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BREAKING THE ADDITION ADDICTION:

CREATING THE CONDITIONS FOR KNOWING-TO ACT IN EARLY ALGEBRA

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Abstract

We use data from a teaching experiment with a group of eight years old students to explore the potential of examining number sentences to promote relational thinking. This type of thinking requires attention to mathematical structure through consideration of relationships between terms contained in the sentence and not just on computation and comparison of the numeric values of each side. We show that children came to "know-to act" in the context of written activities and orchestrated discussions about number sentences, overcoming some of their computational habits and developing new ways to see and more flexibly approach the sentences. The results help to advance the study of young students´ emergent algebraic modes of thinking.

INTRODUCTION

In the last two decades, within the mathematics education research community there has been a strong interest in analyzing and promoting the integration of algebraic thinking in the elementary curriculum. This curricular proposal, called Early Algebra, aims not just to facilitate the later learning of algebra, but to foster students´ conceptual development of deeper and more complex mathematics from very early ages (Blanton & Kaput, 2005; Kaput, 1998). Algebraic ways of thinking are considered to naturally emerge from elementary mathematics and to have the potential to enrich school mathematics activity. This proposal is based on a broad conception of algebra which includes the study of functional relations, the study and generalization of patterns and numeric relations, the study of structures abstracted from computation and relations, the development and manipulation of symbolism, and modelling as a domain of expression and formalization of generalizations (Kaput, 1998).

The Early Algebra view is shared by other researchers (Hewitt, 1998; Mason, Graham, Pimm, & Gowar, 1985) who consider generalization as the root of algebra and highlight the role of algebraic thinking in arithmetic. They argue that arithmetic learning requires students to interiorize generalities about the structure of arithmetic as well as to develop (general) methods to compute and solve problems. All these authors agree that arithmetic teaching needs to provide students opportunities for:

(a) Appreciation of patterns and verbalizing and recording generalizations, as first steps towards symbolically expressing generalizations, and

(b) becoming aware and making explicit the structure of arithmetic, which is required in order to later be able to use arithmetic structure in algebraic contexts.

In essence, these recommendations argue that algebraic thinking requires children to approach numbers and equations from a structural perspective rather than an operational one, treating expressions as objects instead of processes (Sfard, 1991). These claims are based on the recognition of the poor understanding of relations and mathematical structure that students tend to develop as a result of traditional arithmetic teaching (Kieran, 1989).

In this chapter we use Mason & Spence's (1999) view of mathematical thinking as "knowing-to act" to argue that to engage in algebraic thinking students have to break some habits, and that carefully engineered number sentences along with carefully orchestrated discussions can be a means to that end. Some specific elements which promote students´ "knowing-to act" in this context are identified.

RELATIONAL THINKING

We focus our attention on a specific type of algebraic thinking, specifically, *relational thinking*. This term refers to students´ recognition and use of relationships between elements in number sentences and expressions (Carpenter, Franke & Levi, 2003; Molina & Ambrose, 2008; Stephens, 2006). When using relational thinking, students consider the sentence and expressions as wholes (instead of as processes to carry out step by step), analyze them, discern some details and recognize some relations, and finally exploit these relations to construct a solution strategy (In a broader context Hejny, Jirotkova & Kratochvilova (2006) named this approach as conceptual meta-strategies). For example, to determine if number equations such as (a) $7+7+9=14+9$ or (b) $27+48-48=27$ are true or false, instead of doing the computations on both sides and comparing both results, students may solve[1] them by looking at the whole sentence, appreciating its structure (e.g., there are operations on both sides of the equal sign or there aren't) and using perceived relations between its elements (e.g., 9 appears in both sides; $7+7$ on the left side adds to the other term on the right side, 14; the same number is being added to and then subtracted from 27) as well as knowledge of the structure of arithmetic to determine the truth or falseness of the sentence.

The arithmetic expressions involved have to be considered from a structural perspective rather than simply a procedural one. The expression or object "$7+7+9$" is compared to the expression or object "$14+9$" to consider their equivalence rather than acting on each expression to determine its value. This implies a subtle but important change in students´ attention from reading the equation from left to right one piece at a time with a computational perspective, to looking at each side of the equal sign and comparing the two expressions to one another (Mason, Drury, & Bills, 2007).

Mason and Spence (1999) distinguish "knowing-to act" among other less sophisticated ways of knowing: knowing-that, knowing-why, knowing-how, and knowing-about. "Knowing-to act" refers to the use of active knowledge, that is, "*knowledge that enables people to act creatively rather than merely react to stimuli with trained or habituated behaviour*" (p.136). This type of knowledge, which is contrasted to inert knowledge, is characterized by being transferable to other (new) contexts/situations. This happens because something in the new situation resonates with past experience: "*The state of sensitivity-awareness of the individual, combined with elements of the situation which metonymically trigger or metaphorically resonate with experience, are*

[1] Along the paper we use the expression "to solve" the true/false sentence to briefly refer to determine if the sentence is true or false.

what produce the sudden knowing-to act in the moment" (p.147). In order for this transference to occur, some elements in previous experiences must have been labelled or articulated in some way so that later, knowledge from that experience can be triggered in new situations.

Mason & Spence use the idea of "know-to act" in the moment to discuss the need to get students "unstuck" or to use the knowledge from previous experience in fresh situations where it would be helpful. They note that students too often struggle with mathematics because they fail to apply what they have learned. Here, we are concerned with helping students to "see" number sentences in new ways and to develop flexibility in approaching them by using their previous arithmetic experience. In both cases the aim is to help students develop their awareness of the structure of arithmetic.

One of the hurdles to getting students to use relational thinking is overcoming "habituated behaviour" because they have to resist the impulse to compute. This assumes that the students have some ideas about arithmetic operations that will allow them to employ relational thinking. To use Mason and Spence's (1999) terms, they "know about" addition but do not always act on that knowledge. The presence of the equal sign, or just the presence of numbers and operational signs, leads them to make computations and obtain the numeric values, i.e., what the students think of as the "answer". Students have developed the "know-how" of computing and have been practicing it so much that they fail to notice (or to attend to) other aspects of expressions. To promote relational thinking, the teacher has to help children break their computational habits, in other words, to break the addition addiction, so that they can look at equations/expressions differently. We engaged in a teaching experiment to see if this was possible.

THE TEACHING EXPERIMENT

Design of the Experiment

Our teaching experiment shared features of design experimentation[2]. The general aim of this study was to analyze the emergence, development and use of relational thinking in a group of third graders. We worked with a class of 26[3] eight-year old Spanish students (12 male and 14 females) from a state school in the region of Granada (Spain). Three of the students received extra support in mathematics at school. The selection of this group of students was due to its availability to participate in the study. We include below a brief description of the design of each session[4].

Our teaching experiment consisted of six one-hour in-class sessions, over a period of one year. This timeline was chosen intentionally (except from vacation periods) because we wanted (a) our intervention to have a longer effect, (b) to diminish the probability of assessing memory-based learning and (c) to have enough time between sessions to analyze the data of the previous session and take decisions about the next one.

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[2] For further information see Molina, M., Castro, E., & Castro, E. (2007). Teaching Experiments within Design Research. *The International Journal of Interdisciplinary Social Sciences*, 2(4), 435-440.

[3] The results will only refer to twenty-five students, as the other student only attended session 1 and 4, and he did not solve the written assessment of session 4.

[4] For further information about the justification of the design of each session see the first author's PhD thesis at http://cumbia.ath.cx:591/pna/Archivos/MolinaM07-2822.pdf.

We provided students with number sentences in the context of individual written activities, whole group discussions and individual interviews. We included action sentences (i.e., sentences with all the operations on one side of the equation as in $15+5-3=17$) and non-action sentences[5] (i.e, sentences with operational symbols on both sides or with no operational symbols as in $14+6=10+10$ and $12=12$, respectively). They involved numbers of one, two or three digits and the addition and subtraction operations. These sentences (not all of which were true) were based on the following arithmetic properties and relations:

- commutative property of addition (e.g., $10+4=4+10$),
- non-commutability of subtraction (e.g., $15-6=6-15$),
- inverse relation of addition and subtraction (e.g., $100+94-94=100$; $122+35-35=122$),
- compensation relation (e.g., $13+11=12+12$; $78-45=77-44$),
- unity element (e.g., $0+325=326$; $125-0=125$),
- inverse element (e.g., $100-100=1$),
- composition/decomposition relationships (e.g., $7+7+9=14+9$; $78-16=78-10-6$)
- relative size comparisons[6] (e.g., $37+22=300$; $10-7=10-4$; $72=56-14$).

Therefore, all the sentences could be solved by using relational thinking as well as by computing. We wondered if certain kinds of sentences were more likely to promote relational thinking than others.

We chose the context of number sentences because (a) it can be a context very rich in patterns (especially patterns related to arithmetic structure), and (b) it is strongly connected to algebraic symbolism. This context offers the possibility to promote the following algebraic elements:

- The conception of expressions as wholes which can be compared, ordered, made equal, transformed, and therefore, the acceptance of lack of closure (i.e. working with expression without knowing their numeric value or not having it expressed in the sentence).
- The use of horizontal language which traditionally has been more typical of algebra than of arithmetic.
- A two-way interpretation of number sentences as well as their exploration as representations of a static relation between two expressions.

Due to the different objectives of each session (described below) we used open number sentences in session 1 and part of session 2, and true/false sentences in the rest. Open number sentences have proved to be useful for revealing different conceptions and challenging children to reconsider their interpretations of the equal sign, while true/false sentences help to challenge students' computational mindset (Molina & Ambrose, 2008). Students had to complete the open sentences and explain how they solved it. In

[5] This classification of number sentences comes from Behr, M., Erlwanger, S., & Nichols, E. (1980). How children view the equal sign. *Mathematics Teaching, 92*, 13-15.

[6] Here we consider sentences in which students can determine the validity of the sentence by attending to the size of the numbers involved and using knowledge of the effect of operations in the size of numbers.

the true/false sentences, they had to determine if the sentences were true or false and to justify their answers. In the discussions we always encouraged students to share the strategies they used to solve the sentences.

The first two sessions were directed at exploring and extending students' understanding of the equal sign and detecting spontaneous evidence of use of relational thinking. We also analysed students´ difficulties in solving the proposed number sentences. From the third session on, we promoted students´ *use* and *verbalization* of relational thinking by encouraging working on the same sentence in different ways, asking students for ways of solving the sentences without doing all the computations, and by showing a special appreciation of explanations based on relations. We did not promote the learning of specific relational strategies but the development of a habit of looking for relations, trying to help students to make explicit and apply the knowledge of structural properties which they had from their previous experience with arithmetic. Session 3, 4, 5 and 6 also aimed to identify the strategies used by the students when solving the sentences and to detect and analyze students´ difficulties.

We video-recorded the sessions, audio-recorded individual interviews with students and collected the students´ worksheets yielding an exhaustive collection of data about the students´ thinking while solving the proposed number sentences. In between our in-class interventions, the official teacher faithfully followed a textbook which was mainly centred on computational practice. Some mental computation strategies were introduced, but their use was not practiced more than once. The students never had opportunities to work on non-action number sentences.

Students' strategies

As we expected, at the beginning of the experiment the students demonstrated their computational habit. For example, in the sentence $6+4+18=10+18$ a student explained "It is true because $6+4+18=28$ and $10+18=28$"[7]. This student used the vertical standard addition algorithm to compute $6+4+18=28$ and $10+18=28$. Other students did the computation mentally or by counting. When students computed the numeric value of each side of the sentence, their attention seemed to be focused on the numbers and operations to perform on them, considering each side, or even each operation, separately. They did not provide any evidence of noticing any relation or characteristic of the sentence apart from the numbers in it, the operations which combined them and the presence of the equal sign. On a few occasions students attended to the size of the numbers involved to decide on which computation to perform first. In these few cases they used relations to inform how to address the computation. Following this computational habit, when asked to provide a different explanation for the same sentence, students proposed a different order in which the expressions could be computed.

However, relational thinking became relatively frequent as we started to promote the use of this type of thinking. In sessions 1 and 2 we detected the first two examples which were displayed:

> In $12+7=7+\square$, a student explained that the missing number was 12 because "It is the same number [R: Is it the same number? What did happen?] They just changed the order of the numbers".

[7] All the examples of students´ explanations provided in this paper have been translated from Spanish to English.

Another student wrote the following explanation in the sentence $9-4=\square-3$: "I have used my mind to do it because $\underset{-1}{9}-\underset{-1}{4}=8-3$ and it gives the same". Her writing under the sentence suggests that she appreciated that if 1 is subtracted from 4 you get 3 and, therefore, she subtracted one from 9 to get the answer, 8.

These explanations demonstrate "knowing-to act" because the students used their previous arithmetic knowledge in a fresh situation. They broke their computational trained behaviour and used their natural powers to discern patterns and recognize relations. Both strategies were specifically constructed by the students attending to the particular characteristics of these sentences.

These students "knew that" the equal sign is used to express equivalence between numeric expressions, i.e., sameness of numeric value. They "knew how" to compute addition and subtraction expressions (at least involving numbers of less than 5 digits). Therefore, they could successfully address this task following a computational approach but also could use a creative/non-trained approach as in the examples shown above which make the most of the special design of the sentences (rich in relations).

When students used relational thinking, their attention was directed to particularities of the sentences— presence of zero, the particular operational signs involved, the presence of operations on both sides— and relations between their terms such as sameness, lack of sameness, difference of one unit between terms and big differences of size between numbers. These observations resonated with their previous experience about, for example, "adding or subtracting zero", "adding and subtracting the same number", "changing the order of the addends in a sum", "the effect of adding/subtracting on the size of numbers". In this way, some previous (implicit or explicit) "knowledge-about" arithmetic structure was flexibly applied in this new context (see further examples in table 1).

Table 1
Examples of students' explanations evidencing use of relational thinking

Sentences	Examples: Students´ explanations
$122+35-35=122$	"True because if we add 122 to 35 and we take it away, it is as if we don't add anything"
$7+7+9=14+9$	"True. I did it by adding seven and seven…. which is fourteen. The same than there [right side]. Nine, the same than there [right side] too"
$13+11=12+12$	"True because you subtract one to the twelve and you give it to the other twelve, and you get what it is there [i.e., the expression on the left side]"
$75-14=340$	"False because 75 minus 14 is less, it cannot be a bigger number"
$11-6=10-5$	"True because if eleven is higher than ten and you subtract one more than five, you get the same"

Occasions for relational thinking

We distinguish three occasions that we observed relational thinking: without computing, while computing and after computing. In the first case, students approached the sentences by attending to its structure and detected particular characteristics or relations between its elements which they used to conclude their answer. They did not need to perform any computation (e.g., In $75+23=23+75$, a student explained *"True because there are the same numbers in one operation and in the other"*. She didn't do any computation).

In other cases, students initiated some computation to obtain the numeric values of both sides but, suddenly, abandoned the computation and changed their approach after appreciating some characteristic of the sentence or some relations between its terms, not previously noticed. Initiating the computation served to make the student aware of the composition of the sentence and pay attention to each of its elements. For example, in the sentence $51+51=50+52$ a student provided the following explanation: *"It is true, because fifty-one plus fifty-one is one hundred and two, but fifty-one, if you subtract* [one], *fifty, you can add* [it] *to the other fifty-one, one more, and you get fifty-two"*. He initially computed the expression on the left side, but then appreciated a compensation relation between the operations in both sides and used it to conclude his answer, without computing the expression on the right side. In the sentence $75+23=23+75$, a student began by writing the numbers in a vertical format and then did not even start computing as she explained, *"It is true because it is the same"*. Her explanation suggests that she appreciated some sameness between the expressions on each side of the equal sign which allowed her to conclude that the sentence is true without knowing the numeric values of each side.

In other cases, students first solved the sentence by computing and comparing the numeric values of both sides, and afterwards explained another way of concluding the truth or falseness of the sentence which was based on some noticed relations (e.g., In the sentence $7+15=8+15$, having computed the addition on both sides, a student explained "False because you don't get the same [result] and, also [because] seven is lower").

These examples illustrate how shifts in attention happen instantaneously. Mason & Spence (1999) related this to a "bolt of lightening" when patterns emerge suddenly and all of a sudden students "know-to act". Relational thinking does not always precede or is opposite to computation. As the above examples show, it can occur during computation as students have an insight about their computation, allowing them to abandon it. Working on the computations was helpful for some students to become aware of the structure and components of the sentence and perceive relations between them which they may use to solve the sentence. Their "knowing-to act" demonstrates that during the computation process they had some free attention to attend to these details which resonated with past experience.

Most students demonstrated some use of relational thinking, in each of these three ways, at some point during the teaching experiment; however, it was alternated with a computational approach. They advanced in this regard throughout the sessions being strongly influenced by social appreciation of explanations based on this type of thinking.

Students´ "knowing-to act" during the sessions

Of the 26 students, six students used relational thinking frequently from the third session on, and three of them did so in all the types of sentences considered (according to the arithmetic relation used in their design). Another ten students evidenced some use of relational thinking occasionally. It was based only on some specific relations but all of them appreciated sameness or lack of it (the most basic relations). Ruben was one of the ten students whose “knowing-to act” was based on specific relations: sameness and composition. He noticed repetition of numbers in both sides of the sentences $75+23=23+75$ and $18-7=7-18$ that he used to conclude their truth, and in the sentence $6+4+18=10+18$ (in session 6) he appreciated a composition relation that he used: “True because $6+4=10+18=10+18$”. In his written explanation he incorrectly used the equal sign to chain a sequence of operations. We suspect that he was able to apply relational thinking to this sentence because he noticed that the 18 was the same on both sides leading him to look for a relation between 6 + 4 and 10.

Another example is the case of Maite. Her approach was computational in all the sentences except from her work in $18-7=7-18$, $75+23=23+75$ and $7+15=8+15$: “True because both are the same”, “True because it is equal” and “False because it is almost equal but it is not equal”, respectively. In all these sentences she previously did the computation of the right side or wrote it vertically and stopped before computing. She displayed a tendency to calculating when approaching the sentences but in these three non-action sentences her attention was not completely taken by the computations, allowing her to recognize some sameness between both sides of the sentences.

The other six students showed limited use of relational thinking. Three of them noticed sameness between the terms in a sentence. In the other three it was based on noticing an instantiation of the property $a-a=0$ or the properties of zero as identity element. They also applied the restriction of subtraction in the natural numbers to operations where the minuend was not lower than the subtrahend (i.e., $a-b$ when $a\angle b$).

Only two or three students never provided evidence of using relational thinking. One of these students was Beatriz. As shown in the examples provided in Table 2, she typically computed the numeric values of the expressions on each side of the equal sign by using the vertical standard addition and subtraction algorithms. Like Beatriz, the other two students who did approach all the sentences computationally displayed some difficulties in computing and sometimes did not respect the structure of the sentences and performed computations which combined numbers from different sides of the equal sign (e.g., see Beatriz´s work in the sentence $6+4+18=10+18$). We conjecture that their lower mastery of computational methods did not allow them to have free attention while computing to perceive relations between the terms and probably also to follow and benefit from their peers´ explanations in the whole group discussions. It also caused them to struggle to make sense of non-action sentence which were not familiar to them.

Table 2

Beatriz´s responses to some true/false number sentences

Number sentence	Responses
$18-7=7-18$	False because it doesn’t give the same result (She computes $18-7=11$ and $7-18=19$ by using the vertical standard subtraction algorithm)
$75-14=340$	False because I added and it doesn’t give 340. (She computes $75-14=81$ by using the vertical standard subtraction algorithm)

Table 2
Beatriz´s responses to some true/false number sentences

Number sentence	Responses
$17-12=16-11$	True because I subtracted 17 and 12 and later I subtracted $16-11$.
$122+35-35=122$	True because I added $122+35$ and then I subtracted to the result I got $122-157$ (She computes $122+35=157$ and $157-35=122$ by using the vertical standard algorithms).
$6+4+18=10+18$	False because I added $6+4$ and I added the result to $10+18$ (She computes $4+6=10$, $6+4+18=118$, $118+10+18=146$ by using the vertical standard addition algorithm)

Within parenthesis we describe the student´ computations done in the worksheet.

Role of number sentences

In addition to creating an atmosphere where "knowing-to act" was valued and having discussions about children's ways of looking at those sentences, the type of true/false number sentences considered were an important element in helping students "knowing-to act in the moment."

Number sentences which include zero relations ($a+0=a$; $a-0=a$; $a-a=0$) seemed to be effective tasks to interrupt students´ habituated behaviour. In these sentences the use of relational thinking was facilitated by not having to relate both sides of the sentences. Even those students who were the most likely to compute (although not always) tended not to do so on these types of sentences. Only one student vertically wrote the operation $125-125$ to determine if the sentence $125-125=13$ was true or false. In this case the size of the number seemed to be a problem for him to conceive 125 as a number.

Sentences involving the commutative property also seemed to interrupt habituated behaviour for students who otherwise computed, although not always. In each session, more "knowing-to act" than computational approaches was displayed in this type of sentences. In the discussion of session 3, none of the students solved the sentence $10+4=4+10$ by computing. Various students claimed loudly that it was true and explained: "they had turned around the numbers". In sessions 4 and 6, only 5 and 8 students, respectively, solved the sentence $75+23=23+75$ by computing the numeric value of each side, while 15 out of 22 students in session 4 and 9 out of 20 students in session 6 determined the truth of the sentence by "knowing-to act"[8]. They explained that the numbers were the same and some mentioned the change of order.

In some cases (7 out of 24 students, in both sessions 4 and 6), the appreciation of sameness in sentences such as $18-7=7-18$ lead students to reason erroneously as result of having overgeneralized the commutative property of addition to the case of subtraction (e.g., "True because eighteen minus 7 and the other is the same, and if it is the same they are equal"). Students' "knowing-about" this type of expression may have been limited by their lack of experience. They had been told that subtraction cannot be performed when the minuend is lower than the subtrahend. Therefore, encountering an

[8] In sessions 4 and 6 there were two and three students, respectively, whose approach could not be identified due to lack of details in their explanations and written work.

expression in which they couldn't compute may have led some of them to assume that the commutative property could also be applied here.

In the sentences based on composition/decomposition relation (e.g. $6+4+18=10+18$) as well as on the inverse relation of addition and subtraction (e.g. $122+35-35=122$) half of the students proceeded computationally while the other half used relational thinking. However, in the latter we detected more use of computational approaches when the sentences included small numbers.

In the sentences based on "relative size comparisons" initially, during the whole group discussion of session 3, students evidenced both approaches but computational approached became more frequent in the sessions 4 and 6. This tendency was specially appreciated in the action sentences considered ($72=56-14$; $\boxed{}$; $\boxed{}$) probably because they did not include equals numbers in both sides while the others ($10-7=10-4$; $7+15=8+15$) did.

The sentences based on the compensation relation (e.g., $53+41=54+40$) were the one least frequently approached relationally, especially those involving subtraction (e.g., $9-4=8-3$). In the discussion of the sentences $51+51=50+52$ and $13+11=12+12$ in session 3, three students' "knowing-to act" became evident. However, despite having shared these explanations with the whole group, in the later sessions only four students displayed relational thinking in sentences based on the compensation relationship.

Clearly students used relational thinking more readily in some types of sentences and use of relational thinking was sporadic for most students. Except for three to six students, it was not the case that they had an insight which led them to be on the look-out for relations. Instead, sometimes the relations in the sentence jumped out at them and other times they did not. The data above indicate that when at least some of the numbers on each side of the equal sign are identical, students are inclined to notice relations.

To connect with the reader's experience

As John Mason usually does, now we ask the readers to try an example themselves so that they experience the ideas we are trying to describe. We invite you to solve this algebra example $x^2-2x=4x-8$ before going on reading.

In this equation you might have factored both sides or you might have subtracted $4x$ from the left side, and added 8 to both sides. What one decides to do depends on what one notices. If you noticed that it was quadratic, you might realize that there will be two solutions to this equation. If you noticed that (x – 2) is a factor of both expressions, you might have guessed that 2 is one of the solutions. This factorization would avoid having to apply the formula for solving quadratic equations which constitute the trained behaviour in this context. You might have addressed the equation looking at its structure and its components and searching for relations between both sides and its terms which inform your approach. You might have initiated some manipulation before noticing any particular characteristic or have noticed them when checking your solutions after using the formula for quadratic equations.

The flexible thinking involved in "knowing-to act in the moment" would allow you to weigh your options and gain insight into the nature of the equation. Attending to the structure of the equation allow you to enrich your knowledge of the equation at hand in addition to inform your selection of a solution strategy. In addition possible discussions starting from this "knowing-to" may lead to interesting inquiry about other equations which are similar in some way (e.g., what can we change in the sentence so that 2 is still a solution and the other one is 4? Or -7? Or $\frac{9}{5}$?).

Teaching students to follow a series of steps, as in teaching equations solving, does not help them to learn algebra, or any area of mathematics, properly because there are always exceptions to any given series of steps they might try to memorize, and more important, this learning does not allow them to establish connections with other mathematical concepts therefore limiting their understanding.

CONCLUSIONS

There are some mathematical situations when students get stuck when they do not "know-to act." The tasks we proposed did not have this limitation because students could calculate to successfully obtain an answer. This fact made the tasks accessible to all students but caused the necessity of breaking students´ trained behaviour and changing their disposition when approaching the sentences.

Asking students to determine and justify the truth or falseness of number sentences in a classroom atmosphere where focus was not on numeric results nor on calculations but on recognising and expressing relationships was successful in promoting the display of "knowing-to act" and altering how students attended to number sentences. Some students didn't "know-to act" until they encountered relational strategies when listening to their peers, in Mason and Spence's (1999) words until they were aware of a possibility to act. Computational approaches were the most familiar strategies for them (if not the only for some), so the sharing of the students´ strategies and the teacher's special appreciation of relational strategies was essential to support "knowing-to act".

We were completely successful in developing a habit of looking for relations in the case of three to six students. Most of the students become aware of the existence of non-computational strategies and when some particularities of the sentence resonated with their previous experience, usually some sameness, allowed their attention to focus on the noticed relations and triggered their knowledge-about them to solve the sentence.

Some students never used relational thinking probably because they needed to devote all of their attention to the calculation at hand, so there was insufficient free attention for any metonymic trigger or metaphoric resonance between the elements in the sentences. Some may have needed some assistance or teaching to help them to investigate the effect of some arithmetic relations such as composition/decomposition and compensation in the numeric value of expressions. A particular student required further experience with big numbers to be able to conceive them as numbers.

Although the results show that six sessions were not enough to fully reach our objective, they prove that true/false number sentences of the forms described here can be fruitful in helping students break their "addition addiction". These activities provoked a transformation in the structure of students´ attention and made it more aligned with the requirement of algebraic thinking: attention to the structure of the sentence and to relations between its terms. Students sometimes see relations after starting to compute supporting Mason and Spence's assertion that shifts in attention can happen quite rapidly. Students tend to notice some relations (zero and sameness relations) more readily than others so teachers need to be prepared with a variety of sentences to support students at different levels of development to use relational thinking. In some cases the presence of big numbers eased the appreciation of relations.

Students need to appreciate that "knowing-to act" (in this context, relational thinking) requires shifts in attention, a playful approach to a problem and some creativity. They have to avoid reacting to stimuli with habituated behaviour and reserve habituated behaviour until after they have considered alternative possibilities. This disposition is important to all areas of mathematics but we have observed that discussing true/false number sentences provides some unique opportunities for students to recognize the value of shifting the attention.

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