DE GRUYTER OPEN Open Math. 2015; 13: 321–332

Open Mathematics **Open Access**

Research Article

Konstantina Panagiotidou* and Juan de Dios Pérez

Commuting Conditions of the k-th Cho operator with the structure Jacobi operator of real hypersurfaces in complex space forms

Abstract: In this paper three dimensional real hypersurfaces in non-flat complex space forms whose k-th Cho operator with respect to the structure vector field ξ commutes with the structure Jacobi operator are classified. Furthermore, it is proved that the only three dimensional real hypersurfaces in non-flat complex space forms, whose k-th Cho operator with respect to any vector field X orthogonal to structure vector field commutes with the structure Jacobi operator, are the ruled ones. Finally, results concerning real hypersurfaces in complex hyperbolic space satisfying the above conditions are also provided.

Keywords: Real hypersurfaces, Structure Jacobi operator, Cho operator, Non-flat complex space forms

MSC: 53C40, 53C15, 53D15

DOI 10.1515/math-2015-0032
Received November 19, 2014; accepted April 1, 2015.

1 Introduction

A *complex space form* is an n-dimensional Kähler manifold of constant holomorphic sectional curvature c. A complete and simply connected complex space form is analytically isometric to a complex projective space $\mathbb{C}P^n$ if $c > 0$, a complex Euclidean space $\mathbb{C}^n$ if $c = 0$, or a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$. The complex projective and complex hyperbolic spaces are called non-flat complex space forms, since $c \neq 0$, and the symbol $M_n(c)$ is used to denote them when it is not necessary to distinguish them.

A real hypersurface M is an immersed submanifold with real codimension one in $M_n(c)$. The Kähler structure (J, G), where J is the complex structure and G is the Kähler metric of $M_n(c)$, induces on M an almost contact metric structure (ξ, φ, η, g). The vector field ξ is called *structure vector field* and when it is an eigenvector of the shape operator A with corresponding eigenvalue $\alpha = g(A\xi, \xi)$ the real hypersurface is called *Hopf hypersurface*.

The study of real hypersurfaces M in $M_n(c)$ was initiated by Takagi, who in [13] classified homogeneous real hypersurfaces in $\mathbb{C}P^n$ and divided them into six types, namely (A_1), (A_2), (B), (C), (D) and (E). These real hypersurfaces are Hopf ones with constant principal curvatures. In case of $\mathbb{C}H^n$, the study of real hypersurfaces with constant principal curvatures was started by Montiel [7] and completed by Berndt in [1]. They are divided into two types, namely (A) and (B), depending on the number of constant principal curvatures. The real hypersurfaces found by them are homogeneous and Hopf.

Another important class of real hypersurfaces in $M_n(c)$, which are not Hopf, is the *ruled hypersurfaces*. They are constructed in the following way: consider a regular curve γ in $M_n(c)$ with tangent vector field X. Then at each point of γ there is a unique hyperplane of $M_n(c)$ cutting γ in a way to be orthogonal to both X and JX. The union

***Corresponding Author: Konstantina Panagiotidou:** Faculty of Engineering, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece, E-mail: kapanagi@gen.auth.gr
Juan de Dios Pérez: Departmento de Geometria y Topologia, Universidad de Granada, 18071, Granada, Spain, E-mail: jdperez@ugr.es

of all these hyperplanes is the ruled hypersurface. Equivalently, for ruled hypersurfaces in $M_n(c)$ we have that the maximal holomorphic distribution $\mathbb{D}$ of M at any point, which consists of all the vectors orthogonal to ξ is integrable and it has an integrable manifold $M_{n-1}(c)$, i.e $g(A\mathbb{D},\mathbb{D}) = 0$.

Last years many geometers have studied real hypersurfaces in $M_n(c)$ under certain geometric conditions. More precisely, the *structure Jacobi operator* of them plays an important role in the study. Generally, the Jacobi operator with respect to X on M is defined by $R(\cdot, X)X$, where R is the Riemannian curvature of M. For $X = \xi$ the Jacobi operator is called structure Jacobi operator and is denoted by $l = R_\xi = R(\cdot,\xi)\xi$.

Another topic of great importance in the study of real hypersurfaces in non-flat complex space forms is the study of them in terms of their *generalized Tanaka-Webster connection*. The notion of generalized Tanaka-Webster connection was first introduced by Tanno in [14] in case of contact metric manifolds in the following way

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\varphi Y.$$

In [2] Cho extended Tanno's work by defining the generalized Tanaka-Webster connection of real hypersurfaces M in $M_n(c)$ in the following way

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y, \tag{1}$$

where X, Y are tangent to M and k is non-null real number.

The second author in [12] introduced the notion of *k-th Cho operator* corresponding to a vector field X as a tensor field of type (1,1) defined in the following way

$$F^{(k)}_X Y = g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y. \tag{2}$$

So relation (1) due to (2) becomes

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + F^{(k)}_X Y. \tag{3}$$

Notice that if $X \in \mathbb{D}$, the k-th Cho operator does not depend on k, so it is written $F_X Y$ and is called *Cho operator* associated to X.

In [12] the second author began the study of real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, whose k-th Cho operator satisfies commuting conditions with the structure Jacobi operator of them. More precisely, he classifed real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, whose structure Jacobi operator satisifies the commuting relation $F^{(k)}_\xi l = lF^{(k)}_\xi$. Furthemore, he also proved that the structure Jacobi operator commutes with the Cho operator, i.e. $F_X l = lF_X$, only for ruled hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$. The condition $F^{(k)}_X l = lF^{(k)}_X$, for some $X \in TM$ is equivalent to $\nabla_X l = \hat{\nabla}^{(k)}_X l$. Geometrically, this means that any eigenspace of l is preserved by $F^{(k)}_X$.

The purpose of this paper is to extend the previous work in case of three dimensional real hypersurfaces in $M_2(c)$ and in case of real hypersurfaces in $\mathbb{C}H^n$, $n \geq 3$. More precisely the following Theorems are proved:

Theorem 1.1. *Every real hypersurface M in $M_2(c)$, whose k-th Cho operator associated to ξ commutes with the structure Jacobi operator, i.e. $F^{(k)}_\xi l = lF^{(k)}_\xi$ is a Hopf hypersurface. Furthermore, for any non-null constant k M is locally congruent to:*

i) *a real hypersurface of type (A),*
ii) *or to a real hypersurface with $A\xi = 0$.*

Theorem 1.2. *Every real hypersurface M in $\mathbb{C}H^n$, $n \geq 3$, whose k-th Cho operator associated to ξ commutes with the structure Jacobi operator, i.e. $F^{(k)}_\xi l = lF^{(k)}_\xi$ is a Hopf hypersurface. Furthermore, for any non-null constant k M is locally congruent to:*

i) *a real hypersurface of type (A),*
ii) *or to a real hypersurface with $A\xi = 0$.*

Theorem 1.3. *Let M be a real hypersurface in $M_2(c)$, whose Cho operator associated to any vector field X orthogonal to ξ commutes with the structure Jacobi operator, i.e. $F_X l = lF_X$. Then M is locally congruent to a ruled real hypersurface.*

Theorem 1.4. *Let M be a real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, whose Cho operator associated to any vector field X orthogonal to ξ commutes with the structure Jacobi operator, i.e. $F_X l = l F_X$. Then M is locally congruent to a ruled real hypersurface.*

This paper is organized as follows: In Section 2 basic relations and results about real hypersurfaces in $M_n(c)$ are provided. In Section 3 proofs of Theorems 1.1 and 1.2 are provided. Furthermore, a Proposition which holds for non-Hopf hypersurfaces in $M_n(c)$, $n \geq 3$ is also presented. Finally, in Section 4 the proof of Theorem 1.3 is included.

Remark 1.5. *As an immediate consequence of relation* (3) *we have that the condition of commutativity of the k-th Cho operator with respect to any vector field X with the structure Jacobi operator is equivalent with the condition of coincidence of the covariant and generalized Tanaka-Webster derivatives of the structure Jacobi operator, i.e.* $\hat{\nabla}_X^{(k)} l = \nabla_X l$. *Therefore, from Theorems 1.1 and 1.3 there do not exist real hypersurfaces in* $M_2(c)$ *such that* $\hat{\nabla}^{(k)} l = \nabla l$ *for any* $k \in \mathbb{R} - \{0\}$.

2 Preliminaries

Throughout this paper all manifolds, vector fields etc. are assumed to be of class C^∞ and all manifolds are assumed to be connected. Furthermore, the real hypersurfaces M are supposed to be without boundary.

Let M be a real hypersurface immersed in $(M_n(c), G)$ with complex structure J of constant holomorphic sectional curvature c. Let N be a unit normal vector field on M and $\xi = -JN$ the structure vector field of M.

For a vector field X tangent to M relation

$$JX = \varphi X + \eta(X)N$$

holds, where φX and $\eta(X)N$ are respectively the tangential and the normal component of JX. The Riemannian connections $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M by

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

where g is the Riemannian metric induced from the metric G.

The shape operator A of the real hypersurface M in $M_n(c)$ with respect to N is given by

$$\overline{\nabla}_X N = -AX.$$

The real hypersurface M has an almost contact metric structure (φ, ξ, η, g) induced from J on $M_n(c)$, where φ is the *structure tensor* which is a tensor field of type (1,1) and η is an 1-form on M such that

$$g(\varphi X, Y) = G(JX, Y), \qquad \eta(X) = g(X, \xi) = G(JX, N).$$

Moreover, the following relations hold

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta \circ \varphi = 0, \qquad \varphi\xi = 0, \qquad \eta(\xi) = 1,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = -g(\varphi X, Y).$$

The fact that J is parallel implies $\bar{\nabla} J = 0$. The last relation leads to

$$\nabla_X \xi = \varphi A X, \qquad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{4}$$

The ambient space $M_n(c)$ is of constant holomorphic sectional curvature c and this results in the Gauss and Codazzi equations to be given respectively by

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X \tag{5}$$
$$-g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z] + g(AY, Z)AX - g(AX, Z)AY,$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi], \tag{6}$$

where R denotes the Riemannian curvature tensor on M and X, Y, Z are any vector fields on M.

Relation (5) implies that the structure Jacobi operator l is given by

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(AX)A\xi, \tag{7}$$

for any X tangent vector to M, where $\alpha = \eta(A\xi)$.

The tangent space $T_P M$, for every point $P \in M$, can be decomposed as

$$T_P M = span\{\xi\} \oplus \mathbb{D},$$

where $\mathbb{D} = \ker\eta = \{X \in T_P M : \eta(X) = 0\}$ and is called (maximal) *holomorphic distribution*, (if $n \geq 3$). Due to the above decomposition, the vector field $A\xi$ can be written

$$A\xi = \alpha\xi + \beta U,$$

where $\beta = |\varphi\nabla_\xi\xi|$ and $U = -\frac{1}{\beta}\varphi\nabla_\xi\xi \in \ker(\eta)$ is a unit vector field, provided that $\beta \neq 0$.

We provide the following Theorem which in case of $\mathbb{C}P^n$ is owed to Maeda [6] and in case of $\mathbb{C}H^n$ is owed to Montiel [7] (also Corollary 2.3 in [9]).

Theorem 2.1. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$. Then*

i) *α is constant.*

ii) *If W is a vector field which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then*

$$(\lambda - \frac{\alpha}{2})A\varphi W = (\frac{\lambda\alpha}{2} + \frac{c}{4})\varphi W.$$

iii) *If the vector field W satisfies $AW = \lambda W$ and $A\varphi W = \nu\varphi W$ then*

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{8}$$

Remark 2.2. *In case of real hypersurfaces of dimension greater than three the third case of Theorem 2.1 occurs when $\alpha^2 + c \neq 0$, since in this case relation $\lambda \neq \frac{\alpha}{2}$ holds. Furthermore, the first of* (4) *and* (7) *for $X = W$ and $X = \varphi W$ respectively implies*

$$\nabla_W\xi = \lambda\varphi W \ \text{ and } \ \nabla_{\varphi W}\xi = -\nu W, \tag{9}$$

$$lW = (\frac{c}{4} + \alpha\lambda)W \ \text{ and } \ l\varphi W = (\frac{c}{4} + \alpha\nu)\varphi W. \tag{10}$$

Remark 2.3. *In case of three dimensional Hopf hypersurfaces we can always consider a local orthonormal basis $\{W, \varphi W, \xi\}$ at some point $P \in$ M such that $AW = \lambda W$ and $A\varphi W = \nu\varphi W$. So relations* (8)*,* (9) *and* (10) *hold.*

Finally, the following Theorem plays an important role in the study of real hypersurfaces in $M_n(c)$, which is due to Okumura in case of $\mathbb{C}P^n$ (see [10]) and to Montiel and Romero in case of $\mathbb{C}H^n$ (see [8]). It provides the classification of real hypersurfaces in $M_n(c)$, $n \geq 2$, whose shape operator commutes with the structure tensor field φ.

Theorem 2.4. *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$. Then $A\varphi = \varphi A$, if and only if M is locally congruent to a homogeneous real hypersurface of type (A). More precisely:*

In case of $\mathbb{C}P^n$

(A_1) a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,

(A_2) a tube of radius r over a totally geodesic $\mathbb{C}P^k$, $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$.

In case of $\mathbb{C}H^n$

(A_0) a horosphere in $\mathbb{C}H^n$, i.e a Montiel tube,

(A_1) a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}$,

(A_2) a tube over a totally geodesic $\mathbb{C}H^k$ $(1 \leq k \leq n-2)$.

2.1 Auxilary facts about three dimensional real hypersurfaces in complex space forms

Let M be a non-Hopf hypersurface in $M_2(c)$ and $\{U, \varphi U, \xi\}$ be a local orthonormal basis at some point P of M. Then the following Lemma holds

Lemma 2.5. *Let M be a non-Hopf real hypersurface in $M_2(c)$. The following relations hold on M*

$$\begin{aligned} &AU = \gamma U + \delta\varphi U + \beta\xi, \quad A\varphi U = \delta U + \mu\varphi U, \quad A\xi = \alpha\xi + \beta U, \\ &\nabla_U \xi = -\delta U + \gamma\varphi U, \quad \nabla_{\varphi U}\xi = -\mu U + \delta\varphi U, \quad \nabla_\xi \xi = \beta\varphi U, \\ &\nabla_U U = \kappa_1 \varphi U + \delta\xi, \quad \nabla_{\varphi U} U = \kappa_2 \varphi U + \mu\xi, \quad \nabla_\xi U = \kappa_3 \varphi U, \\ &\nabla_U \varphi U = -\kappa_1 U - \gamma\xi, \quad \nabla_{\varphi U}\varphi U = -\kappa_2 U - \delta\xi, \quad \nabla_\xi \varphi U = -\kappa_3 U - \beta\xi, \end{aligned} \tag{11}$$

where α, β, $\gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M and $\beta \neq 0$.

Remark 2.6. *The proof of Lemma 2.5 is included in [11].*

The structure Jacobi operator for $X = U$, $X = \varphi U$ and $X = \xi$, due to (11), implies

$$lU = (\frac{c}{4} + \alpha\gamma - \beta^2)U + \alpha\delta\varphi U, \quad l\varphi U = \alpha\delta U + (\frac{c}{4} + \alpha\mu)\varphi U \ \text{ and } \ l\xi = 0. \tag{12}$$

The Codazzi equation (6) for $X \in \{U, \varphi U\}$ and $Y = \xi$ because of Lemma 2.5 implies the following relations

$$U\beta - \xi\gamma = \alpha\delta - 2\delta\kappa_3, \tag{13}$$

$$\xi\delta = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2, \tag{14}$$

$$U\alpha - \xi\beta = -3\beta\delta, \tag{15}$$

$$\xi\mu = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3, \tag{16}$$

$$(\varphi U)\alpha = \alpha\beta + \beta\kappa_3 - 3\beta\mu, \tag{17}$$

$$(\varphi U)\beta = \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu, \tag{18}$$

and for $X = U$ and $Y = \varphi U$

$$U\delta - (\varphi U)\gamma = \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu, \tag{19}$$

$$U\mu - (\varphi U)\delta = \gamma\kappa_2 + \beta\delta - \kappa_2\mu - 2\delta\kappa_1. \tag{20}$$

Furthermore, combination of the Gauss equation (5) with the formula of Riemannian curvature $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, taking into account relations of Lemma 2.5 implies

$$U\kappa_2 - (\varphi U)\kappa_1 = 2\delta^2 - 2\gamma\mu - \kappa_1^2 - \gamma\kappa_3 - \kappa_2^2 - \mu\kappa_3 - c. \tag{21}$$

3 Proof of Theorems 1.1 and 1.2

3.1 Three dimensional real hypersurfaces in $M_2(c)$

Let M be a three-dimensional real hypersurface in $M_2(c)$ whose k-th Cho operator associated to ξ commutes with the structure Jacobi operator, i.e.

$$F_\xi^{(k)} lY = lF_\xi^{(k)} Y,$$

for any $Y \in TM$. The above relation due to (2) for $X = \xi$ implies

$$k\varphi lY = \beta g(\varphi U, lY)\xi + \beta\eta(Y)l\varphi U + kl\varphi Y. \tag{22}$$

Let $\mathcal{N}$ be the open subset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0 \text{ in a neighborhood of } P\}.$$

On $\mathcal{N}$ relation (22) for $Y = \xi$ taking into account the third of (12) implies $l\varphi U = 0$ and because of the latter relation (22) for $Y = \varphi U$ yields $lU = 0$. So on $\mathcal{N}$ relations $l\xi = lU = l\varphi U = 0$ hold, i.e. $l = 0$ and due to Proposition 8 in [4] we conclude that $\mathcal{N}$ is empty. Thus, the following Proposition is proved:

Proposition 3.1. *Every real hypersurface in $M_2(c)$ whose structure Jacobi operator satisfies relation* (22) *is Hopf.*

Because of the above Proposition relation of Theorem 2.1 and Remarks 2.2 and 2.3 hold.

Relation (22) for $Y = W$ due to (10) results in

$$\alpha(\lambda - \nu) = 0.$$

Thus, locally either $\alpha = 0$ or $\lambda = \nu$. If $\alpha = 0$ in case of $\mathbb{C}P^2$ we have two cases:

1) if $\lambda \neq \nu$ then M is locally congruent to a non-homegeneous real hypersurface considered as a tube of radius $r = \frac{\pi}{4}$ over a holomorphic curve,

2) if $\lambda = \nu$ then M is locally congruent to a geodesic hypersphere of radius $r = \frac{\pi}{4}$.

In case of $\mathbb{C}H^2$ if $\alpha = 0$ M is a Hopf hypersurface with $A\xi = 0$ (for the construction of such real hypersurfaces see [5]).

If $\alpha \neq 0$ then $\lambda = \nu$ and this implies

$$(A\varphi - \varphi A)X = 0$$

for any X tangent to M. So due to Theorem 2.4 M is locally congruent to a real hypersurface of type (A) and this completes the proof of Theorem 1.1.

3.2 Real hypersurfaces in $\mathbb{C}H^n$, $n \geq 3$

First we provide the following Proposition which holds for non-Hopf hypersurfaces in $M_n(c)$, $n \geq 3$.

Proposition 3.2. *There do not exist real hypersurfaces M in $M_n(c)$, $n \geq 3$, whose shape operator is given by*

$$AU = (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})U + \beta\xi, \quad A\varphi U = -\frac{c}{4\alpha}\varphi U \quad \text{and} \quad A\xi = \alpha\xi + \beta U,$$

if $\nabla_\xi U = \kappa_3\varphi U$, where $\kappa_3 = g(\nabla_\xi U, \varphi U)$ and α, β are non-vanishing functions on M.

Proof. The inner product of Codazzi equation, because of the relation for the shape operator yields:

$$\frac{\beta^2\kappa_3}{\alpha} = \beta\kappa_1 + \frac{c}{4\alpha}(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}), \quad \text{for } X = U \text{ and } Y = \xi \text{ with } \varphi U, \tag{23}$$

$$(\varphi U)\beta = \beta^2 + \beta\kappa_1 + \frac{c}{2\alpha}(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}), \quad \text{for } X = \varphi U \text{ and } Y = \xi \text{ with } U \text{ due to (23)}, \tag{24}$$

$$(\varphi U)\alpha = \beta(\alpha + \kappa_3 + \frac{3c}{4\alpha}), \quad \text{for } X = \varphi U \text{ and } Y = \xi \text{ with } \xi, \tag{25}$$

$$\xi\alpha = \frac{4\alpha^2\beta\kappa_2}{c}, \quad \text{for } X = \varphi U \text{ and } Y = \xi \text{ with } \varphi U, \tag{26}$$

$$(\varphi U)(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}) = \beta(\frac{\beta^2}{\alpha} + \frac{\beta\kappa_1}{\alpha} - \frac{3c}{4\alpha}), \quad \text{for } X = U \text{ and } Y = \varphi U \text{ with } U, \tag{27}$$

$$U\alpha = \frac{4\alpha\beta^2\kappa_2}{c}, \quad \text{for } X = U \text{ and } Y = \varphi U \text{ with } \varphi U, \tag{28}$$

$$U\alpha = \xi\beta = \frac{4\alpha\beta^2\kappa_2}{c}, \quad \text{for } X = U \text{ and } Y = \xi \text{ with } \xi \text{ due to (28)}, \tag{29}$$

$$U\beta = \beta\kappa_2(\frac{4\beta^2}{c} + 1), \quad \text{for } X = U \text{ and } Y = \xi \text{ with } U \text{ due to (26) and (29)}, \tag{30}$$

where $\kappa_1 = g(\nabla_U U, \varphi U)$, $\kappa_2 = g(\nabla_{\varphi U} U, \varphi U)$ and $\kappa_3 = g(\nabla_\xi U, \varphi U)$.

Relation (27), because of (23), (25) and (24), yields:

$$\kappa_3 = -4\alpha, \tag{31}$$

and so relation (23) becomes:

$$\beta\kappa_1 = \frac{c}{4\alpha}(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}) - 4\beta^2. \tag{32}$$

The Riemannian curvature on M satisfies relation (5) and on the other hand is given by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. The combination and the inner product of these two relations for $X = Z = U$, $Y = \xi$ with φU and $X = \xi$, $Y = \varphi U$, $Z = U$ with φU, owing to $\nabla_\xi(\varphi U) = (\nabla_\xi\varphi)U + \varphi\nabla_\xi U$ and the second of (4) implies respectively:

$$U\kappa_3 - \xi\kappa_1 = \kappa_2(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} - \kappa_3), \tag{33}$$

$$(\varphi U)\kappa_3 - \xi\kappa_2 = \kappa_1(\kappa_3 + \frac{c}{4\alpha}) + \beta(\kappa_3 - \frac{c}{2\alpha}). \tag{34}$$

Differentiating (31) and (32) with respect to U and ξ respectively and substituting in (33) and due to (29), (26) and (31) we obtain:

$$\kappa_2(c - 2\beta^2 - 4\alpha^2) = 0. \tag{35}$$

Owing to (35), suppose that $\kappa_2 \neq 0$ then $2\beta^2 + 4\alpha^2 = c$. Differentiation of the last relation along ξ and taking into account (29), (26) and $2\beta^2 + 4\alpha^2 = c$ yields $\kappa_2 = 0$, which is a contradiction.

Thus, $\kappa_2 = 0$ and relations (30), (29) and (26) become:

$$U\alpha = U\beta = \xi\alpha = \xi\beta = 0.$$

Using the above relations and (31) we obtain:

$$[U,\xi]\alpha = U(\xi\alpha) - \xi(U\alpha) = 0,$$

$$[U,\xi]\alpha = (\nabla_U\xi - \nabla_\xi U)\alpha = \frac{1}{4\alpha}(4\beta^2 + 16\alpha^2 - c)(\varphi U)\alpha.$$

Combining the last two relations we have:

$$(4\beta^2 + 16\alpha^2 - c)(\varphi U)\alpha = 0.$$

Suppose that $(\varphi U)\alpha \neq 0$ then the above relation implies $16\alpha^2 + 4\beta^2 = c$. Differentiating the last relation with respect to φU and taking into account (25), (24), (31), (32) and $c = 16\alpha^2 + 4\beta^2$, implies: $\alpha^2 = 0$, which is impossible.

So $(\varphi U)\alpha = 0$. Then, relations (25), (31) and (32) imply: $c = 4\alpha^2$ and $\beta\kappa_1 = \alpha^2 - 5\beta^2$. On the other hand from relation (34), because of (31), we obtain: $\kappa_1 = -2\beta$. Substitution of κ_1 in $\beta\kappa_1 = \alpha^2 - 5\beta^2$ yields: $3\beta^2 = \alpha^2$. Taking the covariant derivative along φU of $3\beta^2 = \alpha^2$, because of (24), we conclude: $\beta = 0$ which is a contradiction and this completes the proof of the present Proposition. □

Let M be a real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, whose structure Jacobi operator commutes with the k-th Cho operator associated to ξ. In this case relation (22) also holds. In $\mathbb{C}H^n$ relation $c = -4$ holds.

Let $\mathcal{N}$ be the open subset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0 \text{ in a neighborhood of } P\}.$$

Following similar steps to those in case of three dimensional real hypersurfaces and taking into account relation (7) for $X = U$ and $X = \varphi U$ we obtain $lU = l\varphi U = 0$. If $\alpha \neq 0$ the latter implies

$$AU = (\frac{\beta^2 + 1}{\alpha})U + \beta\xi \text{ and } A\varphi U = \frac{1}{\alpha}\varphi U.$$

The above relation leads to the conclusion that $\mathbb{D}_U$, which is the orthogonal complement to span$\{U, \varphi U, \xi\}$, is A-invariant.

Let $Z \in \mathbb{D}_U$ such that $AZ = tZ$ then relation (22) for $Y = Z$ implies $\varphi lZ = l\varphi Z$. The last one because of relation (7) yields $\alpha(\varphi A - A\varphi)Z = 0$ and this results in $\varphi AZ = A\varphi Z$. Since $AZ = tZ$ we obtain $A\varphi Z = t\varphi Z$, for any $Z \in \mathbb{D}_U$. The inner product of Codazzi equation for any $X = Z \in \mathbb{D}_U$ and $Y = \xi$ with U and ξ respectively implies

$$(\frac{\beta^2 + 1}{\alpha} - t)g(\nabla_\xi U, Z) = 0 \text{ and } Z\alpha = \beta g(\nabla_\xi U, Z).$$

Suppose that $t_1 = g(\nabla_\xi U, Z) \neq 0$ then the above relation implies $t = \frac{\beta^2+1}{\alpha}$. The inner product of the Codazzi equation for $X = \xi$ and $Y = U$ with Z yields $g(\nabla_U U, Z) = g(\nabla_U Z, U) = 0$. Furthermore, the inner product of Codazzi equation for $X = U$ and $Y = Z$ with ξ and U because of the latter yields

$$Z\beta = 0 \text{ and } Z(\frac{\beta^2 + 1}{\alpha}) = 0.$$

The last relation taking into account relations of $Z\beta$ and $Z\alpha$ results in $t_1 = 0$, which is a contradiction. Therefore, we have that $g(\nabla_\xi U, Z) = 0$ and that $\nabla_\xi U = \kappa_3 \varphi U$.

Due to Proposition 3.2 we conclude that on $\mathcal{N}$ relation $\alpha = 0$ holds. Relation (7) for $X = \varphi U$ implies $l\varphi U = -\varphi U$. On the other hand relation (22) for $Y = \xi$ results in $l\varphi U = 0$. Combination of the last two relations leads to a contradiction. Thus, $\mathcal{N}$ is empty and the following Proposition is proved:

Proposition 3.3. *Every real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, whose structure Jacobi operator satisfies relation* (22)*, is Hopf.*

Since M is a Hopf hypersurface we consider two cases
Case I: $\alpha^2 - 4 \neq 0$.
In this case relations of Theorem 2.1 and remark 2.2 hold. Following similar steps to those of the case of three dimensional real hypersurfaces we obtain

$$\alpha(\lambda - \nu) = 0.$$

So we have $A\varphi X = \varphi AX$ and Theorem 2.4 holds.
Case II: $\alpha^2 - 4 = 0$.
Suppose that $\lambda \neq 1$ then $A\varphi W = \nu\varphi W$ and (8) results in $\nu = 1$. Following similar steps as in the previous case we lead to a contradiction.

Therefore, $\lambda = 1$ is the only eigenvalue for all vector fields in $\mathbb{D}$ and M is locally congruent to a horosphere and this completes the proof of Theorem 1.2.

Remark 3.4. *For real hypersurfaces in $\mathbb{C}H^n$ of dimension greater than three it is known that none of real hypersurfaces of type (A) satisfy $\alpha = 0$, but the authors do not know if there exist Hopf hypersurfaces with vanishing α.*

4 Proof of Theorem 1.3

Let M be a three-dimensional real hypersurface in $M_2(c)$ whose k-th Cho operator associated to any vector $X \in \mathbb{D}$ commutes with the structure Jacobi operator, i.e.

$$F_X lY = lF_X Y,$$

for any $Y \in TM$. The above relation due to (2) for $X \in \mathbb{D}$ implies

$$g(\varphi AX, lY)\xi = -\eta(Y)l\varphi AX. \tag{36}$$

Suppose M is a Hopf hypersurface then Theorem 2.1 and remarks 2.2 and 2.3 hold.

Relation (36) for $Y = \xi$ implies

$$l\varphi AX = 0, \quad \text{for } X \in \mathbb{D}. \tag{37}$$

Relation (37) for $X = W$ and $X = \varphi W$ because of (10) yields respectively

$$\lambda(\frac{c}{4} + \alpha\nu) = 0 \text{ and } \nu(\frac{c}{4} + \alpha\lambda) = 0.$$

Combination of the above two relations implies that $\lambda = \nu$ and so $A\varphi X = \varphi AX$, for any $X \in TM$. Thus, because of Theorem 2.4 M is locally congruent to a real hypersurface of type (A). Furthermore, the above relation results in

$$\lambda(\frac{c}{4\alpha} + \alpha\lambda) = 0.$$

Therefore, locally either relation $\lambda = 0$ or relation $\lambda = -\frac{c}{4\alpha}$ holds. Substitution of the previous values in (8) and taking into account $\lambda = \nu$ leads to a contradiction. Therefore, we conclude:

Proposition 4.1. *There do not exist Hopf hypersurfaces in $M_2(c)$ whose Cho operator corresponding to any $X \in \mathbb{D}$ commutes with the structure Jacobi operator.*

Next we examine non-Hopf hypersurfaces. Let $\{U, \varphi U, \xi\}$ be a local orthonormal basis at some point $P \in M$. The shape operator with respect to this basis is given by (11).

Relation (36) for $Y = \xi$ implies

$$l\varphi AX = 0, \quad \text{for } X \in \mathbb{D}. \tag{38}$$

The inner product of relation (38) for $X = \varphi U$ with U due to (11) and (12) yields

$$\alpha\delta = 0 \text{ and } \mu(\frac{c}{4} + \alpha\gamma - \beta^2) = 0.$$

Suppose that $\mu \neq 0$ then the latter implies that $\frac{c}{4} + \alpha\gamma - \beta^2 = 0$ and the first of (12) results in $lU = 0$. Relation (38) for $X = U$ implies $\gamma(\frac{c}{4} + \alpha\mu) = 0$. If $\gamma \neq 0$ the last relastion results in $\frac{c}{4} + \alpha\mu = 0$ and the second of (12) leads to $l\varphi U = 0$. So the structure Jacobi operator vanishes identically and because of Proposition 8 in [4] we obtain a contradiction. Therefore, $\gamma = 0$ and $\beta^2 = \frac{c}{4}$. Differentiation of the latter with respect to φU implies $(\varphi U)\beta = 0$ and relation (18) implies

$$\beta\kappa_1 + \frac{c}{2} + \alpha\mu = 0. \tag{39}$$

Furthermore, differentiating $\gamma = 0$ with respect to φU and taking into account relation (19) yields

$$\kappa_1 = 2\beta. \tag{40}$$

Relations (13), (15) due to $\gamma = 0$ and $\beta^2 = \frac{c}{4}$ implies

$$U\alpha = U\beta = \xi\beta = \xi\gamma = 0.$$

Differentiation of (39) with respect to U because of the above relation yields $\alpha(U\mu) = 0$. Suppose that $U\mu \neq 0$ then the latter implies $\alpha = 0$ and relation (39) because of (40) and $\beta^2 = \frac{c}{4}$ results in $c = 0$, which is a contradiction. So $U\mu = 0$ and relation (20) results in $\kappa_2 = 0$.

Concluding we have the following

$$\beta^2 = \frac{c}{4}, \quad \kappa_1 = 2\beta, \quad \gamma = \kappa_2 = 0 \text{ and } \mu \neq 0.$$

Relation (14), taking into account all the above relations, yields $\mu\kappa_3 + \frac{c}{2} = 0$. Relation (21), taking into account all the previous relations, implies that $c = 0$, which is a contradiction.

So, on M relation $\mu = 0$ holds and relation (38) for $X = U$ implies $\gamma = 0$. Therefore, the shape operator becomes

$$A\xi = \alpha\xi + \beta U, \quad AU = \beta\xi, \quad A\varphi U = 0.$$

So M is a ruled real hypesurface and this completes the proof of Theorem 1.3.

4.1 Real hypersurfaces in $\mathbb{C}H^n, n \geq 3$

Let M be a real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, whose structure Jacobi operator commutes with the Cho operator associated to any vector field $X \in \mathbb{D}$. In this case relation (36) and relation $c = -4$ hold.

Let M be a Hopf hypersurface. We consider the following two cases:

Case I: $\alpha^2 - 4 \neq 0$.

In this case relations of Theorem 2.1 and remark 2.2 hold. Following similar steps to those in the proof of Theorem 1.3 it is proved that there do not exist Hopf hypersurfaces in $\mathbb{C}H^n, n \geq 3$, whose structure Jacobi operator commutes with Cho operator associated to $X \in \mathbb{D}$.

Case II: $\alpha^2 - 4 = 0$.

If $\lambda \neq 1$ then because of Theorem 2.1 we have $A\varphi W = \nu\varphi W$ and relation (8) implies $\nu = 1$. Relation (36) for $Y = \xi$ yields $l\varphi AX = 0$, for any $X \in \mathbb{D}$. The latter for $X = W$ and for $X = \varphi W$ respectively implies $\lambda = 0$ and $\lambda = \frac{1}{2}$. Combination of the last two relations gives a contradiction.

Thus, $\lambda = 1$ is the only eigenvalue for all vector fields $X \in \mathbb{D}$. In this case following similar steps to those of the case of $\lambda \neq 1$ results in $1 = 0$, which is impossible.

Therefore, the following Proposition is proved.

Proposition 4.2. *There do not exist Hopf hypersurfaces in $\mathbb{C}H^n, n \geq 3$, whose structure Jacobi operator commutes with Cho operator associated to any $X \in \mathbb{D}$.*

Next, we examine non-Hopf hypersurface in $\mathbb{C}H^n, n \geq 3$, whose structure Jacobi operator commutes with Cho operator associated to any $X \in \mathbb{D}$, i.e. relation (36) holds. Relation (36) for $Y = \xi$ since $l\xi = 0$ yields $l\varphi AX = 0$, for any $X \in \mathbb{D}$. The latter due to relation (7) implies

$$-\varphi AX + \alpha A\varphi AX + \alpha\beta g(A\varphi U, X)\xi + \beta^2 g(A\varphi U, X)U = 0, \quad \text{for any } X \in \mathbb{D}. \tag{41}$$

The inner product of relation (41) for any X orthogonal to $\{U, \xi\}$ results in

$$g(AX, \varphi X) = 0.$$

The above relation for $X = \varphi U$ implies $g(AU, \varphi U) = g(A\varphi U, U) = 0$. Relation (41) for $X = U$ due to the last relation yields

$$\varphi AU = \alpha A\varphi AU.$$

The inner product of relation (41) for any $X \in \mathbb{D}$ with U because of the above relation results in

$$g((1 + \beta^2)A\varphi U - \varphi AU, X) = 0.$$

So the vector field $(1+\beta^2)A\varphi U - \varphi AU$ has no component in $\mathbb{D}$ and since $g((1+\beta^2)A\varphi U - \varphi AU, \xi) = 0$ we conclude that

$$\varphi AU = (1+\beta^2)A\varphi U. \tag{42}$$

Let $\mathbb{D}_U$ be the orthogonal complement to span$\{U, \varphi U, \xi\}$. The inner product of relation (36) for $X \in \mathbb{D}$ with $Y \in \mathbb{D}_U$ implies $g(A\varphi Y - \alpha A\varphi AY, X) = 0$. So the vector field $A\varphi Y - \alpha A\varphi AY$ has no component on $\mathbb{D}$ and since $g(A\varphi Y - \alpha A\varphi AY, \xi) = \alpha\beta g(A\varphi U, Y)\xi$ we conclude that

$$A\varphi Y - \alpha A\varphi AY = \alpha\beta g(A\varphi U, Y)\xi, \tag{43}$$

for any $Y \in \mathbb{D}_U$. Combination of relation (41) for $Y \in \mathbb{D}_U$ with above relation implies $A\varphi Y + \beta^2 g(A\varphi U, Y)U - \varphi AY = 0$. The inner product of the latter with φU taking into account relation (42) implies

$$(1+\beta^2)g(\varphi A\varphi U, Y) = 0,$$

for any $Y \in \mathbb{D}_U$. The above relation for $Y = \varphi Y$ results in $g(A\varphi U, Y) = 0$. So $A\varphi U$ has no component in $\mathbb{D}_U$ and because of (42) AU also has no component in $\mathbb{D}_U$. The latter implies that $\mathbb{D}_U$ is invariant by A. Furthemore, the shape operator on U and φU takes the form

$$AU = (\beta^2+1)\mu U + \beta\xi \ \text{ and } \ A\varphi U = \mu\varphi U.$$

Relation (41) for $X = \varphi U$ because of the above yields

$$\mu(1-\alpha\mu) = 0.$$

If $(1-\alpha\mu) \neq 0$ then $\mu = 0$ and $A\varphi U = 0$ and $AU = \beta\xi$. Consider a vector field $Z \in \mathbb{D}_U$ such that $AZ = tZ$. Combination of relations (41) and (43) for $X = Z$ results in $A\varphi Z = t\varphi Z$. Thus, relation (41) for $X = Z$ yields $t(1-t\alpha) = 0$.

If $1 - t\alpha \neq 0$ then $t = 0$ and M is a ruled hypersurface.

If $1 - t\alpha = 0$ it is obvious that $\alpha \neq 0$ and this results in $t = \frac{1}{\alpha}$. The inner product of Codazzi equation for $X = Z$ and $Y = \xi$ with U and for $X = \xi$ and $Y = U$ with Z respectively implies

$$Z\beta = \frac{1}{\alpha}g(\nabla_\xi U, Z) \quad Z\beta = \beta g(\nabla_U Z, U).$$

The inner product of Codazzi equation for $X = U$ and $Y = Z$ with ξ due to the above relations results in $Z\beta = 0$ and so $g(\nabla_\xi U, Z) = 0$. The latter implies $\nabla_\xi U$ has only component on φU. The inner product of Codazzi equation for $X = U$ and $Y = \varphi U$ with U implies $g(\nabla_U \varphi U, U) = 0$. So the inner product of Codazzi equation for $X = U$ and $Y = \xi$ with φU results in $\beta^2 + 1 = 0$, which is contradiction.

So the remaining case is that of $\mu = \frac{1}{\alpha}$. The shape operator in this case has the form

$$AU = \frac{1+\beta^2}{\alpha}U + \beta\xi \quad A\varphi U = \frac{1}{\alpha}\varphi U.$$

In this case we also have that $\mathbb{D}_U$ is A-invariant and φ-invariant. So if $AZ = tZ$ then $A\varphi Z = t\varphi Z$. Following similar steps as in the proof of Theorem 1.2 we conclude that the struture Jacobi operator of such real hypersurfaces does not commute with the Cho operator associated to any $X \in \mathbb{D}$ and this completes the proof of Theorem 1.4.

Acknowledgement: This paper is part of the work produced during the first author's visit at the IEMath-Granada and supported by it. The second author is supported by grant Proj. No. NRF-2011-220-C0002 from National Research Foundation of Korea and by MCT-FEDER Grant MTM2010-18099.

References

[1] Berndt J., Real hypersurfaces with constant principal curvatures in complex hyperbolic space, J. Reine Angew. Math., 1989, 395, 132-141

[2] Cho J.T., CR-structures on real hypersurfaces of a complex space form, Publ. Math. Debrecem, 1999, 54, 473-487

[3] Cho J.T., Pseudo-Einstein CR-structures on real hypersurfaces of a complex space form, Hokkaido Math. J., 2008, 37, 1-17

[4] Ivey T.A., Ryan P.J., The structure Jacobi operator for real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$, Results Math., 2009, 56, 473-488

[5] Ivey T.A., Ryan P.J., Hopf hypersurfaces of small Hopf principal curvature in $\mathbb{C}H^2$, Geom. Dedicata, 2009, 141, 147-161

[6] Maeda Y., On real hypersurfaces of a complex projective space, J. Math. Soc. Japan, 1976, 28, 529-540

[7] Montiel S., Real hypersurfaces of a complex hyperbolic space, J. Math. Soc. Japan, 1985, 35, 515-535

[8] Montiel S., Romero A., On some real hypersurfaces of a complex hyperbolic space, Geom. Dedicata, 1986, 20, 245-261

[9] Niebergall R., Ryan P.J., Real hypersurfaces in complex space forms, Math. Sci. Res. Inst. Publ., 1997, 32, 233-305

[10] Okumura M., On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc., 1975, 212, 355-364

[11] Panagiotidou K., Xenos Ph.J, Real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ whose structure Jacobi operator is Lie $\mathbb{D}$-parallel, Note Mat., 2012, 32, 89-99

[12] Pérez J.D., Commutativity of Cho and structure Jacobi operators of a real hypersurface in a complex projective space, Ann. di Mat. (in press), DOI 10.1007/s10231-014-0444-0

[13] Takagi R., On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math., 1973, 10, 495-506

[14] Tanno S., Variational problems on contact Riemannian manifolds, Trans. Am. Math. Soc., 1989, 314, 349-379