estudio de los procesos de toma de decisión ha suscitado mucha atención en la comunidad científica en áreas tan diversas cómo la Economía, ls Psicología, la Sociología, la Inteligencia artificial, y la Ingeniería.

Hemos de tener en cuenta que en el proceso de toma de decisiones, normalmente se proporcionan opiniones y aseveraciones basadas en puntos de vista, experiencia y conocimientos previos. Así pues, una opinión puede considerarse como una aseveración personal del nivel de interés en alguna variable, hecha con una mezcla de información cualitativa y cuantitativa [MW10]. Por tanto, los procesos de toma de decisión están muy lejos de ser precisos y deterministas, por el contrario, son intrínsecamente inciertos y subjetivos. En este sentido, en los procesos teóricos de toma de decisión una cuestión clave es cómo modelar dicha incertidumbre en las opiniones de los expertos, y es, en éste punto, dónde la Teoría de los Conjuntos Difusos [Zad65] propuesta por el Profesor Zadeh adquiere un papel crucial.

Los conjuntos difusos constituyen una extensión de la noción clásica de set, según la cuál la pertenencia de un elemento a un set no es un término binario, sino un valor continuo. Es decir, los conjuntos difusos son conjuntos cuyos elementos tienen grados de pertenencia. Estos grados de permanencia constituyen una herramienta muy valiosa para modelar la incertidumbre en la opiniones y por ello la aplicación de Conjuntos difusos en procesos de toma de decisión fue propuesta por Belman an Zadeh in 1970 [BZ70]. Desde entonces, habida cuenta de su utilidad para modelar la incertidumbre en la opiniones de las personas al tomar una decisión, han suscitado amplio interés en la comunidad científica [HVHCL04, CHHV01, CT94, FR94, HVCFA07, HM00].

De forma general, la toma de decisión requiere del esfuerzo de analizar todas las posibles alternativas y comparar y evaluar cuál de ellas es mejor, dependiendo no solamente de las características de cada opción pero también del entorno y de la situación concreta. Por ello, cuando nos encontramos ante el caso de decisiones complejas , en la mayoría de las ocasiones, la decisión es tomada por un grupo de personas, también denominado grupo de expertos. En estas situaciones, aun cuando los expertos pueden tener sus propias opiniones y bagaje previo, tienen un interés en común que consiste en alcanzar un acuerdo para seleccionar la mejor opción.

Las metodologías de toma de decisión en grupo, GDM, han sido ampliamente estudiadas en las dos últimas décadas, y por tanto varios propuestas se han propuesto en la literatura especializada, entre ellas, [Saa80, HHVV96a, CHHV01, HVACF07]. Sin embargo, hoy en día, gracias al auge de las nuevas tecnologías, las formas de comunicación y colaboración han experimentado un cambio radical y con ellas los procesos de toma de decisión. De hecho, los procesos de toma de decisión actuales en muchos casos pueden involucrar un gran número de expertos que han de tomar una decisión eligiendo entro un variado abanico de alternativas [PCHV10, PCAHV14]. Esta variabilidad en los procesos de toma de decisión conlleva el incremento de la incertidumbre inherente, ya que resulta complejo para los expertos proporcionar opiniones cuando son muchas las opciones que han de considerarse. Por todo ello nuevas propuestas capaces de manejar esta complejidad son deseables.

El principal objetivo de esta Tésis doctoral consiste en el estudio y el desarrollo de nuevas metodologías de toma de decisión en grupo en entornos dónde la incertidumbre es muy alta y por ello hay información desconocida. Para ello hemos de tener en cuenta que en procesos de toma de decisión en grupo en estos nuevos entornos complejos surgen varias desafíos para la investigación. A continuación describimos brevemente dichos desafíos y explicamos de que forma esta tesis propone nuevas soluciones.

- **Representación de preferencias:** La forma mediante la cual los expertos formulan sus opiniones afecta de forma muy directa al proceso de toma de decisión y por ello ha sido objeto de diversos estudios. Existen múltiples formas de formular las preferencias, desde valores discretos, a preferencias linguísticas basadas en conjuntos difusos. Dependiendo del tipo de proceso de toma de decisión y del grado de incertidumbre asociado puede ser mejor usar un tipo u otro. Por ejemplo, las preferencias intuicionistas, que serán definidas ampliamente en la sección 2, permiten al usuario expresar las dudas asociadas al enunciar sus preferencias. Por ello, en situaciones en las que los expertos experimentan incertidumbre acerca de sus opiniones, este tipo de preferencias puede resultar de utilidad. En esta tesis doctoral se ha realizado un análisis crítico de los diferentes tipos de relaciones de preferencia existentes en la literatura mostrando sus principales ventajas e inconvenientes.

- **Información incompleta:** En procesos de toma de decisión en grupo, las situaciones en las que todos los expertos son capaces de expresar sus preferencias respecto a todas las alternativas constituyen la excepción en lugar de la regla. Para ello, se necesitaría que todos los expertos posean un conocimiento preciso y suficiente del problema a evaluar, incluyendo la capacidad de discernir el grado por el cuál unas opciones son mejores que otras. Estos supuestos pueden considerarse irreales en la mayoría de las situaciones, especialmente aquellas en las que se consideren un elevado número alternativas y fuentes de información contradictorias y dinámicas. En esta memoria presentamos un análisis en profundidad de las metodologías existentes para procesar la información incompleta en procesos de toma de decisión en grupo. Así mismo proponemos una nueva metodología para trabajar con información incompleta en entornos de alta incertidumbre.

- **Consenso:** Resulta obvio que cuando varias personas interactúan proporcionando sus opiniones, es natural que cada uno de ellos tenga distintos puntos de vista. Sin embargo, en general, es deseable o incluso o obligatorio alcanzar una decisión que sea aceptada por el grupo al completo. Por ello la inclusión de mecanismos que aseguren alcanzar un cierto consenso entre los expertos está ampliamente justificada. Estas metodologías son conocidas cómo procesos de consenso y generalmente consisten en negociaciones iterativas. En esta memoria analizamos los procesos de consenso existentes en la literatura y presentamos una nueva metodología que emplea la incertidumbre inherente en las opiniones de los expertos modelándolas mediante gránulos de información para incrementar el nivel de acuerdo entre las opiniones de los expertos sin necesidad de llevar a cabo una negociación con varias iteraciones.

- **Agregación de información:** Un aspecto clave, cuando se trabaja con las opiniones de múltiples expertos es cómo combinarlas. Por ejemplo, existen situaciones en las que la misma importancia se le atribuye a cada uno de los miembros involucrados en el procesos de decisión, cómo es el caso de las elecciones políticas. Sin embargo, hay situaciones en las que resulta necesario atribuir mayor importancia a aquellos expertos que presentan opiniones más relevantes. Esto es, por ejemplo, menos contradicción en sus propuestas. En este sentido se han propuesto varias opciones. No obstante, en situaciones de incertidumbre la confianza de los expertos hacia las opiniones proporcionadas también debe de ser tenida en cuenta a la hora de agregar la información. Por ejemplo la opinión de un experto que está completamente seguro de su respuesta puede ser más útil que la de uno que presente más dudas. Por ello, presentamos una nueva metodología de toma de decisión en grupo que calcula el nivel de confianza de los expertos en las opiniones proporcionadas y asigna mayor importancia en la agregación a aquellos expertos que presentan mayor confianza en sus opiniones.

- **Herramientas software para toma de decisión en grupo** Con el auge de las nuevas tecnologías que facilitan la comunicación y la colaboración, la complejidad de los procesos de toma de decisión se ha visto incrementada debido a la facilidad para involucrar múltiples expertos decidiendo sobre un amplio rango de alternativas. Por ello es necesario contar con herramientas de software eficaces, capaces de manejar la información incompleta y al mismo tiempo proporcionar representaciones gráficas del estado del proceso de toma de decisión. En este sentido proponemos una nueva librería de software libre desarrollada en R para desarrollo de procesos de toma de decisión de forma automática.

Esta memoria se compone de dos partes, la primera está destinada a la presentación de los problemas abordados así cómo a la discusión de los resultados obtenidos. La segunda parte contiene una recopilación de de las principales publicaciones en revistas internacionales especializadas que se han realizado cómo fruto del trabajo llevado a cabo a lo largo de esta tesis doctoral.

La primera parte comienza con la sección de Preliminares, Sección 2, en la que se exponen los fundamentos de la Toma de decisión en Grupo y se enuncian las principales herramientas y modelos empleados para el desarrollo de esta tesis doctoral. En la Sección 3, identificamos las líneas de investigación abiertas que justifican el desarrollo de esta tesis. A continuación, en la sección 4 se enuncian los objetivos de esta tesis, y una discusión conjunta de los resultados obtenidos se presenta en la Sección 5. Finalmente en la Sección 6 se presentan las conclusiones del trabajo realizado y en la Sección 7 se pone de manifiesto nuevas lineas de investigación que han surgido fruto de la realización de esta tesis.

1. Introduction

Decision making consists on a thought and cognitive process of selecting a logical and best choice from the set of available options. This is a pervasive task in human beings every day routine. Indeed, we make choices ranging from quotidian elections, such as the type of coffee, to more complex and transcendentals selections, such as the best investment. Therefore the study of decision making mechanisms to obtain the best solution has attracted extensive research attention in very diverse areas ranging from Economy, Psychology and Sociology to Artificial Intelligence, and Engineering.

When making a decision, usually people provide opinions and judgments influenced by their own views, experience and background. Thus, an opinion can be considered as a personal assessment of the level of a variable of interest and is made using a mixture of qualitative and quantitative information [MW10]. Therefore decision making processes are far from being precise and deterministic, on the contrary they are inherently uncertain and subjective. In this sense, in decision making theory, a key issue is how to model this subjectivity in people's opinions and, indeed, in this situations is when the Fuzzy Sets Theory [Zad65] proposed by Prof. Zadeh comes into play.

The Fuzzy sets are an extension of the classical notion of set in which the membership on an element on a set is not binary term but a continuous value. In other words, fuzzy sets are sets whose elements have degrees of membership. These degrees of membership constitute a very useful tool to model the uncertainty in the experts opinions and thus, the application of Fuzzy sets in decision making was firstly proposed by Bellman and Zadeh in 1970 [BZ70]. Since then, as Fyzzy sets have been proved to be very useful to tackle with human uncertainty on decision making, they have received extensive research attention [HVHCL04, CHHV01, CT94, FR94, HVCFA07, HM00].

Generally speaking decision making requires the effort of analyzing the different suitable alternatives and comparing and assessing which one is better depending not only on the inherent characteristics of the feature in question but also on the environment and the concrete situation. When it comes to the case of a complex choices, in the majority of the occasions, the decision is made by a group of people, also known as group of experts. This kind of decision making processes involving more than one person is formally known as Group Decision Making, GDM. In this situations, even though experts, may have their own opinions and background approaching the problem from different perspectives, they share the common interest in reaching agreement on selecting the most suitable options.

GDM methodologies has been extensively studied in the last two decades, and thus, many approaches have been proposed [Saa80, HHVV96a, CHHV01, HVACF07]. However nowadays, due to the apogee of the new technologies, the communication and the collaboration has completely changed, and so the decision making processes. Current GDM processes my involve a large number of experts, choosing from a wide range of alternatives [PCHV10, PCAHV14]. This large number of experts and the wide variety of alternatives makes the uncertainty inherent in the decision process to increase. In other words, it is challenging for the experts to provide opinions from such a spread set of alternatives. Thus new decision making approaches able to deal with uncertainty are desired.

The main aim of this dissertation lies in the study and development of new group decision making approaches under highly uncertainty environments with missing information. When dealing with multiple experts in decision making situations in this new demanding environments some specific research challenges arise. In the following we briefly describe those challenges and explain how this dissertation aims to improve the state of the art of the current research efforts in these lines.

- **Preference representation formats:** The way in which the experts enunciate their opinions highly affects the decision process and so it has attracted extensive research attention. There are multiple ways of enunciating the preferences, ranging from crisp values to linguistic preference relations based on fuzzy sets. Depending on the type of decision making process and the degree of uncertainty involved, it could be better to use one type of preference relation or another. For instance, intuitionist preference relations allow the user to express certain degree of hesitation when enunciating their opinions. Therefore in highly uncertain environments they can be of great help. In this dissertation we will carry out a critical analysis of the different types of preference relations that has been proposed in the literature pointing out their main strengths and weakness.

- **Missing information:** Decision making situations where all experts are able to efficiently express their preferences over all the available options might be considered the exception rather than the rule. Indeed,this scenario requires the experts to posses a precise or sufficient level of knowledge of the whole problem to tackle, including the ability to discriminate the degree up to which some options are better than others. These assumptions can be seen as unrealistic in many decision making situations, especially those involving a large number of alternatives to choose from and/or conflicting and dynamic sources of information. In this contribution we present a thoughtfully review of the main methodologies proposed to deal with missing information in GDM, for the most extended types of preference relations. Moreover a new methodology designed to deal with incomplete information in highly uncertain environments is proposed.

- **Consensus:** When many experts interact providing their opinions it is natural that they have different point of views. However, in general, it is desirable or even mandatory to reach a decision accepted by the whole group. Therefore the inclusion of mechanisms ensuring that some agreement have been obtained is more than justified. These methodologies are known as consensus processes and, in general, are designed as iterative negotiation processes. In this contributions we analyze the proposed consensus approaches under highly uncertainty environments and we introduce a new approach that leverage the uncertainty inherent in the expert's opinion to increase the agreement taking advantage of granular information without the necessity of going over a multi-stage negotiation.

- **Information aggregation:** Obviously a key issue when dealing with the opinions of multiple experts is how to combine them. For instance, there are situations in which the same degree of importance is given to all the people involve in the decision making, this is the case of the political elections. Nevertheless, there are situations in which it makes sense to allocate more importance to those experts that presents more meaningful answers. That is, less contradiction in their opinions. In this sense, various approaches have been proposed [HVCFA07],[CHPHV10]. However under uncertainty situations, the experts confidence on the enunciated opinions also may play a key role. In other words, the opinion of an expert who is hundred percent confident on his/her answer could be more valuable than the opinion of the one that is doubtful. Therefore, in this contribution we present a new GDM approach that calculates the experts degree of confidence on the provided solutions and allocates more importance to those ones that are more confident with their answer.

- **Software tools to automatically carry out GDM approaches**: With the inclusion of new technologies the complexity of the decision making processes have increase involving in many cases a huge number of experts considering a wide set of alternatives. To that aim effective software tools to deal with this complexity, being able to estimate the missing information and at the same time providing meaningful graphical representations needs to be presented. In this sense our aim is to propose a new open source software library that automatically deals with decision making processes.

This dissertation is composed of two main parts. The first one is devoted to the statement of the problems addressed and the discussion of the obtained results. The second part is a compilation of the main publications in highly impact international journals that supports this dissertation.

Part I begins with the Preliminaries in Section 2, that exposes the basis of GDM and enunciates the main tools and models used throughout this contribution. In Section 3, we identify and define the open research challenges that justify the development of this thesis. Section 4 enunciates the proposed objectives of this dissertation. A joint discussion of the main results obtained and of the new approaches proposed is presented in Section 5. Finally, Section 6 draws the conclusion of this dissertation and in Section 7 the open challenges after the completion of this dissertation are pointed out.

2. Preliminaries

2.1. Group decision making processes

A Group decision making situation arises when a group of experts , $E = \{e_1, \ldots, e_m\}$, $(m \geq 2)$, are asked to express their opinions or preferences about the set of available options $X = \{x_1, \ldots, x_n\}$, $(n \geq 2)$. Usually experts may have different background and therefore different points of views but the share the common goal of choosing the best option between all the available ones.

In Classic GDM problems the processes to be carried out to reach the solution are twofold [HHVV96a]: The consensus process and the selection process. On the one hand, the consensus process, is aimed to maximize the agreement between all the experts with the provided solution. On the other hand, the selection process aims to obtain the final solution set of alternatives from the opinions expressed by experts.

A GDM process can be viewed as a dynamic and iterative negotiation in which experts may change their opinions so as to reach a solution accepted by all the group members. First of all, the consensus process is carried out to reach the maximum consensus degree among experts' preferences. In every step of the process the current consensus degree is measured, and if it does not reach an acceptable level, experts are encouraged to discuss their points of view and change their opinions in order to increase the proximity of their preferences. Once a certain level of consensus has been reached the selection process is applied and the final solution is obtained. The main steps in a classical GDM process are illustrated in Fig. 1

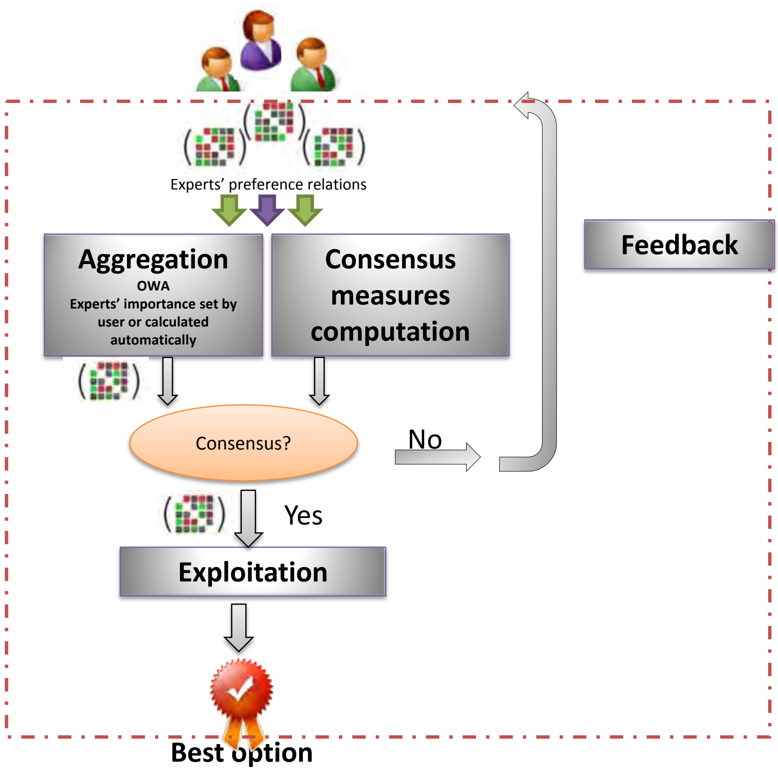


Figure 1: Classical GDM approach

- **Consensus Process:**

  A consensus process is an iterative procedure in which the experts accept to change their preferences following the advice given by a moderator. The moderator knows the agreement in each moment of the consensus process by means of the computation of some consensus measures. As aforementioned, most of the consensus models are guided and controlled by means of consensus measures [HVACF07, CPHV10, HVCKP14b]. The consensus process can be divided in several steps which are graphically depicted in Figure 5:

1. First of all, the problem to be solved is presented to the experts, as well as the different alternatives.
2. Experts provide their preferences about the alternatives in a particular preference representation usually carrying out pairwise comparison.
3. The moderator receives all the experts' preferences and computes some consensus measures which mainly asses the distance between the experts opinions and also the distance to the global solution.
4. If enough agreement has been reached the consensus process stops and the selection process begins. Otherwise, it is possible to provide some feedback to the experts, to help them to reach a consensual solution. To that aim the moderator, with all the information that he/she has (all preferences expressed by experts, consensus measures and so on) can prepare some guidance and advice for experts . Note that this step is optional and is not present in every consensus model.
5. Finally, the advice is given to the experts and the first round of consensus is finished. Again, experts must discuss their opinions and preferences in order to approach their points of view (step b).

As we can observe the majority of the process require of a moderator to help the experts by providing some recommendations to reach a consensual solution. However the inclusion of an expert in the negotiation presents some important drawbacks:

- The moderator might introduce certain degree of subjectivity into the decision process. For example, the moderator can be biased and so He/she may try to manipulate the experts to reach a specific solution which is not necessarily the best one. Therefore some automatic tools that generate the recommendations for the experts and even include this recommendations without the experts intervention are necessaries and so they need to be developed. In sec 5.3 we present a new software tool with this purpose.

- Iterative processes in some cases may not be a feasible alternative due to experts limited time. Therefore new approaches that allows to reach agreement between the experts in a single step are desired. In 5.2.2 we present a new consensus approach with this purpose.

- **Selection Process:**

  Once the desired agreement between the experts has been reached the selection process takes place. At this stage the main aim is to fuse all the experts preferences into one collective preference relation and from it obtaining a ranking of the alternatives [Tan84a, FR94]. Therefore this process is composed of two main steps namely Aggregation and Exploitation.

  1. *Aggregation Phase:* Given a group of experts, their collective preference is obtained by fusing their individual preferences using an appropriate aggregation operator. A widely used aggregation operator in decision making with fuzzy preferences is Yager's Ordered Weighted Averaging (OWA) operator [Yag88], or one of its extended versions such as the Induced OWA (IOWA) [Yag03].

     Definition 1 *An IOWA operator of dimension* m *is a function* $\Phi_W\colon (\mathbb{R}\times\mathbb{R})^m \to \mathbb{R}$*, to which a set of weights or weighting vector is associated,* $W = (w_1,\ldots,w_m)$*, such that* $w_i \in [0,1]$ *and* $\Sigma_i w_i = 1$*, is expressed as follows:*

     $$\Phi_W\left(\langle u_1,p_1\rangle,\ldots,\langle u_m,p_m\rangle\right) = \sum_{i=1}^{m} w_i \cdot p_{\sigma(i)},$$

     being $\sigma\colon \{1,\ldots,m\} \to \{1,\ldots,m\}$ *a permutation such that* $u_{\sigma(i)} \geq u_{\sigma(i+1)}$*,* $\forall i = 1,\ldots,m-1$*.*

     Consistency based IOWA operators have been proposed in literature so that the reordering of arguments to aggregate and the computation of the aggregation weights are obtained using consistency degrees values derived from the preferences experts provide [HVCFA07]. In the case of reciprocal intuitionistic fuzzy preference relation a multiplicative consistency IOWA (MC-IOWA) operator was presented in [WC14].

The general procedure for the inclusion of importance weight values, $\{u_1, \ldots, u_m\}$, in the aggregation process involves the transformation of the values to aggregate under the importance degree to generate a new value and then aggregate these new values using an aggregation operator. In the area of quantifier guided aggregations, Yager provided a procedure to evaluate the overall satisfaction of m important criteria (experts) by an alternative x by computing the weighting vector associated to an OWA operator as follows [YR96]:

$$w_h = Q\left(\frac{S(h)}{S(m)}\right) - Q\left(\frac{S(h-1)}{S(m)}\right)$$

being Q the membership function of the linguistic quantifier, $S(h) = \sum_{k=1}^{h} u_{\sigma(k)}$, and σ the permutation used to produce the ordering of the values to be aggregated. This approach for the inclusion of importance degrees associates a zero weight to those experts with zero importance degree. The linguistic quantifier is a Basic Unit-interval Monotone (BUM) function $Q : [0,1] \to [0,1]$ such that $Q(0) = 0$, $Q(1) = 1$ and if $x > y$ then $Q(x) \geq Q(y)$.

Yager extended this procedure to the case of IOWA operator. In this case, each component in the aggregation consists of a triple, with first element being the argument value to aggregate, the second element the importance weight value associated to the first element and the third element being the order inducing value [Yag03]. The same expression as above is used with σ being the permutation that order the induce values from largest to lowest.

2. *Exploitation Phase:* This final step uses the information produced in the aggregation phase to identify the solution set of alternatives for the problem. To do so we must apply some mechanism to obtain a partial order of the alternatives and thus select the best alternative(s). There are several different ways to do this, but a usual one is to associate a certain utility value to each alternative (based on the aggregated information), thus producing a natural order of the alternatives.

2.2. Preference Relations in Decision Making

In any decision making problem, once the set of feasible alternatives (X) is identified, experts are called to express their opinions or preferences. In this sense Millet [Mil97] conducted a comparison study between different alternative preference elicitation methods, and pairwise comparison approaches were concluded to be more accurate than non-pairwise ones (utilities, orderings, ...) [FR94]. This is specially the case of decision making problems involving a large number of alternatives to choose from and/or conflicting and dynamic sources of information [PCHV10, PCAHV14].

The main advantage of preference relations, which are built by pairwise comparisons, is that focusing exclusively on two options at a time makes easier fir the experts to articulate their preference. However, the drawback is that some experts might not been able to discriminate the degree up to which some of the options are better than others, and as a consequence incomplete preferences are provided [DMO+12].

Two main mathematical models based on the concept of preference relation can be used in this context. In the first one, a preference relation is defined for each one of the above three possible preference states (preference, indifference, incomparability) [Fis79], which is usually referred to as a preference structure on the set of alternatives [RV85]. The second one integrates the three possible preference states into a single preference relation [BSR78]. In this paper, we focus on the second one.

Formally, a preference relation is defined as follows:

Definition 2 (Preference Relation (PR) [ACH+08]) *A preference relation R is a type of binary relation defined on the set X that is characterised by a function $\mu_p : X \times X \to D$, where D is the domain of representation of preference degrees provided by the decision maker for each pair of alternatives. When cardinality of X is small, R may be conveniently represented by an $n \times n$ matrix $R = (r_{ij})$, being $r_{ij} = \mu_p(x_i, x_j)$ the degree or intensity of preference of alternative x_i over x_j.*

The elements of R can be of a numeric or linguistic nature, i.e., could represent numeric or linguistic preferences, respectively.

2.2.1. Numeric Preferences

The main types of numeric preference relations used in decision making are: crisp preference relations, additive preference relations, multiplicative preference relations, interval valued preference relations and intuitionistic preference relations. In the following subsections we analyze each one of these options.

2.2.1.a. Crisp Preference Relation

When an expert is able to compare two alternatives the following broad outcomes are possible: (I) one alternative is preferred ($\succ$) to another; or (II) the two alternatives are indifferent ($\sim$). Using a numerical representation of preferences, any ordered pair of alternatives $(x_i, x_j) \in X \times X$ can be associated a number from the set $D = \{0, \frac{1}{2}, 1\}$ as follows:

$$\begin{array}{lll} r_{ij} = 1 & \Leftrightarrow & x_i \succ x_j \\ r_{ij} = 0 & \Leftrightarrow & x_j \succ x_i \\ r_{ij} = 0{,}5 & \Leftrightarrow & x_j \sim x_i \end{array}$$

The equivalent set of values ($\{1, 0, -1\}$) has also been used in these cases (see [Fis79]). Thus, if $r_{ij} = 1$ the expert prefers alternative x_i to alternative x_j, while if $r_{ij} = \frac{1}{2}$ the expert is indifferent between both alternatives. Moreover, it is always assumed the following 'reciprocity' property: when $r_{ij} = \frac{1}{2}$ it is also $r_{ji} = \frac{1}{2}$; and when $r_{ij} = 1$ then $r_{ji} = 0$. This property guarantees that the preferences are represented by a weak order, i.e. the asymmetric property is verified and 'inconsistent' situations where an expert could prefer two alternatives at the same time are avoided. Formally, the binary preference relation $\succ$ is asymmetric if given two alternatives x_i and x_j, $x_i \succ x_j$ implies that $x_j \not\succ x_i$.

Given three alternatives x_i, x_j, x_k such that x_i is preferred to x_j and x_j to x_k, the question whether the 'degree or strength of preference' of x_i over x_j exceeds, equals, or is less than the 'degree or strength of preference' of x_j over x_k cannot be answered with this crisp preference modeling [Fis79, CHVAH09].

2.2.1.b. Additive Preference Relation

The introduction of the concept of fuzzy set as an extension of the classical concept of set when applied to a binary relation leads to the concept of a fuzzy or [0,1]-valued preference relation, $P = (p_{ij})$, which extends the crisp preference relation in that its elements p_{ij} can take any value from the unit interval $D = [0, 1]$, with the following interpretation:

- $p_{ij} > 0{,}5$ indicates that the expert prefers the alternative x_i to the alternative x_j, with $p_{ij} = 1$ being the maximum degree of preference for x_i over x_j;
- $p_{ij} = 0{,}5$ represents indifference between x_i and x_j.

As in the previous case, the following reciprocity property of preferences is usually assumed as an extension of the crisp asymmetry property described above:

$$\forall i, j \in \{1, \cdots, n\} : \; p_{ij} + p_{ji} = 1. \tag{I.1}$$

This type of preference relations will be referred to as additive preference relations in this paper.

Definition 3 (Additive Preference Relation (APR)) *An APR P on a finite set of alternatives X is characterised by a membership function $\mu_P : X \times X \longrightarrow [0, 1]$, $\mu_P(x_i, x_j) = p_{ij}$, verifying $p_{ij} + p_{ji} = 1 \; \forall i, j \in \{1, \ldots, n\}$.*

An APR can be seen as a particular case of a (weakly) complete fuzzy preference relation, i.e. a fuzzy preference relation satisfying $p_{ij} + p_{ji} \geq 1 \; \forall i, j$.

2.2.1.c. Multiplicative Preference Relation

The measuring of the intensity of preferences can be done using a ratio scale instead, with the most widely ratio scale used being the interval $D = [1/9, 9]$ [Saa80]. In this case, preferences are represented via the so-called multiplicative preference relation, $A = (a_{ij})$, whose element a_{ij} is interpreted as follows: x_i is a_{ij} times as good as x_j [HVHCL04], and in particular:

- $a_{ij} = 1$ indicates indifference between x_i and x_j;
- $a_{ij} = 9$ indicates that x_i is absolutely preferred to x_j;
- $a_{ij} \in \{1, \cdots, 9\}$ indicates intermediate preference evaluations.

Furthermore, the preference relation A is assumed to verify the following multiplicative reciprocity property:

$$\forall i, j \in \{1, \cdots, n\} : \; a_{ij} \cdot a_{ji} = 1 \tag{I.2}$$

This type of preference relations will be referred to as multiplicative preference relations in this paper.

Definition 4 (Multiplicative Preference Relation (MPR)) *A MPR A on a finite set of alternatives X is characterised by a membership function* $\mu_A\colon X \times X \longrightarrow [1/9, 9]$, $\mu_A(x_i, x_j) = a_{ij}$, *verifying* $a_{ij} \cdot a_{ji} = 1\ \forall i, j \in \{1, \ldots, n\}$.

In [CHHV01], it was proved that multiplicative and additive preference relations are isomorphic:

Proposition 1 *Suppose that we have a set of alternatives,* $X = \{x_1, \ldots, x_n\}$, *and associated with it a MPR* $A = (a_{ij})$, *with* $a_{ij} \in [1/9, 9]$ *and* $a_{ij} \cdot a_{ji} = 1, \forall i, j$. *Then the corresponding APR,* $P = (p_{ij})$, *associated to A, with* $p_{ij} \in [0, 1]$ *and* $p_{ij} + p_{ji} = 1, \forall i, j$, *is given as follows:*

$$p_{ij} = f(a_{ij}) = \frac{1}{2}\left(1 + \log_9 a_{ij}\right).$$

The above transformation function is bijective and, therefore, allows to transpose concepts that have been defined for APRs to MPRs, and vice-versa.

2.2.1.d. Interval Valued Preference Relation

Membership functions of fuzzy sets are subject to uncertainty arising from various sources [Men01]. Klir and Folger comment [KF92, page 12]:

> "*... it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers. Although this does not pose a serious problem for many applications, it is nevertheless possible to extend the concept of the fuzzy set to allow the distinction between grades of membership to become blurred.*"

Here Klir and Folger described blurring a fuzzy set to form an *interval valued fuzzy set*:

Definition 5 (Interval Valued Fuzzy Set (IVFS)) *Let* $INT([0, 1])$ *be the set of all closed subintervals of* $[0, 1]$ *and X be a universe of discourse. An* *interval valued fuzzy set (IVFS) $\widetilde{A}$ on X is characterised by a membership function $\mu_{\widetilde{A}} : X \to INT([0,1])$. An IFS $\widetilde{A}$ on X can be expressed as follows:*

$$A = \{(x, \mu_{\widetilde{A}}(x));\ \mu_{\widetilde{A}}(x) \in INT([0,1])\ \forall x \in X\}. \tag{I.3}$$

The application of the concept of IVFS to an APR leads to the concept of interval valued APR (IVAPR) , i.e. a preference relation with domain of representation of preference degrees, $D = INT([0,1])$, is the set of all closed subintervals of $[0,1]$.

Definition 6 (Interval Valued Additive Preference Relation (IVAPR)) *An interval valued additive preference relation (IVAPR) $\widetilde{P}$ on a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ is characterised by a membership function $\mu_{\widetilde{P}} \colon X \times X \longrightarrow INT([0,1])$, with $\mu_{\widetilde{P}}(x_i, x_j) = \widetilde{p}_{ij} = [p^-_{ij}, p^+_{ij}]$, verifying*

$$\forall i, j \in \{1, \ldots, n\} :\ \widetilde{p}_{ji} = 1 - \widetilde{p}_{ij}. \tag{I.4}$$

The above definition of IVAPR can be expressed in terms of the lower and upper bound of the interval valued preference values as follows:

$$\forall i, j = 1, 2, \ldots n :\ p^-_{ij} + p^+_{ji} = p^+_{ij} + p^-_{ji} = 1. \tag{I.5}$$

2.2.1.e. Intuitionistic Preference Relation

The concept of an Intuitionistic Fuzzy Set (IFS) was introduced by [Ata86]:

Definition 7 (Intuitionistic Fuzzy Set (IFS)) *An intuitionistic fuzzy set (IFS) A over a universe of discourse X is represented as $A = \{(x, \langle \mu_A(x), \nu_A(x) \rangle) \,|x \in X\}$ where $\mu_A \colon X \to [0,1]$, $\nu_A \colon X \to [0,1]$ and $0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad \forall x \in X$. For each $x \in X$, the numbers $\mu_A(x)$ and $\nu_A(x)$ are known as the degree of membership and degree of non-membership of x to A, respectively.*

An IFS becomes an FS when $\mu_A(x) = 1 - \nu_A(x)\ \forall x \in X$. However, when there exists at least a value $x \in X$ such that $\mu_A(x) < 1 - \nu_A(x)$, an extra parameter has to be taken into account when working with IFSs: the hesitancy degree, $\tau_A(x) = 1 - \mu_A(x) - \nu_A(x)$, that represents the amount of lacking information in determining the membership of x to A. If the hesitation degree is zero, the reciprocal relationship between membership and non-membership makes the latter one unnecessary in the formulation as it can be derived from the former.

In [SK02] , Szmidt and Kacprzyk defined the intuitionistic fuzzy preference relation (IFPR) as a generalisation of the concept of APR.

Definition 8 (Intuitionistic Fuzzy Preference Relation (IFPR)) *An intuitionistic fuzzy preference relation (IFPR) B on a finite set of alternatives X is characterised by a membership function*

$$\mu_B : X \times X \to [0, 1]$$

and a non-membership function

$$\nu_B : X \times X \to [0, 1]$$

such that

$$0 \leq \mu_B(x_i, x_j) + \nu_B(x_i, x_j) \leq 1 \quad \forall (x_i, x_j) \in X \times X. \tag{I.6}$$

An IFPR can be conveniently represented by a matrix $B = (b_{ij})$ with $b_{ij} = (\mu_{ij}, \nu_{ij})$ $\forall i, j = 1, 2, \ldots, n$. The value $\mu_{ij} = \mu_B(x_i, x_j)$ can be interpreted as the certainty degree up to which x_i is preferred to x_j, while the value $\nu_{ij} = \nu_B(x_i, x_j)$ represents the certainty degree up to which x_i is non-preferred to x_j. When the following additional conditions are imposed:

- $\mu_{ii} = \nu_{ii} = 0{,}5$ $\forall i$.
- $\mu_{ji} = \nu_{ij}, \nu_{ji} = \mu_{ij}$ $\forall i, j$.

we refer to this IFPR as additive and we will denote it as IAPR. Note that when the hesitancy degree function is the null function we have that $\mu_{ij} + \nu_{ij} = 1$ $\forall i, j$, and therefore the IAPR $B = (b_{ij})$ is mathematically equivalent to the APR (μ_{ij}), i.e. $B = (\mu_{ij})$. In any case, given an IAPR, it is always possible to derive an APR via the application of a score function [WC12].

Linguistic label	Meaning	Semantics $(a_i, b_i, \alpha_i, \beta_i)$
C	Certain	(1,1,0,0)
EL	Extremely Likely	(0.98,0.99,0.05,0.01)
ML	Most Likely	(0.78,0.92,0.06,0.05)
MC	Meaningful Chance	(0.63;0.80;0.05;0.06)
IM	It May	(0.41,0.58,0.09,0.07)
SC	Small Chance	(0.22,0.36,0.05,0.06)
VLC	Very Low Chance	(0.1,0.18,0.06,0.05)
EU	Extremely Unlikely	(0.01,0.02,0.01,0.05)
I	Impossible	(0,0,0,0)

Table I.1: A set of nine linguistic labels with its semantics

2.2.2. Linguistic Preferences

Subjectivity, imprecision and vagueness in the articulation of opinions pervade real world decision applications, and individuals usually find it difficult to evaluate their preferences using exact numbers [Zad65]. Individuals might feel more comfortable using words by means of linguistic labels or terms to articulate their preferences [Zad75, Zad65]. Furthermore, human beings exhibit a remarkable capability to manipulate perceptions and other characteristics of physical and mental objects, without any exact numerical measurements and complex computations [HM00, HACHV09].

Usually, in a linguistic approach it is assumed a set of linguistic terms, $\mathcal{L} = \{l_0, \ldots, l_s\}$ ($s \geq 2$), with a ranking relation defined as a linear ordering, i.e., $l_0 < l_1 < \ldots < l_s$, in order to express the expert preferences. For example, in Table I.1 we show an example of a set of nine linguistic labels and their corresponding meanings for the comparison of the ordered pair of alternatives (x_i, x_j). It is also assumed that the number of labels is odd and the central label $l_{s/2}$ stands for the indifference state when comparing two alternatives, the remaining labels are usually located symmetrically around that central assessment, which guarantees that a kind of reciprocity property holds as in the case of numerical preferences previously discussed.

Definition 9 (Linguistic Preference Relation (LPR)) *A LPR P on a fi-*

nite set of alternatives X is characterised by a linguistic membership function $\mu_P\colon X \times X \longrightarrow \mathcal{L},\ \mu_P(x_i, x_j) = p_{ij} \in \mathcal{L}$

The main two methodologies to manage LPRs in decision making are [HACHV09]: (I) the *cardinal* representation model based on the use of fuzzy sets and their associated membership functions, which are mathematically processed using Zadeh's *extension principle* [Zad65]; and (II) the *ordinal* representation model based on the ordered structure defined on the labels [HHVV96a, Yag81].

2.2.2.a. LPR based on cardinal representation

Convex normal fuzzy subsets of the real line, also known as fuzzy numbers, are commonly used to represent linguistic terms. By doing this, each linguistic assessment is represented using a fuzzy number that is characterized by a membership function, with base variable the unit interval $[0, 1]$, describing its semantic meaning. The membership function maps each value in $[0, 1]$ to a degree of performance which represents its compatibility with the linguistic assessment [Zad75, Zad65].

2.2.2.b. LPR based on ordinal representation

In an ordinal linguistic approach the semantics of the linguistic labels is established by assuming that in the set of linguistic terms $\mathcal{L}$ the labels are uniformly and symmetrically distributed around that central assessment $l_{s/2}$, i.e., assuming the same discrimination levels on both sides of $l_{s/2}$ and by considering that both terms l_i and l_{s-i} are equally informative.

Linguistic symbolic computational models are defined to manage the ordinal linguistic information in the decision making problems [HACHV09]. The symbolic models works with the ordinal scales of the set of linguistic terms to combine linguistic information. There exit four different linguistic symbolic computational models based on ordinal scales: (I) a linguistic symbolic computational model based on max-min operators; (II) a linguistic symbolic computational model based on indexes; (III) a linguistic symbolic computational model based on continuous term sets; and (IV) linguistic symbolic model based on the 2-tuple representation.

1. *Linguistic symbolic computational model based on max-min operators* [Yag81]. In this model to combine information expressed as linguistic labels in that ordered linguistic scale $\mathcal{L}$ the following Max, Min and Neg operators are used:

   - $Max(l_i, l_j) = l_i$ if $l_i > l_j$.
   - $Min(l_i, l_j) = l_i$ if $l_i < l_i$.
   - $Neg(l_i) = l_{s-i}$.

2. *Linguistic symbolic model based on convex combination* [HHVV96a]. In this model the aggregation of linguistic information is carried out using a convex combination of linguistic labels acting directly over the label indexes of $\mathcal{L}$ in a recursive way. For this model it may be also necessary to introduce an approximation function to obtain a final label on the $\mathcal{L}$ since the result of the aggregation of labels is not necessary integer, i.e., a label index [HHVV96a].

3. *Linguistic symbolic model based on virtual linguistic term set* [Xu04a]. This model is based on the extension transformation of the original discrete term set $\mathcal{L}$ into a continuous term set $\hat{\mathcal{L}} = \{l_\alpha | \alpha \in [-s, s]\}$ with the following operations :

$$l_\alpha \oplus l_\beta = l_{max\{-s,min\{\alpha+\beta,s\}\}} \tag{I.7}$$

$$\lambda l_\alpha = l_{\lambda\alpha}, \text{ where } \lambda \in [0,1] \tag{I.8}$$

   This model also requires a translation function to express the results of the operations in the original terms set.

4. *Linguistic symbolic model based on the 2-tuple linguistic representation* [HM00]. This model was introduced to avoid the loss of information that appear when we use an approximation function (as the rounding operation) in the linguistic symbolic model based on convex combination. To do that, the linguistic 2-tuple representation model was introduced.

Definition 10 *Let $\mathcal{L}$ be a linguistic term set and $\beta \in [0, s]$ a value supporting the result of a symbolic aggregation operation, then the 2–tuple* *that expresses the equivalent information to β is obtained with the following function:*

$$\begin{aligned} \Delta: \quad & [0,s] \longrightarrow S \times [-0.5, 0.5) \\ & \Delta(\beta) = (l_i, \alpha) \\ & i = round(\beta) \\ & \alpha = \beta - i \end{aligned} \tag{2.1}$$

where "round" is the usual rounding operation, l_i has the closest index label to "β" and "α" is the value of the symbolic translation.

2.3. Consistency in decision making

There are three fundamental and hierarchical levels of rationality assumptions when dealing with preference relations [CHVAH09]:

- The first level of rationality requires indifference between any alternative x_i and itself.
- The second one requires that if an expert prefers x_i to x_j, that expert should not simultaneously prefer x_j to x_i. This asymmetry condition is viewed as an "obvious" condition/criterion of consistency for preferences [Fis79]. This rationality condition is modelled by the property of reciprocity in the pairwise comparison between any two alternatives [CHVAH09], which is seen by Saaty as basic in making paired comparisons [Saa80].
- Finally, the third one is associated with the transitivity in the pairwise comparison among any three alternatives.

A preference relation verifying the third level of rationality is usually called a *consistent preference relation* and any property that guarantees the transitivity of the preferences is called a consistency property [CHVAH09]. The lack of consistency in decision making can lead to inconsistent conclusions; that is why it is important, in fact crucial, to study conditions under which consistency is satisfied [Saa80].

The traditional requirement to characterize consistency in the case of APRs or MPRs are based on the notion of transitivity, in the sense that if x_i is preferred to x_j ($x_i \succ x_j$) and this one to x_k ($x_j \succ x_k$) then alternative x_i should be preferred to x_k ($x_i \succ x_k$), which is normally referred to as *weak* *stochastic transitivity* [LS65]. The main difference in these cases with respect to the classical one is that transitivity has been modeled in many different ways due to the role that intensities of preferences have [FR94, Saa80, Tan84b, HVHCL04, CHVAH09].

Due to the hierarchical structure of the three rationality assumptions for a preference relation, the verification of a particular level of rationality should be a necessary condition in order to verify the next level of rationality. This means that the third level of rationality, transitivity of preferences, should imply or be compatible with the second level of rationality, reciprocity of preferences, and the second level with the first one, indifference of any alternative with itself.

This necessary compatibility between the rationality assumptions can be used as a criterion for considering a particular condition modeling any one of the rationality levels as adequate or inadequate. In the case of additive (multiplicative) preference relations, the indifference between any alternative, x_i, and itself is modeled by associating the preference value $p_{ii} = 0{,}5$ ($a_{ii} = 1$). The reciprocity of additive (multiplicative) preferences is modeled using the property $p_{ij} + p_{ji} = 1,\ \forall i, j$ ($a_{ij} \cdot a_{ji} = 1,\ \forall i, j$). A necessary condition for a preference relation to verify reciprocity should be that indifference between any alternative and itself holds. Because reciprocity property implies the indifference of preferences, we conclude that both properties are compatible.

In the case of MPRs, Saaty means by *consistency* what he calls *cardinal transitivity* in the strength of preferences, which is a stronger condition than the traditional requirement of the transitivity of preferences [Saa80]:

Definition 11 (Consistent MPR) *Given a MPR $A = (a_{ij})$, it is consistent if and only if*

$$a_{ij} \cdot a_{jk} = a_{ik}\ \forall i, j, k = 1, \ldots, n. \tag{I.2}$$

Inconsistency for Saaty is a violation of proportionality which may not entail violation of transitivity [Saa80]. Furthermore, consistency implies reciprocity, and therefore, they are both compatible.

In [Saa80] Saaty shows that a reciprocal MPR is consistent if and only if its maximum or principal eigenvalue λ_{max} is equal to the number of alternatives n. Under this consistency property, Saaty proves that there exists a set of priorities (utilities) $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $a_{ij} = \frac{\lambda_i}{\lambda_j}$. Moreover, this set of values is unique up to positive linear transformation $f(\lambda_i) = \beta \cdot \lambda_i$ with $\beta > 0$. Thus, if a MPR is consistent then it can be represented by a unique (up to positive linear transformations) utility function.

For APRs, there exist many properties or conditions that have been suggested as rational conditions to be verified by a consistent relation, among which we can cite [CHVAH09, HVHCL04]: triangle condition, weak transitivity, max-min transitivity, max-max transitivity, restricted max-min transitivity, restricted max-max transitivity, additive transitivity, and multiplicative transitivity. Among these, the most widely used in the context of incomplete information are the following two [CHVAH09]:

Definition 12 (Additive consistency of APR [Tan84b]) *Given a APR $P = (p_{ij})$ on a finite set of alternatives X, it is additive consistent if and only if*

$$(p_{ij} - 0{,}5) + (p_{jk} - 0{,}5) = p_{ik} - 0{,}5 \quad \forall i, j, k = 1, 2, \cdots, n \tag{I.3}$$

As in the case of MPRs, if an APR is additive consistent then it can be represented by a unique (up to positive linear transformations) utility function. Although equivalent to Saaty's consistency property for MPRs [CHHV01, HVHCL04], additive transitivity is in conflict with the $[0, 1]$ scale used for providing the preference values and therefore, it is not the most appropriate property to model consistency of reciprocal PRs.

Definition 13 (Multiplicative consistency of APR [Tan84b]) *Given a APR $P = (p_{ij})$ on a finite set of alternatives X, it is multiplicative consistent if and only if*

$$p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ik} \cdot p_{kj} \cdot p_{ji} \quad \forall i, k, j \in \{1, 2, \dots n\} \tag{I.4}$$

Multiplicative consistency property was proposed by Tanino for fuzzy preference relations when $p_{ij} > 0 \ \ \forall i, j$ and under reciprocity it is the restriction to the region $[0, 1] \times [0, 1] \backslash \{(0, 1), (1, 0)\}$ of the Cross Ratio uninorm [CHVAH09]:

$$U(x, y) = \begin{cases} 0, & (x, y) \in \{(0, 1), (1, 0)\} \\ \dfrac{xy}{xy + (1 - x)(1 - y)}, & \textit{otherwise} \end{cases} \tag{I.5}$$

Consistency property for the rest of preference relations has been model by extending the above consistency properties via the Zadeh's extension principle. Although it is extended the practice of adapting the above properties no proper justification of its validity has been reported.

Definition 14 (Multiplicative Consistent IVAPR [WC14]) *An IVAPR* $\widetilde{P} = (\widetilde{p}_{ij}) = ([p_{ij}^-, p_{ij}^+])$, *is multiplicative consistent if and only if*

$$\forall i, j, k : \begin{cases} p_{ij}^- \cdot p_{jk}^- \cdot p_{ki}^- = p_{ik}^- \cdot p_{kj}^- \cdot p_{ji}^- \\ p_{ij}^+ \cdot p_{jk}^+ \cdot p_{ki}^+ = p_{ik}^+ \cdot p_{kj}^+ \cdot p_{ji}^+ \end{cases} \quad \text{(I.6)}$$

Definition 15 (Additive Consistent IVAPR [ACH+08]) *An IVAPR* $\widetilde{P} = (\widetilde{p}_{ij}) = ([p_{ij}^-, p_{ij}^+])$, *is additive consistent if and only if*

$$\forall i, j, k : \begin{cases} p_{ik}^- = p_{ij}^- + p_{jk}^- - 0{,}5 \\ p_{ik}^+ = p_{ij}^+ + p_{jk}^+ - 0{,}5 \end{cases} \quad \text{(I.7)}$$

Because the IAPR $R = (r_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$ *is isomorphic to the IVAPR* $B = (b_{ij}) = ([\mu_{ij}, 1-\nu_{ij}])$, *a multiplicative consistent IAPR can be defined as follows [WC14]:*

Definition 16 (Multiplicative Consistent IAPR [WC14]) *An IAPR* $R = (r_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$ *is consistent if and only if*

$$\forall i, j, k : \begin{cases} \mu_{ij}\mu_{jk}\mu_{ki} = \mu_{ik}\mu_{kj}\mu_{ji} \\ (1-\nu_{ij})(1-\nu_{jk})(1-\nu_{ki}) = (1-\nu_{ik})(1-\nu_{kj})(1-\nu_{ji}) \end{cases} \quad \text{(I.8)}$$

Definition 17 (Consistency of LPRs) *In the case of LPRs, the consistency property has been defined with different expressions depending on the linguistic approach used:*

- LPRs in cardinal approach *[WC10]. If we have a LPR,* $\tilde{P} = \tilde{p_{ij}}$ *in which each linguistic preference degree has associated a triangular fuzzy membership function, i.e.,*$\tilde{p_{ij}} = (p_{ij}^L, p_{ij}^M, p_{ij}^R)$, *then* $\tilde{P}$ *is additive consistent if and only if*

$$\forall i, j, k : \begin{cases} p_{ij}^L + p_{jk}^L + p_{ki}^R = \frac{3}{2} \\ p_{ij}^M + p_{jk}^M + p_{ki}^M = \frac{3}{2} \\ p_{ij}^R + p_{jk}^R + p_{ki}^L = \frac{3}{2} \end{cases} \quad \text{(I.9)}$$

- LPRs in ordinal approach: The case of 2-tuple LPRs *[ACC+09]. Given a 2-tuple LPR* $P = (p_{ij})$ *on a set of alternatives* X*, such that,*

$$p_{ij} : X \times X \longrightarrow \mathcal{L} \times [-0{,}5, 0{,}5)$$

then P *will be considered consistent if for every three alternatives* x_i*,* x_j *and* x_k*, the following condition holds*

$$p_{ik} = \Delta(\Delta^{-1}(p_{ij}) + \Delta^{-1}(p_{jk}) - \frac{s}{2}) \quad \forall i, j, k \in \{1, \ldots, n\} \qquad \text{(I.10)}$$

2.4. Granular information

The idea of granularity was firstly introduce by Prof. Zadeh in [Zad96]. This notion serves as a foundation for the later development in computing with words and granular computing.

> "... *There are many situations, however, in which the finiteness of the resolving power of measuring or information gathering devices cannot be dealt with through an appeal to continuity. In such case, the information may be said to be granular in the sense that the data points within in a granule have to be dealt with as a whole rather than individually.* "

In a nutshell, granular computing is geared toward representing and processing basic chunks of information - information granules. Information granules, as the name itself stipulates, are collections of entities, usually originating at the numeric level, that are arranged together due to their similarity, functional adjacency, indistinguishability or alike. The process of forming information granules is referred to as information granulation. No matter how this granulation proceeds and what fundamental technology becomes involved therein, there are several essential factors that drive all pursuits of information granulation [Ped01, Ped11, Ped13b, Ped13a]. These factors include

- A need to split the problem into a sequence of more manageable and smaller subtasks. Here granulation serves as an efficient vehicle to modularize the problem. The primary intent is to reduce an overall computing effort

- A need to comprehend the problem and provide with a better insight into its essence rather than get buried in all unnecessary details. In this sense, granulation serves as an abstraction mechanism that reduces an entire conceptual burden. As a matter of fact, by changing the *size* of the information granules, we can hide or reveal a certain amount of details one intends to deal with during a certain design phase.

Granular computing has attracted extensive research attention, indeed in [PC] it has been reported a review of the application of this technique to the field of decision making in different contexts ranging from the economy to the digital libraries and the consensus situation. In this contribution as we will show in section 5.2.2 we propose a new consensus approach that leverage the power of representation of the information granules to model the experts uncertainty in their opinions and obtain a consensual solution.

2.5. Incomplete information

It is often assumed in GDM that all the experts are able to provide preference degrees between any pair of possible alternatives, which means that complete PRs are assumed. However, as aforementioned, this is not always possible because of time pressure, lack of knowledge, decision maker's limited expertise on the field dealt with, or incapacity to quantify the degree of preference of one alternative over another. Thus, an expert might decide not to guess the preference values in doubt to maintain the consistency of the values already provided. Indeed, a recent study [DMO$^+$12] has reported that increasing the intensity of conflict in a multicriteria comparison increases the likelihood that decision makers consider two alternatives as incomparable, and therefore leading to the expression of incomplete preferences. These results also indicate that a large attribute spread increases the frequency of incomparability statements when allowed, otherwise an increase of indifference statements happens.

To model the situations in which an expert is not able to provide a judgments about all the alternatives the concept of incomplete PR was introduced in [HVCFA07].

Definition 18 *A function* $f: X \longrightarrow Y$ *is* partial *when not every element in the set* X *necessarily maps to an element in the set* Y*. When every element from the set* X *maps to one element of the set* Y *then we have a* total *function.*

Definition 19 *A preference relation P on a set of alternatives X with a* partial *membership function is an* incomplete preference relation.

The concept of incomplete preference relations has attracted the attention of researchers in the last 20 years and therefore specific settings for different types of PRs have been introduced and analyzed in the literature [Xu04a, XCS11]. The majority of this approaches tries to estimate the incomplete information before carrying out the selection of the alternatives. Thus an additional completion step is included in the GDM process as it is depicted in Fig 2.

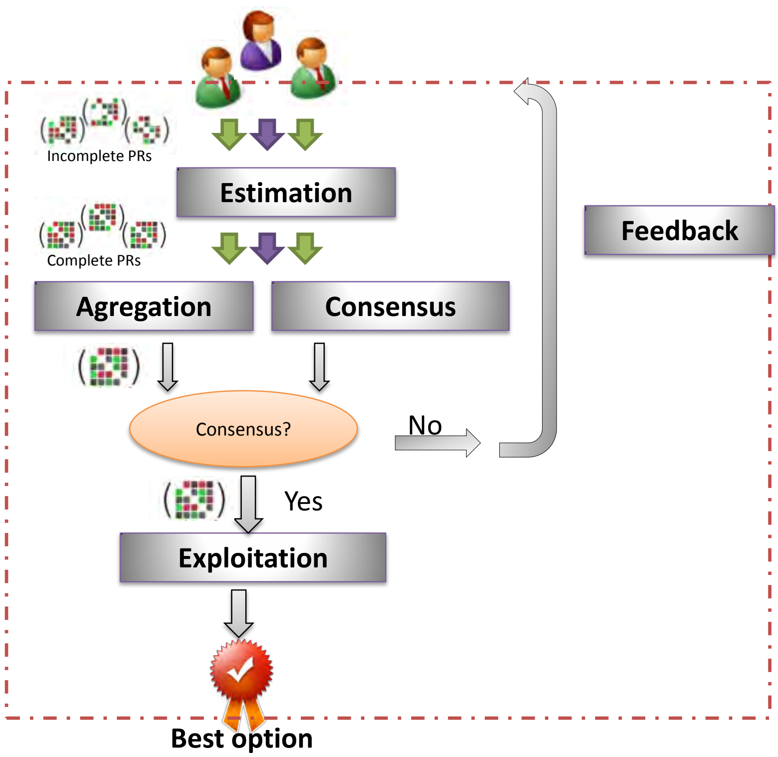


Figure 2: GDM approach when

In this contribution we analyzed the main approaches to estimate the missing information in GDM and present a new methodology to deal with incomplete isolationistic fuzzy preference relations.

3. Justification

After the presentation of all the main concepts related to the topic, we identify the main research challenges that this contribution aims to accomplish:

- **Dealing with missing information in Group decision making situations.**
  Some methodologies widely adopted under incomplete information in GDM rating more negatively or even discarding the opinions of those experts that provide preferences with missing values. However, incomplete information is not equivalent to low quality information, and consequently these methodologies could lead to biased or even bad solutions since useful information might not been taken into account properly in the decision process. Therefore, alternative approaches to manage incomplete preference relations that estimates the missing information in decision making are desirable and possible and thus need to be thoughtfully studied, analyzed and improved.

- **Dealing with uncertainty in decision making processes with incomplete information by means Intuitionistic preference relations.**
  Intuitionistic preference relations constitute a flexible and simple representation format of experts preference on a set of alternative options, while at the same time allowing to accommodate degrees of hesitation inherent to all decision making processes. In comparison with fuzzy preference relations, the use of intuitionistic fuzzy preference relations in decision making is limited, which is mainly due to the computational complexity associated to using membership degree, non-membership degree and hesitation degree to model experts subjective preferences. Therefore the use of this type of preference relations needs to be simplified.

- **Assessing experts confidence in GDM situations**
  Confidence has been defined as a person's belief that a statement represents the best possible response. Obviously the more confident an expert feels with the provided solution the more relevant the opinion can be considered, and thus more importance should be allocated to her/his opinion. However, as far as the author knows, there is no effective way to objectively asses the expert's confidence with the information provided

in decision making. Therefore it is necessary to develop a methodology that objectively asses the experts confidence, and include this valuable information in the decision making process.

- **Dealing with the uncertainty in consensus processes by using granular information.**
  Consensus is defined as a cooperative process in which a group of decision makers develops and agrees to support a decision in the best interest of the whole group. Usually consensus is regarded as a negotiation process in which the members usually modify their choices until a high level of agreement within the group is achieved. However usually due to the uncertainty the experts preferences cannot be modelled by a crisp number but by an information granule. Therefore some consensus models should be developed to leverage the flexibility that this type of preference representation offers.

- **Lack of software tools to automatically carry out DM processes.**
  As pointed out previously new paradigms and ways of making decisions, such as web 2.0 frameworks, social networks and e-democracy, have made the complexity of decision making processes to increase, involving a huge number of decision makers [PCHV10, PCAHV14]. These new scenarios require automatic tools not only to combine the information in the best possible way but also to better analyze the whole context, providing a rapid and complete insight about the current state of the process. In this direction, some efforts have already been made [AHVCH10, PM14, PMH14, PCHV10, PCAHV14], however, these approaches present various deficiencies: (i) do not offer graphical visualizations displaying the evolution of the process, and (ii) do not offer the possibility of creating a data set to test and compare the performance of different approaches. (iii) do no effectively estimate the missing information in the decision making process. (iv) are closed systems and so they are not aimed to be upgraded or extended by other researchers,

Thus, the study of those aspects is a key point to develop reliable and realistic GDM models and processes adapted to the current GDM scenarios.

4. Objectives

The aim of this thesis is to perform an in-depth study of the various methodologies in the literature to deal with decision making situations in environments where the uncertainty is high and therefore the information provided is not complete. This thesis is organized in several objectives which gather the open problems that were described in the previous section.

1. **Analyze the state of the art of completion approaches in Group Decision Making problems and identify new research opportunities**.
   In this dissertation we carry out an in depth review of the various methodologies in the literature available to complete missing preference relations in Group Decision Making processes for the different types of preference relations previously explained. To do so, we first review the most widely used types of preference relations and their main consistency properties and them we analyze the most relevant completion approaches for each type of preference relation, their main common points, differences and weakness. Finally we identify some opens areas of future research, some of them are also addressed in this thesis.

2. **Development of new GDM approaches under highly uncertainty contexts.**

   This objective will be addressed by accomplishing the following main milestones:

   - Reduce the computational complexity when dealing with intuitionistic preference relations.
   - Assessing the experts degree of confidence on the provided solution under highly uncertain situations.
   - Propose a new GDM approach that deals with uncertainty and incomplete information based on the expert's confidence.
   - Propose a new consensus model for GDM that deals with uncertainty using granular information.

3. **Develop an open source software library to carry out GDM processes with incomplete information**

5. Joint discussion of results

In this section, a summary of the proposals included in this Ph.D. dissertation are presented, describing their main contents along with a brief discussion of the obtained results and the associated journal publications.

5.1. Analyzing the main approaches to deal with incomplete information for different types of preference relations.

In these section we analyze the main approaches in the literature to deal with incomplete information for the various types of preference relations previously analyzed. These approaches can be broadly classified into three main groups:

- Methods that directly discard the incomplete information and process only pieces of complete information [Mil97];
- Methods that penalize or rate negatively the experts who provide incomplete preferences [EM00];
- Methods that estimate the missing preference values using the provided ones [HVCFA07, HVACF07].

The first two are based on the assumption that a good solution to a decision making problem cannot be achieved from incomplete information, or that the solution would not be as good as the one that would be obtained using complete information. However, empirical evidence suggests that the incomplete relation derived from the random deletion of as much as 50 % of the elements of a complete pairwise preference relation provides good results without compromising the accuracy [CKZ97]. Therefore, these two groups of methods eliminate or undervalue useful information in the data provided, which could lead to serious biases [KR11]. Indeed, incomplete information is not equivalent to low quality information, and consequently imposing penalties in the decision making processes to experts providing incomplete information could lead to misleading solution, specially when the incomplete information is consistent and the complete information is not. Thus, alternative approaches to manage incomplete information in decision making are desirable. One of these approaches is based on the selection of an appropriate methodology to 'build' the matrix, and/or to assign importance values to experts based not on the amount of information provided but on how consistent the information provided is.

Some of the existing methods that estimate missing preference values in GDM use the information provided by the rest of experts together with aggregation procedures [KCHK98]. The main drawback of this approach is that it requires several experts to estimate the missing values of a particular one, which in conjunction with notable difference between the experts preferences could lead to the estimation of information not naturally compatible with the rest of the expert's information.

An alternative approach here, is to use methods to estimate an expert's missing values using just his/her own assessments and consistency criteria to avoid incompatibility. Indeed this methodology has been extensively applied in decision making contexts under preference relations [HVCFA07, HVACF07, FG07, ACH+08, CHVA01, ACC+09, CHPHV10, GBAX10, LPXY12, Lee12]. An extreme case of incomplete preferences happens when one or more experts in the group do not provide any preference information on at least one of the feasible alternatives. This situations are called in literature *total ignorance* or simply *ignorance* situations, and several approaches to deal with them have been presented in [AHVCH09]. In the following we present an analysis pf the last two methodologies.

5.1.1. Estimation approaches in GDM situations

These approaches mainly use consistency properties to estimate the missing preferences and can be widely classified depending on the mathematical methodology followed to carry out the estimation; Figure 3 depicts a schema of the different approaches existing in the literature to estimate the incomplete information in DM using only the expert information.

1. **Iterative approaches:** This methodologies directly estimate the missing preferences [Xu05a, ACH+08, CHVAH08, BC11, BC12, Lee12]. In general, these approaches use indirect chains of known preference relations to estimate the unknown ones applying consistency properties. The completion in these cases is carried in various stages.

   The most important method in this category was presented by Herrera-Viedma et al. in [HVCFA07]. Given an unknown preference value $p_{ij} (i \neq j)$, this approach uses intermediate alternatives x_k to create an indirect

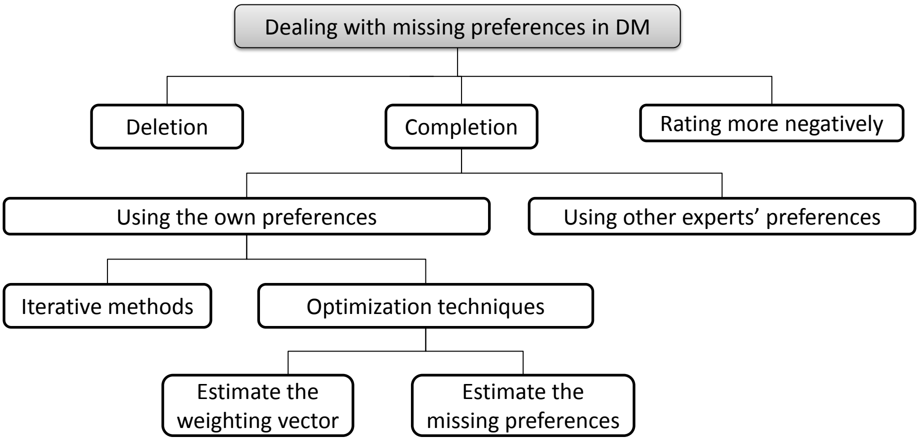


Figure 3: Classification of the different approaches to deal with missing information in DM.

chain of known preference values (p_{ik}, p_{kj}) to derive, using the additive consistency property of an APR, its local consistency based estimated value

$$ep_{ij}^k = p_{ik} + p_{kj} - 0{,}5. \tag{I.11}$$

By averaging all the local estimated values, the overall consistency based estimated value ep_{ij} is obtained:

$$ep_{ij} = \sum_{k=1, k\neq i,j}^{n} \frac{ep_{ij}^k}{n-2} \tag{I.12}$$

In each iteration, the algorithm checks the set of pairs of alternatives for which preference values are unknown and can be estimated using known ones. The algorithm stops when this sets in empty. Note that the cases when an incomplete APR cannot be successfully completed are reduced to those cases when no preference values involving a particular alternative are known, which means that a whole row or column of the APR is completely missing.

In [ACH+08], an extension to deal with MPR, IVPR, and LPR is presented. Since Herrera-Viedma et al.'s approach appeared, many authors have used it to further develop research solutions to different problems of incomplete information. Notable examples can be found in [HVACF07, BC11, BC12, Lee12].

2. **Optimization approaches.** These methodologies use optimization techniques to estimate the missing information or to directly rank the alternatives without previously completing the preference relations. The optimization approaches could also be classified in two subgroups depending on their use:

   - **Methods where missing entries are directly computed:** For example Fedrizzi & Gioves [FG07] propose to use a function that measures the global additive inconsistency of the incomplete APR. To that aim they first define the inconsistency contribution for each triplet of alternatives (x_i, x_j, x_k) as:

     $$L_{ijk} = (p_{ik} + p_{kj} - p_{ij} - 0{,}5)^2 \tag{I.13}$$

     The global inconsistency index of an APR is defined according to the following expression:

     $$\rho = 6 \cdot \sum_{i<k<j} L_{ijk} \tag{I.14}$$

     Missing preference values are the variables in the global inconsistency index. Under this approach, the stationary vector that minimises the global inconsistency function is taken as the estimated values for the unknown preference values. Obviously, these estimated values are the most consistent with respect to the known preference values. Under reciprocity, if a preference value p_{ij} is missing then the value p_{ji} is also missing, and the authors refer to this as the missing comparison $\{x_i, x_j\}$. When a single comparison $\{x_i, x_j\}$ is missing, Fedrizzi and Giove's method produces the following linear equation:

     $$(n-2)p_{ij} - \sum_{k=1,k\neq j}^{n} p_{ik} - \sum_{k=1,k\neq i}^{n} p_{kj} + \frac{n}{2} = 0 \tag{I.15}$$

     with solution

     $$\hat{p}_{ij} = \frac{1}{n-2}\left(\sum_{k=1,k\neq j}^{n} p_{ik} - \sum_{k=1,k\neq i}^{n} p_{kj} - \frac{n}{2}\right) \tag{I.16}$$

A comparison between Fedrizzi and Giove's method and Herrera-Viedma et al.'s in [HVCFA07] is found in [CHVA01]. Both methods are driven by the additive consistency property. Both methods, as originally presented, provide the same set of solutions for independent sets of missing comparisons but not for dependent missing comparisons. This comparative study also shows that a modification of Herrera-Viedma et al.'s coincides with Fedrizzi and Giove's method. However, the main difference between both methods resides in their successful application in reconstructing an incomplete APR. Fedrizzi and Giove's method performs worse than Herrera-Viedma et al.'s method for a large number of alternatives. As mentioned before, Herrera-Viedma et al.'s method fails, as well as Fedrizzi and Giove's method, to complete an incomplete APR only when no preference values are known for at least one of the alternatives. Therefore,. it was concluded that both methods are complementary, rather than antagonistic, in their application, and as such, a new policy for reconstructing incomplete APRs that makes use of both methods was proposed.

- **Methods that estimate the weighting vector**, these methods aim at ranking the alternatives using directly the incomplete APR, and therefore no completion process is needed. They are based on Saaty's assumption for MPR that there is an exact functional relation between the preference values and the priority vector. Two main approaches are used to develop indirect completion models based on the computation of the priority vector: linear based methods [Har87, Xu05b, XC08, Xu10, Xu04b, DRT11], and least square optimisation based methods [Gon08, Xu10, LPXY12, XPW13].

Table I.2 summarizes the different research papers that have been analyzed in this contribution in chronological order.

Analyzing the number of publication/year, it is fair to conclude that the management of incomplete information in DM based on PRs is currently a hot topic that has been disseminated in the most relevant journals including: IEEE Trans. On Systems, Man and Cybernetics, IEEE Trans. on Fuzzy Sets and Systems and Information Sciences.

Reference	Year	Relation	Consistency	Characteristics
[Har87]	1987	APR	Multiplicative	Extension of the Saaty eigenvalue approach to incomplete APR.
[ACHHV04]	2004	APR	Additive	Iterative procedure.
[Xu04b]	2004	MPR	Multiplicative	Goal Programming method to estimate the priority vector.
[Xu05a]	2005	APR	Multiplicative	Interactive approach to reach consensus.
[Xu05b]	2005	MPR	Multiplicative	Eigenvalue approach.
[Xu06]	2006	LPR (Virtual terms set)	Additive	Iterative approach
[HVCFA07]	2007	APR	Additive	Iterative procedure, new consistency measure,new AC-IOWA operator, .
[FG07]	2007	APR	Additive	Minimizes a global additive inconsistency function.
[LS07]	2007	LPR	Additive	Extension of LINMAP to deal with incomplete LPR
[ACH+08]	2008	APR, MPR, IVPR, LPR	Additive	Extension of the iterative procedure in [HVCFA07] to work with more PR.
[XC08]	2008	APR	Additive	Calculates the priority vector solving a linear system of equations.
[Gon08]	2008	APR	Multiplicative	Minimizes the LS dif. between the consensus case and the real case.
[CHVAH08]	2008	APR	Multiplicative	Iterative procedure similar to [HVCFA07].
[CHVA01]	2009	APR	Additive	Comparative study between [FG07] and [HVCFA07].
[YYYW09]	2009	IVPR	—	Dominance based rough set approach.
[ACC+09]	2009	2-tuple LPR	Additive	Iterative procedure similar to [HVCFA07].
[AHVCH09]	2009	APR	Additive	Individual and social strategies for total ignorance situations.
[XDW10]	2010	APR, MPR, LPR	Multiplicative	Minimizes a collective deviation degree to improve the consensus and rank the alternatives.
[Xu10]	2010	APR	Additive	Obtains the weighting vector solving a liner system of equations.
[GBAX10]	2010	IVPR	Multiplicative	Estimates the missing values directly from the known ones.
[CHPHV10]	2010	2-tuple LPR	Additive	Iterative procedure, Proposes consistency measure to set the experts' weight.
[PHV10]	2010	2-tuple LPR	Additive	Estimates preferences in fuzzy linguistic recommender systems using [HVCFA07].
[WC10]	2010	LPR (membership functions)	Additive	Directly estimates the missing preference using the known ones.
[Xu10]	2010	APR, LPR	Additive	Estimates the attribute weights and ranks the options.
[HW11]	2011	LPR (virtual terms set)	Additive	Propose three ways of pairwise comparison
[LPXY12]	2011	APR	Additive	LS minimization method.
[BC11]	2011	APR	Additive	Multicriteria decision framework for sustainable supplier selection.
[DRT11]	2011	APR	Multiplicative	Optimizes a dissimilarity function.
[XCS11]	2011	IPR	Multiplicative	Estimates directly the missing preferences from the known ones
[BC12]	2012	APR	Additive	Aimed to quality function deployment.
[Lee12]	2012	APR	Additive and order	Iterative completion approach.
[XPW13]	2012	MPR	Multiplicative	Logarithmic LS method to rank the options.
[ZDX12]	2012	APR	Additive	Maximizes the consistency level [HVCFA07].
[WC14]	2013	IPR	Multiplicative	Iterative process similar to [HVCFA07].

Table I.2: Summary of the analyzed contributions in chronological order

5.1.2. Approaches dealing with total ignorance situations

The procedures exposed in the previous section cannot be applied successfully when some experts do not provide any information about a particular alternative, which is known as ignorance situations. Alonso et al. [AHVCH09] developed several strategies to deal with ignorance situations in the context of GDM with APRs. These strategies can be broadly classified in two main groups depending on whether the information provided by other experts is used to estimate the missing values, known as *social strategies*, otherwise named *individual strategies*.

- **Individual strategies**
  The proposed ignorance individual strategies (IIS) can be divided in two main steps:

  1. Setting some particular seed values to provide some initial information to the estimation procedure to be able to compute the other missing values. The selection of the seed values can be accomplished using two different methodologies:

     IIS1 Choosing indifference seed values: Let P be an incomplete APR with no preference information on alternative x_i, i.e $p_{ij}^{'}$ and $p_{ji}^{'}$ are unknown for all j. In this strategy, indifference seed values are assumed, i.e. $p_{ij}^{'} = p_{ji}^{'} = 0{,}5 \ \forall j$. This strategy adjusts the estimated preference values to make the APR more consistent with the previously existing information. This approach is particularly useful when there are no external sources of information about the problem and when a high consistency level is required.

     IIS2 Choosing proximity seed values: In this case the seed values are obtained from the preference values given to similar alternatives. This is possible if some extra information or properties about alternatives, which strongly suggest that the ignored alternative is similar to another one, are known. This strategy could be useful in some decision making problems where the alternatives to be evaluated are goods with similar characteristics (similar models).

  2. Estimating the rest of the missing values using the consistency based procedure proposed in [HVCFA07].

- **Social strategies**
  Ignorance social strategies (ISS) are based on the use of the information provided by the set of experts. The authors present three main approaches in this case:

  ISS1 The first social strategy uses consensus preference values of the collective PR, computed by aggregating all the experts' individual PRs. The main advantage of this approach is that it improves the consensus of the set of experts making their opinions close to each other.

  ISS2 The second strategy uses only the consensus preference values provided by those experts nearest to the expert whose PR is incomplete. This strategy is aimed to narrow the differences between the expert with an ignored alternative and those who have a similar opinion about the rest of alternatives.

  ISS3 The third approach integrates the previous two by taking into account both information from the collective preference relation and from the nearest experts. This strategy encompasses the advantages of the previous two social strategies since the estimated information not only helps in the consensus process but also tries to keep a high consistency level in the individual experts' PR. Therefore it is considered by the authors of the proposal as the best strategy to deal with ignorance situations in GDM.

The journal article associated to this part is:

- M.R. Ureña, F. Chiclana, J.A. Morente-Molinera, E. Herrera-Viedma. Managing Incomplete Preference Relations in Decision Making: A Review and Future Trends. Information Sciences 302:1 (2015) 14-32. doi: 10.1016/j.ins.2014.12.061 .

5.2. Developing new group decision making approaches under highly uncertain contexts

5.2.1. Confidence-consistency decision making approaches in highly uncertain contexts

Due to its flexibility in handling vagueness/uncertainty, intuitionistic fuzzy set theory [Ata86] constitute an interesting preference representation format in the field of GDM. However, in this field, much research has been carried out in preferences modeled using fuzzy relations in comparison to the case of using intuitionistic fuzzy relations. This is mainly due to the computational complexity associated to using membership degree, non-membership degree and hesitation degree to model experts' subjective preferences, which duplicate the complexity comparing to the case of fuzzy preference relations where the preferences are modeled by means of only one membership function. Therefore it is desirable to create a methodology that not only simplified the computation with the intuitionistic preference relations but also allows to simply apply the well known GDM approaches to the case of intuitionitic preference relations.

Intuitionistic fuzzy preference relation and asymmetryc fuzzy preference relations

To that aim in this contribution the set of reciprocal intuitionistic fuzzy preference relations and the set of asymmetric fuzzy preference ones are proved to be mathematically isomorphic. Let us define the following mapping between the set set of reciprocal intuitionistic fuzzy preference relations, $\mathcal{B}$, and the set of fuzzy preference relations, $\mathcal{R}$, $f\colon \mathcal{B} \longrightarrow \mathcal{R}$ that associates to a reciprocal intuitionistic fuzzy preference relation $B = (b_{ij}) = (< \mu_{ij}, \nu_{ij} >)$ the following fuzzy preference relation $f(B) = R$:

$$f(B) = (f(b_{ij})) = (\mu_{ij}) = (r_{ij}) = R. \tag{I.17}$$

This result can be exploited to extend the methodologies developed for fuzzy preference relations to the case of intuitionistic fuzzy preference relations and, ultimately, to overcome the computational complexity mentioned above and to extend the use of reciprocal intuitionistic fuzzy preference relations in decision making. In particular, in this contribution, this isomorphic equivalence is used to address the presence of incomplete reciprocal intuitionistic fuzzy preference relations in decision making by developing a multiplicative consistency driven estimation procedure via the corresponding equivalent incomplete asymmetric fuzzy preference relation procedure proposed in [HVCFA07].

Assesing experts confindence in GDM situation with intuitionistic preference relations

On the other hand, expert's confidence in the provided opinion is another relevant issue to bear in mind in the decision making processes. Indeed, some researches have reported that freely interacting groups choose the positions of their most confident members as their group decisions. This phenomenon has been witnessed with groups discussing a mathematical puzzle [JT67], a recall task [SAWW86] and a recognition task [Hin90], concluding that confidence was a significant predictor of influence. Furthermore Guha et al. state in [GC10] that in any real field decision making situation when experts give their responses to a particular alternative, their confidence level regarding the opinions are very much important.

Therefore the hesitancy degree of the reciprocal intuitionistic fuzzy preference relation is used to introduce the concept of expert's confidence from which a GDM procedure, based on a new aggregation operator that takes into account not only the experts' consistency but also their confidence degree towards the opinion provided, is proposed.

Definition 20 *Given a reciprocal intuitionistic fuzzy preference relation* $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$, *the* confidence level associated to the intuitionistic preference value b_{ij} *is measured as*

$$CFL_{ij} = 1 - \tau_{ij},$$

with τ_{ij} *being the hesitancy degree associated to* b_{ij}.

As noted before, $\tau_{ij} = 1 - \mu_{ij} - \nu_{ij}$ and therefore we have that $CFL_{ij} = \mu_{ij} + \nu_{ij}$. In other words, when $CFL_{ij} = 1$ $(\mu_{ij} + \nu_{ij} = 1)$ then $\tau_{ij} = 0$ and there is no hesitation at all. The lower the value of CFL_{ij}, the higher the value of τ_{ij} and the more hesitation is present in the intuitionistic value b_{ij}.

Definition 21 *Given a reciprocal intuitionistic fuzzy preference relation* $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$, *the* confidence level associated to the alternative x_i *is defined as*

$$CFL_i = \frac{\sum\limits_{\substack{j=1 \\ i \neq j}}^{n} \left(CFL_{ij} + CFL_{ji} \right)}{2(n-1)}.$$

Because B is reciprocal, we have that $CFL_{ij} = CFL_{ji}$ $(\forall i, j)$ and therefore it is

$$CFL_i = \frac{\sum\limits_{\substack{j=1 \\ i \neq j}}^{n} CFL_{ij}}{n-1}.$$

A similar interpretation of CFL_i with respect to the confidence on the preference values on the alternative x_i can be done as it was done above with CFL_{ij}.

Definition 22 *The* confidence level associated to a reciprocal intuitionistic fuzzy preference relation $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$ *is measured as*

$$CFL_B = \frac{\sum\limits_{i=1}^{n} CFL_i}{n}.$$

Notice that when $CFL_B = 1$, then the reciprocal intuitionistic fuzzy preference relation B is a reciprocal fuzzy preference relation.

Confidence-consistency guided aggregation

As previously explained Herrera-Viedma's IOWA aggregation operator [HVCFA07] associate a higher degree of importance to the experts who present a better consistency degree. However, there is no operator that takes into consideration the expert's confidence associated to preferences to aggregate. Hence, we proposed a new consistency and confidence IOWA (CC-IOWA) operator, i.e. an IOWA operator that trades off consistency and confidence criteria in both re-ordering the preferences to aggregate and deriving the aggregation weights to use in their fusing to derive the collective preference.

Definition 23 (CC-IOWA operator) *Let a set of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, using the reciprocal intuitionistic fuzzy preference relations, $\{B^1, \ldots, B^m\}$. A consistency and confidence IOWA (CC-IOWA) operator of dimension m, Φ_W^{CC}, is an IOWA operator whose set of order inducing values is the set of consistency/confidence index values, $\{CCI^1, \ldots, CCI^m\}$, associated with the set of experts.*

Therefore, the collective reciprocal intuitionistic fuzzy preference relation $B^{cc} = (b_{ij}^{cc}) = (\langle \mu_{ij}^{cc}, \nu_{ij}^{cc} \rangle)$ is computed as follows:

$$\mu_{ij}^{cc} = \Phi_W^{CC}\left(\langle CCI^1, \mu_{ij}^1 \rangle, \cdots, \langle CCI^m, \mu_{ij}^m \rangle\right) = \sum_{h=1}^{m} w_h \cdot \mu_{ij}^{\sigma(h)} \tag{I.18}$$

$$\nu_{ij}^{cc} = \Phi_W^{CC}\left(\langle CCI^1, \nu_{ij}^1 \rangle, \cdots, \langle CCI^m, \nu_{ij}^m \rangle\right) = \sum_{h=1}^{m} w_h \cdot \nu_{ij}^{\sigma(h)} \tag{I.19}$$

$$CCI^h = (1 - \delta) \cdot CL^h + \delta \cdot CFL^h \tag{I.20}$$

such that $CCI^{\sigma(h-1)} \geq CCI^{\sigma(h)}$, $w_{\sigma(h-1)} \geq w_{\sigma(h)} \geq 0$ ($\forall h \in \{2, \cdots, m\}$) with $\sum_{h=1}^{m} w_h = 1$, CL_{ij}^h the consistency level associated to $R^h = f(B^h)$, CFL^h the confidence level associated to B^h, and $\delta \in [0,1]$ a parameter to control the weight of both consistency and confidence criteria in the inducing variable.

In our case, we propose to use the consistency/confidence values associated with each expert both as an importance weight and as the order inducing values. Thus, the ordering of the preference values is first induced by the ordering of the experts from the most to the least consistent/confident, and the weights of the CC-IOWA operator is obtained as follows:

$$w_h = Q\left(\frac{\sum_{k=1}^{h} CCI^{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=1}^{h-1} CCI^{\sigma(k)}}{T}\right)$$

with $T = \sum_{k=1}^{m} CCI^k$.

The Journal article associated to this part is:

- R. Ureña, F. Chiclana, H. Fujita, E. Herrera-Viedma. Confidence-consistency driven group decision making approach with incomplete reciprocal intuitionistic preference relations. Journal: Knowledge-Based Systems, in press. doi: 10.1016/j.knosys.2015.06.020

5.2.2. Consensus processes where uncertainty is modeled via information granularity

As already explained, in a GDM situation, a consensus process is usually defined as an iteratively negotiation whose member are willing. up to certain point, to modify their opinions following some advice or feedback [BR06, HVCKP14a, SL94] so as to get a solution accepted by the whole group.

In the first consensus approaches in the literature [HHVV96b, HHVV97, KF86, KFN92], this advice was provided by a moderator, who knows the agreement degree at each round by computing some consensus measures. However, as the moderator can introduce some subjectivity in the process, new consensus approaches have been proposed in order to make more effective and efficient the decision making process by replacing the moderator or providing him/her with better analytical tools [CPHV10, HVACF07, HVHC02, KZR10, PCAHV14]. Either way, several consensus rounds are usually required in order to achieve a sufficient agreement. As a result, this process can take a considerable amount of time.

Independently from the source of advice, reaching a desired level of agreement requires certain degree of flexibility in the experts to change their main with the common goal of achieving a solution accepted by the whole group. In this situation is where information granularity [Ped11, Ped13a, Ped13b] comes into play.

In this contribution we present a consensus approach in which each decision maker expresses his/her preferences using a fuzzy preference relation, but the necessary degree of flexibility is included by allowing them to be granular rather than crisp. Thus, if the pairwise comparisons of the fuzzy preference relations are not treated as single numeric values, which are completely defined, but as information granules, this will bring the essential factor of flexibility. It means that the fuzzy preference relation is abstracted to its granular format. The notation $\boldsymbol{G}(PR)$ is used to emphasize the fact that we are interested in granular fuzzy preference relations, where $\boldsymbol{G}(.)$ represents a specific granular formalism being used here (for instance, intervals, fuzzy sets, rough sets, probability density functions, and alike).

In this contribution we introduce the concept of granular fuzzy preference relation being exploited as a way to increase the level of consensus achieved among the decision makers. In summary, the level of granularity is treated as synonymous of the level of flexibility injected into the modeling environment, which makes easy the collaboration. Obviously, the higher the level of granularity offered to the decision maker, the higher the feasibility of arriving at decisions accepted by all members of the group.

This idea can be formalized depending on the form of the information granules being the entries of the fuzzy preference relations. In particular, in this contribution, the granularity of the information is modeled via intervals and, therefore, the length of such intervals determines the level of granularity α.

In the granular model of fuzzy preference relations, it is supposed that each decision maker feels equally comfortable when selecting any fuzzy preference relation whose values are placed within the bounds established by the fixed level of granularity α, which is used to increase the level of consensus within the group. However, we have to take into account that when the entries of the fuzzy preference relations are modified in order to increase the level of agreement, it might produce some inconsistencies in the fuzzy preference relations. In particular, the higher the values of α, the higher the potential of reaching a significant level of consensus in spite of producing inconsistent fuzzy preference relations.

Thus the proposed methodology is based on an optimization approach where the open parameter is the level of granularity α and presents two main goals:

- On the one hand it is used to increase the consensus within the group members by bringing all their preference closer. To that aim the global consensus degree among all the decision makers' opinions is maximized.

$$Q_1 = cr \tag{I.21}$$

- On the other hand it increases the consistency degree of the fuzzy preference relations at the decision maker level.

$$Q_2 = \frac{1}{m} \sum_{l=1}^{m} cd_l \tag{I.22}$$

Considering the scalar version of the optimization problem, it arises as follows:

$$Q = \delta \cdot Q_1 + (1 - \delta) \cdot Q_2 \tag{I.23}$$

being $\delta \in [0, 1]$ a parameter to set up a trade-off between the consensus obtained within the group and the consistency level achieved by each decision maker. The higher the value of δ, the more attention is being paid to the consensus at the group level. Usually, $\delta > 0{,}5$ will be used to give more importance to the consensus criterion.

Therefore the overall optimization problem is formalized as follows:

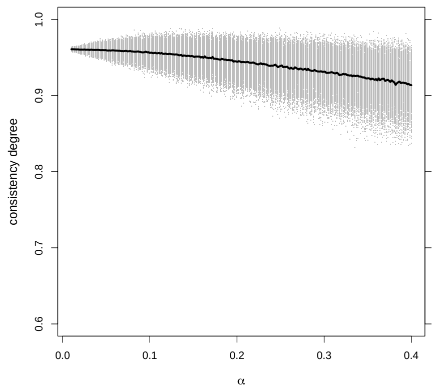
$$\text{Max}_{PR^1, PR^2, \ldots, PR^m \in \boldsymbol{P}(PR)} Q \tag{I.24}$$

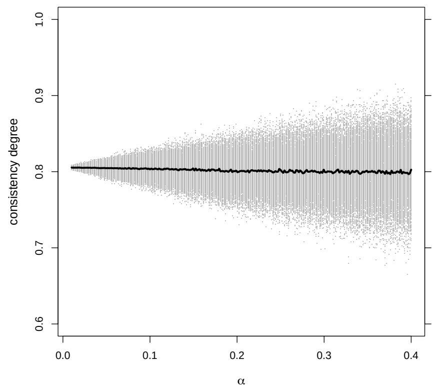
Due to the nature of the indirect relationship between the optimized fuzzy preference relations, selected from a large search space formed by $\boldsymbol{P}(PR)$, the use of advanced global optimization techniques where required. In this contribution we take advantage of the Particle swarm optimization,PSO, [KE95] to carry out this optimization. The PSO is a viable optimization technique for this problem since it offers a substantial level of optimization flexibility without high computational requirements as could be the case of other global optimization approaches such as the genetic algorithms. In the associathe jorunal paper an experimental study has been carried out concluding that bot,h the level of consensus within the group of decision makers and the level of consistency achieved by the individual decision makers, have been significantly increased with the use of the proposed approach.

In the following we depict an and analyze the most relevant results

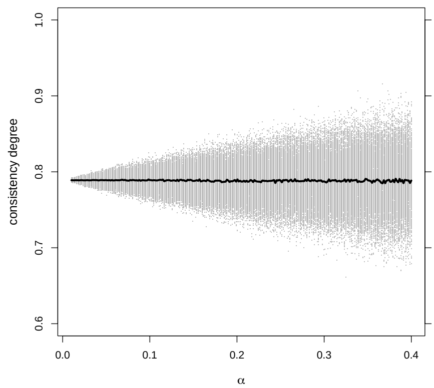
In figure 4 we illustrate how the average consistency evolves with respect to the granularity level α in a consensus process following the proposed approach. In this representation we can clearly observe how the likelihood of arriving at more consistent fuzzy preference relations increases when increasing the values α. This It is not surprising since we have inserted some level of flexibility that we intend to take advantage of. On the other hand, the possibility of generating a very inconsistent fuzzy preference relation increases as well. Despite that, the average value of consistency remains pretty steady with respect to increasing values of the granularity level α, as reported for the fuzzy preference relations. However, there is some slight downward trend for higher values of α. In particular, when the consistency degree of the initial fuzzy preference relation provided by the decision maker is very high, it is very common that its average consistency degree decreases for higher values of the granularity level α

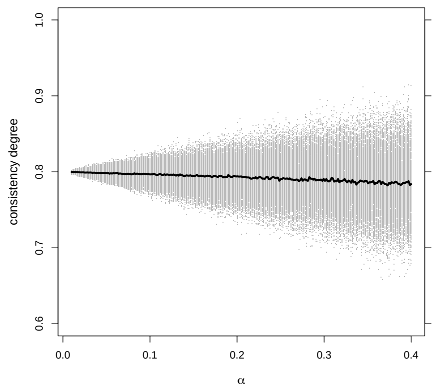
Considering a given level of granularity, Figure 5 depicts the performance of the PSO quantified in terms of the fitness function obtained in successive generations. The most remarkable improvement is appreciated at the very beginning of the optimization, and afterwards, there is a clearly visible stabi-





(a) Consistency degree versus α for PR^1 (b) Consistency degree versus α for PR^2





(c) Consistency degree versus α for PR^3 (d) Consistency degree versus α for PR^4

Figure 4: Consistency degrees versus α for the fuzzy preference relations PR^1–PR^4.

lization, where the values of the fitness function remain constant.

Let us examine an impact of the granularity level α and the parameter δ in the composite fitness function on the performance of the method and the form of the obtained results. For $\delta = 0$, the optimization concerns each of the fuzzy preference relations individually. Here, the increment in the values of α offers

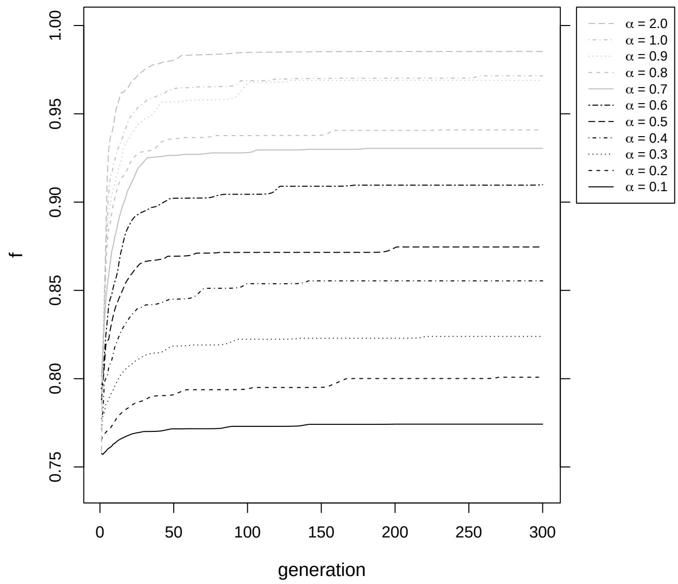
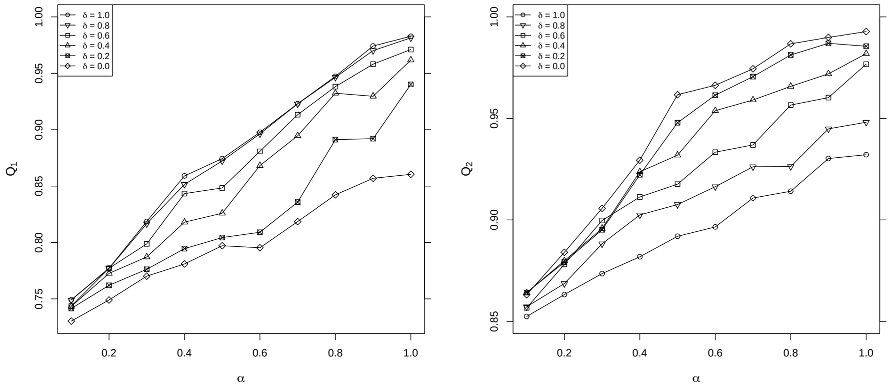


Figure 5: Fitness function f in successive PSO generations for selected values of α (here $\delta = 0{,}75$).

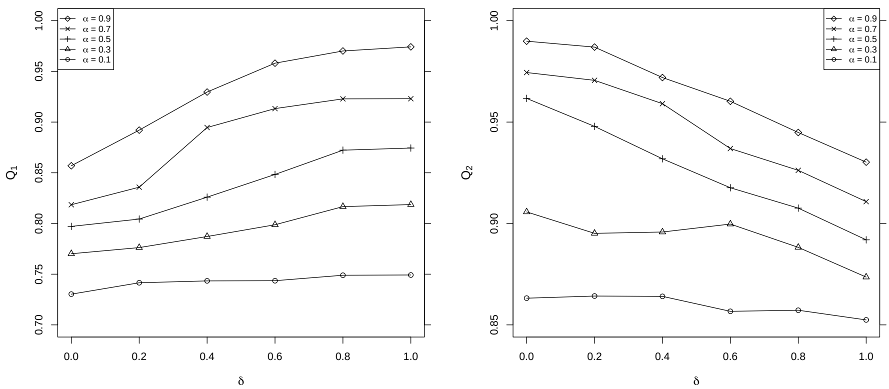
more flexibility, which, if wisely used (optimized by the PSO), produces the fuzzy preference relations of higher consistency. This effect is clearly observable in Fig. 6b (the curve for $\delta = 0$). The beneficial effect of granularity is evident: with the increasing values of α, the fuzzy preference relations become more flexible, which results in higher levels of consistency reached by the decision makers. A similar effect is visible when δ takes nonzero values: if there is some interaction, the impact of introduced granularity is positive (the overall level of consistency quantified by Q_2 is an increasing function of α). The strictly monotonic character of this relationship is not maintained for higher values of δ, as it is again shown in Fig. 6b. However, it is not surprising as the performance criterion optimized by PSO is not Q_2 itself but Q, which incorporates also the effect of the level of consensus achieved within the group of decision makers. On the other hand, Fig. 6a includes the progression of the values of Q_1, which shows the consensus within the group. Again, the advantageous effect of granularity is visible, as higher values of α translate into higher values of Q_1. However, now, higher values of δ produce higher values of Q_1 as more important is assigned



(a) Q_1 versus α for selected values of δ (b) Q_2 versus α for selected values of δ

Figure 6: Plots of Q_1 and Q_2 versus α for selected values of δ.

to Q_1 in the composite criterion Q.



(a) Q_1 versus δ for selected values of α (b) Q_2 versus δ for selected values of α

Figure 7: Plots of Q_1 and Q_2 versus δ for selected values of α.

Fig. 7a includes a number of plots of Q_1 regarded as functions of δ for selected levels of granularity α. Once more, the impact of the granularity level is obvious. However, here, for the fixed value of α, there is a visible saturation effect for higher values of δ: when moving beyond a certain point, the values of Q_1 does not increase. On the other hand, the cumulative level of consistency Q_2 drops quickly with the increasing values of δ, as illustrated in Fig. 7b, and this effect is noticiable for different values of α. However, higher values of the granularity level also result in higher consistency levels in this case.

The journal article associated to this part is:

- F.J. Cabrerizo, R. Ureña, W. Pedrycz, E. Herrera-Viedma. Building consensus in group decision making with an allocation of information granularity. Fuzzy Sets and Systems 255 (2014) 115-127. doi: 10.1016/j.fss.2014.03.016.

5.3. Developing new software tools to deal with Group decision making processes with incomplete information

Nowadays due to the evolution of web 2.0 technologies new paradigms and ways of making decisions, such as web 2.0 frameworks, social networks and e-democracy, have made the complexity of decision making processes to increase involving in many cases a huge number of experts with intermittent and low participation rates [APCHV13]. These new scenarios require automatic tools not only to combine the information in the best possible way but also to better analyze the whole context, providing a rapid and complete insight about the current state of the process.

In this direction some tools have already been presented [AHVCH10, PM14, PMH14, PCHV10, PCAHV14]. However, the majority of them are developed as closed systems and therefore they are not aimed to be upgraded or extended by other researchers, since in most of the cases they do not provide the source code or they are based in proprietary software. Moreover, this tools are extremely dependent of the user interface and so they cannot be adapted to work in other environments such as smart phones. And, most importantly, they do not provide any type of graphical visualizations or output measures illustrating the evolution of the consensus process.

In this contribution we present the **GDM-R framework**. This is a software tool that automatically carries out GDM processes with consensus and dealing with missing information. The proposed system is designed to be useful in both classical DM scenarios and more demanding ones. Indeed the system is completely developed as an open source tool in R and presents the following main features:

- It provides support for both real GDM situations and simulation environments. Being useful not only to assist decision making processes, but also to compare and validate already existing approaches and to develop new ones.
- It carries out a number of consensus iteration to obtain a solution accepted by all the decision makers and provides the best alternative using well known decision making algorithms [HVCFA07, HVACF07].
- At every iteration, it calculates in real time the consensus and the consistency levels for each particular expert and the global measures. Based on this information it provides the experts with recommendations about

how to change their opinions in other to increase both consensus and consistency in the process.

- It deals with the missing information in the decision making process. Estimating, if possible the unknown preferences from the known one as well as increasing the consistency in the experts' preferences.
- It computes and depicts powerful visual analytics to quickly verify the state of the decision process. Among its various representations it depicts experts' preferences 3D maps to quickly detect those experts who are far from the consensus solution and are more reluctant to change their mind. It also allows to identify those ones who provide more contradictory or inconsistent opinions. Besides the system gives the user the facility to visually check the evolution of the global consensus and consistency among the various round of consensus.
- It is an open source framework implemented in R [R15], following a modular architecture which easily enables the extension of the tool to be able to work with other types of preference relations or to be used by other researchers.

5.3.1. System architecture

The proposed framework has been designed following a Model-View-Controller architectural pattern [GHJV95]. Therefore, the logic is completely separated from the data storage requirements and from the user interface. This design eases its adaptation to different interfaces, such us web or mobile environments, since it works totally independently from the user interface. The framework is built from various processing independent modules so it can be easily upgraded and extended just by making changes in a particular module or adding new ones.

The developed framework tries to fill the gap that the other systems leave. To that aim, it includes powerful visualization tools, and enables various working modes. To do so, the system is composed of the following modules: (i) control module, (ii) preference module, (iii) estimation module, (iv) consistency module, (v) aggregation module, (vi) consensus module, (vii) feedback module, (viii) exploitation module, (ix) storage module, and (x) graphical representation module. The framework's architecture is depicted in Fig. 8 which shows the interaction among all the modules.

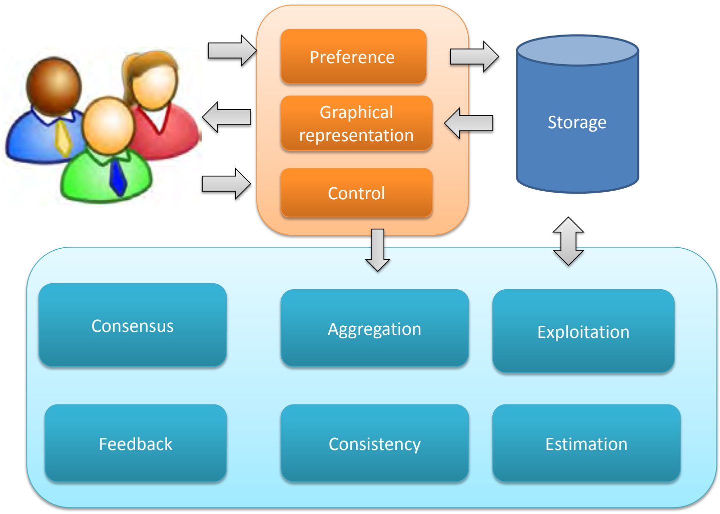


Figure 8: Architecture of the developed framework and the main interactions between modules.

5.3.2. Graphical evolution of the decision making process

One of the main novelties that the developed system presents with respect to the existing ones is the possibility of getting a quick insight in the GDM process by means of diverse graphical representations. All these representations make the system a graphical monitoring tool to support decision makers by providing them with easily understandable visual information about the current status and the evolution of the decision process. This tool eases the analysis of diverse crucial aspects that are common in these problems, among them, we can highlight:

- Monitoring the evolution of the global consensus across the whole GDM process.
- Monitoring the decision makers' consistency along the whole GDM process. This is especially important to make sure that they are keeping an acceptable consistency level in their preferences after the recommendation rounds.

- Detection of the alternatives that are posing more controversy in the GDM process.
- Detection of those decision makers or group of them, whose preferences are further from the consensus solution, or those that are more reluctant to change their point of view.
- Detection of those decision makers that are being influenced or manipulated to provide preferences far from the consensus solution.
- Providing information to the decision makers about the GDM process, and showing them how their preferences are located with respect to the consensus one.

The graphical representations that our system includes can be divided in two wide groups, depending on whether they show the evolution among the various consensus rounds, or they display information related to a single round:

- Representation of the evolution across the consensus rounds:
  - **Consistency vs consensus evolution in the GDM process**. This representation shows the evolution of both global consistency and global consensus in each consensus round. The desirable situation is that most of the point or at least the final ones lie over the diagonal line and the points present a positive tendency. It would mean that the final solution has reached a high level of agreement and that it is consistent. This representation also enables to detect whether the consensus process is not only helping to bring the decision makers' opinions closer but also to keep or increase their consistency.
  - **Decision maker's consistency vs decision maker's consensus in the GDM process**. This representation allows to check how decision makers' consensus and consistency evolves during the GDM process. It also enables to visually check the different decision makers profiles depending on the shape of the curve for each decision maker. Curves with a positive tendency and located over the diagonal represent the desired situation of those decision makers that are more willing to change their opinions in the interest of increasing the global consensus while keeping a highly consistency level. Curves parallel to the y-axis represents those decision makers

which are reluctant to change their mind during the process, and therefore they may require special attention.

- Representation of the consensus state in a single round:

  - **Barplot of each decision maker's proximity to the aggregated solution**. This representation enables to check who are the decision makers whose opinions are closer to achieve a high degree of consensus, and who are those with highly disagree with the proposed solution.
  - **Barplot of the average consensus achieved for each alternative**. This representation allows to quicky identify which alternatives are posing more controversy in the decision process.
  - **Barplot of the average consistency achieved for each decision maker**. This representation provides a quick insight on those decision makers providing more consistent fuzzy preference relations in the decision making process.
  - **2D representation map of the decision makers' fuzzy preference relations and the consensus solution**. This representation provides a quick insight of the current state of the decision process and enables the rapid identification of sub groups of decision makers who share similar opinions. It also eases the detection of conflicts among decision makers. Moreover, it provides the decision makers with a good idea about the status of the consensus process and how far their opinions are from the consensus solution. This 2D representation is obtained after carrying out a classical 2D multidimensional scaling reduction of the decision makers' fuzzy preference relation matrix [Gow66]. In addition, R also offers the possibility of non metric multidimensional scaling.
  - **3D representation of the position of each decision maker with respect to the consensus solution among with their consistency**. This plot easily allows to identify those groups of decision makers that are far from the consensus solution but keep a high degree of consistency, and, therefore, need special attention. To easily visualize this plot, we have also included a interactive representation.

The journal article associated to this part is:

- R. Ureña, F.J. Cabrerizo, J.A. Morente-Molinera, E. Herrera-Viedma, GDM-R: A new framework in R to support fuzzy group decision making processes **Submitted to Information Sciences**.

6. Conclusiones

En esta sección se presentan las conclusiones del trabajo realizado a lo largo de esta tesis doctoral. Estos resultados persiguen el objetivo común de mejorar el Estado del Arte de las investigaciones en metodologías de toma de decisión que trabajan con información incompleta en entornos de incertidumbre. El trabajo presentado auna tanto el desarrollo teórico de nuevas metodologías de toma de decisión en grupo en entornos de incertidumbre, cómo el desarrollo de nuevas herramientas software para la aplicación de las metodologías propuestas en situaciones reales. Las principales contribuciones de esta tesis se presentan a continuación.

6.1. Análisis de las principales metodologías de toma de decisión en grupo que trabajan con información incompleta para distitos tipos de relaciones de preferencia

Hemos revisado y analizado el Estado del arte de las investigaciones llevadas a cabo en toma de decisión en grupo desde la perspectiva de la estimación de la información incompleta para los principales tipos de relaciones de preferencia. Asimismo se han presentado los fundamentos y desarrollos en esta área así como los modelos computacionales mas relevantes que han sido aplicados al contexto de toma de decisión: Relaciones de preferencia Additivas, Relaciones de Preferencia multiplicativas, Relaciones de preferencia Intuicionistas, relaciones de preferencia intervalares y relaciones de preferencia Linguísticas. Tras el estudio realizado concluimos que las técnicas de estimación existentes usan, en su mayoría, las propiedades de consistencia aditiva o multiplicativa para estimar la información incompleta a partir de las relaciones de preferencia proporcionadas, además de incrementar la consistencia global y en muchos casos el consenso. Las metodologías estudiadas pueden dividirse en dos grupos principales: (i) procedimientos iterativos, (ii) procedimientos de optimización. Además se ha proporcionado una amplia lista de las aplicaciones que se han desarrollado más recientemente en éste ámbito.

6.2. Toma de decisión en grupo en contextos de alta incertidumbre

6.2.1. Trabajando con relaciones de preferencia intuicionistas incompletas

Los principales hallazgos y propuestas hechos en este apartado se enumeran a continuación:

- Primeramente hemos probado la equivalencia matemática entre el set de las relaciones de preferencia asímétricas difusas y el set de relaciones de preferencia intuicionistas. Este resultado es de gran importancia ya que permite transponer los conceptos ampliamente definidos para el primer tipo de relaciones de preferencia al otro. Así pues las propuestas hechas para relaciones de preferencia difusas asimétricas pueden aplicarse al caso de relaciones de preferencia intuicionistas, beneficiándose, en este caso, de la flexibilidad que este último tipo de relaciones preferencia proporciona, pero sin el inconveniente asociado de la mayor complejidad computacional.

- De hecho, las relaciones de preferencia intuicionistas incompletas han sido abordadas completando la relación de preferencia difusa equivalente empleando un algoritmo de estimación conocido para este tipo de relaciones preferencia.

- El concepto de nivel de confianza asociado a una relación de preferencia recíproca intuicionista ha sido definido con objeto de atribuir distintos grados de importancia a los expertos en la agregación de las relaciones de preferencias individuales para obtener una relación de preferencia global. Este concepto, junto con el nivel de consistencia ha sido empleado para proponer un nuevo operador de agregación inducido por los grados de confianza-consistencia (CC-IOWA), con objeto de implementar ambas en el procesos de selección en un problema de toma de decisión en grupo/ multicriteria.

6.2.2. Empleando información granular en procesos de consenso bajo incertidumbre

Hemos propuesto una nueva metodología de consenso que utiliza la información granular como un medio indispensable para incrementar el consenso alcanzado por un grupo de expertos en procesos de toma de decisión en grupo. Para ello, la flexibilidad en las opiniones de los expertos, necesaria para alcanzar consenso ha motivado la introducción de las relaciones de preferencia difusas granulares. Sin lugar a dudas este nuevo tipo de relaciones de preferencia constituye una representación mucho más rica, con el potencia de proporcional realizaciones numéricas de acuerdo con las cuales ambos, el nivel de consistencia y el nivel de consenso se vean mejorados en un proceso de toma de decisión en grupo. Para ello se ha demostrado que el algorítmico de optimización por enjambre de partículas, PSO, constituye una adecuada metodología de optimización para tratar con este tipo de relaciones de preferencia. Las ventajas del nuevo proceso de consenso propuesto pueden resumirse en dos puntos fundamentales: (i) el consenso se alcanza en una única iteración en lugar de llevar a cabo varias rondas de consenso, y por tanto el tiempo necesario para obtener el nivel de consenso deseado se reduce. (ii) Al no ser necesario llevar a cabo una negociación entre los expertos existen menos posibilidades de que sean manipulados para cambiar su punto de vista.

6.3. Herramientas software para trabajar con procesos de toma de decisión en grupo con información incompleta

En este punto se ha llevado a cabo una revisión crítica de las herramientas software existentes en la literatura para llevar a cabo procedimientos de toma de decisión en grupo de forma automática, concluyendo que existen pocas herramientas disponibles, y las que existen no son software libre y no permiten llevar a cabo procesos de toma de decisión en grupo que trabajen con información incompleta. Asimismo dichas herramientas presentan arquitecturas cerradas y no modulares que dificultan la ampliación y adaptación de las mismas por parte de otros grupos de investigación.

A lo largo de esta tesis doctoral se ha desarrollado GDM-R una nueva librería open-source completamente implementada en R para llevar a cabo toma de decisión en grupo de forma automática, diseñada con objeto de superar todas las deficiencias que el resto de herramientas ya existentes presentan.

Las principales características de la herramienta desarrollada se enumeran a continuación:

- La herramienta desarrollada muestra representaciones gráficas que proporcionan en un rápido vistazo una visión precisa del estado en el que se encuentra el proceso de toma de decisión y permiten identificar a aquellos expertos cuya opinión discrepa en mayor medida con la opinión del grupo. Asimismo permite detectar de forma visual a aquellos expertos que presentan un comportamiento no cooperativo, así como sub-comunidades de expertos con puntos de vista similares y los expertos que ejercen mayor influencia.

- Proporciona un modo test que permite la creación de un escenario para realizar pruebas de las distintas metodologías propuestas y comparar sus resultados. Resulta de utilidad para validar y comparar de forma objetiva las nuevas metodologías con las ya existentes.

- La herramienta software propuesta puede extenderse de forma fácil para trabajar con otro tipo de relaciones de preferencia o para incluir nuevas metodologías de toma de decisión en grupo. Así pues otros investigadores pueden extenderla y adaptarla a sus propias necesidades.

- La herramienta desarrollada puede adaptarse fácilmente a trabajar con otro tipo de plataformas tales cómo los teléfonos inteligentes, tablets o la web ya que la lógica de la aplicación está completamente separada de la interfaz de usuario.

6. Concluding remarks

In this section we present the results obtained from the research carried out during this PhD dissertation. These results follow the common goal of improving the state of the art in decision making approaches that deals with incomplete information in environments of uncertainty. This study encompass both, the proposal of new theoretical Group decision making methodologies under uncertainty, and the development of a new software framework to ease the application and dissemination of the proposed approaches in real world scenarios. Thus the main contributions are summarized as follows:

6.1. Analyzing the main approaches to deal with incomplete information for different types of preference relations

We have reviewed and analyzed the state-of-the-art research efforts on group decision making from the perspective of the estimation of missing preferences using different types of preference relations. We have presented the foundations and developments in that field along with the most relevant computational models that have been applied to the decision making context: Additive preference relations, Multiplicative preference relations, Intuitionist fuzzy preference relations, Interval-valued preference relations and Linguistic preference relations. These estimation techniques mainly use the additive or the multiplicative consistency properties to calculate the missing preferences from the known ones, as well as increasing the global consistency level and in many cases the consensus. The studied methodologies can be widely classified in two main groups: (i) iterative procedures and (ii) optimization procedures. Furthermore a comprehensive list of the most recent developed applications in the specialized literature has been presented.

6.2. Group decision making under highly uncertain contexts

6.2.1. Dealing with incomplete intuitionistic fuzzy preference relations

The most significant findings and advantages with this regard are listed below:

- Firstly, we have proved the mathematical equivalence between the set of asymmetric fuzzy preference relations and the set of reciprocal intuitionistic fuzzy preference relations. This result is of great importance since it enables to transpose concepts defined for one preference structure to the other one. Therefore very well known approaches developed for the case of Fuzzy prefernce relations, can be applied to the case of intuitionist preference relations, taking advantage of the flexibility that this last type of preference relations provide but without the associated drawback of the complexity.

- Indeed, incomplete reciprocal intuitionistic fuzzy preference relations has been addressed by completing the equivalent incomplete asymmetric fuzzy preference relations using a well known estimation process developed for fuzzy preference relations.

- The concept of confidence level associated to a reciprocal intuitionistic fuzzy preference relation has been defined to associate different importance degrees to experts in the aggregation of individual reciprocal intuitionistic fuzzy preference relations in decision making to derive the collective reciprocal intuitionistic fuzzy preference relation. This concept has been used in conjunction with the consistency level to propose a new consistency and confidence induced ordered weighted averaging (CC-IOWA) operator, in order to implement both consistency and confidence in the resolution process of a group/multicriteria decision making problem.

6.2.2. Using multigranular information in consensus process under uncertainty

We have proposed an approach based on an allocation of information granularity as an indispensable asset to increase the consensus achieved within the group of decision makers. The required flexibility in the opinions provided by the decision makers, necessary to increase the level of consensus, was a motivating factor behind the introduction of the concept of granular fuzzy preference relations. Undoubtedly, the granular fuzzy preference relation conveys a far richer representation which can produce numeric realizations so that both the level of consensus and the level of consistency are improved. To do so, the PSO environment has been proved as a suitable optimization framework. The advantages of the proposed approach with respect to the existing ones are twofold: i) the consensus is achieved in a single step rather than running several consensus rounds, thus, the required time to reach consensus is reduced. ii)Since the experts do not need to negotiate their opinions there are no chances for the experts to be manipulated.

6.3. Software tools to deal with Group decision making processes with incomplete information

In this dissertation we have presented a critical review of the available software frameworks for computer assisted GDM, concluding that there are few available tools an the ones that have already been developed are not open source and are not able to carry out GDM processes including multiple types of preference elicitations and ways of dealing with unknown information. Moreover the majority of these tools present a non modular architecture which makes very complex for other researcher to extend or adapt to their own necessities or with test purposes.

In this contribution, we present GDM-R, a new open source framework fully implemented in R, overcoming the weaknesses of the previous software systems for GDM processes. Its main new and interesting aspects are summarized below:

- It displays various graphical representations which provide a rapid insight in the state and the evolution of the GDM process and enable to identify decision makers whose opinions are far from the group solution and those who present a non cooperative behavior in order to reach an agreement among with experts subcommunities and more influential decision makers.
- It offers a test mode which enables to set a trial scenario to try and compare the performance of different GDM approaches. It is helpful to validate and objectively compare the already existing algorithms and to develop new ones.
- The proposed framework can be easily extended to work with other types of preference relations and to include other methodologies of GDM. Therefore, other researchers can extend and customize it for comparative and test purposes.

7. Future work

From the research work carried out along this PhD dissertation numerous issues have arisen as interesting research lines that worth to be further explored.

7.1. Development of new GDM methodologies dealing with incomplete information modeled with other promising types of preference relations

Recently, hesitant fuzzy PRs (HFPR) based in Hesistan Fuzzy sets proposed by Prof. Torra in [Tor10] becoming widely used in decision making to allow experts to present some flexibility in their opinions, [ZX14]. However, as far as we are aware, there is no approach in the literature able to deal with incomplete HFPRs. A possible approach in these cases would be to extend existing validated approaches for the case of incomplete APRs, IVPRs, IFPRs and LPRs using the multiplicative consistency property of HFPRs introduced in [ZX14] and the iterative procedure developed in [CHVAH09]. However, it remains to develop a formal and theoretical framework to support the validity of the methodology adopted in this area, which consists on the straightforward application of existing mathematical tools and procedure developed specifically for type-1 fuzzy preferences to hesitancy preferences. A possible way to investigate how to tackle this issue might reside in the similarities that exist between the definitions of hesitant fuzzy set and that of type-2 fuzzy set, which can lead to considering the first one as a particular type of the second one.

7.2. Dealing with Linguistic Incomplete Preference relations modeled as Interval-Type-2-Fuzzy-Sets

In decision making a natural alternative for the experts to express their preferences consists in using linguistic terms to describe the desired values. In this sense, as aforementioned, Zadeh proposed the paradigm of **Computing With Words**, CWW, [Zad96, Zad99] which models words by means of type-1 Fuzzy Sets (T1FS), or their extension, type-2 Fuzzy sets (T2FS). For the case of T1FS various authors have pointed out that they presents some limitations [Men09] when modeling words. For instance, Herrera et al. pointed out that it is difficult to accept that all individuals should agree on the same membership function associated to a linguistic tag [HHV97]. Going further Mendel remarks that *words mean different things to different people and so are uncertain* [Men07a, Men07b] and demonstrates that it is scientifically incorrect to model a word using a T1FS, because given a T1FS A for a word, the word is well-defined by its membership function (MF) $\mu A(x)(x \in X)$ which is totally certain once all of its parameters are specified [Men07b]. Hence, a T2FS allows to better model the uncertainty. The boundaries of the T2FS are defined buy two T1FS, the Upper membership, UMF, function and the lower membership function, LMF.

However, in spite of being proved more suitable for dealing with uncertainty, T2FS has not been extensively used in decision making. This is mainly due to its computational complexity, since a T2FS can contains infinite T1FS membership functions with infinite shapes. A simplification of this model consists on considering an uniform distribution of the uncertainty, leading to the concept on Interval Type 2 Fuzzy sets, IT2FS. This model allows an optimal treatment of uncertainty but reduces its computational complexity. Therefore GDM approaches dealing with incomplete information when the linguistic preferences are modeled as IT2FS should be developed. To that aim the first issue to tackle is how to model the consistency. In this sense we propose the extension of Tanino's Multiplicative consistency from the case of reciprocal [0,1] -value fuzzy preference relations to the case of T1FS and IT2FS, based on Zadeh's extension Principle. Once the consistency is properly model both completion and selection approaches can be built upon this.

7.3. Validation frameworks for GDM approaches with incomplete preferences

It is clear that there are many different decision making approaches to tackle incomplete information. However, it is also evident that there is a lack of a comparison frameworks available to evaluate their performance and consequently to help analyze the causes that might affect such performance. This shortage of comparison tools represents an important problem in the decision making field because decision making practitioners are unable to discriminate between the accuracy and the quality of the proposals available to them in the context of incomplete information. Thus, it seems imperative to develop methods to evaluate and validate the different techniques proposed in the literature to estimate the missing preferences. By doing this, it could be possible to compare in a quantitative way the existing GDM methodologies and find out which ones are more suitable depending on the problem to solve and to identify their main advantages and drawbacks.

Some efforts in this direction have been pointed in this dissertation, with the development of GDM-R, an open source library to carry out GDM processes with incomplete information. However the development of methods to evaluate the quality of the different GDM approaches with incomplete preferences is still in early stage and there are many challenges that need to be addressed:

- To create a training and test framework with examples to allow benchmark tests to compare and validate different decision making approaches.
- To find proper metrics to compare different completion approaches.

Part II. Publications: Published, Accepted and Submitted Papers

1. Analyzing the main approaches to deal with incomplete information for different types of preference relations.

The journal paper associated to this part is:

- R. Ureña, F. Chiclana, J.A. Morente-Molinera, E. Herrera-Viedma. Managing Incomplete Preference Relations in Decision Making: A Review and Future Trends. Information Sciences 302:1 (2015) 14-32. doi: 10.1016/j.ins.2014.12.061 .
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Managing Incomplete Preference Relations in Decision Making: A Review and Future Trends

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Abstract

In decision making, situations where all experts are able to efficiently express their preferences over all the available options are the exception rather than the rule. Indeed, the above scenario requires all experts to possess a precise or sufficient level of knowledge of the whole problem to tackle, including the ability to discriminate the degree up to which some options are better than others. These assumptions can be seen unrealistic in many decision making situations, especially those involving a large number of alternatives to choose from and/or conflicting and dynamic sources of information. Some methodologies widely adopted in these situations are to discard or to rate more negatively those experts that provide preferences with missing values. However, incomplete information is not equivalent to low quality information, and consequently these methodologies could lead to biased or even bad solutions since useful information might not being taken properly into account in the decision process. Therefore, alternative approaches to manage incomplete preference relations that estimates the missing information in decision making are desirable and possible. This paper presents and analyses methods and processes developed on this area towards the estimation of missing preferences in decision making, and highlights some areas for future research.

Keywords: Group decision making, Uncertainty, Incomplete information, Fuzzy preferences, Consistency

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1. Introduction

Group decision making (GDM) consists of multiple individuals interacting to choose the best option between all the available ones. Each decision maker (DM) or expert may have his/her own opinions and background and, although they might share a common interest in achieving agreement on selecting the most suitable option, it is expected that they would approach the problem in different ways.

The majority of GDM problems comprise the following phases depicted in Figure 1 [36]: (1) definition of the problem; (2) analysis of the problem; (3) identification of a set of alternatives; (4) identification of the set of criteria and panel of experts; and (5) application of a selection process to derive the solution to the problem.

In GDM systems experts have to express their preferences by means of a set of evaluations over a set of alternatives. To that aim different preference representation formats are available [27]. However, it is common that an expert might not possess a precise or

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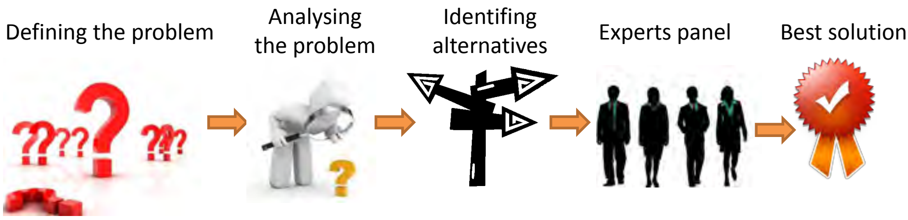

Figure 1: GDM problem resolution steps.

sufficient level of knowledge of part of the problem and, as a consequence, he/she might not provide all the information that is required [2, 14, 24, 44]. Actually, situations where all experts are able to efficiently express their preferences over all the available options might be considered the exception rather than the rule. Indeed, the above scenario requires all experts to possess a precise or sufficient level of knowledge of the whole problem to tackle, including the ability to discriminate the degree up to which some options are better than others. These assumptions can be seen as unrealistic in many decision making situations, especially those involving a large number of alternatives to choose from and/or conflicting and dynamic sources of information. Indeed, a study by Deparis et al. [22] corroborates empirically the following hypothesis: "increasing the intensity of conflict in a multicriteria comparison increases the likelihood that DMs consider two alternatives as incomparable," and therefore leading to the expression of incomplete preferences. Their results indicate that a large attribute spread increases the frequency of incomparability statements when allowed, otherwise an increase of indifference statements happens. Therefore, it becomes necessary to develop decision models to address the presence of incomplete information, i.e. information with missing data.

Different approaches have been developed to deal with incomplete information modelled using different representation formats, which can be broadly classified into three main groups:

(i) methods that directly discard the incomplete information and process only pieces of complete information [52];

(ii) methods that penalise or rate negatively the experts who provide incomplete preferences [24]; and

(iii) methods that estimate the missing preference values using the provided ones [39, 40].

The first two groups of methods are based on the assumption that a good solution to a decision making problem cannot be achieved from incomplete information, or that the solution would not be as good as the one that would derive using complete information. However, empirical evidence suggests that the incomplete relation derived from the random deletion of as much as 50 % of the elements of a complete pairwise preference relation provides good results without compromising accuracy [14]. Therefore, these two groups of methods eliminate or undervalue useful information in the data provided, which could lead to serious biases [43]. Indeed, incomplete information is not equivalent to low quality information, and consequently imposing penalties in the decision making processes to experts providing incomplete information could lead to misleading solution, specially when the incomplete information is consistent and the complete information is not. Thus, alternative approaches to manage incomplete information in decision making are desirable. One of these approaches is based on the selection of an appropriate methodology to 'build' the matrix, and/or to assign importance values to experts based not on the amount of information provided but on how consistent the information provided is.

Some of the existing methods that estimate missing preference values in GDM use the information provided by the rest of experts together with aggregation procedures [44]. The main drawback for this approach is that it requires several experts to estimate the missing values of a particular one, which in conjunction with notable difference between the experts preferences could lead to the estimation of information not naturally compatible with the rest of the expert's information. An alternative approach here is to use methods to estimate an expert's missing values using just his/her own assessments and consistency criteria to avoid incompatibility. This has been a tool extensively applied in decision making contexts under preference relations [1–5, 25, 29, 39, 40, 46, 48, 74]. An extreme case of incomplete preferences happens when one or more experts in the group do not provide any preference information on at least one of the feasible alternatives. This situations are called in literature *total ignorance* or simply *ignorance* situations, and several approaches to deal with them have been presented in [4].

This paper presents a review of the foundations and developments in estimating missing preferences in decision making with the following different kinds of preference relations used as the preference representation format: additive, multiplicative, intuitionistic, interval and linguistic preference relations. A comprehensive analysis of the most recent developed applications in the specialised literature is presented. Finally, some of the current trends and potential future research lines of enquiry on this research topic are also outlined.

The remainder of the paper is set out as follows: In Section 2 the principal types of preference relations used in decision making are reviewed, including a description on the characterisation of their consistency. The main strategies developed to tackle the presence of incomplete preferences for the different types of preference relations will be presented in Section 3. Section 4 focuses on those cases that are being called as ignorance situations in GDM. A discussion on the current trends and future work in this research area is covered in Section 5. In Section 6 conclusions are drawn.

2. Preference Relations in Decision Making

In any decision making problem, once the set of feasible alternatives (X) is identified, experts are called to express their opinions or preferences on such set. Different preference elicitation methods were compared in [52], concluding that pairwise comparison methods are more accurate than non-pairwise methods. A comparison of two alternatives of X by an expert can lead to the preference of one alternative to the other or to a state of indifference between them. Obviously, there is the possibility of an expert being unable to compare them.

Two main mathematical models based on the concept of preference relation can be used in this context. In the first one, a preference relation is defined for each one of the above three possible preference states (preference, indifference, incomparability) [26], which is usually referred to as a preference structure on the set of alternatives [60]. The second one integrates the three possible preference states into a single preference relation [8]. For this second type of mathematical model Xu has carried in [86] a comprehensive review of the different types of preference relations in the literature among with some of their main properties. In this paper, we also focus on this second one.

Formally, a preference relation is defined as follows:

Definition 1 (Preference Relation (PR) [54]). *A preference relation R is a binary relation defined on the set X that is characterised by a function $\mu_p : X \times X \to D$, where D is the domain of representation of preference degrees provided by the decision maker.*

When cardinality of X is small, R may be conveniently represented by an $n \times n$ matrix $R = (r_{ij})$, with $r_{ij} = \mu_p(x_i, x_j)$ being interpreted as the degree or intensity of preference of alternative x_i over x_j. The elements of R can be of a numeric or linguistic nature, i.e., could represent numeric or linguistic preferences, respectively.

2.1. Numeric Preferences

The main types of numeric preference relations used in decision making are: crisp preference relations, additive preference relations, multiplicative preference relations, interval-valued preference relations and intuitionistic preference relations. In the following subsections we analyse each one of these options.

2.1.1. Crisp Preference Relation

When an expert is able to compare two alternatives the following broad outcomes are possible: (i) one alternative is preferred ($\succ$) to another; or (ii) the two alternatives are indifferent ($\sim$). Using a numerical representation of preferences, any ordered pair of alternatives $(x_i, x_j) \in X \times X$ can be associated a number from the set $D = \{0, \frac{1}{2}, 1\}$ as follows [26]:

$$\begin{array}{lcl} r_{ij} = 1 & \Leftrightarrow & x_i \succ x_j \\ r_{ij} = 0 & \Leftrightarrow & x_j \succ x_i \\ r_{ij} = 0.5 & \Leftrightarrow & x_j \sim x_i \end{array}$$

The following 'reciprocity' property is always assumed to avoid 'inconsistent' situations where an expert could prefer two alternatives at the same time: when $r_{ij} = \frac{1}{2}$ it is also $r_{ji} = \frac{1}{2}$; and when $r_{ij} = 1$ then $r_{ji} = 0$.

2.1.2. Additive Preference Relation

The introduction of the concept of fuzzy set as an extension of the classical concept of set when applied to a binary relation leads to the concept of a fuzzy or [0,1]-valued preference relation, $P = (p_{ij})$ [8], referred to as additive preference relation (APR) in this paper:

Definition 2 (Additive Preference Relation (APR) [54]). *An APR P on a finite set of alternatives X is characterised by a membership function*

$$\mu_P \colon X \times X \longrightarrow [0,1],\ \ \mu_P(x_i, x_j) = p_{ij},$$

verifying

$$p_{ij} + p_{ji} = 1 \ \forall i, j \in \{1, \ldots, n\}.$$

The following interpretation is assumed:

- $p_{ij} > 0.5$ indicates that the expert prefers the alternative x_i to the alternative x_j, with $p_{ij} = 1$ being the maximum degree of preference for x_i over x_j;
- $p_{ij} = 0.5$ represents indifference between x_i and x_j.

An APR can be seen as a particular case of a (weakly) complete fuzzy preference relation [27], i.e. a fuzzy preference relation satisfying $p_{ij} + p_{ji} \geq 1 \; \forall i, j$.

2.1.3. Multiplicative Preference Relation

The measuring of the intensity of preferences can be done using a ratio scale instead, with the most widely ratio scale used being the interval $D = [1/9, 9]$ [61].

Definition 3 (Multiplicative Preference Relation (MPR)). *A MPR A on a finite set of alternatives X is characterised by a membership function*

$$\mu_A \colon X \times X \longrightarrow [1/9, 9], \;\; \mu_A(x_i, x_j) = a_{ij},$$

verifying

$$a_{ij} \cdot a_{ji} = 1 \; \forall i, j \in \{1, \ldots, n\}.$$

The following interpretation is assumed: x_i is a_{ij} times as good as x_j, and in particular:

- $a_{ij} = 1$ indicates indifference between x_i and x_j;
- $a_{ij} = 9$ indicates that x_i is absolutely preferred to x_j;

In [16], it was proved that multiplicative and additive preference relations are isomorphic:

Proposition 1. *Suppose that we have a set of alternatives, $X = \{x_1, \ldots, x_n\}$, and associated with it a MPR $A = (a_{ij})$, with $a_{ij} \in [1/9, 9]$ and $a_{ij} \cdot a_{ji} = 1, \forall i, j$. Then the corresponding APR, $P = (p_{ij})$, associated to A, with $p_{ij} \in [0, 1]$ and $p_{ij} + p_{ji} = 1, \forall i, j$, is given as follows:*

$$p_{ij} = f(a_{ij}) = \frac{1}{2} \left(1 + \log_9 a_{ij}\right).$$

The above transformation function is bijective and, therefore, allows to transpose concepts that have been defined for APRs to MPRs, and vice-versa.

2.1.4. Interval-Valued Preference Relation

Membership functions of fuzzy sets are subject to uncertainty arising from various sources [51]. To reflect that Klir and Folger described blurring a fuzzy set to form an *interval-valued fuzzy set* [45]:

Definition 4 (Interval-Valued Fuzzy Set (IVFS)). *Let $INT([0, 1])$ be the set of all closed subintervals of $[0, 1]$ and X be a universe of discourse. An interval-valued fuzzy set (IVFS) $\widetilde{A}$ on X is characterised by a membership function $\mu_{\widetilde{A}} : X \to INT([0, 1])$. An IVFS $\widetilde{A}$ on X can be expressed as follows:*

$$A = \{(x, \mu_{\widetilde{A}}(x)); \;\; \mu_{\widetilde{A}}(x) \in INT([0, 1]) \; \forall x \in X\}.$$

The application of the concept of IVFS to an APR leads to the concept of interval-valued APR (IVPR), i.e. a preference relation with domain of representation of preference degrees is the set of all closed subintervals of $[0,1]$, $D = INT([0,1])$.

Definition 5 (Interval-Valued Additive Preference Relation (IVPR)). *An interval-valued additive preference relation (IVPR) [80] $\widetilde{P}$ on a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ is characterised by a membership function $\mu_{\widetilde{P}}\colon X \times X \longrightarrow INT([0,1])$, with $\mu_{\widetilde{P}}(x_i, x_j) = \widetilde{p}_{ij} = [p_{ij}^-, p_{ij}^+]$, verifying*

$$\forall i, j \in \{1, \ldots, n\} : \; \widetilde{p}_{ji} = 1 - \widetilde{p}_{ij}.$$

The above definition of IVPR can be expressed in terms of the lower and upper bound of the interval-valued preference values as follows:

$$\forall i, j = 1, 2, \ldots n : \; p_{ij}^- + p_{ji}^+ = p_{ij}^+ + p_{ji}^- = 1.$$

2.1.5. Intuitionistic Preference Relation

The concept of an *intuitionistic fuzzy set* (IFS) was introduced by Atanassov [7]:

Definition 6 (Intuitionistic Fuzzy Set (IFS)). *An intuitionistic fuzzy set (IFS) A over a universe of discourse X is represented as $A = \{(x, \langle \mu_A(x), \nu_A(x)\rangle) \,|x \in X\}$ where $\mu_A\colon X \to [0,1]$, $\nu_A\colon X \to [0,1]$ and $0 \le \mu_A(x) + \nu_A(x) \le 1 \quad \forall x \in X$. For each $x \in X$, the numbers $\mu_A(x)$ and $\nu_A(x)$ are known as the degree of membership and degree of non-membership of x to A, respectively.*

An IFS becomes a FS when $\mu_A(x) = 1 - \nu_A(x)\ \forall x \in X$. However, when there exists at least a value $x \in X$ such that $\mu_A(x) < 1 - \nu_A(x)$, an extra parameter has to be taken into account when working with IFSs: the hesitancy degree, $\tau_A(x) = 1 - \mu_A(x) - \nu_A(x)$, that represents the amount of lacking information in determining the membership of x to A. If the hesitation degree is zero, the reciprocal relationship between membership and non-membership makes the latter one unnecessary in the formulation as it can be derived from the former.

In [64], Szmidt and Kacprzyk defined the intuitionistic fuzzy preference relation (IFPR) as a generalisation of the concept of APR.

Definition 7 (Intuitionistic Fuzzy Preference Relation (IFPR)). *An intuitionistic fuzzy preference relation (IFPR) B [87] on a finite set of alternatives X is characterised by a membership function*

$$\mu_B : X \times X \to [0,1]$$

and a non-membership function

$$\nu_B : X \times X \to [0,1]$$

such that

$$0 \le \mu_B(x_i, x_j) + \nu_B(x_i, x_j) \le 1 \;\; \forall (x_i, x_j) \in X \times X.$$

An IFPR can be conveniently represented by a matrix $B = (b_{ij})$ with $b_{ij} = (\mu_{ij}, \nu_{ij})\ \forall i, j = 1, 2, \ldots, n$. The value $\mu_{ij} = \mu_B(x_i, x_j)$ can be interpreted as the certainty degree up to which x_i is preferred to x_j, while the value $\nu_{ij} = \nu_B(x_i, x_j)$ represents the certainty degree up to which x_i is non-preferred to x_j. When the following additional conditions are imposed:

- $\mu_{ii} = \nu_{ii} = 0.5 \ \forall i$.
- $\mu_{ji} = \nu_{ij}, \nu_{ji} = \mu_{ij} \ \forall i, j$.

we refer to this IFPR as additive and we will denote it as IAPR. Notice that when the hesitancy degree function is the null function we have that $\mu_{ij} + \nu_{ij} = 1 \quad \forall i, j$, and therefore the IAPR $B = (b_{ij})$ is mathematically equivalent to the APR (μ_{ij}), i.e. $B = (\mu_{ij})$. Given an IAPR, it is always possible to derive an APR via the application of a score function [71, 72, 87].

2.2. Linguistic Preferences

Subjectivity, imprecision and vagueness in the articulation of opinions pervade real world decision applications, and individuals usually find it difficult to evaluate their preferences using exact numbers [100]. Individuals might feel more comfortable using words by means of linguistic labels or terms to articulate their preferences [101].

In a linguistic context, experts' preferences are usually represented using an ordered set of linguistic terms, $\mathcal{L} = \{l_0, \ldots, l_s | s \geq 2 \ \wedge i < j : \ l_i < l_j\}$. Table 1 provides an example with seven linguistic labels and their corresponding semantic meanings for the comparison of the ordered pair of alternatives (x_i, x_j).

Linguistic label	Semantic meaning
l_0	x_j is absolutely preferred to x_i
l_1	x_j is highly preferred to x_i
l_2	x_j is slightly preferred to x_i
l_3	x_i and x_j are equally preferred
l_4	x_i is slightly preferred to x_j
l_5	x_i is highly preferred to x_j
l_6	x_i is absolutely preferred to x_j

Table 1: Seven linguistic labels and their semantic meanings

An odd number of labels is also assumed, with the central label $l_{s/2}$ standing for the indifference state when comparing two alternatives, and the remaining labels being usually located symmetrically around that central assessment to guarantees that a kind of reciprocity property holds as in the case of numerical preferences previously discussed.

Definition 8 (Linguistic Preference Relation (LPR)). *A LPR P on a finite set of alternatives X is characterised by a linguistic membership function $\mu_P \colon X \times X \longrightarrow \mathcal{L}$, $\mu_P(x_i, x_j) = p_{ij} \in \mathcal{L}$.*

The main two methodologies to manage LPRs in decision making are [36]: (i) the *cardinal* representation model based on the use of fuzzy sets and their associated membership functions, which are mathematically processed using Zadeh's *extension principle* [100]; and (ii) the *ordinal* representation model based on the ordered structure defined on the labels [97].

2.2.1. LPR based on cardinal representation

Convex normal fuzzy subsets of the real line, also known as fuzzy numbers, are commonly used to represent linguistic terms. By doing this, each linguistic assessment is represented using a fuzzy number that is characterised by a membership function, with base variable the unit interval $[0,1]$, describing its semantic meaning. The membership function maps each value in $[0,1]$ to a degree of performance which represents its compatibility with the linguistic assessment [75, 101].

2.2.2. LPR based on ordinal representation

In an ordinal linguistic approach the semantics of the linguistic labels is established by assuming that in the set of linguistic terms $\mathcal{L}$ the labels are uniformly and symmetrically distributed around that central assessment $l_{s/2}$, i.e., assuming the same discrimination levels on both sides of $l_{s/2}$ and by considering that both terms l_i and l_{s-i} are equally informative.

Linguistic symbolic computational models have been defined to manage the ordinal linguistic information in the decision making problems [36]. The symbolic models work with the ordinal scales of the set of linguistic terms to combine linguistic information. There exit four different linguistic symbolic computational models based on ordinal scales:

1. *Linguistic symbolic computational model based on max-min operators* [97], which is based on the application of the following three operators to combine information expressed as linguistic labels in the ordered linguistic set $\mathcal{L}$:
   - $Max(l_i, l_j) = l_i$ if $l_i > l_j$.
   - $Min(l_i, l_j) = l_i$ if $l_i < l_i$.
   - $Neg(l_i) = l_{s-i}$.
2. *Linguistic symbolic model based on convex combination* [37]. This model aggregates the linguistic information using a convex combination of linguistic labels acting directly over their associated indexes $\mathcal{L}$ in a recursive way. Since the result of this aggregation is not necessary integer it is also necessary to introduce an approximation function to obtain a final label in $\mathcal{L}$.
3. *Linguistic symbolic model based on virtual linguistic term set* [79], which extends the original discrete term set $\mathcal{L}$ into a continuous term set $\hat{\mathcal{L}} = \{l_\alpha | \alpha \in [-s, s]\}$ with the following operations :

   $$l_\alpha \oplus l_\beta = l_{max\{-s, min\{\alpha+\beta, s\}\}}$$

   $$\lambda l_\alpha = l_{\lambda\alpha}, \text{ where } \lambda \in [0,1]$$

   This model also requires a translation function to express the results of the operations in the original terms set
4. *Linguistic symbolic model based on the 2-tuple linguistic representation* [38], which was introduced to avoid the loss of information that appears when the mentioned translation function in the linguistic symbolic model based on convex combination is applied. This model is built on the following linguistic 2-tuple representation definition:.

Definition 9. *Let $\mathcal{L}$ be a linguistic term set and $\beta \in [0, s]$ a value supporting the result of a symbolic aggregation operation, then the 2–tuple that expresses the equivalent information to β is obtained with the following function:*

$$\begin{aligned}\Delta : \quad & [0, s] \longrightarrow S \times [-0.5, 0.5) \\ & \Delta(\beta) = (l_i, \alpha) \\ & i = round(\beta) \\ & \alpha = \beta - i\end{aligned}$$

where "round" is the usual rounding operation, l_i has the closest index label to "β" and "α" is the value of the symbolic translation.

In [58], the representation of linguistic preferences using the cardinal approach based on the use of fuzzy sets, and the ordinal approach based on the use of the 2–tuples were proved to be mathematically isomorphic when fuzzy numbers are ranked using their respective centroids.

2.3. Consistency of Preferences

There are three fundamental and hierarchical levels of rationality assumptions when dealing with preference relations [19]:

- The first level of rationality requires indifference between any alternative x_i and itself.
- The second one requires that if an expert prefers x_i to x_j, that expert should not simultaneously prefer x_j to x_i. This asymmetry condition is viewed as an "obvious" condition/criterion of consistency for preferences [26]. This rationality condition is modelled by the property of reciprocity in the pairwise comparison between any two alternatives, which is seen by Saaty as basic in making paired comparisons [61].
- Finally, the third one is associated with the transitivity in the pairwise comparison among any three alternatives. That is, if x_i is preferred to x_j ($x_i \succ x_j$) and this one to x_k ($x_j \succ x_k$) then alternative x_i should be preferred to x_k ($x_i \succ x_k$), which is normally referred to as *weak stochastic transitivity* [49].

A preference relation verifying the third level of rationality is usually called a *consistent preference relation* and any property that guarantees the transitivity of the preferences is called a consistency property [19]. The lack of consistency in decision making can lead to inconsistent conclusions; that is why it is important, in fact crucial, to study conditions under which consistency is satisfied [61].

In the case of MPRs, Saaty means by *consistency* what he calls *cardinal transitivity* in the strength of preferences, which is a stronger condition than the traditional requirement of the transitivity of preferences [61]:

Definition 10 (Consistent MPR). *A MPR $A = (a_{ij})$ is consistent if and only if*

$$a_{ij} \cdot a_{jk} = a_{ik} \ \forall i, j, k = 1, \ldots, n.$$

Inconsistency for Saaty is a violation of proportionality which may not entail violation of transitivity [61]. Furthermore, consistency implies reciprocity, and therefore, they are both compatible.

For APRs, there exist many properties or conditions that have been suggested as rational conditions to be verified by a consistent relation, among which we can cite [19, 41]: triangle condition, weak transitivity, max-min transitivity, max-max transitivity, restricted max-min transitivity, restricted max-max transitivity, additive transitivity, and multiplicative transitivity. Among these, the most widely used in the context of incomplete information are the following two [19]:

Definition 11 (Additive consistency of APR [65]). *An APR $P = (p_{ij})$ on a finite set of alternatives X, it is additive consistent if and only if*

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = p_{ik} - 0.5 \quad \forall i, j, k = 1, 2, \cdots, n$$

Although equivalent to Saaty's consistency property for MPRs [41], additive transitivity is in conflict with the $[0, 1]$ scale used for providing the preference values and therefore, it is not the most appropriate property to model consistency of reciprocal PRs.

Definition 12 (Multiplicative consistency of APR [65]). *An APR $P = (p_{ij})$ on a finite set of alternatives X is multiplicative consistent if and only if*

$$p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ik} \cdot p_{kj} \cdot p_{ji} \quad \forall i, k, j \in \{1, 2, \ldots n\}$$

Multiplicative consistency property was proposed by Tanino for $p_{ij} > 0 \quad \forall i, j$ and under reciprocity it is the restriction to the region $[0, 1] \times [0, 1] \backslash \{(0, 1), (1, 0)\}$ of the Cross Ratio uninorm [19]:

$$U(x, y) = \begin{cases} 0, & (x, y) \in \{(0, 1), (1, 0)\} \\ \dfrac{xy}{xy + (1 - x)(1 - y)}, & \text{otherwise} \end{cases}$$

Definition 13 (Additive Consistent IVAPR [3, 69]). *An IVAPR $\widetilde{P} = (\widetilde{p}_{ij}) = ([p_{ij}^-, p_{ij}^+])$, is additive consistent if and only if*

$$\forall i, j, k : \begin{cases} p_{ik}^- = p_{ij}^- + p_{jk}^- - 0.5 \\ p_{ik}^+ = p_{ij}^+ + p_{jk}^+ - 0.5 \end{cases}$$

A formal approach to modelling the multiplicative consistency property of IVAPR and IAPR, however, can be found in [74].

Definition 14 (Multiplicative Consistent IVAPR [74]). *An IVAPR $\widetilde{P} = (\widetilde{p}_{ij}) = ([p_{ij}^-, p_{ij}^+])$, is multiplicative consistent if and only if*

$$\forall i, j, k : \begin{cases} p_{ij}^- \cdot p_{jk}^- \cdot p_{ki}^- = p_{ik}^- \cdot p_{kj}^- \cdot p_{ji}^- \\ p_{ij}^+ \cdot p_{jk}^+ \cdot p_{ki}^+ = p_{ik}^+ \cdot p_{kj}^+ \cdot p_{ji}^+ \end{cases}$$

Because the IAPR $R = (r_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$ is isomorphic to the IVAPR $B = (b_{ij}) = ([\mu_{ij}, 1 - \nu_{ij}])$, a multiplicative consistent IAPR can be defined as follows:

Definition 15 (Multiplicative Consistent IAPR [74]). *An IAPR* $R = (r_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$ *is consistent if and only if*

$$\forall i,j,k: \begin{cases} \mu_{ij}\mu_{jk}\mu_{ki} = \mu_{ik}\mu_{kj}\mu_{ji} \\ (1-\nu_{ij})(1-\nu_{jk})(1-\nu_{ki}) = (1-\nu_{ik})(1-\nu_{kj})(1-\nu_{ji}) \end{cases}$$

Xu et al. in [96] investigate the consistency of intuitionistic preference relations in GDM concluding that if all individual intuitionistic preference relations are consistent, then the collective intuitionistic preference relation is consistent as well. Moreover they propose an iterative approach to improve the cosistency of this type of preference relations.

In the case of LPRs, the consistency property has been defined with different expressions depending on the linguistic approach used:

Definition 16 (Cardinal Additive Consistency of LPRs [68]). *Given a LPR,* $\tilde{P} = \tilde{p_{ij}}$ *in which each linguistic preference degree has associated a triangular fuzzy membership function, i.e.,*$\tilde{p_{ij}} = (p^L_{ij}, p^M_{ij}, p^R_{ij})$*, then* $\tilde{P}$ *is additive consistent if and only if*

$$\forall i,j,k: \begin{cases} p^L_{ij} + p^L_{jk} + p^R_{ki} = \frac{3}{2} \\ p^M_{ij} + p^M_{jk} + p^M_{ki} = \frac{3}{2} \\ p^R_{ij} + p^R_{jk} + p^L_{ki} = \frac{3}{2} \end{cases}$$

Definition 17 (Ordinal Additive Consistency of LPRs [1]). *Given a 2-tuple LPR* $P = (p_{ij})$ *on a set of alternatives* X*, such that*

$$p_{ij}\colon X \times X \longrightarrow \mathcal{L} \times [-0.5, 0.5)$$

then P *will be considered consistent if for every three alternatives* x_i*,* x_j *and* x_k*, the following condition holds*

$$p_{ik} = \Delta(\Delta^{-1}(p_{ij}) + \Delta^{-1}(p_{jk}) - \frac{s}{2}) \quad \forall i,j,k \in \{1,\ldots,n\}.$$

2.4. Advantages and drawbacks of preference relations

In this subsection we remark some advantages and drawbacks on the use of preference relations in decision making problems.

Millet [52] conducted a comparison study between different alternative preference elicitation methods and pairwise comparison methods were concluded to be more accurate than non-pairwise methods (utilities, orderings, ...) [27]. This is specially the case of decision making problems involving a large number of alternatives to choose from and/or conflicting and dynamic sources of information [56, 57]. The main advantage of preference relations, which are built by pairwise comparisons, is that of focusing exclusively on two options at a time, which facilitates experts when expressing their preferences. However, the drawback is that some experts might not been able to discriminate the degree up to which some of the options are better than others, and as a consequence incomplete preferences are provided [22].

The use of different types of measurement scales to provide assessments on the alternatives lead to different preference relations: numeric or linguistic. The advantage of numeric preference relations is that of providing the preferences in a more precise way, although an associated drawback is that experts are force to assess their preferences by means of numeric assessments, obviating that some of them might feel more comfortable using words (linguistic labels) to articulate their preferences. On the other hand, linguistic preference relations are a more user-friendly representation format to express the preferences in decision making problems when experts' participation is necessary, and thus they are not recommended in decision making problems that not require of user-systems interaction such as automatic classification problems [28].

Regarding numeric preference relations, as it was shown previously, we also have different possibilities. Crisp preference relations are the simplest and easiest to use because they are valued in the simple numerical scale $D = \{0, \frac{1}{2}, 1\}$ whose interpretation is easy to understand. However, the drawback is that of lacking flexibility to express preferences and manage uncertainty in decision making problems. To overcome this problem, APR [8] and MPR [61] were introduced, which use richer numerical scales, i.e. $D = [0, 1]$ and $D = [1/9, 9]$, respectively. Although the interpretation of intensities of preferences are different in these last two types of relation (additive interpretation vs ratio interpretation), it has been proved that are isomorphic [16], and therefore both are admissible to be used in the same problems because concepts that have been defined for APRs can be easily transpose to MPRs, and vice-versa. IVPRs [45] and IFPRs [64] were introduced to express preferences with a greater level of uncertainty in decision making problems, and it is well known that both are mathematically isomorphic. However, their drawback is twofold: experts have more difficulties in providing their preferences with such representations because more numerical parameters are to be provided, and the computation complexity of the decision making processes is higher in comparison to using APRs or MPRs.

As aforementioned, although linguistic preference relations are user-friendly and the provision of preferences by users is mitigated, they suffer the drawback of fixing the adequate linguistic scale to express preferences. Usually, we find that different experts present different conceptions to model the linguistic information and they might choose important parameters to define a linguistic modelling, such as the cardinality of linguistic term sets and the meaning associated with each label [36, 50, 53], differently. In the case of LPRs based on cardinal representation the additional drawbacks that we find are twofold: that of defining the membership functions associated with each label and the known problem of linguistic approximation that sometimes entails loss of information [36]. Decision making approaches that use LPRs based on ordinal representation are easier to define, overcome the problem of linguistic approximation by means of the definition of symbolic computational models [37], and the problem of the loss of information by means of 2-tuple linguistic representation models can be avoided [38, 99].

3. Decision making approaches with incomplete preferences

It is often assumed in GDM that all the experts are able to provide preference degrees between any pair of possible alternatives, which means that complete PRs are assumed. However this is not always possible because of time pressure, lack of knowledge, decision maker's limited expertise on the field dealt with, or incapacity to quantify the degree of preference of one alternative over another. Thus, an expert might decide not to guess the preference values in doubt to maintain the consistency of the values already provided. To model these situations the concept of incomplete PR was introduced in [40].

Definition 18. *A function $f\colon X \longrightarrow Y$ is* partial *when not every element in the set X*

necessarily maps to an element in the set Y. When every element from the set X maps to one element of the set Y then we have a total *function.*

Definition 19. *A preference relation P on a set of alternatives X with a* partial *membership function is an* incomplete preference relation.

The concept of incomplete preference relations has attracted the attention of researchers in the past 20 years and therefore specific settings for different types of PRs have been introduced and analyse in the literature [79, 86, 93].

In this section we analyse the main techniques developed in the literature to deal with incomplete information in decision making for the different types of preference relations reviewed in Section 2. These techniques use consistency properties to estimate the missing preferences and can be divided in two different approaches:

1. **Iterative approaches** to estimate the missing preference values and complete the preference relations [3, 10, 11, 13, 18, 46, 82]. Some approaches also present interactive procedures to increase the consensus degree among the experts [39, 44, 82, 88, 89]
2. **Optimisation approaches** to estimate the missing preference values or to directly rank the alternatives without previously completing the preference relations. Therefore there are two types of these approaches:

   2.1 Methods that estimate the missing preferences [25, 102], and

   2.2 Methods that estimate the weighting vector [23, 30, 35, 48, 78, 81, 83, 88, 88, 94].

Notice that because both the iterative and the optimisation based approaches use consistency criteria, in many cases the corresponding outputs are similar, as it is proved by Chiclana et al. [17] for the case of using additive consistency property and APRs.

Figure 2 depicts a schema of the different approaches existing in the literature to deal with incomplete information in decision-making, which will be analysed in the following subsections for the case of APR and MPR, IVPR and IFPR, and LPR, respectively.

3.1. Managing missing preference values in APRs and MPRs

Notice that the majority of the techniques developed to deal with uncertainty and missing information in GDM are for APRs and MPRs. Recall that in [16] both types of PRs were proved to be isomorphic.

3.1.1. Iterative approaches

Three main iterative approaches to estimate incomplete APRs and MPRs can be found: additive consistency based approaches [3, 10, 11, 40, 46], multiplicative consistency based approaches [82], and its generalisation approach based on the use of uninorm operators [18].

1. **Additive consistency based approaches:** The main additive consistency based method is due to Herrera-Viedma et al. [40], which consists of an iterative procedure to estimate missing preference values followed by a choice process of the solution alternative. The iterative method to estimate missing preference values is summarised below:

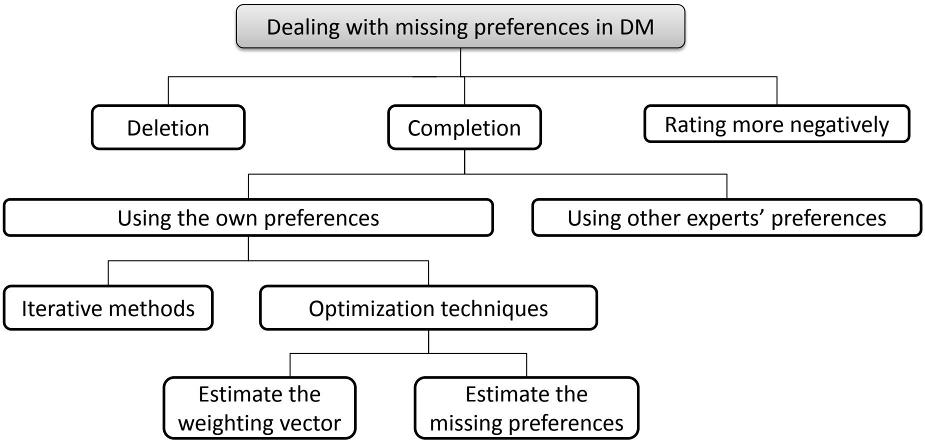

Figure 2: Different approaches to deal with missing information in DM.

Given an unknown preference value p_{ij} $(i \neq j)$ the iterative procedure starts by using intermediate alternatives, x_k, to create indirect chains of known preference values, (p_{ik}, p_{kj}), that will be used to derive, using the additive consistency property, the *local consistency based estimated values*:

$$ep_{ij}^k = p_{ik} + p_{kj} - 0.5.$$

By averaging all the local consistency based estimated values, the *overall consistency based estimated value* is obtained:

$$ep_{ij} = \sum_{k=1, k \neq i,j}^{n} \frac{ep_{ij}^k}{n-2}$$

In each iteration, the algorithm checks the set of pairs of alternatives for which preference values are unknown and can be estimated using known ones. The algorithm stops when this set is empty. Notice that the cases when an incomplete APR cannot be successfully completed are reduced to those cases when no preference values involving a particular alternative are known, which means that a whole row or column of the APR is completely missing.

Because of the conflict between the additive consistency property and the unit scale used to measure preference values [19], the overall consistency based estimated preferences might be greater than 1 or lower than 0, and therefore a normalisation process using the median operator is necessary [20].

In [3], an extension to deal with MPR, IVPR, and LPR is presented. The original approach by Herrera-Viedma et al. has been taken forward by many authors to tackle different research problems with incomplete APRs. Notable examples can be found in [10, 11, 39, 46].

2. **Multiplicative consistency based approaches:** The most relevant method developed using the multiplicative consistency property are presented in [82] and [90]. In [82] each individual incomplete APR is completed using the multiplicative consistency property, followed by their aggregation into a collective preference relation. Based on the deviations between the collective and individuals APRs, the decision makers interact to increase the level of consensus. In [90] it is presented a completion method for MPR based on the multiplicative transitivity. This method estimates the unknown preferences using several pairs of adjoining known elements. To compute the final value it calculates the geometrical mean of all the possible ones.

3. **Uninorms based approaches:** As it has been mentioned before, additive consistency property does not generalise the concept of transitivity of crisp preferences. In [19] it is shown that, under a set of conditions, consistency of APR can be characterised by representable uninorms. In [18], Herrera-Viedma et al's iterative method is adapted to implement the modelling of consistency of preferences using a self-dual almost continuous uninorm operator. Since Tanino's multiplicative transitivity property is an example of such type of uninorms [18, 19], this approach to deal with incomplete information in APRs is more general than the above one.

3.1.2. Optimisation and linear programming based methods

The two optimisation approaches to deal with incomplete PRs are analysed next:

1. **Optimisation methods to estimate missing preference values.** The most relevant of these methods are due to Fedrizzi and Giove [25] and Zhang et al. [102], and they aim to estimate the missing reference values by maximizing the consistency and/or the consensus of the experts' preferences.

   (a) Fedrizzi and Giove [25] propose a model that minimises the *global additive inconsistency* of the incomplete APR

   $$\rho = 6 \cdot \sum_{i<k<j} L_{ijk}$$

   where

   $$L_{ijk} = (p_{ik} + p_{kj} - p_{ij} - 0.5)^2$$

   The missing preference values are the variables in the global inconsistency index. A comparison between this method and Herrera-Viedma et al. [40] is reported in [17]. This study proves that both methods, driven by the additive consistency property, provide the same set of solutions for independent sets of missing comparisons but not for dependent missing comparisons. Fedrizzi and Giove's method performs worse than Herrera-Viedma et al.'s method for a large number of alternatives, and both methods fails to complete an incomplete APR when no preference values are known for at least one of the alternatives. Finally the authors conclude that both methods are complementary and therefore they introduce a new methodology for reconstructing incomplete APRs that encompasses both approaches.

   (b) Zhang et al. [102] propose a model for incomplete APR $F = (f_{ij})_{n\times n}$ that aims to calculate a complete fuzzy preference relation $F' = (f'_{ij})_{n\times n}$ with

$f'_{ij} = f_{ij}$ for non-null entries of F maximising the consistency level proposed by Herrera-Viedma et al. [40]. To increase the individual consistency the following linear optimisation method that minimises the Manhattan distance between the provided preference relation and the completed consistent based one is proposed:

$$\max CL(F') = 1 - \frac{2}{3n(n-1)(n-2)} \sum_{i,k=1;j\neq k}^{n} \sum_{j=1;j\neq ik}^{n} |f'_{ij} + f'_{jk} - f'_{ik} - 0.5|$$
$$\text{s.t. } f'_{ij} \geq 0 \; i,j = 1,2,\cdots,n$$
$$\text{s.t. } f'_{ij} + f'_{ji} = 1 \; i,j = 1,2,\cdots,n$$
$$\text{s.t. } f'_{ij} + f'_{ji} = 1 \; i,j = 1,2,\cdots,n$$
$$\text{s.t. } f'_{ij} = f_{ij} \text{ for } f_{ij} \neq \text{null}$$

2. **Optimisation methods to directly compute the priority weights.** These methods aim to rank the alternatives using directly the incomplete APR, and therefore no completion process is needed. They are based on Saaty's assumption for MPR regarding the exact functional relation between the preference values and the priority vector. Two main approaches are used to develop indirect completion models based on the computation of the priority vector: linear based methods where the unknown variables are the elements of the weighting vector [23, 35, 81, 83, 88, 94], and least square error minimization approaches [30, 48, 78, 88].

   (a) Harker [35] extends the eigenvector approach proposed by Saaty [61] for non-negative quasi reciprocal matrices in order to apply it to the case of incomplete APRs.

   (b) Xu [83] presents a method based on a system of equations to determine the priority vector of an incomplete APR, by replacing a missing preference value p_{ij} using the following priority weighting vector based value: $\frac{w_i}{w_i+w_j}$. With this procedure if there exists a unique solution to this system of equations, then the obtained solution is used to rank the alternatives and to select the most desirable one; otherwise, it requires the experts to provide more evaluation information until the unique priority vector can be obtained.

   (c) Xu and Chen [94] propose a completion method based on the additive transitivity property that requires solving a linear system of equations to rank the alternatives. Shen et al. [62] and Xu [88] subsequently proved that the relation between the original PR and the elements of the priority weight vector postulated by Xu and Chen [94], $r_{ij} = 0.5(w_i - w_j + 1)$, does not always hold and can lead to ambiguous priority vectors. To overcome this drawback, Xu [88] proposed to use the following auxiliary additive transitivity based APR to estimate the missing preferences values, $R' = (r'_{ij})_{n\times n}$:

$$r'_{ij} = r_{ij} \text{ , if } r_{ij} \text{ is known;} \qquad (1)$$
$$r'_{ij} = \frac{n-1}{2}(w_i - w_j) + \frac{1}{2} \text{ , otherwise.}$$

   (d) Xu [81] proposes two goal programming models for obtaining the priority vector of an incomplete APR, and their extension to obtain the collective priority vector.

(e) Dopazo and Ruiz-Tagle [23] propose a parametric goal programming model based on the consistency property of MPR to obtain the weighted priority vector. This model makes use of a dissimilarity function between the ideal case, when the preferences are consistent and there is unanimous consensus among experts, $I^k = \left(\frac{w_i}{w_j}\right)$, and the provided incomplete MPR, M^k. The objective function corresponds to a compromise criterion constructed as a convex combination of the two extreme criteria: to minimise the weighted sum of expert deviations and to minimise the largest weighted deviation. In this model, the relative residual aggregation is modelled by a parameter α used to control the importance given to the most discrepant expert.

(f) Gong [30] presented a multiplicative consistency based least-square model for APRs aiming at maximising the consensus among the experts by minimising the following error function:

$$\min g(w) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{d_{ij}} (r_{ijl} w_j - r_{jil} w_i)^2 \tag{2}$$

$$\text{s.t.} \sum_{i=1}^{n} w_i = 1, \ , w_i > 0, \ i \in n \tag{3}$$

where d_{ij} stands for the number of experts who have provided a preference between the alternatives x_i and x_j. Xu et al. [77] proposed a similar approach that accepts the following three types of incomplete PR: APR, MPR and LPRs. Similar models have been proposed based on the use of logarithmic least squares by Xu et al. [78] and on the additive consistency property by Liu et al. [48], respectively.

3.2. Managing missing preference values in IVPRs an IFPRs

In this subsection we analyse the methods proposed in the literature to deal with incomplete information when the experts' preferences are expressed by means of IVPR and IFPRs. For the case of IVPRs two main approaches are analysed: The first one uses consistency properties to estimate the missing PRs [3, 29] whereas the second one [98] is based on the rough set theory [55]. For the case of IFPR three iterative approaches have been considered [74, 87, 93]. Finally an approach presented by Xu et al. in [92] to deal with missing interval value intuitionistic additive and multiplicative preference relations (IVIFPR) is also analysed.

- Genc et al. [29] extended the optimisation method proposed by Xu and Chen Xu and Chen [95] for deriving the priority weighting vector to the case of incomplete IVPRs. To that aim they also propose the so-called interval multiplicative transitivity property.

- Alonso et al. [3] extend the iterative procedure proposed by Herrera-Viedma et al. [40] to the case of IVPR relations.

- Yang et al. [98] propose a dominance-based rough set approach to estimate missing values in incomplete interval-valued information systems. This approach considers three types of unknown values: (i) IVPR with unknown upper limit and known lower limit, (ii) IVPR with unknown lower limit and known upper limit, and (iii) IVPR data with both unknown lower and upper limits.

- Xu [87] firstly defines the concept of IFPR and introduces an iterative completion method based on the multiplicative consistency.

- Xu et al. [93] presents a completion method based on the multiplicative consistency property for IFPR. This method can be summarised as follows:

  Given an incomplete IFPR $R = (r_{ij})_{n\times n}$ each missing preference value $r_{ij}(i = 1, 2, \cdots, n-1, j = i+1)$ is estimated by $\dot{r}_{ij} = (\dot{\mu}_{ij}, \dot{v}_{ij}, \dot{\pi}_{ij})$ where

$$\begin{aligned} \dot{\mu}_{ij} &= \frac{1}{m_{ij}} \sum_{k\in M_{ij}} \frac{\mu_{ik}\mu_{kj}}{\mu_{ik}\mu_{kj} + (1-\mu_{ik})(1-\mu_{kj})} \\ \dot{v}_{ij} &= \frac{1}{m_{ij}} \sum_{k\in M_{ij}} \frac{v_{ik}v_{kj}}{v_{ik}v_{kj} + (1-v_{ik})(1-v_{kj})} \\ & \text{for all } r_{ik}, r_{kj} \in \Omega, \text{ and } i \le k \le j \end{aligned} \tag{4}$$

  and $\dot{\pi_{ij}} = 1 - \dot{\mu_{ij}} - \dot{v_{ij}}$, where $r_{ik} = (\mu_{ij}, v_{ij}, \pi_{ij})$, and Ω is the set of all the known elements in R, $M_{ij} = \{k | r_{ik}, r_{kj} \in \Omega\}$ and m_{ij} is the number of elements in M_{ij}. If there exists k_0 such that $(\mu_{ik_0}, \mu_{ik_0}) \in \{(0,1),(1,0)\}$ or $(v_{k_0j}, v_{k_0j}) \in \{(0,1),(1,0)\}$, then $\dfrac{\mu_{ik}\mu_{kj}}{\mu_{ik}\mu_{kj} + (1-\mu_{ik})(1-\mu_{kj})} = 0$

- Wu and Chiclana [74] propose a GDM process with consensus in which the missing values of the IFPR are estimated following an iterative procedure that is based on the one proposed by Herrera-Viedma et al. [40]. This method is based on the multiplicative consistency property for IFPRs, which is formally generalised from APR to IFPR by applying Zadeh's *Extension Principle* [101] and *Representation Theorem* [100] .

- Xu et al introduce in [92] the additive and the multiplicative consistent incomplete interval-valued intuitionistic fuzzy preference relations and define the concept of acceptable incomplete interval-valued intuitionistic fuzzy preference relation. In this contribution they also propose two procedures for completing the acceptable incomplete interval-valued intuitionistic based on the arithmetic average and the geometric mean, respectively.

- Wang et al. [70] propose an approach to multiattribute decision making with incomplete attribute weight information where individual assessments are provided as IVIFPRs. By employing a series of optimization models, the proposed approach derives a linear program for determining attribute weights

3.3. Managing missing preference values in LPRs

There are three different methodologies to deal with incomplete LPRs, which are defined according to the three different linguistic decision frameworks: (i) 2-tuple LPRs [3, 12, 59]; (ii) LPRs based on virtual linguistic term sets [42, 85]; and (iii) LPRs based on a cardinal approach [47, 68].

(i) 2-tuple LPRs.

   (a) Alonso et al. [3] propose a method which converts the 2- tuple LPR into an APR and estimates the missing values using the additive transitivity property. Once the APR is completed it is transformed back to the corresponding 2-tuple LPR.

   (b) Alonso et al. [1] apply the linguistic additive consistency property to estimate the missing 2-tuple linguistic values and design an iterative procedure similar to the one proposed by Herrera-Viedma et al. [40]. That was later used by Cabrerizo et al. [12] to define an additive consistency measure of the information provided by each expert to assign importance degrees to experts in the aggregation process. Porcel and Herrera-Viedma [59] present an application in the context of fuzzy linguistic recommender systems that allows incomplete linguistic information.

(ii) LPRs based on Virtual linguistic term sets.

   (a) Xu [85] proposes an additive transitivity property based method to estimate missing LPRs assessed on virtual linguistic term sets. This author also propose in [84] and in [76] completion methods based on the multiplicative transitivity.

   (b) Hsu and Wang [42] present an alternative additive transitivity property based estimation method of missing LPRs assessed on virtual linguistic term sets for which they propose three ways of pairwise comparisons: horizontal, vertical and oblique.

(iii) LPRs based on a cardinal approach.

   (a) Li and Sun [47] propose an extension of the well known LINMAP method [63] to deal with decision making problems with fuzzy linguistic information. Each alternative is assessed on the basis of its distance to a fuzzy positive ideal solution (FPIS) which is unknown, using a new method to calculate the distance between trapezoidal fuzzy number scores. The FPIS and the weights of attributes are then estimated using a linear programming model guided by the consistency and inconsistency criteria. The distance of each alternative to the FPIS is calculated to determine the ranking order of all alternatives.

   (b) Wang and Chen [68] present an approach which uses triangular membership function to model linguistic information and that is driven by the additive consistency property of the reciprocal APR.

3.4. Summary

Table 2 summarises, in chronological order, the main papers dealing with the different approaches to manage incomplete information reviewed and analysed in this contribution. It is fair to conclude that the management of incomplete information in DM based on PRs is currently a relevant topic in fuzzy decision making analysis, and that it has been disseminated in the most important journals on this research area including: IEEE Trans. on Systems, Man and Cybernetics–Part B; IEEE Trans. on Fuzzy Systems; Knowledge-Based Systems; Information Sciences; Information Fusion, Soft Computing and Fuzzy Sets and Systems. Evidence of this is that the scientific database Essential

Science Indicators, provided by Thomson Reuter, is currently listing incomplete information as part of the following Research Front: Incomplete Fuzzy Linguistic Preference Relations; Group Consensus Algorithm Based; Unbalanced Fuzzy Linguistic Information; AHP Group Decision (accessed on 29–10–2014).

4. Processes dealing with ignorance situations in GDM

The procedures exposed in the previous section cannot be applied successfully when some experts do not provide any information about a particular alternative, which is known as ignorance situations. Alonso et al. [4] developed several strategies to deal with ignorance situations in the context of GDM with APRs. These strategies can be broadly classified in two main groups depending on whether the information provided by other experts is used to estimate the missing values, known as *social strategies*, otherwise named *individual strategies.*

4.1. Ignorance individual strategies

The proposed ignorance individual strategies (IIS) can be divided in two main steps:

1. Setting some particular seed values to provide some initial information to the estimation procedure to be able to compute the other missing values. The selection of the seed values can be accomplished using two different methodologies:

   IIS1 Choosing indifference seed values: Let P be an incomplete APR with no preference information on alternative x_i, i.e p'_{ij} and p'_{ji} are unknown for all j. In this strategy, indifference seed values are assumed, i.e. $p'_{ij} = p'_{ji} = 0.5\ \forall j$. This strategy adjusts the estimated preference values to make the APR more consistent with the previously existing information. This approach is particularly useful when there are no external sources of information about the problem and when a high consistency level is required.

   IIS2 Choosing proximity seed values: In this case the seed values are obtained from the preference values given to similar alternatives. This is possible if some extra information or properties about alternatives, which strongly suggest that the ignored alternative is similar to another one, are known. This strategy could be useful in some decision making problems where the alternatives to be evaluated are goods with similar characteristics (similar models).

2. Estimating the rest of the missing values using the consistency based procedure proposed in [40].

4.2. Ignorance social strategies

Ignorance social strategies (ISS) are based on the use of the information provided by the set of experts. The authors present three main approaches in this case:

ISS1 The first social strategy uses consensus preference values of the collective PR, computed by aggregating all the experts' individual PRs. The main advantage of this approach is that it improves the consensus of the set of experts making their opinions close to each other.

Reference	Year	Relation	Consistency	Characteristics
[35]	1987	APR	Multiplicative	Extension of Saaty's eigenvalue approach to incomplete APR.
[2]	2004	APR	Additive	Iterative procedure.
[81]	2004	MPR	Multiplicative	Goal Programming method to estimate the priority vector.
[82]	2005	APR	Multiplicative	Interactive approach to reach consensus.
[83]	2005	MPR	Multiplicative	Eigenvalue approach.
[85]	2006	LPR (Virtual terms set)	Additive	Iterative approach.
[84]	2006	LPR(Virtual terms set)	Multiplicative	Iterative.
[40]	2007	APR	Additive	Iterative procedure, new consistency measure, new AC-IOWA operator.
[25]	2007	APR	Additive	Minimises the global additive inconsistency.
[47]	2007	LPR	Additive	Extension of LINMAP to deal with incomplete LPR.
[87]	2007	IFPR	Multiplicative	Iterative approach.
[3]	2008	APR, MPR, IVPR, LPR	Additive	Extension of the iterative procedure by Herrera-Viedma et al. [40].
[94]	2008	APR	Additive	Calculates the priority vector by solving a linear system of equations.
[30]	2008	APR	Multiplicative	Minimises differences between the consensus case and the real case.
[18]	2008	APR	Multiplicative	Iterative procedure based on [40].
[17]	2009	APR	Additive	Comparative study between [25] and [40].
[98]	2009	IVPR	—	Dominance based rough set approach.
[1]	2009	2-tuple LPR	Additive	Iterative procedure based on [40].
[4]	2009	APR	Additive	Individual and social strategies for total ignorance situations.
[92]	2009	IVIFPR	Additive and Multiplicative	Arithmetic average and the geometric mean.
[70]	2009	IVIFPR	–	linear program for determining attribute weights.
[77]	2010	APR, MPR, LPR	Multiplicative	Minimises a collective deviation degree to improve the consensus and rank the alternatives.
[88]	2010	APR	Additive	Obtains the weighting vector by solving a linear system of equations.
[29]	2010	IVPR	Multiplicative	Estimates the missing values directly from the known ones.
[12]	2010	2-tuple LPR	Additive	Iterative procedure that proposes consistency measure to set experts' weight.
[59]	2010	2-tuple LPR	Additive	Estimates preferences in fuzzy linguistic recommender systems using [40].
[68]	2010	LPR (membership functions)	Additive	Directly estimates the missing preference using the known ones.
[88]	2010	APR, LPR	Additive	Estimates the attribute weights and ranks the options.
[42]	2011	LPR (virtual terms set)	Additive	Propose three ways of pairwise comparison.
[48]	2011	APR	Additive	Least square minimisation method.
[10]	2011	APR	Additive	Multi-criteria decision framework for sustainable supplier selection.
[23]	2011	APR	Multiplicative	Optimises dissimilarity function.
[93]	2011	IFPR	Multiplicative	Estimates directly the missing preferences from the known ones.
[11]	2012	APR	Additive	Aimed to quality function deployment.
[46]	2012	APR	Additive and order	Iterative completion approach.
[78]	2012	MPR	Multiplicative	Logarithmic least square method to rank the options.
[102]	2012	APR	Additive	Maximises the consistency level [40].
[90]	2012	MPR	Multiplicative transitivity	Geometrical mean
[74]	2013	IFPR	Multiplicative	Iterative process similar to [40].
[76]	2014	LPR	Multiplicative	Iterative

Table 2: Summary of the analysed contributions in chronological order

ISS2 The second strategy uses only the consensus preference values provided by those experts nearest to the expert whose PR is incomplete. This strategy is aimed to narrow the differences between the expert with an ignored alternative and those who have a similar opinion about the rest of alternatives.

ISS3 The third approach integrates the previous two by taking into account both information from the collective preference relation and from the nearest experts. This strategy encompasses the advantages of the previous two social strategies since the estimated information not only helps in the consensus process but also tries to keep a high consistency level in the individual experts' PR. Therefore it is considered by the authors of the proposal as the best strategy to deal with ignorance situations in GDM.

4.3. Advantages and drawbacks of ignorance strategies

In this section we will discuss the advantages and drawbacks of each one of the five strategies, and the situations where some of them may be more adequate to be applied than the others.

IIS1 This strategy improves the approach which considers ignorance equivalent to indifference because the initial indifference preference values associated to the unknown alternative is corrected, by means of the consistency property, when there is no indifference between some of the other alternatives. This approach is particularly useful when there are no external sources of information about the problem and when a high consistency level is required in the experts' preference relations.

IIS2 This strategy implies having some additional knowledge about the alternatives of the problem, and as such it is recommended to be use in decision problems where the alternatives to be evaluated share similar characteristics (similar models), which can be exploited to avoid ignorance situations in which an expert is not familiar with one of the alternatives, but has enough knowledge about a similar one.

ISS1 This strategy is appropriate for GDM problems because their resolution process usually requires the computation of the collective preference relation, and it could help to reach a consensus more easily because the unknown preferences are estimated from the collective ones. Additionally, the use of the estimation procedure assures that the loss of consistency will be minimized. Thus, this kind of approach could be useful in problems where a fast and converging consensus process is needed.

ISS2 This strategy also helps the consensus process to converge because an expert's unknown information is obtained from the nearest experts. However, this convergence is achieved in a different way with respect to the previous social strategy because the unknown information is estimated by using only the information of some of the experts. This strategy could prove useful in GDM problems in which the estimated information should be compatible with the information expressed by the expert, which is assured because it is obtained using the information of the nearest experts rather than the information from the whole group of experts.

ISS3 This strategy unifies all the advantages of the previous two social strategies. The estimated information will not only help in the consensus process to converge but also will try to maintain a high consistency level for the expert.

5. Trends and future work

According to the previous analysis we present some current trends on the estimation of information in GDM, along with some open questions and prospects about them. We identify three current trends:

1. Development of management procedures of incomplete preferences in the case of hesitant and type-2 PRs.
2. Development of comparison tools to evaluate and validate the different GDM approaches.
3. Managing incomplete information in Web 2.0 contexts.

5.1. Development of management procedures of incomplete preferences in the case of hesitant and type-2 PR.

Experts can perceive the provision of preferences for real decision making processes as complex because of the multiple alternatives and criteria that they need to be taken into account. Therefore it is natural that experts might present some degree of hesitancy in the expression of their preferences. To that aim the use of preference relations such us the IFPR and the IVPR are attracting the attention of many researchers in the last decade. Additionally, two very promising types of PRs are becoming recently widely used in decision making as well: (i) type-2 fuzzy PR [51] (ii) hesitant fuzzy PRs (HFPR) [66]

The concept of HFPR is captured in the following [91]:

Definition 20. *Let $X = \{x_1, s_2, \cdots, x_n\}$ be a fixed set, a HFPR H on X is presented by a matrix $H = (h_{ij})_{nxn} \subset X \times X$ where $h_{ij} = \{h_{ij}^s, s = 1, 2, \cdots, l_{h_{ij}}\}$ is a HFS indicating al the possible degrees to which x_i is preferred to x_j. Moreover, h_{ij} should satisfy the following conditions:*

$$h_{ij}^{\sigma(s)} + h_{ji}^{\sigma(l_{h_{ij}}-s)+1} = 1, h_{ii} = \{0.5\}, l_{h_{ij}} = l_{h_{ji}}, i, j = 1, 2, \cdots, n \quad (5)$$

The use of HFPRs in GDM have attracted the attention of many researchers in the last few years [91, 103]. However, as far as we are aware, there is no approach in the literature able to deal with incomplete HFPRs. A possible approach in these cases would be to extend existing validated approaches for the case of incomplete APRs, IVPRs, IFPRs and LPRs using the multiplicative consistency property of HFPRs introduced in [103] and the iterative procedure developed in [19]. However, it remains to be developed a formal and theoretical sound framework to support the validity of the methodology adopted in this area, which consists of the straightforward application of existing mathematical tools and procedure developed specifically for type-1 fuzzy preferences to hesitancy preferences. Without tackling this issue in the first place it could well be that some of the approaches already proposed in this area could well been proved to be incorrect, as it has been already the case of the modelling of multiplicative consistency in the case of intuitionistic preference relations put forward in [87] that has been subsequently proved to be incorrect by the author proposing it and others in [74]. A possible avenue to investigate to tackle this issue might reside in the similarities that exist between the definitions of hesitant fuzzy set and that of type-2 fuzzy set, which can lead to considering the first one as a particular type of the second one. In any case, type-2 PRs, i.e. preference relations whose elements are type-2 fuzzy sets, have not been the object of research regarding the estimation of missing type-2 fuzzy preference values in decision making. An explanation for this might reside in the complexity of type-2 computation. In any case, a possible approach to develop in this case would necessarily involve the decomposition of each type-2 fuzzy sets in its associated set of type-1 embedded sets to which type-1 fuzzy approaches are possible to be applied in conduction to the application of Zadeh's extension principle to obtain the type-2 fuzzy set output [21, 31–34].

5.2. Development of comparison tools to evaluate and validate decision making approaches with incomplete preferences

It is clear that there are many different decision making approaches to tackle incomplete information. However, it is also evident that there is a lack of a comparison framework available to evaluate their performance and consequently to help analyse the causes that might affect such performance. This shortage of comparison tools represent an important problem in the decision making field because decision making practitioners are unable to discriminate between the accuracy and the quality of the proposals available to them in the context of incomplete information. Thus, it seems imperative to develop methods to evaluate and validate the different techniques proposed in the literature to estimate the missing preferences. By doing this, it could be possible to compare in a quantitative way the existing GDM methodologies and find out which ones are more suitable depending on the problem to solve and to identify their main advantages and drawbacks.

Some initial efforts in the direction pointed above have been presented in [9] and in [15]. Brunelli et al. [9] conducted a comparative study of seven different methods for reconstructing incomplete fuzzy preference relations in terms of the consistency of the resulting complete preference relation; while Chiclana et al. [15] carried out a statistical comparative study to find out the differences in group consensus that different distance measures could lead to.

However the development of methods to evaluate the quality of the different GDM approaches with incomplete preferences is still in a very early stage and therefore there are many challenges that need to be addressed:

- To create a training and test framework with examples to allow benchmark tests to compare and validate different decision making approaches.
- To find proper metrics to compare different completion approaches.
- To develop software tools to carry out the evaluation and comparison of the different GDM approaches in the literature.

5.3. Managing incomplete information in the new Web 2.0 contexts.

Web 2.0 is the common term for advanced internet technologies and applications including social networks, blogs, wikis, RSS, podcasting and mashups. Web 2.0 content is user generated and it is characterised for the high degree of collaboration among internet users. As a result, these technologies provide an ideal framework to collaborate, negotiate, communicate, and interact while at the same time allowing their users to take advantage of values such as democratic participation, collaboration, collective intelligence and knowledge sharing on a massive scale beyond geographical barriers. All these values are extremely useful in social decision making processes [67]. Thus, it becomes necessary to adapt and develop new and appropriate decision making approaches for these new environments. In any case, it is of special importance to be aware that web 2.0 communities have some peculiarities, among which the most relevant to the efforts in developing tailored decision making models are [6]: (i) it constitutes a large and heterogeneous user base expressing opinions and preferences; and (ii) the low and intermittent participation rate.

Initial research proposals in this area can be found in [6, 59]. Alonso et al. [6] described a consensus approach for web 2.0 technologies, which includes a delegation feedback; while Porcel and Herrera-Viedma [59] developed a method to estimate users' preferences in fuzzy linguistic recommender system. However, due to the inherent characteristics of the Web 2.0 communities in many occasions the information about users' preferences is scarce or incomplete. These situations provide a good opportunity to implement incomplete preference management procedures in web 2.0 context: (1) to extend some of the previous incomplete approaches here reviewed; and (2) to develop new mechanisms to estimate missing information based on new information inherent to web 2.0 context such as trust degree, reputation or new techniques based on social networks analysis [73].

6. Conclusions

In decision making, situations where all experts are able to efficiently express their preferences over all the available options are the exception rather than the rule. Indeed, the above scenario requires from all experts to possess a precise or sufficient level of knowledge of the whole problem to tackle, including the ability to discriminate the degree up to which some options are better than others, which can obviously seen as unrealistic in many decision making situations, especially those involving a considerable large number of alternatives to choose from and/or conflicting and dynamic sources of information.

In this paper we have reviewed and analysed the state-of-the-art research efforts on group decision making from the perspective of the estimation of missing preferences using different types of preference relations. We have presented the foundations and developments in that field along with the most relevant computational models that have been applied to the decision making context: APR, MPR, IFPR, IVPR and LPR. These estimation techniques mainly use the additive or the multiplicative consistency properties to calculate the missing preferences from the known ones, as well as increasing the global consistency level and in many cases the experts' consensus. They can be widely classified in two main groups: (i) iterative procedures, and (ii) optimisation procedures. A comprehensive list of the most recent developed applications in the specialised literature has been presented. Finally, several current trends and prospects about the topic have been introduced.

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2. New group decision making approaches under highly uncertain contexts

2.1. Confidence-consistency driven group decision making approach in uncertainty environments

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Confidence-consistency driven group decision making approach with incomplete reciprocal intuitionistic preference relations

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Abstract

Intuitionistic preference relations constitute a flexible and simple representation format of experts' preference on a set of alternative options, while at the same time allowing to accommodate degrees of hesitation inherent to all decision making processes. In comparison with fuzzy preference relations, the use of intuitionistic fuzzy preference relations in decision making is limited, which is mainly due to the computational complexity associated to using membership degree, non-membership degree and hesitation degree to model experts' subjective preferences. In this paper, the set of reciprocal intuitionistic fuzzy preference relations and the set of asymmetric fuzzy preference relations are proved to be mathematically isomorphic. This result can be exploited to use methodologies developed for fuzzy preference relations to the case of intuitionistic fuzzy preference relations and, ultimately, to overcome the computation complexity mentioned above and to extend the use of reciprocal intuitionistic fuzzy preference relations in decision making. In particular, in this paper, this isomorphic equivalence is used to address the presence of incomplete reciprocal intuitionistic fuzzy preference relations in decision making by developing a consistency driven estimation procedure via the corresponding equivalent incomplete asymmetric fuzzy preference relation procedure. Additionally, the hesitancy degree of the reciprocal intuitionistic fuzzy preference relation is used to introduce the concept of expert's confidence from which a group decision making procedure, based on a new aggregation operator that takes into account not only the experts' consistency but also their confidence degree towards the opinion provided, is proposed.

Keywords: Group decision making, Uncertainty, Incomplete information, Intuitionistic fuzzy preference relations, Asymmetric fuzzy preference relations, Uninorm.

1. Introduction

Intuitionistic fuzzy preference relations are based on the concept of intuitionistic fuzzy set that Atanassov introduced in [3] as an extension of the concept of fuzzy set. Due to its flexibility in handling vagueness/uncertainty, intuitionistic fuzzy set theory [4] has been extensively used in many areas, such as virtual medical diagnosis [11], pattern recognition [26], clustering analysis [30] and decision making [23, 27–29]. For example in [10], Fujita et al. propose to model the user cognitive behaviour on mental cloning-based software using intuitionistic fuzzy sets.

Much research has been carried out in decision making with preferences modelled using fuzzy relations in comparison to using intuitionistic fuzzy relations. This is mainly to the longer existence of the former representation format of preferences in comparison to the second

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one. However, an additional cause for the lesser use of intuitionistic fuzzy preference relations in decision making is the increase computational complexity associated to the use of membership degree, non-membership degree and hesitation degree to model experts' subjective preferences. Notice that intuitionistic fuzzy preference relations are usually assumed to be reciprocal (Section 2).

A first objective of this paper is to prove the mathematical equivalence between the set of reciprocal intuitionistic fuzzy preference relations and the set of asymmetric fuzzy preference relations. This result can thus be exploited to use methodologies developed for fuzzy preference relations to the case of intuitionistic fuzzy preference relations and, ultimately, to extend the use of reciprocal intuitionistic fuzzy preference relations in decision making and to overcome the computation complexity mentioned above. In other word, this result will allow to take advantage of mature and well defined methodologies developed for fuzzy preference relations while leveraging the flexibility of reciprocal intuitionistic fuzzy preference relations to model vagueness/uncertainty. Indeed, an issue that can be addressed using the mentioned equivalence is the presence of incomplete reciprocal intuitionistic fuzzy preference relations in decision making.

Incomplete information as a result from the incapability of experts to provide complete information about their preferences [7, 14] may happen more frequently than expected due to different reasons such as: experts not having a precise or sufficient level of knowledge of part of the problem, lack of time, difficulty to distinguish up to which degree one preference is better than other, or conflicting between alternatives, among others. In the literature, different approaches to deal with missing or incomplete information have been extensively studied for the case of using fuzzy preference relations as the representation format of preferences [25]. Most of the existing approaches are based on the selection of an appropriate methodology to 'build' the matrix, and/or to assign importance values to experts based not on the amount of information provided but on how consistent the information provided is [1, 8, 14, 18, 28].

The case of incomplete intuitionistic fuzzy preference relations has been addressed in literature in [28, 29], where the above mentioned methodology to estimate missing information driven by the consistency was adopted. The main difference between both approaches resides in the way consistency of reciprocal intuitionistic fuzzy preference relations was modelled. On the one hand, in [29] a straight forward transposition of the multiplicative consistency property for fuzzy preference relations was proposed for the case of reciprocal intuitionistic fuzzy preference relations, which has been later proved to be incorrect [28], and publicly acknowledged by the authors that proposed it [31]. On the other hand, in [28] the concept of multiplicative consistency for reciprocal intuitionistic fuzzy preference relations was derived by formally extending the multiplicative transitivity property for fuzzy preference relations via the use of both the *Extension Principle* and *Representation Theorem* [35]. In this contribution, though, a different approach to incomplete reciprocal intuitionistic fuzzy preference relations is presented based on the aforementioned equivalence between the set of reciprocal intuitionistic fuzzy preference relations and the set of asymmetric fuzzy preference relations. The main advantage of the approach put forward here is that the isomorphic relation between reciprocal intuitionistic fuzzy preference relations and asymmetric fuzzy preference relations makes superfluous both the extension principle and the representation theorem that were required in [28], as well as being less computationally complex because there is no need to split the reciprocal intuitionistic fuzzy preference relations into two reciprocal fuzzy preference relations but one single asymmetric fuzzy preference relation.

A second objective of this paper is to develop a fuse approach of the information provided by the experts taking into account the confidence level of each expert in his/her own opinion, which is intrinsically connected to the information he/she provides [12], and which in the case of reciprocal intuitionistic fuzzy preference relations is linked to the associated hesitancy function. Obviously, the more confident the expert feels about his/her opinion the more relevant the opinion can be considered, and thus more importance should be allocated to it. This can be achieved in the aggregation phase of a group decision making model by implementing an appropriate confidence and consistency based induced ordered weighted average to compute the collective preferences [6, 14, 28].

The rest of the paper is set out as follows: Section 2 presents the main mathematical frameworks for representing preferences of interest, while Section 3 deals with the concept of consistency of fuzzy preference relations as needed throughout the rest of the paper. Section 4 demonstrates the mathematical equivalence between the set of reciprocal intuitionistic fuzzy preference relations and the set of asymmetric fuzzy preference relations, which is used in Section 5 to present a methodology to estimate missing values of reciprocal intuitionistic fuzzy preference relations. The hesitancy function is proposed in a confidence-consistency driven group decision making approach with incomplete reciprocal intuitionistic fuzzy preference relations whose application is illustrated with an example in in Section 6,. Finally, Section 7 includes an analysis of the proposed group decision making model, including some future work and draws conclusions.

2. Preference Relations in Decision Making

In any decision making problem, once the set of feasible alternatives (X) is identified, experts are called to express their opinions or preferences on such set. Different preference elicitation methods were compared in [19], concluding that pairwise comparison methods are more accurate than non-pairwise methods since it allows the expert to focus only in two alternatives at a time. A comparison of two alternatives of X by an expert can lead to the preference of one alternative to the other or to a state of indifference between them. Obviously, there is the possibility of an expert being unable to compare them. Two main mathematical models based on the concept of preference relation can be used in this context. In the first one, a preference relation is defined for each one of the above three possible preference states (preference, indifference, incomparability), which is known as a preference structure on the set of alternatives. The second one integrates the three possible preference states into a single preference relation. In this paper, we focus on the second one as per the following definition:

Definition 1 (Preference Relation)**.** *A preference relation P on a set X is a binary relation $\mu_P : X \times X \to D$, where D is the domain of representation of preference degrees provided by the decision maker.*

A preference relation P may be conveniently represented by a matrix $P = (p_{ij})$ of dimension $card(X)$, with $p_{ij} = \mu_P(x_i, x_j)$ being interpreted as the degree or intensity of preference of alternative x_i over x_j. The elements of P can be of a numeric or linguistic nature, i.e., could represent numeric or linguistic preferences, respectively [20]. The main types of numeric preference relations used in decision making are: crisp preference relations, additive preference relations, multiplicative preference relations, interval-valued preference relations and intuitionistic preference relations. In this contribution we are going to focus on the reciprocal intuitionistic fuzzy preference relations and their equivalence to a subclass of asymmetric fuzzy preference relations.

2.1. Fuzzy Set and Fuzzy Preference Relation

Definition 2 (Fuzzy Set)**.** *Let U be a universal set defined in a specific problem, with a generic element denoted by x. A fuzzy set X in U is a set of ordered pairs:*

$$X = \left\{(x, \mu_X(x))|x \in U\right\}$$

where $\mu_X\colon U \to [0,1]$ is called the membership function of A and $\mu_X(x)$ represents the degree of membership of the element x in X.

Notice that the degree of non-membership of the element x in X is here defined as $\nu_X(x) = 1 - \mu_X(x)$. Thus, $\mu_X(x) + \nu_X(x) = 1$.

Definition 3 (Fuzzy Preference Relation)**.** *A fuzzy preference relation $R = (r_{ij})$ on a finite set of alternatives X is a fuzzy relation in $X \times X$ that is characterised by a membership function $\mu_R : X \times X \longrightarrow [0,1]$ with the following interpretation:*

- *$r_{ij} = 1$ indicates the maximum degree of preference for x_i over x_j*
- *$r_{ij} \in]0.5, 1[$ indicates a definite preference for x_i over x_j*
- *$r_{ij} = 1/2$ indicates indifference between x_i and x_j*

When

$$r_{ij} + r_{ji} = 1 \ \forall i, j \in \{1, \ldots, n\}$$

is imposed we have a reciprocal fuzzy preference relation.

2.2. Intuitionistic fuzzy set and Intuitionistic fuzzy preference relation

Definition 4 (Intuitionistic Fuzzy Set)**.** *An intuitionistic fuzzy set X over a universe of discourse U is given by*

$$X = \Big\{ \big(x, \langle \mu_X(x), \nu_X(x) \rangle\big) \big| x \in U \Big\}$$

where $\mu_X\colon U \to [0,1]$,and $\nu_X\colon U \to [0,1]$ verify

$$0 \leq \mu_X(x) + \nu_X(x) \leq 1 \quad \forall x \in U.$$

$\mu_X(x)$ and $\nu_X(x)$ degree of membership and degree of non-membership of x to X.

An intuitionistic fuzzy set becomes a fuzzy set when $\mu_X(x) = 1 - \nu_X(x)\ \forall x \in U$. However, when there exists at least a value $x \in U$ such that $\mu_X(x) < 1 - \nu_X(x)$, an extra parameter has to be taken into account when working with intuitionistic fuzzy sets: the hesitancy degree, $\tau_X(x) = 1 - \mu_X(x) - \nu_X(x)$, that represents the amount of lacking information in determining the membership of x to X. If the hesitation degree is zero, the reciprocal relationship between membership and non-membership makes the latter one unnecessary in the formulation as it can be derived from the former.

In [23], Szmidt and Kacprzyk defined the intuitionistic fuzzy preference relation as a generalisation of the concept of fuzzy preference relation, which is adopted in the following defintion.

Definition 5 (Intuitionistic Fuzzy Preference Relation)**.** *An intuitionistic fuzzy preference relation B on a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ is characterised by a membership function $\mu_B\colon X \times X \to [0,1]$ and a non-membership function $\nu_B\colon X \times X \to [0,1]$ such that*

$$0 \leq \mu_B(x_i, x_j) + \nu_B(x_i, x_j) \leq 1 \ \ \forall (x_i, x_j) \in X \times X.$$

with $\mu_B(x_i, x_j) = \mu_{ij}$ interpreted as the certainty degree up to which x_i is preferred to x_j; and $\nu_B(x_i, x_j) = \nu_{ij}$ interpreted as the certainty degree up to which x_i is non-preferred to x_j.

An intuitionistic fuzzy preference relation can also be represented by a matrix $B = (b_{ij})$ with $b_{ij} = (\mu_{ij}, \nu_{ij})$ $\forall i, j = 1, 2, \ldots, n$. Notice that when the hesitancy function is the null function we have that $\mu_{ij} + \nu_{ij} = 1 \quad \forall i, j$, and therefore the intuitionistic fuzzy preference relation $B = (b_{ij})$ is mathematically equivalent to the reciprocal fuzzy preference relation (μ_{ij}), i.e. $B = (\mu_{ij})$. An intuitionistic fuzzy preference relation is referred to as reciprocal when the following additional conditions are imposed:

- $\mu_{ii} = \nu_{ii} = 0.5$ $\forall i \in \{1, \ldots, n\}$.
- $\mu_{ji} = \nu_{ij} \forall i, j \in \{1, \ldots, n\}$.

Notice that when the hesitancy degree function is the null function in a reciprocal intuitionistic fuzzy preference relation, $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$, then it is $\mu_{ij} + \nu_{ij} = 1$ $\forall i, j$, and B is mathematically equivalent to the reciprocal fuzzy preference relation $R = (r_{ij}) = (\mu_{ij})$.

3. Consistency of fuzzy preference relations

Consistency of fuzzy preference relations has been modeled using the notion of transitivity in the pairwise comparison among any three alternatives: if x_i is preferred to x_j ($x_i \succ x_j$) and this one to x_k ($x_j \succ x_k$) then alternative x_i should be preferred to x_k ($x_i \succ x_k$), which is normally referred to as *weak stochastic transitivity* [5]. Any property that guarantees the transitivity of the preferences is called a consistency property. Clearly, the lack of consistency in decision making can lead to inconsistent conclusions; that is why it is crucial to study conditions under which consistency is satisfied [21].

Different properties or conditions have been suggested as rational conditions to be verified by a consistent fuzzy preference relation [5, 15]: triangle condition, weak transitivity, max-min transitivity, max-max transitivity, restricted max-min transitivity, restricted max-max transitivity, additive transitivity, and multiplicative transitivity. The last two properties, proposed by Tanino in [24], are the most widely used in the context of incomplete information [5].

Definition 6 (Additive transitivity [24]). *A fuzzy preference relation $R = (r_{ij})$ on a finite set of alternatives X is additive transitive if and only if*

$$(r_{ij} - 0.5) + (r_{jk} - 0.5) = r_{ik} - 0.5 \quad \forall i, j, k = 1, 2, \cdots, n$$

Although equivalent to Saaty's consistency property for multiplicative preference relations [15], additive transitivity is in conflict with the $[0, 1]$ scale used for providing the preference values and therefore it is not appropriate to model consistency of fuzzy preference relations [5]. Tanino also proposed the following alternative transitivity property:

Definition 7 (Multiplicative transitivity [24]). *A fuzzy preference relation $R = (r_{ij})$ on a finite set of alternatives X is multiplicative transitive if and only if*

$$r_{ij} \cdot r_{jk} \cdot r_{ki} = r_{ik} \cdot r_{kj} \cdot r_{ji} \quad \forall i, k, j \in \{1, 2, \ldots n\} \tag{1}$$

Chiclana et al. in [5] propose the modeling of the cardinal consistency of reciprocal fuzzy preference relations via a functional equation, and they proved that when such a function is almost continuous and monotonic (increasing) then it must be a representable uninorm. Cardinal consistency with the conjunctive representable cross ratio uninorm

$$U(x, y) = \begin{cases} 0, & (x, y) \in \{(0, 1), (1, 0)\} \\ \dfrac{xy}{xy + (1 - x)(1 - y)}, & \text{otherwise} \end{cases} \tag{2}$$

was proved equivalent to Tanino's multiplicative transitivity property, and because any two representable uninorms are order isomorphic, it was concluded that multiplicative transitivity is the most appropriate property to model consistency of reciprocal fuzzy preference relations. It is worth reminding that multiplicative transitivity property extends weak stochastic transitivity, and therefore extends the classical transitivity property of crisp preference relations. This is why we refer to this property as the *multiplicative consistency* property.

Multiplicative consistency property (1) can be used to estimate the preference value between a pair of alternatives (x_i, x_j) with $(i < j)$ using another different intermediate alternative x_k $(k \neq i, j)$ as follows:

$$mr_{ij}^{k} = \frac{r_{ik} \cdot r_{kj} \cdot r_{ji}}{r_{jk} \cdot r_{ki}} \tag{3}$$

as long as the denominator is not zero. We call mr_{ij}^{k} the partially multiplicative transitivity based estimated fuzzy preference value of the pair of alternatives (x_i, x_j) obtained using the intermediate alternative x_k.

Notice that expression (1) is always true when two of the three subindexes are equal. Let $k = i$ and $r_{ji} \neq 0$ then $mr_{ij}^{i} = r_{ij}$, while if $r_{ij} \neq 0$ then $mr_{ji}^{i} = r_{ji}$. Because $r_{ji} = 1 - r_{ij}$, then we have that: $r_{ji} \neq 0$ if and only if $r_{ji} \neq 1$. Thus, if $k = i$ and $(r_{ij}, r_{ji}) \notin \{(1,0),(0,1)\}$ we have $mr_{ij}^{i} = r_{ij}$ and $mr_{ji}^{i} = r_{ji}$. A similar reasoning and conclusion is obtained when $k = j$. Summarising, although it is possible to obtain the multiplicative transitivity based estimated fuzzy preference value of the pair of alternatives (x_i, x_j) when $k \in \{i, j\}$ and $(r_{ij}, r_{ji}) \notin \{(1,0),(0,1)\}$, it is also true that there is no indirect estimation process as described above. Furthermore, when the fuzzy preference value r_{ij} is unknown its estimation will automatically require that $k \notin \{i, j\}$. Finally, when $i = j$ we have by definition that $r_{ii} = 0.5$ and we would have $mr_{ii}^{k} = r_{ii}$ whenever $r_{ik} \notin \{(0,1), (1,0)\}$. Thus, this last case will not be relevant when having incomplete information, and it is also not assumed from now on.

The average of all possible partially multiplicative transitivity based estimated values of the pair of alternatives (x_i, x_j) can be interpreted as their global multiplicative transitivity based estimated value

$$mr_{ij} = \frac{\sum_{k \in R_{ij}^{01}} mr_{ij}^{k}}{\# R_{ij}^{01}};$$

where $R_{ij}^{01} = \{k \neq i, j | (r_{ik}, r_{kj}) \notin R^{01}\}$, $R^{01} = \{(1,0),(0,1)\}$, and $\# R_{ij}^{01}$ is the cardinality of R_{ij}^{01}. Therefore, given a fuzzy preference relation, $R = (r_{ij})$, the following multiplicative transitivity based fuzzy preference relation, $MR = (mr_{ij})$, can be constructed. Notice that when a fuzzy preference relation $R = (r_{ij})$ is multiplicative transitive then $R = MR$. Indeed, if R is multiplicative transitive then (1) holds $\forall i, j, k$. In particular, we have

$$r_{ij} = \frac{r_{ik} \cdot r_{kj} \cdot r_{ji}}{r_{jk} \cdot r_{ki}};$$

whenever $k \in R_{ij}^{01}$. Consequently, $mr_{ij}^{k} = r_{ij}$ for all i, j and $k \in R_{ij}^{01}$, which proves that $r_{ij} = mr_{ij}$ for all i, j. A fuzzy preference relation R will be referred to as multiplicative consistent from now on when $R = MR$.

Definition 8 (Multiplicative Consistency)**.** *A fuzzy preference relation $R = (r_{ij})$ is multiplicative consistent if and only if $R = MR$.*

The similarity between the values r_{ij} and mr_{ij} is proposed to be used in measuring the level of consistency of a fuzzy preference relation at its three different levels: pair of alternatives, alternatives and relation [14]:

Level 1. *Consistency Index of pair of alternatives.*

$$CL_{ij} = 1 - d(r_{ij}, mr_{ij}) \ \ \forall i, j.$$

Here $d(r_{ij}, mr_{ij})$ represents the distance between the values r_{ij} and mr_{ij}. Obviously, the higher the value of CL_{ij} the more consistent is r_{ij} with respect to the rest of the preference values involving alternatives x_i (row i of the fuzzy preference relation) and x_j (column j of the fuzzy preference relation).

Level 2. *Consistency Level of alternatives.*

$$CL_i = \frac{\sum_{j=1;\ i \neq j}^{n} CL_{ij}}{n-1}.$$

Level 3. *Consistency Level of a fuzzy preference relation.*

$$CL = \frac{\sum_{i=1}^{n} CI_i}{n}.$$

The following result characterises multiplicative consistency of a fuzzy preference relation using its corresponding consistency level.

Proposition 1. *A fuzzy preference relation R is multiplicative consistent if and only if $CL = 1$.*

Proof.

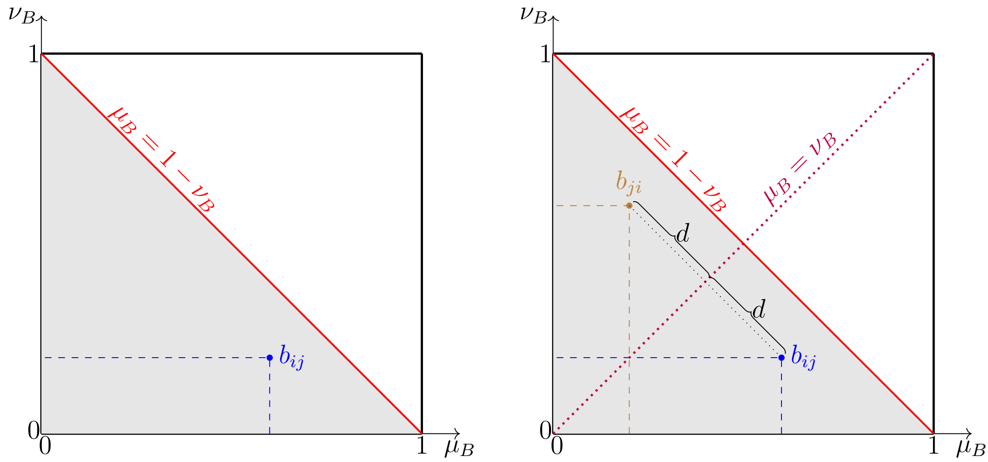
1. *R is multiplicative consistent $\Longrightarrow CL = 1$.* Because R is assumed to be multiplicative consistent the Definition 8 applies and we have that $R = MR$, which means that $d(r_{ij}, mr_{ij}) = 0 \ \forall i, j$. Consequently, $CL = 1$.

2. *$CL = 1 \Longrightarrow R$ is multiplicative consistent.* If $CL = 1$ then $\sum_{i,j=1, i\neq j}^{n} CL_{ij} = n \times (n-1)$. We have that $CL_{ij} \in [0, 1]$ and therefore it is $CL_{ij} = 1 \, \forall i \neq j$ otherwise it would be $\sum_{i,j=1, i\neq j}^{n} CL_{ij} < n \times (n-1)$. Therefore we have that $CL = 1$ if and only if $r_{ij} = mr_{ij} \ \forall i \neq j$. Finally, when $i = j$ we have $mr_{ii}^{k} = r_{ii} = 0.5$ whenever $r_{ik} \notin \{(0, 1), (1, 0)\}$, and therefore $mr_{ii} = 0.5$. Thus, we have that $r_{ij} = mr_{ij} \ \forall i, j$, i.e. $R = MR$, and conclude that R is multiplicative consistent.

□

4. Reciprocal intuitionistic fuzzy preference relations and asymmetric fuzzy preference relations

This section proves the equivalence between the set of asymmetric fuzzy preference relations and the set of reciprocal intuitionistic fuzzy preference relations, and will provide the isomorphism to derive an asymmetric fuzzy preference relation given a reciprocal intuitionistic fuzzy preference relation. This result is exploited Section 5 to tackle the presence of incomplete reciprocal intuitionistic fuzzy preference relations in decision making by developing a consistency driven estimation procedure via the corresponding equivalent incomplete asymmetric fuzzy preference relation.



(a) Intuitionistic fuzzy preference value

(b) Reciprocal intuitionistic fuzzy preference values

Figure 1: Representation of intuitionistic fuzzy preference relations $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$

Given an intuitionistic fuzzy preference relation $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$, the constraint $0 \leq \mu_{ij} + \nu_{ij} \leq 1$ implies that an element of B can be represented by a point in the lower half unit square area as shaded in Fig. 1(a). Notice that in the case of being the intuitionistic fuzzy preference relation B reciprocal, given the element $\langle \mu_{ij}, \nu_{ij} \rangle$ then we have that $\langle \mu_{ji}, \nu_{ji} \rangle = \langle \nu_{ij}, \mu_{ij} \rangle$, i.e. $\langle \mu_{ji}, \nu_{ji} \rangle$ is the mirror image of $\langle \mu_{ij}, \nu_{ij} \rangle$ with respect to the line $\mu_B = \nu_B$ as illustrated in Fig. 1(b). Consequently, a reciprocal intuitionistic fuzzy preference relation

$$B = \begin{pmatrix} \langle \mu_{11}, \nu_{11} \rangle & \langle \mu_{12}, \nu_{12} \rangle & \cdots & \langle \mu_{1i}, \nu_{1i} \rangle & \cdots & \langle \mu_{1n}, \nu_{1n} \rangle \\ \langle \mu_{21}, \nu_{21} \rangle & \langle \mu_{22}, \nu_{22} \rangle & \cdots & \langle \mu_{1i}, \nu_{2i} \rangle & \cdots & \langle \mu_{2n}, \nu_{2n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle \mu_{i1}, \nu_{i1} \rangle & \langle \mu_{i2}, \nu_{i2} \rangle & \cdots & \langle \mu_{ii}, \nu_{ii} \rangle & \cdots & \langle \mu_{in}, \nu_{in} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle \mu_{n1}, \nu_{n1} \rangle & \langle \mu_{n2}, \nu_{n2} \rangle & \cdots & \langle \mu_{ni}, \nu_{ni} \rangle & \cdots & \langle \mu_{nn}, \nu_{nn} \rangle \end{pmatrix}$$

can be completely characterised using just its upper triangular part

$$UB = \begin{pmatrix} \langle \mu_{11}, \nu_{11} \rangle & \langle \mu_{12}, \nu_{12} \rangle & \cdots & \langle \mu_{1i}, \nu_{1i} \rangle & \cdots & \langle \mu_{1n}, \nu_{1n} \rangle \\ & \langle \mu_{22}, \nu_{22} \rangle & \cdots & \langle \mu_{1i}, \nu_{2i} \rangle & \cdots & \langle \mu_{2n}, \nu_{2n} \rangle \\ & & \ddots & \vdots & \ddots & \vdots \\ & & & \langle \mu_{ii}, \nu_{ii} \rangle & \cdots & \langle \mu_{in}, \nu_{in} \rangle \\ & & & & \ddots & \vdots \\ & & & & & \langle \mu_{nn}, \nu_{nn} \rangle \end{pmatrix}$$

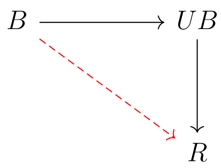
and this one can be represented equivalently as the following fuzzy preference relation

$$R = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1i} & \cdots & \mu_{1n} \\ \nu_{12} & \mu_{22} & \cdots & \mu_{1i} & \cdots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \nu_{1i} & \nu_{2i} & \cdots & \mu_{ii} & \cdots & \mu_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \nu_{1n} & \nu_{2n} & \cdots & \nu_{in} & \cdots & \mu_{nn} \end{pmatrix}.$$

Because $\nu_{ij} = \mu_{ji}$ then we have that

$$R = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1i} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{1i} & \cdots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mu_{i1} & \mu_{i2} & \cdots & \mu_{ii} & \cdots & \mu_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{ni} & \cdots & \mu_{nn} \end{pmatrix}.$$

This is illustrated in the following diagram:



In the following, we formalise the above relationship. Let denote with $\mathcal{B}$ the set of reciprocal intuitionistic fuzzy preference relations:

$$\mathcal{B} = \left\{ B = (b_{ij}) | \forall ij: \ b_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle \, , \ \mu_{ij}, \nu_{ij} \in [0,1], \ \mu_{ii} = \nu_{ii} = 0.5 \, \mu_{ij} = \nu_{ji}, \ 0 \leq \mu_{ij} + \nu_{ij} \leq 1 \right\}$$

and with $\mathcal{R}$ the set of fuzzy preference relations

$$\mathcal{R} = \left\{ R = (r_{ij}) | \forall ij: \ r_{ij} \in [0,1] \right\}$$

Let $f\colon [0,1] \times [0,1] \longrightarrow [0,1]$ be the following function $f(x_1, x_2) = x_1$. We can define the following mapping, $F\colon \mathcal{B} \longrightarrow \mathcal{R}$, between the set of reciprocal intuitionistic fuzzy preference relations, $\mathcal{B}$, and the set of fuzzy preference relations, $\mathcal{R}$,

$$R = F(B) = (f(b_{ij})) = (\mu_{ij}).$$

The following properties can be proved:

Proposition 2. *Function F is well defined, i.e. given $B \in \mathcal{B}$ it is true that $f(B) \in \mathcal{R}$.*

Proof. It is obvious and it is omitted. □

Proposition 3. *Function F is an injection.*

Proof. Let $B_1 = (b^1_{ij})$ and $B_2 = (b^2_{ij})$ two reciprocal intuitionistic fuzzy preference relation such that $F(B_1) = F(B_2)$. Then we have that

$$f(b^1_{ij}) = f(b^2_{ij}) \Leftrightarrow \mu^1_{ij} = \mu^2_{ij} \ \forall i, j.$$

Because $\mu^1_{ij} = \nu^1_{ji}$ and $\mu^2_{ij} = \nu^2_{ji}$ then it is obvious that

$$\nu^1_{ij} = \nu^2_{ij} \ \forall i, j.$$

Therefore we have that

$$b^1_{ij} = \langle \mu^1_{ij}, \nu^1_{ij} \rangle = \langle \mu^2_{ij}, \nu^2_{ij} \rangle = b^2_{ij} \ \forall i, j.$$

Consequently, it is concluded that

$$B_1 = B_2.$$

□

For function F to be a surjection, the following needs to be verified:

$$\forall R \in \mathcal{R}\ \exists B \in \mathcal{B}:\ F(B) = R.$$

However, by definition of $\mathcal{B}$ and F we have that $R = (r_{ij}) = (\mu_{ij})$ verifies:

$$0 \leq r_{ij} + r_{ji} = \mu_{ij} + \mu_{ji} = \mu_{ij} + \nu_{ij} \leq 1.$$

Thus R is an asymmetric fuzzy preference relation [9]. This proves that the range of function F is the subset of fuzzy preference relations that are asymmetric. In other words:

Proposition 4. *The range of function function F is the set of asymmetric fuzzy preference relations, i.e. function F is not a surjection.*

Putting together these results we have:

Theorem 1. *The set of reciprocal intuitionistic fuzzy preference relations is isomorphic to the set of asymmetric fuzzy preference relations.*

Asymmetric fuzzy preference relations guarantee that when $p_{ij} \geq 0.5$ then $p_{ji} \leq 0.5$. In preference modelling it guarantees that both p_{ij} and p_{ji} cannot be high at the same time. In other words, an asymmetric fuzzy preference relation guarantees that when an alternative x is preferred to another alternative y, then alternative y is not preferred to alternative x.

To conclude this section, we notice that when $B \in \mathcal{B}$ has hesitancy degree always zero then we have that:

$$\mu_{ij} + \nu_{ij} = 1\ \forall i, j. \tag{4}$$

In these cases, $F(B) = R$ is also reciprocal, i.e. $r_{ij} + r_{ji} = 1 \forall i, j$. The proof of this is quite simple as we have the following:

$$\forall i, j:\ r_{ij} = f(b_{ij}) = \mu_{ij}\ \wedge\ r_{ji} = f(b_{ji}) = \mu_{ji}.$$

However, because B is reciprocal then we have that $\mu_{ji} = \nu_{ij}\ \forall i, j$ and consequently applying (4) it is:

$$r_{ij} + r_{ji} = \mu_{ij} + \mu_{ji} = \mu_{ij} + \nu_{ij} = 1\ \forall i, j.$$

Thus, as previously mentioned in Section 2 the subset of reciprocal fuzzy preference relations $\{B \in \mathcal{B} | \mu_{ij} + \nu_{ij} = 1\ \forall i, j\}$, highlighted in red in Fig. 1, is invariant under function F, i.e. function F is the identity function when reduced to the subset of reciprocal fuzzy preference relations.

5. Incomplete reciprocal intuitionistic fuzzy preference relations

The previous section main result allows to solve problems associated to reciprocal intuitionistic fuzzy preference relations by solving the corresponding problem to their equivalent asymmetric fuzzy preference relations. Thus, the presence of incomplete reciprocal intuitionistic fuzzy preference relations in decision making can be tackled by using the estimation procedure presented in Section 3 to the corresponding equivalent incomplete asymmetric fuzzy preference relations. Before doing this, we present the formal definition of the concept of incomplete preference relation [14]:

Definition 9. *A function $g : X \longrightarrow Y$ is partial when not every element in the set X necessarily maps to an element in the set Y. When every element from the set X maps to one element of the set Y then we have a total function.*

Definition 10. *A preference relation on a set of alternatives with a* partial *membership function is an* incomplete preference relation.

It is assumed that for incomplete reciprocal intuitionistic fuzzy preference relations, given a pair of alternatives (x_i, x_j) for which b_{ij} is not known, both membership and non-memberships will be unknown. Due to reciprocity, we have that if b_{ij} is not known then b_{ji} is also not known. In general the letter x will be used when a particular entry of an incomplete reciprocal intuitionistic fuzzy preference relation is unknown/missing. Thus, if B is an incomplete reciprocal intuitionistic fuzzy preference relation, then $R = F(B)$ will be an incomplete asymmetric fuzzy preference relation. However, the missing preference value r_{ij} $(i \neq j)$ cannot be partially estimated, using an intermediate alternative x_k, via expression (1) because r_{ij} is also unknown. As we have already mentioned, under reciprocity, Tanino's multiplicative transitivity property (1) cab be rewritten as (2). Thus the missing preference value $r_{ij}(i \neq j)$ can be partially estimated, using an intermediate alternative x_k, with the value:

$$cr_{ij}^k = \begin{cases} 0, & (r_{ik}, r_{kj}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ik} \cdot r_{kj}}{r_{ik} \cdot r_{kj} + (1 - r_{ik}) \cdot (1 - r_{kj})}, & \text{Otherwise.} \end{cases}$$

The following notation is introduced:

$$\begin{aligned} A &= \{(i,j) \mid i,j \in \{1,\ldots,n\} \wedge i \neq j\}\,; \\ MV &= \{(i,j) \mid r_{ij} \text{ is unknown, } (i,j) \in A\}\,; \\ EV &= A \setminus MV. \end{aligned}$$

MV is the set of pairs of different alternatives for which the fuzzy preference degree is unknown or missing; EV is the set of pairs of different alternatives with known fuzzy preference values. The global multiplicative transitivity based estimated value, cr_{ij}, is defined as follows:

$$cr_{ij} = \frac{\sum\limits_{k \in R_{ij}^{01}} cr_{ij}^k}{\# R_{ij}^{01}}$$

where $H_{ij}^{01} = \{k \in R_{ij}^{01} | (i,j) \in MV \ \wedge \ (i,k),(k,j) \in EV\}$.

The iterative procedure to complete reciprocal fuzzy preference relations developed in [14] can be applied to complete R and, consequently, to complete the incomplete reciprocal intuitionistic fuzzy preference relation B as the following example illustrates. Notice that the cases when an incomplete fuzzy preference relation cannot be successfully completed are reduced to those when no preference values involving a particular alternative are known. This is the case when a whole row or column is completely missing. Therefore the general sufficient condition for an incomplete fuzzy preference relation to be completed is that a set of $n-1$ non-leading diagonal preference values with each one of the alternatives compared at least once is known.

Example 1. *Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of alternatives evaluated by a decision maker against a particular criterion using the following incomplete reciprocal intuitionistic fuzzy preference relation [28]:*

$$B = \begin{pmatrix} \langle 0.50, 0.50\rangle & \langle 0.40, 0.30\rangle & x & x \\ \langle 0.30, 0.40\rangle & \langle 0.50, 0.50\rangle & \langle 0.50, 0.40\rangle & x \\ x & \langle 0.40, 0.50\rangle & \langle 0.50, 0.50\rangle & \langle 0.30, 0.40\rangle \\ x & x & \langle 0.40, 0.30\rangle & \langle 0.50, 0.50\rangle \end{pmatrix}$$

The associated incomplete asymmetric fuzzy preference relation is:

$$R = \begin{pmatrix} 0.5 & 0.4 & - & - \\ 0.3 & 0.5 & 0.5 & - \\ - & 0.4 & 0.5 & 0.3 \\ - & - & 0.4 & 0.5 \end{pmatrix}$$

Step 1: *The set of elements that can be estimated at this stage are:*

$$EMV_1 = \{(1,3),(2,4),(3,1),(4,2)\}.$$

Notice that $(1,4)$ *cannot be estimated at this step. Indeed, the estimation of element* $(1,4)$ *requires that at least one of the following pairs of preference values are known:* $\{(1,2),(2,4)\}$, $\{(1,3),(3,4)\}$. *However, the preference values for* $(2,4)$ *and* $(1,3)$ *are unknown. The same applies to* $(4,1)$, *which cannot be estimated at this step because the preference values for* $(4,2)$ *and* $(3,1)$ *are unknown.*

The computation of the estimated values cr_{13} *and* cr_{31} *are given below, which make use of intermediate and different alternatives* k *so that the chain of preference values* $(1,k)$ *and* $(k,3)$ *are known. The only intermediate alternative to use at this step is* $k=2$, *for which we have (rounding to 2 decimal places):*

$$cr_{13}^2 = \frac{r_{12}\cdot r_{23}}{r_{12}\cdot r_{23} + (1-r_{12})\cdot(1-r_{23})} = \frac{0.4\cdot 0.5}{0.4\cdot 0.5 + 0.6\cdot 0.5} = 0.4,$$

and

$$cr_{31}^2 = \frac{r_{32}\cdot r_{21}}{r_{32}\cdot r_{21} + (1-r_{32})\cdot(1-r_{21})} = \frac{0.4\cdot 0.3}{0.4\cdot 0.3 + 0.6\cdot 0.7} = 0.22.$$

The computation of the estimated values cr_{24} *and* cr_{42} *is done using intermediate alternative* $k=3$:

$$cr_{24}^3 = \frac{r_{23}\cdot r_{34}}{r_{23}\cdot r_{34} + (1-r_{23})\cdot(1-r_{34})} = \frac{0.5\cdot 0.3}{0.5\cdot 0.3 + 0.5\cdot 0.7} = 0.3,$$

and

$$cr_{42}^3 = \frac{r_{43}\cdot r_{32}}{r_{43}\cdot r_{32} + (1-r_{43})\cdot(1-r_{32})} = \frac{0.4\cdot 0.4}{0.4\cdot 0.4 + 0.6\cdot 0.6} = 0.31.$$

After the estimation process is applied, we have:

$$R = \begin{pmatrix} 0.5 & 0.4 & \mathbf{0.4} & - \\ 0.3 & 0.5 & 0.5 & \mathbf{0.3} \\ \mathbf{0.22} & 0.4 & 0.5 & 0.3 \\ - & \mathbf{0.31} & 0.4 & 0.5 \end{pmatrix}$$

Step 2: *The remaining unknown elements can be estimated at this stage,* $EMV_2 = \{(1,4),(4,1)\}$. *We elaborate the computation process of the estimated value for* cr_{14} *(rounding to 2 decimal places):*

$$cr_{14}^2 = \frac{r_{12}\cdot r_{24}}{r_{12}\cdot c_{24} + (1-c_{12})\cdot(1-c_{24})} = \frac{0.4\cdot 0.3}{0.4\cdot 0.3 + 0.6\cdot 0.7} = 0.22;$$

$$cr_{14}^3 = \frac{r_{13} \cdot r_{34}}{r_{13} \cdot r_{34} - (1 - r_{13}) \cdot (1 - r_{34})} = \frac{0.4 \cdot 0.3}{0.4 \cdot 0.3 + 0.6 \cdot 0.7} = 0.22;$$

$$cr_{14} = \frac{cr_{14}^2 + cr_{14}^3}{2} = 0.22.$$

For cr_{41} we have:

$$cr_{41}^2 = \frac{r_{42} \cdot r_{21}}{r_{42} \cdot c_{21} + (1 - c_{42}) \cdot (1 - c_{21})} = \frac{0.31 \cdot 0.3}{0.31 \cdot 0.3 + 0.69 \cdot 0.7} = 0.16;$$

$$cr_{41}^3 = \frac{r_{43} \cdot r_{31}}{r_{43} \cdot r_{31} - (1 - r_{43}) \cdot (1 - r_{31})} = \frac{0.4 \cdot 0.22}{0.4 \cdot 0.22 + 0.6 \cdot 0.78} = 0.16;$$

$$cr_{41} = \frac{cr_{41}^2 + cr_{41}^3}{2} = 0.16.$$

Step 3: *Thus, we obtain the following completed asymmetric fuzzy preference relation R:*

$$R = \begin{pmatrix} 0.5 & 0.4 & \mathbf{0.4} & \mathbf{0.22} \\ 0.3 & 0.5 & 0.5 & \mathbf{0.3} \\ \mathbf{0.22} & 0.4 & 0.5 & 0.3 \\ \mathbf{0.16} & \mathbf{0.31} & 0.4 & 0.5 \end{pmatrix}$$

Step 4: *The complete reciprocal intuitionistic fuzzy preference relation is:*

$$B = F^{-1}(R) = \begin{pmatrix} \langle 0.50, 0.50 \rangle & \langle 0.40, 0.30 \rangle & \langle \mathbf{0.40}, \mathbf{0.22} \rangle & \langle \mathbf{0.22}, \mathbf{0.16} \rangle \\ \langle 0.30, 0.40 \rangle & \langle 0.50, 0.50 \rangle & \langle 0.50, 0.40 \rangle & \langle \mathbf{0.30}, \mathbf{0.31} \rangle \\ \langle \mathbf{0.22}, \mathbf{0.40} \rangle & \langle 0.40, 0.50 \rangle & \langle 0.50, 0.50 \rangle & \langle 0.30, 0.40 \rangle \\ \langle \mathbf{0.16}, \mathbf{0.22} \rangle & \langle \mathbf{0.31}, \mathbf{0.30} \rangle & \langle 0.40, 0.30 \rangle & \langle 0.50, 0.50 \rangle \end{pmatrix}$$

Notice that the completed reciprocal intuitionistic fuzzy preference relation obtained coincides with the one in [28], where there was a typo in b_{41} (b_{14}) that appeared as $\langle 0.19, 0.22 \rangle$ ($\langle 0.22, 0.19 \rangle$) *instead of the correct one* $\langle 0.16, 0.22 \rangle$ ($\langle 0.22, 0.16 \rangle$).

For the following additional incomplete reciprocal intuitionistic fuzzy preference relations [28]

$$\mathrm{B}^{\mathbf{2}} = \begin{pmatrix} \langle 0.50, 0.50 \rangle & \langle 0.40, 0.45 \rangle & x & \langle 0.30, 0.40 \rangle \\ \langle 0.45, 0.40 \rangle & \langle 0.50, 0.50 \rangle & \langle 0.45, 0.40 \rangle & x \\ x & \langle 0.40, 0.45 \rangle & \langle 0.50, 0.50 \rangle & \langle 0.40, 0.55 \rangle \\ \langle 0.40, 0.30 \rangle & x & \langle 0.55, 0.40 \rangle & \langle 0.50, 0.50 \rangle \end{pmatrix}$$

$$\mathrm{B}^{\mathbf{3}} = \begin{pmatrix} \langle 0.50, 0.50 \rangle & \langle 0.50, 0.40 \rangle & x & \langle 0.40, 0.30 \rangle \\ \langle 0.40, 0.50 \rangle & \langle 0.50, 0.50 \rangle & \langle 0.60, 0.30 \rangle & \langle 0.50, 0.40 \rangle \\ x & \langle 0.30, 0.60 \rangle & \langle 0.50, 0.50 \rangle & \langle 0.35, 0.40 \rangle \\ \langle 0.30, 0.40 \rangle & \langle 0.40, 0.50 \rangle & \langle 0.40, 0.35 \rangle & \langle 0.50, 0.50 \rangle \end{pmatrix}$$

$$\mathrm{B}^{\mathbf{4}} = \begin{pmatrix} \langle 0.50, 0.50 \rangle & \langle 0.40, 0.50 \rangle & \langle 0.45, 0.40 \rangle & x \\ \langle 0.50, 0.40 \rangle & \langle 0.50, 0.50 \rangle & \langle 0.50, 0.40 \rangle & \langle 0.50, 0.30 \rangle \\ \langle 0.40, 0.45 \rangle & \langle 0.40, 0.50 \rangle & \langle 0.50, 0.50 \rangle & \langle 0.50, 0.40 \rangle \\ x & \langle 0.30, 0.50 \rangle & \langle 0.40, 0.50 \rangle & \langle 0.50, 0.50 \rangle \end{pmatrix}$$

The application of the estimation procedure yields the corresponding completed reciprocal intuitionistic fuzzy preference relations:

$$\mathrm{B}^{\mathbf{2}} = \begin{pmatrix} \langle 0.50, 0.50\rangle & \langle 0.40, 0.45\rangle & \langle \mathbf{0.35}, \mathbf{0.33}\rangle & \langle 0.30, 0.40\rangle \\ \langle 0.45, 0.40\rangle & \langle 0.50, 0.50\rangle & \langle 0.45, 0.40\rangle & \langle \mathbf{0.31}, \mathbf{0.38}\rangle \\ \langle \mathbf{0.33}, \mathbf{0.35}\rangle & \langle 0.40, 0.45\rangle & \langle 0.50, 0.50\rangle & \langle 0.40, 0.55\rangle \\ \langle 0.40, 0.30\rangle & \langle \mathbf{0.38}, \mathbf{0.31}\rangle & \langle 0.55, 0.40\rangle & \langle 0.50, 0.50\rangle \end{pmatrix}$$

$$\mathrm{B}^{\mathbf{3}} = \begin{pmatrix} \langle 0.50, 0.50\rangle & \langle 0.50, 0.40\rangle & \langle \mathbf{0.45}, \mathbf{0.20}\rangle & \langle 0.40, 0.30\rangle \\ \langle 0.40, 0.50\rangle & \langle 0.50, 0.50\rangle & \langle 0.60, 0.30\rangle & \langle 0.50, 0.40\rangle \\ \langle \mathbf{0.20}, \mathbf{0.45}\rangle & \langle 0.30, 0.60\rangle & \langle 0.50, 0.50\rangle & \langle 0.35, 0.40\rangle \\ \langle 0.30, 0.40\rangle & \langle 0.40, 0.50\rangle & \langle 0.40, 0.35\rangle & \langle 0.50, 0.50\rangle \end{pmatrix}$$

$$\mathrm{B}^{\mathbf{4}} = \begin{pmatrix} \langle 0.50, 0.50\rangle & \langle 0.40, 0.50\rangle & \langle 0.45, 0.40\rangle & \langle \mathbf{0.43}, \mathbf{0.30}\rangle \\ \langle 0.50, 0.40\rangle & \langle 0.50, 0.50\rangle & \langle 0.50, 0.40\rangle & \langle 0.50, 0.30\rangle \\ \langle 0.40, 0.45\rangle & \langle 0.40, 0.50\rangle & \langle 0.50, 0.50\rangle & \langle 0.50, 0.40\rangle \\ \langle \mathbf{0.30}, \mathbf{0.43}\rangle & \langle 0.30, 0.50\rangle & \langle 0.40, 0.50\rangle & \langle 0.50, 0.50\rangle \end{pmatrix}$$

6. Confidence/consistency selection approach with incomplete reciprocal intuitionistic fuzzy preference relations

The aim of the selection process of a group decision making model is to choose the best alternatives according to the opinion of the experts. A classical selection process consists of two phases [9]: (1) aggregation and (2) exploitation. The aggregation phase defines a collective fuzzy preference relation, which indicates the global preference between every ordered pair of alternatives, while the exploitation phase transforms the global information about the alternatives into a global ranking of them, from which a selection set of alternatives is derived.

Confidence has been defined in [36] as a person's belief that a statement represents the best possible response. Frequently, researchers have found that freely interacting groups choose the positions of their most confident members as their group decisions. This phenomenon has been witnessed with groups discussing a mathematical puzzle [17], a recall task [22] and a recognition task [16], concluding that confidence was a significant predictor of influence. Furthermore Guha et al. state in [12] that in any real field decision making situation when experts give their responses to a particular alternative, their confidence level regarding the opinions are very much important. Therefore in this section a measurement of the expert's degree of confidence on the opinions provided for reciprocal intuitionistic fuzzy preference relations is defined and used to drive the aggregation of the experts' preferences.

6.1. Expert's degree of confidence

Given a reciprocal intuitionistic fuzzy preference relation, the hesitancy degrees used to define confidence measures at its three different levels: pair of alternatives, alternatives and relation levels, as follows:

Definition 11. *Given a reciprocal intuitionistic fuzzy preference relation $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij}\rangle)$, the* confidence level associated to the intuitionistic preference value b_{ij} *is measured as*

$$CFL_{ij} = 1 - \tau_{ij},$$

with τ_{ij} being the hesitancy degree associated to b_{ij}.

As noted before in Section 2.2, $\tau_{ij} = 1-\mu_{ij}-\nu_{ij}$ and therefore we have that $CFL_{ij} = \mu_{ij}+\nu_{ij}$. In other words, when $CFL_{ij} = 1$ ($\mu_{ij} + \nu_{ij} = 1$) then $\tau_{ij} = 0$ and there is no hesitation at all. The lower the value of CFL_{ij}, the higher the value of τ_{ij} and the more hesitation is present in the intuitionistic value b_{ij}.

Definition 12. *Given a reciprocal intuitionistic fuzzy preference relation* $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$, *the* confidence level associated to the alternative x_i *is defined as*

$$CFL_i = \frac{\sum\limits_{\substack{j=1 \\ i \neq j}}^{n} \left(CFL_{ij} + CFL_{ji} \right)}{2(n-1)}.$$

Because B is reciprocal, we have that $CFL_{ij} = CFL_{ji}$ $(\forall i, j)$ and therefore it is

$$CFL_i = \frac{\sum\limits_{\substack{j=1 \\ i \neq j}}^{n} CFL_{ij}}{n-1}.$$

A similar interpretation of CFL_i with respect to the confidence on the preference values on the alternative x_i can be done as it was done above with CFL_{ij}.

Definition 13. *The* confidence level associated to a reciprocal intuitionistic fuzzy preference relation $B = (b_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$ *is measured as*

$$CFL_B = \frac{\sum\limits_{i=1}^{n} CFL_i}{n}.$$

Notice that when $CFL_B = 1$, then the reciprocal intuitionistic fuzzy preference relation B is a reciprocal fuzzy preference relation.

6.2. Confidence-consistency guided aggregation

Given a group of experts, their collective preference is obtained by fusing their individual preferences using an appropriate aggregation operator. A widely used aggregation operator in decision making with fuzzy preferences is Yager's Ordered Weighted Averaging (OWA) operator [32], or one of its extended versions such as the Induced OWA (IOWA) [33].

Definition 14. *An IOWA operator of dimension* m *is a function* $\Phi_W \colon (\mathbb{R} \times \mathbb{R})^m \to \mathbb{R}$, *to which a set of weights or weighting vector is associated,* $W = (w_1, \ldots, w_m)$, *such that* $w_i \in [0, 1]$ *and* $\Sigma_i w_i = 1$, *is expressed as follows:*

$$\Phi_W \left(\langle u_1, p_1 \rangle, \ldots, \langle u_m, p_m \rangle \right) = \sum_{i=1}^{m} w_i \cdot p_{\sigma(i)},$$

being $\sigma \colon \{1, \ldots, m\} \to \{1, \ldots, m\}$ *a permutation such that* $u_{\sigma(i)} \geq u_{\sigma(i+1)}$, $\forall i = 1, \ldots, m-1$.

Consistency based IOWA operators have been proposed in literature so that the reordering of arguments to aggregate and the computation of the aggregation weights are obtained using consistency degrees values derived from the preferences experts provide [14]. In the case of reciprocal intuitionistic fuzzy preference relation a multiplicative consistency IOWA (MC-IOWA) operator was presented in [28]. These aggregation operators associate higher degree of importance the higher the consistent the preferences are. However, they do not take into consideration the confidence associated to preferences to aggregate, which is proposed to do here by developing a new consistency and confidence IOWA (CC-IOWA) operator, i.e. an IOWA operator that trades off consistency and confidence criteria in both re-ordering the preferences to aggregate and deriving the aggregation weights to use in their fusing to derive the collective preference.

Definition 15 (CC-IOWA operator). *Let a set of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, using the reciprocal intuitionistic fuzzy preference relations, $\{B^1, \ldots, B^m\}$. A consistency and confidence IOWA (CC-IOWA) operator of dimension m, Φ_W^{CC}, is an IOWA operator whose set of order inducing values is the set of consistency/confidence index values, $\{CCI^1, \ldots, CCI^m\}$, associated with the set of experts.*

Therefore, the collective reciprocal intuitionistic fuzzy preference relation $B^{cc} = (b_{ij}^{cc}) = (\langle \mu_{ij}^{cc}, \nu_{ij}^{cc} \rangle)$ is computed as follows:

$$\mu_{ij}^{cc} = \Phi_W^{CC}\left(\langle CCI^1, \mu_{ij}^1 \rangle, \cdots, \langle CCI^m, \mu_{ij}^m \rangle\right) = \sum_{h=1}^{m} w_h \cdot \mu_{ij}^{\sigma(h)} \tag{5}$$

$$\nu_{ij}^{cc} = \Phi_W^{CC}\left(\langle CCI^1, \nu_{ij}^1 \rangle, \cdots, \langle CCI^m, \nu_{ij}^m \rangle\right) = \sum_{h=1}^{m} w_h \cdot \nu_{ij}^{\sigma(h)} \tag{6}$$

$$CCI^h = (1-\delta) \cdot CL^h + \delta \cdot CFL^h \tag{7}$$

such that $CCI^{\sigma(h-1)} \geq CCI^{\sigma(h)}$, $w_{\sigma(h-1)} \geq w_{\sigma(h)} \geq 0$ $(\forall h \in \{2, \cdots, m\})$ with $\sum_{h=1}^{m} w_h = 1$, CL_{ij}^h the consistency level associated to $R^h = F(B^h)$, CFL^h the confidence level associated to B^h, and $\delta \in [0,1]$ a parameter to control the weight of both consistency and confidence criteria in the inducing variable.

The general procedure for the inclusion of importance weight values, $\{u_1, \ldots, u_m\}$, in the aggregation process involves the transformation of the values to aggregate under the importance degree to generate a new value and then aggregate these new values using an aggregation operator. In the area of quantifier guided aggregations, Yager provided a procedure to evaluate the overall satisfaction of m important criteria (experts) by an alternative x by computing the weighting vector associated to an OWA operator as follows [34]:

$$w_h = Q\left(\frac{S(h)}{S(m)}\right) - Q\left(\frac{S(h-1)}{S(m)}\right)$$

being Q the membership function of the linguistic quantifier, $S(h) = \sum_{k=1}^{h} u_{\sigma(k)}$, and σ the permutation used to produce the ordering of the values to be aggregated. This approach for the inclusion of importance degrees associates a zero weight to those experts with zero importance degree. The linguistic quantifier is a Basic Unit-interval Monotone (BUM) function $Q : [0,1] \to [0,1]$ such that $Q(0) = 0$, $Q(1) = 1$ and if $x > y$ then $Q(x) \geq Q(y)$.

Yager extended this procedure to the case of IOWA operator. In this case, each component in the aggregation consists of a triple, with first element being the argument value to aggregate, the second element the importance weight value associated to the first element and the third element being the order inducing value [33]. The same expression as above is used with σ being the permutation that order the induce values from largest to lowest. In our case, we propose to use the consistency/confidence values associated with each expert both as an importance weight and as the order inducing values. Thus, the ordering of the preference values is first induced by the ordering of the experts from the most to the least consistent/confident, and the weights of the CC-IOWA operator is obtained as follows:

$$w_h = Q\left(\frac{\sum_{k=1}^{h} CCI^{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=1}^{h-1} CCI^{\sigma(k)}}{T}\right)$$

with $T = \sum_{k=1}^{m} CCI^k$.

Example 2 (Continuation of Example 1)**.** *The first step to compute the collective preference relation is to get the individual consistency and confidence levels.*

Consistency level computation. *For each matrix R^i the matrix MR^i is obtained by applying the estimation procedure using equation (3) and the similarity values between the elements of R^i and MR^i as described in Section 3. To illustrate this procedure we show here only the computation of the estimate of the preference value of the pair of alternatives $(1,2)$ of R^1, $(mr_{12})^1$, and the corresponding consistency index of such pair of alternatives, $(CL_{12})^1$.*

The expression of value $(mr_{12})^1$ is

$$(mr_{12})^1 = \frac{\sum_{k \in R_{12}^{01}} (mr_{12}^k)^1}{\# R_{12}^{01}};$$

where $R_{12}^{01} = \{k \neq i, j | ((r_{ik})^1, (r_{kj})^1) \notin R^{01}\}$, $R^{01} = \{(1,0),(0,1)\}$, $\# R_{12}^{01}$ is the cardinality of R_{12}^{01} and

$$(mr_{12}^k)^1 = \frac{(r_{1k})^1 \cdot (r_{k2})^1 \cdot (r_{21})^1}{(r_{2k})^1 \cdot (r_{k1})^1}$$

The values of $(mr_{12}^3)^1$ and $(mr_{12}^4)^1$ are:

$$(mr_{12}^3)^1 = \frac{(r_{13})^1 \cdot (r_{32})^1 \cdot (r_{21})^1}{(r_{23})^1 \cdot (r_{31})^1} = \frac{0.4 \cdot 0.4 \cdot 0.3}{0.5 \cdot 0.22} = 0.43636$$

$$(mr_{12}^4)^1 = \frac{(r_{14})^1 \cdot (r_{42})^1 \cdot (r_{21})^1}{(r_{24})^1 \cdot (r_{41})^1} = \frac{0.22 \cdot 0.31 \cdot 0.3}{0.3 \cdot 0.16} = 0.42625$$

Thus, with two decimal places we have:

$$(mr_{12})^1 = \frac{(mr_{12}^3)^1 + (mr_{12}^4)^1}{2} = 0.43$$

and

$$(CL_{12})^1 = 1 - |0.4 - 0.43| = 0.97$$

The consistency levels of each individual expert are:

$$CL^1 = 0.99,\ CL^2 = 0.99,\ CL^3 = 0.92,\ CL^4 = 0.98$$

Confidence level computation. *For each reciprocal intuitionistic fuzzy preference relation, B^i, its confidence level, CFL^i as described in Section 6.1 is computed, resulting in the following values:*

$$CFL^1 = 0.65,\ CFL^2 = 0.79,\ CFL^3 = 0.80,\ CFL^4 = 0.85$$

Aggregation. *The completed reciprocal intuitionistic fuzzy preference relations are fuses into a collective preference relation by means of the CC-IOWA defined in expressions (5) and (6) using the experts' consistency-confidence levels CCI as the order inducing variable. To that aim we calculate each expert's confidence-consistency level following expression (7) with a value of $\delta = 0.5$*

$$CCI^1 = 0.82,\ CCI^2 = 0.89,\ CCI^3 = 0.86,\ CCI^4 = 0.91$$

In order to generate the weighting vector we use the linguistic quantifier "most of" using $Q(r) = r^{1/2}$ [6], and with $\sigma(1) = 4,\ \sigma(2) = 2,\ \sigma(3) = 3,\ \sigma(4) = 1$ the following weights are obtained:

$$w_1 = 0.13,\ w_2 = 0.21,\ w_3 = 0.15,\ w_4 = 0.51.$$

The collective preference relation B^{cc} is:

$$\mathrm{B}^{\mathbf{cc}} = \begin{pmatrix} \langle 0.50, 0.50\rangle & \langle 0.42, 0.45\rangle & \langle 0.42, 0.33\rangle & \langle 0.37, 0.33\rangle \\ \langle 0.45, 0.42\rangle & \langle 0.50, 0.50\rangle & \langle 0.51, 0.38\rangle & \langle 0.44, 0.33\rangle \\ \langle 0.33, 0.42\rangle & \langle 0.38, 0.51\rangle & \langle 0.50, 0.50\rangle & \langle 0.43, 0.43\rangle \\ \langle 0.30, 0.37\rangle & \langle 0.33, 0.44\rangle & \langle 0.43, 0.43\rangle & \langle 0.50, 0.50\rangle \end{pmatrix}$$

6.3. Exploitation phase

At this point, in order to select the alternative(s) 'best' acceptable for the majority of the experts and taking advantage of the equivalence between reciprocal intuitionistic fuzzy preference relations and asymmetric fuzzy preference relations we propose the following two quantifier-guided choice degrees of alternatives for the collective reciprocal intuitionistic fuzzy preference relations [13].

1. *The Intuitionistic Fuzzy Quantifier Guided Dominance Degree (IFQGDD)* for the alternative x_i quantifies the dominance that alternative x_i has over the fuzzy majority of the remaining alternatives:

   $$IFQGDD_i = \phi_Q\left(r_{ij}^{cc}, j = 1, \ldots, n\right),$$

   with $r_{ij}^{cc} = f(b_{ij}^{cc})$ and ϕ_Q is an OWA operator guided by the linguistic quantifier represented by the BUM function Q.

2. *The Intuitionistic Fuzzy Quantifier Guided Non Dominance Degree (IFQGNDD)* for the alternative x_i quantifies the degree up to which such alternative is not dominated by a fuzzy majority of the remaining alternatives:

   $$IFQGNDD_i = \phi_Q\left(1 - r_{ji}^{s}, j = 1, \ldots, n\right),$$

   with $r_{ji}^{s} = \max\left\{r_{ji}^{cc} - r_{ij}^{cc}, 0\right\}$ representing the degree up to which x_i is strictly dominated by x_j.

Example 3 (End of Example 1). *Using the same linguistic quantifier "most of", the resultant weighting vector $W_{exp} = (w_{exp1}, w_{exp2}, w_{exp3})$ and quantifier guided dominance and non-dominance degrees are:*

$$w_{exp1} = Q(1/3) - Q(0) = 0.58 - 0 = 0.58.$$
$$w_{exp2} = Q(2/3) - Q(1/3) = 0.82 - 0.58 = 0.24.$$
$$w_{exp3} = Q(1) - Q(2/3) = 1 - 0.82 = 0.18.$$

	x_1	x_2	x_3	x_4
$IFQGDD_i$	*0.41*	*0.48*	*0.40*	*0.38*
$IFQGNDD_i$	*0.99*	*1*	*0.95*	*0.96*

Rankings of alternatives can be produced based on each choice degree and the best alternative according to both choice degrees for "most of" the experts can be selected.

7. Conclusions

Uncertainty, hesitation and fuzziness is inherent to all the human being decisions. Therefore in GDM situations it might well be the case of the experts not being able to provide an accurate degree of preference. In these situations reciprocal intuitionistic fuzzy preference relations play a key role since they are able to represent both uncertainty and hesitation, which can be seen as one of the reasons many researchers have turned their research effort to develop theoretical framework for using them n decision making context under uncertainty, of which this paper contributes towards.

The most significant findings and and advantages of this contribution are listed below:

- Firstly, we have proved the mathematical equivalence between the set of asymmetric fuzzy preference relations and the set of reciprocal intuitionistic fuzzy preference relations, which can be used to transpose concepts defined in for one preference structure to the other one.
- Indeed, in this paper incomplete reciprocal intuitionistic fuzzy preference relations has been addressed by completing the equivalent incomplete asymmetric fuzzy preference relations using a well known estimation process developed for fuzzy preference relations.
- The concept of confidence level associated to a reciprocal intuitionistic fuzzy preference relation has been defined to associate different importance degrees to experts in the aggregation of individual reciprocal intuitionistic fuzzy preference relations in decision making to derive the collective reciprocal intuitionistic fuzzy preference relation. This concept has been used in conjunction with the consistency level to propose a new consistency and confidence induced ordered weighted averaging (CC-IOWA) operator, in order to implement both consistency and confidence in the resolution process of a group/multicriteria decision making problem.

This contribution opens the door to the development of new methodologies for group decision making that will be addressed in future contributions:

- Development of a consensus approach in which the experts' confidence level will be taken into account to provide recommendations to increase the consensus as it could be providing recommendations coming from those experts with higher confidence levels.
- Development of a new methodology based on confidence and proximity to estimate the missing preferences in cases where a particular expert is not able to provide any preference for one or more alternatives, a situation that is being described as of total ignorance [2].

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2.2. Consensus processes where uncertainty is modeled via information granularity

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Building consensus in group decision making with an allocation of information granularity

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Abstract

Consensus is defined as a cooperative process in which a group of decision makers develops and agrees to support a decision in the best interest of the whole. It is a questioning process, more than an affirming process, in which the group members usually modify their choices until a high level of agreement within the group is achieved. Given the importance of forming an accepted decision by the entire group, the consensus problem has attained a great attention as it is a major goal in group decision making. In this study, we propose the concept of the information granularity being regarded as an important and useful asset supporting to reach consensus in group decision making. By using fuzzy preference relations to represent the opinions of the decision makers, we develop a concept of a granular fuzzy preference relation where each pairwise comparison is formed as a certain information granule (say, a interval, fuzzy set, rough set, and alike) instead of a single numeric value. As being more abstract, the granular format of the preference model offers the required flexibility to increase the level of agreement within the group using the fact that we select the most suitable numeric representative of the fuzzy preference relation.

Keywords:
Group decision making, consensus, consistency, granularity of information, particle swarm optimization

1. Introduction

Group Decision Making (GDM) is a pervasive and critical activity within companies and organizations both in the public and private sector [26]. Policies, budget plans, and other organizational tasks frequently involve group discussions or meetings due to their effectiveness in making decisions. Research in social psychology on group performance suggests that group tend to be more effective than direct aggregation of individual group members' choices and make better decisions than the most highly skilled individual in a group [40].

A GDM situation involves multiple decision makers interacting to reach a decision. To do this, decision makers have to convey their preferences or opinions by means of a set of evaluations over a set of possible alternatives. An important issue here is the level of agreement achieved among the group members before making the decision. It is worth noting that when decisions are made by a

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group of decision makers, it is recommendable that they are engaged in a consensus process [4, 37], in which all group members discuss their reasons for making decisions in order to arrive at a sufficient agreement that is acceptable (to the highest possible extent) to all. In essence, consensus aims at attaining the consent, not necessarily the agreement, of the decision makers by accommodating views of all parties involved to accomplish a decision that will yield. This decision will be beneficial to the whole group, not necessarily to the particular decision makers who may give consent to what will not necessarily be their first choice but because, for instance, they wish to cooperate with the group. The full consent, however, does not mean that each decision maker is in full agreement [4]. Therefore, reaching consensus does not assume that everyone must be in complete agreement, a highly unlikely situation in a group of intelligent, creative individuals.

In a GDM situation, a consensus process is usually defined as a negotiation process developed iteratively and composed by several consensus rounds, where the decision makers accept to change their preferences following some advice [4, 17, 37]. In the first consensus approaches proposed in the literature [9, 14, 15, 22, 23], the advice was provided by a moderator, which knows the agreement degree in each round of the consensus process by means of the computation of some consensus measures. However, as the moderator can introduce some subjectivity in the process, new consensus approaches have been proposed in order to make more effective and efficient the decision making process by substituting the moderator figure or providing to the moderator with better analysis tools [7, 16, 19, 24, 32]. Either way, several consensus rounds are usually required in order to achieve a sufficient agreement. As a result, the process of building consensus can take a considerable amount of time.

Independently of the source of the advice, it is easy to see that consensus requires that each member of the group has to allow a certain degree of flexibility and be ready to make an adjustment of his/her first choices and, here, information granularity [29, 30, 31] may come into play. Information granularity is an important design asset and may offer to each decision maker a real level of flexibility using some initial opinions expressed by each decision maker that can be modified with the intent to reach a higher level of consensus. Assuming that each decision maker expresses his/her preferences using a fuzzy preference relation, this required flexibility is brought into the fuzzy preference relations by allowing them to be granular rather than numeric. That is, we consider the entries of the fuzzy preference relations are not plain numbers but information granules, say intervals, fuzzy sets, rough sets, probability density functions, etc. In summary, information granularity that is present here serves as an important modeling asset, offering an ability of the decision maker to exercise some flexibility to be used in adjusting his/her initial position when becoming aware of the opinions of the other group members. To do so, the fuzzy preference relation is elevated (abstracted) to its granular format.

The aim of this study is to propose an allocation of information granularity as a key component to facilitate the achievement of consensus. In such a way, in the realization of the granular representation of the fuzzy preference relations, we introduce a certain level of granularity supplying the required flexibility to increase the level of consensus among the decision makers. This proposed concept of granular fuzzy preference relation is used to optimize a performance index, which comes as an additive combination of two components: (i) the first one quantifies the level of consensus within the group, and (ii) the second one expresses the level of consistency of the individual decision makers. Given the nature of the required optimization, the ensuing optimization problem is solved by engaging a machinery of population-based optimization, namely Particle Swarm Optimization (PSO) [25].

The study is arranged into five sections. We start with the presentation of the GDM scenario considered in this study. Furthermore, in this section, we describe both the method to obtain the level of consensus reached within the group and the procedure to obtain the consistency level achieved by an individual decision maker when expressing his/her opinions using fuzzy preference relations. Section 3 is concerned with the building of consensus through an allocation of information granularity. In addition, the use of PSO as the underlying optimization tool is described; strong attention is given to the content of the particles utilized in the method and a way in which the information granularity component is used in the adjustment of the single numeric values of the original fuzzy preference relations. To illustrate the method, an experimental study is reported in Section 4. Finally, we offer some conclusions and future works in Section 5.

2. Group decision making

In a classical GDM situation, there is a problem to solve, a solution set of possible alternatives, $X = \{x_1, x_2, \ldots, x_n\}$, $(n \geq 2)$, and a group of two or more decision makers, $E = \{e_1, e_2, \ldots, e_m\}$, $(m \geq 2)$, characterized by their own motivations, attitudes, ideas and knowledge, who express their opinions about this set of alternatives to achieve a common solution [21]. The objective is to classify the alternatives from best to worst, associating with them some degrees of preference.

Among the different representation formats that decision makers may use to express their opinions, fuzzy preference relations [21, 27, 33] are one of the most used because of their effectiveness as a tool for modelling decision processes and their utility and easiness of use when we want to aggregate decision makers' preferences into group ones [21, 38].

Definition 2.1. A fuzzy preference relation PR on a set of alternatives X is a fuzzy set on the Cartesian product $X \times X$, i.e., it is characterized by a membership function $\mu_{PR} : X \times X \rightarrow [0, 1]$.

A fuzzy preference relation PR may be represented by the $n \times n$ matrix $PR = (pr_{ij})$, being $pr_{ij} = \mu_{PR}(x_i, x_j)$ $(\forall i, j \in \{1, \ldots, n\})$ interpreted as the preference degree or intensity of the alternative x_i over x_j: $pr_{ij} = 0.5$ indicates indifference between x_i and x_j $(x_i \sim x_j)$, $pr_{ij} = 1$ indicates that x_i is absolutely preferred to x_j, and $pr_{ij} > 0.5$ indicates that x_i is preferred to x_j $(x_i \succ x_j)$. Based on this interpretation we have that $pr_{ii} = 0.5$ $\forall i \in \{1, \ldots, n\}$ $(x_i \sim x_i)$. Since pr_{ii}'s (as well as the corresponding elements on the main diagonal in some other matrices) do not matter, we will write them as '–' instead of 0.5 [18, 21]. When it is assumed that $pr_{ij} + pr_{ji} = 1$ $(\forall i, j \in \{1, \ldots, n\})$ the preference relation is called reciprocal preference relation and it is more easily interpreted as a stochastic relation [8, 12, 13, 34]. However, as it is always not the case [3, 18], this assumption is not made in this study.

In what follows, we are going to describe two important aspects which have to be taken into account when dealing with GDM situations: (i) the level of agreement or consensus achieved among the group of decision makers, and (ii) the level of consistency achieved by each decision maker in his/her opinions.

2.1. Level of agreement

Usually, GDM problems are faced by applying two different processes before a final solution can be given [2, 23]: (i) the consensus process, which refers to how to obtain the maximum degree of consensus or agreement within the group of decision makers, and (ii) the selection process, which obtains the final solution according to the preferences given by the decision makers. The selection process involves two different steps [5, 35]: aggregation of individual preferences and exploitation of the collective preference. Clearly, it is preferable that the decision makers had achieved a high level of consensus concerning their preferences before applying the selection process.

In order to evaluate the agreement achieved among the decision makers, we need to compute coincidence existing among them. Usual consensus approaches determine consensus degrees, which are used to measure the current level of consensus in the decision process, given at three different levels of a preference relation [6, 14]: pairs of alternatives, alternatives, and relation. In such a way, once the fuzzy preference relations have been provided by the decision makers, the computation of the consensus degrees is carried out as follows:

1. For each pair of decision makers (e_k, e_l) $(k = 1, \ldots, m-1,\ l = k+1, \ldots, m)$ a similarity matrix, $SM^{kl} = (sm^{kl}_{ij})$, is defined as:
$$sm^{kl}_{ij} = 1 - |pr^k_{ij} - pr^l_{ij}|$$

2. A consensus matrix, $CM = (cm_{ij})$, is calculated by aggregating all the $(m-1) \times (m-2)$ similarity matrices using the arithmetic mean as the aggregation function, ϕ, although different aggregation operators could be used depending on the nature of the GDM problem to solve:

$$cm_{ij} = \phi(sm_{ij}^{kl}),\ k = 1, \ldots, m-1,\ l = k+1, \ldots, m$$

3. Once the consensus matrix has been computed, the consensus degrees are obtained at three different levels:

   (a) *Consensus degree on pairs of alternatives.* The consensus degree on a pair of alternatives (x_i, x_j), called cp_{ij}, is defined to measure the consensus degree among all the decision makers on that pair of alternatives. In this case, this is expressed by the element of the collective similarity matrix CM:

   $$cp_{ij} = cm_{ij}$$

   (b) *Consensus degree on alternatives.* The consensus degree on the alternative x_i, called ca_i, is defined to measure the consensus degree among all the decision makers on that alternative:

   $$ca_i = \frac{\sum_{j=1;j\neq i}^{n}(cp_{ij} + cp_{ji})}{2(n-1)}$$

   (c) *Consensus degree on the relation.* The consensus degree on the relation, called cr, expresses the global consensus degree among all the decision makers' opinions. It is computed as the average of all the consensus degree for the alternatives:

   $$cr = \frac{\sum_{i=1}^{n} ca_i}{n}$$

The consensus degree of the relation, cr, is the value used to control the consensus situation. The closer cr is to 1, the greater the agreement among all the decision makers' opinions.

2.2. Level of consistency

When information is provided by individuals, an important issue to bear in mind is that of consistency [1, 10, 18]. Due to the complexity of most decision making problems, decision makers' preferences may not satisfy formal properties that fuzzy preference relations are required to verify. Consistency is one of them, and it is associated with the transitivity property.

Definition 2.1 dealing with a preference relation does not imply any kind of consistency property. In fact, preference values of a fuzzy preference relation can be contradictory. However, the study of consistency is crucial for avoiding misleading solutions in GDM [18].

To make a rational choice, properties to be satisfied by such fuzzy preference relations have been suggested [20]. In this paper, we make use of the additive transitivity property which facilitates the verification of consistency in the case of fuzzy preference relations. As it is shown in [20], additive transitivity for fuzzy preference relations can be seen as the parallel concept of Saaty's consistency property for multiplicative preference relations [36]. The mathematical formulation of the additive transitivity was given by [38]:

$$(pr_{ij} - 0.5) + (pr_{jk} - 0.5) = (pr_{ik} - 0.5), \forall i, j, k \in \{1, \ldots, n\} \quad (1)$$

Additive transitivity implies additive reciprocity. Indeed, because $pr_{ii} = 0.5\ \forall i$, if we make $k = i$ in Eq. (1) then we have: $pr_{ij} + pr_{ji} = 1, \forall i, j \in \{1, \ldots, n\}$.

Eq. (1) can be rewritten as follows:

$$pr_{ik} = pr_{ij} + pr_{jk} - 0.5, \forall i, j, k \in \{1, \ldots, n\} \quad (2)$$

A fuzzy preference relation is considered to be "additively consistent" when for every three options encountered in the problem, say $x_i, x_j, x_k \in X$, their associated preference degrees, $pr_{ij}, pr_{jk}, pr_{ik}$, fulfil Eq. (2).

Given a fuzzy preference relation, Eq. (2) can be used to calculate an estimated value of a preference degree using other preference degrees. Indeed, using an intermediate alternative x_j, the following estimated value of pr_{ik} ($i \neq k$) can be obtained in three different ways [18]:

- From $pr_{ik} = pr_{ij} + pr_{jk} - 0.5$ we obtain the estimate

$$(ep_{ik})^{j1} = pr_{ij} + pr_{jk} - 0.5 \tag{3}$$

- From $pr_{jk} = pr_{ji} + pr_{ik} - 0.5$ we obtain the estimate

$$(ep_{ik})^{j2} = pr_{jk} - pr_{ji} + 0.5 \tag{4}$$

- From $pr_{ij} = pr_{ik} + pr_{kj} - 0.5$ we obtain the estimate

$$(ep_{ik})^{j3} = pr_{ij} - pr_{kj} + 0.5 \tag{5}$$

Then, we can estimate the value of a preference p_{ik} according to the following expression:

$$ep_{ik} = \frac{\sum_{\substack{j=1 \\ j \neq i,k}}^{n} \left((ep_{ik})^{j1} + (ep_{ik})^{j2} + (ep_{ik})^{j3} \right)}{3(n-2)} \tag{6}$$

When information provided is completely consistent then $(ep_{ik})^{jl} = pr_{ik} \ \forall j, l$. However, because decision makers are not always fully consistent, the assessment made by an decision maker may not verify Eq. (2) and some of the estimated preference degree values $(ep_{ik})^{jl}$ may not belong to the unit interval $[0, 1]$. We note, from (3)–(5), that the maximum value of any of the preference degrees $(ep_{ik})^{jl}$ ($l \in \{1, 2, 3\}$) is 1.5 while the minimum one is −0.5. Taking this into account, the error between a preference value and its estimated one in $[0, 1]$ is computed as follows [18]:

$$\varepsilon p_{ik} = \frac{2}{3} \cdot |ep_{ik} - pr_{ik}| \tag{7}$$

Thus, it can be used to define the consistency degree cd_{ik} associated to the preference degree pr_{ik} as follows:

$$cd_{ik} = 1 - \varepsilon p_{ik} \tag{8}$$

When $cd_{ik} = 1$, then $\varepsilon p_{ik} = 0$ and there is no inconsistency at all. The lower the value of cd_{ik}, the higher the value of εp_{ik} and the more inconsistent is pr_{ik} with respect to the rest of information.

In the following, we define the consistency degrees associated with individual alternatives and the overall fuzzy preference relation:

- The consistency degree, $cd_i \in [0, 1]$, associated to a particular alternative x_i of a fuzzy preference relation is defined as:

$$cd_i = \frac{\sum_{k=1; i \neq k}^{n} (cd_{ik} + cd_{ki})}{2(n-1)} \tag{9}$$

- The consistency degree, $cd \in [0, 1]$, of fuzzy preference relation is defined as follows:

$$cd = \frac{\sum_{i=1}^{n} cd_i}{n} \tag{10}$$

When $cd = 1$, the fuzzy preference relation is fully consistent. Otherwise, the lower cd the more inconsistent the fuzzy preference relation is.

3. Building consensus through an allocation of information granularity

Building consensus is about arriving a solution that each decision maker is comfortable with. It is needless to say that this state calls for some flexibility exhibited by all members of the group, who in the name of cooperative pursuits give up their initial opinions and show a certain level of elasticity.

In a GDM problem in which the decision makers communicate their opinions using fuzzy preference relations, these changes of opinions are expressed through alterations of the entries of the fuzzy preference relations. That is, if the pairwise comparisons of the fuzzy preference relations are not treated as single numeric values, which are rigid, but rather as information granules, this will bring the essential factor of flexibility. It means that the fuzzy preference relation is abstracted to its granular format. The notation $\boldsymbol{G}(PR)$ is used to emphasize the fact that we are interested in granular fuzzy preference relations, where $\boldsymbol{G}(.)$ represents a specific granular formalism being used here (for instance, intervals, fuzzy sets, rough sets, probability density functions, and alike). In this manner, we introduce the concept of granular fuzzy preference relation and accentuate a role of information granularity being regarded here as an important conceptual and computational resource which can be exploited as a means to increase the level of consensus achieved among the decision makers. In summary, the level of granularity is treated as synonymous of the level of flexibility injected into the modeling environment, which makes easy the collaboration.

Obviously, the higher level of granularity is offered to the decision maker, the higher the feasibility of arriving at decisions accepted by all members of the group. Here, we appeal to the intuitive concept of granularity by trying to present a qualitative nature of the process in which the asset of granularity is involved. This idea can be formalized depending on the form of information granules being the entries of the fuzzy preference relations. In particular, in this study, the granularity of information is articulated through intervals and, therefore, the length of such intervals (entries of the fuzzy preference relations) can be sought as a level of granularity α. As here we are using interval-valued fuzzy preference relations, $\boldsymbol{G}(PR) = \boldsymbol{P}(PR)$, where $\boldsymbol{P}(.)$ denotes a family of intervals. The flexibility offered by the level of granularity can be effectively used to optimize a certain optimization criterion to capture the essence of the reconciliation of the individual preferences.

The formulation of the optimization problem needs to be now specified so that all technical details are addressed. In what follows, the optimization criterion which has to be optimized is given and its optimization using the PSO framework is described.

3.1. The optimization criterion

In the granular model of fuzzy preference relations, it is supposed that each decision maker feels equally comfortable when selecting any fuzzy preference relation whose values are placed within the bounds established by the fixed level of granularity α, which is used to increase the level of consensus within the group. However, we have to take into account that when the entries of the fuzzy preference relations are adjusting within the bounds offered by the admissible level of granularity in order to increase the level of agreement, it can produce some inconsistencies in the fuzzy preference relations. In particular, the higher the values of α, the higher the potential to reach a significant level of consensus and the higher the potential of producing some quite inconsistent fuzzy preference relations at the level of individual decision maker. Therefore, the level of granularity α is employed in two ways:

- It is used to increase the consensus within the group members by bringing all preference close to each other. This goal is realized by maximizing the global consensus degree among all the decision makers' opinions, which is quantified in terms of the consensus degree on the relation described in Section 2.1:

$$Q_1 = cr \tag{11}$$

- It is used to increment the consistency of the fuzzy preference relations. This improvement is effectuate at the level of individual decision maker. The following performance index quantifies this effect:

$$Q_2 = \frac{1}{m}\sum_{l=1}^{m} cd_l \tag{12}$$

These are the two objectives to be maximized. If we consider the scalar version of the optimization problem, it arises in the following form:

$$Q = \delta \cdot Q_1 + (1-\delta) \cdot Q_2 \tag{13}$$

being $\delta \in [0,1]$ a parameter to set up a tradeoff between the consensus obtained within the group and consistency level achieved at the individual decision maker. The higher the value of δ, the more attention is being paid to the consensus at the group level. In the limit, when $\delta = 0$, we are concerned with the consistency achieved at the level of individual decision maker only. Usually, $\delta > 0.5$ will be used to give more importance to the consensus criterion.

The overall optimization problem now reads as follows:

$$\text{Max}_{PR^1, PR^2, \ldots, PR^m \in \boldsymbol{P}(PR)} Q \tag{14}$$

The aforementioned maximization problem is carried out for all interval-valued fuzzy preference relations admissible because of the introduced level of information granularity α. This fact is underlined by including a granular form of the fuzzy preference relations allowed in the problem, i.e., PR^1, PR^2, …, PR^m, are elements of the family of interval-valued fuzzy preference relations, namely, $\boldsymbol{P}(PR)$.

This optimization task is not an easy one. Because of the nature of the indirect relationship between optimized fuzzy preference relations, which are selected from a quite large search space formed by $\boldsymbol{P}(PR)$, it calls for the use of advanced techniques of global optimization, such as, e.g., genetic algorithms, evolutionary optimization, PSO, simulated annealing, ant colonies, and the like. In particular, here the optimization of the fuzzy preference relations, coming from the space of interval-valued fuzzy preference relations, is realized by means of the PSO, which is a viable optimization alternative for this problem, as it offers a substantial level of optimization flexibility and does not come with a prohibitively high level of computational overhead as this is the case of other techniques of global optimization (say, genetic algorithms). Obviously, one could think of the usage of some other optimization mechanisms as well.

In what follows, we briefly recall the essence of the method and associate the generic representation scheme of the PSO with the format of the problem at hand.

3.2. PSO as a vehicle of optimization of fuzzy preference relations

PSO is a population-based stochastic optimization technique developed by Kennedy and Eberhart [25], which is inspired by social behavior of bird flocking or fish schooling. A particle swarm is a population of particles, which are possible solutions to an optimization problem located in the multidimensional search space [11, 25, 39]. Each particle explores the search space and during this search adheres to some quite intuitively appealing guidelines navigating the process: (i) it tries to follow its own previous direction, and (ii) it looks back at the best performance reported both at the level of the individual particle as well as the entire population. Based on the history, it changes its velocity and moves to the next position, which looks the most promising. In this search, the algorithm exhibits some societal aspects meaning that there is some collective search of the problem space. The method is equipped with some component of memory (expressed in terms of the previous velocity) incorporated as an integral part of the search mechanism.

The optimization of the fuzzy preference relations coming from the space of interval-valued fuzzy preference relations is realized by means of the PSO. In the following, we elaborate on the fitness function, its realization, and the PSO optimization along with the corresponding formation of the components of the swarm.

3.2.1. Particle

In a PSO algorithm, an important point is finding a suitable mapping between problem solution and the particle's representation. Here, each particle represents a vector whose entries are located in the interval [0, 1]. Basically, if there is a group of m experts and a set of n alternatives, the number of entries of the particle is $m \cdot n(n-1)$.

Starting with the initial fuzzy preference relation provided by the expert and assuming a given level of granularity α (located in the unit interval), let us consider an entry pr_{ij}. The interval of admissible values of this entry of $\boldsymbol{P}(PR)$ implied by the level of granularity is equal to:

$$[a, b] = [\max(0, pr_{ij} - \alpha/2), \min(1, pr_{ij} + \alpha/2)] \quad (15)$$

Let assume that the entry of interest of the particle is x. It is transformed linearly according to the expression $z = a + (b - a)x$. For example, consider that pr_{ij} is equal to 0.7, the admissible level of granularity $\alpha = 0.1$, and the corresponding entry of the particle is $x = 0.4$. Then, the corresponding interval of the granular fuzzy preference relation computed as given by Eq. (15) becomes equal to $[a, b] = [0.65, 0.75]$. Subsequently, $z = 0.69$, and, therefore, the modified value of pr_{ij} becomes equal to 0.69.

The overall particle is composed of the individual segments, where each of them is concerned with the optimization of the parameters of the fuzzy preference relations.

3.2.2. Fitness function

In the PSO, the performance of each particle during its movement is assessed by means of some performance index (fitness function). Here, the aim of the PSO is the maximization both the consensus achieved among the decision makers and the individual consistency achieved by each decision maker. Therefore, the fitness function, f, associated with the particle is defined as:

$$f = Q \quad (16)$$

being Q the optimization criterion presented in Section 3.1. The higher the value of f, the better the particle is.

3.2.3. Algorithm

In this study, the generic form of the PSO algorithm is used. Here, the updates of the velocity of a particle are realized in the form $\mathbf{v}(\mathrm{t}+1) = w \times \mathbf{v}(\mathrm{t}) + c_1\mathbf{a} \cdot (\mathbf{z}_p - \mathbf{z}) + c_2\mathbf{b} \cdot (\mathbf{z}_g - \mathbf{z})$ where "t" is an index of the generation and $\cdot$ denotes a vector multiplication realized coordinatewise. $\mathbf{z}_p$ denotes the best position reported so far for the particle under discussion while $\mathbf{z}_g$ is the best position overall and developed so far across the entire population. The current velocity $\mathbf{v}(\mathrm{t})$ is scaled by the inertia weight (w) which emphasizes some effect of resistance to change the current velocity. The value of the inertia weight is kept constant through the entire optimization process and equal to 0.2 (this value is commonly encountered in the existing literature [28]). By using the inertia component, we form the memory effect of the particle. The two other parameters of the PSO, that is $\mathbf{a}$ and $\mathbf{b}$, are vectors of random numbers drawn from the uniform distribution over the [0, 1] interval. These two update components help form a proper mix of the components of the velocity. The second expression governing the change in the velocity of the particle is particularly interesting as it nicely captures the relationships between the particle and its history as well as the history of overall population in terms of their performance reported so far. The next position (in iteration step "t+1") of the particle is computed in a straightforward manner: $\mathbf{z}(\mathrm{t}+1) = \mathbf{z}(\mathrm{t}) + \mathbf{v}(\mathrm{t}+1)$.

When it comes to the representation of solutions, the particle $\mathbf{z}$ consists of "$m \cdot n(n-1)$" entries positioned in the [0,1] interval that corresponds to the search space. Finally, one should note that while PSO optimizes the fitness function, there is no guarantee that the result is optimal, rather than that we can refer to the solution as the best one being formed by the PSO.

4. Experimental study

In this section, we report on a experimental study, which helps quantifying the performance of the proposed approach. In particular, we highlight the advantages, which are brought by an effective allocation of information granularity in the building of consensus.

Proceeding with the details of the optimization environment, we set up the values of the parameters, which are typically encountered in the literature. The standard PSO version is being used with the value of the parameters in the update equation for the velocity of the particle set as $c_1 = c_2 = 2$. The population size was set to 100 individuals and the method was run for 300 generations. These values were set up experimentally through a trial-and-error process.

Let us suppose four fuzzy preference relations coming from four decision makers $E = \{e_1, e_2, e_3, e_4\}$. The entries of these fuzzy preference relations are reflective of the pairwise comparisons of four alternatives $X = \{x_1, x_2, x_3, x_4\}$.

$$PR^1 = \begin{pmatrix} - & 0.1 & 0.6 & 0.4 \\ 0.8 & - & 0.8 & 0.7 \\ 0.4 & 0.1 & - & 0.2 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix} \quad PR^2 = \begin{pmatrix} - & 0.2 & 0.7 & 0.6 \\ 0.6 & - & 0.9 & 0.3 \\ 0.3 & 0.3 & - & 0.5 \\ 0.1 & 0.7 & 0.5 & - \end{pmatrix}$$

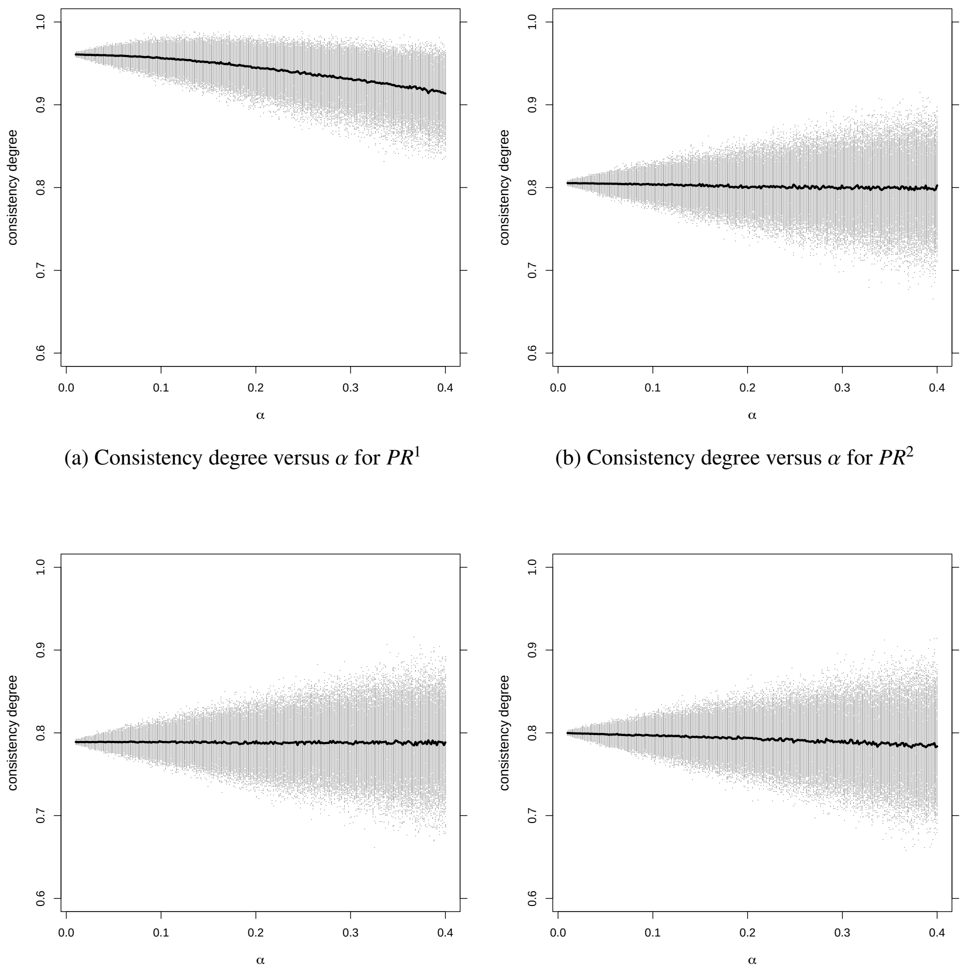
$$PR^3 = \begin{pmatrix} - & 0.7 & 0.5 & 0.3 \\ 0.3 & - & 0.6 & 0.8 \\ 0.5 & 0.4 & - & 0.9 \\ 0.6 & 0.1 & 0.3 & - \end{pmatrix} \quad PR^4 = \begin{pmatrix} - & 0.8 & 0.2 & 0.6 \\ 0.4 & - & 0.6 & 0.2 \\ 0.8 & 0.4 & - & 0.5 \\ 0.4 & 0.8 & 0.5 & - \end{pmatrix}$$

The corresponding consistency degrees of the four fuzzy preference relations are $cd_1 = 0.96$, $cd_2 = 0.81$, $cd_3 = 0.79$, and $cd_4 = 0.80$. All the fuzzy preference relations exhibit a similar level of consistency degree, with an exception of the fuzzy preference relation PR^1, whose consistency degree is higher than for the rest of the fuzzy preference relations. In case no granularity is admitted, the consensus degree achieved among the group of decision makers is $cr = 0.72$.

Before proceeding with the PSO optimization of the fuzzy preference relations when supplied with the required granularity level, it becomes instructive to analyze an impact of the improvement or deterioration of consistency of the fuzzy preference relations. For a given fuzzy preference relation PR, we allow a certain value of the granularity level α to quantify the effect of the imposed granularity. Then, for this specific value, a fuzzy preference relation is randomly generated coming from a granular representation of PR, $\boldsymbol{P}(PR)$, and its associated consistency degree is computed. The calculations are repeated 500 times for each value of α. The corresponding plots of the consistency degree cd versus the imposed granularity level α are shown in Fig. 1. In addition, in these plots, we visualize average values of the consistency degrees.

On the one hand, the likelihood of arriving at more consistent fuzzy preference relations increases when increasing the values of the granularity level α. It is not surprising as we have inserted some level of flexibility that we intend to take advantage of. On the other hand, the possibility of generating a very inconsistent fuzzy preference relation increases as well. Despite that, the average value of consistency remains pretty steady with respect to increasing values of the granularity level α, as reported for the fuzzy preference relations. However, there is some slight downward trend for higher values of α. In particular, when the consistency degree of the initial fuzzy preference relation provided by the decision maker is very high, it is very common that its average consistency degree decreases for higher values of the granularity level α (see Fig. 1a).

Once we have analyzed the impact of the given granularity level in the improvement or deterioration of the consistency, we run the optimization of the entries of the fuzzy preference relations. Considering a given level of granularity α, Fig. 2 illustrates the performance of the PSO quantified in terms of the fitness function obtained in successive generations. The most notable improvement is noted as the very beginning of the optimization, and afterwards, there is a clearly visible stabilization,



(a) Consistency degree versus α for PR^1

(b) Consistency degree versus α for PR^2

(c) Consistency degree versus α for PR^3

(d) Consistency degree versus α for PR^4

Fig. 1: Plots of consistency degrees versus α for the fuzzy preference relations PR^1–PR^4.

where the values of the fitness function remain constant. It is also interesting to analyze the computing time required by the proposed approach in order to measure its efficiency. In this study, the average running time per run of the method is 0.394 seconds and, therefore, the computational cost of our approach is low.

To put the obtained optimization results in a certain context, we report the performance obtained when considering the entries of the fuzzy preference relations are single numeric values, that is, when no granularity is allowed ($\alpha = 0$). In such a case, the value of the fitness function f is 0.74 (considering $\delta = 0.75$). Comparing with the values obtained by the PSO, the fitness function f takes on now lower values. As we can see in Fig. 2, the higher the admitted level of granularity α, the higher the values obtained by the fitness function f. It is due to the fact that the higher the level of granularity α, the higher the level of flexibility introduced in the fuzzy preference relations and, therefore, the possibility of realizing decisions with higher level of consensus and consistency increases. In particular, when each entry of the granular preference relation is treated as the whole $[0, 1]$ interval (it occurs when $\alpha = 2.0$), the value of the fitness function is near to the maximum one, which is 1. However, when

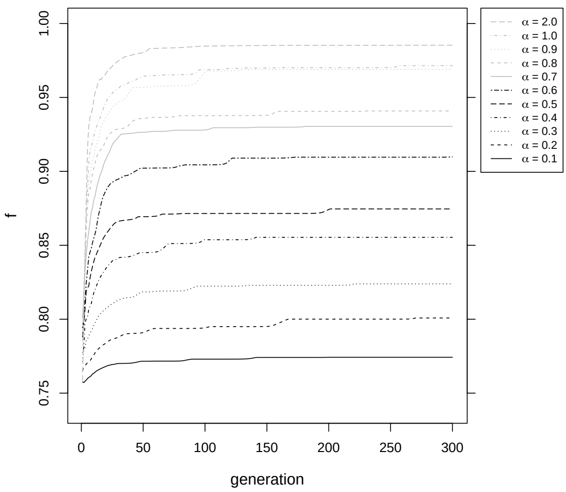
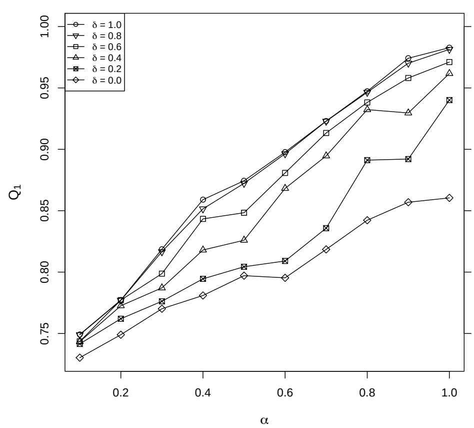
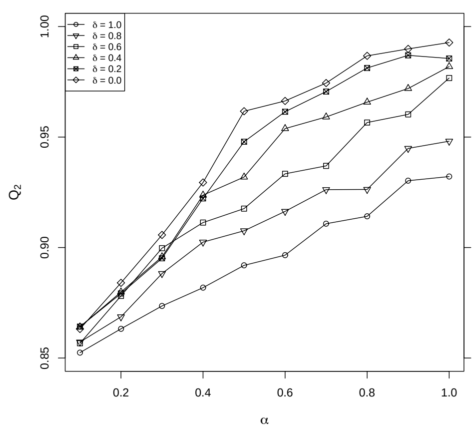


Fig. 2: Fitness function f in successive PSO generations for selected values of α (here $\delta = 0.75$).

the level of granularity is very high, the values of the entries of the fuzzy preference relation could be very different in comparison with the original values provided by the decision maker and, therefore, he/she could reject them.



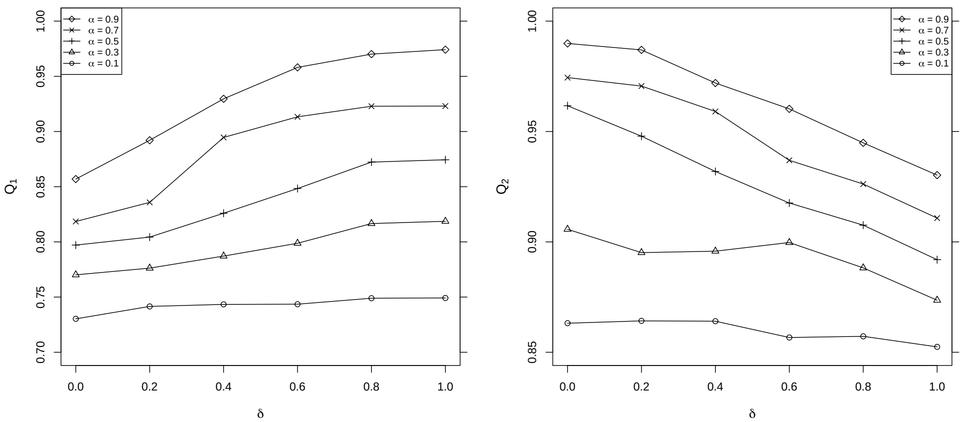
(a) Q_1 versus α for selected values of δ



(b) Q_2 versus α for selected values of δ

Fig. 3: Plots of Q_1 and Q_2 versus α for selected values of δ.

Let us examine an impact of the granularity level α and the parameter δ in the composite fitness function on the performance of the method and the form of the obtained results. For $\delta = 0$, the optimization concerns each of the fuzzy preference relations individually. Here, the increment in the values of α offers more flexibility, which, if wisely used (optimized by the PSO), produces the fuzzy preference relations of higher consistency. This effect is clearly observable in Fig. 3b (the curve for $\delta = 0$). The beneficial effect of granularity is evident: with the increasing values of α, the fuzzy preference relations become more flexible, which results in higher levels of consistency reached by the decision makers. A similar effect is visible when δ takes nonzero values: if there is some interaction, the impact of introduced granularity is positive (the overall level of consistency quantified by Q_2 is an increasing function of α). The strictly monotonic character of this relationship is not maintained for higher values of δ, as it is again shown in Fig. 3b. However, it is not surprising as the performance criterion optimized by PSO is not Q_2 itself but Q, which incorporates also the effect of the level of consensus achieved within the group of decision makers. On the other hand, Fig. 3a includes the progression of the values of Q_1, which shows the consensus within the group. Again, the advantageous effect of granularity is visible, as higher values of α translate into higher values of Q_1. However, now, higher values of δ produce higher values of Q_1 as more important is assigned to Q_1 in the composite criterion Q.



(a) Q_1 versus δ for selected values of α

(b) Q_2 versus δ for selected values of α

Fig. 4: Plots of Q_1 and Q_2 versus δ for selected values of α.

Fig. 4a includes a number of plots of Q_1 regarded as functions of δ for selected levels of granularity α. Once more, the impact of the granularity level is obvious. However, here, for the fixed value of α, there is a visible saturation effect for higher values of δ: when moving beyond a certain point, the values of Q_1 does not increase. On the other hand, the cumulative level of consistency Q_2 drops quickly with the increasing values of δ, as illustrated in Fig. 4b, and this effect is noticiable for different values of α. However, higher values of the granularity level also result in higher consistency levels in this case.

In summary, as it has been shown in this experimental study, we can conclude that both the level of consensus within the group of decision makers as well as the level of consistency achieved by the individual decision makers have been significantly increased with the use of the method proposed in this study, which speaks to the important role played by information granularity in the building of consensus.

5. Conclusions and future works

In this study, we have developed a method based on an allocation of information granularity as an important asset to increase the consensus achieved within the group of decision makers in group decision making situations. The required flexibility in the opinions provided by the decision makers, which is necessary to increase the level of consensus, was a motivating factor behind the introduction of the concept of granular fuzzy preference relations. Undoubtedly, the granular fuzzy preference relation conveys a far richer representation which can produce numeric realizations so that both the level of consensus and the level of consistency are improved. To do so, the PSO environment has been shown to serve a suitable optimization framework. Using this approach, the consensus is built in a single step rather than running several consensus rounds. On the one hand, it reduces the amount of time required for building consensus. On the other hand, negotiations among the decision makers are not included and, therefore, the decision makers influencing each other are not considered.

In the future, it is worth continuing this research in several directions:

- While the study presented here was focused on interval type of information granulation, different formalisms of information granulation such as fuzzy sets or rough sets can be incorporated into the discussed method.
- In the scenario analyzed in this study, a uniform allocation of granularity has been discussed, where the same level of granularity α has been allocated across all the fuzzy preference relations. However, a nonuniform distribution of granularity could be considered, where these levels are also optimized so that each decision maker might have an individual value of α becoming available to his/her disposal.

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3. Developing new software tools to deal with Group decision making processes with incomplete information

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GDM-R: A new framework in R to support fuzzy group decision making processes

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Abstract

With the incorporation of web 2.0 frameworks the complexity of decision making situations has exponentially increased, involving in many cases a huge number of decision makers, and many different alternatives. In the literature we can find a great variety of methodologies to assist multi-person decision making. However these classical approaches are not suitable to deal with such complexity since there are no tools able to carry out automatically the decision making processes, providing graphical information about its evolution.

The main objective of this contribution is to present an open source framework fully developed in R to carry out consensus guided decision making processes using fuzzy preference relations and providing mechanism to deal with missing information. The system includes tools to visualize the evolution of the decision making process and presents various operation modes, including a test operation one which automatically creates a customized decision scenario to validate, test and compare among various decision making approaches.

Keywords: Group decision making, fuzzy preference modeling, software development, R

1. Introduction

It has been traditionally assumed that knowledge is a sparse "commodity" in the sense that some specific individuals of the society own it, not everybody. Furthermore, it is divided in the sense that not all individuals of the "chosen ones" mentioned above have knowledge on some topic of interest or relevance to the same degree. Hence, a subgroup of individuals (experts) should be chosen to most efficiently and effectively employ that knowledge [31], and their opinions should be considered to arrive at a consensus solution accepted by the group as a whole [8]. Group decision making (GDM) consists of multiple decision makers, with different knowledge and points of view, interacting to choose the best option among all the available ones [12, 30].

GDM processes have attracted research attention in the last ten years and therefore a wide range of different methodologies have been proposed [26, 34]. However, new paradigms and ways

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of making decisions, such as web 2.0 frameworks, social networks and e-democracy, have made the complexity of decision making processes to increase, involving a huge number of decision makers [5]. These new scenarios require automatic tools not only to combine the information in the best possible way but also to better analyze the whole context, providing a rapid and complete insight about the current state of the process. In this direction, some efforts have already been made [4, 35, 36, 37, 38], however, these approaches present various deficiencies: (i) they do not make available graphical visualizations or output measures displaying the evolution of the process, and (ii) they do not offer the possibility of creating a data set to test and compare the performance of different approaches. (iii) They are developed as closed systems and, hence, they are not aimed to be upgraded or extended by other researchers,

In this paper, we present a new framework to carry out GDM processes, both in classical and current scenarios, whose main features are:

- The proposed system provides support for both real GDM situations and simulation environments. Being useful not only to assist decision making processes, but also to compare and validate already existing approaches and to develop new ones.
- The system provides powerful visualizations tools to quickly verify the state of the decision process. Among its various representations it depicts experts' preferences 3D maps to quickly detect those experts who are far from the consensus solution and are more reluctant to change their mind and also to detect those ones who provide more contradictory or inconsistent opinions. The system also allows the user to visually check the evolution of the global consensus and consistency among the various round of consensus.
- In many GDM situations, especially those involving a large number of alternatives to choose from conflicting and dynamic sources of information, some of the decision makers could not efficiently express their opinions over all the available options, and sometimes it is necessary to deal with incomplete information [44], being necessary to try to estimate the missing information since it could be very valuable for the decision making process. In such a way, the system is able to deal with this uncertainty.
- It carries out a number of consensus round to obtain a solution accepted by all the decision makers [11, 26, 34] and provides the best alternative using well known decision making algorithms [9, 22].
- It is an open source framework implemented in R [1], following a modular architecture which easily enables the extension of the tool by other researchers.

The rest of the paper is set out as follows: In Section 2 we carry out a review of the main concepts of GDM, including consensus and mechanism to estimate missing preference relations. Section 3 reviews and analyzes the existing tools available carry out GDM. In Section 4, we present our new R framework to support decision making. A practical example is included in Section 5 to illustrate how the proposed system works, its usefulness and all its capabilities and visualizations tools. Finally Section 6 closures this work pointing out future research lines and summarizing the main novelties and features of the proposed framework.

2. Background

In order to make this paper as self contained as possible, in this section we briefly introduce the main concepts used along the paper. First, we describe a classical GDM situation and, second, we focus on their different steps. Finally a brief description of R, the software environment and programming language, used to develop the proposed framework is presented.

2.1. GDM problems

A classical GDM problem may be defined as a decision situation where [19]: (i) there exists a group of two or more decision makers, $E = \{e_1, \ldots, e_m\}$ $(m \geq 2)$, (ii) there is a problem to solve in which a solution must be chosen among a set of possible alternatives, $X = \{x_1, \ldots, x_n\}$ $(n \geq 2)$, and (iii) the decision makers try to achieve a common solution. In a fuzzy context, the objective is to classify the alternatives from best to worst, associating with them some degrees of preference expressed in the $[0, 1]$ interval.

There are various preference representation formats which can be used by decision makers to provide their testimonies [28]. Among them, preference relations are one of the commonly used because decision makers have much more freedom when expressing their opinions and they can gain in expressivity. In particular, the fuzzy preference relations are the most used in the literature [30, 33, 43].

Definition 1. A fuzzy preference relation P^h on a set of alternatives X, given by a decision maker e_h, is a fuzzy set on the Cartesian product $X \times X$, i.e., it is characterized by a membership function $\mu_P \colon X \times X \longrightarrow [0, 1]$.

A fuzzy preference relation P^h may be represented by the $n \times n$ matrix $P^h = (p^h_{ik})$, being $p^h_{ik} = \mu_{P^h}(x_i, x_k)$ $(\forall i, k \in \{1, \ldots, n\})$ interpreted as the degree or intensity of preference of alternative x_i over x_k: $p^h_{ik} = 1/2$ indicates indifference between x_i and x_k $(x_i \sim x_k)$; $p^h_{ik} = 1$ indicates that x_i is absolutely preferred to x_k; $p^h_{ik} > 1/2$ indicates that x_i is preferred to x_k $(x_i \succ x_k)$. Obviously, we have that $p^h_{ii} = 1/2$ $\forall i \in \{1, \ldots, n\}$ $(x_i \sim x_i)$.

In what follows, we are going to describe two important aspects which have to be taken into account when dealing with fuzzy preference relations in GDM problems.

2.1.1. Consistency

Due to the complexity of most GDM problems, decision makers' preferences may not satisfy formal properties that fuzzy preference relations are required to verify. Actually, the preference values can be contradictory. In [29], it was presented some properties that need to be satisfied by fuzzy preference relations to make a rational choice. Consistency is one of them, which is crucial for avoiding misleading solutions [3, 10, 17].

Consistency can be interpreted as a measure of the self-contradiction expressed in the preference relation and is related to the concept of transitivity [13]. A preference relation is considered consistent when the pairwise comparisons among every three alternatives satisfy a particular transitivity property. For fuzzy preference relations, there exist many properties or conditions that have been suggested as rational conditions to be verified by a consistent relation [14]. Here, we take advantage of the additive transitivity property. As it is shown in [27], additive transitivity for fuzzy preference relations can be seen as the parallel concept of Saaty's consistency property for multiplicative preference relations [40]. The mathematical formulation of the additive transitivity was given by [43]:

$$(p^h_{ij} - 0.5) + (p^h_{jk} - 0.5) = p^h_{ik} - 0.5, \quad \forall i, j, k \in \{1, \ldots, n\} \tag{1}$$

Additive transitivity implies additive reciprocity. Indeed, because $p^h_{ii} = 0.5$, $\forall i$, if we make $k = i$ in Eq. (1), then we have: $p^h_{ij} + p^h_{ji} = 1$, $\forall i, j \in \{1, \ldots, n\}$. Eq. (1) can be rewritten as follows:

$$p_{ik}^h = p_{ij}^h + p_{jk}^h - 0.5, \quad \forall i, j, k \in \{1, \ldots, n\} \tag{2}$$

A fuzzy preference relation is considered to be "additively consistent" when for every three options encountered in the problem, say x_i, x_j, $x_k \in X$, their associated preference degrees, p_{ij}^h, p_{jk}^h, p_{ik}^h, fulfill Eq. (2).

Given a fuzzy preference relation, Eq. (2) can be used to calculate an estimated value of a preference degree using other preference degrees. Indeed, using an intermediate alternative x_j, the estimated value of p_{ik}^h $(i \neq k)$ can be obtained in three different ways (see [27]).

2.1.2. Incomplete information

Missing information is a problem that we have to consider as decision makers are not always able to provide preferences degrees between every pair of possible alternatives. It might be due to a number factor such us time pressure, lack of knowledge or data, or limited expertise related to the problem domain [44]. In order to model these situations, the following definitions express the concept of an incomplete fuzzy preference relation [27].

Definition 2. A function $f\colon X \longrightarrow Y$ is *partial* when not every element in the set X necessarily maps to an element in the set Y. When every element from the set X maps to one element of the set Y then we have a *total* function.

Definition 3. A fuzzy preference relation P on a set of alternatives X with a *partial* membership function is an incomplete fuzzy preference relation.

According to it, the completeness level C^h for the preference relation P^h given by decision maker e_h is computed as:

$$C^h = \frac{\#EV^h}{n \cdot (n-1)} \tag{3}$$

where $\#EV^h$ is the number of preference values provided by the decision maker e_h. When $C^h = 1$ then the fuzzy preference relation is complete (all values are known).

2.2. GDM steps

The solution for a GDM problem is derived either from the individual preferences provided by the decision makers, without constructing a social opinion, or by computing first a social opinion and then using it to find a solution [30]. Here, we focus on the second one, since we are interested in obtain a solution accepted by the whole group of decision makers (see Fig. 1). In the following, we describe in more details these steps and, in the next section, we will explain how they have been implemented in the proposed framework.

2.2.1. Aggregation step

In order to obtain a collective fuzzy preference relation, the aggregation step of a GDM problem consists in combining all the preferences given by the decision makers into only one preference structure that summarizes or reflects the properties contained in all the individual preferences. This aggregation can be carried out by means of particular aggregation operators that are usually defined for this purpose [48]. Among them, the Ordered Weighted Averaging (OWA) operator proposed by Yager [46] and the Induced Ordered Weighted Averaging (IOWA) operator [47] are the most widely used.

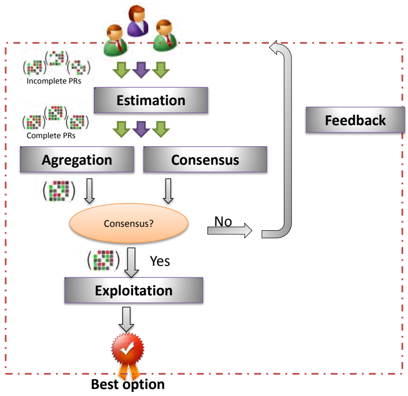

Figure 1: Steps of a GDM process.

2.2.2. Exploitation step

In order to identify the solution set of alternatives, the exploitation step uses the information produced in the aggregation step. Here, some mechanism must be applied to obtain a partial order of the alternatives and thus select the best one(s). There are several ways to do this. A usual one is to associate a certain utility value to each alternative, based on the aggregated information, producing a natural order of the alternatives. To do so, two quantifier-guided choice degrees of alternatives can be used: a dominance and a non-dominance degree [27].

2.2.3. Consensus

The steps above, aggregation and exploitation, compose the selection process for reaching a solution of any GDM problem [19, 39] without requiring any consensus among the decision makers. However, this could led to situations in which some decision makers do not agree with the final solution, since they may consider that their opinions have not been taken into account [8, 41]. To avoid these situations, it is preferable to include mechanisms, which are widely known as consensus processes [26], to check the agreement among the decision makers before obtaining a solution. A consensus process is a negotiation process composed by several consensus rounds, where the decision makers agree to change their testimonies following the advice given by a moderator, which knows the agreement degree in each round of the consensus process by means of the computation of some consensus measures [11]. If an enough consensus state has been reached, the consensus process stops and the above selection process begins. Otherwise, a feedback step is applied, where the moderator, with all the available information (all preferences given by the decision makers, consensus measures and so on), can prepare some advice for the decision makers to more easily reach consensus.

2.3. R software environment

R [1] is a free and open source software environment for statistical computing and graphics, which includes a free implementation of the high-level language S [6] originally created and distributed by Bell Labs. R runs on all major operating systems, i.e., Windows, GNU/Linux, and Mac OS X and it can be considered as an alternative to traditional statistical packages such as SPSS, SAS, and Stata. R main advantages are that it allows for the user to freely distribute, study, change, and improve the software under the Free Software Foundation's GNU General Public License and that performs a wide variety of basic to advanced statistical and graphical techniques. These advantages over other statistical software encourage the growing use of R in many well known research groups and universities, and there is a extensive research community developing frameworks in this language.

R is a functional programming language whose main data structure is the data-frame, which consist in a matrix that supports different types of values and whose rows and columns can be accessed by both index and name.

Additionally R provides very powerful tools to carry out graphical representations. Among these tools, we highlight the ones used in this framework. The lattice library [42] have been used to represent statistical plots such us barplots and scattered plots. The library scatterplot3D [32] has been used to represent 3D plots. And the library rgl3 [2] has been used to represent 3D interactive graphics.

For all the reasons explained above (multi platform, open-source, extensive use in the research community and powerful visualizations tools) we have selected R as the only language to develop our framework. Moreover as far a we know there is no tools developed in this widely used language to support fuzzy group decision making processes.

3. Related work

In this section, we review the existing computerized tools to assist GDM processes pointing out their main weaknesses.

- In [4], it is presented a web based consensus support system dealing with different types of incomplete preference relations. It is developed to work with web environments and, to that aim, it is fully implemented using a LAMP stack (GNU/Linux operating system, Apache web server, MySQL database server and PHP programming language). This system implements the iterative decision making process proposed in [27], among with the consensus reaching process proposed in [25].

- In [37], it is presented a prototype of a decision support system (DSS) designed for dynamic mobile systems. It carries out an iterative consensus process, offering the possibility to express preferences in various representation formats such as preferences orderings, utility functions, fuzzy preference relations and multiplicative preference relations. It also presents a new approach for dealing with dynamics alternatives, that is, it is able to include new alternatives during the decision process, or to remove the old ones. It is implemented using a "client/server" architecture, being the client a mobile device sending the preferences to the server and receiving the results, whereas the server carries out the data aggregation and computation. The technologies used comprise Java and Java MIDlets for the client software, PHP for the server functions, and MySQL for the database management.

- In [38], it is presented an ontology based consensus approach an its web implementation as a tool to select wines. It is aimed to deal with a large set of alternatives by defining a fuzzy ontology which selects an smaller sub-set of the most likely ones which fulfill the decision makers' preferences, reducing the complexity of the decision process.

- In [36], a graphical monitoring tool based on Self-Organizing Maps (SOMs) is proposed. It provides a 2-D graphical interface showing the temporal evolution of the decision makers' preferences. This system is aimed to ease the analysis of information in GDM processes involving a large number of decision makers. It also provides important information such as the detection of agreement/disagreement positions within the group, the evolution of decision makers' preferences, or the level of closeness among decision makers' opinions. This tool uses both JAVA to generate the data sets from the preference relations and Matlab to compute the SOMs and obtaining the graphical representations.

- In [35], it is presented a multiagent approach of a consensus system to deal with GDM processes involving a large number of decision makers. It aims to overcome the problem of the human intervention, presenting a semisupervised operation mode in which there is no need to use a human moderator in the different consensus rounds.

From this review, the main weaknesses identified in the above tools are summarized as follows:

1. The majority of the already available tools are developed as closed systems and therefore they are not aimed to be upgraded or extended by other researchers, since in most of the cases they do not provide the source code or they are based in proprietary software.

2. They are extremely dependent of the user interface and so they cannot be adapted to work in other environments such as smart phones.

3. The available DSSs do not provide any type of graphical visualizations or output measures illustrating the evolution of the consensus process.

4. There are not many methodologies or tools that easily creates a dataset to test and compare the performance of various applications. Some initial efforts in this direction have been presented in [7], [16] and [34]. In [7] it is conducted a comparative study of seven different methods for reconstructing incomplete fuzzy preference relations in terms of the consistency of the resulting complete preference relation; In [16] it has been carried out a statistical comparative study to find out the differences in group consensus that different distance measures could lead to. In [34] it has been carried out a review of various consensus methodologies and a framework in Java to compare them has been proposed.

As we will explain in the next section, the proposed framework aims to overcome the main weaknesses that these tools present as well as encompassing their main strengths in just one open source GDM framework.

4. A new framework in R to support GDM processes in a fuzzy environment

In this section, we present a new open source framework fully developed in R to automatically support GDM processes. The proposed system named as GDM-R, has been designed following a Model-View-Controller architectural pattern [20]. Therefore, the logic is completely separated from the data storage requirements and from the user interface. This design eases its adaptation to different interfaces, such us web or mobile environments, since it works totally independently from the user interface. The framework is built from various processing independent modules so it can be easily upgraded and extended just by making changes in a particular module or adding new ones.

The developed framework tries to fill the gap that the other systems leave. To that aim, it includes powerful visualization tools, and enables various working modes. To do so, the system is composed of the following modules: (i) control module, (ii) preference module, (iii) estimation module, (iv) consistency module, (v) aggregation module, (vi) consensus module, (vii) feedback module, (viii) exploitation module, (ix) storage module, and (x) graphical representation module. The framework's architecture is depicted in Fig. 2 which shows the interaction among all the modules.

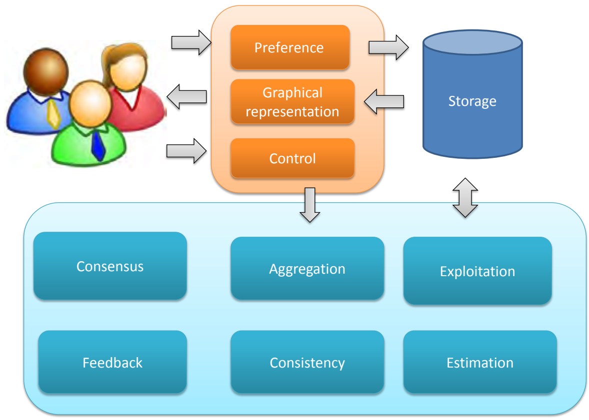

Figure 2: Architecture of the developed framework and the main interactions between modules.

In the following subsections, we describe the different modules, focusing on their characteristics and in relevant implementation details.

4.1. Control module

The framework presents a centralized architecture where the control module manages the whole system, and works as its starting point. This module carries out the following four main functionalities:

1. *Configuration parameters setting.* The system offers an user interface in which the user can set the main parameters of the GDM process (the meaning of most of this parameters will be explained in detail in the following subsections):

    - Working mode: test, normal.
    - Number of decision makers.
    - Path where the decision makers' preferences are located. This path could point to the computer file system or could be an URL.
    - Consensus threshold: γ parameter.
    - Max number of consensus round: $MaxIter$.
    - Feedback mechanism: automatic, semisupervised, non-supervised.
    - Type of distance used in the consensus process: Euclidean, Manhattan, Jaccard, Cosine, Dice.
    - Weighting vector for the aggregation.

- Exploitation type: dominance or non-dominance degree.
- Consistency vs Consensus: δ parameter.

2. *Communication with other modules.* In order to make the system fully upgradeable and extensible, the control module coordinates and initializes all the other modules.

3. *Control of the consensus rounds and the feedback mechanism.* This module also checks if enough consensus degree has been reached. Otherwise, it starts the feedback mechanism and, if necessary, asks to the decision makers to provide new fuzzy preference relations following some advice.

4. *Access to the data storage module.* This module has fully access to all the data frames in the system.

4.2. Preference module

This module is in charge of obtaining and adapting the decision makers' fuzzy preference relations. To that aim, we can distinguish two operation ways: test and normal modes.

- *Normal mode.* In this operation mode, the decision makers have to provide their complete or incomplete fuzzy preference relations by means of a CSV file, one file per decision makers, receiving the system one or more paths where the fuzzy preference relations are located. These paths can point to a file in the computer's file system where the program is running or to an URL where the files are located. In the last case, all the files will be automatically downloaded.

- *Test mode.* In this case a data set with the fuzzy preferences relations is automatically generated and the user only has to set the number of decision makers and the number of alternatives. The system can build both consistent and non-consistent fuzzy preference relations. Moreover, to test the quality of the available completion algorithms, incomplete fuzzy preference relations can be also generated.

Once all the fuzzy preference relations have been provided by the decision makers or generated by the system, they are included in an unique R data frame to be used in the next steps of the GDM process. Each row in the generated data frame corresponds to the preferences of one decision maker, not including the diagonal elements of the fuzzy preference relation. It is important to note that the system can work with any number of decision makers and any number of alternatives.

4.3. Estimation module

Prior to any other computation, the system needs to make sure that all the provided fuzzy preference relations are complete. To do so, this module carries out the iterative completion approach proposed in [27], which is based on the additive consistency property. This process is as follows: given an unknown preference value p_{ik}^h $(i \neq k)$, the iterative procedure starts by using intermediate alternatives, x_j, to create indirect chains of known preference values, (p_{ij}^h, p_{jk}^h), that will be used to derive, using the additive consistency property, the local consistency based estimated values:

$$ep_{ik}^{hj} = p_{ij}^h + p_{jk}^h - 0.5 \quad (4)$$

By averaging all the local consistency based estimated values, the overall consistency based estimated value is obtained:

$$ep_{ik}^{h} = \sum_{j=1, j\neq i,k}^{n} \frac{ep_{ik}^{hj}}{n-2} \tag{5}$$

In each iteration, the algorithm checks the set of pairs of alternatives for which the fuzzy preference values are unknown and can be estimated using known ones. It stops when this set is empty. Notice that the cases when an incomplete fuzzy preference relation cannot be successfully completed are reduced to those cases when no preference values involving a particular alternative are known, which means that a whole row or column of the fuzzy preference relation is completely missing. Finally, it is important to note that, although the approach proposed in [27] is used, any other algorithm of incomplete information [44] can be easily added to the framework.

4.4. Consistency module

This module calculates the self-contradiction level for each decision maker taking as a input his/her fuzzy preference relation. To that aim, it implements the consistency level based on the additive consistency proposed in [27], which defines the consistency level as the error between the provided preference relation and its estimated one.

The error between a preference value of a fuzzy preference relation P^h and its estimated one, computed in Eq. (5) is:

$$\varepsilon p_{ik}^{h} = |ep_{ik}^{h} - p_{ik}^{h}| \tag{6}$$

This definitions can be extended to calculate the consistency degree at three different levels, namely, pair of alternatives, alternatives and relation:

- Given a fuzzy preference relation P^h, the consistency level associated to the preference value p_{ik}^h is defined as:

  $$cl_{ik}^{h} = 1 - \varepsilon p_{ik}^{h} \tag{7}$$

  The lower the value of cl_{ik}^h, the higher the value of εp_{ik}^h and the more inconsistent is p_{ik}^h with respect to the rest of information.

- Given a fuzzy preference relation P^h, the consistency level associated to a particular alternative x_i is defined as:

  $$cl_{i}^{h} = \frac{\sum_{k=1, i\neq k}^{n} (cl_{ik}^{h} + cl_{ki}^{h})}{2(n-1)} \tag{8}$$

  The lower the value of cl_i^h, the more inconsistent these preference values are.

- The consistency level of a fuzzy preference relation P^h is defined as follows:

  $$cl^{h} = \frac{\sum_{i=1}^{n} cl_{i}^{h}}{n} \tag{9}$$

  When $cl^h = 1$, the preference relation P^h is fully consistent. Otherwise, the lower cl^h the more inconsistent is P^h.

Finally, in a GDM problem, the global consistency measure is computed as follows:

$$CL = \frac{\sum_{h=1}^{m} cl^{h}}{m} \tag{10}$$

When $CL = 1$, all the decision makers are completely consistent. The lower CL is, the more inconsistent the group of decision makers is.

4.5. Aggregation module

This module is in charge of fusing all the fuzzy preference relations, $\{P^1, \ldots, P^m\}$, given by the decision makers into a group one, P^c. To so, this module receives the data frame with all the preferences and stores the aggregated matrix in a separated data frame. The aggregation can be done in various ways:

- *Using an OWA operator* [46]. The OWA operator, ϕ_Q, carries out the aggregation as follows:

$$p_{ik}^c = \phi_Q \left(p_{ik}^1, \ldots, p_{ik}^m\right) = \sum_{j=1}^{m} w_j \cdot p_{ik}^{\sigma(j)}, \tag{11}$$

  where σ is a permutation function such that $p_{ik}^{\sigma(j)} \geq p_{ik}^{\sigma(j+1)}$, $\forall k = 1, \ldots, m-1$; Q is a fuzzy linguistic quantifier [49] that represents the concept of fuzzy majority [30] and it is used to calculate the weighting vector of ϕ_Q, $W = (w_1, \ldots, w_n)$ such that, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following expression [46]:

$$w_j = Q\left(j/n\right) - Q\left((j-1)/n\right), \ \forall j \in \{1, \ldots, n\} \tag{12}$$

- *Using an IOWA operator* [47]. Yager and Filev defined the IOWA operator as an extension of the OWA operator to allow a different reordering of the values to be aggregated. In this sense, in [27], an additive consistency based IOWA operator (AC-IOWA) was presented, where the ordering is induced depending on each decision maker's consistency, from the most to the least consistent one. In this case, the system automatically sets the decision makers' weights according to the consistency of the opinions provided by them in each round of consensus. To compute it, the system includes an implementation of the AC-IOWA operator presented in [27], although any other IOWA operator could be added [15].

- *Using a weighting vector set by the user.* This weighting vector can be used to set the decision makers' degree of importance in the aggregation. This way, the user can set the importance of each decision maker's opinion. This is specially useful in situations where the information handled by the decision makers is not equally relevant [15]. For example, when a group of medical experts expresses their opinions on the possible illness that a patient presents, its diagnostics must not be considered with equal relevance, given that, there will be more experienced medical experts, and, hence, all the opinions shall not be equally reliable.

4.6. Consensus module

This module receives as an input the data frame with all the decision makers' fuzzy preference relations and the aggregated matrix, and it calculates at each step of the process both the consensus degree, which measure the current level of agreement among all the decision makers, and the proximity measures, which quantify how far is each decision maker from the group opinion.

In order to reach agreement achieved among all the decision makers, the system computes coincidence existing among them [24]. To do so, the system, as most of the consensus approaches proposed in the literature, determines consensus degrees given at three different levels of a fuzzy preference relation [11, 23]: pairs of alternatives, alternatives, and relation.

1. For each pair of decision makers (e_h, e_l) $(h = 1, \ldots, m-1, \ l = h+1, \ldots, m)$ a similarity matrix, $SM^{hl} = (sm_{ik}^{hl})$, is defined as:

$$sm_{ik}^{hl} = 1 - d(p_{ik}^h, p_{ik}^l) \tag{13}$$

where $d : [0,1] \times [0,1] \to [0,1]$ is a distance function [18]. The closer sm_{ik}^{hl} to 1, the more similar p_{ik}^h and p_{ik}^l.

2. A consensus matrix, $CM = (cm_{ik})$, is calculated by aggregating all the $(m-1) \times (m-2)$ similarity matrices by means of an aggregation function, ϕ:

$$cm_{ik} = \phi(sm_{ik}^{hl}), \; h = 1, \dots, m-1, \; l = h+1, \dots, m \tag{14}$$

3. Once the consensus matrix has been computed, the consensus degrees are obtained at three different levels:

   (a) *Consensus degree on pairs of alternatives*, cp_{ik}. It is defined to measure the consensus degree among all the decision makers on the pair of alternatives (x_i, x_k). This is expressed by the element of the collective similarity matrix CM:

$$cp_{ik} = cm_{ik} \tag{15}$$

   (b) *Consensus degree on alternatives*, ca_i. It is defined to measure the consensus degree among all the decision makers on the alternative x_i, and it is obtained by aggregating the consensus degrees of all the pair of alternatives involving it:

$$ca_i = \phi(cp_{ik}), \; k = 1, \dots, n \wedge k \neq i \tag{16}$$

   (c) *Consensus degree on the relation*, CR. It expresses the global consensus degree among all the decision makers' opinions, and it is obtained by aggregating all the consensus degrees at the level of alternatives:

$$CR = \phi(ca_i), \; i = 1, \dots, n \tag{17}$$

It is clear that in any decision making process a high level of both consensus and consistency are necessary. To that aim, the system computes a *consensus/consistency level* (CCL) that needs to surpass a minimum threshold $\gamma \in [0,1]$ set as a control parameter in order to continue with the exploitation phase:

$$CCL = (1-\delta) \cdot CL + \delta \cdot CR \tag{18}$$

The parameter $\delta \in [0,1]$ is set by the user depending on how important is consistency and consensus in the final solution. If this minimum threshold γ is not surpassed, the consensus-feedback process will keep running until the minimum threshold γ is surpassed or the maximum number of consensus rounds $maxIter$ have been reached. This maximum number of iterations is incorporated in order to avoid that the consensus process does not converge after several rounds of discussion.

The proximity measures for each decision maker are calculated based on the collective preference relation, P^c:

1. The proximity measure of a decision maker e_h on the pair of alternatives (x_i, x_k) to the group one, denoted as pp_{ik}^h, is calculated as:

$$pp_{ik}^h = 1 - d(p_{ik}^h, p_{ik}^c) \tag{19}$$

2. The proximity measure of a decision maker e_h on alternative x_i to the group one, denoted as pa_i^h, is calculated as:

$$pa_i^h = \phi(pp_{ik}^h),\ k = 1,\ldots,n \wedge k \neq i \tag{20}$$

3. The proximity measure of a decision maker e_h on his/her preference relation to the group one, denoted as pr^h, is calculated as:

$$pr^h = \phi(pa_i^h),\ i = 1,\ldots,n \tag{21}$$

These proximity measures will be used by the feedback module to identify those decision maker who are far from the collective solution and give them some recommendations about how they should change their preferences to reach an acceptable level of consensus.

The developed module enables to calculate the distance among the decision makers' preferences following various distance functions: Manhattan, Euclidean, Dice, Cosine, and Jarccard distance [16, 18]. That way the user can set the most suitable distance depending on the characteristics of the GDM process, such as the number of decision makers or the maximum number of possible rounds. For example, in [16], it is proved that the Manhattan and the Euclidean distances increase consensus level as the number of decision makers increases and help the consensus process to converge faster than the other ones. On the other hand, the Cosine and the Dice distances results in a fairly similar consensus levels regardless of the number of decision makers, whereas the Jaccard distance function contributes the least to the speed of convergence of the consensus processes. In addition, although the arithmetic mean is used by the system an aggregation function, ϕ, different aggregation operators could be used depending on the nature of the GDM problem to solve [16].

The results of the computation of both the consensus degrees and the proximity measures in the different consensus rounds are stored in a data frame, one data frame per consensus/proximity level. That way, the system keeps track of all the intermediate results generated during all the rounds of the process.

4.7. Feedback module

The aim of the feedback mechanism is to provide advice to the decision makers using consensus/consistency criteria to easily reach the desired consensus level while keeping a high consistency level in the decision makers' fuzzy preference relations. To do so, this module carries out two main tasks: (i) identification of the preference values, and (ii) generation of advice.

A three step process is carried out to identify the decision makers, the alternatives and, finally, the particular preference values, that contribute less to the consensus/consistency level.

- To identify the fuzzy preference relations that need to be modified, the system first identifies the decision makers whose consensus/consistency level of the fuzzy preference relation is lower than the threshold value γ:

$$EXPCH = \{h \mid (1-\delta)\cdot cl^h + \delta \cdot pr^h < \gamma\} \tag{22}$$

- Then, the system selects among those decision makers' alternatives with a consensus/consistency level lower than the threshold value γ:

$$ALT = \{(h,i) \mid h \in EXPCH \ \wedge \ (1-\delta)\cdot cl_i^h + \delta \cdot pa_i^h < \gamma\} \tag{23}$$

- Finally, the fuzzy preference values to be modified are those with an associated consensus/consistency level lower than the threshold value γ:

$$APS = \{(h,i,k) \mid (h,i) \in ALT \ \wedge (1-\delta) \cdot cl_{ik}^{h} + \delta \cdot pp_{ik}^{h} < \gamma\} \tag{24}$$

Once the decision makers' preferences, which need to be modified in order to increase the consensus/consistency, have been detected, the system has to carry out those changes. To that aim, this module is able to work following three different operation modes depending on the degree of human intervention [35]:

- *Automatic mode.* In this mode, the system automatically changes the decision makers' fuzzy preference relations according to the recommendations, rp_{ik}^{h}, which are generated using the following equation:

$$rp_{ik}^{h} = (1-\delta) \cdot cp_{ik}^{h} + \delta \cdot p_{ik}^{c} \tag{25}$$

- *Semi-supervised operation mode.* In this case, the system carries out an aggregation of the original values of the decision makers' fuzzy preference relations and the recommended one. The weight value to carry out this combination can be set in the control module. There can be set one value for each decision maker.

- *Fully-supervised operation mode.* In this case, the system provides easy to follow rules. The system saves the recommendations for each decision maker in one text file per decision maker. That way each decision maker can access confidentially to the system's recommendations.

4.8. Exploitation module

This module receives as an input the matrix in which the opinions of all the decision makers have been aggregated, and provides a global ranking of the alternatives. The global ranking can be calculated following one of the following two choice degrees [22]: the quantifier guided dominance degree ($QGDD$) and the quantifier guided non-dominance degree ($QGNDD$).

- For the alternative x_i, the system computes the quantifier guided dominance degree, $QGDD_i$, which quantifies the dominance that alternative the x_i has over all the others in a fuzzy majority sense as follows:

$$QGDD_i = \phi_Q(p_{i1}^{c}, p_{i2}^{c}, \ldots, p_{i(i-1)}^{c}, p_{i(i+1)}^{c}, \ldots, p_{in}^{c}) \tag{26}$$

- For the alternative x_i, the system computes the quantifier guided non-dominance degree, $QGNDD_i$, which gives the degree in which the alternative x_i is not dominated by a fuzzy majority of the remaining alternatives. It is defined as follows:

$$QGNDD_i = \phi_Q(1-p_{1i}^{s}, 1-p_{2i}^{s}, \ldots, 1-p_{(i-1)i}^{s}, 1-p_{(i+1)i}^{s}, \ldots, 1-p_{ni}^{s}) \ , \tag{27}$$

where $p_{ki}^{s} = max\{p_{ki}^{c} - p_{ik}^{c}, 0\}$ represents the degree in which x_i is strictly dominated by x_k.

4.9. Storage module

Various storage data structures have been developed to easily store and manage all the information produced by the system. To that aim, we have take advantage of the R built-in data structure, the data frame. As it has been metioned previously an R data frame consist in a 2D matrix whose elements can be of any type. Various data frames have been implemented for the storage module. That way, each one of the modules above stores their information in its corresponding data frame. Therefore, it is pretty easy to retrieve the necessary information for each module or to include new modules that use this information. The main data frames implemented are the following:

- *CurrentPreferences.* In each row of this data frame are stored each decision maker's fuzzy preference relation. The size of this data frame is $m \times n(n-1)$.
- *PreferenceList.* This is an array of data frames, which store the *CurrentPreferences* in each round of the consensus process.
- *ConsensusLevel1.* In each row of this data frame are stored the computation of the consensus at the preferences level.
- *ConsensusLevel2.* In each row of this data frame are stored the computation of the consensus at the alternative level.
- *GlobalConsensusPerRound.* This data frame has only one column and stores in each row the global consensus level obtained in each round.
- *ProximityLevel1.* In each row of this data frame are stored the computation of the proximity between the decision maker's fuzzy preference relation and the aggregated matrix at the preferences level.
- *ProximityLevel2.* In each row of this data frame are stored the computation of the proximity at the alternative level.
- *GlobalProximityDecisionMakers.* This data frame stores in each row the global proximity between each decision maker and the aggregated matrix.
- *GlobalProximityPerRound.* This data frame stores in each column the decision makers' average proximity for each consensus round.
- *GlobalConsistencyPerRound.* This data frame stores in each column the decision makers' average consistency in each round.

4.10. Graphical representation module

One of the main novelties that the developed system presents with respect to the existing ones is the possibility of getting a quick insight in the GDM process by means of diverse graphical representations. All these representations make the system a graphical monitoring tool to support decision makers by providing them with easily understandable visual information about the current status and the evolution of the decision process. This tool eases the analysis of diverse crucial aspects that are common in these problems, among them, we can highlight:

- Monitoring the evolution of the global consensus across the whole GDM process.

- Monitoring the decision makers' consistency along the whole GDM process. This is especially important to make sure that they are keeping an acceptable consistency level in their preferences after the recommendation rounds.
- Detection of the alternatives that are posing more controversy in the GDM process.
- Detection of those decision makers or group of them, whose preferences are further from the consensus solution, or those that are more reluctant to change their point of view.
- Detection of those decision makers that are being influenced or manipulated to provide preferences far from the consensus solution.
- Providing information to the decision makers about the GDM process, and showing them how their preferences are located with respect to the consensus one.

In the following the graphics that our system includes and how they have been developed and integrated in this framework are detailed. The graphical representations that our system includes can be divided in two wide groups, depending on whether they show the evolution among the various consensus rounds, or they display information related to a single round:

- Representation of the evolution across the consensus rounds:
  - *Consistency vs consensus evolution in the GDM process.* This representation shows the evolution of both global consistency and global consensus in each consensus round. The desirable situation is that most of the point or at least the final ones lie over the diagonal line and the points present a positive tendency. It would mean that the final solution has reached a high level of agreement and that it is consistent. This representation also enables to detect whether the consensus process is not only helping to bring the decision makers' opinions closer but also to keep or increase their consistency.
  - *Decision maker's consistency vs decision maker's consensus in the GDM process.* This representation allows to check how decision makers' consensus and consistency evolves during the GDM process. It also enables to visually check the different decision makers profiles depending on the shape of the curve for each decision maker. Curves with a positive tendency and located over the diagonal represent the desired situation of those decision makers that are more willing to change their opinions in the interest of increasing the global consensus while keeping a highly consistency level. Curves parallel to the y-axis represents those decision makers which are reluctant to change their mind during the process, and therefore they may require special attention.
- Representation of the consensus state in a single round:
  - *Barplot of each decision maker's proximity to the aggregated solution.* This representation enables to check who are the decision makers whose opinions are closer to achieve a high degree of consensus, and who are those with highly disagree with the proposed solution.
  - *Barplot of the average consensus achieved for each alternative.* This representation allows to quicky identify which alternatives are posing more controversy in the decision process.

– *Barplot of the average consistency achieved for each decision maker.* This representation provides a quick insight on those decision makers providing more consistent fuzzy preference relations in the decision making process.

– *2D representation map of the decision makers' fuzzy preference relations and the consensus solution.* This representation provides a quick insight of the current state of the decision process and enables the rapid identification of sub groups of decision makers who share similar opinions. It also eases the detection of conflicts among decision makers. Moreover, it provides the decision makers with a good idea about the status of the consensus process and how far their opinions are from the consensus solution. This 2D representation is obtained after carrying out a classical 2D multidimensional scaling reduction of the decision makers' fuzzy preference relation matrix [21]. In addition, R also offers the possibility of non metric multidimensional scaling.

– *3D representation of the position of each decision maker with respect to the consensus solution among with their consistency.* This plot easily allows to identify those groups of decision makers that are far from the consensus solution but keep a high degree of consistency, and, therefore, need special attention. To easily visualize this plot, we have also included a interactive representation.

5. Illustrative example

In this section, we include two illustrative examples to show how the GDM-R works and its wide range of possibilities as well as its usefulness in practice. The first example illustrates how the test environment works and explains the different graphical representations available in the system. The second example includes real data taken from the experts and deals with incomplete information in the decision makers opinions as shows how the supervised operation mode works.

5.1. Example illustrating the consensus process and the graphics analytics

5.1.1. Problem definition and parameter setting

In order to show the test capabilities of the developed framework, the test module is used to generate a data set with the decision makers' preferences. The configuration parameters are set as follows: In this example a GDM situation involving 20 decision makers and 4 different alternatives is considered. The minimum consensus threshold to be achieved is 0.8, and the maximum number of consensus rounds is 4. In addition, the initial average level of consistency of the fuzzy preference relations is 0.8 and the initial average level of consensus is 0.6.

Configuration parameters

```
M=20 #Number of decision makers
N=4 #Number of alternatives
prefererecies_file=''#File with the decision makers' preferences
consensusThreshold=0.8
numberOfRounds=4
distance='euclidean'
quantifierAggregation='most'
dominance="QGDD"
quantifierExploitation='most'
feedback="automatic"
```

```
operationMode="Simulation"
initialConsensus=0.6
initialConsistency=0.8
```

5.1.2. Graphical representations for each round of consensus

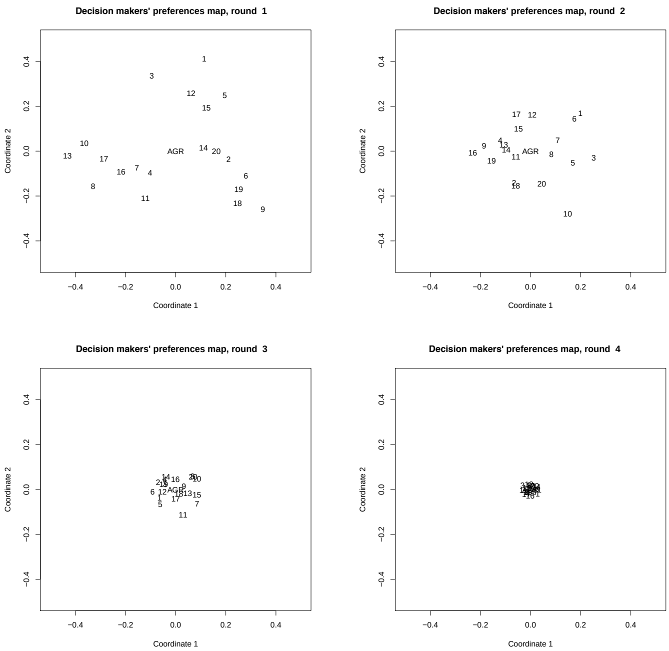

Table 1: Evolution of the decision makers' preferences among the consensus rounds.

In the following we show some of the most relevant representations along with their utility to increase the quality of the GDM process are explained.

First of all, a 2D map with the position of each decision maker with respect to the aggregated solution for each consensus round is depicted in Table 1. The global solution is always displayed in the center of the plot. This type of visualization allows to ease the rapid detection of those decision makers whose opinions are far from the global solution as it is the case of decision maker 1. Hence, in real case situations, some especial actions can be taken depending on the characteristics of the process, such as discarding their opinions as they can be considered as outliers. Furthermore, it is possible to recognize how in the first round, the preferences are in general pretty spread up, but after each round of recommendations the opinions of the decision makers get closer and closer verifying that the decision making process is going on the right direction. It worths to point out that this type of maps also allow to easily recognize those decision makers reluctant to change their opinions in order to achieve a solution accepted by the whole group by keeping track of those whose position in the map does not get closer to the global solutions with the iterations. In this

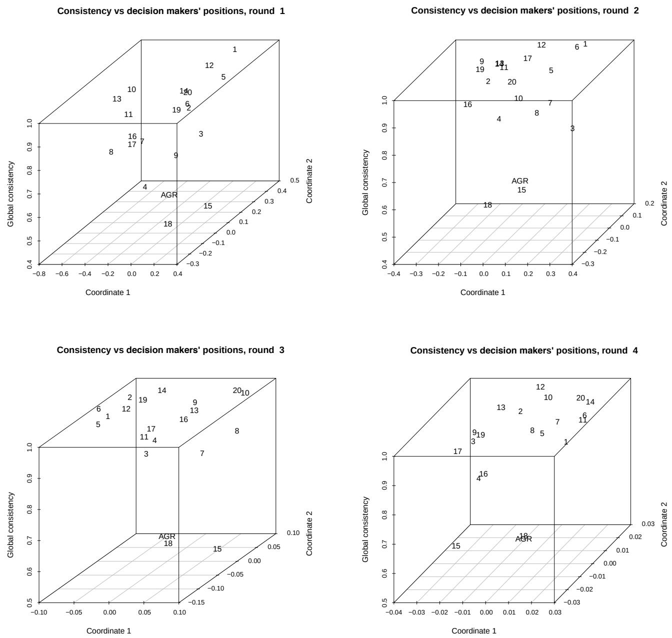

Table 2: Evolution of the decision makers preferences among the consensus rounds.

sense small sub-communities of decision makers that share similar opinions can be identified as well as well as and those who exert a greater influence on their sub-communities.

A 3D map of the decision makers preferences among with the degree of consistency for each decision maker is shown in Table 2. This type of map allow to recognize the decision makers whose preferences are more consistent and her distance to the global solution. For example, in the first iteration the decision maker number 1, presents a very high level of consistency even though his/her preferences are far from the consensus solution. Therefore, this decision maker's opinions are worth to be taken into consideration. It also allows to quickly recognize communities of decision makers who share the same points of views, and also identify those decision makers who have more influence or more persuasion power over the group. They can be recognized easily since they do not change their opinions with the time, but they attract others forming small clusters in the map that become bigger with the time. Usually, the most influential decision makers also present a high level of consistency.

In Table 3 it is presented a barplot with the decision makers average consensus and consistency degree per round, along with both lines showing the global average consensus and consistency degrees. These plots easily allow to asses the evolution of both consensus and consistency and recognize those decision makers that may present more controvert opinions, or less consistent ones, and take especial actions with those ones.

5.1.3. Results of the GDM process

The decision making process finishes when the maximum number of rounds has been overpassed or when the desired consensus degree has been achieved. In Fig. 3 it is depicted the evolution

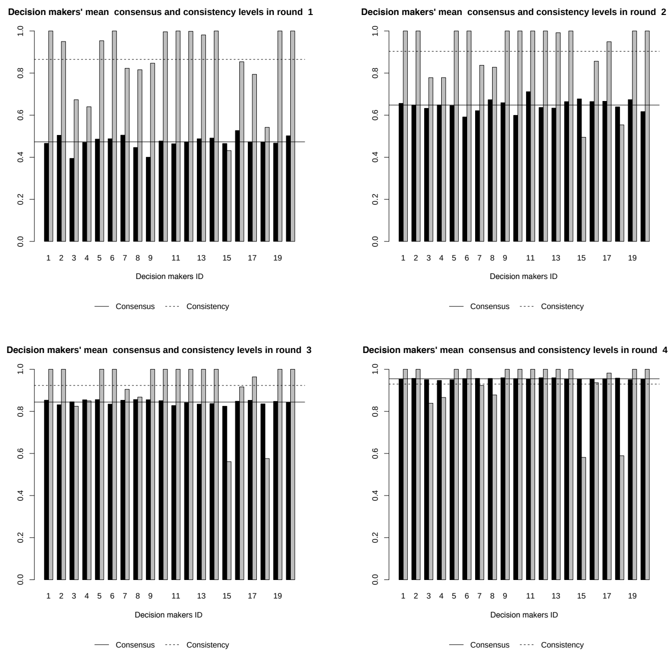

Table 3: Evolution of the decision makers' consistency and consensus in each round.

of the degree of consensus vs the degree of consistency for each iteration. Notice that the line slope in this plot allows to easily recognize how the decision process has gone. For example, if the line is almost parallel to the x-axis it means that the iterations of the decision process have only contributed to increase the global consistency. That is, in average the decision makers' opinions have become more consistent with the time, but the decision makers had not change their mind to increase the consensus. This type of line means that the decision makers are very committed to provide non contradictory solutions to the problem, but they present a non cooperative behavior towards achieving a solution accepted for the whole group.

A similar situation would happen if the line is parallel to the y-axis, but in this case it would mean that the consensus has improved whereas the decision makers consistency has barely changed.In this case that would mean that the decision makers are easily manipulated to change their minds, without caring about the quality of the provided solution.

The most desired solution is having a line with positive slope, like the one in Fig. 3, that means that the different rounds have contributed to positive increase both the consensus and the consistency of the decision makers. Also the average slope of this line also provide us with a general measure of how fast the consensus increase vs the consistency, this measurement can be leverage to test the performance of different decision making approaches.

Finally, the system provides a graphical representation with the ranking of the alternatives using both the dominance and the non-dominance degrees as the one presented in Table 4. In this concrete case it was clear that the most desired alternative was the number two.

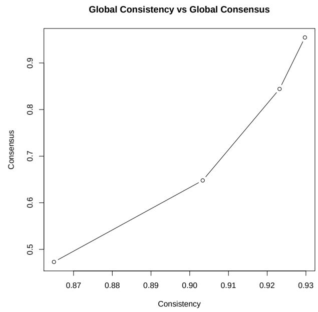

Figure 3: Global consensus and consistency evolution along the consensus rounds.

5.2. GDM process with incomplete information

In this section we analyze a real example in which 4 experts have been asked to provide their opinions to choose the best mobile phone from four different models taking into account the relationship between quality and price.

- Huawai P8
- Nexus 6
- iphone 6
- Samsung Galaxy S6

In this case the system works in supervised mode, that means that all the changes proposed by the framework needs to be accepted by the decision maker before being included in the preferences. The minimum consensus threshold to be achieved is 0.8, and the maximum number of consensus rounds is 4. The configuration parameters are as follows:

Configuration parameters

```
M=4 #Number of decision makers
N=4 #Number of alternatives
consensusThreshold=0.8
numberOfRounds=4
distance='euclidean'
quantifierAggregation='most'
dominance="QGDD"
quantifierExploitation='most'
feedback='supervised'
```

In this case we are going to focus on the interaction of decision maker e_1 with the system. The same can be applied to the rest of the experts.

First of all, the expert e_1 provides the following preference values: $p^1_{12} = 0.6$ and $p^1_{13} = 0.6$:

$$P^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ x & - & x & x \\ x & x & - & x \\ x & x & x & - \end{pmatrix}$$

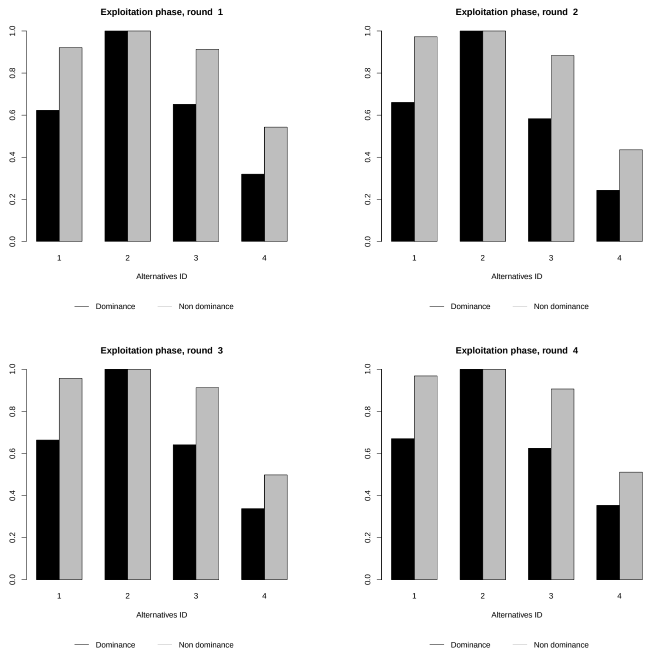

Table 4: Dominance and non-dominace degrees in the exploitation phase.

The system, using these values estimates the values p^1_{23}, p^1_{24}, p^1_{32} , p^1_{34}, p^1_{42}, p^1_{43}, and the system presents the following matrix to the user to be approved.

$$P^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ x & - & 0.9 & 0.7 \\ x & 0.1 & - & 0.3 \\ x & 0.3 & 0.7 & - \end{pmatrix}$$

Expert e_1 considers the estimated value p^1_{32} does not reflect her real preference value and $p^1_{32} = 0.8$ is inserted instead. The system alerts that there is a high inconsistency associated to this new value, and consequently e_1 realizes that there is contradiction in her preference relation ($p^1_{23} = 0.9 \Rightarrow x_1 \succ x_2$ and $p^1_{32} = 0.8 \Rightarrow x_2 \succ x_1$) and changes p^1_{32} to the value that the system initially suggested ($p^1_{32} = 0.1$).

Finally e_1 completes her preference relation accepting the values estimated by the system.

$$P^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix}$$

For the rest of the experts the system follows a similar behavior. Firstly, the experts provide the following incomplete matrices:

$$P^2 = \begin{pmatrix} - & x & 0.7 & x \\ 0.4 & - & x & 0.7 \\ 0.3 & x & - & x \\ x & 0.4 & x & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & x & 0.7 & x \\ 0.4 & - & x & 0.7 \\ 0.3 & x & - & x \\ x & 0.4 & x & - \end{pmatrix}$$

$$P^4 = \begin{pmatrix} - & x & 0.7 & x \\ 0.4 & - & x & 0.7 \\ 0.3 & x & - & x \\ x & 0.4 & x & - \end{pmatrix}$$

Which are automatically completed by the system as follows:

$$P^2 = \begin{pmatrix} 0.50 & 0.4 & 0.30 & 0.25 \\ 0.62 & 0.5 & 0.40 & 0.40 \\ 0.70 & 0.6 & 0.50 & 0.45 \\ 0.80 & 0.7 & 0.57 & 0.50 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.50 & 0.60 & 0.46 & 0.30 \\ 0.30 & 0.50 & 0.31 & 0.40 \\ 0.54 & 0.69 & 0.50 & 0.27 \\ 0.75 & 0.87 & 0.73 & 0.50 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} 0.5 & 0.4 & 0.5 & 0.7 \\ 0.6 & 0.5 & 0.6 & 0.7 \\ 0.6 & 0.4 & 0.5 & 0.7 \\ 0.3 & 0.2 & 0.3 & 0.5 \end{pmatrix}$$

In figure 4 a map with the experts preferences with respect to the global solution for the first iteration is depicted:

In this first iteration the consensus level reached $CR = 0.76$ and the Global Consistency $CL = 0.97$.

Since the required consensus level has not been achieved the system suggests to the experts various changes to increase the consensus level, for example for the case of expert 1 the recommendations are as follows:

```
Provide a value for (2, 1) close to 0.52
Provide a value for (2, 3) close to 0.8
Provide a value for (2, 4) close to 0.76
Provide a value for (3, 1) close to 0.44
Provide a value for (3, 2) close to 0.2
Provide a value for (3, 4) close to 0.42
Provide a value for (4, 1) close to 0.38
Provide a value for (4, 2) close to 0.35
Provide a value for (4, 3) close to 0.6
```

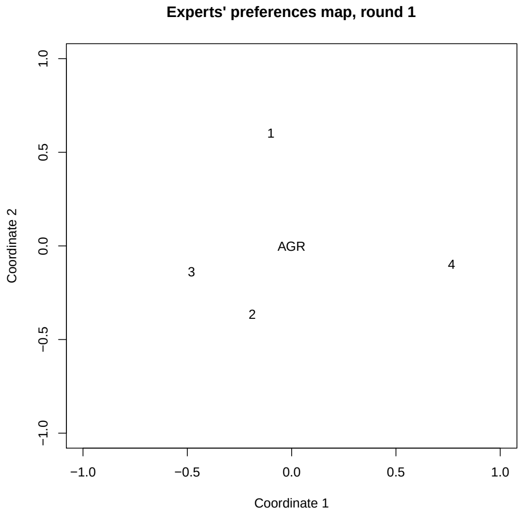

Figure 4: Experts preferences with respect the global one after the first round of consensus

Since the system is working on the supervised mode the experts are the ones that decide whether they accept the proposed recommendations or not. In figure 5 a map with the experts preferences after the feedback round is depicted.

In this second iteration the consensus level reached is $CR = 0.88$ and the Global Consistency $CL = 0.91$. Therefore enough level of agreement is achieved and no more iterations are required. Finally the result of the decision process indicates that the winner alternative is the number 3, see figure 6.

6. Conclusion and future work

In this contribution we have presented a critical review of the available software frameworks for computer assisted GDM, concluding that there are few available tools an the ones that have already been developed are not open source and are not able to carry out GDM processes including multiple types of preference elicitations and ways of dealing with unknown information. Moreover the majority of these tools present a non modular architecture which makes very complex for other researcher to extend or adapt to their own necessities or with test purposes.

In this contribution, we present GDM-R, a new open source framework fully implemented in R, overcoming the weaknesses of the previous software systems for GDM processes. Its main new and interesting aspects are summarized below:

- It displays various graphical representations which provide a rapid insight in the state and the evolution of the GDM process and enable to identify decision makers whose opinions are far from the group solution and those who present a non cooperative behavior in order to reach an agreement among with experts subcommunities and more influential decision makers.

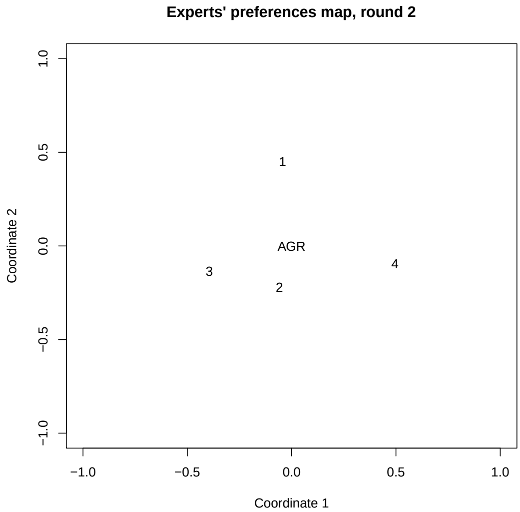

Figure 5: Experts preferences with respect the global one after the first round of consensus

- It offers a test mode which enables to set a trial scenario to try and compare the performance of different GDM approaches. It is helpful to validate and objectively compare the already existing algorithms and to develop new ones.
- The proposed framework can be easily extended to work with other types of preference relations and to include other methodologies of GDM. Therefore, other researchers can extend and customize it for comparative and test purposes.
- The developed system can be easily adapted to work in other environments such as smartphones, tablets and web, since the logic of the application is totally independent from the graphical user interface. .

As future work, we point out several directions as the extension of this framework to work with different platforms such as mobile and web based environments, allowing to carry out GDM processes in environments in which the decision makers can access to the decision process from different clients. In addition, more complex approaches based on ontologies [38] and trust networks [45] will be validated and incorporated.

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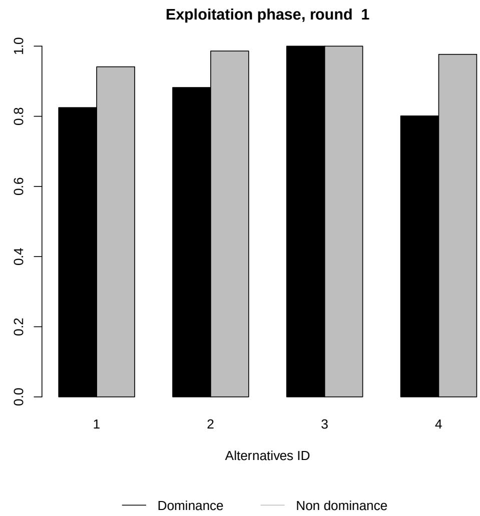

Figure 6: Results of the GDM process

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